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DOUBLE TELESCOPES

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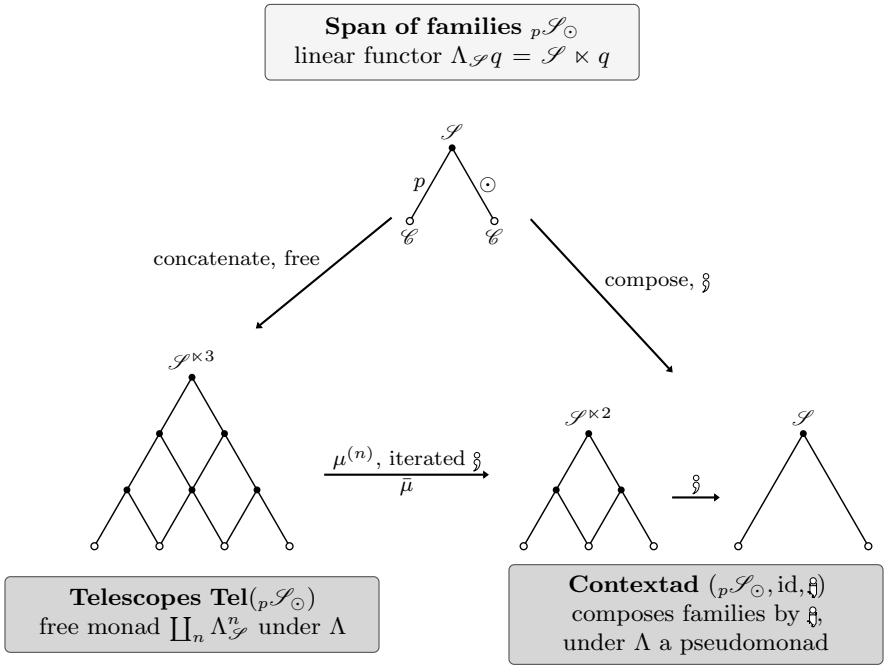
Abstract. A telescope is a list of families, each over the total category of the previous ones. This paper relates three ways of handling telescopes through the Grothendieck construction, for spans with a fibred left leg, and tags each statement with its level. Concatenating telescopes is the telescope monad, the classical free monoid on spans. Composing them is a contextad, a pseudomonoid of spans with a fibration as left leg, which the linear functor of spans turns into a pseudomonad whose Kleisli arrows are the contextful arrows. The triangle of the three holds for every contextad whose families form a split fibration. Its instances are families of categories and doubly indexed categories in spans. The latter define double telescopes and their contextad.



The three constructions and their edges (Table 2) for the span of families ${}_p\mathscr{S}_\odot$ of a contextad, with pasting diagrams of spans in **CAT** read as in Table 4.

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1. Introduction

A telescope is a list of families in which each family lies over the total category of the previous ones. Categorically, a telescope is an iterated Grothendieck construction, because each family is an indexed category over the total category of the previous ones. This paper relates three constructions on telescopes through the Grothendieck construction for spans with a fibred left leg, first for indexed categories and then for the doubly indexed categories in spans of Cruttwell, Lambert, Pronk and Szyld [CLPS22].

The Grothendieck construction works at several levels at once, and Table 1 keeps them apart for families, for double families and in general. Spans with a fibred left leg, the telescope monad and contextads live in the hom 2-categories $\mathbf{H}_X$ of the 3-category $\mathbf{f\mathfrak{S}pan(CAT)}^{\Rightarrow}$ of Capucci and Myers [CM24]. Every definition, result and figure below carries its level, such as L1, and a passage between two levels carries both, such as L2→1. Figure 1 draws one telescope at levels 2 to 0. The span of families of a contextad is $\mathscr{S}$ on page 1, in Figure 1 and Table 1, and $\mathscr{N}$ in Sections 2 and 3. The label K marks a known construction.

Level	Single, $\mathscr{C} = \mathsf{Cat}$	Double, $\mathscr{C} = \mathsf{Dbl}_\mathrm{s}$	General, $\mathscr{C} = X$
L0 objects	family $F\colon B^{\mathrm{op}} \to \mathbf{Cat}$, its fibres $F(b)$ at $b \in B$; unit: terminal; sum $(F \mathbin{⅋} G)(b) = \Sigma_{x \in F(b)} G(b,x)$; fibration $\int F\colon B \odot F \to B$; telescopes (Def. 3.7, Rem. 4.3)	double family $\mathbb{F}$, its fibres $\mathbb{F}(b)$, $\mathbb{F}(m)$ at objects and loose morphisms of $\mathbb{B}$ (Def. 4.5); unit: terminal; sum (4.3); double fibration $\int\!\!\!\int \mathbb{F}$; double telescopes (Thm. 4.7)	$F \in \mathscr{S}_B$, telescopes (Def. 3.7)
L1 categories	$\mathsf{ICat} = \mathrm{dom} \int_{\mathsf{Cat}} \mathsf{Psd}((-)^{\mathrm{op}}, \mathbf{Cat})$, split over Cat	$\mathsf{DblICat}$, split over Dbl_s (Sec. 4.3)	families $p\colon \mathscr{S} \to X$, split
L2 2-categories	the contextad $({}_p\mathsf{ICat}_\odot, \mathrm{id}, ⅋)$ in $\mathbf{H}_{\mathsf{Cat}}$, **Tel**, Λ, $\mathbb{K}$l, co$\mathbb{K}$l; Thm. 3.8	the contextad of double telescopes $({}_p\mathsf{DblICat}_\odot, \mathrm{id}, ⅋)$ in $\mathbf{H}_{\mathsf{Dbl}_\mathrm{s}}$; Thm. 4.7	$({}_p\mathscr{S}_\odot, \mathrm{id}, ⅋)$ in $\mathbf{H}_X$; Thm. 3.8
L3 3-categories	the 3-category $\mathbf{f\mathfrak{S}pan(CAT)}^{\Rightarrow}$ of fibred spans and its composition $\ltimes$		

TABLE 1. The four levels of the two examples and the general setting.

The *telescope monad* **Tel** is the free-monoid 2-monad on spans $X \xleftarrow{p} \mathscr{M} \xrightarrow{\odot} X$, written ${}_p\mathscr{M}_\odot$. It is to spans what the list monad is to sets, and so it concatenates but never composes, while composing is the structure map of an algebra. A *contextad* $({}_p\mathscr{N}_\odot, \mathrm{id}, ⅋)$ of Capucci and Myers is a dependently graded comonad on $\mathscr{C}$ [CM24], and the *linear functor* Λ sends a span with split left leg to an endo-2-functor $\Lambda_{\mathscr{N}}$ of $[\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}]$. Table 2 lists the three vertices and the three edges of the triangle of page 1, which Theorem 3.8 proves at level 2.

The Grothendieck–contextad edge has two faces, the Kleisli arrows of Λ and the *contextful arrows* $(F, f)\colon B \nrightarrow B'$ of a family F on B and any functor $f\colon B \odot F \to B'$, which agree by Yoneda (Proposition 2.10).

Double categories enter through the arenas of categorical systems theory. An arena of Myers is a double category whose tight morphisms are lenses and whose loose morphisms are charts, and it is the Grothendieck double construction of an indexed category [Mye23]. A family over an arena therefore has a double category as its base and takes values in spans, and a telescope of such families is a double telescope. The second part shows that the whole triangle holds for double categories, where the families are doubly indexed categories

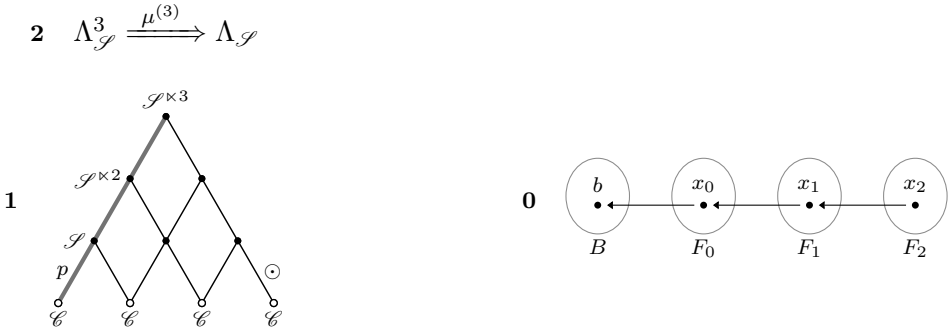


FIGURE 1. L2→0 One telescope $t = (B \mid F_0, F_1, F_2)$ of a contextad with families $\mathscr{S}$ at levels 2 to 0 of Table 1. At level 2, $\mu^{(3)}$ composes t into $F_0 \fatsemi (F_1 \fatsemi F_2)$ (Theorem 3.8(iii)). At level 1, $\mathscr{S}^{\ltimes 3} \cong \operatorname{dom} \int_{\mathscr{C}} \Lambda^3_{\mathscr{S}} 1$ (Theorem 3.8(i)), whose thick left edge consists of split fibrations forgetting the last family (Lemma 3.2(ii)), and t lies in its fibre at B. At level 0, for $\mathscr{C} = \mathsf{Cat}$, the objects of the last base $B^{(3)}$ of t are the dependent tuples (b, x_0, x_1, x_2) with $x_i \in F_i(b, x_0, \ldots, x_{i-1})$.

	L2→1 Theorem 3.8
Grothendieck construction	linear functor $\Lambda_{\mathscr{N}} q = \mathscr{N} \ltimes q$
Telescope monad	$\mathbf{Tel}(\mathscr{M}) = \coprod_n \mathscr{M}^{\ltimes n}$, concatenates
Contextad	$({}_p\mathscr{N}_{\odot}, \mathrm{id}, \fatsemi)$, composes
Grothendieck–telescope	$\Lambda_{\mathbf{Tel}(\mathscr{N})} \cong \coprod_n \Lambda^n_{\mathscr{N}}$, the free monad
Grothendieck–contextad	the pseudomonad $\Lambda_{\mathscr{N}}$, $\mathbb{K}\mathrm{l}(\Lambda_{\mathscr{N}}) \simeq \mathrm{co}\mathbb{K}\mathrm{l}(\fatsemi)$
telescope–contextad	$\coprod_n \mu^{(n)}$ and $\bar{\mu}\colon \mathbf{Tel}(\mathscr{N}) \to \mathscr{N}$, which restricts to id and $\fatsemi$

TABLE 2. The triangle of the Grothendieck construction, the telescope monad and contextads: its vertices and edges.

in spans that are neither discrete nor valued in sets. Theorem 3.8 asks only for a contextad whose families form a split fibration, and Theorem 4.7 builds one at level 2 from formulas at level 0, the contextad of double telescopes, whose telescopes are the double telescopes. All three edges of Theorem 3.8 therefore hold for double telescopes. K *Construction.* A double telescope concatenates double Grothendieck constructions (4.4), whose dependent sums are tuples in the loose direction as in the tight one.

Table 3 sets the paper against the literature. Spans and the telescope monad need only strict pullbacks, as in a paradise of Capucci and Myers [CM24, Def. 2.3], whereas the fibred left legs serve the contextads, whose right-colax maps compose along chosen lifts [CM24, Sec. 1.3]; on the indexed side a lift is the reindexing $u^*F = u^{\mathrm{op}} \fatsemi F$, and so the paper speaks of base change (Lemma 2.3).

The reader is assumed to know 2-monads [BKP89], pseudomonads [Lac00], double categories and double fibrations. Section 2 recalls fibred spans, contextads, contextful arrows and the linear functor. Section 3 builds the telescope monad and proves Theorem 3.8. Section 4 gives the two examples, families in Section 4.1 and double families in Sections 4.2–4.3, which define double telescopes with their contextad, and The appendix recalls the conventions, the diagram calculi of Table 4, fibrations and the double-categorical definitions.

	Known	Here
Free monoids	coproducts of powers [Kel80, Thm. 23.4]; in endo-spans, free internal categories [Lac10, Sec. 5]; monoids monadic by Beck	**Tel** on spans with a fibred left leg (Def. 3.4)
Kleisli and coKleisli	from a polynomial monad to its contextad, sketched for parametric right adjoints [CM24, Thm. A.4, Rem. A.5]	from a contextad to $\Lambda_{\mathscr{N}}$, which runs opposite to the one stated by Capucci and Myers, on representables by Yoneda (Prop. 2.10)
Linear functor	$\Sigma_p\Delta_\odot$ of a span [Web15, Thm. 4.1.4], Street's 2-monads of fibrations [Web15, Prop. 4.2.3]	its action on right-colax maps (Lemma 2.8)
Unit and sums	discrete: the polynomial of a natural model is a pseudomonad [AN18, Thm. 4.1(a)]; composing with a span adds sums [vG15, Prop. 1.33]	split families that need not be discrete (Prop. 2.9)
Telescopes	de Bruijn [dB91]; pyramids of spans by the Segal condition [Bar14, Def. 3.3], [Hau18, Def. 5.5]; telescopes of presheaves as contexts, collapsed by Σ into one presheaf, an equivalence [Pal16, Thms. 2.1, 3.2]	the free monad on $\Lambda_{\mathscr{N}}$ (Thm. 3.8(i))
Double families	$\int\colon \mathsf{Dbl2Cat}(\mathbb{B}^{\mathrm{op}}, \mathbb{S}\mathrm{pan}(\mathbf{Cat})) \simeq \mathsf{DblFib}(\mathbb{B})$ [CLPS22]; pseudonatural in the base for discrete double fibrations [FM24, Thm. 6.2.1]	the contextad of double telescopes (Thm. 4.7)

TABLE 3. The literature and this paper.

2. Preliminaries

This section recalls the 3-category of fibred spans and the contextads and contextful arrows of Capucci and Myers [CM24], and it introduces the linear functor of a span, which turns a contextad into a pseudomonad.

2.1. Fibred spans

This subsection works at levels 3 and 2 over a fixed category X. Spans on X compose by $\ltimes$.

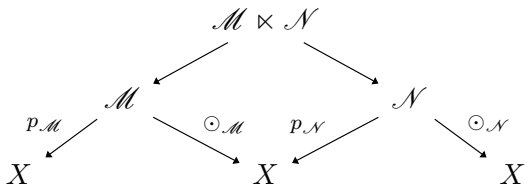
Definition 2.1 (Fibred spans L3→2, [CM24, Def. 3.10])**.** The 3-category $\mathbf{f\mathfrak{S}pan}(\mathbf{CAT})^{\Rightarrow}$ has categories as 0-cells. Its hom 2-category $\mathbf{H}_X$ on a category X has the following parts.

(i) *Objects.* The spans $X \xleftarrow{p} \mathscr{M} \xrightarrow{\odot} X$ of categories whose left leg p is a cloven fibration. Such a span is written ${}_p\mathscr{M}_\odot$, or $\mathscr{M}$ when its legs are understood. The fibre of p at $A \in X$ is $\mathscr{M}_A$, and the right leg gives $A \odot P := \odot(P)$ for $P \in \mathscr{M}_A$.

(ii) *1-cells.* The right-colax maps $(k, \varphi)\colon \mathscr{M} \to \mathscr{M}'$ of a functor k with $k\,;p' = p$ and a natural $\varphi\colon k\,;\odot' \Rightarrow \odot$ with components $\varphi_P\colon A \odot' kP \to A \odot P$. The map is *strict* when φ is the identity, and the strict maps form the sub-2-category $\mathbf{H}^{\mathrm{s}}_X$.

(iii) *2-cells.* The transformations $\sigma\colon k \Rightarrow k'$ with $\sigma\,;p' = \mathrm{id}$ and $\varphi_P = \odot'(\sigma_P)\,;\varphi'_P$.

(iv) *Composition.* The composite $\mathscr{M} \ltimes \mathscr{N}$ of spans, which is the strict pullback



(2.1)

with the outer legs, whose left leg is a fibration by Lemmas A.2 and A.3. Hence its objects are the pairs (P, Q) with $A \in X$, $P \in \mathscr{M}_A$ and $Q \in \mathscr{N}_{A \odot P}$. On strict maps k and h the composite $k \ltimes h$ is induced by k and h between the pullbacks.

(v) *Unit.* The span $X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$, written ${}_1X_1$.

Composing right-colax maps by $\ltimes$ moves the second entry along chosen lifts [CM24, Rem. 2.20, Lemma 3.11]. Strict maps need no lifts, and the telescope monad and its algebras need only them.

Lemma 2.2 (Strict maps form a monoidal 2-category L2)**.** $(\mathbf{H}^{\mathrm{s}}_X, \ltimes, {}_1X_1)$ *is a monoidal 2-category whose coproducts are those of* **CAT***, and* $\ltimes$ *preserves them in each variable.*

Proof. The strictly commuting map (k, id_X, h) of cospans induces $k \ltimes h$ on strict pullbacks. Likewise, the coherence isomorphisms are the canonical isomorphisms of iterated pullbacks, which are unique and hence coherent. Finally, a coproduct of fibrations is a fibration, and $\ltimes$ preserves coproducts in each variable because pullback in **CAT** preserves them. □

A monoid $(\mathscr{M}, I, \otimes)$ in $(\mathbf{H}^{\mathrm{s}}_X, \ltimes, {}_1X_1)$ is a strict double category with tight category X, loose category $\mathscr{M}$, source p, target $\odot$, loose units I and loose composition $\otimes$ [CM24, Sec. 1.3], whose source is a fibration.

2.2. Contextads and contextful arrows

This subsection stays at level 2. Let $p\colon \mathscr{N} \to \mathscr{C}$ be a cloven fibration. *Base change* along $u\colon A \to B$ is the reindexing $u^*\colon \mathscr{N}_B \to \mathscr{N}_A$ by the chosen lifts $u^*F \to F$ above u. These are *cartesian* (Appendix 4.3), and so a morphism $G \to F$ above u is a unique $\tau\colon G \to u^*F$ in $\mathscr{N}_A$ followed by the chosen lift, written (u, τ); the chosen lift is (u, id_{u^*F}). A functor $k\colon \mathscr{N} \to \mathscr{N}'$ over $\mathscr{C}$ into a cloven fibration p' then has a *comparison* $\kappa_{u,F}\colon k(u^*F) \to u^*(kF)$ in $\mathscr{N}'_A$, given by $k(u, \mathrm{id}_{u^*F}) = (u, \kappa_{u,F})$, and k is *cartesian*, that is, it sends cartesian morphisms to cartesian morphisms, exactly when every comparison is invertible.

Lemma 2.3 (Base change L1)**.** *Let* $\mathscr{N}$ *and* $\mathscr{N}'$ *be split fibrations over* $\mathscr{C}$*. Then* $\mathscr{N} \cong \mathrm{dom} \int_{\mathscr{C}} \mathscr{N}_{(-)}$*, where* $\mathscr{N}_{(-)}\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{CAT}$ *sends* A *to the fibre* $\mathscr{N}_A$*, and the functors* $\mathscr{N} \to \mathscr{N}'$ *over* $\mathscr{C}$ *that are cartesian are the Grothendieck constructions over* $\mathscr{C}$ *of the pseudonatural transformations* $\mathscr{N}_{(-)} \Rightarrow \mathscr{N}'_{(-)}$*.*

Proof. This is the Grothendieck construction over $\mathscr{C}$, a 2-equivalence between pseudofunctors and fibrations with the functors that are cartesian [MV20, Thm. 2.3], restricted to split fibrations, and its pseudonaturality constraints are the comparisons $\kappa_{u,F}$ [MV20, Sec. 2.1]. □

A contextad is a pseudomonoid in $(\mathbf{H}_{\mathscr{C}}, \ltimes, {}_1\mathscr{C}_1)$ whose unit and multiplication are cartesian [CM24, Def. 3.15], and the paper uses its form as a dependently graded comonad [CM24, Sec. 3.5, Def. 3.26].

Definition 2.4 (Contextad L2, [CM24, Def. 3.15, 3.26])**.** A *contextad* $\odot$ on $\mathscr{C}$ has the following parts.

(i) *Families.* A cloven fibration $p\colon \mathscr{N} \to \mathscr{C}$, whose fibre $\mathscr{N}_A$ consists of the families at $A \in \mathscr{C}$.
(ii) *Action.* A functor $\odot\colon \mathscr{N} \to \mathscr{C}$, which sends a family F at A to $A \odot F$.
(iii) *Counit.* Unit families id_A, given by a functor $\mathrm{id}_{(-)}\colon \mathscr{C} \to \mathscr{N}$ over $\mathscr{C}$ that is cartesian, and $\varepsilon\colon \mathrm{id}_{(-)} \mathbin{;} \odot \Rightarrow \mathrm{id}_{\mathscr{C}}$ with $\varepsilon_A\colon A \odot \mathrm{id}_A \to A$.
(iv) *Comultiplication.* Composite families $F \fatsemi G \in \mathscr{N}_A$, the *dependent sums*, for $F \in \mathscr{N}_A$ and $G \in \mathscr{N}_{A \odot F}$, given by a functor $\fatsemi\colon \mathscr{N} \ltimes \mathscr{N} \to \mathscr{N}$ over $\mathscr{C}$ that is cartesian, and a natural $\delta\colon \fatsemi \mathbin{;} \odot \Rightarrow \odot_{\mathscr{N} \ltimes \mathscr{N}}$ with $\delta_{F,G}\colon A \odot (F \fatsemi G) \to (A \odot F) \odot G$. Like ;, the white semicolon $\fatsemi$ composes in diagrammatic order.
(v) *Coherence.* Isomorphisms $\lambda_F\colon \mathrm{id}_A \fatsemi \varepsilon_A^* F \cong F$, $\rho_F\colon F \fatsemi \mathrm{id}_{A \odot F} \cong F$ and $\alpha_{F,G,H}\colon F \fatsemi (G \fatsemi H) \cong (F \fatsemi G) \fatsemi \delta_{F,G}^* H$ in $\mathscr{N}_A$, natural over $\mathscr{C}$, subject to the axioms of [CM24, Def. 3.15, (3.5.2)–(3.5.4)].

It is *strict* when the cleavage is normal and ε, δ, λ, ρ and α are identities. The span ${}_p\mathscr{N}_\odot$ carries it, but the contextad is named by its action, and with its span it is written $({}_p\mathscr{N}_\odot, \mathrm{id}, \fatsemi)$.

A strict contextad $({}_p\mathscr{N}_\odot, \mathrm{id}, \fatsemi)$ is a monoid in $(\mathbf{H}^{\mathrm{s}}_{\mathscr{C}}, \ltimes, {}_1\mathscr{C}_1)$, and hence an algebra of **Tel** by Corollary 3.6.

Definition 2.5 (Contextful arrows L2, [CM24, Def. 3.17])**.** The *coKleisli double category* of a contextad $({}_p\mathscr{N}_\odot, \mathrm{id}, \fatsemi)$ on $\mathscr{C}$ is the pseudo double category $\mathrm{co}\mathbb{K}\mathrm{l}(\fatsemi)$ of contextful arrows, whose loose units hold only up to the unitors, and it has the following parts.

(i) *Objects and tight morphisms.* Those of $\mathscr{C}$.
(ii) *Loose morphisms.* A loose morphism $A \nrightarrow B$ is a *contextful arrow* (F, f) of a family $F \in \mathscr{N}_A$ and a *body* $f\colon A \odot F \to B$.
(iii) *Cells.* A cell from $(F, f)\colon A \nrightarrow B$ to $(F', f')\colon A' \nrightarrow B'$ over $u\colon A \to A'$ and $v\colon B \to B'$ is a $\theta\colon F \to F'$ in $\mathscr{N}$ above u with $f \mathbin{;} v = \odot(\theta) \mathbin{;} f'$.
(iv) *Loose composition.* For $(F, f)\colon A \nrightarrow B$ and $(G, g)\colon B \nrightarrow D$,

$$(F, f) \mathbin{⨟} (G, g) := \big(F \fatsemi f^* G,\ \delta_{F, f^*G} \mathbin{;} \odot(f, \mathrm{id}_{f^*G}) \mathbin{;} g\big),$$

where f^*G is the base change of G along the body f and $\odot(f, \mathrm{id}_{f^*G})\colon (A \odot F) \odot f^*G \to B \odot G$ is the image of its chosen lift. The coKleisli composite ⨟ composes the families white and the bodies black.
(v) *Loose identities.* $(\mathrm{id}_A, \varepsilon_A)\colon A \nrightarrow A$.
(vi) *Associator and unitors.* The globular cells given by α, λ, ρ and the comparisons of $\mathrm{id}_{(-)}$ and $\fatsemi$ with base change, together with the comparisons of composite chosen lifts when the cleavage is not split.

Capucci and Myers denote it by $\mathrm{Ctx}(\odot)$ and obtain it as a wreath product.

2.3. The linear functor

Construction 2.6 (Linear functor L2)**.** Let $[\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}]$ be the 2-category of functors $\mathscr{C}^{\mathrm{op}} \to \mathbf{CAT}$, pseudonatural transformations and modifications, with terminal object 1 and Yoneda embedding y. A span ${}_p\mathscr{N}_\odot$ whose left leg p is a split fibration then has a *linear functor* $\Lambda_{\mathscr{N}}$, which is the endo-2-functor of $[\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}]$ with the following parts.

(i) *Values.* At $Y \in [\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}]$ and $B \in \mathscr{C}$, the value of $\Lambda_{\mathscr{N}}$ is the Grothendieck construction over the fibre $\mathscr{N}_B$

$$(\Lambda_{\mathscr{N}} Y)(B) = \Sigma_{F \in \mathscr{N}_B} Y(B \odot F),$$

whose objects are the pairs (F, y) of $F \in \mathscr{N}_B$ and $y \in Y(B \odot F)$ and whose morphisms $(F, y) \to (F', y')$ are the pairs (θ, γ) of $\theta\colon F \to F'$ in $\mathscr{N}_B$ and $\gamma\colon y \to Y(\odot\theta)y'$.

(ii) *Base change.* Along $u\colon A \to B$, $u^*(F, y) = (u^*F,\ Y(\odot(u, \mathrm{id}_{u^*F}))y)$.

(iii) *On 1-cells and 2-cells.* A pseudonatural $\varphi\colon Y \Rightarrow Y'$ goes to $(F, y) \mapsto (F, \varphi(y))$, with the constraints of φ, and a modification goes to the identity on the first component.

$\Lambda_{\mathscr{N}}$ is restriction along $\odot$ followed by the fibrewise Grothendieck construction along p, and this is its indexed form. Along the Grothendieck construction over $\mathscr{C}$, which takes $[\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}]$ to the split fibrations over $\mathscr{C}$, it pulls a fibration q back along $\odot$ and composes with p, so that $\Lambda_{\mathscr{N}} q = \Sigma_p \odot^* q$. Reading $q\colon \mathscr{Q} \to \mathscr{C}$ as the span $\mathscr{C} \xleftarrow{q} \mathscr{Q} \to 1$ into the terminal category, this is the composite of spans $\Lambda_{\mathscr{N}} q = \mathscr{N} \ltimes q$, the strict pullback of $\odot$ and q with the left leg p [CM24, Lemma 3.11]. For spans of categories it is the linear 2-functor $\Sigma_p \Delta_{\odot}$ [Web15, Thm. 4.1.4], [GK13, Ex. 1.7], restricted to split fibrations.

Proposition 2.7 (The linear functor is monoidal L2)**.** *On spans with split left legs, Λ has the following properties.*

(i) Composition. $\Lambda_{\mathscr{M} \ltimes \mathscr{N}} \cong \Lambda_{\mathscr{N}} ; \Lambda_{\mathscr{M}}$ *and* $\Lambda_{{}_1\mathscr{C}_1} \cong \mathrm{id}$ *by 2-natural isomorphisms that satisfy the coherence of a monoidal functor.*

(ii) Coproducts. $\Lambda_{\coprod_i \mathscr{N}_i} \cong \coprod_i \Lambda_{\mathscr{N}_i}$*, and each* $\Lambda_{\mathscr{N}}$ *preserves coproducts.*

Proof. In fibred form $\Lambda_{\mathscr{N}} q = \mathscr{N} \ltimes q$. (i) The isomorphisms are the canonical isomorphisms $(\mathscr{M} \ltimes \mathscr{N}) \ltimes q \cong \mathscr{M} \ltimes (\mathscr{N} \ltimes q)$ and ${}_1\mathscr{C}_1 \ltimes q \cong q$ of iterated strict pullbacks, which are unique and hence 2-natural, compatible with base change and coherent. The left legs stay split by Lemmas A.2 and A.3. (ii) $\ltimes$ preserves coproducts in each variable by Lemma 2.2. □

The linear functor also acts on the right-colax maps of Definition 2.1(ii) whose functor is cartesian, and this is what turns a contextad into a pseudomonad [CM24, Sec. 1.3.2, Def. 2.16, Lemma 3.11].

Lemma 2.8 (The linear functor on right-colax maps L2)**.** *Let $\mathscr{N}$ and $\mathscr{N}'$ have split left legs.*

(i) 1-cells. *A right-colax map (k, φ) whose functor is cartesian gives the pseudonatural transformation $\Lambda_{(k,\varphi)}\colon \Lambda_{\mathscr{N}} \Rightarrow \Lambda_{\mathscr{N}'}$ with the components $(F, y) \mapsto (kF,\ Y(\varphi_F)y)$.*

(ii) 2-cells. *A 2-cell σ gives the modification Λ_σ with the components (σ_F, id).*

(iii) Functoriality. *Λ preserves the composition of right-colax maps and of 2-cells, and up to the isomorphisms of Proposition 2.7(i) it preserves their composites by $\ltimes$.*

Proof. (i) Each component is a functor because φ is natural. It is pseudonatural in B with the constraints $(\kappa_{u,F}, \mathrm{id})$, because k is cartesian and φ is natural at the chosen lifts. It is moreover pseudonatural in Y, and at a pseudonatural $\psi\colon Y \Rightarrow Y'$ its constraint is $(\mathrm{id}, \psi_{\varphi_F})$. (ii) (σ_F, id) is a morphism from $(kF, Y(\varphi_F)y)$ to $(k'F, Y(\varphi'_F)y)$ because $\varphi_F = \odot'(\sigma_F) ; \varphi'_F$. (iii) The composite colaxator is $\varphi'_{kF} ; \varphi_F$, and Y is a functor. Composites of 2-cells are computed on first components. The composite $k \ltimes h$ of right-colax maps moves the second family along the chosen lift of φ_F [CM24, Lemma 3.11], and so does the base change of $\Lambda_{\mathscr{N}} Y$ in Construction 2.6(ii). □

Hence Λ is a monoidal 2-functor

$$\Lambda\colon \mathbf{H}^{\mathrm{split}}_{\mathscr{C}} \to \mathbf{Hom}_{\mathfrak{Gray}}\big([\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}], [\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}]\big)$$

from the spans with split left legs and the right-colax maps that are cartesian to the endo-2-functors, pseudonatural transformations and modifications, and it sends $\ltimes$ to composition.

2.4. The pseudomonad of a contextad

Proposition 2.9 (The pseudomonad of a contextad L2)**.** *Let* $({}_p\mathscr{N}_\odot, \mathrm{id}, \fatsemi)$ *be a contextad on* $\mathscr{C}$ *whose families form a split fibration.*

(i) Unit. *The counit* $(\mathrm{id}_{(-)}, \varepsilon)$ *gives* $\eta\colon \mathrm{id} \Rightarrow \Lambda_{\mathscr{N}}$ *with the components* $\eta_Y(y) = (\mathrm{id}_B,\ Y(\varepsilon_B)y)$ *for* $y \in Y(B)$.

(ii) Multiplication. *The comultiplication* $(\fatsemi, \delta)$ *gives* $\mu\colon \Lambda_{\mathscr{N}}\Lambda_{\mathscr{N}} \Rightarrow \Lambda_{\mathscr{N}}$ *with the components*

$$\mu_Y(F,(G,y)) = (F \fatsemi G,\ Y(\delta_{F,G})y).$$

Here $F \in \mathscr{N}_B$ *and* $G \in \mathscr{N}_{B\odot F}$*, while* $y \in Y((B \odot F) \odot G)$.

(iii) Pseudomonad. $(\Lambda_{\mathscr{N}}, \eta, \mu)$ *is a pseudomonad on* $[\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}]$*, the image of the contextad under* Λ*. Its associativity and unit constraints are* (α, id)*,* (λ, id) *and* (ρ, id).

(iv) At 1. $\Lambda_{\mathscr{N}}1 = \mathscr{N}_{(-)}$ *is the functor of fibres, with* $\eta_1 = \mathrm{id}_{(-)}$ *and* $\mu_1 = \fatsemi$.

Proof. (iv) holds because the Grothendieck construction of the terminal functor is the fibre and takes no second component. The counit and the comultiplication are right-colax maps $(\mathrm{id}_{(-)}, \varepsilon)\colon {}_1\mathscr{C}_1 \to \mathscr{N}$ and $(\fatsemi, \delta)\colon \mathscr{N} \ltimes \mathscr{N} \to \mathscr{N}$ that are cartesian, and Lemma 2.8(i) turns them into η and μ after the isomorphisms $\Lambda_{{}_1\mathscr{C}_1} \cong \mathrm{id}$ and $\Lambda_{\mathscr{N}\ltimes\mathscr{N}} \cong \Lambda_{\mathscr{N}}\Lambda_{\mathscr{N}}$ of Proposition 2.7. This proves (i) and (ii). (iii) By Proposition 2.7 and Lemma 2.8, Λ is a monoidal 2-functor. It sends pseudomonoids to pseudomonoids, since the three pseudomonoid axioms are preserved by the coherence of its monoidal structure (a direct check). It therefore sends the contextad, which is a pseudomonoid in $(\mathbf{H}_{\mathscr{C}}, \ltimes, {}_1\mathscr{C}_1)$ [CM24, Def. 3.15], to a pseudomonad, whose constraints are the images (α, id), (λ, id) and (ρ, id) of Lemma 2.8(ii). □

When p is the identity of $\mathscr{C}$, the families are trivial, and each construction of this section reduces to a classical one.

	Contextad $({}_p\mathscr{N}_\odot, \mathrm{id}, \fatsemi)$	Comonad W, $p = \mathrm{id}_{\mathscr{C}}$
structure	$\odot$, $(\mathrm{id}_{(-)}, \varepsilon)$, $(\fatsemi, \delta)$	W, ε, δ
arrows (Def. 2.5)	(F, f), $f\colon A \odot F \to B$	coKleisli maps $WA \to B$
linear functor (Prop. 2.9)	$\Lambda_{\mathscr{N}}$, η, μ	W^*, $Y(\varepsilon)$, $Y(\delta)$

Proposition 2.10 (The two faces L2)**.** *Let* $({}_p\mathscr{N}_\odot, \mathrm{id}, \fatsemi)$ *be a contextad on* $\mathscr{C}$ *whose families form a split fibration, and let* $\mathbb{K}\mathrm{l}(\Lambda_{\mathscr{N}})$ *be the Kleisli construction of* $\Lambda_{\mathscr{N}}$ *on representables: its loose morphisms* $B \nrightarrow D$ *are the 1-cells* $s\colon \mathrm{y}B \to \Lambda_{\mathscr{N}}\mathrm{y}D$*, its cells over* $u\colon B \to B'$ *and* $v\colon D \to D'$ *the modifications* $s\,;\Lambda_{\mathscr{N}}\mathrm{y}v \Rightarrow \mathrm{y}u\,;s'$*, its loose composite* $s\,;\Lambda_{\mathscr{N}}t\,;\mu_{\mathrm{y}Z}$ *and its loose identities* $\eta_{\mathrm{y}B}$*. Then* $\mathbb{K}\mathrm{l}(\Lambda_{\mathscr{N}}) \simeq \mathrm{co}\mathbb{K}\mathrm{l}(\fatsemi)$ *by an equivalence that is the identity on objects and tight morphisms and sends* s *to a contextful arrow* (F, f) *with* $f\colon B \odot F \to D$*. A family* F *is the loose morphism* $(F, \mathrm{id})\colon B \nrightarrow B \odot F$ *(here* id *is the identity functor of* $B \odot F$*), and for a family* G *at* $B \odot F$

$$(F, \mathrm{id}) \fatsemi (G, \mathrm{id}) = \big(F \fatsemi G,\ \delta_{F,G}\colon B \odot (F \fatsemi G) \to (B \odot F) \odot G\big). \tag{2.2}$$

Proof. By the 2-Yoneda lemma, $[\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}](\mathrm{y}B, \Lambda_{\mathscr{N}}\mathrm{y}D) \simeq (\Lambda_{\mathscr{N}}\mathrm{y}D)(B)$, whose objects are the pairs (F, f) with $f \in \mathrm{Hom}_{\mathscr{C}}(B \odot F, D)$. Let $s = (F, f)$ and $t = (G, g)$ with $G \in \mathscr{N}_D$ and $g\colon D \odot G \to Z$. At the generic element, $\Lambda_{\mathscr{N}}t$ sends (F, f) to $(F, (f^*G, \odot(f, \mathrm{id}_{f^*G})\,; g))$, which $\mu_{\mathrm{y}Z}$ then sends to $(F \fatsemi f^*G,\ \delta_{F,f^*G}\,; \odot(f, \mathrm{id}_{f^*G})\,; g)$. Likewise η gives $(\mathrm{id}_B, \varepsilon_B)$, and so these are the loose composition and identities of $\mathrm{co}\mathbb{K}\mathrm{l}(\fatsemi)$. At the generic element, a modification $s\,;\Lambda_{\mathscr{N}}\mathrm{y}v \Rightarrow \mathrm{y}u\,;s'$ is a morphism from $(F, f\,;v)$ to $u^*(F', f')$, which amounts to a $\theta\colon F \to F'$ above u with $f\,;v = \odot(\theta)\,;f'$. The globular constraints agree because both sides are given by α, λ, ρ and the comparisons with base change. □

3. Telescopes and the triangle

This section builds the telescope monad on spans and proves the triangle at level 2 (Theorem 3.8).

3.1. The telescope monad

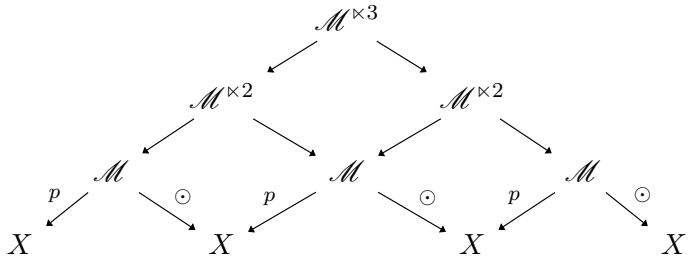
This subsection works at level 2 over a fixed category X. The telescope monad concatenates spans, and its algebras compose them.

Definition 3.1 (Powers and chains of a span L2)**.** For a span ${}_p\mathscr{M}_\odot$ on X and $n \geq 0$, the n-th power $\mathscr{M}^{\ltimes n}$ is the span with the following parts.

(i) *Apex.* $\mathscr{M}^{\ltimes 0} = X$ and $\mathscr{M}^{\ltimes(n+1)} = \mathscr{M}^{\ltimes n} \ltimes \mathscr{M}$.

(ii) *Chains.* The objects of $\mathscr{M}^{\ltimes n}$ are the *chains* $(A \mid \vec{P}) = (A \mid P_0, \dots, P_{n-1})$ with $A \in X$ and $P_i \in \mathscr{M}_{A \odot P_0 \odot \cdots \odot P_{i-1}}$ for $0 \leq i < n$, where the iterated $\odot$ is bracketed to the left.

(iii) *Legs.* By pasting of pullbacks, $\mathscr{M}^{\ltimes n}$ is the apex of a pyramid of pullbacks, shown for $n = 3$,



(3.1)

The legs are the outer edges of the pyramid, and hence for $n \geq 1$ the right leg is the projection to the last factor $\mathscr{M}$ followed by $\odot$. The pyramid is *cartesian*, the right Kan extension of its bottom row, as for spans under the Segal condition [Bar14, Def. 3.3], [Hau18, Def. 5.5]; here its pullbacks are the pairs of Definition 2.1(iv).

Lemma 3.2 (Powers of a span L2)**.**

(i) Concatenation. $\mathscr{M}^{\ltimes n} \ltimes \mathscr{M}^{\ltimes m} \cong \mathscr{M}^{\ltimes(n+m)}$.

(ii) Split powers. *If p is a split fibration, then so is each functor $\mathscr{M}^{\ltimes(n+1)} \to \mathscr{M}^{\ltimes n}$ on the left edge of the pyramid (3.1), which forgets the last entry of a chain. Hence the left leg of $\mathscr{M}^{\ltimes n}$, their composite with p, is a split fibration, and its base change acts on the entries of a chain one after the other.*

Proof. Part (i) is the canonical isomorphism of iterated pullbacks. Part (ii) follows from Lemmas A.2 and A.3 because $\mathscr{M}^{\ltimes(n+1)} = \mathscr{M}^{\ltimes n} \ltimes \mathscr{M} \to \mathscr{M}^{\ltimes n}$ is the pullback of p along the right leg of $\mathscr{M}^{\ltimes n}$. □

Proposition 3.3 (Free monoids as coproducts of powers L2)**.** *Let $(\mathbf{K}, \boxtimes)$ be a monoidal 2-category with countable coproducts, preserved by $\boxtimes$ in each variable, and let $\mathbf{Mon}(\mathbf{K}, \boxtimes)$ be the 2-category of monoids in $\mathbf{K}$, the 1-cells preserving unit and multiplication and the compatible 2-cells. The forgetful 2-functor $U\colon \mathbf{Mon}(\mathbf{K}, \boxtimes) \to \mathbf{K}$ has a left 2-adjoint* Fr*, and the free monoid* $\mathrm{Fr}(Y)$ *is $\coprod_{n \geq 0} Y^{\boxtimes n}$, with concatenation as multiplication and the injection of $Y^{\boxtimes 0}$ as unit. When $\mathbf{K}$ consists of coproduct-preserving endo-2-functors, 2-natural transformations and modifications under composition, the monoids are the 2-monads and* $\mathrm{Fr}(Y)$ *is the* free monad *on Y.*

Proof. The free monoid is Kelly's construction [Kel80, Thm. 23.4], and among the endo-spans of an object it is the free internal category [Lac10, Sec. 5]. Its universal property extends to 2-cells. Indeed, a 2-cell out of $\coprod_n Y^{\boxtimes n}$ compatible with the multiplication is

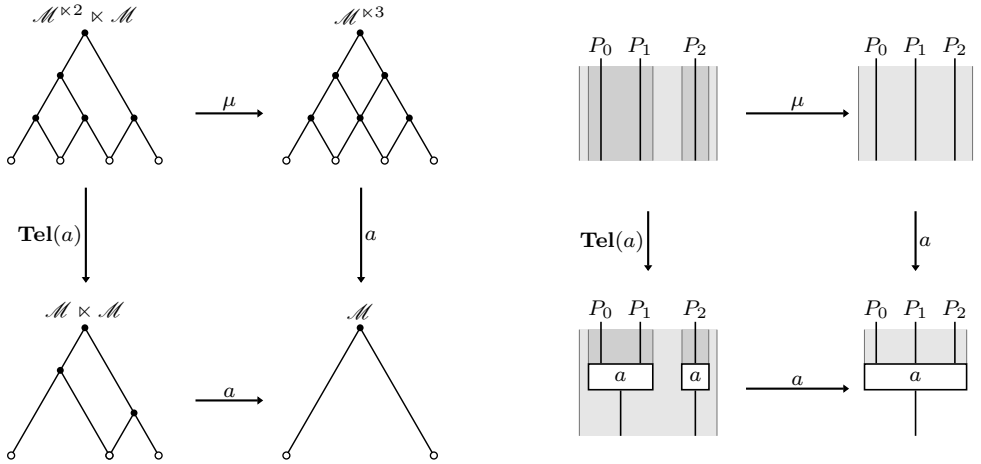


FIGURE 2. L2 Concatenating and composing for an algebra $(\mathscr{M}, a)$ of **Tel**, as pasting diagrams in **CAT** (left) and stripe diagrams (right, Table 4). From the summand $\mathscr{M}^{\ltimes 2} \ltimes \mathscr{M}$ of $\mathbf{Tel}^2(\mathscr{M})$, μ concatenates before a composes, and $\mathbf{Tel}(a)$ composes the blocks first.

determined by its restriction σ along the injection of Y, and conversely every σ extends by $\sigma^{\boxtimes n}$ on the n-th summand. Hence the adjunction is a 2-adjunction. □

Definition 3.4 (Telescope monad L2)**.** The *telescope monad* $(\mathbf{Tel}, \eta, \mu)$ is the 2-monad on $\mathbf{H}^{\mathrm{s}}_X$ of the 2-adjunction $\mathrm{Fr} \dashv U$ for $(\mathbf{H}^{\mathrm{s}}_X, \ltimes)$, and it consists of the following.

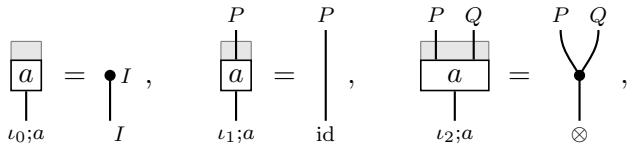
(i) *Objects.* $\mathbf{Tel}({}_p\mathscr{M}_\odot) = \coprod_{n\geq 0} \mathscr{M}^{\ltimes n}$ with the legs of the summands, which is also called the Kleene star of $\mathscr{M}$.
(ii) *1-cells and 2-cells.* $\mathbf{Tel}(k) = \coprod_n k^{\ltimes n}$ and $\mathbf{Tel}(\sigma) = \coprod_n \sigma^{\ltimes n}$.
(iii) *Unit.* η is the coproduct injection ι_1 of $\mathscr{M} = \mathscr{M}^{\ltimes 1}$.
(iv) *Multiplication.* μ is the coproduct of the concatenation isomorphisms $\mathscr{M}^{\ltimes n_1} \ltimes \cdots \ltimes \mathscr{M}^{\ltimes n_r} \cong \mathscr{M}^{\ltimes (n_1+\cdots+n_r)}$ of Lemma 3.2(i), whose domains are the summands of $\mathbf{Tel}^2(\mathscr{M})$ by Lemma 2.2. It concatenates chains of chains and never composes them.

Remark 3.5 (Lists and telescopes L2). The telescope monad is the list monad of spans, except that $\ltimes$ puts each entry over the previous ones. Its free monoid is $(\mathbf{Tel}({}_p\mathscr{M}_\odot), (- \mid), +\!+)$, whose unit is the injection of the length 0, and so a telescope $(B \mid)$ of length 0 has no family and both its feet are B.

	Lists	Telescopes
on	$(\mathsf{Set}, \times, 1)$	$(\mathbf{H}^{\mathrm{s}}_X, \ltimes, {}_1X_1)$
monad, η, μ	List, length 1, concatenate	**Tel**, length 1, concatenate
free monoid	$(\mathrm{List}(A), [\,], +\!+)$	$(\mathbf{Tel}({}_p\mathscr{M}_\odot), (- \mid), +\!+)$
its unit, at $n = 0$	$[\,]\colon 1 = A^0 \to \mathrm{List}(A)$	$(- \mid)\colon {}_1X_1 = \mathscr{M}^{\ltimes 0} \to \mathbf{Tel}(\mathscr{M})$
algebras	$\mathbf{EM}(\mathrm{List}) \cong \mathbf{Mon}(\mathsf{Set}, \times)$	$\mathbf{EM}(\mathbf{Tel}) \cong \mathbf{Mon}(\mathbf{H}^{\mathrm{s}}_X, \ltimes)$

Corollary 3.6 (Algebras of the telescope monad L2)**.** *An algebra $(\mathscr{M}, a)$ of* **Tel** *is a 1-cell $a\colon \mathbf{Tel}(\mathscr{M}) \to \mathscr{M}$ of $\mathbf{H}^{\mathrm{s}}_X$ with the unit law $\eta\,;a = \mathrm{id}$ and the multiplication law $\mu\,;a = \mathbf{Tel}(a)\,;a$. The algebras of* **Tel** *are the monoids $(\mathscr{M}, I, \otimes)$ in $(\mathbf{H}^{\mathrm{s}}_X, \ltimes)$, and hence the strict double categories with tight category X whose source is a fibration. In fact, the 2-category* $\mathbf{EM}(\mathbf{Tel})$ *of algebras, maps of algebras and compatible 2-cells is isomorphic to*

the 2-category $\mathbf{Mon}(\mathbf{H}^{\mathrm{s}}_X, \ltimes)$ *of Proposition 3.3. Along the coproduct injections* $\iota_n \colon \mathscr{M}^{\ltimes n} \to \mathbf{Tel}(\mathscr{M})$*, an algebra and its monoid correspond by*



(3.2)

and $\iota_n \,;\, a = (\mathscr{M}^{\ltimes(n-2)} \ltimes \otimes) \,;\, \cdots \,;\, (\mathscr{M} \ltimes \otimes) \,;\, \otimes$ *for* $n \geq 3$*, as Figure 2 draws for* $n = 3$*.*

Proof. By Beck's theorem the forgetful 2-functor $U \colon \mathbf{Mon}(\mathbf{H}^{\mathrm{s}}_X, \ltimes) \to \mathbf{H}^{\mathrm{s}}_X$ is strictly monadic, since a split fork stays split after $\ltimes$ in each variable [ML98, Thm. VI.7.1]. The comparison sends a monoid to its iterated multiplications, as for unbiased monoidal categories [Lei04, after Cor. 3.2.5], and its inverse restricts a along ι_0 and ι_2. Figure 2 draws the multiplication law $\mu \,;\, a = \mathbf{Tel}(a) \,;\, a$ on the summand $\mathscr{M}^{\ltimes 2} \ltimes \mathscr{M}$ of $\mathbf{Tel}^2(\mathscr{M})$. □

3.2. The triangle

Definition 3.7 (Telescopes L0)**.** Let $\mathscr{N}$ be the families of a contextad. A *telescope* of length n at B is a chain $(B \mid F_0, \ldots, F_{n-1})$ of $\mathscr{N}$ in the sense of Definition 3.1(ii), whose entries are families F_i at the bases $B^{(0)} := B$ and $B^{(i+1)} := B^{(i)} \odot F_i$. A morphism $(u_0 \mid \vec{\tau})$ of telescopes is a list of morphisms $\tau_i \colon G_i \to F_i$ above u_i with $u_{i+1} := \odot\tau_i$, and $\mathbf{Tel}({}_p\mathscr{N}_\odot)$ is the span of telescopes, whose fibre $\mathbf{Tel}(\mathscr{N})_B$ has the morphisms with $u_0 = \mathrm{id}_B$. A telescope is to **Tel** what a list is to the list monad.

For $n \geq 2$ let $\mu^{(n)} := \Lambda^{n-2}_{\mathscr{N}}\mu \,;\, \cdots \,;\, \Lambda_{\mathscr{N}}\mu \,;\, \mu \colon \Lambda^n_{\mathscr{N}} \Rightarrow \Lambda_{\mathscr{N}}$, where $\Lambda^k_{\mathscr{N}}\mu$ is the 2-functor $\Lambda^k_{\mathscr{N}}$ applied to μ. Moreover, let $\mu^{(0)} := \eta$ and $\mu^{(1)} := \mathrm{id}$. The n-fold composite, or n-fold dependent sum, of a telescope $(B \mid F_0, \ldots, F_{n-1})$ is the family $F_0 \mathbin{⨟} (F_1 \mathbin{⨟} (\cdots \mathbin{⨟} F_{n-1}))$ at B, which is id_B for $n = 0$ and F_0 for $n = 1$.

Theorem 3.8 (The triangle at level 2 L2→1)**.** *Let* $({}_p\mathscr{N}_\odot, \mathrm{id}, ⨟)$ *be a contextad on* $\mathscr{C}$ *whose families form a split fibration.*

(i) Grothendieck–telescope. $\Lambda_{\mathbf{Tel}(\mathscr{N})} \cong \coprod_n \Lambda^n_{\mathscr{N}}$ *is the free monad on* $\Lambda_{\mathscr{N}}$*. At* 1 *it gives the telescopes:* $\Lambda^n_{\mathscr{N}}1 \cong (\mathscr{N}^{\ltimes n})_{(-)}$ *is the functor of telescopes of length* n*, and so* $\mathrm{dom} \int_{\mathscr{C}} \Lambda^n_{\mathscr{N}}1 \cong \mathscr{N}^{\ltimes n}$ *over* $\mathscr{C}$*. Moreover, the left leg of* $\mathbf{Tel}(\mathscr{N})$ *is a split fibration over* $\mathscr{C}$*.*

(ii) Grothendieck–contextad. Λ *sends the contextad to its pseudomonad* $\Lambda_{\mathscr{N}}$ *of Proposition 2.9, whose Kleisli construction on representables is* $\mathbb{K}\mathrm{l}(\Lambda_{\mathscr{N}}) \simeq \mathrm{co}\mathbb{K}\mathrm{l}(⨟)$ *(Proposition 2.10).*

(iii) Telescope–contextad. $\coprod_n \mu^{(n)}$ *runs from the free monad* $\coprod_n \Lambda^n_{\mathscr{N}}$ *to the pseudomonad* $\Lambda_{\mathscr{N}}$*, and its domain is* $\Lambda_{\mathbf{Tel}(\mathscr{N})}$ *by (i). Since* η *and* μ *are only pseudonatural, it is defined directly and not by the universal property of the free monad. At* 1 *it sends a telescope to its* n*-fold composite, and its Grothendieck construction over* $\mathscr{C}$ *is a functor* $\bar{\mu} \colon \mathbf{Tel}(\mathscr{N}) \to \mathscr{N}$ *over* $\mathscr{C}$ *that is cartesian. Along* ι_0 *and* ι_2 *it restricts to the unit* $\mathrm{id}_{(-)}$ *and the comultiplication* $⨟$*, and for a strict contextad it is the structure map of the algebra of* **Tel** *in Corollary 3.6.*

Proof. (i) Since the left leg of $\mathscr{N}$ is split, the left legs of all powers $\mathscr{N}^{\ltimes n}$ are split by Lemma 3.2(ii), and Proposition 2.7 gives $\Lambda_{\mathbf{Tel}(\mathscr{N})} \cong \coprod_n \Lambda^n_{\mathscr{N}}$. The coproduct-preserving endo-2-functors with 2-natural transformations and modifications form a monoidal 2-category under composition, whose tensor preserves coproducts in each variable, and there $\coprod_n \Lambda^n_{\mathscr{N}}$ is the free monad on $\Lambda_{\mathscr{N}}$ by Proposition 3.3. At 1, $\Lambda_{\mathscr{N}}1$ is the functor of fibres of p, and hence $\Lambda^n_{\mathscr{N}}1$ is the functor of fibres of $\mathscr{N}^{\ltimes n}$, the telescopes of length n. The left

leg of $\mathbf{Tel}(\mathscr{N})$ is split by Lemma 3.2(ii), and so it is $\mathrm{dom}\int_{\mathscr{C}}\mathbf{Tel}(\mathscr{N})_{(-)}$ by Lemma 2.3. (ii) This is Propositions 2.9 and 2.10. (iii) Part (i) identifies $\coprod_n \Lambda^n_{\mathscr{N}}$ with $\Lambda_{\mathbf{Tel}(\mathscr{N})}$. At 1, $\mu_1 = \fatsemi$ by Proposition 2.9(iv), and $\Lambda^k_{\mathscr{N}}\mu$ applies it to the last two families of a telescope of length $k+2$. Hence $\mu_1^{(n)}$ composes the families from the right, and it sends a telescope to its n-fold composite. As a 1-cell $\Lambda^n_{\mathscr{N}}1 \to \Lambda_{\mathscr{N}}1$ of $[\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}]$ it is a pseudonatural transformation between functors of fibres. Its Grothendieck construction over $\mathscr{C}$ is therefore a functor $\mathscr{N}^{\ltimes n} \to \mathscr{N}$ over $\mathscr{C}$ that is cartesian by (i) and Lemma 2.3. At lengths 0 and 2, $\mu_1^{(0)} = \eta_1 = \mathrm{id}_{(-)}$ and $\mu_1^{(2)} = \mu_1 = \fatsemi$ by Proposition 2.9(iv). For a strict contextad, $\mathrm{id}_{(-)}$ and $\fatsemi$ are the unit and multiplication of a monoid in spans, and $\bar{\mu}$ composes chains from the right as in (3.2), and so it is the structure map of Corollary 3.6. □

4. Examples

The triangle has two examples, families of categories over $\mathscr{C} = \mathsf{Cat}$ in Section 4.1 and double families over $\mathscr{C} = \mathsf{Dbl}_{\mathrm{s}}$ in Sections 4.2–4.3.

4.1. Families

Example 4.1 (The contextad of families $\boxed{\mathsf{L0}\to\mathsf{1}}$)**.** Let $\mathscr{C} = \mathsf{Cat}$, the category of small categories, and read the single column of Table 1. A *family* on B is an indexed category $F\colon B^{\mathrm{op}} \to \mathbf{Cat}$ with the fibration $\int F\colon B \odot F \to B$, $B \odot F := \mathrm{dom}\int F$, whose objects are the pairs (b, x) with $x \in F(b)$. A morphism $(u, \tau)\colon (A, G) \to (B, F)$ of ICat is a pair of $u\colon A \to B$ and $\tau\colon G \Rightarrow u^{\mathrm{op}}\,;F$, and $p\colon \mathsf{ICat} \to \mathsf{Cat}$ is a split fibration with base change $u^*F = u^{\mathrm{op}}\,;F$ and chosen lifts (u, id_{u^*F}) [MV20, Rem. 2.2], while $p = \int;\mathrm{cod}$ and $\odot = \int;\mathrm{dom}$ for $\int\colon \mathsf{ICat} \to \mathsf{Cat}^{\to}$. The unit family id_B is terminal, and the dependent sum is $(F \fatsemi G)(b) = \Sigma_{x\in F(b)}G(b, x)$. The counit ε_B, the comultiplication $\delta_{F,G}$ and the coherence isomorphisms are the canonical isomorphisms of iterated Grothendieck constructions, which rebracket the pairs (Lemma 4.2). This is the contextad $({}_p\mathsf{ICat}_{\odot}, \mathrm{id}, \fatsemi)$ of families on Cat, to which Theorem 3.8 applies. Here ICat and its fibres are the underlying 1-categories of the 2-categories of [MV20].

Lemma 4.2 (Grothendieck constructions of families $\boxed{\mathsf{L0}}$)**.** *Let F be a family on a category B.*

(i) Base change. *For a functor $f\colon E \to B$, the image $\odot(f, \mathrm{id}_{f^*F})\colon E \odot (f^{\mathrm{op}}\,;F) \to B \odot F$ of the chosen lift sends (e, x) to (fe, x). Then $(\int(f^{\mathrm{op}}\,;F),\ \odot(f, \mathrm{id}_{f^*F}))$ is an isomorphism from $E \odot (f^{\mathrm{op}}\,;F)$ onto the strict pullback of f and $\int F$, and so $\int(f^{\mathrm{op}}\,;F)$ is a strict pullback of $\int F$ along f.*

(ii) Fibrations compose. *For a family G on $B \odot F$, $\delta_{F,G}\colon B \odot (F \fatsemi G) \to (B \odot F) \odot G$, which sends $(b, (x, y))$ to $((b, x), y)$, is an isomorphism of categories with $\delta_{F,G}\,;\int G\,;\int F = \int(F \fatsemi G)$. Hence $\int(F \fatsemi G) \cong \int G\,;\int F$.*

Proof. Both are the standard computations of the Grothendieck construction [Jac99, Sec. 1.10]. (i) Both sides have the pairs (e, x) with $x \in F(fe)$ and the morphisms and constraints of F whiskered by f^{op}. (ii) Both sides have the triples (b, x, y), and their morphisms differ by a composition constraint of G, which $\delta_{F,G}$ inserts while keeping b. □

Remark 4.3 (Families as fibrations $\boxed{\mathsf{L1}\to\mathsf{0}}$). The Grothendieck construction represents families by fibrations, since $\int\colon \mathsf{ICat} \simeq \mathsf{Fib}$ is an equivalence over Cat onto the fibrations and the functors between them that are cartesian [MV20, Thm. 2.3], and for double families the double Grothendieck construction of Section 4.2 gives $\int\colon \mathsf{Dbl2Cat}(\mathbb{B}^{\mathrm{op}}, \mathbb{S}\mathrm{pan}(\mathbf{Cat})) \simeq \mathsf{DblFib}(\mathbb{B})$ [CLPS22, Thm. 3.45, 3.51]. Under $\int$ a telescope $(B \mid F_0, \dots, F_{n-1})$ becomes

the tower of fibrations $\int F_i \colon B^{(i+1)} \to B^{(i)}$, and iterating Lemma 4.2(ii) turns its n-fold dependent sum into the composite of the tower, $\int (F_0 \fatsemi \cdots \fatsemi F_{n-1}) \cong \int F_{n-1} \,;\cdots;\, \int F_0$, a fibration by [Jac99, Lemma 1.5.5]. For presheaves these are the telescopes of Palmgren and their collapse by Σ [Pal16, Thms. 2.1, 3.2]. This is the special case of representing the families of a contextad by display maps (a class of maps closed under pullback), through a morphism of contextads into display maps [CM24, Ex. 3.21, Def. 4.25], and the paper does not use it further.

4.2. Doubly indexed categories in spans

This subsection collects the ingredients of the double case, and Section 4.3 builds from them the contextad of double telescopes.

Double categories with loose units y_b, double fibrations and $\mathsf{DblFib}(\mathbb{B})$ follow Definitions A.4 and A.5, and Dbl_s is the category of small double categories and strict double functors. The families of a double fibration $\mathbb{P} \colon \mathbb{E} \to \mathbb{B}$ take values in spans of categories, because its fibre $\mathbb{E}_b$ at an object b of $\mathbb{B}$ is a category and its fibre $\mathbb{E}_m$ at $m \colon b \nrightarrow b'$ is a category with functors to $\mathbb{E}_b$ and $\mathbb{E}_{b'}$ induced by src and tgt. These spans live in the following double 2-category.

Definition 4.4 (Double 2-category of spans L2, [CLPS22, Def. 3.2, Constr. 3.6])**.** The double 2-category $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ is the pseudo category in 2-categories with tight 2-category $\mathbf{Cat}$ and loose 2-category $\mathbf{Cat}^{\mathsf{Sp}}$ of 2-functors from the walking span $\mathsf{Sp} = \{0 \leftarrow s \rightarrow 1\}$ to $\mathbf{Cat}$, whose source and target evaluate at 0 and 1, composed by strict pullback with the identity spans as units.

Definition 4.5 (Doubly indexed category in spans L0, [CLPS22, Def. 3.12])**.** A *doubly indexed category in spans* on $\mathbb{B}$, or *double family*, is a normalised lax double pseudofunctor

$$\mathbb{F} = (F_0, F_1, \phi, \iota, \Phi) \colon \mathbb{B}^{\mathrm{op}} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$$

with the following parts.

(i) *Tight part.* A pseudofunctor $F_0 \colon \mathbb{B}_0^{\mathrm{op}} \to \mathbf{Cat}$, with $f^* \colon \mathbb{F}b' \to \mathbb{F}b$ for $f \colon b \to b'$ in $\mathbb{B}_0$.

(ii) *Loose part.* A pseudofunctor $F_1 \colon \mathbb{B}_1^{\mathrm{op}} \to \mathbf{Cat}^{\mathsf{Sp}}$ over F_0, with $\mathbb{F}b \leftarrow \mathbb{F}m \rightarrow \mathbb{F}b'$ for $m \colon b \nrightarrow b'$ and $\theta^* \colon \mathbb{F}n \to \mathbb{F}m$ for $\theta \colon m \to n$.

(iii) *Comparison maps.* $\phi_{m,n} \colon \mathbb{F}m \times_{\mathbb{F}b'} \mathbb{F}n \to \mathbb{F}(m \,;\, n)$ for $m \colon b \nrightarrow b'$ and $n \colon b' \nrightarrow b''$, and $\iota_b \colon \mathbb{F}b \to \mathbb{F}(\mathrm{y}_b)$, pseudonatural in cells.

(iv) *Associativity.* An invertible modification Φ comparing the two composites of ϕ, subject to the axioms of Definition A.7.

Double families on $\mathbb{B}$ and the transformations of Definition A.6 with invertible modifications T and J form a category $\mathsf{Dbl2Cat}(\mathbb{B}^{\mathrm{op}}, \mathbb{S}\mathrm{pan}(\mathbf{Cat}))$. When they take values in spans of sets, they recover the lax double presheaves of [Par11].

Figure 3 draws the comparison map $\phi_{m,n}$, while ι_b fills the degenerate triangle at b and Φ the tetrahedra.

By [CLPS22, Prop. 3.35, Cor. 3.44] a double family is a pseudo monoid in spans of indexed categories whose multiplication has the comparison maps ϕ as its apex, and such a pseudo monoid is a pseudo category in indexed categories with 2-natural source and target. In this description the laxness of ϕ lies in the 1-cells of indexed categories rather than in the internal structure, because these 1-cells are pseudonatural transformations with non-invertible components. Although Myers' doubly indexed categories [Mye23, Def. 4.3.0.1] are covariant and take values in profunctors, two-sided categories of elements turn them into double families.

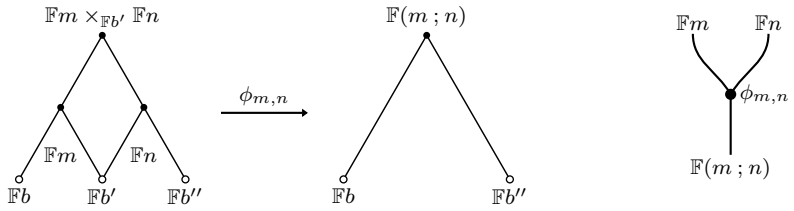


FIGURE 3. L0 The comparison map of a double family for $m\colon b \nrightarrow b'$ and $n\colon b' \nrightarrow b''$, as a pasting diagram of spans in **CAT** and as a string diagram of spans of categories.

The double Grothendieck construction takes $\mathbb{F}$ to the double fibration $\int\mathbb{F}\colon \mathbb{B}\odot\mathbb{F} \to \mathbb{B}$ from its total double category $\mathbb{B}\odot\mathbb{F} := \mathrm{dom}\int\mathbb{F}$, and it is an equivalence

$$\textstyle\int\colon \mathsf{Dbl2Cat}(\mathbb{B}^{\mathrm{op}}, \mathbb{S}\mathrm{pan}(\mathbf{Cat})) \simeq \mathsf{DblFib}(\mathbb{B})$$

with the following parts.

Construction 4.6 (Double Grothendieck construction L0→1, [CLPS22, Thm. 3.45, 3.51])**.** Let $\mathbb{F}$ and $\mathbb{G}$ be double families on $\mathbb{B}$. Then $\mathbb{B}\odot\mathbb{F}$ has parts (i) and (ii), $\int\mathbb{F}$ takes first components, and a morphism $\tau\colon \mathbb{F} \to \mathbb{G}$ of $\mathsf{Dbl2Cat}(\mathbb{B}^{\mathrm{op}}, \mathbb{S}\mathrm{pan}(\mathbf{Cat}))$ gives the pseudo double functor $\int\tau$ of part (iii).

(i) *Tight and loose categories.* The Grothendieck constructions of F_0 and F_1: objects (b, X) with $X \in \mathbb{F}b$ and tight morphisms $(f, \bar{f})$ with $\bar{f}\colon X \to f^*Y$, loose morphisms $(m, \bar{m})$ with $\bar{m} \in \mathbb{F}m$ and cells $(\theta, \bar{\theta})$ with $\bar{\theta}\colon \bar{m} \to \theta^*\bar{n}$.

(ii) *Loose structure.* $(m,\bar{m})\,;(n,\bar{n}) = \big(m\,;n,\ \phi(\bar{m},\bar{n})\big)$, $\mathrm{y}_{(b,X)} = \big(\mathrm{y}_b,\ \iota_b X\big)$, and the associators $(\alpha^{\mathbb{B}}, \Phi)$.

(iii) *Transformations.* $\int\tau\colon \mathbb{B}\odot\mathbb{F} \to \mathbb{B}\odot\mathbb{G}$ lies over $\mathbb{B}$, since it is the identity on first components and τ on second components,

$$\begin{aligned} \textstyle\int\tau(b,X) &= (b, \tau_b X), \qquad & \textstyle\int\tau(f,\bar{f}) &= \big(f, \tau_b(\bar{f})\,;\tau_f\big), \\ \textstyle\int\tau(m,\bar{m}) &= (m, \tau_m\bar{m}), & \textstyle\int\tau(\theta,\bar{\theta}) &= \big(\theta, \tau_m(\bar{\theta})\,;\tau_\theta\big). \end{aligned} \tag{4.1}$$

4.3. The contextad of double telescopes

The double families on $\mathbb{B}$ with the strict transformations of Definition A.6 form the fibre

$$\mathsf{DblICat}_{\mathbb{B}} := \mathsf{Dbl2Cat}_{\mathrm{s}}(\mathbb{B}^{\mathrm{op}}, \mathbb{S}\mathrm{pan}(\mathbf{Cat})).$$

Precomposition $\mathbb{U}^{\mathrm{op}}\,;\mathbb{F}$ with a strict $\mathbb{U}\colon \mathbb{A} \to \mathbb{B}$ reindexes them, and it is strictly functorial in $\mathbb{U}$. Hence their Grothendieck construction

$$\mathsf{DblICat} := \mathrm{dom}\textstyle\int_{\mathsf{Dbl}_{\mathrm{s}}} \mathsf{Dbl2Cat}_{\mathrm{s}}((-)^{\mathrm{op}}, \mathbb{S}\mathrm{pan}(\mathbf{Cat}))$$

over $\mathsf{Dbl}_{\mathrm{s}}$ has a split fibration p to $\mathsf{Dbl}_{\mathrm{s}}$, with base change $\mathbb{U}^*\mathbb{F} := \mathbb{U}^{\mathrm{op}}\,;\mathbb{F}$. A morphism $(\mathbb{U},\tau)\colon (\mathbb{A},\mathbb{G}) \to (\mathbb{B},\mathbb{F})$ of $\mathsf{DblICat}$ consists of $\mathbb{U}$ and a strict $\tau\colon \mathbb{G} \to \mathbb{U}^{\mathrm{op}}\,;\mathbb{F}$, and it determines the strict double functor

$$\textstyle\odot(\mathbb{U},\tau) := (\int\tau)\,;(\mathbb{U}\odot\mathbb{F})\colon \mathbb{A}\odot\mathbb{G} \to \mathbb{B}\odot\mathbb{F}, \tag{4.2}$$

which composes $\int\tau$ of Construction 4.6(iii) with the strict double functor $\mathbb{U}\odot\mathbb{F}\colon \mathbb{A}\odot(\mathbb{U}^{\mathrm{op}}\,;\mathbb{F}) \to \mathbb{B}\odot\mathbb{F}$ that is $\mathbb{U}$ on first components and the identity on second components.

A dependent sum restricts a double family $\mathbb{G}$ on $\mathbb{B}\odot\mathbb{F}$ to the fibres of $\int\mathbb{F}$. For an object b of $\mathbb{B}$, $\mathbb{G}(b,-)$ restricts G_0 along the inclusion of $\mathbb{F}b$ into the tight category of $\mathbb{B}\odot\mathbb{F}$.

Likewise, for $m\colon b \nrightarrow b'$, $\mathbb{G}(m,-)$ restricts the apices of G_1 along the inclusion of $\mathbb{F}m$ into the loose category. The objects of their Grothendieck constructions $\Sigma_{X\in\mathbb{F}b}\mathbb{G}(b,X)$ and $\Sigma_{\bar{m}\in\mathbb{F}m}\mathbb{G}(m,\bar{m})$ are the pairs (X,y) of $X\in\mathbb{F}b$ and $y\in\mathbb{G}(b,X)$ and the pairs $(\bar{m},\bar{y})$ of $\bar{m}\in\mathbb{F}m$ and $\bar{y}\in\mathbb{G}(m,\bar{m})$.

Theorem 4.7 (The contextad of double telescopes L0→2)**.** *The double families form a contextad* $({}_p\mathsf{DblIICat}_\odot, \mathrm{id}, \fatsemi)$ *on* Dbl_s*, the* contextad of double telescopes*, which has the following parts.*

(i) Action. $\mathbb{B}\odot\mathbb{F} := \mathrm{dom}\int\mathbb{F}$ *on objects and* (4.2) *on morphisms.*

(ii) Unit families. $\mathrm{id}_\mathbb{B}$ *is the terminal double family, whose values are the terminal category and whose comparison maps are identities, and* $\varepsilon_\mathbb{B} := \int\mathrm{id}_\mathbb{B}\colon \mathbb{B}\odot\mathrm{id}_\mathbb{B}\to\mathbb{B}$ *is invertible.*

(iii) Dependent sums. *For* $\mathbb{F}$ *on* $\mathbb{B}$ *and* $\mathbb{G}$ *on* $\mathbb{B}\odot\mathbb{F}$*, the double family* $\mathbb{F}\fatsemi\mathbb{G}$ *on* $\mathbb{B}$ *has the values and comparison maps*

$$\begin{aligned}&(\mathbb{F}\fatsemi\mathbb{G})(b) := \Sigma_{X\in\mathbb{F}b}\mathbb{G}(b,X), \qquad (\mathbb{F}\fatsemi\mathbb{G})(m) := \Sigma_{\bar{m}\in\mathbb{F}m}\mathbb{G}(m,\bar{m}),\\ &\phi^{\mathbb{F}\fatsemi\mathbb{G}} := (\phi^\mathbb{F},\phi^\mathbb{G}), \qquad \iota^{\mathbb{F}\fatsemi\mathbb{G}} := (\iota^\mathbb{F},\iota^\mathbb{G}), \qquad \Phi^{\mathbb{F}\fatsemi\mathbb{G}} := (\Phi^\mathbb{F},\Phi^\mathbb{G}).\end{aligned} \tag{4.3}$$

Its legs, reindexing and constraints are those of $\mathbb{F}$ *on first components and those of* $\mathbb{G}$ *along the chosen lifts* (f,id) *and* (θ,id) *of* $\int\mathbb{F}$ *on second components. Thus* $\phi^{\mathbb{F}\fatsemi\mathbb{G}}$ *sends a composable pair of* $(\bar{m},\bar{y})$ *and* $(\bar{n},\bar{z})$ *to* $\big(\phi^\mathbb{F}(\bar{m},\bar{n}),\ \phi^\mathbb{G}(\bar{y},\bar{z})\big)$*, where* $\phi^\mathbb{G}$ *is taken at* $(m,\bar{m})$ *and* $(n,\bar{n})$*.*

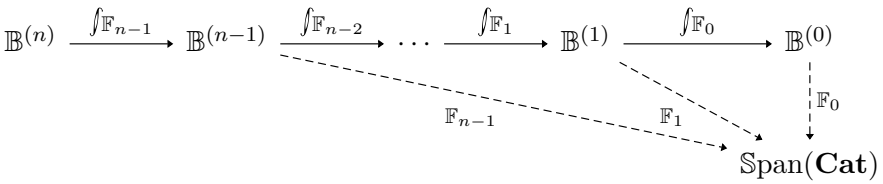
(iv) Comultiplication. $\delta_{\mathbb{F},\mathbb{G}}\colon \mathbb{B}\odot(\mathbb{F}\fatsemi\mathbb{G})\to(\mathbb{B}\odot\mathbb{F})\odot\mathbb{G}$ *is the invertible strict double functor that rebrackets the triples* (b,X,y) *into the pairs* $((b,X),y)$*, and likewise on tight morphisms, loose morphisms and cells.*

(v) Coherence. $\lambda\colon \mathrm{id}_\mathbb{B}\fatsemi\varepsilon_\mathbb{B}^*\mathbb{F}\cong\mathbb{F}$, $\rho\colon\mathbb{F}\fatsemi\mathrm{id}_{\mathbb{B}\odot\mathbb{F}}\cong\mathbb{F}$ *and* $\alpha\colon\mathbb{F}\fatsemi(\mathbb{G}\fatsemi\mathbb{H})\cong(\mathbb{F}\fatsemi\mathbb{G})\fatsemi\delta_{\mathbb{F},\mathbb{G}}^*\mathbb{H}$ *are the canonical isomorphisms of iterated Grothendieck constructions, which drop the entries of terminal families and rebracket tuples.*

Proof. Families. p is a split fibration over Dbl_s, because precomposition is strictly functorial. *Action.* The composite of $(\mathbb{U},\tau)$ and $(\mathbb{V},\sigma)$ in $\mathsf{DblIICat}$ is $\big(\mathbb{U}\,;\mathbb{V},\ \tau\,;(\mathbb{U}^\mathrm{op}\,;\sigma)\big)$, and by (4.2) and (4.1) its image and $\odot(\mathbb{U},\tau)\,;\odot(\mathbb{V},\sigma)$ agree on objects, $(d,X)\mapsto\big(\mathbb{V}\mathbb{U}d,\ \sigma_{\mathbb{U}d}\tau_dX\big)$, and likewise on morphisms, loose morphisms and cells, since whiskering along strict double functors is functorial. *Dependent sums are double families.* The tight and loose parts of $\mathbb{F}\fatsemi\mathbb{G}$ are pseudofunctors, because they are Grothendieck constructions of restrictions of G_0 and G_1 and reindex along chosen lifts. By Construction 4.6(ii) the loose composite of $(m,\bar{m})$ and $(n,\bar{n})$ in $\mathbb{B}\odot\mathbb{F}$ is $\big(m\,;n,\ \phi^\mathbb{F}(\bar{m},\bar{n})\big)$, and so $\phi^\mathbb{G}(\bar{y},\bar{z})$ lies in $\mathbb{G}\big(m\,;n,\phi^\mathbb{F}(\bar{m},\bar{n})\big)$. Hence $\phi^{\mathbb{F}\fatsemi\mathbb{G}}$ lands in $(\mathbb{F}\fatsemi\mathbb{G})(m\,;n)$, and likewise $\iota^{\mathbb{F}\fatsemi\mathbb{G}}$ lands over the loose unit (y_b,ι_bX) of $\mathbb{B}\odot\mathbb{F}$. By Construction 4.6(ii), $\Phi^\mathbb{G}$ lies over the associator of $\mathbb{B}\odot\mathbb{F}$, whose second component is $\Phi^\mathbb{F}$. Hence $\Phi^{\mathbb{F}\fatsemi\mathbb{G}}$ is well typed, and it satisfies the axioms of Definition A.7 componentwise because $\Phi^\mathbb{F}$ and $\Phi^\mathbb{G}$ do. *Counit.* $\mathrm{id}_{(-)}$ is cartesian and keeps the chosen lifts on the nose, since $\mathbb{U}^\mathrm{op}\,;\mathrm{id}_\mathbb{B}=\mathrm{id}_\mathbb{A}$, and ε is natural because (4.2) is $\mathbb{U}$ on first components. *Comultiplication.* The functor $\fatsemi$ acts on morphisms componentwise, and it is cartesian and keeps the chosen lifts on the nose because both $\mathbb{U}^\mathrm{op}\,;(\mathbb{F}\fatsemi\mathbb{G})$ and $(\mathbb{U}^\mathrm{op}\,;\mathbb{F})\fatsemi\big((\mathbb{U}\odot\mathbb{F})^\mathrm{op}\,;\mathbb{G}\big)$ have the same pairs and the same comparison maps by (4.3). By Construction 4.6(ii) the loose composites in $\mathbb{B}\odot(\mathbb{F}\fatsemi\mathbb{G})$ and in $(\mathbb{B}\odot\mathbb{F})\odot\mathbb{G}$ are $\big(m\,;n,(\phi^\mathbb{F}(\bar{m},\bar{n}),\phi^\mathbb{G}(\bar{y},\bar{z}))\big)$ and $\big((m\,;n,\phi^\mathbb{F}(\bar{m},\bar{n})),\phi^\mathbb{G}(\bar{y},\bar{z})\big)$, and the same holds for units, associators and the composition of tight morphisms, which is the canonical isomorphism of iterated Grothendieck constructions of Lemma 4.2(ii). Hence δ is an invertible strict double functor, and it is natural because $\odot$ and $\fatsemi$ act componentwise. *Coherence and axioms.* λ,

ρ and α are strict transformations that keep the entries of tuples, natural including the chosen lifts, and each axiom of a contextad equates two composites of ε, δ, λ, ρ and α, which only drop entries of terminal families or rebracket tuples, and so both sides send a tuple to the same tuple. □

A *double telescope* $(\mathbb{B} \mid \vec{\mathbb{F}}) = (\mathbb{B} \mid \mathbb{F}_0, \dots, \mathbb{F}_{n-1})$ is a telescope of the contextad of double telescopes (Definition 3.7), that is, an object of $\mathbf{Tel}({}_p\mathsf{DblICat}_\odot)$, the telescope monad on $\mathbf{H}^{\mathrm{s}}_{\mathsf{Dbl}_\mathrm{s}}$ applied to the span of double families. Its entries $\mathbb{F}_k$ are double families on $\mathbb{B}^{(k)}$, where $\mathbb{B}^{(0)} := \mathbb{B}$ and $\mathbb{B}^{(k+1)} := \mathbb{B}^{(k)} \odot \mathbb{F}_k$, joined by the double fibrations $\int \mathbb{F}_k \colon \mathbb{B}^{(k+1)} \to \mathbb{B}^{(k)}$,



(4.4)

The families of Theorem 4.7 form a split fibration, and so Theorem 3.8 applies to double telescopes: $\Lambda_{\mathbf{Tel}(\mathsf{DblICat})}$ is the free monad on $\Lambda_{\mathsf{DblICat}}$, $\mathbb{K}\mathrm{l}(\Lambda_{\mathsf{DblICat}}) \simeq \mathrm{co}\mathbb{K}\mathrm{l}(⨟)$, and $\bar{\mu}$ sends a double telescope to its n-fold dependent sum.

By induction on n, (4.3) also computes the n-fold dependent sum of a double telescope $(\mathbb{B} \mid \mathbb{F}_0, \dots, \mathbb{F}_{n-1})$: up to rebracketing, its loose morphisms over $m \colon b \nrightarrow b'$ are the tuples $(\bar{m}_0, \dots, \bar{m}_{n-1})$ and its comparison maps act componentwise, exactly as on objects.

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Appendix

Conventions

(a) Double categories are pseudo and normalised unless called strict. Their tight morphisms are vertical, whereas their loose morphisms are horizontal and drawn $\nrightarrow$.

(b) Composition is diagrammatic, and $f \,;\, g$ means f then g. Whiskering is likewise denoted $*$ or ;. Families compose by the white semicolon ⨟, and contextful arrows by ⨾, which composes their families white and their bodies black.

(c) **CAT** and **Cat** are the 2-categories of large and small categories, Cat is the 1-category of small categories, and Set is the category of small sets.

(d) Typefaces follow the levels. Named 1-categories such as Cat, ICat, Fib and Dbl_s are sans-serif, 2-categories such as **Cat** and $\mathbf{H}_X$ are bold, and 3-categories such as $\mathbf{f\mathfrak{G}pan}(\mathbf{CAT})^{\Rightarrow}$ and $\mathfrak{G}\mathbf{ray}$ carry a fraktur capital. Double categories, double functors and double families such as $\mathbb{B}$, $\mathbb{F}$ and $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ carry a blackboard capital, and category variables such as $\mathscr{C}$ and $\mathscr{M}$ are script.

(e) A span is written ${}_p\mathscr{M}_\odot$ with its legs at the sides of its apex, and ${}_1X_1$ is the identity span. For a pseudofunctor $G \colon A^{\mathrm{op}} \to \mathbf{CAT}$, $\Sigma_{x \in A} G(x) := \mathrm{dom} \int G$ is the total category of its Grothendieck construction $\int G \colon \mathrm{dom} \int G \to A$. Isomorphisms are written $\cong$ and equivalences $\simeq$.

(f) Levels follow Table 1: $\boxed{\mathsf{Ln}}$ names the level of the objects a statement is about, and $\boxed{\mathsf{Lm} \to \mathsf{n}}$ marks a passage from level m to level n. $\int F$ of a family is at level 0, and $\int_{\mathscr{C}}$ over the category of bases at level 1.

Diagram calculi

Pasting diagrams and string diagrams are the two kinds of diagram in this paper, matched by Table 4, and triangle pictures draw pasting diagrams of spans. A dot with only an output wire is the unit I of a monoid.

In string diagrams **Tel** is a shaded stripe around the wires it is applied to, as McCurdy drew monoidal functors [McC12, Sec. 3]: μ merges two nested stripes, $\eta = \iota_1$ and more generally ι_n open a stripe around 1 or n wires, the box of an algebra a closes one, and the inner stripes of $\mathbf{Tel}^2(\mathscr{M})$ are its blocks.

In triangle pictures objects are drawn in the base, never as dots of a triangle, and a vertical arrow between pictures on different feet is a map of spans over the functors of the feet.

Proposition A.1 (Soundness of the diagrams L2). (i) String diagrams. *A string diagram is read as the composite of its layers from top to bottom, where each layer is the $\ltimes$ of the boxes and wires at one height. In $\mathbf{H}^{\mathrm{s}}_X$ any two layerings of a diagram give the same 1-cell up to the canonical isomorphisms of iterated pullbacks.*

(ii) Stripes. *For a choice of lengths of the stripes, erasing the stripes repeats a box inside a stripe on each of its wires and reads η as ι_1 and μ as concatenation. Two stripe diagrams with the same boundary then denote the same 1-cell of $\mathbf{H}^{\mathrm{s}}_X$ when they agree after erasing the stripes for every choice of lengths.*

Proof. (i) A layering is a linear extension of the order of the boxes along the wires, and two of them are joined by exchanges of adjacent boxes on disjoint wires. Each exchange is an equality $(k \ltimes \mathscr{N})\,;(\mathscr{M}' \ltimes h) = k \ltimes h = (\mathscr{M} \ltimes h)\,;(k \ltimes \mathscr{N}')$ because $\ltimes$ is a 2-functor on $\mathbf{H}^{\mathrm{s}}_X$ (Lemma 2.2), and the canonical isomorphisms of iterated pullbacks are coherent. This is the planar case of the theorem of Joyal and Street [JS91, Thm. 1.2], [Sel11, Thm. 3.1]. (ii) Since $\ltimes$ preserves coproducts in each variable (Lemma 2.2), a $\ltimes$ of iterated **Tel**s is the coproduct of the $\ltimes$-powers indexed by the lengths of the stripes, and a 1-cell out of it is determined by its restrictions along the injections, along which $\mathbf{Tel}(k)$ restricts to $k^{\ltimes n}$, μ to the concatenation isomorphism of Lemma 3.2(i), η to ι_1 and a stripe opened by ι_n to its summand. These restrictions are the diagrams of the statement, and (i) compares them. □

	Pasting diagram	Triangle picture, pasting in **CAT**	String diagram
Lives in	$\mathscr{C}$, **CAT** or a 2-category	**CAT**, over the feet	$(\mathbf{H}^{\mathrm{s}}_X, \ltimes, {}_1X_1)$
Span $\mathscr{M}$	span $X \xleftarrow{p} \mathscr{M} \xrightarrow{\odot} X$	apex with legs p, $\odot$, unfilled	wire
Composite $\mathscr{M} \ltimes \mathscr{N}$	strict pullback (2.1)	lattice of pullback squares (3.1)	juxtaposition
1-cell k	map of spans	arrow between pictures on the same feet	box
Multiplication $\otimes$	map of spans from $\mathscr{M} \ltimes \mathscr{M}$	arrow from a pyramid of two to one triangle	dot with two inputs
Telescope monad **Tel**	$\coprod_n \mathscr{M}^{\ltimes n}$	a pyramid for each summand	shaded stripe
Used for	pullbacks and squares	spans, their composites and 2-cells	laws of monads and algebras
Justified by	pasting theorem [JY21, Thm. 3.3.7]		Proposition A.1

TABLE 4. L0→2 The two kinds of diagram, where triangle pictures draw pasting diagrams of spans.

Fibrations

A functor $P\colon \mathscr{X} \to \mathscr{B}$ is read as a family of categories indexed by $\mathscr{B}$, whose member at b is the *fibre* $\mathscr{X}_b$, and fibrations follow [LR20, Def. 2.2.1]. A morphism $g\colon x \to y$ of $\mathscr{X}$ is *cartesian* when every $h\colon z \to y$ with $Ph = w\,;Pg$ factors uniquely as $h = h'\,;g$ with $Ph' = w$. For $u\colon c \to Py$, a cartesian g with $Pg = u$ is a *lift* of u at y, unique up to a unique isomorphism in the fibre at c, and the main text calls cartesian morphisms lifts. P is a *fibration* when every such u has a lift at y. A *cleavage* chooses lifts $u^*y \to y$, which make reindexing $u^*\colon \mathscr{X}_b \to \mathscr{X}_c$ a functor. It is *normal* when the lifts of identities are identities, and *split* when moreover the lift of $v\,;u$ is the composite of those of v and u, so that $(v\,;u)^* = u^*\,;v^*$. Any cartesian morphism differs from the chosen lift by an isomorphism above an identity [Str23, Sec. 2], and so a functor over $\mathscr{B}$ between cloven fibrations preserves cartesian morphisms exactly when its comparisons are invertible (Section 2.2).

Lemma A.2 (Composite fibrations L1, [Jac99, Lemma 1.5.5])**.** *For cloven fibrations $\mathscr{X} \xrightarrow{P} \mathscr{D} \xrightarrow{Q} \mathscr{B}$, $P\,;Q$ is a fibration with the composite cleavage, which is split if P and Q are. A morphism g of $\mathscr{X}$ is then $(P\,;Q)$-cartesian if and only if g is P-cartesian and $P(g)$ is Q-cartesian.*

Lemma A.3 (Pullback fibrations L1, [Jac99, Lemma 1.5.1])**.** *For a cloven fibration $P\colon \mathscr{X} \to \mathscr{B}$ and $U\colon \mathscr{A} \to \mathscr{B}$, $\mathscr{A} \times_{\mathscr{B}} \mathscr{X} \to \mathscr{A}$ is a fibration that is split if P is, and its cartesian morphisms are the (u, g) with g cartesian.*

Double categories, double fibrations and double families

Definition A.4 (Double category L0, [GP99, Sec. 7.1])**.** A *double category* $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1, \mathrm{src}, \mathrm{tgt}, -\,;-, \mathrm{y}, \alpha)$ has a category $\mathbb{D}_0$ of objects and tight morphisms, a category $\mathbb{D}_1$ of loose morphisms and cells with $\mathrm{src}, \mathrm{tgt}\colon \mathbb{D}_1 \to \mathbb{D}_0$, loose composition $-\,;-\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ and units $\mathrm{y}\colon \mathbb{D}_0 \to \mathbb{D}_1$ over src and tgt, and natural invertible associators $\alpha_{m,n,l}\colon (m\,;n)\,;l \cong m\,;(n\,;l)$ with identity boundaries, which satisfy the pentagon and are trivial at the strict units.

A *pseudo double functor* has tight and loose functors commuting with src and tgt and invertible comparison cells for loose composition and units [GP99, Sec. 7], and it is *strict* when the comparison cells are identities.

Definition A.5 (Double fibration L0, [CLPS22, Def. 2.25, Cor. 4.7])**.** A *double fibration* is a strict double functor $\mathbb{P}\colon \mathbb{E} \to \mathbb{B}$ whose tight and loose functors P_0 and P_1 are fibrations with cleavages, whose lifts are its *chosen lifts*, such that src and tgt preserve the cleavages and y and loose composition preserve cartesian morphisms. The category $\mathsf{DblFib}(\mathbb{B})$ has the double fibrations into $\mathbb{B}$ as objects and the pseudo double functors over $\mathbb{B}$ with cartesian tight and loose functors as morphisms [CLPS22, Def. 3.38].

Definition A.6 (Strict transformations L0)**.** A *transformation* $\tau\colon \mathbb{F} \to \mathbb{G}$ of double families on $\mathbb{B}$ consists of pseudonatural $\tau_0\colon F_0 \Rightarrow G_0$ and $\tau_1\colon F_1 \Rightarrow G_1$ over τ_0, with constraints $\tau_f\colon \tau_b(f^*Y) \cong f^*(\tau_{b'}Y)$ for $f\colon b \to b'$ and τ_θ for cells θ, and of comparison modifications $T_{m,n}\colon (\tau_m \times \tau_n)\,;\phi^{\mathbb{G}} \Rightarrow \phi^{\mathbb{F}}\,;\tau_{m;n}$ and $J_b\colon \tau_b\,;\iota^{\mathbb{G}} \Rightarrow \iota^{\mathbb{F}}\,;\tau_{\mathrm{y}_b}$ subject to [CLPS22, Def. 3.20]. It is *strict* when T and J are identities.

Definition A.7 (Coherence of a double family L0)**.** For $\mathbb{F}$ on $\mathbb{B}$ as in Definition 4.5, the coherence data are an invertible modification $\Phi\colon (\phi \times \mathrm{id})\,;\phi \Rightarrow (\mathrm{id} \times \phi)\,;\phi\,;(\alpha^{\mathbb{B}})^*$ for the associator $\alpha^{\mathbb{B}}$ of $\mathbb{B}$ and identity unitors, and Φ satisfies the pentagon and the unit coherence of [CLPS22, Def. 3.12]. Hence $\phi(\iota_b X, x) = x = \phi(x, \iota_{b'} Y)$, and Φ is an identity at such arguments.

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