

BOUNDS FOR ERDŐS PROBLEM #188: $7 \leq k \leq 60$

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ABSTRACT. Let ℓ_k denote k collinear points with consecutive distances 1. Erdős Problem #188 asks for the least k such that the plane can be colored red and blue with no two red points at distance 1 and no blue copy of ℓ_k ; the bounds $6 \leq k \leq 6330$ were known. We prove $7 \leq k \leq 60$. For the lower bound we show that every red/blue coloring of the plane contains two red points at distance 1 or a blue ℓ_6 , following the skeleton of Arman and Tsaturian's proof in three dimensions; four distance lemmas are reduced to explicit unsatisfiable propositional formulas, certified by DRAT proofs. For the upper bound we give an explicit periodic coloring whose red set consists of 90 closed regular 24-gons of 48 different sizes, all of diameter less than 0.999, in each period, with rational centers and sizes. Any two cells are either far apart (all distances between their points exceed 1) or close together (all distances are less than 1), which is checked in exact arithmetic and rules out red unit pairs. That every ℓ_{60} contains a red point is proved by a computer program that shows that 60 translates of the red set cover the torus, with every inclusion checked in ball arithmetic. A second, separately written program, based on a reduction to one-dimensional covering problems solved in outward-rounded interval arithmetic, confirms this. The computations, the verification code and drafts of the text were produced with AI tools under the author's direction; all proof certificates and verification programs are public.

Changes in this version (2026-10-15). Corrections to the text only; no theorem, bound, certificate or program changes. Section 5.5 now states that the value of the coloring is exactly 60, with the blue ℓ_{59} that the package already contained (the previous version said the exact value was not known), and Section 5.6 refers to it. The definition of the search score in Section 5.6 was off by one (it counted translates instead of one less than that count); the definition is corrected and called a numerical proxy, and the numbers in that section are unchanged. Section 5.2 now says that the 24 boxes refined twice are main-pass boxes. The package DOI cited is now the all-versions DOI.

1. INTRODUCTION

For finite configurations $K, K' \subset \mathbb{E}^n$, write $\mathbb{E}^n \rightarrow (K, K')$ if every red/blue coloring of $\mathbb{E}^n$ contains a red congruent copy of K or a blue congruent copy of K' , and $\mathbb{E}^n \not\rightarrow (K, K')$ otherwise. Here ℓ_k is the configuration of k collinear points with distance 1 between consecutive points; a copy of ℓ_2 is a pair of points at distance 1. Erdős Problem #188 [4] asks for the smallest k such that the plane can be colored red and blue with no two red points at distance 1 and no blue ℓ_k ; that is, the smallest k with $\mathbb{E}^2 \not\rightarrow (\ell_2, \ell_k)$.

Before this note the known bounds were $6 \leq k \leq 6330$: Tsaturian [26] proved $\mathbb{E}^2 \rightarrow (\ell_2, \ell_5)$, and Currier, Mody, Xie and Zhang proved $\mathbb{E}^2 \not\rightarrow (\ell_2, \ell_{6330})$ [10, Theorem 1.2]. Their planar coloring is built from the Voronoi cells of a hexagonal lattice, scaled to have diameter slightly less than 1, with red cells chosen at random subject to a local rule that rules out red unit pairs. Section 1.1 places the problem among the known results on relations $\mathbb{E}^n \rightarrow (\ell_s, \ell_t)$ and their negations. We prove the following.

Theorem 1. *Every red/blue coloring of $\mathbb{E}^2$ contains two red points at distance 1 or a blue ℓ_6 . That is, $\mathbb{E}^2 \rightarrow (\ell_2, \ell_6)$.*

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Theorem 2. *There is a red/blue coloring of $\mathbb{E}^2$, periodic with respect to a lattice, in which no two red points are at distance 1 and every copy of ℓ_{60} contains a red point. That is, $\mathbb{E}^2 \not\rightarrow (\ell_2, \ell_{60})$.*

Corollary 3. *The answer k to Erdős Problem #188 satisfies $7 \leq k \leq 60$.*

For Theorem 1 we follow Arman and Tsaturian [2]: first exclude red pairs at a few distances, then find a monochromatic circle. Their distance lemmas use three-dimensional circles. In the plane we obtain the distance lemmas instead from the unsatisfiability of four propositional formulas on explicit finite point sets built from rotated patches of the triangular lattice (Sections 2 and 3). Remark 6 records what we know about the next case, $\mathbb{E}^2 \rightarrow (\ell_2, \ell_7)$.

For Theorem 2 the red set is a periodic union of convex cells of diameter less than 1, as in [10], where the cell framework and the hexagonal cells come from. Our cells are nearly round (regular 24-gons), they have 48 different sizes, and their centers are not on a lattice: 90 cells at explicit rational centers in each period of a lattice Γ (Section 4). That no two red points are at distance 1 follows from a simple rule, checked in exact arithmetic: every two cells are either far apart or close together (Lemma 8). That there is no blue ℓ_{60} is proved by computer (Section 5): the space of all copies of ℓ_{60} modulo the period is compact, and a program checks, with every inclusion certified in ball arithmetic, that finitely many translates of the red set cover the torus (Method 2). A second, separately written program with a different reduction (Method 1) confirms the result.

History. An earlier draft of this note proved $k \leq 166$ with a coloring of the kind used in [10]: 36 red closed regular hexagons of diameter $19/20$ among the Voronoi cells of a hexagonal lattice, in a 19×19 period. Cells that are nearly disks of diameter just below 1, centers that are free rational points instead of lattice points, and the insertion of cells into the largest holes then gave $k \leq 107$ and $k \leq 85$ (42 equal cells per period). Red cells of different sizes, with smaller cells in the gaps between the large ones, gave $k \leq 69$ and then $k \leq 64$, and further search in the same setting gave $k \leq 63$. The step from 63 to 62 comes from relaxing the rule that all cells are far apart: a small cell (a *satellite*) may sit close to another cell, so close that all distances between the two are less than 1 (Lemma 8). The targeted insertion of one further cell gave $k \leq 61$, and a search in which satellites are inserted, moved and resized gave $k \leq 60$. The searches are described in Section 5.6; they play no role in the proofs.

1.1. Background. Relations of the form $\mathbb{E}^n \rightarrow (K, K')$ were introduced by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus in the series [15, 16, 17]; the case of ℓ_2 against a set K is their question of which K must appear in blue when the red points span no unit distance. (With more colors and ℓ_2 in every color one obtains the Hadwiger–Nelson problem; at least five colors are needed for the plane [13].) A coloring of $\mathbb{E}^n$ restricts to every plane, or subspace, of $\mathbb{E}^n$, so a relation $\mathbb{E}^d \rightarrow (K, K')$ implies $\mathbb{E}^n \rightarrow (K, K')$ for all $n \geq d$, and $\mathbb{E}^n \not\rightarrow (K, K')$ implies $\mathbb{E}^d \not\rightarrow (K, K')$ for $d \leq n$. Table 1 lists the results we know of; most are collected in the introduction of [10], to which we refer for further discussion.

ℓ_2 against ℓ_k in fixed dimension. In the plane, Erdős et al. [16] proved $\mathbb{E}^2 \rightarrow (\ell_2, K)$ for every 3-point set K and $\mathbb{E}^2 \rightarrow (\ell_2, \ell_4)$; Juhász [23] extended the first result to every 4-point set, and Tsaturian [26] proved $\mathbb{E}^2 \rightarrow (\ell_2, \ell_5)$. For general sets there are small exceptions: Juhász [23] gave a 12-point set K with $\mathbb{E}^2 \not\rightarrow (\ell_2, K)$, and Csizmadia and Tóth [9] an 8-point one, a regular heptagon with its center. In three dimensions Iván [21] proved, as reported in [2], $\mathbb{E}^3 \rightarrow (\ell_2, K)$ for every 5-point set K , and Arman and Tsaturian proved $\mathbb{E}^3 \rightarrow (\ell_2, \ell_6)$ [2] and $\mathbb{E}^n \rightarrow (\ell_2, \ell_{n+3})$ for $n \geq 4$ [1].

Growth with the dimension. Szlam [25] (see [5]) showed, using the Frankl–Wilson theorem [18], that every red/blue coloring of $\mathbb{E}^n$ contains a red ℓ_2 or a blue translate of any given set of at most 2^{cn} points; with Raigorodskii’s bound [24] this gives $\mathbb{E}^n \rightarrow (\ell_2, K)$ for $|K| \leq (1.239 \dots + o(1))^n$ [10, Section 1]; here and below $o(1)$ is as $n \rightarrow \infty$. Conversely, Conlon and Fox [5] proved $\mathbb{E}^n \not\rightarrow (\ell_2, K)$

TABLE 1. Known results on $\mathbb{E}^n \rightarrow (\ell_s, K)$ and $\mathbb{E}^n \nrightarrow (\ell_s, K)$ (red ℓ_s , blue K). Positive results in $\mathbb{E}^d$ hold in all $\mathbb{E}^n$, $n \geq d$. Rows with $o(1)$ are asymptotic statements, $o(1) \rightarrow 0$ as $n \rightarrow \infty$.

Relation	Dimension	Reference
$\rightarrow (\ell_2, K)$, every K with $ K = 3$; $\rightarrow (\ell_2, \ell_4)$	$n = 2$	[16]
$\rightarrow (\ell_2, K)$, every K with $ K = 4$	$n = 2$	[23]
$\rightarrow (\ell_2, \ell_5)$	$n = 2$	[26]
$\rightarrow (\ell_2, \ell_6)$	$n = 2$	Theorem 1
$\nrightarrow (\ell_2, K)$ for some K with $ K = 12$; $ K = 8$	$n = 2$	[23]; [9]
$\nrightarrow (\ell_2, \ell_{6330})$	$n = 2$	[10, Thm. 1.2]
$\nrightarrow (\ell_2, \ell_{60})$	$n = 2$	Theorem 2
$\rightarrow (\ell_2, K)$, every K with $ K = 5$	$n = 3$	[21] (see [2])
$\rightarrow (\ell_2, \ell_6)$	$n = 3$	[2]
$\rightarrow (\ell_2, \ell_{n+3})$	$n \geq 4$	[1]
$\rightarrow (\ell_2, K)$, every K with $ K \leq (1.239 \dots + o(1))^n$	$n \rightarrow \infty$	[25, 24, 10]
$\nrightarrow (\ell_2, \ell_m)$, $m \geq 10^{5n}$	all n	[5]
$\nrightarrow (\ell_2, \ell_m)$, $m > (5 + o(1))^n$	$n \rightarrow \infty$	[10, Thm. 1.1]
$\rightarrow (\ell_3, \ell_3)$	$n = 2$	[11]
$\nrightarrow (\ell_3, \ell_{10^{50}})$; $\nrightarrow (\ell_3, \ell_{1177})$; $\nrightarrow (\ell_3, \ell_{20})$	all n	[6]; [19]; [12]
$\nrightarrow (\ell_4, \ell_{14})$; $\nrightarrow (\ell_5, \ell_8)$	all n	[12]
$\nrightarrow (\ell_6, \ell_6)$	all n	[15, Thm. 12]

for every 1-separated set K of diameter at most $R - 1$ with $|K| > 10^{4n} \log_2 R$ ($R > 2$), in particular $\mathbb{E}^n \nrightarrow (\ell_2, \ell_m)$ for $m \geq 10^{5n}$; this answered a question of Erdős et al. [16]. Currier, Mody, Xie and Zhang improved this to $m > (5 + o(1))^n$ ([10, Theorem 1.1] and the remark following it). In the plane, Erdős and Graham [14] remarked without proof that some m “at most 10000000” has $\mathbb{E}^2 \nrightarrow (\ell_2, \ell_m)$ (we quote this from [10, Section 1]; see also [4]); the argument of [5] gives m of order 10^{10} , and [10, Theorem 1.2] gives $m = 6330$.

Longer red progressions. Currier, Moore and Yip [11] proved that every red/blue coloring of the plane contains a monochromatic ℓ_3 , that is, $\mathbb{E}^2 \rightarrow (\ell_3, \ell_3)$. Conlon and Fox, and independently Arman and Tsaturian, asked whether there is an m with $\mathbb{E}^n \nrightarrow (\ell_3, \ell_m)$ for every n ; Conlon and Wu [6] showed that there is, with $m = 10^{50}$ (as stated in [19]), Fűhrer and Tóth [19] lowered m to 1177, and Currier, Moore and Yip [12] to 20. The latter also proved $\mathbb{E}^n \nrightarrow (\ell_4, \ell_{14})$ and $\mathbb{E}^n \nrightarrow (\ell_5, \ell_8)$ for all n , while $\mathbb{E}^n \nrightarrow (\ell_6, \ell_6)$ goes back to [15, Theorem 12].

Colorings. The negative results use geometric coloring constructions, sometimes with probabilistic choices. Those of [15] color a point x by the residue of an integer part of a multiple of $|x|^2$. In [5, 10] space is partitioned into cells of diameter less than 1, cells are colored red at random subject to a rule that prevents red unit pairs, and a probabilistic argument shows that some such coloring has no blue copy of the forbidden configuration. The coloring of the present note is explicit: its red set is a periodic union of cells of several sizes (regular 24-gons, which do not partition the plane), any two of which are far apart or close together, and that every copy of ℓ_{60} meets a red cell is checked by computer; [12] gives a general framework for avoiding short progressions. Erdős Problem #188 is the planar case of the (ℓ_2, ℓ_k) question: before this note $6 \leq k \leq 6330$ was known, and Theorems 1 and 2 give $7 \leq k \leq 60$.

2. PROOF OF THE LOWER BOUND

Call a red/blue coloring of the plane *good* if it contains no two red points at distance 1 and no blue ℓ_6 . We show that no good coloring exists.

Lemma 4. *In a good coloring, no two red points are at distance d , for each $d \in \{4, 3, 2, 6\}$.*

The four cases are proved in the order $d = 4, 3, 2, 6$, and the case d uses only the cases before it.

Proof. Identify the plane with $\mathbb{C}$. For each $d \in \{4, 3, 2, 6\}$ let $D'(d)$ be the set of earlier distances: $D'(4) = \emptyset$, $D'(3) = \{4\}$, $D'(2) = \{4, 3\}$, $D'(6) = \{4, 3, 2\}$. In Section 3 we give a finite set $S_d \subset \mathbb{C}$ containing 0 and d together with a propositional formula F_d in one variable r_x (“ x is red”) for each $x \in S_d$. The formula F_d is a conjunction of clauses of the following four kinds:

- (i) the unit clauses r_0 and r_d ;
- (ii) $\neg r_x \vee \neg r_y$ for certain pairs $x, y \in S_d$ with $|x - y| = 1$;
- (iii) $r_x \vee r_{x+u} \vee \dots \vee r_{x+5u}$ for certain $x \in S_d$ and $|u| = 1$ with all six points in S_d ;
- (iv) $\neg r_x \vee \neg r_y$ for certain pairs $x, y \in S_d$ with $|x - y| \in D'(d)$.

Each F_d is unsatisfiable (Section 3).

Suppose the case d fails, so a good coloring has red points p, q with $|p - q| = d$, and assume the earlier cases hold. The map $\phi(z) = p + (q - p)z/d$ is an orientation-preserving isometry of $\mathbb{C}$ with $\phi(0) = p$ and $\phi(d) = q$. Set r_x true if and only if $\phi(x)$ is red. Clauses (i) hold because p, q are red. Clauses (ii) hold because the coloring has no red pair at distance 1. Clauses (iii) hold because ϕ maps $\{x, x + u, \dots, x + 5u\}$ onto a copy of ℓ_6 , which is not all blue. Clauses (iv) hold by the earlier cases. So F_d is satisfiable, a contradiction. $\square$

Proof of Theorem 1. Suppose a good coloring exists. Not every point is blue, since the plane contains an ℓ_6 ; let A be red. Let u be any unit vector. The points $A + u, A + 2u, \dots, A + 6u$ form an ℓ_6 , so one of them is red. Its distance from A is $j \in \{1, \dots, 6\}$. The case $j = 1$ is excluded because the coloring has no red pair at distance 1, and $j \in \{2, 3, 4, 6\}$ is excluded by Lemma 4. Hence $A + 5u$ is red for every unit vector u , so the whole circle of radius 5 about A is red. This circle contains two points at distance 1, a contradiction. $\square$

3. THE POINT SETS AND THE CERTIFICATES FOR THE LOWER BOUND

Let $\omega = e^{i\pi/3}$, so $\omega^2 = \omega - 1$, and let

$$H_R = \{a + b\omega : a, b \in \mathbb{Z}, \max(|a|, |b|, |a + b|) \leq R\},$$

a hexagonal patch of the triangular lattice with $3R^2 + 3R + 1$ points. We use two rotations by unit complex numbers,

$$\rho_M = \frac{5 + \sqrt{-11}}{6} \quad (\cos = 5/6), \quad \rho_7 = \frac{5 + 3\omega}{7} \quad (\cos = 13/14).$$

The first is the Moser-spindle angle, the rotation about the origin that moves a lattice point at distance $\sqrt{3}$ by exactly 1. The second moves a lattice point at distance $\sqrt{7}$ by exactly 1. With $\bar{\rho}$ denoting complex conjugation, define the Minkowski unions

$$\begin{aligned} S &= H_6 \cup (H_6 + \rho_M H_3) \cup (H_6 + \bar{\rho}_M H_3), \\ S' &= S \cup (H_6 + \rho_7 H_3) \cup (H_6 + \bar{\rho}_7 H_3). \end{aligned}$$

We take $S_4 = S_3 = S$ (8,899 points) and $S_2 = S_6 = S'$ (11,797 points). All points lie in the field $K = \mathbb{Q}(\omega, \sqrt{-11})$ of degree 4 over $\mathbb{Q}$; each point is stored exactly as an integer 4-tuple (a, b, c, e)

representing $((a + b\omega) + (c + e\omega)\sqrt{-11})/42$. Since $\{1, \omega, \sqrt{-11}, \omega\sqrt{-11}\}$ is a $\mathbb{Q}$ -basis of K , equal tuples are equal points, and with the embedding $\sqrt{-11} \mapsto i\sqrt{11}$,

$$4 \cdot 42^2 |z|^2 = (2a + b)^2 + 3b^2 + 11(2c + e)^2 + 33e^2 + 4(bc - ae)\sqrt{33},$$

so every distance condition in the formulas can be checked in integer arithmetic.

The clauses use a finite set U of unit vectors (up to sign): all unit differences between points of the set within distance 2.2 of the origin, each verified to have norm exactly 1 in K . This gives $|U| = 15$ for S and $|U| = 21$ for S' . The clauses of type (ii) are the pairs $\{x, x + u\}$, type (iii) the progressions $x, x + u, \dots, x + 5u$, and type (iv) the pairs $\{x, x + d'u\}$ for $d' \in D'(d)$, over all $u \in U$ and all x such that the points lie in the set. Using only some of the valid constraints is harmless: any clause set in which every clause is of type (i)–(iv) suffices for the proof of Lemma 4.

Lemma	Points	$ U $	Clauses (ii)	(iii)	(iv)	Total	DRAT proof
$d = 4$	8,899	15	62,004	16,380	—	78,386	57.6 MB
$d = 3$	8,899	15	62,004	16,380	22,863	101,249	3.3 MB
$d = 2$	11,797	21	79,368	27,672	89,181	196,223	3.8 kB
$d = 6$	11,797	21	79,368	27,672	150,624	257,666	1.9 kB

Lemma	Earlier distances	Solve time	Check time	Clauses in core
$d = 4$	—	1854 s	464 s	46,695
$d = 3$	4	106 s	107 s	18,776
$d = 2$	4, 3	3.4 s	68 s	117
$d = 6$	4, 3, 2	5.4 s	85 s	108

TABLE 2. The four formulas. Totals include the two unit clauses (i). The type (iv) counts split as $35,559 + 53,622$ (distances 4, 3) for $d = 2$ and $35,559 + 53,622 + 61,443$ (distances 4, 3, 2) for $d = 6$. Proofs are binary DRAT produced by CaDiCaL 1.5.3 [3] through PySAT [20]; checks are by `drat-trim` [27]. Times are wall-clock on a shared laptop and are only indicative; with local copies of the files, `drat-trim` checks $d = 4$ in about 260 s and each of the others in under 5 s.

Remark 5. The two smallest proofs have unsatisfiable cores of 117 and 108 clauses, small enough in principle to be written out as a case analysis by hand. The case $d = 4$ is the hard one: its core has 46,695 clauses, and it is the only lemma that uses no earlier distance.

The certificates were checked by the construction pipeline; by a separate verifier written from the data files alone, with its own field arithmetic and its own solvers (Glucose and Lingeling), which also produced and checked its own DRAT proofs for $d = 2, 3, 6$; by a separate AI-assisted review, which re-ran `drat-trim` on all four proofs; and by an independent re-check with a different AI system, which regenerated the two point sets, checked every clause in exact arithmetic and re-solved all four formulas as unsatisfiable with its own SAT solver. None found an error. We also converted the four DRAT proofs to LRAT with `drat-trim -L` and checked them with `lrat-check`; the LRAT files are in the public package, so they can also be checked by a formally verified LRAT checker [8], which we have not done.

Remark 6 (Towards $k \geq 8$). For $t > 0$ let L_t be the statement: a red/blue coloring of the plane with no red pair at distance 1 and no blue ℓ_7 has no two red points at distance t . With ℓ_7 in place of ℓ_6 in the clauses (iii), the method of Lemma 4 gives the following conditional statements; for each, the clauses were checked in exact arithmetic by a separate script and a DRAT (or

DRUP) proof was verified by `drat-trim`. On the set S' , L_3 implies each of L_2, L_4, L_6, L_7 , and L_7 implies L_3 ; and a formula on 43 points (a red point 0 and the points ju , $1 \leq j \leq 7$, for six exact unit vectors u) shows that a coloring with no red pair at distance 1, 2, 3, 4 or 6 has a blue ℓ_7 . Hence $k \geq 8$ would follow from the single lemma L_3 . In the larger $\mathbb{Z}$ -module $M = \mathbb{Z}[\omega] + \sum_{\rho} (\rho \mathbb{Z}[\omega] + \bar{\rho} \mathbb{Z}[\omega])$, $\rho \in \{\rho_M, \rho_7, \rho_{10}\}$ with $\rho_{10} = (1 + 3\sqrt{-11})/10$ (cosine $1/10$), which lies in K and contains 63 unit vectors up to sign, the same method proves $L_{\sqrt{21}} \Rightarrow L_3$, $L_3 \Rightarrow L_{\sqrt{21}}$ and $L_{\sqrt{19}} \wedge L_{\sqrt{21}} \Rightarrow L_3$ (on sets of 20,941, 20,941 and 11,503 points built like S'). So $L_{\sqrt{21}}$ would also suffice. We have proved neither L_3 nor $L_{\sqrt{21}}$. The $\mathbb{Z}$ -module M' generated by S' has a homomorphism $\phi: M' \rightarrow \mathbb{Z}/5$ that is nonzero on every unit vector of M' (given explicitly, with a checker, in the package [7]), and coloring $x \in M'$ red if and only if $\phi(x) = 0$ gives no red unit pair and no blue ℓ_7 inside M' , since five consecutive points of a unit progression in M' meet every residue class. So no finite subset of M' can yield a contradiction from the constraints “no red unit pair” and “no blue ℓ_7 ” together with one prescribed red point; in particular $\mathbb{E}^2 \rightarrow (\ell_2, \ell_7)$ cannot be proved in this way inside M' . In M this particular obstruction disappears: $\rho_{10} + \bar{\rho}_{10} = 1/5$ lies in M , so every homomorphism $M \rightarrow \mathbb{Z}/5$ vanishes on the unit vector 1. This does not show that some finite subset of M gives an unsatisfiable formula.

4. THE COLORING FOR THE UPPER BOUND

Let

$$P_1 = \left(\frac{15620641}{10^6}, 0 \right), \quad P_2 = \left(\frac{7980039}{10^6}, \frac{13537743}{10^6} \right), \quad \Gamma = \mathbb{Z}P_1 + \mathbb{Z}P_2,$$

and write $L = 15.620641$, $H = 13.537743$ and $P_2 = (P_{2x}, H)$ with $P_{2x} = 7.980039$; so P_2 points in the direction $59.48 \dots^\circ$. The fundamental parallelogram $F = \{\alpha P_1 + \beta P_2 : \alpha, \beta \in [0, 1)\}$ spanned by P_1 and P_2 is a fundamental domain of Γ , of area $LH = 211.468 \dots$.

For $0 \leq k \leq 11$ let $n_k = (\cos \psi_k, \sin \psi_k)$ with $\psi_k = (2k + 1)\pi/24$, and for $h > 0$ let

$$C(h) = \{q \in \mathbb{R}^2 : |n_k \cdot q| \leq h \text{ for } k = 0, \dots, 11\},$$

the closed regular 24-gon with center 0, inradius (apothem) h and circumradius $R(h) = h/\cos(\pi/24)$, whose vertices point in the directions $15m^\circ$. Thus $D(0, h) \subseteq C(h) \subseteq D(0, R(h))$, where $D(z, r)$ is the closed disk of radius r about z . Let $c_1, \dots, c_{90}$ be the rational points and $h_1, \dots, h_{90}$ the rational apothems listed in Appendix A, put $R_i = R(h_i)$, and define

$$\text{RED} = \bigcup_{g \in \Gamma} \bigcup_{i=1}^{90} (c_i + g + C(h_i)), \quad \text{BLUE} = \mathbb{R}^2 \setminus \text{RED}.$$

We call the sets $c_i + g + C(h_i)$ the *red cells* and $c_i + g$ their *centers*. The coloring is invariant under translation by Γ . There are 48 different apothems: 39 of the cells have apothem $h = 619/1250 = 0.4952$, and the others have apothems between 0.0302 and 0.466. Every h_i is at most 0.4952, so $R_i < 0.4994731$ and every cell has diameter $2R_i < 0.998947$. Figure 1 shows one period. The same data are in the file `cfg_k60.json` of the package [7] (SHA-256 `b15251de...7b2e`).

Remark 7 (Other cell shapes). Claim A below uses only $c_i + g + C(h_i) \subseteq D(c_i + g, R_i)$, and the second proof of Claim B (Section 5.3) uses only $D(c_i + g, h_i) \subseteq c_i + g + C(h_i)$. So Theorem 2, with the bound 60, holds for every coloring whose red set is $\bigcup_{g,i} (C_i + g)$ with $D(c_i, h_i) \subseteq C_i \subseteq D(c_i, R_i)$, for instance for the 24-gons in any orientation or for the closed disks of radius h_i . The inclusion tests of the second proof are strict, so it in fact uses only the open disks: $\text{RED} \supseteq \bigcup_{g,i} D^\circ(c_i + g, h_i)$, where $D^\circ(z, r)$ is the open disk (Section 5.3). Hence the same conclusion holds when $D(c_i, h_i)$ is replaced by $D^\circ(c_i, h_i)$ in the hypothesis, for instance for the open disks of radius h_i . The first proof of Claim B (Section 5.2) is written for the 24-gons $C(h_i)$ above.

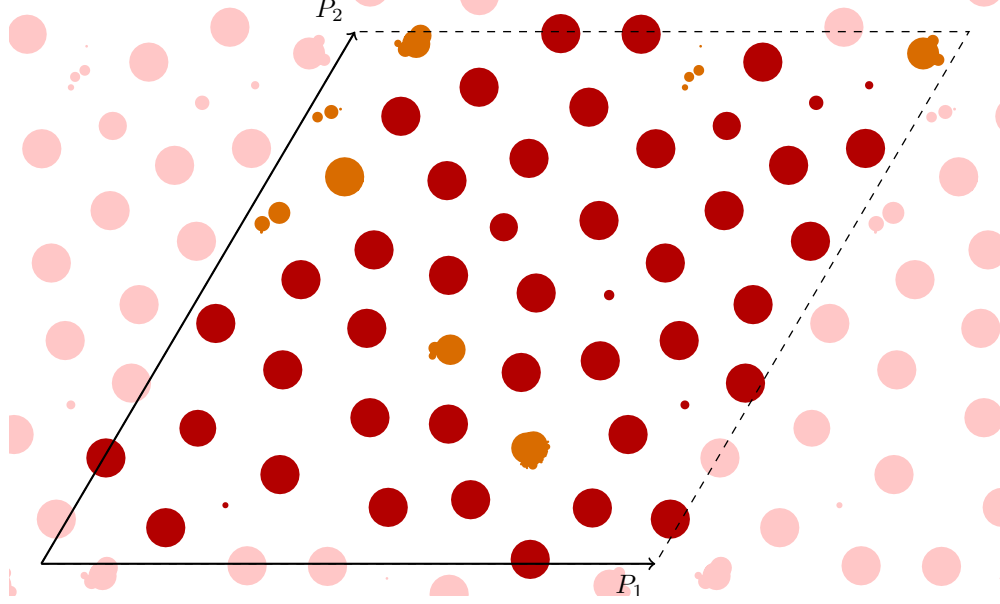


FIGURE 1. Illustration of one period of the coloring, drawn in floating point from `cfg_k60.json`. The dashed parallelogram is the fundamental domain F spanned by P_1 and P_2 ; the dark cells are the cells $c_i + g_i + C(h_i)$, with $g_i \in \Gamma$ chosen so that the center lies in F , and the light ones are some of their other translates. The 44 cells that belong to near pairs are drawn in orange.

Claim A: no two red points are at distance 1. The cells of $[10]$ are pairwise far apart. We also allow pairs of cells that are close together.

Lemma 8 (Near or far). *Let $\text{RED} = \bigcup_{\alpha \in A} Q_\alpha$, where each piece Q_α lies in a closed disk $D(z_\alpha, r_\alpha)$ with $2r_\alpha < 1$. Call two distinct indices α, β far if $|z_\alpha - z_\beta| > 1 + r_\alpha + r_\beta$, and near if $|z_\alpha - z_\beta| + r_\alpha + r_\beta < 1$. If every two distinct indices are far or near, then no two points of RED are at distance 1.*

Proof. Let $p \in Q_\alpha$ and $q \in Q_\beta$. If $\alpha = \beta$, then $|p - q| \leq 2r_\alpha < 1$. Otherwise $|p - z_\alpha| \leq r_\alpha$ and $|q - z_\beta| \leq r_\beta$, so $|z_\alpha - z_\beta| - r_\alpha - r_\beta \leq |p - q| \leq |z_\alpha - z_\beta| + r_\alpha + r_\beta$. If α, β are far, the left-hand side exceeds 1; if they are near, the right-hand side is less than 1. $\square$

So the distances between the points of two far pieces are all greater than 1, and those between the points of two near pieces are all less than 1. Near pieces may overlap; the lemma does not need disjoint pieces. The extra freedom matters: a small cell can sit next to a large one, inside the annulus of width 1 around it from which all far cells are excluded.

Proposition 9 (Claim A). *The red cells, with $z = c_i + g$ and $r = R_i$, satisfy the hypothesis of Lemma 8. Hence no two red points are at distance 1. There are 204 unordered near pairs among the 90 listed cells $c_i + C(h_i)$ of Appendix A, and none of them involves a nonzero lattice translate (all have $g = 0$). They involve 44 of the 90 cells and fall into 8 groups. In 7 of them any two cells are near, so any two points of such a group are at distance less than 1; the largest of these groups has 19 cells, the others 3 to 5. The eighth group consists of the cells 43, 56, 58, 61, in which every pair except $\{58, 61\}$ is near. Every other pair of cells is far.*

Proof. Let c^+ be a rational number with $c^+ > 1/\cos(\pi/24)$ (checked in Arb) and put $R_i^+ = h_i c^+$, so $R_i < R_i^+$, and $2R_i^+ < 0.998947$ is checked exactly. Two cells that are far, or near, with the

radii R_i^+ are far, or near, with the radii R_i . By periodicity it suffices to consider pairs of cells with centers $c_i + g$ and c_j , $g \in \Gamma$, $(i, g) \neq (j, 0)$. Let X and Y be the exact widths of the ranges of the x - and y -coordinates of the centers $c_1, \dots, c_{90}$ and put $D = X + Y + 1 + 2 \max_i R_i^+$. If $|c_i + g - c_j| \leq 1 + R_i^+ + R_j^+$, then $|g| \leq |c_i - c_j| + 1 + 2 \max_i R_i^+ \leq D$; all other pairs are far. A program (`claimA_nf.py`) lists the 59 translates $g = aP_1 + bP_2$ with $|b| \leq \lfloor D/H \rfloor + 1$ and, for each such b , $\lfloor (-D - bP_{2x})/L \rfloor - 1 \leq a \leq \lfloor (D - bP_{2x})/L \rfloor + 1$; these contain every g whose two coordinates are at most D in absolute value, hence every g with $|g| \leq D$. For each such g and each pair (i, j) with $(i, g) \neq (j, 0)$ (477,810 triples) it computes $q = |c_i + g - c_j|^2$ in exact rational arithmetic. It classifies the pair as near if $1 - R_i^+ - R_j^+ > 0$ and $q < (1 - R_i^+ - R_j^+)^2$, and otherwise requires $q > (1 + R_i^+ + R_j^+)^2$, that is, that the pair is far. All triples pass. The smallest margin among the far pairs is attained for $(i, j) = (58, 61)$ and $g = 0$, with $|c_i - c_j|^2 = 626837873549/5000000000000$, so $|c_i - c_j| > 1.119676$, against $1 + R_{58}^+ + R_{61}^+ < 1.114581$. The smallest near margin is $(1 - R_6^+ - R_{80}^+)^2 - d^2 > 4.27 \cdot 10^{-3}$, for the cells 6 and 80 (apothems 0.4952 and 0.0793, $d = 0.41542\dots$). $\square$

A second program, `indep_nearfar.py`, which was written separately, in an AI-assisted review of `claimA_nf.py`, checks the same statement in exact rational arithmetic only: it bounds π by Machin's formula and $\cos(\pi/24)$ from below by a truncated Taylor series, so it uses its own rational upper bound for the R_i ; and it uses its own window: for each pair (i, j) , every $g \in \Gamma$ for which both coordinates of $c_i + g - c_j$ are at most $1 + 2 \max_i R_i$ in absolute value. It finds no pair that is neither near nor far, the same 204 near pairs and the same closest far pair.

5. CLAIM B: EVERY ℓ_{60} CONTAINS A RED POINT

Proposition 10 (Claim B). *For every $x \in \mathbb{R}^2$ and every unit vector u , at least one of the points $x, x + u, \dots, x + 59u$ is red. Hence the coloring of Section 4 has no blue ℓ_{60} .*

Proposition 10 is proved by computation with Method 2 (Section 5.3). Method 1 (Section 5.2) is a separately written second check with a different reduction; it confirms the proposition. We first state the common reduction, which is rigorous, and then the two methods and what exactly each of them checks.

5.1. Common reduction. Write $u(\theta) = (\cos \theta, \sin \theta)$ and, for an integer M , let $B(M)$ be the statement: for every $x \in \mathbb{R}^2$ and every $\theta \in [0, \pi]$ some point $x + ju(\theta)$ with $0 \leq j \leq M$ is red. A copy of ℓ_{M+1} with direction $\theta + \pi$ is also a copy with direction θ (start from its other end), so $B(M)$ implies that every copy of ℓ_{M+1} contains a red point. We show $B(59)$. The following reformulation is immediate, since $x + ju \in \text{RED}$ if and only if $x \in \text{RED} - ju$.

Lemma 11. *Let u be a unit vector and $M \geq 0$. There is a start x with $x, x + u, \dots, x + Mu$ all blue if and only if the $M + 1$ translates $\text{RED} - ju$, $0 \leq j \leq M$, do not cover $\mathbb{R}^2$. Since RED is Γ -periodic, this holds if and only if their images do not cover the torus $\mathbb{R}^2/\Gamma$, that is, if and only if they do not cover some (equivalently, every) fundamental domain of Γ .*

Both methods cover the directions by finitely many closed intervals (“boxes”) and, for each box, cover a fundamental domain of Γ by finitely many pieces, each of which is shown to lie in $\text{RED} - ju(\theta)$ for one $j \leq 59$ and all θ in the box. They differ in the direction boxes, the fundamental domain, the shape of the pieces and the enclosure of the red set, and in the code.

5.2. Method 1: strips and one-dimensional coverings. This is the method of an earlier draft of this note, generalized to a rational period lattice, to 24-gonal cells and to cells of different sizes.

Direction boxes. Fix N , let $\eta = \pi/(2N)$ and, for $0 \leq i < N$, let $\Theta_i = [2i\eta, (2i+2)\eta]$ with center $\theta_i = (2i+1)\eta$; these cover $[0, \pi]$. Put $e = u(\theta_i)$, $n = u(\theta_i + \pi/2)$ and write points in the frame (e, n) : $x = se + tn$. If $\theta = \theta_i + \varphi$ with $|\varphi| \leq \eta$, then $x + ju(\theta) = (s + j \cos \varphi)e + (t + j \sin \varphi)n$, with $0 \leq 1 - \cos \varphi \leq \eta^2/2$ and $|\sin \varphi| \leq \eta$. Hence, if $t \in [t_0, t_1]$, the j -th point lies in the closed rectangle

$$Q_j(s) = [s + j - j\eta^2/2, s + j] \times [t_0 - j\eta, t_1 + j\eta] \quad (\text{frame coordinates}).$$

Rectangles in cells. Let $Z = \sigma_Z e + \tau_Z n$ be the center of a red cell, with apothem h_Z , and let $A_k = n_k \cdot e = \cos(\psi_k - \theta_i)$ and $B_k = n_k \cdot n = \sin(\psi_k - \theta_i)$. (The program checks that its enclosure of every A_k excludes 0 and reports a failure otherwise.) For a relative height τ , the constraints $|\sigma A_k + \tau B_k| \leq h_Z$ say that σ lies in the closed interval with endpoints $(\pm h_Z - \tau B_k)/A_k$; let $L(\tau)$ be the largest of the twelve lower endpoints and $U(\tau)$ the smallest of the twelve upper endpoints. Then the horizontal section of $Z + C(h_Z)$ at height $\tau_Z + \tau$ is $[\sigma_Z + L(\tau), \sigma_Z + U(\tau)]$ (empty if $L > U$). Since $Z + C(h_Z)$ is convex, a rectangle $[\sigma', \sigma''] \times [\tau', \tau'']$ lies in it if and only if its four corners do, that is, if and only if $[\sigma', \sigma'']$ lies in both sections at the heights τ' and τ'' . For each cell and each strip $[t_0, t_1]$ (below) the program first determines a candidate window $j_0 \leq j \leq j_1$ of indices (omitting an index is conservative, see below), and then uses the single enlarged height range $[\tau', \tau''] = [t_0 - j_1\eta, t_1 + j_1\eta]$, which contains $[t_0 - j\eta, t_1 + j\eta]$ for every $j \leq j_1$. With $a_Z = \max(L(\tau' - \tau_Z), L(\tau'' - \tau_Z))$ and $b_Z = \min(U(\tau' - \tau_Z), U(\tau'' - \tau_Z))$ put

$$I_{Z,j} = [\sigma_Z + a_Z - j + j\eta^2/2, \sigma_Z + b_Z - j] \quad (j_0 \leq j \leq j_1).$$

If $s \in I_{Z,j}$ then $Q_j(s) \subseteq Z + C(h_Z)$, so for every start $x = se + tn$ with $t \in [t_0, t_1]$ and every direction $\theta \in \Theta_i$ the point $x + ju(\theta)$ is red. (The enlarged range only shrinks the intervals, so it is conservative; empty intervals are discarded.)

Strips. Let $[t_{\min}, t_{\max}]$ contain the range of t over the fundamental parallelogram F (Section 4), and cover it by consecutive closed strips $[t_0, t_1]$ of width W (the last one possibly longer), each starting exactly where the previous one ends. For each strip let $[s_{\min}, s_{\max}]$ contain the range of s over the part of the closure $\bar{F}$ in the strip. These enclosures are computed as follows. Since t is linear, its range over $\bar{F}$ is attained at the vertices $0, P_1, P_1 + P_2, P_2$. The set $\bar{F} \cap \{t_0 \leq t \leq t_1\}$ is a convex polygon whose vertices lie on the edges of $\bar{F}$, so its range of s is attained on the edges: a point $v + \lambda(v' - v)$ of the edge from v to v' with $t \in [t_0, t_1]$ has $\lambda \in [0, 1] \cap ([t_0, t_1] - t(v))/(t(v') - t(v))$ (or $\lambda \in [0, 1]$ if the enclosure of $t(v') - t(v)$ contains 0), and then $s = s(v) + \lambda(s(v') - s(v))$; $[s_{\min}, s_{\max}]$ is the hull of these four enclosures, and the strip is skipped only if all four are empty, in which case it does not meet $\bar{F}$. For $s \in [s_{\min}, s_{\max}]$ let $f(s)$ be the least $j < j_{\text{cap}}$ such that $s \in I'_{Z,j}$ for some red cell Z near the strip, where $I'_{Z,j} \subseteq I_{Z,j}$ are the computed intervals. The intervals are closed, so $f_{\max} = \max_s f(s)$ is found exactly by sorting their endpoints; if some piece of $[s_{\min}, s_{\max}]$ is not covered, the program reports a failure. Then every progression with start in the part of F over the strip and direction in Θ_i has a red point with index at most $f_{\max}$. Taking the maximum over all strips and boxes gives an M for which $B(M)$ holds.

Arithmetic. All quantities on which the conclusion depends—the frame coordinates of the centers and of the vertices of F , the range $[t_{\min}, t_{\max}]$, the ranges $[s_{\min}, s_{\max}]$ of the strips, A_k , B_k , the sections L , U , the losses $j\eta$ and $j\eta^2/2$, and the endpoints of $I'_{Z,j}$ —are computed in interval arithmetic with IEEE-754 double precision and outward rounding after every operation, starting from enclosures of $\cos \theta_i$, $\sin \theta_i$, $\cos(\psi_k - \theta_i)$, $\sin(\psi_k - \theta_i)$ and η computed with the ball arithmetic library Arb [22] (via python-flint) at 128 bits and converted to doubles from their exact endpoints, and from outward-rounded enclosures of the exact rationals P_1 , P_2 , c_i and h_i ; each cell uses the enclosure of its own apothem. The program checks that every enclosure is finite, that each strip starts at the endpoint of the previous one (the same double), the first at $t_{\min}$, and that the last ends at or above $t_{\max}$; otherwise it reports a failure. The enclosures

of $t_{\min}$, $t_{\max}$, $s_{\min}$, $s_{\max}$ are enlarged by a further 10^{-7} , which is not needed for rigour. The remaining choices are conservative: the program decides by floating-point estimates which cells and which values of j it tries for a strip (using the bounding radius $1/2$ for every cell, which is valid because every $R_i < 1/2$, checked in Arb), and omitting a cell or a value of j can only increase f or cause a failure; a strip that would need $j \geq j_{\text{cap}}$ is reported as a failure.

Refinement and result. The main pass uses $N = 20000$, $W = 0.04$ and $j_{\text{cap}} = 130$. Each box that fails or gives a value above 59 is replaced by its four sub-boxes in the tiling with $4N$ boxes (which are nested and cover it exactly), recomputed with $W = 0.005$ and $j_{\text{cap}} = 160$; a sub-box that still fails or exceeds 59 is replaced in the same way by its four sub-boxes in the tiling with $16N$ boxes, recomputed with $W = 0.002$. A final script (`final.m.py`) reads all records, checks for each one the SHA-256 of the configuration file and its position in the nested tilings, and prints the maximum of the values at the finest level used; $B(M)$ holds for this maximum M . For the coloring of this note the refinement went one level further. The main pass ($N = 20000$, $W = 0.04$, $j_{\text{cap}} = 130$) had no failures and maximum 86; 448 boxes gave a value above 59. After the first refinement ($4N$ boxes, $W = 0.005$), 24 of these main-pass boxes still had sub-boxes with a value above 59. Each of these 24 main-pass boxes was replaced by its 64 sub-boxes in the tiling with $64N = 1,280,000$ boxes, again with $W = 0.005$; this left 210 sub-boxes with the value 60, which were recomputed with the narrower strip width $W = 0.002$ and then all gave at most 47. These 210 sub-boxes lie in the thin range of directions from 21.24° to 21.36° (boxes 2360–2373 of the main pass), the same region in which Method 2 needed its deepest bisections (five levels, near 21.35°). A separate script (`combine.k60.py`, in place of `final.m.py`) reads every record of all levels, checks in each the SHA-256 of the configuration file, the parameters N and W and the position in the nested tilings, and gives the certified value $M = 59$; so Method 1 confirms Proposition 10.

5.3. Method 2: covering the torus by disks. The second program was written from the definition of the coloring alone, without access to the code of Method 1, and uses Lemma 11 directly, with only the inscribed disks: $\text{RED} \supseteq \bigcup_{g,i} D(c_i + g, h_i)$. (Its generalization to cells of different sizes was also made without access to Method 1.)

Direction boxes. The interval $[0, 3.1416] \supseteq [0, \pi]$ is split into 40,000 consecutive boxes $[\theta_l, \theta_h]$ whose endpoints are binary floating-point numbers; let θ_m be the (floating-point) midpoint and $\delta = \max(\theta_h - \theta_m, \theta_m - \theta_l)$, computed exactly. For every θ in the box, $|u(\theta) - u(\theta_m)| \leq |\theta - \theta_m| \leq \delta$, hence

$$D(c_i + g - ju(\theta_m), h_i - j\delta) \subseteq D(c_i + g, h_i) - ju(\theta) \subseteq \text{RED} - ju(\theta).$$

The same holds for the open disks: $D^\circ(c_i + g - ju(\theta_m), h_i - j\delta) \subseteq D^\circ(c_i + g, h_i) - ju(\theta)$. By Lemma 11, $B(59)$ holds for all directions of the box if the disks $D(c_i + g - ju(\theta_m), h_i - j\delta)$, $g \in \Gamma$, $1 \leq i \leq 90$, $0 \leq j \leq 59$, cover a fundamental domain. A box that fails is replaced by its two halves $[\theta_l, \theta_m]$ and $[\theta_m, \theta_h]$.

Squares. As an implementation detail, this program covers the rectangle $F_0 = [0, L) \times [0, H)$ instead of F , which is also a fundamental domain of Γ because P_1 is horizontal (any fundamental domain works by periodicity, Lemma 11). The rectangle lies in the grid $G = [0, 63/4] \times [0, 55/4]$ (checked exactly). The program divides G into 63×55 squares of side $1/4$ and subdivides each as a quadtree (depth at most 14) until every leaf square Q , with center q and half-diagonal r_Q , lies in one of the disks $D(z, r)$ above; the test is $s = r - r_Q > 0$ and $s^2 - |q - z|^2 > 0$.

Arithmetic. A first stage in floating point (not trusted) builds the tree and records for each leaf one disk, named by (i, j, a, b) with $g = aP_1 + bP_2$. A second stage (trusted) walks the tree again from the root grid, so that the leaves tile G by construction and independently of the first stage; recomputes every leaf square exactly from integers; asserts $1 \leq i \leq 90$ and $0 \leq j \leq 59$; and checks both inequalities of the test, with the radius $h_i - j\delta$ of the disk's own cell i , as certain

statements in Arb ball arithmetic at 96 bits, with P_1, P_2, c_i, h_i as exact rationals, $\theta_l, \theta_h, \theta_m$ as exact binary numbers and $\cos \theta_m, \sin \theta_m$ as Arb balls. It checks that $\theta_l \leq \theta_m \leq \theta_h$ and that $\delta = \max(\theta_h - \theta_m, \theta_m - \theta_l)$ is computed exactly (an Arb ball of radius 0), so that $h_i - j\delta$ is an Arb enclosure of the exact radius. Finally, the trusted stage checks in exact rational arithmetic that the boxes it has certified tile the direction interval: sorted, the first starts at 0, each starts exactly where the previous one ends, and the last ends at the double nearest to 3.1416, which exceeds π (checked in Arb). This also checks that every bisection replaced a box by two halves covering it. No floating-point error bound enters the trusted stage. The inequalities are strict: they give $|q - z| < r - r_Q$, so the closed leaf square lies in the open disk $D^\circ(z, r)$, and therefore the open disks $D^\circ(c_i + g, h_i)$ already suffice (Remark 7).

Result. With 60 translates ($0 \leq j \leq 59$), all but 73 of the 40,000 boxes passed at once, and those 73 passed after at most five bisections (40,198 certified boxes): 147,421,290 leaves, every one verified in Arb, with smallest certified margin $s^2 - |q - z|^2 \geq 1.85 \cdot 10^{-9}$ (1049 s on 8 processes in our run).

5.4. What is rigorous. In both methods the reduction (periodicity, reversal, direction boxes and the enclosures of the moving points) is exact mathematics, stated above. In Method 1 every quantity that credits a point as red, and every range of s and t over the fundamental domain and its strips, is enclosed in outward-rounded interval arithmetic, and the strips are checked to tile the range of t ; the only remaining floating-point decisions are which cells and which values of j are tried, and omitting one can only lose red points; in Method 2 every inclusion is certified in ball arithmetic and the covering is by construction, so no floating-point error analysis is needed. Method 2 also makes Claim B independent of the shape of the cells (Remark 7). Neither method uses Claim A, so the near pairs play no role in Claim B. Both programs were written with the same AI tool in the same project, but by separate agents, the second one seeing only the data of the coloring; the pipeline of Method 1 was also checked, in a separate AI-assisted review, on a 39-cell predecessor coloring with equal cells, where no error was found and its certified value was matched by a rigorously checked blue progression. A further AI-assisted review, of the near/far rule, of `claimA_nf.py` and of the way both programs treat cells of different sizes (each cell with its own apothem, and no separation assumption in Claim B), found no error; it also produced the second check of Claim A described after Proposition 9. Independent checks of the reductions and of the code by others are very welcome.

5.5. The exact value of this coloring. The value of this coloring is exactly 60. Theorem 2 shows that it has no blue ℓ_{60} , and it does contain a blue ℓ_{59} : the points $x + ju$, $j = 0, \dots, 58$, with $x = (9461029/10^6, 264991/25000)$ and u an explicit rational unit vector at about 149.53° (file `extra_checks/blind_verifier_V3/blue59_exact.txt` of the package), are all blue. This is checked in exact rational arithmetic, with rigorous enclosures of the unit normals n_k : for each point $p = x + ju$ and each translate $c = c_i + g$ of a cell center with $|p - c| \leq 0.51$, the certified lower bound on $\max_k |n_k \cdot (p - c)| - h_i$ exceeds $4.4 \cdot 10^{-4}$, so p is at Euclidean distance more than $4.4 \cdot 10^{-4}$ from that cell; every other cell is more than $0.51 - 0.4995 = 0.0105$ away. The next point $x + 59u$ is red. Thus this coloring cannot witness $k \leq 59$.

5.6. How the coloring was found. The searches play no role in the proofs, but we describe them, including one failure, because a search that optimizes a proxy for the verified quantity can over-fit. The configurations were found by simulated annealing over the centers and apothems of the cells (and, in later runs, the period lattice), with moves that shift, resize, insert or delete a cell, each followed by a relaxation that restores the separation rule with a safety margin of 0.005. Up to 2 October 2026 a candidate was scored by the longest blue progression found on a sample of lines. On that day the search reported a configuration whose longest blue progression found

on that sample had 63 points, which would have given $k \leq 64$; Method 2 with 64 translates failed on it in 14,013 direction boxes, for instance near 25.5° , 47.4° and 59.7° , and it contains blue progressions of at least 68 points: the search had over-fitted its sample of lines. We then made the score the verifier itself (`hon.py`): for each direction θ in a fixed set of a few thousand directions (for the last runs 3,930), let $n(\theta)$ be the least number of translates $\text{RED} - ju(\theta)$, $j = 0, \dots, n(\theta) - 1$, for which the floating-point first stage of Method 2 reports that they cover the torus, which accounts for every offset of the line rather than a sample; the score is $\max_\theta n(\theta) - 1$, a numerical proxy for the longest blue progression in the tested directions. A score of s therefore suggests $k \leq s + 1$, which Method 2 must then prove with $s + 1$ translates; and a new best configuration is accepted only after a floating-point sweep over all 20,000 direction boxes, with bisection, in which every box is covered. Directions where the sweep fails are added to the scoring set. With this score (`mix10.py` and its successors, the last being `mix13.py`) the score went down from 68 to 62 in steps of one. A greedy step that inserts one satellite, near one existing cell and far from all others, without moving any other cell (`T188/q3.py`), took the coloring with score 62 to one with score 61 ($k \leq 62$ by Method 2). The annealing variant `mix14.py` builds near pairs into the search, with moves that insert a satellite, resize a cell that has near partners, and turn one pair from near to far or back. From a coloring with score 61, the insertion of one cell at a point of the period chosen to repair the worst directions (`tgt.py`) gave a coloring of 90 cells with score 60 (Method 2 then proved $k \leq 61$ with 61 translates), and `mix14.py`, started from that coloring, gave the coloring used here, whose score is 59. These scores come from the search and are not certified values; the exact value of the final coloring, 60, is established separately in Section 5.5. The bound claimed in this note is for the final coloring, which passes both verifiers of Section 5 in their rigorous mode; the intermediate bounds $k \leq 62$ and $k \leq 61$ above were checked with Method 2 only.

6. VERIFICATION

The public package [7] contains all proof certificates and verification programs; a `README.txt` describes them, with the exact commands and expected outputs.

Lower bound. The package contains, for each lemma, the point list, the formula in DIMACS format, the DRAT proof and the LRAT proof. The command `bash verify_lower_bound.sh` first runs `verify_ladder.py` (Python standard library only), which checks every clause in exact integer arithmetic: the variable count equals the number of points, the points are distinct, the only unit clauses are r_0 and r_d , every binary clause is a pair at distance exactly 1 or exactly an earlier lemma distance (the order 4, 3, 2, 6 is hard-coded), and every 6-clause is an exact unit-step progression. It then runs `drat-trim` on each proof and, if `lrat-check` is installed, checks the LRAT proofs. It exits with status 0 only if everything passes; the whole run takes about 5–11 minutes on a laptop, almost all of it for $d = 4$. The certificates for Remark 6 (formulas, point lists, proofs and an exact clause checker for each group) are in the folder `lower_bound_remark`.

Upper bound. The programs and the candidate colorings are in the folder `upper_bound` (Python 3 with numpy, numba and python-flint; Linux or WSL recommended). From the top folder of the package, the single command `bash verify_all.sh` checks Claim A with both programs and Claim B by Method 2 for the coloring `cfg_k60.json` and prints one PASS or FAIL summary; `bash verify_all.sh full` also runs Method 1 and the checks of the lower bound. Since the computation of Method 1 for this coloring takes several hours, the package contains all its records; `full` re-validates and combines them with `combine_k60.py` and recomputes three of them from scratch (`recheck_records.py`), which must give identical values.

Program	Checks (runtime)
<code>claimA_nf.py</code> <code>cfg_k60.json</code>	Claim A, exact rationals and Arb (minutes)
<code>indep_nearfar.py</code> <code>cfg_k60.json</code>	Claim A, exact rationals only (minutes)
<code>claimB_arb.py</code> 40000 60 14 10 1	Claim B, Method 2 (about 10–20 min, 8 processes)
<code>combine_k60.py</code> , <code>recheck_records.py</code>	Claim B, Method 1, shipped records (about 5 min)
<code>cert.py</code> , <code>refine_boxes.py</code> , <code>vpoly2.py</code>	Claim B, Method 1, recomputed (several hours, 8 cores)

The full command lines are in `README.txt`; each program exits with a nonzero status on failure.

USE OF AI TOOLS

The computations, the verification programs and drafts of this text were produced with AI tools (Claude Code, by Anthropic) under the author’s direction. The two upper-bound verifiers were written in separate Claude Code sessions, the second one from the data of the coloring alone, and the earlier upper-bound pipeline was checked in a further session; these are not independent in the sense of different authors or different AI systems. A different AI system (ChatGPT, by OpenAI) was used to check the work: it reviewed an earlier version of the manuscript and of the upper-bound code, and for the lower bound it regenerated the point sets of Section 3, checked every clause of the four formulas in exact arithmetic and re-solved all four as unsatisfiable with its own SAT solver. The geometric reductions in Sections 2, 4 and 5 can be checked by hand; the finite computational assertions require the public certificates and programs, which check them independently of how they were found. Independent checks by others are very welcome. The author takes responsibility for the content.

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We thank Thomas Bloom for maintaining <https://www.erdosproblems.com>, where we learned of the problem. The lower bound rests on the work of Tsaturian [26] and of Arman and Tsaturian [2]. The upper bound builds on the cell colorings of Currier, Mody, Xie and Zhang [10], from whom the framework of red convex cells of diameter less than 1 and the hexagonal cells of our first coloring come. We are grateful to Gabriel Currier for the suggestion to use red cells of different sizes, with smaller cells in the gaps (private communication, 1 October 2026), and for helpful comments on the background section and the exposition. We are not aware of the bounds $k \geq 7$ or $k \leq 60$ in the literature and would be grateful to be corrected.

DATA AVAILABILITY

The proof certificates, verification programs and verification logs are available at Zenodo [7], doi:10.5281/zenodo.23151058, as a single archive. The search programs named in Section 5.6 are not part of the proof; they are available on request.

APPENDIX A. THE CENTERS AND APOTHEMS OF THE RED CELLS

The 90 centers $c_i = (x_i, y_i)$ and apothems h_i of Section 4. All decimals are exact (every coordinate is a rational number with denominator dividing 10^6 , and every apothem one with denominator dividing 10^4); the same numbers are given as fractions in `cfg_k60.json` in the package.

i	x_i	y_i	h_i	i	x_i	y_i	h_i	i	x_i	y_i	h_i
1	1.643662	2.69228	0.4952	31	18.108003	6.593659	0.4952	61	16.773541	13.165869	0.0336
2	16.00296	1.135099	0.4952	32	17.914106	4.596132	0.4952	62	4.684073	1.489866	0.0767
3	4.438848	6.114078	0.4952	33	19.567752	8.204132	0.4952	63	7.027833	11.36304	0.1347
4	3.981979	3.44783	0.466	34	20.9685	10.566792	0.4952	64	7.612312	11.566914	0.036
5	6.605805	7.228759	0.4952	35	22.43376	12.982768	0.4048	65	12.490386	2.776296	0.2729
6	7.716638	9.844303	0.4952	36	10.922264	1.630544	0.4952	66	22.680805	13.305316	0.1505
7	9.146902	11.384647	0.4952	37	12.209147	4.867126	0.4952	67	9.402424	13.362098	0.1654
8	3.165274	0.918178	0.4952	38	17.371304	8.984784	0.4952	68	22.81991	12.821319	0.1538
9	6.142905	4.933719	0.4952	39	13.930584	11.615075	0.4952	69	12.389463	2.538846	0.0462
10	6.0717	2.26965	0.4952	40	8.464311	7.98506	0.4952	70	12.663654	2.79917	0.0745
11	11.137547	12.123226	0.4952	41	16.228019	5.679142	0.4952	71	12.306428	2.522171	0.0793
12	8.280466	5.989564	0.4952	42	10.406841	5.4456	0.385	72	12.769293	2.712419	0.036
13	10.320321	9.74786	0.4952	43	16.727153	12.552956	0.1361	73	12.208966	2.535072	0.0316
14	8.360547	3.716225	0.4952	44	11.767153	8.562078	0.3569	74	12.524255	2.49096	0.0521
15	10.360885	7.337907	0.4952	45	17.439105	11.138879	0.3569	75	12.504037	2.51555	0.1181
16	13.215777	13.4872	0.4952	46	7.37859	11.493868	0.1778	76	12.748516	2.64957	0.0337
17	12.40706	10.313278	0.4952	47	12.522407	3.011397	0.357	77	7.816839	10.27831	0.0441
18	15.26011	13.471304	0.4952	48	16.373436	4.042663	0.1123	78	8.050466	9.538085	0.0421
19	8.825516	1.435327	0.4952	49	19.71351	11.727297	0.1824	79	9.950073	5.287121	0.0948
20	10.357865	3.553369	0.4952	50	9.648667	13.436412	0.2773	80	8.12398	9.762741	0.0793
21	12.589453	6.888756	0.4952	51	21.060902	12.174294	0.1047	81	12.715871	2.594633	0.0408
22	14.188351	8.733841	0.4952	52	14.445994	6.835849	0.1315	82	12.867842	2.977564	0.0793
23	15.633475	10.558684	0.4952	53	6.055626	8.927469	0.2787	83	12.841407	2.804441	0.0361
24	14.220512	5.163513	0.4952	54	12.352055	2.943816	0.3943	84	12.81595	2.756065	0.0433
25	18.355625	12.762509	0.4952	55	5.619897	8.651821	0.1952	85	12.844534	2.860493	0.0483
26	15.875031	7.651755	0.4952	56	16.481337	12.386684	0.1277	86	12.890076	3.104215	0.0312
27	12.438138	0.112866	0.4952	57	12.362684	2.653382	0.178	87	9.072667	13.238957	0.0933
28	19.011327	10.136359	0.4952	58	16.375648	12.119276	0.08	88	12.356009	2.437378	0.0302
29	14.927775	3.288526	0.4952	59	10.009891	5.485076	0.1602	89	5.600397	8.434584	0.0372
30	14.019288	1.419433	0.4952	60	9.259388	13.084942	0.1891	90	9.538187	13.225333	0.3555

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