

A Counterexample to Beauville's Stability Conjecture

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Abstract

We construct a morphism $f : X \rightarrow Y$ from a genus-two curve to an elliptic curve such that $f_*\mathcal{L}$ is unstable for every $\mathcal{L} \in \text{Pic}^3(X)$. This gives a counterexample to a conjecture of Beauville.

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1 Introduction

Let $f : X \rightarrow Y$ be a finite morphism of smooth projective curves. The stability of vector bundles under pullback and direct image has been studied extensively; see [4, 5, 10, 11, 12, 15, 16, 17]. In particular, direct images need not preserve semistability. In [3], Beauville proposed a conjecture concerning the stability of $f_*\mathcal{E}$ for a *general* vector bundle $\mathcal{E}$ on X and reduced it to the line-bundle case. Namely:

Conjecture 1.1 (Beauville [3]). *Let $f : X \rightarrow Y$ be a finite morphism of smooth projective connected complex curves. For every $d \in \mathbb{Z}$ and a general line bundle $\mathcal{L} \in \text{Pic}^d(X)$, the direct image $f_*\mathcal{L}$ is stable if $g(Y) \geq 2$ and semistable if $g(Y) = 1$.*

Since being stable/semistable is an open condition, it suffices to prove there exists one such $\mathcal{L} \in \text{Pic}^d(X)$. Beauville proved stability/semi-stability in a range controlled by the Euler characteristic, together with several further cases [3, § 3–4]. The conjecture has recently been established when the cover itself is general: Coskun, Larson, and Vogt proved it for f that is *general* in any component of the relevant Hurwitz space [6]. On the other side, we will give a concrete counterexample to Conjecture 1.1 in this paper.

Theorem 1.2. *There exist a smooth projective curve X of genus 2, an elliptic curve Y , and a ramified finite morphism $f : X \rightarrow Y$ of degree 7 such that $f_*\mathcal{L}$ is unstable for every $\mathcal{L} \in \text{Pic}^3(X)$.*

This counterexample is a $(2, 7)$ -elliptic curve of genus 2, in the sense studied by Franciosi, Pardini, and Rollenske [8]. The construction is essentially motivated by the Frey–Kani construction of genus-two curves from isogenies and anti-symplectic torsion gluings [9]. The curve X is equipped with another morphism $b : X \rightarrow B$ to an elliptic curve B of degree 2. These

curves are fitted into the following commutative diagram of fibre products, where the vertical maps v , p , and q are étale covers of degree 3.

$$\begin{array}{ccccc} B' & \xleftarrow{b'} & X' & \xrightarrow{f'} & Y' \\ q \downarrow & & \downarrow v & & \downarrow p \\ B & \xleftarrow{b} & X & \xrightarrow{f} & Y \end{array}$$

We will show in § 3.3 that for every $\mathcal{L} \in \text{Pic}^3(X)$, there are $\mathcal{N} \in \text{Pic}^1(Y')$ and $\mathcal{M} \in \text{Pic}^1(B')$ such that

$$v^*\mathcal{L} = f'^*\mathcal{N} \otimes b'^*\mathcal{M}.$$

Then we will prove in § 3.4 that $\mathcal{A}_{\mathcal{L}} := p_*\mathcal{N}$ admits a nonzero morphism to $f_*\mathcal{L}$. By [1, 3], $\mathcal{A}_{\mathcal{L}} = p_*\mathcal{N}$ is stable. A simple calculation gives:

$$\mu(\mathcal{A}_{\mathcal{L}}) = \frac{1}{3} > \frac{2}{7} = \mu(f_*\mathcal{L}).$$

Therefore, the saturated image of $\mathcal{A}_{\mathcal{L}}$ destabilizes $f_*\mathcal{L}$.

Although the example above disproves Conjecture 1.1, we also prove a complementary positive result for Galois covers. The precise statement is the following.

Theorem 1.3. *Let $h: X \rightarrow Y$ be a finite ramified Galois morphism between smooth projective connected complex curves. Assume that $g(Y) \geq 1$, then there is a nonempty Zariski open dense subset $U_d \subseteq \text{Pic}^d(X)$ for every integer d such that $h_*\mathcal{L}$ is semistable for $\mathcal{L} \in U_d$.*

If h is unramified, then the semistability is already obtained by [3, § 4].

This paper is organized as follows: § 2 provides necessary preliminaries, § 3 gives the degree-7 counterexample and § 4 proves the Galois-cover theorem. Experts can skip § 2 at first.

2 Preliminaries

We work over $\mathbb{C}$. All curves are smooth, projective and connected unless otherwise stated.

2.1 Stability of vector bundles on curves

For a nonzero vector bundle $\mathcal{V}$ on a smooth projective curve, its *slope* is $\mu(\mathcal{V}) := \deg(\mathcal{V})/\text{rk}(\mathcal{V})$.

Definition 2.1. A vector bundle $\mathcal{V}$ is *stable* (respectively, *semistable*) if

$$\mu(\mathcal{V}') > \mu(\mathcal{V}), \quad (\text{respectively, } \mu(\mathcal{V}') \geq \mu(\mathcal{V}))$$

for every proper nonzero quotient bundle $\mathcal{V} \twoheadrightarrow \mathcal{V}'$. We say $\mathcal{V}$ is *unstable*, if it is not semistable.

The following results on stability are well known.

Proposition 2.2. (1) *Let $f: C \rightarrow D$ be a separable morphism of curves, $\mathcal{E}$ be a vector bundle on D , then $\mathcal{E}$ is semistable if and only if $f^*\mathcal{E}$ is so. In particular, $\mathcal{E}$ is unstable if $f^*\mathcal{E}$ is unstable.*

(2) (see [3, § 4]) *Let $f: C \rightarrow D$ be an étale morphism of curves, then for any semistable vector bundle $\mathcal{E}$, $f_*\mathcal{E}$ is semistable.*

2.2 On Abelian varieties

Let A be an Abelian variety, $\hat{A}$ be the dual Abelian variety. Denote by $e_n : A[n] \times \hat{A}[n] \rightarrow \mu_n$ the Weil pair.

2.2.1 On theta group of Abelian varieties

We recall the theory of theta group on Abelian varieties. The standard reference is [13, Chapter IV]. Given an ample line bundle $\mathcal{M}$ on A , translation defines the polarization

$$\lambda_{\mathcal{M}} : A \longrightarrow \hat{A} = \text{Pic}^0(A), \quad a \longmapsto t_a^* \mathcal{M} \otimes \mathcal{M}^{-1},$$

and we write $K(\mathcal{M}) = \ker \lambda_{\mathcal{M}}$ the translation stabilizer. The theta group $\mathcal{G}(\mathcal{M})$ of $\mathcal{M}$ is the central extension

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(\mathcal{M}) \longrightarrow K(\mathcal{M}) \longrightarrow 1,$$

classifying the pair (a, τ) , where $a \in K(\mathcal{M})$ and $\tau : T_a^* \mathcal{M} \simeq \mathcal{M}$ is an isomorphism. Its commutators define the nondegenerated bilinear alternating pairing, known as the *theta pairing*

$$e^{\mathcal{M}} : K(\mathcal{M}) \times K(\mathcal{M}) \longrightarrow \mathbb{C}^*, \quad (a, b) \mapsto \bar{a} \cdot \bar{b} \cdot \bar{a}^{-1} \cdot \bar{b}^{-1},$$

for any lifting $\bar{a}, \bar{b} \in \mathcal{G}(\mathcal{M})$ of $a, b \in K(\mathcal{M})$. A finite subgroup $H \subseteq K(\mathcal{M})$ is *isotropic* if $e^{\mathcal{M}}|_{H \times H} = 1$.

Theorem 2.3 (Chapter IV, §23 [13]). *(1) If $H \subseteq K(\mathcal{M})$ is isotropic, then the line bundle $\mathcal{M}$ descends to the quotient $\pi : A \rightarrow A/H$. If H is furthermore maximal isotropic, the descended polarization is principal.*

(2) We have

$$e_n(a, \lambda_{\mathcal{M}}(b)) = e^{\mathcal{M}^{\otimes n}}(a, b),$$

for any $a \in A[n], b \in [n]^{-1}(K(\mathcal{M}))$.

Remark 2.4. It is well known that the $\lambda_{\mathcal{M}}, K(\mathcal{M}), \mathcal{G}(\mathcal{M})$ and $e^{\mathcal{M}}$ all depend only on the numerical class of $\mathcal{M}$. In particular, in the elliptic curve case, all these depend only on $\deg \mathcal{M}$.

2.3 On elliptic curves

Given an elliptic curve (E, o) , denote by λ_E the canonical principal polarization associated to the line bundle $\mathcal{M} = \mathcal{O}_E(o)$:

$$\lambda_E = \lambda_{\mathcal{M}} : E \simeq \hat{E}, \quad x \mapsto t_x^* \mathcal{M} \otimes \mathcal{M}^{-1} = \mathcal{O}_E(x - o).$$

We denote by $e_{n,E}$ the theta pairing on $E[n]$ giving by any line bundle of degree n (see Remark 2.4). Following from Theorem 2.3(2),

$$e_{n,E}(a, b) = e_n(a, \lambda_E(b)), \quad \forall a, b \in E[n].$$

By abuse of language, we also call this e_n the Weil pairing.

2.4 On the decomposition of cyclic cover of Elliptic curves

Let $\pi_n = [n] : A \rightarrow A$ be the multiplication-by- n map. Then π_n is a Galois cover with Galois group $A[n]$. Following [7, Example 11.18], we have the following decomposition:

$$(\pi_n)_* \mathcal{O}_A = \bigoplus_{\chi} \mathcal{L}_{\chi},$$

where

- χ runs through all multiplicative character of $A[n]$;
- $\mathcal{L}_{\chi}$ is the line subbundle of $(\pi_n)_* \mathcal{O}_A$ so that the translation operator

$$t_a^*|_{\mathcal{L}_{\chi}} = \chi(a) \cdot \text{id}, \quad \forall a \in A[n].$$

- $\mathcal{L}_{\chi} \in \widehat{A}[n]$ is such that $e_n(a, \mathcal{L}_{\chi}) = \chi(a)$, $\forall a \in E[n]$.

Let $u : F \rightarrow E$ be a finite homomorphism of elliptic curves of degree n . Then the multiplication-by- n map of E factors through u as:

$$\pi_n : E \xrightarrow{v} F \xrightarrow{u} E.$$

And thus v is a Galois cover with Galois group $G := \text{Ker}(v)$ and u with Galois group $\text{Ker}(u) = E[n]/G$. The above decomposition of $(\pi_n)_* \mathcal{O}_E$ then gives:

$$u_* \mathcal{O}_F = \bigoplus_{\chi, \chi|_G=1} \mathcal{L}_{\chi} = \left(\bigoplus_{\chi} \mathcal{L}_{\chi} \right)^G \subseteq \bigoplus_{\chi} \mathcal{L}_{\chi} = (\pi_n)_* \mathcal{O}_E.$$

It is classical that v is dual to u in the sense of canonical principal polarizations:

$$v = \lambda_F^{-1} \cdot \widehat{u} \cdot \lambda_E.$$

In other words, given $a \in E[n]$, $\mathcal{M} := \lambda_E(a) = \mathcal{O}_E(a - o_E)$, then

$$u^* \mathcal{M} = \mathcal{O}_F(v(a) - o_F) = \lambda_F(v(a)).$$

In summary, we have

Proposition 2.5. *Given any $\mathcal{M} \in \widehat{E}[n]$, and $b \in \text{Ker}(u)$ then $u^* \mathcal{M} = \lambda_F(b)$ if and only if*

$$\chi(b) = e_{n, \widehat{E}}(\mathcal{L}_{\chi}, \mathcal{M})$$

for all χ with $\chi|_G \equiv 1$.

Proof. We can write $\mathcal{M} = \mathcal{O}_E(a - o_E) = \lambda_E(a)$ for some $a \in E[n]$, then $u^* \mathcal{M} = \lambda_F(b)$ if and only if $b = v(a)$. That is equivalent to

$$\chi(b) = \chi(a) = e_n(\mathcal{L}_{\chi}, a) = e_{n, \widehat{E}}(\mathcal{L}_{\chi}, \lambda_E(a)) = e_{n, \widehat{E}}(\mathcal{L}_{\chi}, \mathcal{M}),$$

for all χ with $\chi_G \equiv 1$. □

3 The counterexample to Conjecture 1.1

3.1 A genus-two Jacobian with two elliptic directions

We now construct the geometric input to Theorem 1.2. The construction is a specialization of the Frey–Kani construction as organized in [8, Example 3.9]. We include the intersection and torsion calculations because they are essential for the counterexample.

Choose an elliptic curve F with $\text{End}(F) = \mathbb{Z}$, and choose a cyclic subgroup $\Gamma_5 \subset F$ of order 5. Put

$$E := F, \quad E' := F/\Gamma_5,$$

and consider

$$\varphi = [3]: F \longrightarrow E, \quad \varphi': F \longrightarrow E' = F/\Gamma_5.$$

Thus

$$\deg \varphi = 9, \quad \deg \varphi' = 5, \quad \ker \varphi \cap \ker \varphi' = F[3] \cap \Gamma_5 = \{0\}.$$

The homomorphism

$$i = (\varphi, \varphi'): F \hookrightarrow E \times E'$$

is therefore an embedding.

Set

$$D_1 := E \times \{0\}, \quad D_2 := \{0\} \times E', \quad H := i(F[2]) \subset (E \times E')[2].$$

and $\mathcal{L}_{2,2} := \mathcal{O}(2D_1 + 2D_2)$ on $E \times E'$.

Lemma 3.1. *We have $K(\mathcal{L}_{2,2}) = (E \times E')[2]$ and the subgroup H is maximal isotropic for the theta pairing $e^{\mathcal{L}_{2,2}}$.*

Proof. The line bundle $\mathcal{L}_{2,2}$ is of type $(2, 2)$ on $E \times E'$, so $K(\mathcal{L}_{2,2}) = (E \times E')[2]$ and

$$e^{\mathcal{L}_{2,2}}(i(a), i(b)) = e_{2,E}(\varphi(a), \varphi(b)) \cdot e_{2,E'}(\varphi'(a), \varphi'(b)).$$

for any $a, b \in F[2]$. On the other hand, following the functoriality of theta pairing, we have

$$e_{2,E}(\varphi(a), \varphi(b)) = e_{2 \cdot \deg \varphi, F}(a, b) = e_{2,F}(a, b)^{\deg \varphi} = e_{2,F}(a, b)^9 = e_{2,F}(a, b),$$

$$e_{2,E'}(\varphi'(a), \varphi'(b)) = e_{2 \cdot \deg \varphi', F}(a, b) = e_{2,F}(a, b)^{\deg \varphi'} = e_{2,F}(a, b)^5 = e_{2,F}(a, b).$$

Thus $e^{\mathcal{L}_{2,2}}(i(a), i(b)) = e_{2,F}(a, b)^2 \equiv 1$. Namely, H is isotropic. It is maximal since H has half $\mathbb{Z}/2\mathbb{Z}$ dimension of $K(\mathcal{L}_{2,2})$. $\square$

Denote by

$$\pi: E \times E' \rightarrow A := (E \times E')/H$$

the quotient Abelian variety by H . Following Theorem 2.3 and Lemma 3.1, $\mathcal{L}_{2,2}$ descends to a principal polarization Θ on A . This is the degree-2 Frey–Kani construction [8, Proposition 2.3].

Lemma 3.2. *The principal polarization Θ is indecomposable. This theta divisor is a smooth curve X of genus 2, and $(A, \Theta) \simeq (J(X), \Theta_X)$.*

Proof. The standard classification of principally polarized abelian surfaces gives two possibilities: either the wanted case: (1) Θ is indecomposable, and $(A, \Theta) \simeq (J(X), \Theta_X)$ for a smooth genus-two curve X ; or the case to be excluded: (2) $\Theta = C_1 + C_2$ were reducible, the C_i would be elliptic curves with $C_1 \cdot C_2 = 1$.

Let $Z_i = \pi(D_i)$, $i = 1, 2$. By construction, the map $\pi : D_i \rightarrow Z_i$ is an isomorphism, hence $\pi^*Z_i \equiv 4D_i$ as $\deg \pi = 4$. So

$$Z_1 \cdot Z_2 = \frac{\pi^*Z_1 \cdot \pi^*Z_2}{4} = 4D_1 \cdot D_2 = 4,$$

and

$$2\Theta \equiv Z_1 + Z_2, \quad \text{since} \quad 2\pi^*\Theta = 4(D_1 + 2D_2) \equiv \pi^*(Z_1 + Z_2).$$

If $\Theta = C_1 + C_2$ were reducible, intersecting with $2\Theta \equiv Z_1 + Z_2$ so that $(C_1 + C_2) \cdot (Z_1 + Z_2) = 4$ forces either $C_1 \cdot Z_1 = C_1 \cdot Z_2 = 1$ or $C_1 \cdot Z_1 = 0, C_1 \cdot Z_2 = 2$. In the first case, quotient by C_1 gives isomorphisms $A/C_1 \simeq Z_i$ for both $i = 1, 2$. A contradiction to the fact $Z_1 \simeq E$ and $Z_2 \simeq E'$ are not isomorphic, see the Lemma 3.3 in the following. In the latter case, we have $C_1 \equiv Z_1$. A contradiction to that $Z_1 \cdot Z_2 = 4, C_1 \cdot Z_2 = 2$. We are done. $\square$

Lemma 3.3. *The elliptic curves E and E' are not isomorphic.*

Proof. Otherwise, an isomorphism $E \simeq E'$ would produce an endomorphism of F of degree 5, impossible because $\text{End}(F) = \mathbb{Z}$. $\square$

From now on, we identify $X = \Theta$ and $J(X) = A$.

3.2 Elliptic directions of degrees 2 and 7

Next we construct the two morphisms from X to two elliptic curves. Given an elliptic curve T , and a non-zero homomorphism $\tau : T \rightarrow J(X)$, its dual homomorphism $\hat{\tau} : J(X) \rightarrow \hat{T}$ induces a finite map $j : X \hookrightarrow J(X) \xrightarrow{\hat{\tau}} \hat{T}$ onto an elliptic curve.

Proposition 3.4. *We have identity $\deg j = \deg \tau^*\Theta$.*

Proof. The map $\hat{\tau}$ can be identified with $\text{Alb}(j)$ while τ is identified with $j^* : T = \text{Pic}^0(\hat{T}) \rightarrow J(X) = \text{Pic}^0(X)$. In particular, it is classical that $\hat{\tau} \cdot \lambda_\Theta \cdot \tau = [\deg j] \cdot \lambda_T$, see [2, §3, proof of Theorem 3.7, pp. 161–162]. On the other hand, following from the functoriality of the λ map, we have $\hat{\tau} \cdot \lambda_\Theta \cdot \tau = \lambda_{j^*\Theta}$.

$$\begin{array}{ccc} T & \xrightarrow{\tau} & J(X) \\ \lambda_{\tau^*\Theta} = [j] \cdot \lambda_T \downarrow & & \downarrow \lambda_\Theta \\ \hat{T} & \xleftarrow{\hat{\tau}} & \hat{J}(X) \end{array}$$

Counting degrees then gives:

$$(\deg j)^2 [\deg j] \cdot \lambda_T = \lambda_{j^*\Theta} = (\deg j^*\Theta)^2.$$

We are done. $\square$

Inside $A = J(X)$, we have two elliptic subcurves:

$$S_2 = \pi(D_2), \quad S_7 = \pi(i(F)).$$

The restriction of π to D_2 is injective, while its restriction to $i(F)$ has kernel $i(F[2])$ and degree 4. The projection formula gives

$$\Theta \cdot S_2 = \pi^* \Theta \cdot D_2 = 2,$$

and

$$4(\Theta \cdot S_7) = \pi^* \Theta \cdot i(F) = 2(D_1 + D_2) \cdot i(F) = 2(\deg \varphi' + \deg \varphi) = 2(5 + 9) = 28.$$

Thus

$$\Theta \cdot S_2 = 2, \quad \Theta \cdot S_7 = 7.$$

By the beginning of this subsection, we therefore obtain elliptic curves $B = \widehat{S}_2$, $Y = \widehat{S}_7$ and finite morphisms

$$b: X \longrightarrow B, \quad \deg b = 2, \quad f: X \longrightarrow Y, \quad \deg f = 7,$$

chosen so that we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Pic}^0(B) & \xrightarrow{b^*} & \mathrm{Pic}^0(X) & \xleftarrow{f^*} & \mathrm{Pic}^0(Y) \\ \downarrow \simeq & & \parallel & & \downarrow \simeq \\ S_2 & \hookrightarrow & J(X) & \hookleftarrow & S_7 \end{array}$$

Moreover, these two elliptic curves S_2 and S_7 share their 3-torsion.

Proposition 3.5. *The two elliptic directions satisfy*

$$S_2 \cap S_7 = \pi(i(F[3])) = S_2[3] = S_7[3].$$

In particular, there is a unique isomorphism

$$\alpha: \mathrm{Pic}^0(Y)[3] \xrightarrow{f^*} S_7[3] = S_2[3] \xrightarrow{b^{*-1}} \mathrm{Pic}^0(B)[3]$$

such that

$$f^* a = b^* \alpha(a) \quad \text{for every } a \in \mathrm{Pic}^0(Y)[3]. \quad (3.1)$$

This isomorphism is anti-symplectic with respect to the pairing e_{3,S_7} and e_{3,S_2} .

Proof. For $x \in F$, $i(x) = (3x, \bar{x}) \in E \times E'$. By construction $\pi(i(x)) \in S_7$ belongs to S_2 if $\pi(i(x)) = \pi(0, \bar{y})$ for some $\bar{y} \in E'$. Namely,

$$(3x, \bar{x} - \bar{y}) = i(x') = (3x', \bar{x}')$$

for some $x' \in F[2]$. Namely x is a 3-torsion after modulo $F[2]$. In other words, $S_2 \cap S_7 = S_7[3]$. The other equality $S_2 \cap S_7 = S_2[3]$ follows immediately then.

Note that the two natural projections $F \rightarrow S_i, i = 2, 7$ has degree 5, 4 coprime to 3, so the natural map $\pi \cdot i: F[3] \rightarrow S_2[3] = S_7[3]$ is an isomorphism. For any $(a, b) \in S_2[3] = S_7[3]$, choose $a', b' \in F[3]$ so that $a = \pi(a'), b = \pi(i(b'))$. Then the functoriality of theta pairing gives:

$$\begin{aligned} e_{3,S_7}(a, b) &= e_{12,F}(a', b') = e_{3,F}(a', b')^4 = e_{3,F}(a', b'), \\ e_{3,S_2}(a, b) &= e_{15,F}(a', b') = e_{3,F}(a', b')^5 = e_{3,F}(a', b')^{-1}. \end{aligned}$$

Hence the isomorphism α is anti-symplectic. $\square$

3.3 The common cover

Fixing any $\eta_Y \in \text{Pic}^0(Y)[3]$, set $\eta_B := \alpha(\eta_Y) \in \text{Pic}^0(B)[3]$, then $\eta := f^*\eta_Y = b^*\eta_B$ in $\text{Pic}^0(X)[3]$. In particular, the étale cyclic covers $p : Y' \rightarrow Y$ and $q : B' \rightarrow B$ defined by

$$\begin{aligned}\mathcal{O}_{Y'} &= \mathcal{O}_Y \oplus \eta_Y \oplus \eta_Y^{\otimes 2} \\ \mathcal{O}_{B'} &= \mathcal{O}_B \oplus \eta_B \oplus \eta_B^{\otimes 2}\end{aligned}$$

pull back to a common étale cover on X . Namely, we have the following commutative cartesian diagram

$$\begin{array}{ccccc} B' & \xleftarrow{b'} & X' & \xrightarrow{f'} & Y' \\ q \downarrow & & \downarrow v & & \downarrow p \\ B & \xleftarrow{b} & X & \xrightarrow{f} & Y \end{array}$$

We then fix a generator σ (resp. σ_B, σ_Y) of the cyclic Galois group of the étale covers v (resp. q, p) such that it acts on η (resp. η_B, η_Y) by a fixed primitive cubic root μ_3 .

Given elliptic curve structures on Y' and B' so that p, q are homomorphisms, then the Galois automorphism σ_Y, σ_B are realized by translations $\sigma_Y = t_{\bar{y}}^*$ and $\sigma_B = t_{\bar{b}}^*$ for unique $\bar{y} \in \text{Ker}(p)$ and $\bar{b} \in \text{Ker}(q)$. Thus for any line bundle $\mathcal{N}_1$ of degree 1 on Y' , we have

$$\sigma_Y^* \mathcal{N}_1 \otimes \mathcal{N}_1^{-1} = t_{\bar{y}}^* \mathcal{N}_1 \otimes \mathcal{N}_1^{-1} = \mathcal{O}_{Y'}(\bar{y} - o_{Y'}) = \lambda_Y(\bar{y}).$$

Lemma 3.6. *For $\mathcal{P} \in \text{Pic}^0(Y)[3]$, $p^*\mathcal{P} \simeq \sigma^* \mathcal{N}_1 \otimes \mathcal{N}_1^{-1}$ if and only if $e_{3,\hat{Y}}(\eta_Y, \mathcal{P}) = \mu_3$.*

Proof. By construction,

$$t_{\bar{y}}^*|_{\eta_Y} = \mu_3 \cdot \text{id}, \quad t_{\bar{y}}^*|_{\eta_Y^{\otimes 2}} = \mu_3^2 \cdot \text{id}.$$

It follows from Proposition 2, $p^*\mathcal{P} \simeq \sigma^* \mathcal{N}_1 \otimes \mathcal{N}_1^{-1} = \lambda_Y(\bar{y})$ if and only if

$$e_{3,\hat{Y}}(\eta_Y, \mathcal{P}) = \mu_3, \quad e_{3,\hat{Y}}(\eta_Y^{\otimes 2}, \mathcal{P}) = \mu_3^2.$$

We are done. □

Now fixing one such $\mathcal{P}$ and take $\mathcal{Q} = \alpha(\mathcal{P}) \in \text{Pic}^0(B)[3]$. Then for any $\mathcal{M}_1$ of degree one on B' we have

$$q^*\mathcal{Q}^{-1} \simeq \sigma_B^* \mathcal{M}_1 \otimes \mathcal{M}_1^{-1}.$$

In fact, it follows from the fact α is anit-symplectic that $e_{3,\hat{B}}(\eta_B, \mathcal{Q}^{-1}) = e_{3,\hat{Y}}(\eta_Y, \mathcal{P}) = \mu_3$.

Lemma 3.7. *Take $\mathcal{R} := f'^*\mathcal{N}_1 \otimes b^*\mathcal{M}_1$, with $\mathcal{N}_1, \mathcal{M}_1$ line bundles on Y', B' of degree one as above, then $\sigma^*\mathcal{R} \simeq \mathcal{R}$.*

Proof. We have:

$$\begin{aligned}\sigma^*\mathcal{R} \otimes \mathcal{R}^{-1} &\simeq f'^*(\sigma_Y^* \mathcal{N}_1 \otimes \mathcal{N}_1^{-1}) \otimes b'^*(\sigma_B^* \mathcal{M}_1 \otimes \mathcal{M}_1^{-1}) \\ &\simeq f'^*p^*\mathcal{P} \otimes b'^*q^*\mathcal{Q}^{-1} \\ &\simeq v^*(f^*\mathcal{P} \otimes b^*\mathcal{Q}^{-1}) \\ &\simeq v^*\mathcal{O}_X \\ &\simeq \mathcal{O}_{X'}.\end{aligned}$$

□

Since $v : X' \rightarrow X$ is a cyclic étale cover with generator σ , this $\mathcal{R}$ can descend to a line bundle $\mathcal{L}_1$ on $X = X'/\langle \sigma \rangle$. Computation gives its degree:

$$\deg \mathcal{L}_1 = \frac{\deg(\mathcal{R})}{3} = \frac{2+7}{3} = 3.$$

Corollary 3.8. *For any line bundle $\mathcal{L} \in \text{Pic}^3(X)$ of degree 3, there are $\mathcal{N} \in \text{Pic}^1(Y')$ and $\mathcal{M} \in \text{Pic}^1(B')$ such that $v^*\mathcal{L} \simeq f'^*\mathcal{N} \otimes b'^*\mathcal{M}$.*

Proof. By the previous lemma, $v^*\mathcal{L}_1 = f'^*\mathcal{N}_1 \otimes b'^*\mathcal{M}_1$. Note $\mathcal{L}$ differs from $\mathcal{L}_1$ by a line bundle in $\text{Pic}^0(X)$ and the homomorphism

$$\Phi = (f^*, b^*) : \text{Pic}^0(Y) \times \text{Pic}^0(B) \longrightarrow \text{Pic}^0(X)$$

is an isogeny, in particular surjective. The corollary then follows. $\square$

3.4 Instability of the direct image

We can now prove the instability

Proof of Theorem 1.2. Fix $\mathcal{L} \in \text{Pic}^3(X)$ and choose $\mathcal{N}, \mathcal{M}$ as in Corollary 3.8. A simple calculation gives that $f_*\mathcal{L}$ has degree 2 and rank 7. Set $\mathcal{A}_{\mathcal{L}} := p_*\mathcal{N}$. It is a semistable bundle (see [3, § 4]) of degree 1 and rank 3.

Noticing a degree-one line bundle on an elliptic curve has a section, so $H^0(X', b'^*\mathcal{M}) \neq 0$. Recall in finite étale case, p^* is also the right adjoint of p_* , we have:

$$\begin{aligned} \text{Hom}_Y(\mathcal{A}_{\mathcal{L}}, f_*\mathcal{L}) &= \text{Hom}_Y(p_*\mathcal{N}, f_*\mathcal{L}) \\ &\simeq \text{Hom}_{Y'}(\mathcal{N}, p^*f_*\mathcal{L}) \\ &\simeq \text{Hom}_{Y'}(\mathcal{N}, f'_*v^*\mathcal{L}) \\ &\simeq \text{Hom}_{X'}(f'^*\mathcal{N}, v^*\mathcal{L}) \\ &= H^0(X', v^*\mathcal{L} \otimes f'^*\mathcal{N}^{-1}) \\ &= H^0(X', b'^*\mathcal{M}) \neq 0. \end{aligned}$$

But

$$\mu(\mathcal{A}_{\mathcal{L}}) = \frac{1}{3} > \frac{2}{7} = \mu(f_*\mathcal{L}).$$

So $f_*\mathcal{L}$ is not semistable. $\square$

Remark 3.9. One can also argue as: $p^*f_*\mathcal{L} = f'_*(v^*\mathcal{L}) = \mathcal{N} \otimes f'_*b'^*\mathcal{M}$. As $\mathcal{M}$ is effective, there is an embedding $\mathcal{N} \rightarrow p^*f_*\mathcal{L}$. As

$$\mu(\mathcal{N}) = 1 > \mu(p^*f_*\mathcal{L}) = 6/7,$$

$p^*f_*\mathcal{L}$ is unstable. So is $f_*\mathcal{L}$, see Proposition 2.2.

4 Generic semistability for Galois covers

The counterexample above concerns a fixed non-Galois cover. We now prove Theorem 1.3, beginning with two auxiliary lemmas. The argument shows that the instability constructed here depends on the special cover and on the chosen degree component of its Picard variety.

Definition 4.1. A finite morphism $h: X \rightarrow Y$ is *genuinely ramified* if it does not factor as $X \rightarrow Y' \rightarrow Y$, where $Y' \rightarrow Y$ is a connected finite étale morphism of degree greater than one.

By definition, any $h: X \rightarrow Y$ factors as

$$X \xrightarrow{h'} Y' \xrightarrow{\pi} Y$$

where h' is genuinely ramified and π is étale. In case h is Galois, h' is automatically Galois while π may not.

Lemma 4.2. *If $h: X \rightarrow Y$ is genuinely ramified, then the pullback*

$$h^*: \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Pic}^0(X)$$

is injective.

Proof. If $h^*\mathcal{M} \simeq \mathcal{O}_X$, then $\mathcal{M}^{\deg h} = Nm_h(h^*\mathcal{M}) = Nm_h(\mathcal{O}_X) = \mathcal{O}_Y$ by the norm identity. If $n := \mathrm{ord}(\mathcal{M})$ in $\mathrm{Pic}^0(Y)$ is greater than one, the cyclic cover

$$Y' = \mathrm{Spec}_Y \left(\bigoplus_{i=0}^{n-1} \mathcal{M}^i \right) \longrightarrow Y$$

is finite étale and connected and trivialized over $h: X \rightarrow Y$. In particular, h factors through Y' , a contradicting genuine ramification. Thus $\alpha = \mathcal{O}_Y$. $\square$

Lemma 4.3. *Let $h: X \rightarrow Y$ be a finite Galois morphism with group G . For every $\mathcal{L} \in J(X)$,*

$$h^* \mathrm{Nm}_h(\mathcal{L}) \simeq \bigotimes_{\gamma \in G} \gamma^* \mathcal{L}.$$

Proof. For a divisor D representing $\mathcal{L}$, the Galois identity $h^*h_*D = \sum_{\gamma \in G} \gamma D$ gives

$$h^* \mathrm{Nm}_h(B) \simeq \bigotimes_{\gamma \in G} \gamma^* B.$$

$\square$

We first prove the generic semistability for genuinely ramified covers.

Theorem 4.4. *Let $h: X \rightarrow Y$ be a genuinely ramified Galois cover, then for any $d \in \mathbb{Z}$, there is an open dense subset $U_d \subseteq \mathrm{Pic}^d(X)$ such that $h_*\mathcal{L}$ is semistable for all $\mathcal{L} \in U_d$.*

Proof. Fix any $d \in \mathbb{Z}$ and set $G = \text{Gal}(X/Y)$ with $g := |G| = \deg h$. The pushforward $(f \times \text{id})_* \mathcal{P}$ of the Poincaré bundle on $X \times \text{Pic}^d(X)$ to $Y \times \text{Pic}^d(X)$ gives a flat family of vector bundles on Y parametrized by $\text{Pic}^d(X)$ whose fiber at $\mathcal{L} \in \text{Pic}^d(X)$ is $h_* \mathcal{L}$.

Suppose $f_* L$ is unstable for any $\mathcal{L}$. By the relative Harder–Narasimhan theorem of Shatz [14], the Harder–Narasimhan type is constant on a nonempty open subset $U \subseteq \text{Pic}^d(X)$. Take e and r to be the degree and rank of the first Harder–Narasimhan term over U . In particular, the first term of the relative filtration is a subbundle of $(f \times \text{id})_* \mathcal{P}$ on $Y \times U$. Take e and r to be the degree and rank of the first Harder–Narasimhan term over U . Taking determinant over U then define a morphism $\delta : U \rightarrow \text{Pic}^e(Y)$. Following the celebrated Weil’s trick, the map δ actually extends to a genuine morphism $\delta : \text{Pic}^d(X) \rightarrow \text{Pic}^e(Y)$ because its target is Abelian. The projection formula gives

$$\delta(\mathcal{L} \otimes h^* \mathcal{N}) = \delta(\mathcal{L}) \otimes \mathcal{N}^{\otimes r} \quad (\mathcal{N} \in \text{Pic}^0(Y)).$$

and the natural identity $h_*(\gamma^* \mathcal{L}) = h_* \mathcal{L}$ gives $\delta \circ \gamma^* = \delta$ for any $\gamma \in G$. Choose $\mathcal{L}_0 \in \text{Pic}^d(Y)$ and define

$$\psi : \text{Pic}^0(X) \rightarrow \text{Pic}^0(Y), \quad \psi(\mathcal{M}) = \delta(\mathcal{L}_0 \otimes \mathcal{M}) \otimes \delta(\mathcal{L}_0)^{-1}.$$

Then $\psi : \text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$ is a homomorphism satisfying $\psi \circ h^* = [r]$. Take $\mathcal{N}_\gamma := (\gamma^{-1})^* \mathcal{L}_0 \otimes \mathcal{L}_0^{-1} \in \text{Pic}^0(X)$, then for any $\mathcal{M} \in \text{Pic}^0(X)$,

$$\begin{aligned} \psi(\gamma^* \mathcal{M}) &= \delta(\mathcal{L}_0 \otimes \gamma^* \mathcal{M}) \otimes \delta(\mathcal{L})^{-1} \\ &= \delta((\gamma^{-1})^*(\mathcal{L}_0 \otimes \gamma^* \mathcal{M})) \otimes \delta(\mathcal{L})^{-1} \\ &= \delta(\mathcal{L}_0 \otimes \mathcal{N} \otimes \mathcal{M}) \otimes \delta(\mathcal{L})^{-1} \\ &= \psi(\mathcal{N} \otimes \mathcal{M}) \\ &= \psi(\mathcal{N}) \otimes \psi(\mathcal{M}). \end{aligned}$$

The special case $\mathcal{M} = \mathcal{O}_X$ gives $\psi(\mathcal{N}) = \mathcal{O}_Y$ and thus $\psi \circ \gamma^* = \psi$. The pullback–norm identity then gives

$$h^* \circ \text{Nm}_h = \sum_{\gamma \in G} \gamma^*$$

as homomorphisms from $J(X)$ to $J(X)$. Since $h_* \gamma^* \mathcal{L} \simeq h_* \mathcal{L}$ for every $\gamma \in G$, the determinant of the maximal destabilizing subbundle is invariant under γ^* . After identifying $\text{Pic}^d(X)$ with $J(X)$, this gives

$$\psi \circ \gamma^* = \psi.$$

Consequently,

$$\begin{aligned} [r] \circ \text{Nm}_h &= \psi \circ h^* \circ \text{Nm}_h \\ &= \psi \circ \sum_{\gamma \in G} \gamma^* \\ &= \sum_{\gamma \in G} \psi \circ \gamma^* \\ &= [g] \circ \psi. \end{aligned}$$

Write $g = g_0 \cdot N$, $r = r_0 \cdot N$ with $\gcd(g_0, r_0) = 1$. Since $\text{Hom}(\text{Pic}^0(X), \text{Pic}^0(Y))$ is torsion-free, there is an $\varrho \in \text{Hom}(\text{Pic}^0(X), \text{Pic}^0(Y))$ so that

$$\text{Nm}_h = [g_0] \circ \varrho, \quad \psi = [r_0] \circ \varrho.$$

As $r < g$ by the unstable assumption, $g_0 > 1$. It is classical that Nm_h can be identified with the induced map Alb_h on Albanese varieties through the canonical principal polarizations, and h^* is then its dual map, see [2, §3, proof of Theorem 3.7, pp. 161–162]. Following $\mathrm{Nm}_h = [g_0] \circ \varrho$, h^* factor through the multiplicative-by- g_0 map, too. A contradiction to Lemma 4.2. We are done. $\square$

Proof of Theorem 1.3. In general, if $h : X \rightarrow Y$ is Galois, then it factor through $h = \pi \circ h'$ with h' genuinely ramified Galois and π étale. For any d , it is proved in Theorem 4.4, $h'_* \mathcal{L}$ is semistable for $\mathcal{L}$ belonging to an open dense subset $U_d \in \mathrm{Pic}^d(X)$. It follows from [3, § 4], that π_* preserves semistability. Thus, for $\mathcal{L} \in U_d$, $h_* \mathcal{L} = \pi_* h'_* \mathcal{L}$ is semistable. We are done. $\square$

Remark 4.5. The theorem proves generic semistability and makes no assertion of generic stability. The Galois hypothesis enters through the pullback–norm identity and invariance under deck transformations. The degree-7 map constructed in this paper is not Galois: by Riemann–Hurwitz its total ramification degree is 2, whereas a ramified Galois cover of prime degree 7 has total ramification degree divisible by 6. Thus Theorem 1.3 does not conflict with Theorem 1.2.

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