

Low-rank Grothendieck rings: an AI-assisted classification project, constructions, and open problems

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Abstract

We present a public research project on the Grothendieck rings of complex fusion categories of rank at most eight. The project reports unrestricted classification theorems in ranks two, three, and four, and classifications within specified fusion-multiplicity bounds in ranks five through eight, with two rank-six categorification questions unresolved. Its resources include research manuscripts, an exact fusion-ring census, verification code, and an interactive atlas. We describe seven constructions that are promising candidates for new tensor categories, subject to a separate assessment of their novelty up to tensor equivalence. We also state the two open problems explicitly. This announcement records the project's claims and their verification status; it invites independent mathematical review and does not provide a new proof of the complete classification.

AI-generated research · independent verification pending. The research manuscripts, arguments, code, and data have not yet been independently human-verified or Lean-formalized. Please treat the reported results with caution. Exact computations establish only the mathematical and computational statements within their documented scope.

1 The classification question and its scope

Let $\mathcal{C}$ be a fusion category over $\mathbb{C}$, with simple-object classes $b_0 = 1, b_1, \dots, b_{r-1}$. Its Grothendieck ring has multiplication

$$b_i b_j = \sum_{k=0}^{r-1} N_{ij}^k b_k, \quad N_{ij}^k \in \mathbb{Z}_{\geq 0}.$$

Here r is the *rank*, and throughout this announcement

$$M(R) = \max_{i,j,k} N_{ij}^k$$

is the *fusion multiplicity*. Thus every bound in Table 1 is a bound on fusion multiplicity, unless explicitly stated otherwise. A fusion ring is counted up to based-ring isomorphism, preserving the unit and distinguished basis. To be *realized* means to admit at least one complex fusion category with that Grothendieck ring. This is a ring-level classification problem; it does not classify all associators or all tensor equivalence classes realizing a given ring.

The rank-one case is the ring of Vec . In rank two the familiar family is $X^2 = 1 + nX$ with $n \geq 0$, and only $n = 0, 1$ admit complex categorifications [1]. Both have $M = 1$, because the unit coefficients are included in the maximum. The higher-rank account below is taken from the fixed project snapshot specified in Section 5.

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Rank	Census through M	Reported general classification	Realized	Open
2	all	Unrestricted theorem	2	0
3	1,000	Unrestricted theorem	5	0
4	128	Unrestricted theorem	15	0
5	20	Complete through 20	17	0
6	9	Complete through 3; two open through 9	57	2
7	4	Complete through 4	29	0
8	2	Complete through 2	77	0

Table 1: Reported status at the pinned snapshot. For ranks five through eight, realized and open counts refer only to the displayed census range. The census bounds in ranks three and four describe retained enumeration data; the reported classification theorems are unrestricted in multiplicity.

2 Reported results and the role of exact computation

The [unrestricted rank-four manuscript](#) reports fifteen possible Grothendieck rings, without assuming pivotality or a multiplicity bound, and states that every categorification admits a spherical structure. The stated sharp bounds are

$$\mathrm{FPdim}(\mathcal{C}) \leq 24 + 12\sqrt{3}, \quad M(K_0(\mathcal{C})) \leq 6.$$

The dimension bound is attained by the near-group ring $\mathrm{NG}(C_3, 6)$. The proof presented there combines a theoretical reduction to $\mathrm{FPdim}(\mathcal{C}) < 1000$ and $M \leq 31$ with exact finite verification of 1,692 remaining rings, partitioned into 1,677 exclusions and 15 models. The separate census through $M = 128$ has 889,530 rings. Related work of Dong, Palcoux, and Plessis treats the pivotal rank-four case [2]; its hypothesis should be distinguished from the unrestricted claim of this project. For ranks five through eight, the complete census totals and recorded decisions are respectively

r	census total	excluded	realized	open
5	40,999	40,982	17	0
6	25,421	25,362	57	2
7	4,201	4,172	29	0
8	973	896	77	0

All four partitions concern their stated multiplicity ranges. At rank six, the only unresolved multiplicities are 4 and 8; all 8,809 rings of exact multiplicity 9 are recorded as excluded. The 57 realizations comprise 56 unitary models and one spherical model whose ring has no pseudo-unitary categorification. Thus the target problem is not restricted to unitary or pseudo-unitary fusion categories.

The approach combines structural constraints on fusion categories with exact applications to enumerated rings. Its criteria include arithmetic and cyclotomic restrictions, formal codegrees, induction to the Drinfeld center, Frobenius–Schur indicators, fusion subcategory and module-category constraints, and associator or tensor-functor obstructions. Positive decisions require a categorical model and a comparison of its entire fusion tensor with the census representative.

These are several distinct verification tasks. A tensor check can establish associativity of a based ring, a full comparison of structure constants can identify two based rings, and a finite certificate can establish a specified obstruction. None of these alone constructs compatible categorical associators or proves that the census is exhaustive. The manuscripts and the [verification guide](#) state the relevant hypotheses, source identifications, and replay domains. The overall project therefore combines theoretical arguments and exact finite verification.

3 Seven constructions to examine for novelty

To the best of our current knowledge, these seven constructions are promising candidates for new tensor categories. Their novelty up to tensor equivalence remains subject to independent verification. They are a selected shortlist of reported constructions from known input categories, rather than a count of established new tensor equivalence classes. In particular, identifying a census ring with an already known category is not itself a discovery of a new category.

In the notation `Rr_m_i`, the first two indices record the rank and exact maximum fusion multiplicity, and the last is a census identifier. Write C_n for the cyclic group of order n , and $\text{NG}(G, k)$ for a near-group category with invertible group G and noninvertible simple ρ satisfying $\rho^2 = \sum_{g \in G} g + k\rho$.

1. **R6_4_249: sign-swap equivariantization.** The reported construction takes the diagonal-fermion quotient of two identical half- E_6 categories and then equivariantizes a factor-swap action with a sign on the condensed algebra. The known fermionic inputs are discussed in [3]. See the [rank-six construction chapter](#), theorem labelled `thm:sign-swap`.
2. **R6_4_322: mixed diagonal-fermion quotient.** Let $\mathcal{E}$ denote the half- E_6 category. The construction uses the diagonal central boson in $\mathcal{E} \boxtimes \text{PSU}(2)_6$ and the corresponding module category. Both factors are known inputs [3]. See the same chapter, proposition labelled `prop:mixed-fermion`.
3. **R6_6_1712: a relative Fibonacci center.** The proposed identification is $Z_{\text{Fib}}(\mathcal{TT}_3)$, where the known category $\mathcal{TT}_3$ is the dual even part of the $2D2$ subfactor [4]. The relative-center framework is standard [5]. See the same chapter, corollary labelled `cor:tt3-relative`.
4. **R7_4_35: an inversion and Morita construction.** Starting from the $\text{NG}(C_4 \times C_4, 16)$ input of Yu–Zhang [6], the project constructs an orbifold, descends a coherent inversion action, and takes the Morita dual associated to the reflection algebra. Its four invertibles form $C_2 \times C_2$, and all simples are self-dual. See the [construction note](#).
5. **R7_4_2665: the cyclic near-group analogue.** The analogous construction starts from Budinski’s $\text{NG}(C_{16}, 16)$ input [7], using the near-group orbifold and Morita machinery of [8]. The invertibles form C_4 ; the two large noninvertibles are mutually dual. Both rank-seven cases have simple dimensions $(1, 1, 1, 1, 2, 4 + 2\sqrt{5}, 4 + 2\sqrt{5})$. See the [construction note](#).
6. **R8_2_37: a nonsplit C_2 -extension of $2D2$.** Using the constructions in [9, 10], the project transports an odd automorphism through a crossed product. The odd invertible a satisfies $a^2 = g$, where g is the nontrivial invertible in the neutral $2D2$ category. The simple dimensions are four copies of 1 and four copies of $2 + \sqrt{5}$. See the [construction note](#).
7. **R8_2_575: a self-dual Hopf extension.** An explicit symmetric Hopf pairing on Kashina’s 16-dimensional self-dual Hopf algebra [11] is used to apply the Davydov–Runkel extension construction [12]. The simple dimensions are $(1, 1, 1, 1, 2, 2, 2, 4)$, and the global Frobenius–Perron dimension is 32. See the [construction note](#).

The most useful checks here concern both the existence of the stated coherence data and comparison with the prior literature. Known input categories and general extension theorems do not by themselves settle whether the resulting examples have appeared before.

4 The two unresolved categorification problems

The following two based rings remain open in the stated project range. They have resisted repeated attempts at both categorical construction and exclusion. The displayed presentations specify concrete targets; the existence of their fusion tensors does not imply categorifiability.

R6_4_155: a generalized near-group ring

The commutative based ring has basis $1, h, g, hg, X, Y$, with

$$\begin{aligned} g^2 &= h^2 = 1, & gh &= hg, & hX &= X, & gX &= Y = X^*, \\ XY &= 1 + h + 4X + 4Y, & X^2 &= g + hg + 4X + 4Y. \end{aligned}$$

The remaining products follow from these relations. Its four invertibles form $C_2 \times C_2$, the two noninvertibles are mutually dual, and

$$\text{FPdim}(X) = \text{FPdim}(Y) = 4 + 3\sqrt{2}, \quad \text{FPdim}(R) = 72 + 48\sqrt{2}.$$

Its maximum fusion multiplicity is 4. The ring is already the explicit candidate $R'(4)$ in Meng–Yu [13, Theorem 1.1]; it is not claimed here as a newly discovered based ring.

Problem 1. *Does $R6_4_155$ admit a fusion-category realization over $\mathbb{C}$? Construct compatible categorical data or prove that no such category exists.*

The [current frontier account](#) records a necessary rank-52 center profile with a semion factor and a rank-26 centralizer candidate. It also records a rank-22 based module satisfying the stated fermion Azumaya linking identities. These are necessary-condition calculations: they supply neither a coherent module category nor associators for a categorification.

R6_8_1660: an Ising-based ring

The commutative based ring has basis $1, g, X, gX, \sigma, T$, where $T = \sigma X$, and relations

$$g^2 = 1, \quad g\sigma = \sigma, \quad \sigma^2 = 1 + g, \quad X^2 = 1 + (3(1 + g) + 4\sigma)X.$$

All basis elements are self-dual, and $\{1, g, \sigma\}$ is an Ising fusion subring. Writing

$$d = 3 + 2\sqrt{2} + 2\sqrt{3} + \sqrt{6},$$

the simple dimensions, in the displayed basis, are $(1, 1, d, d, \sqrt{2}, \sqrt{2}d)$ and $\text{FPdim}(R) = 4(1 + d^2)$. Its maximum fusion multiplicity is 8, attained in T^2 .

Problem 2. *Does $R6_8_1660$ admit a fusion-category realization over $\mathbb{C}$? Determine whether the necessary center and module-category data can be completed to an actual category.*

The [current frontier account](#) records relative-center, Morita, and indicator restrictions. For the positive spherical representative, a bounded necessary-condition domain contains 640 labelled restriction profiles; this is not a list of categories or associators. A particular rank-37 candidate fails Verlinde integrality, while other profiles remain open. Neither this rejection nor the survival of other profiles decides the general categorification question.

5 Resources, provenance, and independent review

The project has three complementary public entry points:

- **Classification manuscripts, models, and decision records:**
<https://github.com/sebastienpalcoux/grothendieck-ring-classification>.
- **Fusion-ring census and enumeration sources:**
<https://github.com/sebastienpalcoux/fusion-ring-census>.
- **Interactive atlas:**
<https://sebastienpalcoux.github.io/grothendieck-ring-classification/>.

The [one-page construction shortlist](#) provides a quick entry into Section 3. The [two exact open fusion tensors](#) are available as Python lists with convention $N[i][j][k] = N_{ij}^k$, zero-based unit index 0, explicit duality maps, and tensor checksums. The symbolic basis orders in Section 4 need not coincide with the canonical file order; the first case is obtained from canonical indices $(0, 3, 4, 5, 1, 2)$, and the second already has the displayed order.

Fixed research snapshot. This announcement describes classification commit [e34396cd95af](#) and census-repository commit [d4c36cc2ee5e](#). Rank-specific source manifests retain the exact census inputs used for individual certificates; the displayed census-repository snapshot does not replace those source bindings. Live pages may evolve, whereas these commit links identify the resources summarized here.

The research collection at this snapshot comprises eight classification manuscripts and six census manuscripts: 14 research manuscripts and 485 pages in total, including references and appendices. Verification guides and this announcement are separate documents. The [classification reading guide](#) identifies the unrestricted rank-four paper as the primary rank-four account.

AI provenance. The project was developed through AI-assisted research under the contributor’s direction; its research text, proposed arguments, code, and data are primarily AI-generated. This announcement was also prepared with ChatGPT. Automated rechecking and AI-assisted revision have been used, but do not constitute independent human verification. No Lean formalization of the full project is claimed. Historical novelty of the shortlisted categorical constructions is a separate question from their mathematical validity. **AI-generated research · independent verification pending.**

Readers can select a focused task: verify a stated categorical construction and its coherence data; compare a model with known tensor equivalence classes; inspect a classification reduction and its finite certificate scope; or construct or exclude one of the two open rings. For code and data, the [replay guide](#) and [source-provenance record](#) identify the relevant inputs and exact checks. We welcome comparisons with previously known tensor categories, independent scrutiny of the proposed constructions and classification arguments, and progress on the two remaining categorification problems.

References

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