

Almost all simple lengths are unique up to isometry

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Abstract

For every fixed hyperbolic torus with one cusp or one geodesic boundary, we prove that, among simple curves of length at most L , the number of isometry orbits and the number of distinct lengths differ by $o(L^2)$. Thus almost every complete length level is a single isometry orbit of at most six curves, and multiplicity converges in mean to the generic orbit size. The modular specialization is exactly Chen’s density theorem for Markoff uniqueness. We also prove the full Schmutz–Schaller bound for every punctured torus having a simple trace that is not an algebraic integer, including any transcendental trace, and for dense families with integral traces. More strongly, every rational length class on these surfaces contains at most two curves. Transcendental twists give multiplicity one, while symmetric families with geodesic boundary attain the sharp bounds four and six. The density proof extends Chen’s method using the stable-norm estimates of Doan–Li–Nguyen.

1 Introduction

1.1 The multiplicity problem

How often can distinct simple curves on one hyperbolic torus have the same length? We study this question both asymptotically, as the length grows, and pointwise under arithmetic hypotheses. A one-holed torus means a complete finite-area once-punctured torus or a compact hyperbolic torus with one geodesic boundary. Curves are essential, nonperipheral, unoriented simple closed geodesics in its interior. For such a surface X and $t > 0$, put

$$m_X(t) = \#\{\gamma : \ell_X(\gamma) = t\}.$$

For punctured tori, the Schmutz–Schaller conjecture, in the form stated in [MP08, Conjecture 1], asserts that

$$m_X(t) \leq 6$$

for every X and every $t > 0$. Schmutz Schaller’s modular-torus paper [Sch96a] and survey [Sch98] provide the historical background. The constant six is forced by the symmetries of the modular torus. The question concerns all curves at an exact length on one surface, and is finer than cumulative curve counting.

McShane and Parlier [MP08, Theorem 1.1] prove that all simple curves have different lengths on a dense set with meagre complement in Teichmüller space. This set also has full measure: each equality between two distinct simple lengths is the zero set of a nonzero real-analytic function. Pinching one curve and using the collar inequality for the other proves nonidentity. Such generic statements do not control a specified exceptional surface. Our density theorem keeps the surface fixed and counts its curves or distinct lengths.

1.2 The density theorem

From now on let X be any fixed one-holed torus, allowing either type of boundary. Fix an integral homology basis. Primitive vectors in $\mathbb{Z}^2$, up to sign, represent curves on X . The simple lengths extend to a strictly convex norm N_X on $\mathbb{R}^2$. This is the McShane–Rivin norm in the punctured case [MR95a, MR95b]; Sorrentino and Veselov [SV19] also treat the geodesic-boundary case. Section 2 recalls the construction and proves the corner and Farey-sector estimates needed here. Write

$$B_X = \{v : N_X(v) \leq 1\},$$

and normalize area so that $\mathbb{Z}^2$ has covolume one. Let $C_X(L)$ count curves of length at most L , let $O_X(L)$ count their orbits under the full isometry group, including orientation-reversing isometries, and let $D_X(L)$ count their distinct lengths. Thus C_X counts multiplicity and D_X does not.

The homology action of the full isometry group is

$$K_X = \{G \in \mathrm{GL}_2(\mathbb{Z}) : N_X \circ G = N_X\}.$$

This group contains $-I$ and has order at most twelve. Every orbit of unoriented curves therefore has at most

$$h_X = |K_X|/2 \leq 6$$

elements, and all but finitely many orbits have exactly h_X elements; see Lemma 3.3. Primitive lattice counting gives

$$\begin{aligned} C_X(L) &= \frac{\mathrm{area}(B_X)}{2\zeta(2)} L^2 + O_X(L \log(2+L)), \\ O_X(L) &= \frac{\mathrm{area}(B_X)}{|K_X|\zeta(2)} L^2 + O_X(L \log(2+L)). \end{aligned}$$

Our result compares the second count with the number of distinct lengths.

Theorem 1.1 (Asymptotic rigidity on every fixed torus). *For every hyperbolic one-holed torus X , with a cusp or a geodesic boundary and with no algebraicity assumption on its traces,*

$$E_X(L) := O_X(L) - D_X(L) = o_X(L^2).$$

In particular,

$$D_X(L) \sim \frac{\mathrm{area}(B_X)}{|K_X|\zeta(2)} L^2.$$

Let $A_X(L)$ count curves of length at most L whose complete length level contains a curve not isometric to them. Then

$$A_X(L) \leq 2h_X E_X(L) = o_X(L^2).$$

All but $o_X(L^2)$ distinct simple lengths at most L have multiplicity exactly h_X . Moreover,

$$\sum_{\substack{t \leq L \\ m_X(t) > 0}} |m_X(t) - h_X| = h_X E_X(L) + O_X(1) = o_X(L^2).$$

The difference $E_X(L)$ measures coincidences that are not explained by isometries. More precisely, if $r_X(t)$ is the number of isometry orbits at length t , then

$$E_X(L) = \sum_{\substack{t \leq L \\ m_X(t) > 0}} (r_X(t) - 1).$$

A level with $r_X(t) \geq 2$ contains at most $h_X r_X(t) \leq 2h_X(r_X(t) - 1)$ curves. This explains the bound for $A_X(L)$ and why the theorem controls complete length levels.

Corollary 1.2 (Density-one Schmutz–Schaller). *For every fixed one-holed torus X , the following limits hold. For a punctured torus they give a density-one form of the Schmutz–Schaller conjecture:*

$$\lim_{L \rightarrow \infty} \frac{\#\{\gamma : \ell_X(\gamma) \leq L, m_X(\ell_X(\gamma)) > 6\}}{C_X(L)} = 0,$$

and

$$\lim_{L \rightarrow \infty} \frac{\#\{t \leq L : m_X(t) > 6\}}{D_X(L)} = 0.$$

The two limits correspond to sampling a curve or a distinct length uniformly below L . If $K_X = \{I, -I\}$, the stronger conclusion of Theorem 1.1 is multiplicity one on almost every level. No effective convergence rate or pointwise cutoff is asserted. Indeed, McShane and Parlier [MP08, Theorem 7.1 and Remark 7.2] construct tori with geodesic boundary having twelve simple curves of equal length. Our result allows such exceptional levels.

For punctured tori the possible limiting multiplicities are $h_X \in \{1, 2, 4, 6\}$; the values four and six occur respectively at the punctured square torus and the modular torus (Proposition 6.3). The same comparison of counting functions also determines the homology distribution under sampling by distinct lengths.

Corollary 1.3 (Distribution under distinct-length sampling). *Choose uniformly a distinct simple length at most L , then uniformly a curve in its complete level, and then uniformly one of its two homology orientations. If v is the resulting primitive vector, the distribution of v/L converges to normalized area measure on B_X .*

The proof is given in Corollary 6.2.

1.3 The modular torus and Markoff uniqueness

The connection with Markoff numbers is described geometrically by Cohn [Coh71] and Series [Ser85a]. On the modular torus M , simple curves have positive trace $3m$, where m is a Markoff number, and their lengths are

$$2 \operatorname{arcosh}(3m/2).$$

Isometry orbits correspond to sorted positive integer triples

$$a \leq b \leq c, a^2 + b^2 + c^2 = 3abc,$$

with $c = m$. Frobenius uniqueness asks that the maximum coordinate determine the sorted triple; equivalently, equal simple lengths on M should be explained by isometries [MP08, Section 7]. Prime-power uniqueness was proved by Button and Schmutz [But01, Sch96b]; Lang–Tan [LT07] give a short proof. Further arithmetic cases appear in [But98, But01, Zha07].

Let $T(R)$ count sorted Markoff triples with maximum at most R , and let $D(R)$ count their distinct maxima. Zagier [Zag82] proved a quadratic logarithmic asymptotic for $T(R)$. Chen [Che26, Theorem 1.1] proves

$$T(R) - D(R) = o((\log R)^2).$$

Consequently, nonunique maxima have density zero among Markoff numbers. With $L = 2 \operatorname{arcosh}(3R/2)$, this is the modular case $O_M(L) - D_M(L) = o(L^2)$ of Theorem 1.1. Since $|K_M| = 12$, almost every complete level on M has six curves. More precisely,

$$O_M(2 \operatorname{arcosh}(3R/2)) = T(R), D_M(2 \operatorname{arcosh}(3R/2)) = D(R)$$

by [Che26, Lemma 2.1], including the two short exceptional orbits. Thus the modular specialization is exactly Chen’s theorem; our extension is to arbitrary one-holed tori.

Dang–Dao–Doan [DDD26, Theorem 1.1] prove that the modular torus uniquely minimizes $\operatorname{area}(B_X)$ among complete finite-area hyperbolic punctured tori, resolving the McShane–Rivin area conjecture. Their comparison concerns the leading coefficient for counting curves. Combined with the density theorem, it also gives a comparison for distinct lengths.

Corollary 1.4 (Fewest distinct simple lengths asymptotically). *Let M be the modular torus and let X be a fixed nonmodular complete finite-area hyperbolic punctured torus. Then*

$$\lim_{L \rightarrow \infty} \frac{D_X(L)}{D_M(L)} = \frac{12 \operatorname{area}(B_X)}{|K_X| \operatorname{area}(B_M)} > 1.$$

In particular, $D_M(L) < D_X(L)$ for every sufficiently large L , with the threshold depending on X .

Indeed, Theorem 1.1 gives the displayed limit; $|K_X| \leq 12$ and the strict area inequality of [DDD26, Theorem 1.1] give strictness. The minimizing surface is therefore the same whether one counts curves or distinct simple lengths, although their asymptotic coefficients differ by the generic multiplicity.

1.4 Stable-norm geometry and earlier bounds

The unit ball B_X converts equal simple lengths into primitive lattice points on the boundary of a dilated convex body. Fock’s convex function [Foc98, Section 7.3] gives a related description of the Markoff length data. Sorrentino and Veselov [SV19] explain its connection with stable norms and Mather’s β -function and study differentiability at irrational and rational directions. O’Sullivan [OS26] develops the corresponding convex and concave encoding functions for the generalized trace equation $x^2 + y^2 + z^2 - xyz = \kappa$. One-sided derivatives at rational directions are computed in the work of Hines [Hin20] and Gaster [Gas22, Propositions 1.8 and 1.10].

Doan, Li and Nguyen [DLN26, Theorem 1.1 and Propositions 3.1, 3.2 and 4.1] establish the corner, Farey-sector and total-turn estimates for every complete finite-area hyperbolic once-punctured torus. These results contain the full geometric content of [Che26, Proposition 3.1], including parts (b) and (c). Lemma 2.2 recalls this input and extends it to geodesic boundary. Doan, Li and Nguyen also show that differentiability at an irrational direction does not imply infinite-order flatness there: the flatness depends on the rate of rational approximation. We use exponential normal-turn estimates on Farey sectors, rather than any uniform assertion of infinite-order flatness at irrational directions. Their boundary-counting theorem [DLN26, Theorem 1.4] gives

$$m_X(t) \leq C_X(\log t)^2(t \geq 2).$$

This bounds every individual level. Theorem 1.1 instead gives a sharp constant on a density-one set of levels; neither conclusion alone eliminates isolated large multiplicities. Theorem 2.4 gives a self-contained proof of the logarithmic-square bound using the Farey-sector estimate and extends it to each fixed geodesic-boundary torus.

1.5 Pointwise bounds and arithmetic families

A *rational length class* is an equivalence class of curves under the relation $\ell_X(\gamma)/\ell_X(\delta) \in \mathbb{Q}_{>0}$. A bound on its size bounds equal-length multiplicity and also restricts rational relations between distinct lengths. Write

$$T_X(\gamma) = 2 \cosh(\ell_X(\gamma)/2), \kappa = 2 - 2 \cosh(b/2),$$

where $b = 0$ denotes a cusp. A positive Farey trace triple satisfies $x^2 + y^2 + z^2 - xyz = \kappa$. Its trace and squared-trace fields are $\mathbb{Q}(x, y, z)$ and $\mathbb{Q}(x^2, y^2, z^2)$.

Theorem 1.5 (Pointwise arithmetic rigidity). *The following statements hold.*

1. *If one simple trace of a one-holed torus is not integral over $\mathbb{Z}[\kappa]$, every rational length class has at most two curves. For a punctured torus, this includes every algebraic trace that is not an algebraic integer and every transcendental trace. At each fixed boundary length, only countably many marked tori can have a rational length class containing three curves.*
2. *For a punctured torus with algebraic traces, the same bound two holds if a real embedding of its trace field changes one of x^2, y^2, z^2 . If the trace field is totally real, the bound holds outside the four unmarked tori listed below in Table 1.*
3. *Fix a curve with expanding eigenvalue $\lambda > 1$, and write the punctured-torus twist family as*

$$(x, y, z) = (\lambda + \lambda^{-1}, c(t + t^{-1}), c(\lambda t + \lambda^{-1}t^{-1})), c = \frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}}.$$

If t is transcendental over $\mathbb{Q}(\lambda)$, every rational length class has one curve. For any fixed λ , the parameters failing this conclusion form a countable set.

The first part follows from Theorem 7.3 and Corollary 7.5. A valuation detecting nonintegrality gives pole orders of the form $|f(v)|$, with f a nonzero homomorphism on integral homology. After orienting and scaling rationally related curves, their vectors lie on one affine line and one norm level; strict convexity permits at most two. The second part follows from Theorems 9.3 and 9.5, and the third from Theorem 8.2 and Corollary 8.3.

The bound two also holds on large families whose traces are algebraic integers. The next theorem states the density and exact-degree conclusions explicitly. Here density refers to the chosen real embedding and to marked Teichmüller space.

Theorem 1.6 (Integral families). *1. Let $F \subset \mathbb{R}$ be a number field with at least two real embeddings. Fix a positive integral punctured-torus trace triple in F and a nonzero ideal $I \subset \mathcal{O}_F$. Within its congruence class modulo I , the triples with both trace fields equal to F and at most two curves in every rational length class are dense in marked Teichmüller space.*

2. *For every $d \geq 2$, there are infinitely many pairwise nonisometric punctured tori with integral traces, both trace fields of degree d , and maximum multiplicity exactly two. Explicitly, for $n \geq 3$ take the root $r \in (n-1, n)$ of $r^d - nr^{d-1} + 1 = 0$ and the triple*

$$(r^2 + 2, r + 2/r, r + 2/r).$$

Every rational length class is one reflection orbit.

3. For every number field F as in the first part, a dense set of the symmetric punctured-torus locus has integral traces, trace field of degree $4[F : \mathbb{Q}]$, squared-trace field of degree $2[F : \mathbb{Q}]$, and every rational length class equal to one reflection orbit. The maximum multiplicity is exactly two.

These are Theorems 10.4, 9.7 and 9.8. Theorem 10.2 proves fixed-trace density as well: if an integral trace τ is fixed, one needs either $\tau^2 - 4$ nonsquare in F , or at least three infinite places of F , with a complex conjugate pair counted once. In the split real-quadratic case the integral points are locally finite instead (Proposition 10.3).

Theorem 1.7 (Sharp symmetry families with geodesic boundary). *1. If $t > 2\sqrt{2}$ is not an algebraic integer, the torus with triple $(t, t, t^2/2)$ has maximum multiplicity exactly four. Every rational length class is one orbit under its four effective symmetries, and has two or four curves.*

- 2. If $t > 3$ is not an algebraic integer or has a real conjugate less than -2 , the torus with triple (t, t, t) has maximum multiplicity exactly six. Every rational length class is one orbit under its six effective symmetries, and has three or six curves. This includes every transcendental $t > 3$ and every $t = \sqrt{d}$ with $d > 9$ a nonsquare integer.*

Theorems 13.6 and 13.7 prove these statements. McShane and Parlier [MP10, Section 6.2] already observed the transcendental diagonal equal-length result. Polynomial methods for related uniqueness questions also appear in McGinn’s study of generalized Markoff trees [McG15]. The rational-class conclusions above also exclude unequal rationally proportional lengths.

Theorem 1.8 (Three length ratios and triple levels). *1. The length ratios of any three prescribed distinct simple curves determine a marked punctured torus. These curves admit equal lengths if and only if their three pairwise intersection numbers satisfy the strict triangle inequalities; the marked torus is then unique.*

- 2. Integral punctured tori with three rationally related simple lengths are dense. Thus density also holds for this locus, in addition to the families in Theorem 1.6.*
- 3. On the reflection walls of a reduced punctured-torus trace chamber, there is exactly one unmarked torus whose N th distinct simple length has multiplicity three for $N = 1, 2$ and for every $N \geq 4$, and none for $N = 3$. The resulting tori for $N \geq 4$ have integral traces, coinciding trace and squared-trace fields with exactly one real embedding, and unbounded field degree.*

The first part is Theorem 9.1 and Corollary 9.2; the second is Proposition 12.1; the third is Theorem 15.1 and Corollary 15.3. On the symmetric locus, the bound two holds outside countably many algebraic parameters, yet parameters admitting a level of at least four curves are dense (Theorem 16.2). These statements concern different subsets of the same locus and are compatible.

1.6 Multiplicity bounds in the different cases

Table 1 records whole-spectrum conclusions. A class bound applies to every rational length class on a surface satisfying the stated hypothesis; it therefore bounds every complete equal-length level. An exact maximum is attained on at least one level, and need not be attained on every level. The hypotheses on the selected level in Table 2 are essential. Both tables refer to the proofs in the body of the paper. In the valuation rows, $P(s) = \max\{0, -v(s)\}$, where v is a valuation nonnegative on the integers and Γ is its ordered value group. The pole-order criteria for bordered tori use real-valued valuations, and a, b, c denote the pole orders of the chosen Farey triple.

Table 1: Whole-spectrum multiplicity and rational-length bounds.

Surface or hypothesis	Conclusion	Result
Every fixed one-holed torus, with either type of boundary.	$m_X(t) \leq C_X \log^2(2+t)$ for every t .	2.4
<i>Punctured tori: arithmetic and twist criteria</i>		
One trace is not an algebraic integer, including any transcendental trace.	Every class has at most 2 curves.	7.3

Table 1continued

Surface or hypothesis	Conclusion	Result
A real embedding of $\mathbb{Q}(x, y, z)$ changes a squared trace.	Every class has at most 2 curves.	9.3
The trace field is totally real, excluding the four tori represented by $(\sqrt{5}, 2\sqrt{5}, 5)$, $(\sqrt{6}, 2\sqrt{3}, 3\sqrt{2})$, $(2\sqrt{2}, 2\sqrt{2}, 4)$ and $(3, 3, 3)$.	Every class has at most 2 curves. The four exclusions remain possible exceptions to this bound.	9.5
The pole homomorphism $f : \mathbb{Z}^2 \rightarrow \Gamma$ associated with a valuation nonnegative on $\mathbb{Z}[\kappa]$ is injective.	Every class has 1 curve.	7.4
The twist parameter t is transcendental over $\mathbb{Q}(\lambda)$ in Theorem 1.5.	Every class has 1 curve.	8.2
Any fixed-length twist family, outside a countable set of parameters.	Every class has 1 curve.	8.3
Rational twists $t = t_0 a/b$ of an algebraic seed, outside the explicit finite divisor list in Theorem 8.4.	Every class has at most 2 curves.	8.4
The quartic twist family with $\lambda = 1 + \sqrt{2}$, $t_0 = \lambda^{-1/2}$ and rational $t/t_0 \neq 1$.	Every class has at most 2 curves; the fixed curve's class is a singleton.	8.6, 8.7
The triple $(a, b, (ab + \sqrt{D})/2)$, where $a, b > 2$ are in a real number field F , and $D = a^2 b^2 - 4a^2 - 4b^2 > 0$ is nonsquare in F .	Every class has at most 2 curves.	9.6
The degree- d integral family $(r^2 + 2, r + 2/r, r + 2/r)$ from Theorem 1.6.	Every class is a reflection orbit; exact maximum 2.	9.7
The dense integral families with exact trace fields and prescribed represented congruences, constructed over fields with at least two real embeddings.	Every class has at most 2 curves.	10.4, 10.2
The dense symmetric integral families with trace-field degree $4[F : \mathbb{Q}]$ and squared-trace-field degree $2[F : \mathbb{Q}]$.	Every class is a reflection orbit; exact maximum 2.	9.8
The reflection family $(t + 2/t, t + 2/t, t^2 + 2)$ outside a countable set of algebraic parameters, in particular for every transcendental $t > 0$.	Every class is a reflection orbit; exact maximum 2.	16.2
<i>One-holed tori: relative integrality, genericity and boundary criteria</i>		
One trace is not integral over $\mathbb{Z}[\kappa]$; in particular, it is transcendental over $\mathbb{Q}(\kappa)$.	Every class has at most 2 curves.	7.3
Each fixed-boundary marked space, outside a countable set of surfaces.	Every class has at most 2 curves.	7.5
A dense residual set of full measure in each fixed-boundary marked space, or in the variable-boundary space.	Every class has 1 curve.	16.1
Geodesic boundary and z transcendental over $\mathbb{Q}(x, y)$.	Every class has at most 4 curves.	13.2
Positive pole orders a, b, c at a Farey trace triple satisfy all strict triangle inequalities.	Every class has at most 6 curves.	13.3
Positive pole orders satisfy $c > a + b$, or $c = a + b$ and the opposite trace $xy - z$ also has pole order c .	Every class has at most 4 curves.	13.3
The pole orders are $(g, g, 0)$, with $g > 0$ and $\max\{0, -v(\kappa)\} < 2g$.	Every class has at most 2 curves.	13.3
A valuation has a positive pole at every simple trace and $\max\{0, -v(\kappa)\} > 0$.	Every class has at most 6 curves.	13.4
A real specialization of the trace ring sends a Farey triple to values of absolute value greater than 2 and negative product.	Every class has at most 6 curves.	13.5
$(t, t, t^2/2)$, where $t > 2\sqrt{2}$ is not an algebraic integer.	Classes have 2 or 4 curves; exact maximum 4.	13.6
(t, t, t) , where $t > 3$ is not an algebraic integer or has a real conjugate less than -2 .	Classes have 3 or 6 curves; exact maximum 6.	13.7

The first row gives a bound depending on length, not a uniform constant. The density theorem gives multiplicity $h_X \leq 6$ on almost every level of each fixed surface, while the subsequent rows give pointwise constants under additional hypotheses. In particular, the unperturbed quartic seed is not covered by the whole-spectrum bound two in its rational-twist row.

For the quartic rows in Table 2, let Q have triple $(2\sqrt{2}, 2\theta, 2\theta)$ with $\theta = \sqrt{1 + \sqrt{2}}$. Put $E = \mathbb{Q}(\sqrt{2})$

and $\mathcal{R} = \mathbb{Z}[\sqrt{2}]$, and write r' for the conjugate of $r \in E$. Prime factors are counted in $\mathcal{R}$ up to units. For the intersection rows, c denotes the intersection number of two curves in the selected level.

Table 2: Bounds requiring a hypothesis on a selected level or collection.

Hypothesis	Conclusion	Result
Any fixed one-holed torus; count all distinct levels below L .	All but $o_X(L^2)$ have multiplicity $h_X \leq 6$.	1.1
In a punctured twist family, $[\mathbb{Q}(\lambda, t) : \mathbb{Q}(\lambda)] > 2MH$.	No relation $n\ell(\gamma) = m\ell(\delta)$ for distinct curves meeting the fixed curve at most H times and $1 \leq n, m \leq M$.	8.2
A punctured-torus level contains two curves with $c = 1$.	Multiplicity at most 3; equality only at the modular systole.	14.1
A punctured-torus level contains two curves with $c = 2$.	Multiplicity exactly 2.	14.1
A punctured-torus level contains two curves with $c = 3$ or 6.	Multiplicity at most 6.	14.1
A punctured-torus level contains two curves with $c = 4$.	Multiplicity at most 4.	14.1
A punctured-torus level contains three curves with all pairwise intersections equal to 3.	Multiplicity exactly 3.	14.1
A level on any one-holed torus contains two curves of intersection c .	Multiplicity at most $4c$, or $2c - 2$ if c is even; the theorem gives a sharper greatest-common-divisor bound.	14.1
An odd-multiplicity level on a nonmodular punctured torus on the reflection walls of a reduced chamber.	Multiplicity at most 3. Even multiplicities are unrestricted by this statement.	15.1
The N th distinct level on those reflection walls has multiplicity 3.	Exactly one unmarked torus for $N = 1, 2$ or $N \geq 4$; none for $N = 3$.	15.1
A level on Q has trace $2r$, where $r \in \mathcal{R}$ and $ r' < 1$; $r/\sqrt{2}$ has ω distinct prime factors.	Multiplicity at most $2^{\max\{1, \omega\}}$. In particular, at most 2 for a unit or a prime power times a unit.	17.1
In the preceding quartic setting, select k prime divisors with their full powers; let C be the product of the others. Assume $ \text{Norm}_{E/\mathbb{Q}}(C) ^4 < (2r + 2)(1 - (r')^2)$.	Multiplicity at most $2^{\max\{1, k\}}$.	17.2

The fixed-trace results give a further finite-exception principle. If an integral trace τ lies in a totally real field E of degree at least two, $\tau > 2$ at the chosen real embedding and $|\sigma(\tau)| < 2$ at every other real embedding, then in any finite real extension K/E whose E -embeddings are real, only finitely many unmarked tori with trace triple (τ, x, y) in K can have a rational length class of size at least three. Theorem 11.1 gives a finite algebraic procedure listing all possible exceptions. This controls an entire fixed-trace locus but does not certify the bound two for the listed exceptions.

These results leave the pointwise punctured-torus conjecture open on the remaining algebraic-integral locus. In particular, neither the asymptotic theorem nor the conditional quartic rows prove a uniform bound for every level on the original quartic torus, or full Markoff uniqueness.

1.7 Proof and attribution

The density proof follows Chen’s geometric reduction, fixed-map argument and summation over rational maps [Che26, Sections 3–6]. The geometric input for every punctured torus is already supplied by Doan–Li–Nguyen [DLN26, Theorem 1.1 and Propositions 3.1, 3.2, 4.1]; it contains all three parts of Chen’s Proposition 3.1. We prove the corresponding estimates with geodesic boundary.

After removing an arbitrarily small proportion of primitive homology vectors, equal-length pairs lie on graphs $w = Gv$ with $G \in \text{GL}_2(\mathbb{Q})$. The exact corner formula shows that a rational linear isometry of the length norm is integral and hence comes from a surface isometry. For every other fixed G , Chen’s analytic zero-one argument gives only $o(L^2)$ collisions. To include arbitrary real traces, we embed

their finitely generated field injectively into a finite extension of $\mathbb{Q}_p$, keeping the initial traces integral. Polynomial mapping-class actions [Gol09], p -adic interpolation [Poo14], and the cusp-orbit count of Burrin–Nevo–Rühr–Weiss [BNRW20] then apply to one paired orbit.

A separate estimate is needed because the graph matrices vary with L . We control both their lattice indices and their entries; a strip count replaces the step using modular rotational symmetry. This gives a summable bound for the maps outside any fixed finite set. Counting the excess of isometry orbits over lengths before restoring the removed curves yields the theorem for complete length levels. The pointwise arguments are independent of this summation: valuations reduce rational length classes to intersections of a norm level with affine lines, and real conjugates are controlled by the three-length rigidity theorem.

The interpolation background includes Bell [Bel06, Bel08], Bell–Ghioca–Tucker [BGT10], and Cantat–Xie [CX18]; related orbit and flow methods appear in [CJ24, Jan25]. Strong approximation and finite Markoff graphs are studied in [BGS16, BGS26, Che24, Bro24, EFLMT25], and Markoff–Hurwitz counting in [GMR19]. Our proof does not assume transitivity on the full Markoff surface modulo a prime. Appendix B records the broader arithmetic, geometric and dynamical background.

1.8 Statement on artificial intelligence use

Substantial parts of the arguments and text were generated with OpenAI’s ChatGPT through Codex. The starting points were the papers of Chen [Che26] and Doan–Li–Nguyen [DLN26], together with the author’s earlier drafts motivated by Hugo Parlier’s question about once-punctured tori with bounded simple-length multiplicity. Assistance included literature searches, proof development, drafting, revision, source checks, LaTeX compilation, and internal cross-checks by parallel model agents. These checks do not constitute independent peer review or verification by a proof assistant. The author does not claim to have independently verified every generated argument.

1.9 Organization and conventions

Section 2 gives the geometric estimates and excision. Section 3 identifies the rational norm symmetries and records the counting consequences of a bound for $E_X(L)$. Section 4 proves the fixed-graph estimate. Section 5 carries out the uniform summation, and Section 6 completes the main proof and the mean and distributional consequences. The following sections prove the pointwise results stated above, their reflection and quartic special cases, and the transfer to four-punctured spheres. Appendix A gives an alternative large-matrix estimate for arbitrary planar norms. Constants may depend on the fixed surface X ; further dependencies are indicated by subscripts. No uniformity over Teichmüller space is asserted. Euclidean norms of vectors and linear forms are denoted by absolute values.

2 The length norm and geometric reduction

For punctured tori, the corner and Farey-sector estimates used here are already contained in [DLN26, Theorem 1.1 and Propositions 3.1, 3.2 and 4.1]. In particular, DLN supplies all three parts of [Che26, Proposition 3.1], for arbitrary punctured tori. We give the calculation with geodesic boundary and then use Chen’s geometric reduction and graph summation [Che26, Sections 3, 5 and 6]. The entry-height estimate in Section 5 replaces the step using the modular torus’s order-six rotation.

Let N be the length norm on $\mathbb{R}^2$, and write $B = \{x : N(x) \leq 1\}$. Thus $N(v)$ is the simple length associated to a primitive integral vector v , and v and $-v$ represent the same unoriented curve. We use Euclidean absolute values for vectors and linear forms. Constants c, C, ρ are positive, may change at each occurrence, and depend only on N . In particular,

$$c|x| \leq N(x) \leq C|x|.$$

For an integer $H \geq 1$, the Farey fan $\mathcal{F}_H$ consists of the rays through primitive $u \in \mathbb{Z}^2$ with $\text{ht}(u) = \max(|u_1|, |u_2|) \leq H$. Adjacent rays bound an open sector. Its positively ordered primitive boundary vectors u, v satisfy

$$\det(u, v) = 1, \text{ht}(u + v) > H.$$

The fan has $O(H^2)$ rays and sectors.

Lemma 2.1 (The length norm with geodesic boundary). *Let X be a compact hyperbolic torus with one geodesic boundary component. The lengths of its essential nonperipheral simple closed geodesics extend*

uniquely to a norm N_X on $H_1(X; \mathbb{R})$, homogeneous on integral rays. Strict convexity follows from the corner calculation below.

Proof. For an integral homology class a , minimize total length among oriented multicurves representing a , allowing integer multiplicities. Compactness and the usual shortening argument give a minimizing geodesic multicurve. One may equivalently first straighten each component: a fixed length bound leaves only finitely many closed geodesics and bounded multiplicities, so the infimum is attained. Contractible and peripheral components are homologically zero and can be discarded. A transverse intersection can be resolved in the oriented way and then shortened, preserving homology and strictly reducing length. Hence a minimizer has disjoint essential components. On a torus with one boundary component any two such components are parallel. Oppositely oriented components cancel and cannot occur in a minimizer. Consequently, if $a = kv$ with v primitive, the minimum equals $|k|\ell_X(v)$.

Taking the union of representatives, resolving intersections in the oriented way and tightening preserves the summed homology class and does not increase length. This proves the triangle inequality. Homogeneity therefore defines a norm on rational homology. To check positive definiteness uniformly, choose two smooth closed one-forms dual to an integral homology basis. Their bounded pointwise norms on the compact surface bound the absolute homology coordinates of any cycle by a constant times its length. A fixed pair of basis cycles gives the reverse bound. Thus the rational norm is equivalent to a Euclidean norm and has a unique continuous extension to real homology.

The cusp case is supplied by [MR95a, MR95b]. $\square$

For a boundary length $b \geq 0$, with $b = 0$ denoting a cusp, set

$$\kappa = 2 - 2 \cosh(b/2), d = 4 - \kappa,$$

$$f_X(t) = \frac{1}{2} \log \left(1 + \frac{d}{4 \sinh^2(t/2)} \right).$$

Then $f_X(t) > 0$ and $f_X(t) \sim (d/2)e^{-t}$ as $t \rightarrow \infty$. For a cusp $d = 4$ and $f_X(t) = \log \coth(t/2)$.

Lemma 2.2 (Corners and exponentially flat Farey sectors). *After choosing an integral homology basis, the following estimates hold. For every primitive vector u and $h \in \mathbb{R}^2$,*

$$D_h N(u) + D_{-h} N(u) = 4 |\det(u, h)| f_X(N(u)).$$

Here D_h is the one-sided directional derivative. The angle of the normal cone at the ray of a primitive vector of height at most H is at least ce^{-CH} . On each open sector of $\mathcal{F}_H$, all supporting functionals of N are within $Ce^{-\rho H}$ of one another. The sum of the angular widths of their normal-direction arcs, over all open sectors, is at most $Ce^{-\rho H}$.

Proof. The norm exists by the preceding construction and [MR95a, MR95b]. For a cusp, the full statement is contained in DLN: the corner formula and lower bound follow from [DLN26, Proposition 4.1 and equation (9)]; endpoint closeness and the sector estimate follow from [DLN26, Propositions 3.1 and 3.2]; and the sum of normal turns is [DLN26, Theorem 1.1]. The supporting functionals throughout a sector are controlled by monotonicity of normal directions and their Lipschitz parametrization on the dual unit circle. Polynomial factors in H are absorbed by decreasing the exponential rate, and the finitely many initial levels by increasing the constant. Thus every part of [Che26, Proposition 3.1] is already supplied by DLN in the punctured case. We include the trace calculation to prove the version with geodesic boundary, deriving the corner identity before using strict convexity.

Choose a reduced positive Fricke triple

$$2 < x \leq y \leq z, x^2 + y^2 + z^2 - xyz = \kappa, z \leq xy/2.$$

Such a triple is obtained by repeatedly replacing the largest trace z by $xy - z$ whenever this decreases it. The process terminates: there are only finitely many simple curves below any fixed length, and each step decreases the sorted triple. If the largest coordinate is already at most the half-product, stop. The three traces are assigned to $e_1, e_2, e_1 + e_2$. Put $w = xy - z$, so $w \geq z \geq y \geq x$. Up to simultaneous sign, the four adjacent pairs of rays in $\mathcal{F}_1$ have trace pairs $(x, z), (z, y), (y, w), (w, x)$; their difference traces are respectively y, x, x, y . Thus the difference trace never exceeds the larger boundary trace. If a pair has boundary traces $a, b > 2$ and difference trace $c \leq \max(a, b)$, then its median trace $ab - c$ exceeds $\max(a, b)$. The same inequality holds for both child sectors. Induction proves that the trace at $u + v$ is the larger Fricke root in every sector of every $\mathcal{F}_H$.

Fix adjacent boundary vectors with $\det(u, v) = 1$. Put $\lambda = e^{N(u)/2}$ and $\mu = e^{N(v)/2}$ and choose holonomy matrices

$$U = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, V = \begin{pmatrix} a & b \\ c & a' \end{pmatrix}.$$

The commutator trace is $\kappa - 2$, so

$$bc = \frac{4 - \kappa}{(\lambda - \lambda^{-1})^2}, aa' = 1 + \frac{4 - \kappa}{(\lambda - \lambda^{-1})^2}.$$

The diagonal entries a, a' are positive. Since

$$\text{tr}(U^n V) = \lambda^n a + \lambda^{-n} a', \text{tr}(U^n V^{-1}) = \lambda^n a' + \lambda^{-n} a,$$

homogeneity and convexity give

$$D_v N(u) = 2 \log a, D_{-v} N(u) = 2 \log a'.$$

Their sum is $4f_X(N(u))$. Decomposing h in the basis (u, v) proves the displayed corner formula.

If the unit ball contained a nontrivial line segment, its interior rays would form an open interval and include a primitive rational ray. The norm would be linear near that ray and have zero directional jump, contradicting the positive formula just proved. Thus the unit ball is strictly convex, including with geodesic boundary.

Let α^+, α^- be the extreme supporting functionals at u . They have uniformly bounded norms, bounded away from zero, and

$$|\det(\alpha^+, \alpha^-)| = 4N(u)f_X(N(u)).$$

Using $0 < \text{sys}(X) \leq N(u) \leq CH$ and $f_X(t) \geq 2e^{-t}$ gives the lower bound for the normal-cone angle.

For flatness set

$$P = (\lambda - \lambda^{-1})(\mu - \mu^{-1}), \xi = \frac{P - \sqrt{P^2 - 4d}}{2} = \frac{2d}{P + \sqrt{P^2 - 4d}}.$$

The larger-root property gives $P \geq 2\sqrt{d}$ and

$$T(u + v) = \lambda\mu + (\lambda\mu)^{-1} - \xi, a = \mu - \frac{\xi}{\lambda - \lambda^{-1}}.$$

The positive systole of the fixed torus bounds λ, μ away from one. Therefore

$$a = \mu(1 - \delta), 0 < \delta \leq Ce^{-N(u) - N(v)}.$$

As $N(u) + N(v) \geq cH$, for all sufficiently large H we have $\delta \leq 1/2$ and hence

$$|D_v N(u) - N(v)| \leq Ce^{-N(u) - N(v)}.$$

Interchanging u, v gives the analogous estimate. For the inward supporting functionals α at u and β at v this yields

$$|(\alpha - \beta)(u)| + |(\alpha - \beta)(v)| \leq Ce^{-N(u) - N(v)}.$$

The dual basis has norm $O(H)$, so $|\alpha - \beta| \leq CH e^{-cH} \leq Ce^{-\rho H}$. Monotonicity of supporting directions around a strictly convex body shows that the normal directions inside the sector lie on the arc between these two inward directions. This arc and the arc of the opposite sector are antipodal and have disjoint interiors, so its angular width is at most π . Its width is consequently bounded by a constant times the distance between its endpoint unit normals. The corresponding supporting functional is a Lipschitz function of its unit normal: it is $\theta/h_B(\theta)$, where the support function h_B is positive and Lipschitz on the unit circle. This proves the sector estimate. There are $O(H^2)$ sectors; summing their angular widths and decreasing ρ proves the total-turning estimate. Increasing C absorbs the finitely many smaller levels H . $\square$

Lemma 2.3 (Removal of a small set of directions). *For $M \geq 1$ and $H \geq C \log(2 + M)$, remove every ray of $\mathcal{F}_H$ and every sector whose closure contains the contact point on ∂B of a nonzero integral form β with*

$|\beta| \leq M$. The removed sectors number $O(\log^2(2+M))$ and have total angular width $O(\log(2+M)/H)$. The removed primitive vectors with $N(v) \leq L$ number at most

$$C \left(\frac{\log(2+M)}{H} L^2 + L \log^2(2+M) + H^2 \right).$$

For every unremoved vector v , every supporting functional γ at v , and every such integral form β ,

$$\angle(\gamma, \{\beta, -\beta\}) \geq ce^{-C_{\text{sep}}H}.$$

Proof. This is the excision argument of [Che26, Lemmas 3.3 and 3.5], applied with Lemma 2.2. Integral forms of size at most M that are not positive multiples of one another make an angle at least M^{-2} . At level $H_0 = C \log(2+M)$ the total normal turning inside open sectors is less than M^{-2} , by Lemma 2.2. There is therefore at most one contact ray per open sector at that level, in addition to the $O(H_0^2)$ boundary rays. Thus all relevant contact points lie on $O(\log^2(2+M))$ rays. Each removes at most two sectors at level H .

For completeness, a union of k level- H sectors has angular width $O(\sqrt{k}/H)$ if $\sqrt{k} \leq cH$. In the octant $0 \leq p \leq q$, a sector has slope width $1/(bd)$, where $b, d \leq H$ and $b+d > H$. Sectors with an endpoint denominator at most $n \leq H/2$ have total width $O(n/H)$; all others have width $O(1/(nH))$ each. Take $n \asymp \sqrt{k}$. Applying this with $k = O(\log^2(2+M))$ proves the angular estimate.

A sector of angular width θ , intersected with LB , has area $O(\theta L^2)$ and perimeter $O(L)$. Its lattice-point count is $O(\theta L^2 + L + 1)$. Summation proves the count; each removed ray contains at most one primitive vector. Finally, between an unremoved sector and either contact point for $\pm\beta$ there is a complete corner at a boundary vector of height at most H . Monotonicity of normals and Lemma 2.2 give the claimed separation. $\square$

Theorem 2.4 (A bound at every length). *For each fixed hyperbolic torus with one cusp or one geodesic boundary component there is a constant C_X such that, for every $t > 0$,*

$$\#(\mathbb{Z}^2 \cap t\partial B_X) \leq C_X (\log(2+t))^2.$$

In particular,

$$m_X(t) \leq C_X (\log(2+t))^2.$$

For a cusp, both bounds are due to Doan–Li–Nguyen [DLN26, Theorem 1.4], with the bounded initial range absorbed into the constant. The proof below also treats a fixed positive boundary length.

Proof. Let $H = \lceil A \log(2+t) \rceil$, where A is a sufficiently large constant depending on X . On any open sector of the Farey fan $\mathcal{F}_H$, fix one inward supporting functional α . If $v \in t\partial B_X$ lies in that sector and γ supports the norm at v , then $\gamma(v) = t$ and the flatness estimate gives

$$|\alpha(v) - t| \leq |\alpha - \gamma||v| \leq C_X t e^{-\rho H}.$$

Also $|v| \leq C_X t$ and $|\alpha| \geq c_X > 0$. Three such points would therefore form a triangle of area at most $C_X t^2 e^{-\rho H}$. Choose A so that this is less than $1/2$ for all $t > 0$, increasing the fixed fan level if necessary. A nondegenerate triangle with integral vertices has area at least $1/2$. Thus three integral points in this sector would have to be collinear. Strict convexity of B_X excludes three distinct collinear points on $t\partial B_X$. Each open sector therefore contains at most two integral boundary points. Each boundary ray contains at most one such point, by homogeneity of the norm. The fan has $O_X(H^2)$ sectors and rays, proving the estimate. $\square$

3 Rational symmetries and the exceptional count

Let $N = N_X$ be the stable norm of a hyperbolic one-holed torus. For a primitive $v \in \mathbb{Z}^2$ and $h \in \mathbb{R}^2$, write

$$J_N(v; h) = D_h N(v) + D_{-h} N(v).$$

We use the corner identity, with $d = 4 - \kappa$ as above,

$$J_N(v; h) = 4 |\det(v, h)| f(N(v)), f(t) = f_X(t) = \frac{1}{2} \log \left(1 + \frac{d}{4 \sinh^2(t/2)} \right).$$

The directional derivatives exist by convexity. The identity holds for all h , by homogeneity and the formula on a unimodular complement of v .

Lemma 3.1 (Rational norm symmetries are integral). *If $G \in \mathrm{GL}_2(\mathbb{Q})$ satisfies $N(Gx) = N(x)$ for every $x \in \mathbb{R}^2$, then $G \in \mathrm{GL}_2(\mathbb{Z})$.*

Proof. Put $\delta = |\det G|$. For each primitive v , write $Gv = c_v w$ with $c_v > 0$ rational and w primitive. Choose a positive integer D such that $A = DG$ is integral. Then $c_v = \gcd(Av)/D$, and $\gcd(Av)$ divides $|\det A|$: indeed the adjugate identity gives $\mathrm{adj}(A)Av = (\det A)v$, and v is primitive. Thus the c_v belong to a fixed finite set.

The directional jump is unchanged on a positive ray. Applying the corner identity at v and w , and using $N(w) = N(v)/c_v$, gives, with $l = N(v)$,

$$f(l) = \frac{\delta}{c_v} f(l/c_v).$$

Fix any primitive v_0 and put $\Delta = |\det A|$. The content $\gcd(Av)$ equals the greatest common divisor of the coordinates of Av together with Δ , because it divides Δ . Thus c_v depends only on v modulo Δ . Choose u with $\det(v_0, u) = 1$. The vectors

$$v_k = v_0 + k\Delta u$$

are primitive, have $c_{v_k} = c_{v_0}$, and satisfy $N(v_k) \rightarrow \infty$. Since $f(t) \sim (d/2)e^{-t}$, the displayed jump identity along this sequence forces $c_{v_0} = 1$ and $\delta = 1$. The primitive vector v_0 was arbitrary. Consequently G maps every primitive vector to a primitive integral vector. In particular its columns are integral, and $|\det G| = 1$ proves the assertion. $\square$

Remark 3.2. The arithmetic-progression argument avoids any systole restriction. In particular, no monotonicity of $t \log \coth(t/2)$ near zero is required.

Put

$$K_X = \{G \in \mathrm{GL}_2(\mathbb{Z}) : N_X \circ G = N_X\}.$$

This is the action on homology of the full isometry group of X . Indeed an isometry preserves every simple length. Conversely an integral norm symmetry preserves the three simple lengths in a Farey basis, hence the corresponding positive Fricke triple; the marked torus is determined by that triple [Gol09]. Thus the corresponding change of marking fixes X and is represented by an isometry.

Lemma 3.3 (Size of a symmetry orbit). *The group K_X is finite, contains $-I$, and has order at most 12. In particular each orbit of unoriented simple curves has at most six elements. All but finitely many such orbits have exactly*

$$h_X = |K_X|/2$$

elements.

Proof. Equivalence of N_X with the Euclidean norm bounds the matrix entries of every element of K_X , so integrality makes the group finite. Average a Euclidean inner product over this finite group. In the averaged metric it is a subgroup of $O(2)$. Its orientation-preserving subgroup is cyclic, say of order n . If a generator has rotation angle $2\pi/n$, its trace is the integer $2 \cos(2\pi/n)$, forcing $n \in \{1, 2, 3, 4, 6\}$. The orientation-preserving subgroup has index at most two, proving $|K_X| \leq 12$. The norm is even, so $-I \in K_X$.

A primitive vector gives an unoriented curve modulo multiplication by -1 . A nontrivial element of $K_X/\{\pm I\}$ fixes such a curve only if its representative has eigenvalue 1 or -1 on the corresponding rational line. Each such eigenline contains only one primitive direction modulo sign. Since K_X is finite, only finitely many curves have nontrivial stabilizer in this quotient. $\square$

For $L > 0$ let $C_X(L)$ count unoriented simple curves of length at most L , let $O_X(L)$ count their K_X -orbits, and let $D_X(L)$ count their distinct lengths. Put $B_X = \{x : N_X(x) \leq 1\}$. Primitive lattice counting and the previous lemma give

$$C_X(L) = \frac{\mathrm{area}(B_X)}{2\zeta(2)} L^2 + O_X(L \log(2 + L)),$$

$$O_X(L) = \frac{\mathrm{area}(B_X)}{|K_X|\zeta(2)} L^2 + O_X(L \log(2 + L)).$$

Define the nonnegative symmetry-orbit excess

$$E_X(L) = O_X(L) - D_X(L).$$

If a distinct length t is represented by $r_X(t)$ symmetry orbits, then

$$E_X(L) = \sum_{t \leq L} (r_X(t) - 1).$$

Lemma 3.4 (Deletion and density conversion). *Suppose that, for each L , one deletes $d_X(L)$ curves from the curves of length at most L , and let $P_X(L)$ be the number of unordered equal-length pairs of retained curves which belong to distinct K_X -orbits. Then*

$$E_X(L) \leq d_X(L) + P_X(L).$$

If $E_X(L) = o(L^2)$, then all the following hold:

1. $D_X(L) \sim \text{area}(B_X)L^2/(|K_X|\zeta(2))$;
2. the number of lengths at most L represented by two distinct symmetry orbits is $o(L^2)$ and has density zero among distinct simple lengths;
3. the number of curves of length at most L whose complete level contains two distinct symmetry orbits is at most $2h_X E_X(L) = o(L^2)$;
4. the number of curves of length at most L whose complete level has multiplicity greater than six is $o(L^2)$;
5. all but $o(L^2)$ distinct lengths at most L have multiplicity exactly h_X .

Proof. Fix a length t represented by r symmetry orbits, and let s of these orbits contain a retained curve. The $r - s$ orbits containing no retained curve contribute at least $r - s$ deleted curves. If $s \geq 1$, selecting one retained curve in each surviving orbit produces at least $\binom{s}{2} \geq s - 1$ pairs counted in $P_X(L)$. If $s = 0$, all r orbits contribute deleted curves. In both cases $r - 1$ is bounded by the deleted curves at that level plus its retained pairs. Sum over levels.

The first conclusion follows by subtracting $E_X(L)$ from $O_X(L)$. For the second, each level with $r_X(t) \geq 2$ contributes at least one to the excess. For the third, a level with $r = r_X(t) \geq 2$ has at most $h_X r \leq 2h_X(r - 1)$ curves. The fourth follows since one orbit has at most six curves. Finally, a level consisting of one symmetry orbit has multiplicity h_X unless that orbit is among the finitely many exceptional orbits in Lemma 3.3. $\square$

Remark 3.5. The deletion argument avoids an unjustified passage from few retained collision pairs to few bad curves. A retained curve may share its length only with deleted curves. The symmetry-orbit excess accounts for every such level before converting to a curve-density statement.

4 A fixed rational graph on an arbitrary one-holed torus

Let X be a hyperbolic one-holed torus. Choose its positive trace coordinates $o = (x, y, z)$ with

$$x^2 + y^2 + z^2 - xyz = \kappa,$$

with no algebraicity restriction. Write N for its stable norm and $B = \{v : N(v) \leq 1\}$. For a primitive integral vector v , let $T_v \in \mathbb{Z}[X, Y, Z]$ be its signed simple trace polynomial for a fixed lift of the character, and put $P_v = T_v^2$. Then

$$\begin{aligned} P_v(o) &= 4 \cosh^2(N(v)/2), \\ P_v(P * \gamma) &= P_{\gamma v}(P). \end{aligned}$$

Squaring removes every choice of sign in a holonomy lift. Here $*$ is the right polynomial action of $\text{GL}_2(\mathbb{Z})$ on the affine three-dimensional character space. Each action map and its inverse have integer coefficients. The element $-I$ acts trivially. These assertions hold on the entire affine character space, not merely on the Fricke surface; they follow from the usual trace-coordinate Nielsen action [Gol09]. This section adapts Chen's fixed-graph argument [Che26, Section 4].

Theorem 4.1 (Fixed rational graph). *If $G \in \text{GL}_2(\mathbb{Q})$ is not a linear isometry of N , then*

$$\#\{v \in \mathbb{Z}_{\text{prim}}^2 : Gv \in \mathbb{Z}_{\text{prim}}^2, N(v) = N(Gv) \leq L\} = o_{X,G}(L^2).$$

4.1 Realizing arbitrary trace fields p -adically

Lemma 4.2 (Integral embedding of finitely generated fields). *Let F be a finitely generated field extension of $\mathbb{Q}$, and let $S \subset F$ be finite. There are an odd prime p , a finite extension $K/\mathbb{Q}_p$, and an injective field homomorphism $\iota : F \rightarrow K$ such that $\iota(S) \subset \mathcal{O}_K$.*

Proof. Choose a transcendence basis $t = (t_1, \dots, t_r)$ of $F/\mathbb{Q}$. Since characteristic zero extensions are separable, the primitive element theorem gives $F = \mathbb{Q}(t)(\alpha)$. Let

$$f(t, A) = A^e + \sum_{i < e} c_i(t) A^i$$

be the monic minimal polynomial of α . Every $s \in S$ has an expression $\sum_{i < e} c_{s,i}(t) \alpha^i$ with coefficients in $\mathbb{Q}(t)$. Choose a nonzero integer polynomial $D(t)$ which is a common denominator of the finitely many rational functions c_i and $c_{s,i}$. Enlarge it, if necessary, so that every coefficient is an integer polynomial divided by D . The discriminant of f is a nonzero rational function of t .

Choose $a \in \mathbb{Q}^r$ at which D and the numerator of this discriminant are nonzero. The specialized polynomial $f(a, A)$ is separable. Choose a root α_0 and put $E = \mathbb{Q}(\alpha_0)$. Select an odd prime p outside a finite set so that the coordinates of a are p -integral, $D(a)$ is a p -adic unit, and $f_A(a, \alpha_0)$ is a unit at every place of E above p . The last exclusion is legitimate because this derivative is a nonzero algebraic number. At any chosen place above p , put $K = E_p$. The element α_0 is integral there, because $f(a, A)$ is monic with integral coefficients.

For every $b \in a + p\mathcal{O}_K^r$, the denominator $D(b)$ remains a unit, all the coefficients above are integral, and Hensel's lemma gives a unique root

$$\alpha(b) \in \alpha_0 + p\mathcal{O}_K$$

of $f(b, A)$: the value at α_0 is in $p\mathcal{O}_K$ and its derivative is a unit. This is also the usual implicit analytic root, although its analytic dependence is not needed here.

Choose b in this ball whose coordinates are algebraically independent over $\mathbb{Q}$. Such a choice exists by the Baire category theorem: there are countably many nonzero rational polynomials, and the zero set of each has empty interior in K^r . The latter assertion follows by induction on r from the fact that a nonzero univariate polynomial has finitely many roots. Evaluation $t \mapsto b$ therefore embeds $\mathbb{Q}(t)$ into K . Under this field isomorphism onto its image, $f(b, A)$ is still irreducible over $\mathbb{Q}(b)$. Sending α to $\alpha(b)$ consequently extends evaluation to an injective homomorphism $F \rightarrow K$. The chosen expressions for the elements of S have integral coefficients and integral argument $\alpha(b)$, so their images are integral.

When $r = 0$, F is a number field; any odd prime avoiding the finitely many denominators of the elements of S gives the conclusion directly. $\square$

4.2 An ambient interpolation lemma

Fix an odd prime p . If a polynomial automorphism g of $\mathbb{Z}_p^d$, with polynomial inverse over $\mathbb{Z}_p$, fixes every point modulo p^2 , then on every residue ball $a + p\mathbb{Z}_p^d$ its expression in the coordinates $a + pu$ is

$$g_a(u) = u + pR(u), R \in \mathbb{Z}_p[u_1, \dots, u_d]^d.$$

Indeed, Taylor expansion gives

$$\frac{g(a + pu) - a}{p} = \frac{g(a) - a}{p} + Dg(a)u + \sum_{j \geq 2} p^{j-1} G_j(u),$$

where the G_j are homogeneous integral polynomials. The constant term is divisible by p , and reduction of the displayed affine expression modulo p is the identity function on $\mathbb{F}_p^d$. Consequently $Dg(a) \equiv I \pmod{p}$.

Put $\Delta f = f \circ g_a - f$ on the restricted power-series algebra $\mathbb{Z}_p\langle u \rangle$. Taylor expansion gives $\Delta(p^r \mathbb{Z}_p\langle u \rangle) \subset p^{r+1} \mathbb{Z}_p\langle u \rangle$. Hence

$$\Phi_g(t, u) = \sum_{k \geq 0} \binom{t}{k} \Delta^k u$$

converges in $\mathbb{Z}_p\langle t, u \rangle^d$, since the coefficient valuation in its k th term is at least $k - v_p(k!) \geq k - k/(p-1)$, which tends to infinity for odd p . For nonnegative integers n , the binomial theorem gives $\Phi_g(n, u) = g_a^n(u)$. Continuity then gives the flow law and shows that $\Phi_g(t, u)$ belongs to the closure of the integer orbit. This is the interpolation argument of Poonen [Poo14, Theorem 1].

4.3 A zero-one lemma

We use the argument of [Che26, Lemma 4.3], written here for any odd prime and a finite extension of the coefficient field. Let a finitely generated group J act by integral polynomial automorphisms on $\mathbb{Z}_p^d$, and suppose every element acts trivially modulo p^2 . Let Y be the closure of one orbit. The uniform measures on its finite residue images are compatible, because each image is a transitive J -set and every equivariant reduction map has constant fibre cardinality. They define an invariant probability measure μ with full support on Y .

If F is a polynomial, with values in a finite extension of $\mathbb{Q}_p$, then

$$\mu\{F = 0\} \in \{0, 1\}.$$

To prove this, first observe that the action is ergodic. The conditional expectation of a J -invariant square-integrable function on each finite residue partition is constant, and these conditional expectations converge to the original function.

For a generator g of J , the interpolation just constructed preserves μ for every $t \in \mathbb{Z}_p$, by uniform approximation by integer iterates. For each $y \in Y$, Strassmann's theorem [Str28] says that $t \mapsto F(\Phi_g(t, y))$ is identically zero or has finitely many zeros. Thus the indicator of $E = \{F = 0\}$ satisfies

$$\int_{\mathbb{Z}_p} |1_E(\Phi_g(t+1, y)) - 1_E(\Phi_g(t, y))| dt = 0.$$

Fubini's theorem and invariance under the flow give $\mu(E \triangle g^{-1}E) = 0$. This holds for every generator; ergodicity proves the dichotomy. If $\mu(E) = 1$, closedness and full support imply $F = 0$ on all of Y .

4.4 Proof of the fixed-graph theorem

Set $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and

$$H = \Gamma \cap G^{-1}\Gamma G.$$

This has finite index in Γ , and the primitive vectors v for which Gv is primitive form finitely many H -orbits. For completeness, write $A = DG$ integral and $\delta = |\det A|$. The subgroup $\Gamma(\delta)$ lies in H . Moreover $\gcd(Av)$ divides δ for primitive v , and Gv is primitive exactly when $\gcd(Av) = D$; this is a condition modulo δ . Primitive vectors in one residue class modulo δ form one $\Gamma(\delta)$ -orbit.

Apply Lemma 4.2 to the finitely generated field $F_0 = \mathbb{Q}(x, y, z)$ and the finite set $\{x, y, z\}$. It gives an injective homomorphism $\iota : F_0 \rightarrow K$, where $K/\mathbb{Q}_p$ is finite for an odd prime p , and $\iota(o) \in \mathcal{O}_K^3$. Write $\mathcal{O} = \mathcal{O}_K$ and continue to denote the image of o by o when working p -adically. A $\mathbb{Z}_p$ -basis of $\mathcal{O}$ identifies $\mathcal{O}^6$ with $\mathbb{Z}_p^{6d}$. The paired action

$$(P, P') \star h = (P \star h, P' \star (GhG^{-1}))$$

is an action of H by polynomial automorphisms with integral $\mathbb{Z}_p$ coefficients and integral inverses. In particular these maps are isometries for the coordinate p -adic metric.

Let J be its kernel on the entire finite quotient $\mathcal{O}^6/p^2\mathcal{O}^6$. This is a finite-index normal subgroup of H , is finitely generated and contains $-I$. There are finitely many J -orbits of admissible primitive vectors, so fix one, with representative u and $w = Gu$ primitive. Set

$$z_0 = (o, o), F(P, P') = P_u(P) - P_w(P').$$

Then

$$F(z_0 \star j) = P_{ju}(o) - P_{Gju}(o).$$

Let $Y = \overline{z_0 \star J}$ and let μ be its orbit measure. The zero-one lemma applies. If F vanished on Y , the injectivity of the embedding of F_0 would give $N(ju) = N(Gju)$ for every $j \in J$. The directions of Ju are dense: for every primitive vector a , a positive power of the parabolic $v \mapsto v + \det(a, v)a$ lies in J , and its iterates on u approach the direction of a unless u already has that direction. Homogeneity and continuity would therefore make G a linear isometry of N , contrary to the hypothesis. It follows that

$$\mu\{F \equiv 0 \pmod{p^n \mathcal{O}}\} \rightarrow 0.$$

We give the conversion from this measure statement to the required lattice count, including the cusp weights. For fixed n , let U_n be the kernel of the action of J on the whole finite ambient quotient $\mathcal{O}^6/p^n\mathcal{O}^6$.

Then U_n is normal in J and contains $-I$; put $Q_n = J/U_n$. Let s generate the stabilizer of u in J , and let ω be the cusp width of u relative to J . Write r_n for the order of s in Q_n . The distinct U_n -orbits in Ju correspond to the sets $\bar{j}\langle\bar{s}\rangle$ in Q_n , that is, the orbits of right multiplication by $\bar{s}$; each has r_n elements. Every one of these orbits has cusp width ωr_n relative to U_n , by normality.

For a finite-index subgroup $V \leq \Gamma$ containing $-I$, a primitive vector a , and a bounded star-shaped region Ω with Lipschitz radial function, the cusp-orbit count is

$$\#(Va \cap L\Omega) = \frac{\omega(a; V)}{[\Gamma : V]} \frac{\text{area}(\Omega)}{\zeta(2)} L^2 + o(L^2).$$

This follows from Burrin–Nevo–Rühr–Weiss [BNRW20, Theorem 2.7 and Section 5.2], with the normalization used in [Che26, Theorem 4.7]. To specify the constant, the stabilizer of a in Γ is an infinite cyclic unipotent group. Passing to V multiplies the length of its stabilizer fundamental interval by $\omega(a; V)$ and multiplies the ambient covolume by $[\Gamma : V]$. The orbit-map counting formula in [BNRW20, Lemma 5.7] therefore multiplies the leading density by $\omega(a; V)/[\Gamma : V]$. For $V = \Gamma$ the normalization is the primitive lattice density $1/\zeta(2)$. Only fixed V, a, Ω are used; no uniform error in the group index is asserted.

The subset

$$E_n = \{\bar{j} \in Q_n : F(z_0 \star j) \equiv 0 \pmod{p^n \mathcal{O}}\}$$

is well-defined and invariant under right multiplication by s , because $su = u$ and $(GsG^{-1})w = w$. Uniform measure on Q_n pushes forward to uniform measure on the state orbit, whence

$$\frac{|E_n|}{|Q_n|} = \mu\{F \equiv 0 \pmod{p^n \mathcal{O}}\}.$$

There are $|E_n|/r_n$ selected U_n -orbits, each contributing

$$\frac{\omega r_n}{[\Gamma : J]|Q_n|} \frac{\text{area}(B)}{\zeta(2)} L^2 + o_n(L^2).$$

Their union contains every exact collision in Ju . Consequently

$$\limsup_{L \rightarrow \infty} \frac{\#\{v \in Ju : N(v) = N(Gv) \leq L\}}{L^2} \leq \frac{\omega}{[\Gamma : J]} \frac{\text{area}(B)}{\zeta(2)} \mu\{F \equiv 0 \pmod{p^n \mathcal{O}}\}.$$

Let n tend to infinity, and sum over the finitely many J -orbits. This proves the theorem.

5 Rational graphs and their summation

The truncation and summation follow [Che26, Sections 5–6]. We include matrix entries in the complexity and use a strip count; these replace the step relying on modular rotational symmetry.

5.1 A complexity bound without modular symmetry

For $G \in \text{GL}_2(\mathbb{Q})$ put

$$\Lambda_{\text{in}} = \mathbb{Z}^2 \cap G^{-1}\mathbb{Z}^2, \Lambda_{\text{out}} = G\Lambda_{\text{in}},$$

$$I(G) = \max\{[\mathbb{Z}^2 : \Lambda_{\text{in}}], [\mathbb{Z}^2 : \Lambda_{\text{out}}]\}, h(G) = \max_{i,k} |G_{ik}|, J(G) = \max\{1, I(G), h(G)\}.$$

Lemma 5.1 (Uniform graph population). *Let $\mathcal{P}$ be a set of pairs (v, Gv) with both entries integral and $N(v), N(Gv) \leq L$. If the first entries contain two linearly independent vectors, then*

$$|\mathcal{P}| \leq C \left(\frac{L^2}{J(G)} + L + 1 \right).$$

If $q(G)$ is the least positive integer making $q(G)G$ integral, then $q(G) \leq I(G)$. For $Q \geq 1$, the number of rational matrices with $J(G) \leq Q$ is $O(Q^9)$, and every such graph plane has a basis of integral vectors of norm $O(Q^2)$.

Proof. Let v_1, v_2 be independent first entries. Since $|v_i| \leq CL$, the covering radius of Λ_{in} is $O(L)$: this is true for the sublattice generated by v_1, v_2 , and hence for the larger lattice. The Voronoi cells centered at the first entries are disjoint, have area $[\mathbb{Z}^2 : \Lambda_{\text{in}}]$, and lie in a disc of radius $O(L)$. Applying the same argument to the independent images Gv_1, Gv_2 gives

$$|\mathcal{P}| \leq CL^2/I(G).$$

For the height bound choose a row of G of Euclidean norm at least $h(G)$. The first entries lie in a disc of radius CL and in a strip of width at most $CL/h(G)$, since $|Gv| \leq CL$. Their containing convex body therefore has area at most $CL^2/h(G)$ and perimeter at most CL . Covering its integral points by centered unit squares proves

$$|\mathcal{P}| \leq C(L^2/h(G) + L + 1).$$

Combining the two estimates proves the assertion for $J(G)$.

The index $n = [\mathbb{Z}^2 : \Lambda_{\text{in}}]$ annihilates $\mathbb{Z}^2/\Lambda_{\text{in}}$. Thus nG is integral and $q(G) \mid n$. For $J(G) \leq Q$, write $G = A/q$ with $q \leq Q$. Every integral entry of A has absolute value at most Q^2 , giving $O(Q^9)$ choices in total. The vectors (qe_i, Ae_i) , $i = 1, 2$, form the required integral basis of the graph plane over $\mathbb{R}$. $\square$

5.2 Reduction to finitely many graphs up to small error

Let K be a finite group of rational linear isometries of N , containing $-I$, and suppose it contains every rational linear isometry of N . A collision is an ordered pair of primitive vectors (v, w) with $N(v) = N(w) \leq L$ and $w \notin Kv$.

Theorem 5.2 (Geometric summation principle). *Let N be the length norm fixed above, with the corner and Farey estimates of Lemma 2.2 and the removal estimate of Lemma 2.3. Suppose that for every fixed $G \in \text{GL}_2(\mathbb{Q}) \setminus K$,*

$$\#\{v \in \mathbb{Z}_{\text{prim}}^2 : Gv \in \mathbb{Z}_{\text{prim}}^2, N(v) = N(Gv) \leq L\} = o_G(L^2).$$

Then the number of K -orbits of primitive vectors with $N(v) \leq L$ minus the number of distinct values of N on these vectors is $o(L^2)$. Consequently only $o(L^2)$ primitive vectors have a norm level containing more than one K -orbit.

Proof. We provide the truncation argument in detail, since convergence for each fixed graph does not itself justify summation over changing graphs.

Fix a sufficiently large constant A . Set

$$H_c = \lceil A \log(2 + L) \rceil, H_f = \lceil B \log(2 + L) \rceil,$$

where B is a fixed multiple of A so large that $\rho B > C_{\text{sep}}A + 5$. Apply Lemma 2.3 with $M = C_1 L^3$, $H = H_c$, where C_1 is fixed below, and also remove the rays of $\mathcal{F}_{H_f}$. This deletes at most

$$CL^2/A + O_A(L \log^2(2 + L))$$

primitive vectors. Call the others retained.

Fix two fine sectors Σ, Σ' . Choose their inward supporting functionals λ, λ' and put $\ell = (\lambda, -\lambda')$. For every collision vector $z = (v, w) \in \mathbb{Z}^4$ in these sectors,

$$|z| \leq CL, |\ell(z)| \leq CL e^{-\rho H_f}, c \leq |\ell| \leq C.$$

Four independent collision vectors are impossible: their nonzero integral determinant has absolute value at least one, whereas expansion in the ℓ direction bounds it by $CL^4 e^{-\rho H_f} < 1$.

If three independent collision vectors span W , their primitive integral normal $\nu = (\alpha, \beta)$ has size at most $C_1 L^3$. Their three-dimensional volume is at least one. Cramer's rule, using vectors of size $O(L)$, bounds the restriction of ℓ to W by $CL^3 e^{-\rho H_f}$. Thus ℓ makes angle at most this quantity with $\mathbb{R}\nu$. Comparing its second block shows $\beta \neq 0$ and that λ' is within angle $CL^3 e^{-\rho H_f}$ of $\{\beta, -\beta\}$. A supporting functional at a retained second entry differs from λ' by $C e^{-\rho H_f}$. This contradicts the separation $c e^{-C_{\text{sep}} H_c}$ in Lemma 2.3. Hence the collision vectors in each fine sector pair span a space of dimension at most two.

If this space is a plane and both coordinate projections have rank two, it is the graph of a single rational matrix G . Since every collision excludes K -partners, this matrix is outside K . Otherwise there are $O(L + 1)$ collisions in this sector pair. Indeed, a rank-one projection has at most two primitive

vectors; for each fixed first or second entry the partners lie on a norm circle, whose integral points number $O(L+1)$: centered unit squares lie in a fixed-width neighborhood of that convex boundary, of area $O(L+1)$. A space of dimension at most one contributes $O(1)$. There are $O_A(\log^4(2+L))$ sector pairs, so all these degenerate pieces contribute $o_A(L^2)$.

Every graph piece contains two pairs with independent first entries. By Lemma 5.1, the pieces with $J(G) > (\log L)^5$ contribute at most

$$O_A\left(\frac{L^2}{\log L} + L \log^4 L\right) = o_A(L^2).$$

It remains to sum graphs of complexity at most $(\log L)^5$.

For integers $j \geq 1$ put $Q_j = 2^j$, and let

$$\mathcal{G}_j = \{G \notin K : Q_j < J(G) \leq 2Q_j\}.$$

Then $|\mathcal{G}_j| \leq CQ_j^9$. Each graph plane has an integral basis of vectors of norm at most CQ_j^2 . Consequently the primitive integral normal to a three-dimensional sum of two distinct such planes has norm at most

$$M_j = CQ_j^6.$$

Fix a sufficiently large j_0 , independent of L , and put $j_{\max} = \lceil 5 \log_2 \log L \rceil$. For $j_0 \leq j \leq j_{\max}$, make the additional excision of Lemma 2.3 with $M = M_j, H = j^3$, and delete the rays of the finer fan of level $\lceil B_1 j^3 \rceil$, where $\rho B_1 > C_{\text{sep}} + 1$. These extra deletions number at most

$$CL^2/j_0 + CL(\log \log L)^3 + C(\log \log L)^7.$$

This follows by summing $C(j^{-2}L^2 + Lj^2 + j^6)$ over j . Retained vectors now also satisfy

$$\angle(\gamma, \{\beta, -\beta\}) \geq ce^{-C_{\text{sep}}j^3} (0 < |\beta| \leq M_j).$$

Fix j and a pair of sectors of level $\lceil B_1 j^3 \rceil$. For $G \in \mathcal{G}_j$, let P_G count retained collisions on its graph inside this sector pair. Call a piece large if $P_G \geq L^2/Q_j^{20}$. At most one large piece occurs in a given sector pair. To prove this, first observe that a finite integral set V in a disc of radius CL , with $D = \max_{v, v' \in V} |\det(v, v')| > 0$, satisfies

$$|V| \leq 4D + CL + C.$$

Choose a determinant-maximizing pair; every point lies in its coefficient parallelogram $[-1, 1]^2$, of area $4D$ and perimeter $O(L)$. For a large piece we therefore find two first entries with

$$|\det(v_1, v_2)| \geq L^2/(8Q_j^{20}),$$

since $Q_j \leq 2(\log L)^5$ makes the threshold dominate L uniformly.

Let λ, λ' be supporting functionals for this sector pair and $\ell = (\lambda, -\lambda')$. The functional $\theta_G = \lambda - \lambda' \circ G$ has values at v_1, v_2 of size $CLe^{-\rho B_1 j^3}$. The dual-basis estimate gives

$$|\theta_G| \leq CQ_j^{20} e^{-\rho B_1 j^3},$$

and hence the restriction of ℓ to the graph plane has the same bound. If two distinct large pieces existed, their graph planes would span a space W of dimension three or four. Choose an integral basis of W from their graph bases, each vector of size at most CQ_j^2 . The basis volume is at least one, so Cramer's rule bounds the restriction of ℓ to W by

$$\varepsilon_j = CQ_j^{28} e^{-\rho B_1 j^3}.$$

In dimension four this contradicts $|\ell| \geq c$ for large j_0 . In dimension three its primitive normal (α, β) has size at most M_j . As in the initial rank reduction, $\beta \neq 0$, and a supporting functional at a retained second entry makes angle at most $C\varepsilon_j$ with $\{\beta, -\beta\}$. The extra excision gives the contradictory lower bound $ce^{-C_{\text{sep}}j^3}$, since

$$2^{28j} e^{-(\rho B_1 - C_{\text{sep}})j^3} \longrightarrow 0.$$

This proves uniqueness of a large band piece in each sector pair.

There are $O(j^{12})$ pairs of sectors at this level. All small pieces in band j contribute at most

$$CL^2 j^{12} Q_j^{9-20} = CL^2 j^{12} 2^{-11j}.$$

For the large pieces, Lemma 5.1 gives total contribution at most

$$Cj^{12}(L^2 2^{-j} + L + 1).$$

Summing from j_0 to $j_{\max}$, we obtain

$$CL^2 \tau(j_0) + O(L(\log \log L)^{13}), \tau(j_0) = \sum_{j \geq j_0} j^{12}(2^{-11j} + 2^{-j}),$$

where $\tau(j_0) \rightarrow 0$. The matrices with $J(G) \leq 2^{j_0}$ form a fixed finite set. By the hypothesis of the theorem, their total collision count is $o_{j_0}(L^2)$.

Let $A_N(L)$ be the excess of K -orbits over distinct norm values. Deleting d primitive vectors deletes at most d orbits. Among the remaining orbits, the excess in a norm level is bounded by the number of retained non- K collision pairs. Therefore the estimates above give

$$\limsup_{L \rightarrow \infty} \frac{A_N(L)}{L^2} \leq C/A + C/j_0 + C\tau(j_0).$$

First let $L \rightarrow \infty$ with A, j_0 fixed, then let $A, j_0 \rightarrow \infty$. This proves $A_N(L) = o(L^2)$. If a level contains $r \geq 2$ K -orbits, then $r \leq 2(r-1)$ and it contains at most $|K|r$ oriented primitive vectors. Thus the number of vectors in such levels is at most $2|K|A_N(L) = o(L^2)$. $\square$

6 Completion of the density theorem

Proof of Theorem 1.1 and Corollary 1.2. Fix X . Lemma 2.2 and Lemma 2.3 give the geometric hypotheses used in Theorem 5.2. Lemma 3.1 identifies every rational linear norm isometry with an element of K_X . Theorem 4.1 supplies the fixed-graph estimate for every $G \notin K_X$, using the injective embedding in Lemma 4.2 for arbitrary real initial traces. Theorem 5.2 consequently gives $E_X(L) = o_X(L^2)$. Lemma 3.4 proves the density assertions; Corollary 6.1 below proves the mean statement, and division by the positive quadratic asymptotics for $C_X(L)$ and $D_X(L)$ proves both density limits. $\square$

For the modular torus, $|K_X| = 12$ and $h_X = 6$. The exact identities in the introduction identify this specialization with [Che26, Theorem 1.1]. For a torus whose only homology symmetries are $\{I, -I\}$, it gives density-one multiplicity one on that fixed torus, even if exceptional coincidences occur.

The conclusion is asymptotic. The proof removes a set of directions whose proportion can be made arbitrarily small; it does not remove the possibility of a particular collision. Likewise the analytic zero-one lemma permits isolated zeros along interpolated orbits. Thus neither the pointwise Schmutz–Schaller conjecture nor full Markoff uniqueness is claimed here. The limiting argument gives no explicit convergence rate and no cutoff above which every length is good.

Corollary 6.1 (Mean multiplicity and distribution of distinct levels). *Assume $E_X(L) = o_X(L^2)$, as proved above, put $h = h_X$ and let $\mathcal{L}_X(L)$ be the set of distinct simple lengths at most L . Then*

$$\sum_{t \in \mathcal{L}_X(L)} |m_X(t) - h| = hE_X(L) + O_X(1) = o_X(L^2).$$

In particular, the multiplicity of a uniformly sampled distinct length at most L converges to h in L^1 , and

$$\frac{1}{D_X(L)} \sum_{t \in \mathcal{L}_X(L)} m_X(t) \rightarrow h.$$

For every real $R > h$,

$$\begin{aligned} \#\{t \leq L : m_X(t) \geq R\} &\leq \frac{hE_X(L)}{R - h}, \\ \#\{\gamma : \ell_X(\gamma) \leq L, m_X(\ell_X(\gamma)) \geq R\} &\leq \frac{hR}{R - h} E_X(L). \end{aligned}$$

Proof. At a length t let $r(t)$ be the number of symmetry orbits and write $\delta(t) = hr(t) - m_X(t) \geq 0$. Only finitely many symmetry orbits have fewer than h curves, so $\sum_t \delta(t) = O_X(1)$. Since

$$m_X(t) - h = h(r(t) - 1) - \delta(t),$$

the inequality $||a - b| - a| \leq b$ for $a, b \geq 0$ gives the first formula after summing. Divide by the positive quadratic asymptotic for $D_X(L)$ to obtain the two convergence claims. If $m_X(t) \geq R > h$, then $r(t) - 1 \geq R/h - 1$ and

$$m_X(t) \leq hr(t) \leq \frac{hR}{R-h}(r(t) - 1).$$

Summing these inequalities proves the tail bounds. $\square$

Corollary 6.2 (Homology distribution under distinct-length sampling). *Choose a length uniformly from $\mathcal{L}_X(L)$, then a curve uniformly from that complete length level, and finally one of its two homology orientations uniformly. Let $V_L \in \mathbb{Z}^2$ be the resulting primitive vector. Then V_L/L converges in distribution to normalized area measure on B_X . Equivalently, for every continuous function f on B_X ,*

$$\frac{1}{D_X(L)} \sum_{t \in \mathcal{L}_X(L)} \frac{1}{2m_X(t)} \sum_{\substack{v \in \mathbb{Z}_{\text{prim}}^2 \\ N_X(v)=t}} f(v/L) \longrightarrow \frac{1}{\text{area}(B_X)} \int_{B_X} f(x) dx.$$

For every fixed $0 \leq a < b$, all but $o_X(L^2)$ levels in $(aL, bL]$ are single symmetry orbits, and

$$D_X(bL) - D_X(aL) \sim \frac{\text{area}(B_X)(b^2 - a^2)}{|K_X|\zeta(2)} L^2.$$

Proof. Put $h = h_X$. Replacing each factor $1/(2m_X(t))$ by $1/(2h)$ changes the unnormalized sum by at most

$$\frac{\|f\|_\infty}{h} \sum_{t \in \mathcal{L}_X(L)} |m_X(t) - h| = o_X(L^2).$$

Primitive lattice equidistribution gives

$$\sum_{\substack{v \in \mathbb{Z}_{\text{prim}}^2 \\ N_X(v) \leq L}} f(v/L) = \frac{L^2}{\zeta(2)} \int_{B_X} f(x) dx + o_f(L^2).$$

For completeness, ordinary lattice Riemann sums and $1_{\gcd(v_1, v_2)=1} = \sum_{d|v_1, v_2} \mu(d)$ prove this: truncate the divisor sum at a fixed D , pass to the limit in the finitely many Riemann sums, and bound the tail by $O(L^2/D + L \log L)$ before letting $D \rightarrow \infty$. Division by $D_X(L) \sim \text{area}(B_X)L^2/(2h\zeta(2))$ proves the first assertion. Subtraction of the counting asymptotics proves the interval count. Its exceptional levels are bounded by the exceptional levels up to bL , which number $o_X(L^2)$. $\square$

Proposition 6.3 (Possible limiting multiplicities for a cusp). *For a once-punctured torus, the number h_X belongs to $\{1, 2, 4, 6\}$. The cases $h_X = 4$ and $h_X = 6$ are respectively the punctured square torus and the modular torus.*

Proof. The conformal compactification of X is a complex torus $E = \mathbb{C}/\Lambda$, with the puncture marked as the origin. Every conformal automorphism fixing that point has the form $z \mapsto az$ with $a\Lambda = \Lambda$. Such multipliers form a finite cyclic group. They have modulus one and integral real trace $2 \operatorname{Re}(a)$ on the lattice, so the group has order $n \in \{2, 4, 6\}$; it always contains -1 .

If $n = 4$, the lattice is invariant under multiplication by i . After scaling it is the square lattice: as a rank-one torsion-free module over the Euclidean ring $\mathbb{Z}[i]$, it is principal. If $n = 6$, the same argument over the Euclidean ring of Eisenstein integers gives the hexagonal lattice. Both lattices are invariant under complex conjugation. Thus in these two cases the full group of conformal and anticonformal automorphisms fixing the puncture has order $2n$. When $n = 2$, it has order either two or four, since any two anticonformal automorphisms differ by a conformal one. The complete finite-area hyperbolic metric is determined by the conformal structure, so these automorphisms are exactly the isometries of X . Their homology action is faithful. Division by the ubiquitous involution $-I$ gives respectively $h_X \in \{1, 2\}$, $h_X = 4$, and $h_X = 6$. The punctured hexagonal torus is the modular torus, as is also seen from its order-six conformal symmetry. $\square$

7 Pointwise bounds from nonintegral and transcendental traces

In this section a one-holed torus may have a cusp or a geodesic boundary. Its boundary parameter is

$$\kappa = 2 - 2 \cosh(b/2) \leq 0,$$

where $b = 0$ in the cusped case. Choose a positive trace triple (x, y, z) corresponding to the slopes $(1, 0), (0, 1), (1, 1)$. Then

$$x^2 + y^2 + z^2 - xyz = \kappa.$$

Every positive simple trace $T_X(v) = 2 \cosh(\ell_X(v)/2)$ is obtained by repeated Farey moves $z \mapsto z' = xy - z$, and

$$zz' = x^2 + y^2 - \kappa.$$

These are the standard trace coordinates of [Gol09]. Rational length classes are the equivalence classes defined in Section 1.5. The results below bound their entire cardinalities, hence also equal-length multiplicities.

Lemma 7.1 (Strict convexity on rational homology). *Extend simple lengths to integral homology by $N_X(nv) = |n|\ell_X(v)$ for primitive v . If u, w are nonparallel rational vectors, then*

$$N_X(u + w) < N_X(u) + N_X(w).$$

An affine line contains at most two rational vectors on a positive N_X -level.

Proof. Strict convexity of the length-norm ball was proved in Lemma 2.2. Apply it to the two unit vectors $u/N_X(u)$ and $w/N_X(w)$ to obtain the strict triangle inequality. If three rational points on an affine line had the same positive norm, the middle one would be a proper convex combination of the others and would have smaller norm. A line through the origin meets a positive norm level in exactly two points by homogeneity. $\square$

A valuation v takes values in a totally ordered abelian group Γ ; real or discrete value groups are not required. Put

$$P(a) = \max\{0, -v(a)\}.$$

Lemma 7.2 (Linear pole orders). *Suppose v is nonnegative on $\mathbb{Z}[\kappa]$. There is a homomorphism $f : \mathbb{Z}^2 \rightarrow \Gamma$ such that*

$$P(T_X(u)) = |f(u)|$$

for every primitive vector u . If some simple trace has a pole, then $f \neq 0$.

Proof. Write $A = P(x), B = P(y), C = P(z)$ and temporarily reorder them so that $A \leq B \leq C$. If $C = B > 0$ and $A > 0$, then xyz has strictly larger pole order than every other term of the trace equation, which is impossible. Thus $A = 0$ in this case. If $C > B$, then z^2 is the unique square of largest pole order. It must cancel with xyz , giving $-v(x) - v(y) = C$. Both $v(x), v(y)$ are negative, since otherwise the other pole order would be at least C . Hence $C = A + B$. If $C = 0$, all three orders vanish. In every case one of the three pole orders is the sum of the other two.

Return to the original ordering and put $C' = P(xy - z)$. If $A \neq B$, the two triples just considered show that $C, C' > 0$; the product identity has pole order $2 \max\{A, B\}$, so

$$C + C' = 2 \max\{A, B\}.$$

If $A = B > 0$, the triple rule gives $C \in \{0, 2A\}$. For $C = 0$, the term xy dominates z , so $C' = 2A$. For $C = 2A$, the product identity gives $v(xy - z) \geq 0$, so $C' = 0$. If $A = B = 0$, both C and C' vanish. Thus the displayed identity always holds, including valuations above 2 and cases of cancellation.

Choose $f(1, 0) = A$ and $f(0, 1) = B$ or $-B$ so that $|f(1, 1)| = C$. The identity

$$|a + b| + |a - b| = 2 \max\{|a|, |b|\}$$

holds in every totally ordered abelian group. Consequently $|f|$ and the pole orders obey the same Farey recursion and agree at the initial triple. They agree everywhere. If $f = 0$, every simple trace has pole order zero, proving the final assertion. $\square$

Theorem 7.3 (A nonintegral trace gives the bound two). *If one simple trace of a hyperbolic one-holed torus is not integral over $\mathbb{Z}[\kappa]$, every rational length class has at most two curves. In particular, for a punctured torus, one simple trace that is not an algebraic integer suffices. This includes every punctured torus with a transcendental simple trace.*

Proof. Let a be a trace not integral over $R = \mathbb{Z}[\kappa]$ and let $F = \mathbb{Q}(x, y, z)$. The ideal $a^{-1}R[a^{-1}]$ is proper: an expression for 1 in this ideal would give a monic equation for a over R . Localize $R[a^{-1}]$ at a maximal ideal containing this ideal and choose a valuation ring of F dominating that local ring. The existence of such a ring is the valuation domination lemma [Stacks, Tag 00IA]. Its valuation is nonnegative on R and positive on a^{-1} , so a has a pole. Lemma 7.2 therefore supplies a nonzero f .

Define the integer trace polynomials, monic for positive indices, by

$$C_0(s) = 2, C_1(s) = s, C_{n+1}(s) = sC_n(s) - C_{n-1}(s).$$

They satisfy $C_n(2 \cosh t) = 2 \cosh(nt)$ and $P(C_n(a)) = nP(a)$ for $n \geq 1$: at positive pole order the leading term dominates, and at pole order zero every term is integral. Take any finite collection u_i in one rational length class and choose positive integers n_i such that $n_i \ell_X(u_i) = L_0$. Their polynomial traces $C_{n_i}(T_X(u_i))$ are equal, whence

$$|f(n_i u_i)| = h$$

for a common $h \geq 0$. If $h = 0$, all directions lie in $\ker f$, a subgroup of rank at most one, and there is at most one primitive unoriented direction. If $h > 0$, orient each u_i so that $f(n_i u_i) = h$. The vectors $n_i u_i$ lie in a single coset of $\ker f$, hence on one affine line, and all have norm L_0 . Lemma 7.1 permits at most two. Since this argument applies to every finite subcollection, the entire class has size at most two. $\square$

Corollary 7.4 (An injective pole homomorphism). *Under the hypotheses of Lemma 7.2, if f is injective, any two distinct simple curves have irrational length ratio. In particular this holds if $f(1, 0)$ and $f(0, 1)$ are linearly independent over $\mathbb{Q}$ in $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. After orienting and scaling two rationally related curves as in the proof of Theorem 7.3, their vectors have equal f -values. Injectivity makes the vectors equal, so their primitive unoriented directions agree. $\square$

Corollary 7.5 (Countable arithmetic exceptional set). *For each fixed boundary parameter $\kappa \leq 0$, all but countably many marked one-holed tori have at most two curves in each rational length class. If three distinct curves have rationally related lengths, every simple trace is integral over $\mathbb{Z}[\kappa]$. Thus every counterexample to the pointwise Schmutz–Schaller bound on a punctured torus has an algebraic-integer initial triple.*

Proof. The integrality statement is the contrapositive of Theorem 7.3. For fixed κ , the ring $\mathbb{Z}[\kappa]$ is countable, as is its set of monic polynomials. Each such polynomial has finitely many real roots. There are therefore only countably many integral trace triples, and a positive trace triple determines the marked torus. $\square$

Remark 7.6. For geodesic boundary, a trace merely transcendental over $\mathbb{Q}$ need not satisfy the hypothesis: it can be algebraic over $\mathbb{Q}(\kappa)$. A trace transcendental over $\mathbb{Q}(\kappa)$ always does satisfy it. On the punctured locus $\kappa = 0$, a coordinate cannot be transcendental over the field generated by the other two coordinates, because the trace equation is quadratic in each coordinate.

8 Twist families: simple spectra and finite exceptional lists

Fix $\lambda > 1$ and set

$$c = \frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}}, d = \frac{2}{\lambda - \lambda^{-1}}, A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, B(t) = \begin{pmatrix} ct & d \\ d & c/t \end{pmatrix} (t > 0).$$

Since $c^2 - d^2 = 1$, these matrices have determinant one and give positive punctured-torus trace coordinates

$$x = \lambda + \lambda^{-1}, y = c(t + t^{-1}), z = c(\lambda t + \lambda^{-1}t^{-1}).$$

The curve $\alpha = (1, 0)$ has fixed length $2 \log \lambda$. Every fixed-length twist family is represented in this form, since solving for the diagonal entries of B gives positive numbers with product c^2 , and diagonal conjugation makes the two off-diagonal entries equal to d .

Lemma 8.1 (Extreme coefficients of the twist polynomial). *For a primitive slope (p, q) with $q > 0$, its trace is a Laurent polynomial $T_{p,q}(t) \in \mathbb{Q}(\lambda)[t, t^{-1}]$ whose highest and lowest terms are*

$$c^q \lambda^p t^q, c^q \lambda^{-p} t^{-q}.$$

Proof. The standard crossing word of a simple slope in the square model has the form

$$A^{a_1} B(t) \cdots A^{a_q} B(t), a_j \in \mathbb{Z}, \sum_j a_j = p.$$

Expand the trace by following the two matrix indices through the product. To obtain degree q , every factor B must use its upper-left entry; this unique term has coefficient $c^q \lambda^{\sum a_j}$. Similarly only the all-lower path has degree $-q$. All other paths have degrees strictly between these extremes. This works for negative p as well, since the diagonal entries of every integer power of A are positive. $\square$

Theorem 8.2 (Transcendental twists and effective degree bounds). *If t is transcendental over $\mathbb{Q}(\lambda)$, any two distinct simple curves have irrational length ratio. Hence the simple length spectrum is simple. More quantitatively, if*

$$[\mathbb{Q}(\lambda, t) : \mathbb{Q}(\lambda)] > 2MH,$$

then distinct curves meeting α at most H times have no relation $n\ell(\gamma) = m\ell(\delta)$ with $1 \leq n, m \leq M$. In particular, relative degree greater than $2H$ gives a simple spectrum on the entire infinite collection of curves meeting α at most H times.

Proof. A proposed rational relation gives

$$C_n(T_{p,q}(t)) = C_m(T_{p',q'}(t)).$$

Orient the slopes so that $q, q' \geq 0$. If just one is zero, one side is constant and the other has nonzero degree. If both are positive, an identity of Laurent polynomials would first give $nq = mq'$ by the highest degree. The corresponding coefficients then give $\lambda^{np} = \lambda^{mq'}$, hence $np = mq'$ because $\lambda > 1$. Primitivity gives the same curve and $n = m$. There cannot be two distinct curves with $q = q' = 0$. Thus distinct curves always give a nonzero Laurent polynomial. It cannot vanish at transcendental t . Under the stated bounds its exponents lie between $-MH$ and MH ; multiplying by t^{MH} gives a nonzero polynomial of degree at most $2MH$, which cannot vanish at an element of larger relative degree. $\square$

Corollary 8.3 (Genericity on every fixed-length twist family). *For every fixed $\lambda > 1$, the positive parameters having a rational relation between two distinct simple lengths form a countable set. Its complement has full Lebesgue measure and is a dense G_δ subset of $(0, \infty)$. For prescribed distinct slopes and prescribed positive integers n, m , there are at most $2 \max\{nq, mq'\}$ positive parameters satisfying $n\ell(p, q) = m\ell(p', q')$.*

Proof. The proof of Theorem 8.2 gives a nonzero polynomial of the asserted degree for each proposed relation. There are countably many slopes and integer pairs. A countable union of finite sets is countable; its complement is a dense G_δ by the Baire theorem and has full measure. $\square$

Theorem 8.4 (An explicit finite list of rational twists). *Fix an algebraic punctured torus, write its twist parameter as $t = t_0 r$, and let $K \subset \mathbb{R}$ be a number field containing λ, t_0 . Put $k = [K : \mathbb{Q}]$, $u = ct_0$, $v = c/t_0$. Choose an integer $D > 0$ with $Du, Dv \in \mathcal{O}_K$, and define*

$$U = |\text{Norm}_{K/\mathbb{Q}}(Du)|, V = |\text{Norm}_{K/\mathbb{Q}}(Dv)|,$$

$$M_u = \prod_{p|U} p^{\lfloor v_p(U)/k \rfloor}, M_v = \prod_{p|V} p^{\lfloor v_p(V)/k \rfloor}.$$

Every reduced positive rational $r = a/b$ outside the finite list

$$a \mid M_v, b \mid M_u, \gcd(a, b) = 1$$

has at most two curves in each rational length class. If $\lambda + \lambda^{-1}$ is not an algebraic integer, this conclusion holds for every positive rational r .

Proof. The middle trace is $y_r = ua/b + vb/a$. If it is integral, then

$$(Du)a^2 + (Dv)b^2 = Daby_r.$$

Reducing in $\mathcal{O}_K$ modulo $a\mathcal{O}_K$ gives $(Dv)b^2 \in a\mathcal{O}_K$. Bezout's identity for a, b^2 implies $Dv/a \in \mathcal{O}_K$. Similarly $Du/b \in \mathcal{O}_K$. Taking field norms yields $a^k \mid V$ and $b^k \mid U$, which are precisely the stated restrictions. Outside them y_r is not integral, so Theorem 7.3 applies. A fixed nonintegral first trace gives the last assertion independently of r . $\square$

Corollary 8.5 (Density among rational parameters). *For an integer $H \geq 1$, let $E = \tau(M_u)\tau(M_v)$ and*

$$\mathcal{R}_H = \{a/b : a, b \in \mathbb{Z}_{>0}, a, b \leq H, \gcd(a, b) = 1\}.$$

At most E members fail to be certified by Theorem 8.4. Their proportion is at most $3E/H^2$. The certified rational parameters are dense in every real twist interval and have density one in every fixed open interval when ordered by the numerator and denominator bound.

Proof. There are at most E fractions in the finite list, whereas

$$\#\mathcal{R}_H = \frac{H^2}{\zeta(2)} + O(H \log(2 + H)) > \frac{H^2}{3}.$$

For the last inequality, the number of noncoprime pairs is at most $\sum_{j=2}^H [H/j]^2$, so the proportion of coprime pairs exceeds $2 - \pi^2/6 > 1/3$. On each fixed nonempty open interval, primitive lattice counting in $\{(a, b) \in (0, H]^2 : a/b \in I\}$ gives a positive multiple of H^2 with error $O_I(H \log(2 + H))$, by M"obius inversion. Deleting a fixed finite list preserves density one and density in the real interval. $\square$

Corollary 8.6 (The original quartic field). *Take $\lambda = 1 + \sqrt{2}$ and $t_0 = \lambda^{-1/2}$. Every positive rational $r \neq 1$ has at most two curves in each rational length class. The only possible exception in the finite list is the original torus with triple*

$$(2\sqrt{2}, \sqrt{4 + 4\sqrt{2}}, \sqrt{4 + 4\sqrt{2}}).$$

The trace field of every member is $\mathbb{Q}(\sqrt{4 + 4\sqrt{2}})$, and its squared-trace field is $\mathbb{Q}(\sqrt{2})$.

Proof. Put $w = \sqrt{\lambda}$. The polynomial $w^4 - 2w^2 - 1$ is irreducible: λ has norm -1 in $\mathbb{Q}(\sqrt{2})$ and cannot be a square there. Thus $K = \mathbb{Q}(w)$ has degree four; both w and w^{-1} are integral units. Here $u = \sqrt{2}/w$, $v = \sqrt{2}w$ are integral with absolute norms four. Taking $D = 1$ gives $M_u = M_v = 1$, so the finite list is $r = 1$. At $r = 1$, $y = z = u + v$ and $(u + v)^2 = 4 + 4\sqrt{2}$.

For the field claims put $h = u + v = \sqrt{4 + 4\sqrt{2}}$. Then

$$\lambda = h^2/4, u = (12h - h^3)/8, v = (h^3 - 4h)/8.$$

Thus all chosen traces lie in $\mathbb{Q}(h)$. Moreover

$$y_r^2 = 4 + 2(r^{-2} - r^2) + 2\sqrt{2}(r^2 + r^{-2}),$$

and the corresponding formula for z_r^2 reverses the sign of the middle term. These squares generate $\mathbb{Q}(\sqrt{2})$. The involution $h \mapsto -h$ fixes the first trace and negates the other two. The same sign pattern propagates by the Farey recursion, so every squared simple trace lies in $\mathbb{Q}(\sqrt{2})$. For $r \neq 1$, the system $y_r = ur + v/r$, $z_r = vr + u/r$ has nonzero determinant $r^2 - r^{-2}$ and recovers u, v , hence h . At $r = 1$ the trace h is already present. This proves both field assertions. $\square$

Corollary 8.7 (The fixed curve in the rational quartic family). *For rational $r \neq 1$ in Corollary 8.6, the curve α of trace $2\sqrt{2}$ is the only simple curve with length rationally related to $\ell(\alpha)$.*

Proof. Choose a rational prime p with $j = v_p(r) \neq 0$ and extend its valuation to K , normalized by $v(p) = 1$. Since w is a unit, the two summands of $y_r = \sqrt{2}(r/w + w/r)$ have unequal values; the same is true for z_r . Hence their common pole order is $g = |j|$ if $p \neq 2$ and $g = |j| - 1/2$ if $p = 2$. In either case $g > 0$, whereas $2\sqrt{2}$ is integral. Lemma 7.2 gives $P(T_X(p', q')) = g|q'|$. A rational relation with the length of α gives equal monic polynomial traces, one of pole order zero, so $q' = 0$. This is precisely the unoriented primitive direction of α . $\square$

Proposition 8.8 (A collision occurs at only finitely many common twists). *Fix distinct simple curves γ, δ and a simple curve α on a punctured torus. Put $Q = \max\{i(\gamma, \alpha), i(\delta, \alpha)\}$. Then*

$$\ell_X(D_\alpha^n \gamma) = \ell_X(D_\alpha^n \delta)$$

holds for at most $2Q$ integers n . If the two intersection numbers are equal, it holds for at most one integer n .

Proof. In the preceding coordinates, a common twist replaces t by $\lambda^n t$, after fixing the sign of the twist convention. By Lemma 8.1, the two trace polynomials are distinct. Clearing negative powers gives a nonzero polynomial of degree at most $2Q$. The positive parameters $\lambda^n t$ are distinct, proving the first bound. For equal intersection $q > 0$, orient the slopes as $(p, q), (p', q)$ with $p > p'$. The function $s \mapsto N_X((s, q))$ is strictly convex because $q > 0$. Its increment over any fixed positive interval is strictly increasing, so

$$s \mapsto N_X((s + p, q)) - N_X((s + p', q))$$

is strictly increasing and has at most one zero. Put $s = nq$. The case $q = 0$ cannot give two distinct primitive unoriented curves. $\square$

9 Three projective lengths and real conjugates

Throughout this section the surfaces are complete finite-area once-punctured tori. We retain the equivalence-class convention for rational length classes from Section 1.5.

Theorem 9.1 (Global coordinates from any three length ratios). *Let u, v, w be three distinct unoriented simple curves, and put*

$$a = |\det(v, w)|, b = |\det(w, u)|, c = |\det(u, v)|.$$

The map

$$X \mapsto \frac{(a\ell_X(u), b\ell_X(v), c\ell_X(w))}{a\ell_X(u) + b\ell_X(v) + c\ell_X(w)}$$

is a real-analytic diffeomorphism from marked Teichmüller space onto

$$\Delta = \{(s_1, s_2, s_3) : s_i > 0, s_1 + s_2 + s_3 = 1, s_i < 1/2\}.$$

In particular, proportional length triples determine the marked torus.

Proof. We make explicit the geometric inputs. Thurston's convexity theorem [Thu98, Theorem 5.1] says that, at a marked torus X ,

$$[\lambda] \mapsto p_\lambda = d_X \log \ell_\lambda$$

embeds projective measured lamination space as the boundary of a convex body in the two-dimensional cotangent plane. Each simple curve gives an extreme point, as can be checked directly. For a simple curve α and a curve β crossing it once, use length and twist coordinates $a_0 > 0, t \in \mathbb{R}$ with holonomies

$$A = \begin{pmatrix} e^{a_0} & 0 \\ 0 & e^{-a_0} \end{pmatrix}, B = \begin{pmatrix} e^t \coth a_0 & \operatorname{csch} a_0 \\ \operatorname{csch} a_0 & e^{-t} \coth a_0 \end{pmatrix}.$$

Their commutator has trace -2 and $\ell_\alpha = 2a_0$. The twist curve β_n has trace

$$2 \coth a_0 \cosh(t + na_0).$$

If $k(a_0) = \log \coth a_0$, then

$$\ell_{\beta_{\pm n}} = 2na_0 + 2(k \pm t) + O(e^{-2na_0}).$$

The first derivatives of the error are locally $O(ne^{-2na_0})$. Consequently

$$n(p_{\beta_{\pm n}} - p_\alpha) \longrightarrow v_\pm = \frac{1}{a_0} \left[\left(k' - \frac{k}{a_0} \mp \frac{t}{a_0} \right) da_0 \pm dt \right].$$

Their determinant in the (da_0, dt) basis is

$$-\frac{2}{a_0^2} \left(k' - \frac{k}{a_0} \right) > 0.$$

Here $k > 0$ and $k' < 0$. A point in the relative interior of a segment of a planar convex boundary has only collinear boundary secants approaching it. The two noncollinear limits exclude this possibility; thus p_α is extreme. Three distinct extreme points are noncollinear. Hence the derivative of

$$X \mapsto \left(\log \frac{\ell_X(v)}{\ell_X(u)}, \log \frac{\ell_X(w)}{\ell_X(u)} \right)$$

is nonsingular everywhere.

The signed determinant identity for u, v, w , together with strict convexity of the homological length norm, gives all three strict triangle inequalities for $a\ell_X(u), b\ell_X(v), c\ell_X(w)$. The stated map Φ therefore takes values in Δ and is a local real-analytic diffeomorphism.

To prove properness, let X_j escape every compact subset. The projective-length compactification of Teichmüller space [FLP79, Thu98] gives, after passage to a subsequence, positive numbers r_j and a nonzero measured lamination λ with $r_j \ell_{X_j}(q) \rightarrow i(\lambda, q)$ for every simple curve q . For a punctured torus this intersection function is $|f(q)|$ for a nonzero real linear functional on homology. Applying f to the signed determinant identity shows that one of $a|f(u)|, b|f(v)|, c|f(w)|$ is the sum of the other two. Their total is positive because u, v are independent. Thus every such limiting image belongs to $\partial\Delta$. This proves properness. A proper local diffeomorphism has open and closed image, hence is onto the connected triangle Δ , and is a covering map. The domain is connected and Δ is simply connected, so the covering has one sheet. Its local analytic inverses give the asserted global analytic inverse. $\square$

Corollary 9.2 (Three specified equal lengths). *Three specified distinct simple curves admit a common length on a punctured torus if and only if their three pairwise intersection numbers satisfy the strict triangle inequalities. The marked torus is then unique.*

Proof. The desired image in Theorem 9.1 is $(a, b, c)/(a + b + c)$, which belongs to Δ exactly under those inequalities. $\square$

Theorem 9.3 (A real-conjugate obstruction). *Let $K = \mathbb{Q}(x, y, z)$ be the trace field of an algebraic positive punctured-torus seed. If a real embedding $\sigma : K \rightarrow \mathbb{R}$ changes one of x^2, y^2, z^2 , every rational-length class contains at most two simple curves.*

Proof. Every nonzero real solution (u, v, w) of the punctured-torus equation has $uvw > 0$ and $|u|, |v|, |w| > 2$. Indeed, if $|u| \leq 2$, then

$$u^2 + v^2 + w^2 - uvw \geq u^2 + (|v| - |w|)^2 + (2 - |u|)|vw| > 0.$$

An embedding cannot send the nonzero trace x to zero. Thus, after changing two lift signs if needed, the conjugate seed defines a positive marked torus Y .

Let $C_0(T) = 2$, $C_1(T) = T$, and $C_{n+1}(T) = TC_n(T) - C_{n-1}(T)$. Then $C_n(2 \cosh t) = 2 \cosh(nt)$ and C_{2n} is even. If three distinct curves have rationally related lengths, choose positive integers n_i with $n_i \ell_X(\gamma_i) = L$. Their polynomial values $C_{2n_i}(T_X(\gamma_i))$ are equal. These identities survive under σ ; even indices remove the lift signs. Consequently $n_i \ell_Y(\gamma_i)$ also have one common value. The length triples on X and Y are proportional. Theorem 9.1 forces $X = Y$, contradicting the changed squared trace. $\square$

9.1 Trace fields and totally real exceptions

Set $K_0 = \mathbb{Q}(x^2, y^2, z^2)$. The identity $xyz = x^2 + y^2 + z^2$ implies $K = K_0(x, y)$, a multiquadratic Galois extension of degree at most four. Its automorphisms change signs according to

$$(x, y, z) \mapsto (\epsilon_1 x, \epsilon_2 y, \epsilon_1 \epsilon_2 z).$$

The trace recursion gives the sign $\epsilon_1^p \epsilon_2^q$ on the trace of slope (p, q) . In particular, K_0 is precisely the field of all squared simple traces.

Corollary 9.4 (Arithmetic restriction on a triple class). *If an algebraic punctured torus has three curves with rationally related lengths, every real embedding of K fixes K_0 , and*

$$\text{Emb}_{\mathbb{R}}(K) = \text{Gal}(K/K_0), r_1(K) = [K : K_0] \in \{1, 2, 4\}.$$

If K is totally real, any three distinct squared simple traces generate K_0 .

Proof. The first statement follows from Theorem 9.3. Every embedding fixing K_0 changes only signs and has real image, so the converse inclusion also holds. For the last assertion let F be the field of the three squares. Every F -embedding of K is real and preserves the three individual lengths after lift signs are removed. Theorem 9.1 forces it to fix K_0 . Thus $[K : F] \leq [K : K_0]$. Since $F \subset K_0$, equality holds and $F = K_0$. $\square$

Theorem 9.5 (Four possible totally real exceptions). *A punctured torus with totally real trace field has at most two curves in each rational-length class, unless it is isometric to one of the four tori with seeds*

$$(\sqrt{5}, 2\sqrt{5}, 5), (\sqrt{6}, 2\sqrt{3}, 3\sqrt{2}), (2\sqrt{2}, 2\sqrt{2}, 4), (3, 3, 3).$$

This is an exception list for the bound two, not a claim that these four tori have uniformly bounded multiplicities by this argument.

Proof. If a rational-length class has at least three curves, Corollary 9.4 gives $K_0 = \mathbb{Q}$. The nonintegral-trace obstruction (Theorem 7.3) also makes every simple trace an algebraic integer. Thus $a = x^2, b = y^2, c = z^2$ are integers and

$$(a + b + c)^2 = abc.$$

Order them so that $5 \leq a \leq b \leq c$. A trace mutation has

$$c' = ab - 2a - 2b - c = (a + b)^2/c,$$

a positive integer at least five. Whenever $c > a + b$, this decreases the sum. Repeated descent reaches $c \leq a + b$. In trace coordinates $2 < x \leq y \leq z \leq xy/2$, whence

$$0 \leq x^2 - (x - 2)y^2 \leq x^2(3 - x).$$

Thus $5 \leq a \leq 9$ and $b \leq a/(\sqrt{a} - 2)$. The discriminant of the quadratic equation for c is $ab((a - 4)b - 4a)$. Nonnegativity and the upper bound on b leave exactly

a	b	$ab((a - 4)b - 4a)$
5	20, 21	0, 105
6	12, 13	0, 156
7	10	140
8	8, 9	0, 288
9	9	729

Only the square discriminants give solutions, namely $(5, 20, 25), (6, 12, 18), (8, 8, 16), (9, 9, 9)$. Their positive square roots are the listed seeds. They are distinct unmarked tori: outward mutation from a reduced seed never lowers its minimum trace, and their minimum traces $\sqrt{5}, \sqrt{6}, 2\sqrt{2}, 3$ differ. $\square$

9.2 Explicit algebraic-integral examples

Theorem 9.6 (A quadratic extension of two prescribed traces). *Let $a, b > 2$ belong to a real number field F . Suppose*

$$D = a^2b^2 - 4a^2 - 4b^2 > 0$$

is nonsquare in F . The seed

$$\left(a, b, \frac{ab + \sqrt{D}}{2}\right)$$

defines a punctured torus with at most two curves in each rational-length class. The two chosen curves are the only simple curves whose squared traces belong to F . If a, b are algebraic integers, all simple traces are algebraic integers.

Proof. Both roots of $Z^2 - abZ + a^2 + b^2$ exceed two: the vertex lies to the right of two, the polynomial at two is $(a - b)^2 + 4 > 0$, and the roots are real and positive. The real conjugation of $F(\sqrt{D})/F$ fixes a, b and exchanges two distinct positive roots. It changes z^2 , so Theorem 9.3 applies. A third squared trace in F would give three unchanged lengths on the two conjugate marked tori, contradicting Theorem 9.1. The monic quadratic and the integer-polynomial trace recursion prove integrality. $\square$

For example, every integer $n \geq 4$ gives the quadratic seed

$$\left(n, n, \frac{n^2 + n\sqrt{n^2 - 8}}{2}\right).$$

The maximum multiplicity is exactly two. The reflection interchanging the first two curves accounts for every repeated length. Its two fixed unoriented slopes have the two distinct roots of the quadratic as traces, and hence have unique lengths. Different n give nonisometric tori: their only rational-trace curves both have trace n .

Theorem 9.7 (An integral family in every degree). *For integers $d \geq 2$ and $n \geq 3$, let $r \in (n-1, n)$ be the root of $r^d - nr^{d-1} + 1 = 0$. The symmetric seed*

$$(s, h, h) = \left(r^2 + 2, r + \frac{2}{r}, r + \frac{2}{r}\right)$$

has integral simple traces. Its trace field and squared-trace field are both $\mathbb{Q}(r)$, of degree d . Every rational-length class is exactly an orbit of the reflection $(p, q) \mapsto (q - p, q)$, and its maximum length multiplicity is two. The tori are pairwise nonisometric as (d, n) varies.

Proof. On $|z| = 1$, the term $-nz^{d-1}$ dominates $z^d + 1$. Rouché's theorem gives exactly $d - 1$ roots inside the unit circle and one outside. The latter lies in $(n-1, n)$. A proper monic integer factor not containing that root would have all its roots of modulus less than one, contradicting its nonzero integer constant coefficient. Thus the polynomial is irreducible. It has a second real root in $(0, 1)$, since its values at zero and one have opposite signs. The constant coefficient makes r a unit. The displayed seed is integral and satisfies $s^2 + 2h^2 = sh^2$. The second real embedding changes s^2 , so Theorem 9.3 gives the bound two for rational-length classes.

The field generated by this triple equals $\mathbb{Q}(r)$ because $r = s/h$. In fact s^2 already generates this field. If an embedding fixed s^2 , it would send r to a root r' with $((r')^2 + 2)^2 = (r^2 + 2)^2$. Every root other than r has modulus less than one and hence $|(r')^2 + 2| < 3$, whereas $r^2 + 2 > 6$. No other embedding fixes s^2 .

The reflection has two fixed slopes, of traces s and $U(s) = 2s/(s-2)$. Every nonfixed orbit already exhausts its rational-length class. The fixed curves cannot have a rational length ratio either: even Chebyshev identities would preserve that ratio at the second real conjugate $s' \neq s$, whereas

$$s \mapsto \frac{\operatorname{arcosh}(s/2)}{\operatorname{arcosh}(U(s)/2)}$$

is strictly increasing on $(2, \infty)$. This proves the orbit assertion. Here $s > 6$ and $2 < U(s) < 3 < h$; descent of positive trace triples shows that $U(s)$ is the unique minimum trace. It determines r . Finally a root shared by two pairs $(d, n), (d', n')$ would give $n - n' = r^{1-d} - r^{1-d'}$, whose absolute value is less than one. Thus $n = n'$ and then $d = d'$, proving nonisometry. $\square$

Theorem 9.8 (Dense integral tori in the symmetric locus). *Let $F \subset \mathbb{R}$ have at least two real embeddings. Let $n \in \mathcal{O}_F$ generate F and assume $n^2 + 16$ is nonsquare in F . Set*

$$s = \frac{4 - n + \sqrt{n^2 + 16}}{2}, h = \frac{s}{\sqrt{s - 2}}.$$

Then (s, h, h) is an integral punctured-torus seed. Its trace field has degree $4[F : \mathbb{Q}]$, its squared-trace field has degree $2[F : \mathbb{Q}]$, and every rational-length class is exactly a reflection orbit. Qualifying parameters n are dense at the chosen real embedding of F , so these tori are dense in the entire symmetric locus $s > 2$.

Proof. The identities

$$s^2 + (n - 4)s - 2n = 0, h^2 = n + 2s = \frac{s^2}{s - 2}$$

prove the trace equation and integrality. The chosen root has $s > 2$. A second real embedding changes n and may be extended by taking the root $s' > 2$ and a real square root of h'^2 . The inverse relation $n = s(4 - s)/(s - 2)$ ensures $s' \neq s$. Theorem 9.3 gives two. The same reflection and strictly monotone fixed-length-ratio argument used in Theorem 9.7 identifies every rational-length class with a reflection orbit.

The nonsquare hypothesis makes $F(s)/F$ quadratic. The other extension of the chosen real embedding of F sends s to a number less than two, so $s^2/(s-2) < 0$ there. Thus adjoining h doubles the degree again. Since $n = s(4-s)/(s-2)$, the actual trace field is $F(s, h)$. Also $s = 2 + s^2/h^2$ belongs to the squared-trace field, which is exactly $F(s)$.

To prove density, choose an integral primitive generator ξ of F . Choose an even integer divisible by every ramified rational prime of F , and choose any prime p dividing its square plus one. Then p is odd, unramified in F , and $p \equiv 1 \pmod{4}$. There is an integer a with $v_p(a^2 + 16) = 1$: first solve the congruence modulo p , then choose any lift other than its unique root modulo p^2 . For integers m, l with $l \neq 0$, put

$$n = a + p^2(m + l\xi).$$

This is integral and generates F . At every prime of F over p , its discriminant $n^2 + 16$ has valuation one and hence is nonsquare. The values $m + l\xi$ with $l \neq 0$ are dense in $\mathbb{R}$. Finally $n \mapsto s$ is a continuous bijection from $\mathbb{R}$ onto $(2, \infty)$, proving the assertion. $\square$

An explicit collection in unbounded degree is obtained by taking $F = \mathbb{Q}(\xi)$, where ξ is a root of

$$\prod_{j=1}^d (X - 10j) - 5, d \geq 2,$$

and setting $n = 2 + \xi$. The polynomial is Eisenstein at five and has d real roots: its value at $10j$ is -5 , whereas its value at $10j + (-1)^{d-j}$ is positive, on disjoint intervals. At the unique place above five, normalized by $v(\xi) = 1$, we have $v(5) = d$, so

$$v(n^2 + 16) = v(20 + 4\xi + \xi^2) = 1.$$

Thus Theorem 9.8 gives degree- $4d$ trace fields of signature $(2d, d)$ and the complete rational-length bound two.

Corollary 9.9 (Algebraic or transcendental length ratios). *On a hyperbolic one-holed torus with algebraic simple traces, every ratio of two simple lengths is either rational or transcendental. Thus on any algebraic family proved here to have no rational length relations outside its symmetry orbits, every length ratio outside those orbits is transcendental.*

Proof. For each simple curve the expanding eigenvalue

$$\lambda_\gamma = \frac{T_X(\gamma) + \sqrt{T_X(\gamma)^2 - 4}}{2} > 1$$

is algebraic and $\ell_X(\gamma) = 2 \log \lambda_\gamma$. The Gelfond–Schneider theorem gives $\log \lambda_\gamma / \log \lambda_\eta \in \mathbb{Q}$ or makes this ratio transcendental; see [Gel34, p. 628]. Indeed, an algebraic irrational ratio would express the algebraic number λ_γ as an algebraic base different from zero and one raised to an algebraic irrational power. $\square$

10 Dense integral families, with fixed trace and congruences

The integral examples here have a fixed number field. Density is under the chosen real embedding; it does not mean that their conjugate coordinates are bounded.

Lemma 10.1 (Following a direction in a lattice). *Let $\Lambda \subset \mathbb{R}^m$ be a full lattice and let π be a linear coordinate whose image on Λ is dense. Given $d \in \ker \pi$ and a nonempty open interval J , there exist $z_T \in \Lambda$ with $z_T = Td + O(1)$ and $\pi(z_T) \in J$ as $T \rightarrow \infty$.*

Proof. Rounding in a lattice basis gives $w_T = Td + O(1)$. The values $\pi(w_T)$ lie in a fixed compact interval. Density of $\pi(\Lambda)$ supplies, for each point of that interval, a lattice translation carrying a neighborhood into J . A finite subcover gives finitely many such translations. Adding one of them to each w_T proves the assertion. $\square$

Theorem 10.2 (Fixed-trace density with exact fields). *Let $K \subset \mathbb{R}$ have at least two real embeddings and let*

$$\tau, x_0, y_0 \in \mathcal{O}_K, \tau, x_0, y_0 > 2, \tau^2 + x_0^2 + y_0^2 = \tau x_0 y_0.$$

Let I be a nonzero ideal of $\mathcal{O}_K$. Assume that either $\tau^2 - 4$ is nonsquare in K , or K has at least three infinite places, a complex conjugate pair counting as one place. Then the positive fixed- τ conic contains a dense set of integral seeds congruent to (τ, x_0, y_0) modulo I , with both trace fields exactly K and at most two curves in each rational-length class.

Proof. Put

$$\lambda = \frac{\tau + \sqrt{\tau^2 - 4}}{2}, d_\lambda = \lambda - \lambda^{-1}, \alpha_0 = x_0 - \lambda y_0 < 0.$$

The conjugate amplitude is $\alpha_0^* = x_0 - \lambda^{-1}y_0 > 0$ and $\alpha_0\alpha_0^* = -\tau^2$. The positive conic is parametrized by $b > 0$ through

$$y = \frac{b + \tau^2/b}{d_\lambda}, x = \frac{\lambda^{-1}b + \lambda\tau^2/b}{d_\lambda}.$$

Both coordinates are at least $2\tau/d_\lambda > 2$, and $b = \lambda y - x$, so this is a homeomorphism.

Every real embedding of K sends τ to a value with absolute value greater than two, by the positivity argument in Theorem 9.3. Suppose first that the discriminant is nonsquare. Set $L = K(\lambda)$ and $R = \mathcal{O}_K[\lambda]$. This is an order free over $\mathcal{O}_K$ with basis $1, \lambda$. If K has p infinite places, L has $2p$: each real place splits into two real places, and each complex place into two complex places. Dirichlet's unit theorem [Mil20, Theorems 5.1 and 5.9] gives ranks $2p - 1$ and $p - 1$ for $\mathcal{O}_L^\times$ and $\mathcal{O}_K^\times$. The same rank holds for $R^\times$: choose a positive integer m with $m\mathcal{O}_L \subset R$. Every unit of $\mathcal{O}_L$ congruent to one modulo $m\mathcal{O}_L$, together with its inverse, belongs to R . This congruence subgroup has finite index in $\mathcal{O}_L^\times$, so $R^\times$ has rank $2p - 1$. The relative norm image has rank $p - 1$, since it contains the squares of the latter units. The norm-one group therefore has rank p , and its logarithms form a full lattice in the relative logarithmic space $\mathbb{R}^p$.

Restrict to norm-one units positive at all real places and congruent to one modulo IR . These are finite-index conditions, so their logarithms still form a full lattice. The logarithmic coordinate at the chosen real place is dense: two multiplicatively independent positive units have logarithms with irrational ratio at an injective real embedding. Choose a relative logarithmic direction with that coordinate zero and every other coordinate nonzero. Lemma 10.1 gives units u_T whose logarithms at the chosen real place lie in any prescribed open interval, while their other place logarithms tend to positive or negative infinity. Set

$$\alpha_T = \alpha_0 u_T = x_T - \lambda y_T.$$

Its coefficients are integral and congruent to x_0, y_0 modulo I , and its norm is $-\tau^2$. Reconstruction puts the pair at the chosen embedding into any desired open neighborhood on the positive conic.

At every other embedding $\eta : K \rightarrow \mathbb{C}$, the magnitude of one of the two relative amplitudes tends to zero and the other to infinity. In

$$\eta(x_T) = \frac{\eta(\lambda)\eta(\alpha_T^*) - \eta(\lambda)^{-1}\eta(\alpha_T)}{\eta(d_\lambda)},$$

the two nonzero-coefficient terms consequently cannot cancel; $|\eta(x_T)| \rightarrow \infty$. The values at the chosen embedding remain in a bounded interval. For large T , no other embedding fixes x_T^2 . Thus $\mathbb{Q}(x_T^2) = K$: a proper subfield would admit more than one extension of its chosen embedding to K . Both trace fields are exactly K . A second real embedding changes x_T^2 , so Theorem 9.3 gives the rational-length bound.

If the discriminant is square, λ is an integral unit in K . Use totally positive units u congruent to one modulo the nonzero ideal $d_\lambda I$. Their logarithms form a full lattice in the product-formula hyperplane of dimension $p - 1 \geq 2$. Their coordinate at the chosen real place is dense for the same reason. That hyperplane has a direction with that coordinate zero and all other place coordinates nonzero, because $p \geq 3$. Apply Lemma 10.1 and set

$$\alpha_T = (x_0 - \lambda y_0)u_T, \alpha_T^* = (x_0 - \lambda^{-1}y_0)u_T^{-1}.$$

The reconstruction is integral and preserves the required congruences: the changes in both numerators are divisible by $d_\lambda I$. The same dominant-term argument at every other place proves the exact-field and rational-length assertions. $\square$

Proposition 10.3 (The split real-quadratic obstruction). *If K is real quadratic, $\tau \in \mathcal{O}_K$, and $\tau^2 - 4$ is square in K , integral points on the positive fixed- τ conic are locally finite. Thus the nonsquare condition in Theorem 10.2 is necessary in this case.*

Proof. Now λ, λ^{-1} are integral in K . For an integral pair put $\alpha = x - \lambda y$ and $\beta = x - \lambda^{-1}y$. Since $\alpha\beta = -\tau^2$, the principal ideal (α) is one of finitely many divisors of (τ^2) . Every admissible α is a generator of one of these principal ideals times a unit. The positive units in a real quadratic field form $\{\varepsilon^j : j \in \mathbb{Z}\}$ for one $\varepsilon > 1$. Thus $-\alpha$ lies in finitely many geometric progressions, locally finite in $(0, \infty)$. The conic parametrization in the preceding proof is a homeomorphism. $\square$

Theorem 10.4 (Global integral density with arbitrary represented congruences). *Let $K \subset \mathbb{R}$ have at least two real embeddings. Fix any positive integral punctured-torus seed in K and any nonzero ideal $I \subset \mathcal{O}_K$. Integral seeds congruent to this seed modulo I , with both trace fields exactly K and at most two curves in each rational-length class, are dense in marked Teichmüller space.*

Proof. For degree at least three, K has at least three infinite places, so Theorem 10.2 applies to every seed trace. In the quadratic case some simple trace has nonsquare discriminant. To see this, suppose a trace s has square discriminant. Its expanding eigenvalue is a positive unit $\lambda = \varepsilon^m$, where $\varepsilon > 1$ generates the positive units. Its neighboring fan traces have the form

$$u_n = A\lambda^n + B\lambda^{-n}, A, B > 0, AB = \frac{s^2}{s^2 - 4} > 1.$$

If infinitely many u_n as $n \rightarrow \infty$ had square discriminant, they could be written $u_n = \varepsilon^{k_n} + \varepsilon^{-k_n}$. Then $k_n - mn \rightarrow \log A / \log \varepsilon$, so this integer difference is eventually constant on those indices. Comparing the two expressions first gives $A = \varepsilon^j$ and then $B = \varepsilon^{-j}$, a contradiction. Choose a suitable seed curve with trace τ , changing the marking and congruences together if needed.

Apply Theorem 10.2 while holding τ fixed. On this conic, a neighboring trace ranges over $[2\tau/\sqrt{\tau^2 - 4}, \infty)$, so the produced neighbor traces are dense in its interior. In the real quadratic case the split traces form the locally finite set $\{\varepsilon^j + \varepsilon^{-j} : j \geq 1\}$; discard them. For each remaining neighbor trace apply Theorem 10.2 again, holding that curve fixed and preserving the original congruences. The union is dense in the open region where this second seed trace exceeds $2\tau/\sqrt{\tau^2 - 4}$.

The trace action of the mapping class group permutes finitely many residue triples modulo I . Its kernel has finite index and contains a positive power of each Dehn twist. Such a power can make the second seed trace arbitrarily large on any marked torus. Move a target into the preceding open region by a kernel element, approximate there, and change the marking back. The congruences, exact fields, and rational-length bound are preserved. $\square$

Taking the initial seed $(3, 3, 3)$ and $I = \mathcal{O}_K$ proves unconstrained density in every such field, including every real quadratic field. One explicit quadratic example uses

$$K = \mathbb{Q}(\sqrt{2}), \lambda = (3 + \sqrt{5})/2, \varepsilon = 19 + 6\sqrt{10}.$$

Expand

$$3(1 - \lambda)\varepsilon^a \lambda^{2b} = x_{a,b} - \lambda y_{a,b}, a, b \in \mathbb{Z}.$$

The coefficients belong to $\mathcal{O}_K$ since $\varepsilon = 19 - 18\sqrt{2} + 12\sqrt{2}\lambda$ belongs to $\mathcal{O}_K[\lambda]$ and has relative norm one. For $a \neq 0$ the seeds $(3, x_{a,b}, y_{a,b})$ form a dense set on the fixed-three conic, with both trace fields K and the bound two. Indeed the conjugation of K fixes λ and replaces ε by its inverse. Equality of the squared positive characters would force $a \log \varepsilon = -a \log \varepsilon$. Also $\log \varepsilon / \log \lambda$ is irrational, since a power relation and its conjugate would force that power to be trivial. The first seed is $(3, 57 - 18\sqrt{2}, 57 + 18\sqrt{2})$.

11 An entire fixed-trace locus with finitely many possible exceptions

Theorem 11.1 (Effective finite exception list). *Let $E \subset \mathbb{R}$ be totally real of degree at least two, and let $\tau \in \mathcal{O}_E$ satisfy $\tau > 2$ at the chosen real embedding and $|\sigma(\tau)| < 2$ at every other embedding. Let $K \subset \mathbb{R}$ be a finite extension of E such that every embedding of K fixing E has real image. Among positive punctured-torus seeds (τ, x, y) with $x, y \in K$, only finitely many unmarked tori can have a rational-length class of size at least three. There is a finite algebraic procedure listing all possible exceptions.*

Proof. The nonintegral-trace obstruction (Theorem 7.3) makes every trace of a possible exception integral. Each embedding of K fixing E has real image. Theorem 9.3 forces it to fix x^2, y^2 , so these squares belong to E . The trace equation also gives $xy \in E$. The coordinates x, y cannot belong to E , since at

a conjugate with $|\tau'| < 2$ the real quadratic form $x'^2 + y'^2 - \tau'x'y'$ is positive definite. Consequently $E(x) = E(y)$ is quadratic over E .

Choose an integral $h > 0$ generating this extension with $h^2 = d \in \mathcal{O}_E$; clearing denominators in a square-root generator suffices. Then $x, y \in hE$. At every other embedding $d' < 0$, for otherwise that embedding would again give a nonzero real solution with $|\tau'| < 2$. A finite extension has only finitely many quadratic subfields, so there are finitely many possible choices of $E(h)$ to test.

For one choice, the neighboring traces around τ obey $u_{j+1} = \tau u_j - u_{j-1}$ and have the form $A\lambda^j + B\lambda^{-j}$ with $A, B > 0$. Choose a least neighbor x and the smaller of its adjacent neighbors y . Twisting and reversing the order gives $x \leq y \leq \tau x/2$. The quadratic equation for y then implies

$$\frac{2\tau}{\sqrt{\tau^2 - 4}} \leq x \leq \frac{\tau}{\sqrt{\tau - 2}}.$$

The upper bound follows by evaluating $Y^2 - \tau x Y + x^2 + \tau^2$ at $Y = x$; the lower bound is its nonnegative discriminant. The recurrence preserves hE .

Set $A = hx, B = hy$, integral elements of E . They satisfy

$$\begin{aligned} A^2 + B^2 - \tau AB &= -d\tau^2, \\ \frac{2h\tau}{\sqrt{\tau^2 - 4}} &\leq A \leq B \leq \frac{\tau A}{2}, A \leq \frac{h\tau}{\sqrt{\tau - 2}}. \end{aligned}$$

At every other embedding write $d' = -e < 0$, $x = ha, y = hb$. The conic becomes $a'^2 + b'^2 - \tau'a'b' = \tau'^2/e$. Completing the square bounds

$$|A'|, |B'| \leq \frac{2|\tau'|\sqrt{e}}{\sqrt{4 - \tau'^2}}.$$

The Minkowski image of $\mathcal{O}_E$ is a lattice, and these bounded boxes contain finitely many points. Test those pairs for the exact equation and for integrality of $A/h, B/h$. This lists representatives of all possible exceptions. $\square$

For example, for every integer $n \geq 4$ set

$$s = \frac{4 - n + \sqrt{n^2 + 16}}{2}, h = \sqrt{4 + \sqrt{n^2 + 16}}, \tau = n + s.$$

Then $E = \mathbb{Q}(\sqrt{n^2 + 16})$ is real quadratic, $K = E(h)$ has signature $(2, 1)$, and the other conjugate of τ belongs to $(0, 2)$. The seed (τ, h, h) is integral and positive. Theorem 11.1 gives a finite possible exception list on the entire K -valued fixed- τ locus. Theorem 10.2 simultaneously gives a dense integral subset of that locus with both trace fields K and rational-length classes of size at most two. Neither statement removes the original symmetric torus from the possible exception list.

12 A dense family with three rationally related lengths

Proposition 12.1 (A contrasting dense integral family). *For positive integers a, b, c satisfying the strict triangle inequalities, there is exactly one $u > 1$ such that*

$$(x, y, z) = (u^a + u^{-a}, u^b + u^{-b}, u^c + u^{-c})$$

is a positive punctured-torus seed. The number u is an algebraic unit. These integral seeds are dense in marked Teichmüller space and have three rationally related simple lengths.

Proof. Write $u = e^t$, $t > 0$. The trace equation is equivalent to

$$4 \sinh \frac{(a+b+c)t}{2} \sinh \frac{(a+b-c)t}{2} \sinh \frac{(a-b+c)t}{2} \sinh \frac{(-a+b+c)t}{2} = 1.$$

This follows by expanding the hyperbolic addition formulas. All four factors have positive coefficients of t , so the left side increases strictly from zero to infinity. Thus the root is unique. Put $N = a + b + c$ and let $F(u) = x^2 + y^2 + z^2 - xyz$ using the Laurent expressions. Because $N > 2 \max(a, b, c)$, $-u^N F(u)$ is a monic integer polynomial of degree $2N$ and constant coefficient one. Hence u is a unit, and all trace polynomials are integral. The lengths are $2a \log u, 2b \log u, 2c \log u$. The same increasing equation and the implicit function theorem apply to positive real triangle ratios; rational ratios are dense. This gives density of the seeds. $\square$

Thus dense sets with and without triple rational-length classes coexist. Arithmetic density alone cannot settle a missing pointwise multiplicity bound by continuity.

13 Pointwise bounds for tori with geodesic boundary

Put $\rho(t) = \text{arcosh}(t/2)$ for $t > 2$. This section gives pointwise results, including bounds on rationally related lengths, for the positive-boundary case. Let X be a hyperbolic one-holed torus, with geodesic boundary of length $b > 0$. A positive trace triple (x, y, z) satisfies

$$x^2 + y^2 + z^2 - xyz = \kappa, \kappa = 2 - 2 \cosh(b/2) < 0.$$

Write $T(v) = 2 \cosh(N_X(v)/2)$ for the positive trace of a primitive unoriented slope v . Farey neighbors satisfy

$$T(u+v) + T(u-v) = T(u)T(v).$$

We use the trace polynomials obtained by this recurrence, with positive values on the chosen torus, and retain the definition of rational length classes from Section 1.5.

13.1 Two elementary counting principles

Lemma 13.1. *An affine line in real homology contains at most two rational vectors of any fixed positive N_X -length. Let C be a cone of angular width less than π , and let P be a positive, homogeneous, concave function on its rational vectors. Then two prescribed positive levels $N_X = L$ and $P = h$ contain at most two rational directions in C .*

Proof. The first assertion is Lemma 7.1, which also gives strict subadditivity for nonparallel rational vectors. For the second assertion, order three distinct directions and write the middle vector as $v = au + bw$, with $a, b > 0$ rational. The two level conditions and strict subadditivity give $L < (a+b)L$, whereas concavity gives $h \geq (a+b)h$. These require respectively $a+b > 1$ and $a+b \leq 1$. $\square$

Let v be a real-valued non-Archimedean valuation on the trace field with $v(n) \geq 0$ for every nonzero integer n , and put

$$P(t) = \max\{0, -v(t)\}.$$

The valuation at infinity on a rational function field is also allowed. For instance, if s is transcendental over a coefficient field, then $v(f/g) = \deg_s g - \deg_s f$ is such a valuation.

The integer trace polynomials

$$C_0(t) = 2, C_1(t) = t, C_{n+1}(t) = tC_n(t) - C_{n-1}(t)$$

satisfy $C_n(2 \cosh a) = 2 \cosh(na)$ and are monic for $n \geq 1$. For these positive indices, if $P(t) > 0$, the leading term is strictly dominant and

$$P(C_n(t)) = nP(t).$$

If $P(t) = 0$, then $P(C_n(t)) = 0$. Thus if finitely many curves have rationally related lengths, choose positive integers n_i with $n_i N_X(u_i) = L$. The vectors $n_i u_i$ lie both on $N_X = L$ and on a common level of every homogeneous pole-order formula proved below. This observation upgrades each proof from equal lengths to entire rational-length classes. Bounding every finite subset bounds the class.

13.2 A freely varying trace

Theorem 13.2. *If z is transcendental over $\mathbb{Q}(x, y)$, then every rational-length class on X has at most four curves.*

Proof. For a primitive vector (p, q) , its trace has degree

$$\deg_z T(p, q) = \min\{|p|, |q|\}.$$

Here degree zero occurs exactly at the two axis directions. To prove the formula for $p \geq q > 0$, begin with

$$T(n, 1) = xT(n-1, 1) - T(n-2, 1), T(0, 1) = y, T(1, 1) = z.$$

The coefficients of z satisfy $c_0 = 0$, $c_1 = 1$ and $c_n = xc_{n-1} - c_{n-2}$. Since $x > 2$, induction gives $c_n > c_{n-1} > 0$ for $n \geq 2$. Thus every $T(n, 1)$ with $n \geq 1$ has degree one. In the Farey cone between $(n, 1)$ and $(n+1, 1)$, the opposite axis has degree zero. The first mediant consequently has degree two. Thereafter the product of two positive-degree parents strictly dominates the removed term, so

degrees add under every outward mediant. They agree with the second coordinate q . These cones cover $p \geq q > 0$. Interchanging x and y gives degree p when $q \geq p > 0$.

For opposite signs, replace one generator by its inverse. The mixed trace becomes $xy - z$, which has degree one in z and remains transcendental over $\mathbb{Q}(x, y)$. The same argument gives the absolute-value formula. Opposite vectors represent the same curve.

For a rational-length class, scale its primitive representatives to one common length on X . The monic polynomial identities above give one common degree $h \geq 0$. If $h > 0$, the signed scaled vectors lie on

$$p = h, p = -h, q = h, q = -h.$$

Each affine line contains at most two such vectors by Lemma 13.1. There are at most eight signed vectors, and hence at most four unoriented curves. If $h = 0$, only the two axis directions occur. $\square$

13.3 Piecewise linear pole formulas

Theorem 13.3. *Suppose a valuation gives positive pole orders a, b, c to a positive trace triple (x, y, z) .*

1. *If a, b, c satisfy every strict triangle inequality, each rational-length class has at most six curves.*
2. *If $c > a + b$, each rational-length class has at most four curves.*
3. *If $c = a + b$ and $xy - z$ also has pole order c , the same bound four holds.*

If instead the initial pole orders are $(g, g, 0)$ with $g > 0$, and $\max\{0, -v(\kappa)\} < 2g$, each rational-length class has at most two curves.

Proof. In the strict triangle case, the product term is strictly dominant at the first outward Farey mutation in each of the three projective cones with endpoints

$$(1, 0), (1, 1); (1, 1), (0, 1); (0, 1), (-1, 0).$$

Every subsequent outward mutation has the same property: the new pole order is the sum of two positive parent orders and exceeds either. For $q \geq 0$, induction gives

$$P(T(p, q)) = \begin{cases} (p - q)a + qc, & p \geq q \geq 0, \\ (q - p)b + pc, & q \geq p \geq 0, \\ -pa + qb, & p \leq 0 \leq q. \end{cases}$$

Extend this homogeneously and by simultaneous sign reversal. A positive level consists of six straight segments, with successive vertices

$$\frac{h(1, 0)}{a}, \frac{h(1, 1)}{c}, \frac{h(0, 1)}{b}, -\frac{h(1, 0)}{a}, -\frac{h(1, 1)}{c}, -\frac{h(0, 1)}{b}.$$

There are at most twelve signed vectors at common length and pole levels, hence at most six curves.

If $c > a + b$, the opposite diagonal $xy - z$ has pole order c as well. The same outward induction in the four projective cones with successive endpoints

$$(1, 0), (1, 1), (0, 1), (-1, 1), (-1, 0)$$

gives, on every quadrant, with $A = |p|$ and $B = |q|$,

$$P(p, q) = \min\{aA + (c - a)B, (c - b)A + bB\}.$$

Indeed, the first expression is smaller when $A \geq B$, and the second when $B \geq A$, because their difference is $(a + b - c)(A - B)$. This is a positive, homogeneous, concave function on each quadrant. Lemma 13.1 allows at most two directions in each of two quadrants representing all unoriented directions, giving four. When $c = a + b$ and both diagonal pole orders are c , the same induction gives $a|p| + b|q|$, with the same bound.

For the final assertion put $D = \max\{0, -v(\kappa)\} < 2g$ and $f(p, q) = g(p - q)$. We prove by Farey mutation that

$$P(T(p, q)) = |f(p, q)|.$$

It holds initially. If parent pole orders are A, B and the old diagonal has order C , the induction hypothesis says $C \in \{A + B, |A - B|\}$. The parent vectors form a lattice basis, so $\max\{A, B\} \geq g$. If $A \neq B$, the identity

$$zz' = x^2 + y^2 - \kappa$$

has right-hand pole order $2 \max\{A, B\} > D$, uniquely attained by the larger square. Both possible diagonal orders are positive, and the new one is the other value in $\{A + B, |A - B|\}$. If $A = B > 0$ and $C = 0$, the identity $z' = xy - z$ gives new order $2A$. If $C = 2A$, the product identity gives new pole order zero, since its right side has pole order at most $2A$. The case $A = B = 0$ is impossible for a lattice basis. This completes the induction. A positive level $|f| = h$ is the union of two affine lines, allowing at most four signed vectors, or two curves. The zero level has only the single rational direction $p = q$. The common-level argument preceding the theorem applies in every case to rationally related lengths. $\square$

Corollary 13.4. *If a valuation has a positive pole at every simple trace and has $D = \max\{0, -v(\kappa)\} > 0$, each rational-length class has at most six curves.*

Proof. Order a triple's positive pole orders as $a \leq b \leq c$. Unless $c = a + b$ and the opposite diagonal has order $0 < c' < c$, one of the three cases of Theorem 13.3 already applies. At a continuing step the trace equation implies $D \leq 2c$. If $D < 2b$, the product identity gives $c' = b - a$; $a = b$ would give $c' \leq 0$ and is excluded. If $D = 2b$, it gives $c' \leq b - a$; a strict inequality makes $b > a + c'$, a terminal strict-dominance case. If $D > 2b$, then $c' = D - c > b - a$, while $c' < c = a + b$, so the new triple satisfies every strict triangle inequality. All continuing steps are therefore the subtractive Euclidean algorithm on a positive pair. Grouping subtractions into Euclidean divisions, every two remainder steps reduce the maximum by at least a factor of two. An infinite descent would give $c \rightarrow 0$, contradicting $D \leq 2c$. Thus the process terminates in a proved case. $\square$

For example, the rational triples

$$\left(\frac{10}{3}, \frac{13}{3}, \frac{11}{3}\right), \left(\frac{10}{3}, \frac{13}{3}, \frac{100}{27}\right)$$

have respective pole triples $(1, 1, 1)$ and $(1, 1, 3)$ at three, and boundary parameters $-260/27$ and $-7211/729$. They give bounds six and four. The triple

$$\left(\frac{17}{4}, \frac{17}{4}, \frac{42}{17}\right)$$

has pole orders $(2, 2, 0)$ at two and boundary parameter $-1385/578$, whose pole order is one. Its bound two is attained by the first two curves. More generally, for positive algebraic A, B, C and transcendental s satisfying

$$As, Bs, Cs > 2, s > \frac{A^2 + B^2 + C^2}{ABC},$$

the triple (As, Bs, Cs) has boundary parameter less than zero, and the degree valuation in s gives the bound six.

13.4 Negative-product real conjugates

Theorem 13.5. *Suppose a ring homomorphism from the trace ring to $\mathbb{R}$ sends a chosen trace triple to three numbers of absolute value greater than two and negative product. Then every rational-length class on X has at most six curves.*

Proof. Changing the signs of two generators if necessary gives conjugate labels $(-a, -b, -c)$, with $a, b, c > 2$. The three projective Farey cones with endpoint pairs

$$(e_1, -e_2), (e_1, e_1 + e_2), (e_2, e_1 + e_2)$$

cover all unoriented directions. In each cone, change lift signs so the endpoints have positive labels, say a, b . Its first mediant then has label $ab + c$. Every subsequent outward label exceeds both parents: if the removed label is smaller than the larger parent and both parents exceed two, their product minus the removed label exceeds either parent and half their product. The invariant throughout is

$$D = a^2 + b^2 + c^2 + abc > 4.$$

The concavity below is the rising-tree case of O'Sullivan's encoding theorem [OS26, Theorem 1.1]. We include its Farey calculation, so no conjugate geometric interpretation is required. Write $\rho(t) = \text{arcosh}(t/2)$. For a primitive vector $(n-k)e + kf$ in one of the cones, let $m_{k/n}$ be its positive conjugate label and set

$$\phi(k/n) = \rho(m_{k/n})/n.$$

For an interior rational $q = k/n$, let $q_- = k_-/n_-$ and $q_+ = k_+/n_+$ be its Farey parents. Write their labels s, t , and the label at q as u . It is the larger root

$$u = \frac{st + \sqrt{(s^2 - 4)(t^2 - 4) + 4(D - 4)}}{2},$$

so $\rho(u) > \rho(s) + \rho(t)$.

The neighbors on the left of q are $q_j^- = (k_- + jk)/(n_- + jn)$. Their labels obey

$$v_{j+1} = uv_j - v_{j-1}, v_0 = s, v_1 = us - t.$$

Put $\lambda = (u + \sqrt{u^2 - 4})/2$. Then

$$v_j = F_- \lambda^j + H_- \lambda^{-j}, F_- = \frac{s\lambda - t}{\sqrt{u^2 - 4}} > 0.$$

The right coefficient is

$$F_+ = \frac{t\lambda - s}{\sqrt{u^2 - 4}} > 0.$$

Positivity follows from $u > \max\{s, t\}$, $s, t > 2$ and $\lambda > u/2$. Thus $\rho(v_j) = j\rho(u) + \log F_- + o(1)$. The slope S_j^- of the secant from q_j^- to q satisfies

$$S_j^- = (n_- + jn)\rho(u) - n\rho(v_j).$$

Since each subsequent mediant again has its larger-root value, $S_{j+1}^- < S_j^-$. The right secant slopes S_j^+ increase. Their limits are

$$S^- = n_- \rho(u) - n \log F_-, S^+ = n \log F_+ - n_+ \rho(u).$$

Using $s^2 + t^2 + u^2 - stu = D$ and $n = n_- + n_+$ gives

$$S^+ - S^- = n \log \frac{F_- F_+}{\lambda} = n \log \frac{u^2 - D}{u^2 - 4} < 0.$$

The logarithm is defined because $F_-, F_+ > 0$. Every left secant slope therefore exceeds every right one. Each Farey neighbor of an interior rational is one of its parents plus a nonnegative integral multiple of its primitive vector, so these estimates apply to consecutive points in every finite Farey sequence. In a Farey sequence containing any three prescribed rational points, consecutive secant slopes decrease; weighted averaging proves concavity of ϕ on all rational points. Consequently the homogeneous function

$$G(Ae + Bf) = (A + B)\phi\left(\frac{B}{A + B}\right)$$

is positive and concave on the rational vectors of each cone. At a primitive curve, $G(v) = \rho(|\sigma T(v)|)$.

For curves with rationally related lengths, choose positive integers n_i with $n_i N_X(u_i) = L$. Apply the ring homomorphism to $C_{2n_i}(T(u_i)) = 2 \cosh L$. Even indices remove conjugate signs, so $2 \cosh(2n_i G(u_i))$ is independent of i , and the scaled vectors also have one common positive G -level. Lemma 13.1 allows at most two directions per cone. The three cones give the bound six. $\square$

Let d be a nonsquare positive integer, and let a, b, c be positive integers. The integral triple $(a\sqrt{d}, b\sqrt{d}, c\sqrt{d})$ satisfies this criterion whenever its coordinates exceed two and

$$\sqrt{d} > \frac{a^2 + b^2 + c^2}{abc}.$$

The conjugation $\sqrt{d} \mapsto -\sqrt{d}$ supplies the required negative product. An example is $(\sqrt{10}, 2\sqrt{10}, 3\sqrt{10})$, with boundary parameter $140 - 60\sqrt{10} < 0$.

13.5 Exact fourfold and sixfold symmetry families

Theorem 13.6. *Let $t > 2\sqrt{2}$ be a real number which is not an algebraic integer. For the torus with positive triple $(t, t, t^2/2)$, each rational-length class is exactly an orbit of the four transformations of unoriented slopes generated by*

$$(p, q) \mapsto (q, p), (p, q) \mapsto (p, -q).$$

The axis orbit and the diagonal orbit have two curves each, with traces t and $t^2/2$. Every other orbit has four curves. In particular, the maximum simple-length multiplicity is exactly four.

Proof. The boundary parameter is $2t^2 - t^4/4 < 0$. Equality of the first two traces gives the first symmetry; $xy - z = z$ gives the second. Farey recursion extends both identities to every curve. The two reflections fix only the axis and diagonal directions, and their product has no real fixed direction. Thus precisely the two indicated orbits have size two; the others have size four.

If t is algebraic, its failure to be an algebraic integer gives a finite-place valuation with $g = -v(t) > 0$. The initial pole orders are

$$g, g, 2g + v(2), v(2) \geq 0.$$

Strict dominance applies if $v(2) > 0$; otherwise both diagonal orders are $2g$ and the equal-sum case applies. If t is transcendental, degree in t gives the equal-sum case directly. Theorem 13.3 bounds each rational-length class by four, so every four-element symmetry orbit fills its class.

Only the two exceptional orbits could still share a rational-length class. Their half-length ratio lies strictly between one and two, because

$$t < t^2/2 < t^2 - 2 = 2 \cosh(2\rho(t)).$$

A rational ratio would force the same ratio of pole orders, namely $2 + v(2)/g \geq 2$, or the degree ratio two in the transcendental case. This is impossible, so the two exceptional classes are distinct. $\square$

Theorem 13.7. *Let $t > 3$. Suppose either that t is not an algebraic integer, or that t has a real conjugate less than -2 . On the torus with positive triple (t, t, t) , every rational-length class is exactly an orbit of the six symmetries of the initial Farey triangle. The two exceptional orbits have three curves each, with traces t and $t^2 - t$. Every other orbit has six curves. In particular, the maximum simple-length multiplicity is exactly six.*

Proof. The boundary parameter is $3t^2 - t^3 < 0$. Permutations of the three initial directions preserve traces by Farey recursion. Their order-three elements have no real fixed direction, and the reflections fix just the directions in the two orbits

$$\{(1, 0), (0, 1), (1, 1)\}, \{(1, -1), (1, 2), (2, 1)\}.$$

The traces of these orbits are t and $t^2 - t$. Every other orbit has six curves.

If t is algebraic but not integral, choose a finite-place pole; if it is transcendental, use degree in t . The three initial orders are equal and positive, so the strict-triangle case of Theorem 13.3 gives at most six curves in every rational-length class. If t instead has a real conjugate less than -2 , Theorem 13.5 gives the same bound. Every six-element symmetry orbit therefore fills its class.

For the two exceptional orbits, the half-length quotient on X lies strictly between one and two, since

$$t < t^2 - t < t^2 - 2 = 2 \cosh(2\rho(t)).$$

At a pole, $t^2 - t$ has twice the pole order of t ; a rational length quotient would have to be two. The same statement holds for the transcendental degree valuation. In the remaining case write $\sigma(t) = -a$, $a > 2$. The conjugate magnitudes are a and $a^2 + a$, and their half-length quotient exceeds two because

$$a^2 + a > a^2 - 2 = 2 \cosh(2\rho(a)).$$

An even Chebyshev identity expressing rational proportionality would force this quotient to equal the quotient on X , which is less than two. All cases give a contradiction. $\square$

The sixfold theorem includes every transcendental $t > 3$, every nonintegral rational $t > 3$, and every $t = \sqrt{d}$ with nonsquare integer $d > 9$. At each nonintegral algebraic boundary parameter $\kappa < 0$, the unique $t > 3$ satisfying $3t^2 - t^3 = \kappa$ is not an algebraic integer, so the same conclusion holds there. The monotonicity of $3t^2 - t^3$ on $(3, \infty)$ gives uniqueness. The transcendental diagonal case was already observed to have the Markoff isometry property by McShane and Parlier [MP10, Section 6.2]. The theorem above also classifies rational-length classes, excluding unequal but rationally proportional lengths.

14 Small intersections in a complete length level

Theorem 14.1 (An arithmetic intersection bound). *Suppose a complete simple-length level on a hyperbolic one-holed torus contains slopes $u = (1, 0)$ and $v = (r, c)$, with $c > 0$, $0 \leq r < c$, and $\gcd(r, c) = 1$. Write $g_- = \gcd(r - 1, c)$ and $g_+ = \gcd(r + 1, c)$. Then*

$$m_X(L) \leq \frac{2c}{g_-} + \frac{2c}{g_+} - 2 + \mathbf{1}_{g_-=1} + \mathbf{1}_{g_+=1}.$$

In particular $m_X(L) \leq 4c$, and if c is even then $m_X(L) \leq 2c - 2$. On a punctured torus the sharper conclusions are:

- *an intersection-one pair belongs to a level of size at most three; equality occurs only at the modular systole;*
- *an intersection-two pair fills its length level;*
- *intersection three or six gives at most six curves;*
- *intersection four gives at most four curves;*
- *three curves with all pairwise intersections three fill their level.*

Thus a level of size at least seven has no pairwise intersection in $\{1, 2, 3, 4, 6\}$.

Proof. We use the strict triangle inequality and the affine-line bound from Lemma 7.1.

For a third curve choose $w = (a, b)$ with $b > 0$. The determinant identity and the strict triangle inequality imply that $c, b, |rb - ca|$ satisfy all strict triangle inequalities. In particular $||rb - ca| - b| < c$, placing w in one of the strips

$$|f_-(a, b)| < c, |f_+(a, b)| < c, f_{\pm}(a, b) = (r \pm 1)b - ca.$$

Neither chosen curve lies in either open strip. The image of $f_{\pm}$ on the lattice is $g_{\pm}\mathbb{Z}$. Each pair of nonzero levels $f_{\pm} = \pm jg_{\pm}$, with $0 < j < c/g_{\pm}$, contributes at most two unoriented curves, by central symmetry and line intersection. The primitive kernel direction has determinant $c/g_{\pm}$ with both u and v . It can belong to the same length level only if $c < 2c/g_{\pm}$, hence only if $g_{\pm} = 1$. Adding the two chosen curves proves the displayed bound. If c is even, r is odd and both gcds are at least two.

For $c = 1$, a third primitive slope must be $(1, 1)$ or $(1, -1)$: the determinant triangle inequalities give $||a| - |b|| < 1$. Three equal Farey traces t satisfy $3t^2 = t^3$, hence $t = 3$. The opposite trace is 6, and outward Farey flips increase the largest trace. Thus exactly the three modular systoles occur.

For $c = 2, 3, 4$, a shear and an orientation change reduce r to one. If $b - ca \geq 0$, the determinant inequalities force $a = 0$, then primitivity gives $b = 1$ and contradicts $c < 2b$. Therefore $|ca - 2b| < c$. For $c = 2$, the sole possibility is the kernel direction $(1, 1)$, excluded by $2 < 1 + 1$. For $c = 4$ the nonzero possibilities are $4a - 2b = \pm 2$, whose two lines contribute at most two additional unoriented curves; the kernel direction is again excluded.

For $c = 3$, the possibilities for $f = 3a - 2b$ are $0, \pm 1, \pm 2$. If the kernel direction $(2, 3)$ is absent, the two pairs of nonzero lines give at most four additional curves. If it is present, apply the determinant inequalities with this direction as well. For $|f| = 1$ they require $b = 3$, inconsistent with $3a = 2b + f$. For $|f| = 2$ they require $b \in \{2, 3, 4\}$; the only integer solutions are $(a, b) = (2, 2), (2, 4)$, both nonprimitive. There is no fourth curve. This also proves the assertion about a triple of pairwise intersection three. Finally, for $c = 6$ the two gcds are 2 and 6, and the general formula gives six. $\square$

15 Exact triple levels on the reflection walls

Write $m_N(X)$ for the multiplicity of the N th distinct simple trace. A reduced trace chamber is

$$\mathcal{F} = \{(x, y, z) : 2 < x \leq y \leq z \leq xy - z, x^2 + y^2 + z^2 = xyz\}.$$

Here the three traces are ordered; changing to these coordinates may permute the initially chosen curves. Its reflection sides are $x = y$, $y = z$, and $z = xy - z$.

Theorem 15.1 (Complete boundary classification of triple levels). *Let b_N be the number of unmarked tori on these reflection walls with $m_N = 3$. Then*

$$b_N = \begin{cases} 1, & N = 1, 2, \\ 0, & N = 3, \\ 1, & N \geq 4. \end{cases}$$

For $N \geq 4$, put $k = N - 2$, define

$$P_0(X) = 1, P_1(X) = X - 1, P_{j+1}(X) = XP_j(X) - P_{j-1}(X),$$

and let $r_k \in (0, 1)$ be the unique solution of

$$r_k P_k(r_k^2 + 2) = 2.$$

The corresponding reduced trace triple is

$$(r_k^2 + 2, r_k + 2/r_k, r_k + 2/r_k).$$

Every odd multiplicity on the reflection walls away from the modular point is at most three. This statement does not bound even multiplicities.

Proof. First consider the side with two equal larger root traces. Use the marking

$$(T_\infty, T_0, T_1) = (a, c, a), c = r^2 + 2, a = r + 2/r, 0 < r < 1.$$

The reflection is $\sigma(p/q) = p/(p - q)$; its fixed slopes are 0 and 2. The fan about 0 consists of the reflected pairs

$$\{-1/j, 1/(j + 1)\}, j \geq 0,$$

where $-1/0 = \infty$. Their traces are $F_j = aP_j(c)$. The second fixed trace is

$$Q = T_2 = a^2 - c = 2 + 4/r^2.$$

Induction in the defining recurrence gives

$$P_j(2 + u) = \sum_{h=0}^j \binom{j+h}{2h} u^h.$$

Consequently $rP_k(r^2 + 2)$ increases strictly from zero, and for $k \geq 2$ it exceeds two at $r = 1$. This proves existence and uniqueness of r_k .

The fan traces satisfy $c < F_0 < F_1 < \dots$. The first slopes leaving its two arms have traces

$$G_j = F_{j-1}F_j - c, j \geq 1.$$

They increase with j and $G_1 - Q = c^2 > 0$. Each G_j is larger than both traces on its shared fan edge, so all descendants in these branches are still larger. The remaining component starts at slope 2 with trace Q ; its first descendants form one reflected pair of trace $H = F_0(Q - 1) > Q$. Outward flip growth again makes their descendants larger. These components exhaust the complement of the fan because the dual Farey graph is a tree.

Thus Q can agree only with one of the strictly ordered fan traces, and the equation $F_k = Q$ is exactly $rP_k(c) = 2$. At r_k , the lower distinct traces are

$$c, F_0, \dots, F_{k-1},$$

so the common value has rank $k + 2$ and consists of precisely one fixed slope and one reflected pair. The parameter r is recovered from the unique systole trace $c = r^2 + 2$, so different parameters on this open side give different unmarked tori. No other odd collision occurs on this open side: slope 0 is the unique systole, and every nonfixed slope belongs to a two-element orbit.

On the open side $x = y = t$ of the reduced chamber, $2\sqrt{2} < t < 3$ and its opposite traces satisfy $t < z < w = t^2 - z$. The fixed slopes of reciprocal reflection have traces z, w . The first other pair beyond z has trace $U = t(z - 1)$, with

$$U - w = (t + 1)(z - t) > 0.$$

The first pair beyond w has trace $V = t(w - 1) > U$. All remaining dual-tree branches grow outward from these pairs. Neither fixed slope therefore shares its trace with any other curve, and odd multiplicity three never occurs there.

On the open side $z = xy - z$, the reflection fixes the slopes of traces x, y and pairs the opposite slopes of trace $z = xy/2$. Here $2 < x < y < z$. The first outside traces are $xz - y$ and $yz - x$, both greater than z ; for example

$$xz - y - z = \frac{y(x - 2)(x + 1)}{2} > 0.$$

Their branches grow outward, so this side also has no odd collision above one.

At the square vertex $(2\sqrt{2}, 2\sqrt{2}, 4)$, the additional projective rotation has order two and no fixed real direction; it pairs all unoriented slopes, so every multiplicity is even. At the modular vertex the root permutation group has two exceptional slope orbits of size three, with traces 3 and 6. Every other orbit has size six: a shorter orbit would have a slope fixed by one of the three projective reflections, whose six fixed directions are exactly the two listed orbits. Outward Farey growth shows that no other curve has trace at most 6. Thus the two exceptional orbits give exactly the first and second triple levels, and every higher level is a union of six-element orbits. This proves the classification and the odd-multiplicity statement. $\square$

Corollary 15.2 (Cusp position of the triple levels). *As $k \rightarrow \infty$,*

$$r_k \sim \frac{W(4k)}{k}, x_k - 2 \sim \frac{W(4k)^2}{k^2}, y_k = z_k \sim \frac{2k}{W(4k)},$$

where $W(s)e^{W(s)} = s$ is the positive Lambert function.

Proof. Put $r = 2 \sinh(s/2)$, so $c = 2 \cosh s$. Solving the recurrence gives

$$P_k(c) = \frac{\cosh((2k + 1)s/2)}{\cosh(s/2)}.$$

The collision equation is $\tanh(s/2) \cosh((2k + 1)s/2) = 1$. It forces $s \rightarrow 0$ and $u = (2k + 1)s/2 \rightarrow \infty$. Hence $ue^u \sim 4k$, so $u \sim W(4k)$ and $r \sim s \sim W(4k)/k$. Substitution proves the other formulas. $\square$

Corollary 15.3 (Integral triple levels of unbounded field degree). *The tori with the triple levels constructed above have algebraic integer simple traces. Their trace and squared-trace fields coincide, have exactly one real embedding, and have unbounded degree as $k \rightarrow \infty$.*

Proof. Put $c_k = r_k^2 + 2$ and $a_k = c_k/r_k$. The equation $r_k P_k(c_k) = 2$ gives

$$F_k(c_k) = 0, F_k(S) = (S - 2)P_k(S)^2 - 4.$$

This is a monic integer polynomial, so c_k is integral. The monic polynomial $r P_k(r^2 + 2) - 2$ similarly makes r_k integral, and

$$a_k = r_k + 2/r_k = r_k + P_k(c_k)$$

is integral. All simple traces are therefore integral by recursion. Also $r_k = 2/P_k(c_k)$, so the trace field is $\mathbb{Q}(c_k)$. The identity $a_k^2 = c_k^2/(c_k - 2)$ gives $c_k = 2 + c_k^2/a_k^2$, proving equality with the squared-trace field.

For real $S \leq 2$, $F_k(S) < 0$. For $S > 2$, the formula $P_k(2 + u) = \sum_{h=0}^k \binom{k+h}{2h} u^h$ shows that $(S - 2)P_k(S)^2$ increases strictly from zero to infinity. Thus F_k has exactly one real root. The minimal polynomial of c_k divides F_k , so $\mathbb{Q}(c_k)$ has exactly one real embedding.

All roots of P_k lie in $(-2, 2)$, as follows from

$$P_k(2 \cos \theta) = \frac{\cos((2k + 1)\theta/2)}{\cos(\theta/2)}.$$

If $|S| \geq 4$, this factorization gives $|(S - 2)P_k(S)^2| \geq 2^{2k+1} > 4$, excluding roots of F_k . All conjugates of c_k consequently have modulus less than four. The c_k are distinct and tend to two. If their degrees were bounded, the elementary symmetric formulas would bound all coefficients of their monic minimal polynomials, allowing only finitely many such polynomials and roots. This is a contradiction. $\square$

16 Generic reflection symmetry and dense higher multiplicities

Proposition 16.1 (Generic absence of rational relations). *In the marked space of one-holed hyperbolic tori with any fixed boundary length, including zero, the surfaces on which distinct simple curves have irrational length ratio form a dense residual set of full measure. The same holds if the boundary length varies.*

Proof. For distinct curves α, β and positive integers n, m , the real-analytic function $n\ell_\alpha - m\ell_\beta$ is nonzero: pinching α sends its length to zero while the collar inequality forces ℓ_β to infinity, since the two curves intersect. Its zero set has empty interior and measure zero. Take the complement of the countable union of these zero sets. $\square$

Theorem 16.2 (Reflection axes). *On the punctured-torus family*

$$(x, y, z) = (t + 2/t, t + 2/t, t^2 + 2), t > 0,$$

let $R(p, q) = (q, p)$. *Outside a countable set of algebraic parameters, every rational-length class is exactly one R -orbit and has at most two curves. In particular this holds at every transcendental t , when every simple trace is transcendental. Nevertheless the parameters having a length level of size at least four form a countable dense subset of the lower branch*

$$(x, y, z) = (u, u, z_-(u)), z_-(u) = \frac{u^2 - u\sqrt{u^2 - 8}}{2}, u > 2\sqrt{2}.$$

Every open interval on this branch contains such levels with arbitrarily large common length.

Proof. The two quantities $m(v) = |p - q|$ and $n(v) = |p + q|$ determine the R -orbit of an unoriented slope $v = (p, q)$. Work first over $\mathbb{Q}(t)$. At $t = 0$ the initial pole orders are $(1, 1, 0)$; at $t = \infty$ they are $(1, 1, 2)$. Lemma 7.2 therefore gives, for every simple trace,

$$P_0(T_v) = |p - q|, P_\infty(T_v) = |p + q|.$$

The traces are integer Laurent polynomials and are positive for $t > 0$. Since $2 \operatorname{arcosh}(T/2) = 2 \log T + o(1)$ as $T \rightarrow \infty$, these pole orders give

$$\ell_t(v) = -2m(v) \log t + O_v(1)(t \rightarrow 0),$$

$$\ell_t(v) = 2n(v) \log t + O_v(1)(t \rightarrow \infty).$$

When the relevant pole order is zero the trace, and hence its length, stays bounded, so the same formulas apply. If $j\ell_t(v) = k\ell_t(w)$ holds identically, with positive integers j, k , the two asymptotics force $jm(v) = km(w)$ and $jn(v) = kn(w)$. The two absolute-value coordinates then identify ju with kw or kRw , up to sign. Primitivity forces $j = k$ and puts the curves in the same reflection orbit. Every trace is an integer Laurent polynomial in t , by the seed formulas and Farey recursion. Define $C_0(T) = 2$, $C_1(T) = T$, and $C_{n+1}(T) = TC_n(T) - C_{n-1}(T)$. A rational relation $j\ell_t(v) = k\ell_t(w)$ is equivalent to $C_j(T_v(t)) = C_k(T_w(t))$. Outside a common reflection orbit this is a nonzero integer Laurent polynomial equation, by the preceding argument, so it has only finitely many positive roots, all algebraic. Taking the countable union over slopes and integers proves the first claim and countability in the second. Each simple trace is a nonconstant Laurent polynomial, since at least one of its two pinching growth coefficients is positive. Its value at a transcendental parameter is therefore transcendental.

For density put $A(u) = \ell_u(1, 0)$ and $B(u) = \ell_u(1, 1)$ on the lower branch. The first increases strictly, while the second decreases strictly, since $u = z_-(u)/\sqrt{z_-(u) - 2}$ and $2 < z_-(u) < 4$. Thus B/A decreases strictly. In a compact interval $[a, b]$ choose positive integers h, k with

$$\frac{B(b)}{A(b)} < \frac{h}{k} < \frac{B(a)}{A(a)}.$$

For a twist fan about η , with the starting curve meeting η once, diagonalizing its holonomy gives

$$T(T_\eta^j \xi) = P\lambda^j + Q\lambda^{-j}, \lambda = e^{\ell(\eta)/2} > 1, P > 0.$$

The coefficients vary continuously, so on a compact family $\ell(T_\eta^j \xi) = j\ell(\eta) + O(1)$ uniformly. Apply this to

$$v_n = (nh + 1, 1), w_n = (nk, nk + 1).$$

Their lengths are $nhA(u) + O(1)$ and $nkB(u) + O(1)$ respectively. Their difference changes sign between a and b for every sufficiently large n , hence vanishes at an interior parameter. The slopes v_n, Rv_n, w_n, Rw_n are distinct for sufficiently large n and all have this length. The lengths tend to infinity uniformly on $[a, b]$, completing the proof. $\square$

17 A complete-level bound on an integral quartic torus

Let Q be the complete cusped torus with positive Fricke triple

$$(x, y, z) = (2\sqrt{2}, 2\theta, 2\theta), \theta = \sqrt{1 + \sqrt{2}}.$$

These coordinates satisfy $x^2 + y^2 + z^2 = xyz$. Put

$$E = \mathbb{Q}(\sqrt{2}), \mathcal{R} = \mathbb{Z}[\sqrt{2}], \lambda = 1 + \sqrt{2} = \theta^2.$$

A prime factor in $\mathcal{R}$ is counted up to association by a unit. The ring $\mathcal{R}$ is Euclidean for the absolute field norm: rounding the two coefficients of an element of E gives a remainder $a + b\sqrt{2}$ with $|a|, |b| \leq 1/2$ and $|a^2 - 2b^2| \leq 1/2$. In particular, factorization into primes is unique. Write r' for the conjugate of $r \in E$.

Theorem 17.1 (Quartic levels with elliptic conjugate). *Suppose that $\tau = 2r > 2$ is a simple trace on Q , where $r \in \mathcal{R}$ and $|r'| < 1$. Then $r/\sqrt{2} \in \mathcal{R}$. If this element has ω distinct prime factors, the complete simple-length level with trace τ has multiplicity at most*

$$2^{\max\{1, \omega\}}.$$

Thus the multiplicity is at most two if $r/\sqrt{2}$ is a unit or a unit times a prime power, and at most four if it has two distinct prime factors.

Proof. Choose a homology basis for which the initial traces of slopes $(1, 0), (0, 1), (1, 1)$ are $2\sqrt{2}, 2\theta, 2\theta$. Every Farey triangle has one vertex with even second coordinate and two with odd second coordinate. The positive trace recurrence is $T_{u+v} = T_u T_v - T_{u-v}$. Induction under Farey mutation gives

$$T_{p,q} = 2r, r \in \sqrt{2} + 2\mathcal{R} (q \text{ even}),$$

$$T_{p,q} = 2\theta u, u \in 1 + 2\mathcal{R} (q \text{ odd}).$$

For the even coordinate the mutation is $r \mapsto 2\lambda uv - r$, and for an odd coordinate it is $u \mapsto 2rv - u$; these preserve the stated congruences. Writing $r = a + b\sqrt{2}$, they say that a is even and b is odd. Consequently r has exactly one factor $\sqrt{2}$, and $r/\sqrt{2}$ has no factor above two. Since $\text{Norm}_{E/\mathbb{Q}}(\lambda) = -1$, the element λ is not a square in E . The automorphism $\theta \mapsto -\theta$ fixes every even trace and negates every odd trace. Thus every curve at the trace under consideration has even second coordinate.

Choose neighboring curves around a target curve with trace $2r$. Their traces are $2\theta u, 2\theta v$ with $u, v > 0$, and the Fricke equation becomes

$$u^2 + v^2 + \lambda^{-1}r^2 = 2ruv.$$

The elements u, v are relatively prime to each other and to r in $\mathcal{R}$. Indeed, divide all traces by two. Farey mutation then has the form $(a, b, c) \mapsto (a, b, 2ab - c)$ and preserves pairwise generation of the unit ideal. The initial coordinates $(\sqrt{2}, \theta, \theta)$ have this property in $\mathbb{Z}[\theta]$, because θ is a unit, satisfying $\theta^4 - 2\theta^2 - 1 = 0$. A nonunit prime of $\mathcal{R}$ remains a nonunit in the integral extension $\mathbb{Z}[\theta]$, by lying over. Hence no such prime can divide any of the indicated pairs in $\mathcal{R}$.

For each odd prime π dividing r , the ratio u/v modulo π is one of the two square roots of -1 . Assign this choice to the neighboring pair. Changing that pair by a twist around the target replaces it by

$$(u, v) \mapsto (v, 2rv - u).$$

Modulo π , this replaces u/v by $-(u/v)^{-1} = u/v$, so every choice is preserved. Interchanging the neighbors reverses all choices simultaneously. Thus there are at most $\max\{1, 2^{\omega-1}\}$ choice classes for target curves.

Consider two target curves in the same class, and order their neighboring pairs (u, v) and (s, t) so that their choices agree at every prime. Set

$$D = vs - ut, F = us - vt.$$

Subtracting the two conic equations gives

$$DF = \lambda^{-1}r^2(st - uv).$$

At any odd $\pi \mid r$, matching choices give $\pi \mid D$ and $F/(vt) = -2$ modulo π . Therefore F is prime to π , and the full π -power dividing r^2 divides D . Moreover $u, v, s, t \equiv 1 \pmod{2}$, so $2 \mid D$, supplying the square of the unique factor above two in r . Hence

$$D/r^2 \in \mathcal{R}.$$

This remains true after arbitrary twists of either neighboring pair, since the prime choices are invariant.

Put $L = \operatorname{arcosh} r$ and $q = e^{-L}$. The quantities

$$\alpha = u - qv, \beta = s - qt$$

are positive, since $(u - qv)(u - q^{-1}v) = -\lambda^{-1}r^2 < 0$, and the first factor is the larger. A twist multiplies α or β by q . We may therefore twist one pair so that $|\log \alpha - \log \beta| \leq L/2$. The conic also shows that every forward or backward twist keeps both coordinates positive. Solving for the coordinates in terms of α and β gives

$$\frac{|D|}{r^2} = \frac{|\sinh(\log \alpha - \log \beta)|}{\lambda \sinh L} \leq \frac{1}{\lambda \sqrt{2r + 2}}.$$

After quadratic conjugation, the conic equation is

$$(u')^2 + (v')^2 - 2r'u'v' = \lambda(r')^2.$$

Its quadratic form is positive definite, with determinant $1 - (r')^2$. Applying its positive square root maps both conjugate neighboring vectors to a circle of squared radius $\lambda(r')^2$. The determinant bound for two points on that circle yields

$$\frac{|D'|}{(r')^2} \leq \frac{\lambda}{\sqrt{1 - (r')^2}}.$$

Consequently

$$|\operatorname{Norm}_{E/\mathbb{Q}}(D/r^2)| \leq \frac{1}{\sqrt{(2r+2)(1-(r')^2)}} < 1.$$

Here the final strict inequality follows from integrality. The two numbers

$$P = \operatorname{Norm}_{E/\mathbb{Q}}(r+1), R = -\operatorname{Norm}_{E/\mathbb{Q}}(r-1)$$

are positive integers, with

$$P + R = 2(r + r'), PR = (r^2 - 1)(1 - (r')^2).$$

Since $r + r' = 2a > 0$ and a is even, $a \geq 2$ and $r > 3$. It follows that

$$PR \geq P + R - 1 > 2r - 3 > (r - 1)/2,$$

which is the required inequality. A nonzero element of $\mathcal{R}$ has nonzero integral norm, so $D = 0$. The two positive neighboring pairs are now proportional. Their conic equations, with the same nonzero constant term, make the proportionality factor one. They are equal.

Equal ordered Farey trace triples determine the same marked hyperbolic torus. Hence the two target curves differ by an isometry. Each prime-choice class therefore contains only one isometry orbit of target curves.

Finally, that orbit contains at most two curves. To see this without a multiplicity assumption, the curve of trace $2\sqrt{2}$ is the unique shortest curve. Each mutation out of the initial triangle creates a trace larger than its previous maximum, and this continues away from that triangle in the dual Farey tree: if $2 < a \leq b \leq c$, replacing either smaller coordinate gives $bc - a > c$ or $ac - b > c$. The initial three mutations also increase the maximum, since $2\sqrt{2} > 2$ and $2\theta > 2\sqrt{2}$. Thus all isometries fix the rational homology direction of this shortest curve. A finite subgroup of $\operatorname{GL}_2(\mathbb{Z})/\{\pm I\}$ fixing a rational direction has order at most two. Indeed, after choosing that primitive vector as the first basis vector and choosing signs, its matrices have the form

$$\begin{pmatrix} 1 & b \\ 0 & \varepsilon \end{pmatrix}, b \in \mathbb{Z}, \varepsilon \in \{1, -1\}.$$

The only finite-order matrix with $\varepsilon = 1$ is the identity, and the product of two distinct matrices with $\varepsilon = -1$ is a nontrivial unipotent matrix of infinite order. Multiplying the number of choice classes by two proves the theorem. $\square$

Corollary 17.2 (A dominant-prime criterion). *Under the hypotheses of Theorem 17.1, select k distinct prime divisors of $r/\sqrt{2}$, retaining their full powers, and let C be the product of all unselected prime powers. If*

$$|\text{Norm}_{E/\mathbb{Q}}(C)|^4 < (2r+2)(1-(r')^2),$$

then the complete level has multiplicity at most $2^{\max\{1,k\}}$. In particular, one sufficiently dominant prime-power factor forces multiplicity at most two, regardless of the number of unselected prime factors.

Proof. Match the root choices only at the selected primes. The divisibility argument in the theorem now gives $DC^2/r^2 \in \mathcal{R}$, because C^2 supplies every omitted prime power. The same twists preserve these selected choices and give

$$|\text{Norm}_{E/\mathbb{Q}}(DC^2/r^2)| \leq \frac{|\text{Norm}_{E/\mathbb{Q}}(C)|^2}{\sqrt{(2r+2)(1-(r')^2)}} < 1.$$

Thus $D = 0$, and the two curves are isometric. There are at most $\max\{1, 2^{k-1}\}$ selected-choice classes modulo simultaneous reversal, and each isometry orbit has at most two curves. $\square$

18 Four-punctured spheres

The same density conclusion holds for every complete finite-area four-punctured sphere. We give the trace correspondence explicitly, including its effect on isometry orbits.

Proposition 18.1 (Length doubling). *There is a real-analytic bijection between the marked spaces of once-punctured tori and four-punctured spheres, after compatible slope markings are chosen, with*

$$\ell_S(v) = 2\ell_X(v)$$

for every primitive unoriented slope v . The correspondence preserves the effective isometry actions on slopes. In particular,

$$m_S(L) = m_X(L/2), C_S(L) = C_X(L/2), D_S(L) = D_X(L/2), O_S(L) = O_X(L/2).$$

Here the effective action means the image of the full isometry group in the permutation group of simple curve classes; its order need not be the order of the isometry group itself.

Proof. Use positive absolute traces at a Farey triangle. The torus equation is $x^2 + y^2 + z^2 = xyz$, and the sphere equation is

$$X^2 + Y^2 + Z^2 - XYZ + 8(X + Y + Z) + 28 = 0.$$

Their trace flips are $z \mapsto xy - z$ and $Z \mapsto XY - Z - 8$, respectively. These coordinates and recursions are given by the explicit gluing constructions in [Han16, Corollary 2.8, Lemma 3.1 and Section 3.1].

Set $X = x^2 - 2$, $Y = y^2 - 2$, $Z = z^2 - 2$. If $s = x^2 + y^2 + z^2$, substitution in the sphere equation gives $(s - xyz)(s + xyz) = 0$. Conversely, for a sphere triple take the positive square roots of $X + 2, Y + 2, Z + 2$. They exceed two and satisfy the torus equation, since $s + xyz > 0$. For completeness, the positive sphere gluing coordinates $a, b > 0$ give the absolute trace triple

$$X + 2 = \frac{(a+1)^2(b+1)^2}{b}, Y + 2 = \frac{(a+1)^2(b+1)}{ab}, Z + 2 = \frac{(a+1)^2(b+1)}{a}.$$

For a given torus triple these expressions agree with x^2, y^2, z^2 when $a = x^2/(y^2 + z^2)$ and $b = z^2/y^2$. Thus both directions belong to the required hyperbolic components. All maps involved are real analytic.

The identity

$$(xy - z)^2 - 2 = (x^2 - 2)(y^2 - 2) - (z^2 - 2) - 8$$

uses the torus equation. Hence the trace correspondence commutes with every Farey flip and holds at every slope. Finally, $T_X(v)^2 - 2 = 2 \cosh \ell_X(v)$ proves length doubling.

We also verify the orbit assertion. Both sets of slope classes carry the natural action of $\text{PGL}_2(\mathbb{Z})$. On the sphere this can be seen by taking the topological double cover of the compactified sphere branched at its four marked points: it is a closed torus, and a simple sphere curve lifts to two parallel primitive torus curves. Homeomorphisms lift, while integral linear maps of the torus descend through its elliptic involution. Thus the full mapping class group has precisely this effective slope action. The analogous

action for the punctured torus is immediate from homology. An element of either effective mapping class group is realized by an isometry precisely when it fixes the marked character. Since a Farey trace triple determines that character, the length-doubling map conjugates these stabilizers. This proves the orbit statement and all counting identities. $\square$

Corollary 18.2 (Density one on every four-punctured sphere). *For every complete finite-area four-punctured sphere S , all but $o_S(L^2)$ simple curves of length at most L belong to length levels consisting of a single full isometry orbit. The same holds when counting distinct levels. If X is its corresponding punctured torus, then the generic orbit size is $h_X \leq 6$ and*

$$D_S(L) \sim \frac{\text{area}(B_X)}{4|K_X|\zeta(2)} L^2.$$

Consequently the multiplicity of a density-one set of levels is exactly h_X . Every pointwise multiplicity or rational-length-class bound proved here for X transfers to S .

Proof. Apply Theorem 1.1 at $L/2$ and use Proposition 18.1. $\square$

A An alternative bound for large matrix entries

The summation above uses joint complexity $J(G) = \max\{1, I(G), h(G)\}$. The following elementary statement also replaces the symmetry-dependent large-entry step in [Che26, Lemma 5.3]. Here height means actual absolute matrix entries, not numerator-denominator height.

Lemma A.1 (Large matrix distortion gives exactly two directions). *Let N be any norm on $\mathbb{R}^2$. Choose $0 < r \leq R$ so that its closed unit ball K satisfies*

$$rB_2 \subset K \subset RB_2,$$

where B_2 is the closed Euclidean unit disk, and put $\rho = R/r$. Let G be a real 2×2 matrix whose singular values satisfy

$$s_1 > 2\rho, s_2 < \frac{1}{2\rho}.$$

Then the equation

$$N(Gv) = N(v), v \neq 0,$$

holds in exactly two unoriented projective directions. In particular, writing $h(G) = \max_{i,j} |G_{ij}|$, this conclusion holds whenever

$$h(G) > 2\rho(1 + |\det G|).$$

No differentiability or strict convexity of N is needed.

Proof. We first record a uniform graph estimate. For any orthogonal coordinates on $\mathbb{R}^2$, the upper and lower boundaries of K above the interval $[-r, r]$ are respectively the graphs of a concave function f_+ and a convex function f_- . The disk inclusions give

$$0 \leq f_+(x) \leq R, -R \leq f_-(x) \leq 0 (|x| \leq r).$$

Each function is $2R/r$ -Lipschitz on $[-r/2, r/2]$. For example, if $-r/2 \leq x < y \leq r/2$, concavity and the values at $-r, r$ give

$$-\frac{2R}{r} \leq \frac{f_+(y) - f_+(x)}{y - x} \leq \frac{2R}{r}.$$

The lower graph follows by applying this argument to $-f_-$. The identical estimate holds for right and left boundaries written as graphs of the second coordinate. All these facts hold after any rotation or reflection of K .

Take a singular-value decomposition

$$G = U \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} V^\top.$$

By homogeneity, it suffices to count solutions with $N(v) = 1$. Write $v = V(x, y)^\top$. Since $v \in K$ and $Gv \in K$, the Euclidean bound by R implies

$$|x| \leq R/s_1 < r/2, |s_2 y| \leq s_2 R < r/2.$$

In V -coordinates the vector v is on one of the two graphs $y = f_\tau(x)$, where $\tau \in \{+, -\}$. In U -coordinates the vector $Gv = (s_1 x, s_2 y)^\top$ is on one of the right or left graphs $s_1 x = g_\sigma(s_2 y)$, where $\sigma \in \{+, -\}$. The central disk excludes vertical segments in the first graph region and horizontal segments in the second; thus every solution has this form. Equivalently, each solution belongs to one of the four equations

$$x = \frac{1}{s_1} g_\sigma(s_2 f_\tau(x)).$$

For two solutions x, x' with the same pair (σ, τ) , the graph estimates yield

$$|x - x'| \leq \frac{(2\rho)^2 s_2}{s_1} |x - x'|.$$

Our singular-value assumptions make the displayed coefficient strictly less than one. Hence $x = x'$, and then $y = y'$. There are at most four vectors on ∂K satisfying the equality. Central symmetry pairs them as $v, -v$, so there are at most two unoriented directions.

There are also at least two. On the two singular directions Ve_1 and Ve_2 , respectively, the norm ratios satisfy

$$\frac{N(GVe_1)}{N(Ve_1)} \geq \frac{s_1}{\rho} > 1, \frac{N(GVe_2)}{N(Ve_2)} \leq \rho s_2 < 1.$$

The two complementary arcs between these directions in $\mathbb{RP}^1$ each contain an equality direction by continuity. Thus there are exactly two.

Finally, $s_1 \geq h(G)$ and $s_1 s_2 = |\det G|$. If the stated height inequality holds, then $s_1 > 2\rho$ and

$$s_2 = \frac{|\det G|}{s_1} < \frac{1}{2\rho}.$$

This proves the final assertion. □

B Additional literature

The arithmetic background begins with Markoff's work on indefinite binary quadratic forms [Mar79, Mar80] and Frobenius's study of Markoff numbers [Fro13]; Aigner's monograph [Aig13] gives a modern account of the uniqueness problem. The geometric interpretation developed through Cohn's work [Coh55, Coh71, Coh72], Gorshkov's arithmetic hyperbolic geometry [Gor81], and Series's exposition [Ser85a]. Borosh [Bor75] gives early computational evidence, while Baragar [Bar96] studies the unicity conjecture. Arithmetic approaches to special cases include the papers of Schmutz [Sch96b], Button [But98, But01], Lang–Tan [LT07], F.-J. Chen–Y.-G. Chen [CC13], Zhang [Zha07], and Srinivasan [Sri09]. These works provide the pointwise arithmetic background to the density question.

Orderings and convexity of Markoff numbers are studied by Rabideau–Schiffler [RS20], Lee–Li–Rabideau–Schiffler [LLRS23], Lagisquet–Pelantová–Tavenas–Vuillon [LPTV21], McShane [McS21], and Gaster [Gas22]. Other aspects of the equation include integral necklaces [Fis25], Lyapunov spectra of the Markov and Euclid trees [SV17], its relation to T-singularities [Per22], and partition regularity [TY25]. The BIRS report [FGMR25] surveys current perspectives. Generalized Markoff equations and their orderings are developed by Gyoda–Matsushita [GM23], Banaian–Sen [BS24], and Banaian [Ban26]. Our multiplicity statements are proved for the hyperbolic trace families specified in the theorems.

Stable norms belong to a broader theory of minimizing geodesics and action-minimizing measures. Relevant foundations include Federer's flat chains [Fed74], Gromov's metric geometry [Gro81], Mather's minimizing measures [Mat91], Bangert's work on twist maps, minimal geodesics and Busemann functions [Ban88, Ban90, Ban94], and Massart's description at rational directions [Mas97]. Makover–Parlier–Sutton [MPS12] construct torus metrics with partially prescribed stable norms. Rockafellar [Roc70] and Schneider [Sch14] provide the convex-geometric background; Jarník [Jar26] and Andrews [And63] are classical sources on boundary lattice points.

For surfaces of higher complexity, Thurston's measured-lamination framework [FLP79] and Bonahon's geodesic currents [Bon88] replace the two-dimensional homology model. Rivin [Riv01], Mirzakhani

[Mir04, Mir08], Erlandsson–Souto [ES16, ES22], and Eskin–Mirzakhani–Mohammadi [EMM22] develop curve counting in this broader setting. McShane’s identity [McS98] and Bowditch’s Markoff-triple approach [Bow96, Bow98] give a complementary use of the same recursive curve structure. The surveys of Schmutz Schaller [Sch98], Farb [Far06], and Parlier [Par14] place simple-length questions in the study of surfaces and moduli.

The dependence on rational approximation discussed in [DLN26] also has broader dynamical context: Series relates the modular surface to continued fractions [Ser85b]; Sullivan [Sul82], Dani [Dan85], and Kleinbock–Margulis [KM98] study approximation through homogeneous or geodesic dynamics. Avila’s almost-reducibility and global-theory papers [Avi10, Avi15] concern one-frequency Schrödinger operators. They are contextual comparisons here; our normal-turn estimates come from the explicit Farey geometry. Classical references for the continued-fraction, arithmetic, local-field and free-group background are [Khi97, Cas57, HW08, Cas86, MKS76].

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