

A CONVERSE TO THE DISCRETE FOUR-VERTEX THEOREM

KRISTOFFER JOSEFSSON

ABSTRACT. Tabachnikov asked whether every cyclic sequence of positive numbers with at least two local maxima and two local minima is the sequence of radii of the circles through consecutive vertex triples of a convex polygon whose circumcenters lie inside the corresponding vertex angles. We prove this for every $n \geq 4$, with extrema counted as plateaus. This is the sharp form: by Dahlberg’s discrete four-vertex theorem the condition is also necessary for nonconstant sequences. Locally near every nonconstant admissible sequence the polygon can be chosen to depend continuously (indeed C^1) on the sequence, but in general not globally: we give an explicit path of admissible hexagon data along which no continuous choice of realizing polygons exists. The existence and continuity results are formalized in Lean 4 with Mathlib. We also prove the corresponding characterization for circles tangent to three consecutive supporting lines, using a separate tangential-polygon construction and Wegner’s forward theorem.

1. INTRODUCTION

The four-vertex theorem says that the curvature of a simple closed plane curve that is not a circle has two local maxima separated by two local minima, with the values at both minima strictly smaller than those at both maxima. Gluck [5, pp. 295–296] proved the converse for continuous strictly positive curvature. Dahlberg [2, Theorem 1.1] proved it for smooth signed curvature, provided the two selected maxima are positive; see [4] for the history.

The four-vertex theorem has many discrete versions [7, 8, 10, 11, 3, 9]. In the one studied here the curvature at a vertex of a polygon is the reciprocal radius of the circle through that vertex and its two neighbours. Let P be a convex n -gon, $n \geq 4$, with vertices $x_1, \dots, x_n$ in cyclic order. Following Tabachnikov, call P *coherent* if for every i the center of the circle through x_{i-1}, x_i, x_{i+1} lies inside the angle of P at x_i . If P is coherent and no four consecutive vertices lie on a circle, then the cyclic sequence $r_1, \dots, r_n$ of the radii of these circles has at least two local maxima and two local minima. This is due to Musin (see [11, 7]); Dahlberg [3] proved a version for simple, locally regular polygons, not necessarily convex, extending Bilinski’s result for convex polygons with obtuse angles [1].

In [12, Problem 2] Tabachnikov posed the converse: a cyclic sequence $r_1, \dots, r_n$ with at least two local maxima and two local minima should be the radius sequence of a coherent convex polygon. Pak remarks that he is not aware of a nontrivial discrete version of the converse four-vertex theorem [9, §21.10, p. 206]. We prove Tabachnikov’s conjecture.

Local extrema are counted as by Dahlberg [3]. A *plateau maximum* of a cyclic sequence is a maximal run of equal consecutive terms, not the whole sequence, whose two outside neighbours are both strictly smaller. Plateau minima are defined in the same way. Two extrema are *distinct* if their runs are disjoint.

Theorem 1.1. *Let $n \geq 4$ and let $r_1, \dots, r_n > 0$ be a cyclic sequence with at least two distinct plateau maxima and two distinct plateau minima. Then there is a simple, strictly convex, coherent polygon $x_1, \dots, x_n$, listed counterclockwise, whose circle through x_{i-1}, x_i, x_{i+1} has radius r_i for every i .*

Strict convexity means that, for each edge, all other vertices lie in the same open half-plane bounded by its supporting line; coherence places each circumcenter in the interior of the corresponding vertex angle. The same polygon has signed Menger curvature $1/r_i$. Moreover, whenever $r_i \neq r_{i+1}$, the four vertices $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ do not lie on a common circle, independently of whether other adjacent radii are equal.

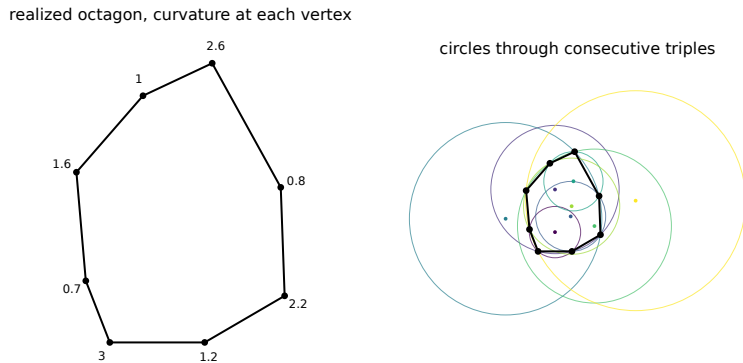


FIGURE 1. A coherent convex octagon realizing the curvatures $(3, 1.2, 2.2, 0.8, 2.6, 1, 1.6, 0.7)$, and its eight circles through consecutive vertex triples, whose centers lie inside the vertex angles. Computed numerically and checked to double precision.

Two distinct plateau maxima already force two distinct plateau minima, and conversely (Lemma 3.7), so either half of the hypothesis suffices.

Figure 1 shows an example. The theorem is sharp:

Corollary 1.2. *Let $n \geq 4$, and let $r_1, \dots, r_n$ be positive numbers that are not all equal. Then r is the radius sequence of a coherent convex polygon if and only if r has two distinct plateau maxima and two distinct plateau minima.*

Proof. One direction is Theorem 1.1. Conversely, a coherent convex polygon is simple, and its circumcenters lie in the closed vertex sectors, which is Dahlberg’s local regularity [3]. It is not inscribed in a circle, since otherwise all the circles through consecutive triples would coincide and r would be constant. Dahlberg’s discrete four-vertex theorem [3], whose extrema are plateau extrema with distinct runs, gives the two maxima and two minima. Only its convex case is needed here, since the polygon is strictly convex. The formalization includes this forward implication and the equivalence. $\square$

Remark 1.3 (The wording of the problem). Read with strict extrema, $r_{i-1} < r_i > r_{i+1}$, Tabachnikov’s hypothesis is a special case of Theorem 1.1. Read with weak extrema, $r_{i-1} \leq r_i \geq r_{i+1}$, the statement is false: $r = (2, 1, 1, 1)$ has the weak local maxima r_1, r_3 and minima r_2, r_4 but one plateau maximum, so by Corollary 1.2 it is not a radius sequence. Constant sequences are excluded by our plateau convention and are realized by regular n -gons. Tabachnikov’s non-concyclicity hypothesis holds for our polygon exactly when $r_i \neq r_{i+1}$ for all i : one implication was just noted, and if $r_i = r_{i+1}$ in a coherent convex polygon, the circles at x_i and x_{i+1} share the chord $x_i x_{i+1}$, the radius and the side of their centers, so they coincide.

It is convenient to work with the curvatures $\kappa_i = 1/r_i$. The hypothesis of Theorem 1.1 is then equivalent to a *strict four-vertex window*: indices $i_1 < i_2 < i_3 < i_4 < i_1 + n$ and a level c with

$$(1) \quad \kappa_{i_2}, \kappa_{i_4} < c < \kappa_{i_1}, \kappa_{i_3}.$$

Organization. Section 2 explains the construction informally; the proof occupies Sections 3–6 and Appendix B. Section 7 shows that the polygon can be chosen to depend in a C^1 way on the data near every nonconstant admissible sequence (Theorem 7.1), but that a global continuous choice does not exist, even along a path of admissible nonconstant hexagon data (Theorem 7.2); the obstruction is the radius of the smallest enclosing circle. Theorems 1.1, 7.1 and 7.2 are formalized in Lean 4 with Mathlib [6]; Appendix A lists the formal statements.

Tabachnikov also conjectured a converse for the radii of the circles tangent to three consecutive sides, the discrete four-vertex theorem of Wegner [13, 11]; the companion conjecture is also stated in [12, Problem 2]. Section 8 proves this companion converse for inward circles tangent to the three supporting lines, including their extensions. It does not use the four-length system of

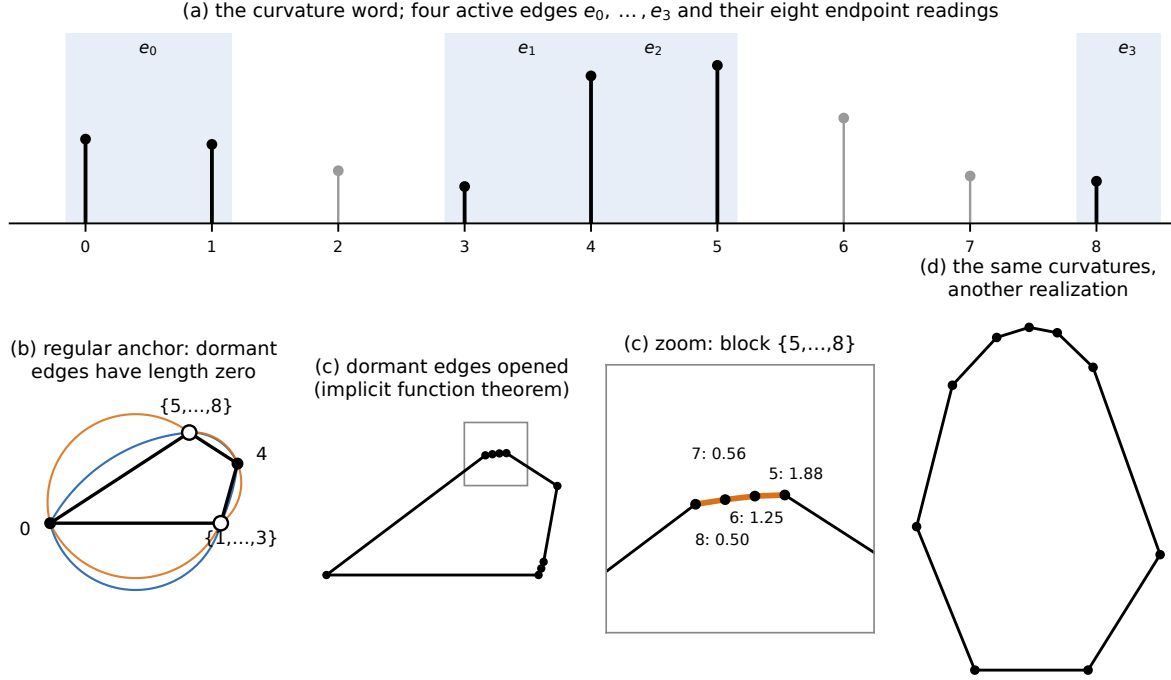


FIGURE 2. The method, computed for the word $\mu = (1, 0.94, 0.63, 0.44, 1.75, 1.88, 1.25, 0.56, 0.5)$. (a) The frame of four active edges $e_0, \dots, e_3$ (shaded) and their eight endpoint readings (black); the other readings (grey) are dormant. (b) A regular anchor: the dormant edges have length zero, each block of vertices between two active edges is a single point (open circles), and the polygon is a quadrilateral. (c) The implicit function theorem opens the dormant edges to small positive lengths, keeping every prescribed curvature. (d) Another realization of the same word, for comparison.

the main theorem. Section 9 gives a counterexample to the four-vertex theorem for turning-angle curvature. It also gives a counterexample to the naive converse for signed Menger curvature and asks what additional hypotheses are needed for that converse.

2. THE METHOD

We never deform a known polygon. Instead we build the polygon from its edge lengths. Write $2h_i$ for the length of the edge from vertex i to vertex $i + 1$. A chord of length $2h$ in a circle of curvature κ makes the angle $\arcsin(\kappa h)$ with the tangent at either end, so if the circle through three consecutive vertices is to have curvature κ_i , the polygon must turn at vertex i by $\theta_i = \arcsin(\kappa_i h_{i-1}) + \arcsin(\kappa_i h_i)$. Laying the edges end to end with these turns, the polygon closes up after one revolution exactly when three real equations hold: the two coordinates of the closure gap and the total turn $\sum \theta_i = 2\pi$. It therefore suffices to solve three equations in the n positive half-lengths h_i , subject to the bounds $\kappa_i h_{i-1} < 1$ and $\kappa_i h_i < 1$. Section 3 proves that every such solution gives a simple, strictly convex, coherent polygon.

The main idea is to put all the length on four edges (Figure 2). If every other edge has length zero, the vertices between two consecutive long edges collapse to one point. The turn at such a collapsed block involves only the curvatures at its two ends, and the three equations become three equations in four unknowns that involve only the eight curvature readings at the ends of the four chosen edges. A positive solution at which the three equations have full rank is a *regular anchor*. It is a closed quadrilateral whose corners carry two circles each. The implicit function theorem then opens every edge of length zero to a small positive length while keeping

all the prescribed curvatures, and the result is a realization (Section 4). The same argument, with the curvatures as parameters, gives the local C^1 dependence.

It remains to choose the four edges so that a regular anchor exists. After removing repeated consecutive curvatures and rescaling, this is a combinatorial problem about the word $\kappa_1, \dots, \kappa_n$, solved by one of three explicit constructions, each on a frame of the shape $(0, i-1, i, n-1)$: the two edges at a peak with curvature 1 at vertex 0 and the two edges at a vertex i further on. The resulting four-length system depends on five numbers only (Section 5).

- (R): If the maximum of κ is attained twice, take the edges at the two maxima. An explicit quadrilateral with perpendicular diagonals, deformed in two parameters, gives the anchor.
- (F): Otherwise let 0 be the highest peak other than the global maximum H . A number φ built from the readings next to 0 and H , the *forward flag*, decides the construction. If it is nonnegative, take the edges at the *first transition*: the first vertex i on the way up to H at which the flag of (κ_i, κ_{i+1}) becomes nonnegative.
- (N): If the forward flag and its mirror image, the backward flag, are both negative, take the edges at the two peaks 0 and H and again an orthogonal quadrilateral.

Each construction finds a zero by the intermediate value theorem or the Poincaré–Miranda theorem, and proves the rank condition by an explicit computation (Section 6).

3. DEVELOPMENT

Throughout, $n \geq 3$, indices of vertices and edges lie in $\mathbb{Z}/n$, and $\kappa = (\kappa_i)_{i \in \mathbb{Z}/n}$ is a *profile* of prescribed curvatures. Edge i joins vertex i to vertex $i+1$; it has length $\ell_i = 2h_i$, and h_i is its *half-length*. For $X, Y, W \in \mathbb{C}$ we write $[X, Y, W] = \text{Im}(\overline{(Y-X)}(W-X))$ for twice the signed area of the triangle XYW , and

$$(2) \quad M(X, Y, W) = \frac{2[X, Y, W]}{|X-Y||Y-W||W-X|}$$

for its signed Menger curvature; $|M|$ is the reciprocal circumradius.

Definition 3.1. Let $\kappa \in \mathbb{R}^n$, and let D_κ be the open set of $h \in \mathbb{R}^n$ with $|\kappa_i h_{i-1}| < 1$ and $|\kappa_i h_i| < 1$ for all i . For $h \in D_\kappa$ the *turning angle* at vertex i is

$$(3) \quad \theta_i = \arcsin(\kappa_i h_{i-1}) + \arcsin(\kappa_i h_i).$$

The *headings* are $\psi_j = \theta_0 + \theta_1 + \dots + \theta_j$ for $j \in \mathbb{N}$ (indices of θ read mod n), the *developed vertices* are

$$(4) \quad P_0 = 0, \quad P_k = \sum_{j=0}^{k-1} \ell_j e^{i\psi_j} \quad (k \in \mathbb{N}),$$

and the *closure gap* and *total turn* are

$$(5) \quad \mathcal{G}(\kappa, h) = P_n \in \mathbb{C}, \quad \mathcal{T}(\kappa, h) = \sum_{i \in \mathbb{Z}/n} \theta_i \in \mathbb{R}.$$

For $\kappa > 0$, h is *moderate* if $h_i > 0$, $\kappa_i h_{i-1} < 1$ and $\kappa_i h_i < 1$ for all i .

Figure 3 illustrates the definition for positive curvature and moderate half-lengths.

Both $\mathcal{G}$ and $\mathcal{T}$ are real-analytic in (κ, h) on the open set where $h \in D_\kappa$, including where some h_i vanish. For moderate h and $\kappa > 0$ both arcsines in (3) lie in $(0, \pi/2)$, so $\theta_i \in (0, \pi)$.

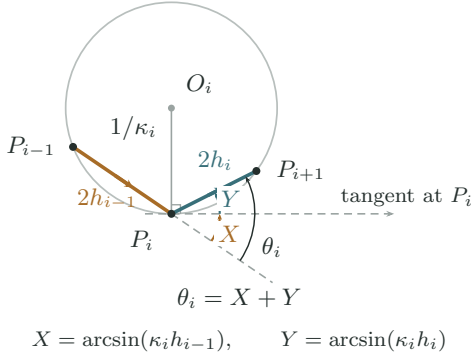
Lemma 3.2 (Tangent–chord). *Let $\kappa > 0$ and let h be moderate. Then $M(P_k, P_{k+1}, P_{k+2}) = \kappa_{k+1}$ for every $k \in \mathbb{N}$.*

Proof. Put $i = k+1$, $a = \ell_{i-1}$, $b = \ell_i$, $X = \arcsin(\kappa_i a/2)$ and $Y = \arcsin(\kappa_i b/2)$, so $\theta_i = X + Y$ and $X, Y \in (0, \pi/2)$. The edges at P_i are $u = ae^{i\psi}$ and $w = be^{i(\psi+\theta_i)}$ with $\psi = \psi_{i-1}$. Hence $[P_{i-1}, P_i, P_{i+1}] = \text{Im}(\bar{u}w) = ab \sin \theta_i$ and, substituting $a = 2 \sin X / \kappa_i$, $b = 2 \sin Y / \kappa_i$,

$$\frac{\kappa_i^2}{4} |P_{i+1} - P_{i-1}|^2 = \sin^2 X + \sin^2 Y + 2 \sin X \sin Y \cos(X + Y) = \sin^2(X + Y).$$

So $\kappa_i |P_{i+1} - P_{i-1}| = 2 \sin \theta_i$ and $M = 2ab \sin \theta_i / (ab |P_{i+1} - P_{i-1}|) = \kappa_i$. $\square$

(a) The turn at one vertex



(b) The developed edges

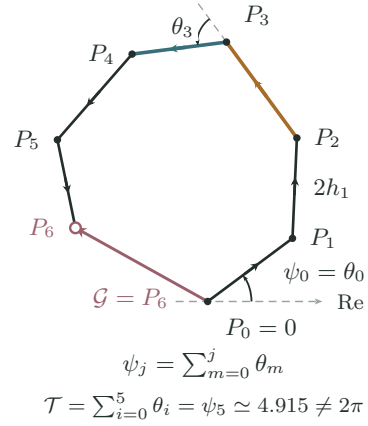


FIGURE 3. The quantities in Definition 3.1. (a) The tangent to the circumcircle at P_i splits the exterior turn into X and Y ; the incoming edge is continued by a dashed ray. (b) Edges of length $2h_j$ are placed at headings ψ_j . The highlighted triple P_2, P_3, P_4 is the one magnified and rotated in (a), with $i = 3$. The gap arrow runs from P_0 to P_6 ; the horizontal reference shows why the first heading is $\psi_0 = \theta_0$. Here $\mathcal{G} \neq 0$ and $\mathcal{T} \neq 2\pi$, so these moderate half-lengths do not yet give a realization.

If $\mathcal{G}(\kappa, h) = 0$ and $\mathcal{T}(\kappa, h) = 2\pi$, then $\psi_{j+n} = \psi_j + 2\pi$, so $\ell_{k+n} e^{i\psi_{k+n}} = \ell_k e^{i\psi_k}$ and, by induction, $P_{k+n} = P_k$. The development then descends to a closed polygon $(P_i)_{i \in \mathbb{Z}/n}$.

Definition 3.3. Let $\kappa \in \mathbb{R}^n$ be positive. Half-lengths h realize κ if h is moderate, $\mathcal{G}(\kappa, h) = 0$ and $\mathcal{T}(\kappa, h) = 2\pi$.

Proposition 3.4. If h realizes κ , then $[P_a, P_{a+1}, P_c] > 0$ whenever $c \notin \{a, a+1\}$. In particular the polygon $(P_i)_{i \in \mathbb{Z}/n}$ is simple and strictly convex.

Proof. Since every $\theta_i > 0$ and $\psi_{j+n} = \psi_j + 2\pi$, we have $\psi_k < \psi_m < \psi_k + 2\pi$ for $k < m < k+n$. Fix k . For $k \leq j$,

$$(6) \quad \operatorname{Im}(e^{-i\psi_k}(P_j - P_k)) = \sum_{m=k}^{j-1} \ell_m \sin(\psi_m - \psi_k),$$

and the full sum over $k \leq m < k+n$ vanishes. Let $k+1 < j < k+n$. If $\psi_{j-1} - \psi_k \leq \pi$, every term in (6) is ≥ 0 and the term $m = k+1$ is $\ell_{k+1} \sin \theta_{k+1} > 0$. Otherwise $\psi_m - \psi_k \in (\pi, 2\pi)$ for $j \leq m < k+n$, and (6) equals $-\sum_{m=j}^{k+n-1} \ell_m \sin(\psi_m - \psi_k) > 0$. Either way $[P_k, P_{k+1}, P_j] > 0$. So every vertex off an edge lies in the open left half-plane of that edge; two non-adjacent edges are separated by the line of one of them, and consecutive edges meet only in their common vertex. $\square$

A polygon is *coherent* if for every i the triple P_{i-1}, P_i, P_{i+1} is non-collinear and its circumcenter O_i lies in the open vertex cone $\{P_i + s(P_{i-1} - P_i) + t(P_{i+1} - P_i) : s, t > 0\}$.

Proposition 3.5. If h realizes κ , the polygon $(P_i)_{i \in \mathbb{Z}/n}$ is coherent.

Proof. Fix i , write $u = P_{i-1} - P_i$, $w = P_{i+1} - P_i$, and let a, b, X, Y be as in the proof of Lemma 3.2. Then $\bar{u}w = -abe^{i\theta_i}$, so $\delta = \operatorname{Im}(\bar{u}w) = -ab \sin \theta_i \neq 0$ and $\mathcal{C} = \operatorname{Re}(\bar{u}w) = -ab \cos \theta_i$. The circumcenter is $O_i = P_i + su + tw$ with $s = |w|^2(|u|^2 - \mathcal{C})/(2\delta^2)$ and $t = |u|^2(|w|^2 - \mathcal{C})/(2\delta^2)$: indeed $\langle su + tw, u \rangle = |u|^2(|u|^2|w|^2 - \mathcal{C}^2)/(2\delta^2) = |u|^2/2$, and likewise for w . So O_i lies in the open cone if and only if $\mathcal{C} < \min(|u|^2, |w|^2)$. Now

$$a^2 - \mathcal{C} = \frac{4}{\kappa_i^2} \sin X (\sin X + \sin Y \cos(X + Y)) = \frac{4}{\kappa_i^2} \sin X \sin(X + Y) \cos Y > 0,$$

since $X, Y \in (0, \pi/2)$; the moderate-arc inequalities are used exactly here, to get $\cos Y > 0$. The inequality $b^2 - \mathcal{C} > 0$ is symmetric. $\square$

If $\kappa \equiv c > 0$, then $h_i = \sin(\pi/n)/c$ realizes κ : every $\theta_i = 2\pi/n$, and the polygon is the regular n -gon.

3.1. Four-vertex windows and plateau extrema.

Definition 3.6. A profile $\kappa \in \mathbb{R}^n$ has a *strict four-vertex window* at level $c \in \mathbb{R}$, written $\text{DFV}(\kappa, c)$, if there are integers $i_1 < i_2 < i_3 < i_4 < i_1 + n$ satisfying (1), indices read mod n . A *plateau maximum run* of κ is a cyclic interval $I = \{a, \dots, a + k\}$ with $k < n - 1$ on which κ is constant, such that $\kappa_{a-1} < \kappa_a > \kappa_{a+k+1}$; plateau minimum runs are defined with the inequalities reversed.

A window forces $n \geq 4$ and, for $\kappa > 0$, $c > 0$. The runs are the plateau extrema of Section 1: the strict drops at both ends force maximality.

Lemma 3.7. For $\kappa \in \mathbb{R}^n$ the following are equivalent: (a) $\text{DFV}(\kappa, c)$ for some c ; (b) κ has two disjoint plateau maximum runs; (c) κ has two disjoint plateau minimum runs.

Proof. (a) $\Rightarrow$ (b),(c). On the integer interval $[i_4 - n, i_2]$ the endpoints are $< c < \kappa_{i_1}$, so a maximum of κ on it is attained at an interior point, and the maximal run of equal values through that point lies strictly inside the interval. This is a plateau maximum run in $(i_4 - n, i_2)$. The interval $[i_2, i_4]$ gives a disjoint one in (i_2, i_4) . The minimum runs come from $[i_1, i_3]$ and $[i_3, i_1 + n]$.

(b) $\Rightarrow$ (a). Let $I_1 = \{a, \dots, a + k\}$ and I_2 be disjoint plateau maximum runs with values $v_1 \leq v_2$, and let $a + j$, $0 \leq j < n$, be the initial point of I_2 . Then $j > k$, and $j \neq k + 1$, $j \neq n - 1$, since otherwise a strict drop at an end of I_1 would give $v_2 < v_1$. For c strictly between $\max(\kappa_{a-1}, \kappa_{a+k+1})$ and v_1 , the indices $a < a + k + 1 < a + j < a + n - 1$ form a window. Apply this to $-\kappa$ for (c). $\square$

Inversion $r \mapsto 1/r$ exchanges plateau maxima and minima. So the hypothesis of Theorem 1.1 says exactly that $\kappa = 1/r$ has a strict four-vertex window at some level $c > 0$.

Theorem 3.8. Every positive profile with a strict four-vertex window has a realization.

Theorem 3.8 is proved in Sections 4–6.

Proof of Theorem 1.1. Let $\kappa = 1/r$ and let h be a realization from Theorem 3.8, with closed polygon P . The polygon is simple and strictly convex by Proposition 3.4, and coherent by Proposition 3.5. Lemma 3.2 and periodicity give $M(P_{i-1}, P_i, P_{i+1}) = \kappa_i$, hence the prescribed circumradii. For the stated local non-concyclicity consequence, if $P_{i-1}, P_i, P_{i+1}, P_{i+2}$ lie on a circle with center O , then O is the circumcenter of both non-collinear triples P_{i-1}, P_i, P_{i+1} and P_i, P_{i+1}, P_{i+2} , so $r_i = |OP_i| = r_{i+1}$. $\square$

4. FOUR ACTIVE EDGES

In this section $\kappa \in \mathbb{R}^n$ is a positive profile. We put all the length on four edges and give the remaining edges length zero.

4.1. The controller.

Definition 4.1. A *frame* is a choice of edges $0 \leq e_0 < e_1 < e_2 < e_3 \leq n - 1$. Its *start* and *finish* readings are $s_j = \kappa_{e_j}$ and $f_j = \kappa_{e_j+1}$ ($j \in \mathbb{Z}/4$); the other readings are *dormant*. On the open set $U = \{L \in \mathbb{R}^4 : |s_j L_j| < 1, |f_j L_j| < 1\}$ put $a_j = \arcsin(s_j L_j)$, $b_j = \arcsin(f_j L_j)$, define the *block turns* $\beta_j = b_{j-1} + a_j$ (indices mod 4) and the *controller* $\Phi = (\Phi_1, \Phi_2): U \rightarrow \mathbb{C} \times \mathbb{R}$,

$$(7) \quad \Phi(L) = \left(L_0 + L_1 e^{i\beta_1} + L_2 e^{i(\beta_1+\beta_2)} + L_3 e^{i(\beta_1+\beta_2+\beta_3)}, \quad \beta_0 + \beta_1 + \beta_2 + \beta_3 - 2\pi \right).$$

The *slot embedding* $\iota: \mathbb{R}^4 \rightarrow \mathbb{R}^n$ sets $(\iota L)_{e_j} = L_j$ and $(\iota L)_i = 0$ for $i \notin \{e_0, \dots, e_3\}$.

The map Φ is real-analytic on U , and $\iota(U) \subset D_\kappa$. In the development of ιL the vertices $e_{j-1} + 1, \dots, e_j$ of a block coincide, and only the two outer readings f_{j-1} and s_j contribute to the turn β_j between the active edges e_{j-1} and e_j .

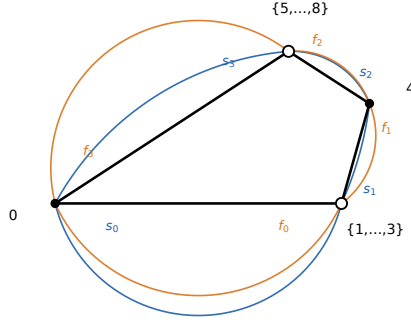


FIGURE 4. The regular anchor of Figure 2(b). Only the four active edges have positive length; the blocks $\{1, 2, 3\}$ and $\{5, \dots, 8\}$ have collapsed to points (open circles). On active edge e_j the two arcs have curvatures s_j (blue) and f_j (orange), the readings at its two ends. The turn at each corner is the sum of the two chord angles of the arcs meeting there, as in (7).

Proposition 4.2. *For every $L \in U$ (no closure hypothesis), with $h = \iota L$,*

$$(8) \quad \mathcal{G}(\kappa, h) = 2e^{i\psi_{e_0}(h)} \Phi_1(L), \quad \mathcal{T}(\kappa, h) - 2\pi = \Phi_2(L).$$

Proof. For $h \in D_\kappa$ let $\gamma_e = \arcsin(\kappa_e h_e) + \arcsin(\kappa_{e+1} h_{e+1})$. Regrouping the arcsines of (3) by edge gives $\mathcal{T} = \sum_e \gamma_e$, and induction on k gives $\psi_k = \arcsin(\kappa_0 h_{n-1}) + \sum_{e < k} \gamma_e + \arcsin(\kappa_k h_k)$ for $0 \leq k < n$. For $h = \iota L$ we have $\gamma_e = 0$ for dormant e and $\gamma_{e_j} = a_j + b_j$. Hence $\mathcal{T} = \sum_j \beta_j$ and $\psi_{e_j} - \psi_{e_0} = \sum_{i < j} (a_i + b_i) + a_j - a_0 = \beta_1 + \dots + \beta_j$, so $\mathcal{G} = \sum_j 2L_j e^{i\psi_{e_j}} = 2e^{i\psi_{e_0}} \Phi_1(L)$. $\square$

4.2. Regular anchors.

Definition 4.3. A *regular anchor* for κ is a frame $e_0 < \dots < e_3$ together with $L \in \mathbb{R}^4$ such that

- (A1) $L_j > 0$, $s_j L_j < 1$ and $f_j L_j < 1$ for $j = 0, \dots, 3$;
- (A2) $\Phi(L) = 0$;
- (A3) the derivative $D\Phi(L): \mathbb{R}^4 \rightarrow \mathbb{C} \times \mathbb{R}$ is onto.

Condition (A2) says that the quadrilateral with half-sides L_j and exterior angles β_j closes with total turn 2π (Figure 4), and (A3) says that the 3×4 Jacobian has rank 3. The system depends only on L and the eight readings.

Theorem 4.4. *Let $\kappa^0 > 0$ admit a regular anchor $(e, \bar{L})$. There are an open neighbourhood $V \subset (0, \infty)^n$ of κ^0 and a C^1 map $h: V \rightarrow \mathbb{R}^n$ such that $h(\kappa)$ realizes κ for every $\kappa \in V$. In particular κ^0 has a realization.*

Proof. Let $\bar{h} = \iota \bar{L}$, let $\mathcal{A} = \{e_0, \dots, e_3\}$ and $\mathcal{Z} = \mathbb{Z}/n \setminus \mathcal{A}$, and let $G(\kappa, h) = (\mathcal{G}(\kappa, h), \mathcal{T}(\kappa, h) - 2\pi)$, a real-analytic map on the open set $\{(\kappa, h) : h \in D_\kappa\}$. By (A1), $\bar{L} \in U$, so $\bar{h} \in D_{\kappa^0}$, and $G(\kappa^0, \bar{h}) = 0$ by (8) and (A2). Write $G_0 = D_h G(\kappa^0, \bar{h})$. Differentiating $G(\kappa^0, \iota L) = (g \Phi_1(L), \Phi_2(L))$, with $g = 2e^{i\psi_{e_0} \iota} \neq 0$, at $\bar{L}$, the derivative of g drops out because $\Phi_1(\bar{L}) = 0$: $G_0(\iota v) = (g(\bar{L}) D\Phi_1(\bar{L})v, D\Phi_2(\bar{L})v)$. By (A3), G_0 maps $\mathbb{R}^4$ onto $\mathbb{C} \times \mathbb{R}$.

So there are $b \in \mathbb{R}^4$ with $G_0 b = -G_0 \mathbf{1}_{\mathcal{Z}}$ and a linear map $N: \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{R}^4$ with $G_0 N = \text{id}$. Put $v = \mathbf{1}_{\mathcal{Z}} + b$ and $\tilde{G}(\kappa, t, y) = G(\kappa, \bar{h} + tv + Ny)$ for (κ, t, y) near $(\kappa^0, 0, 0)$ in $\mathbb{R}^n \times \mathbb{R} \times (\mathbb{C} \times \mathbb{R})$. Then $\tilde{G}(\kappa^0, 0, 0) = 0$ and $D_y \tilde{G}(\kappa^0, 0, 0) = \text{id}$. The implicit function theorem gives a neighbourhood $\mathcal{N}$ of $(\kappa^0, 0)$ and a C^1 map y on $\mathcal{N}$ with $\tilde{G}(\kappa, t, y(\kappa, t)) = 0$. Let $H(\kappa, t) = \bar{h} + tv + Ny(\kappa, t)$. Because N takes values in $\mathbb{R}^4$, $H_i(\kappa, t) = t$ for $i \in \mathcal{Z}$. At $(\kappa^0, 0)$ the active entries of H are $\bar{L}_j > 0$, and the inequalities $\kappa_i H_{i-1} < 1$, $\kappa_i H_i < 1$ hold strictly: they read $s_j \bar{L}_j < 1$ or $f_j \bar{L}_j < 1$ at the ends of an active edge and $0 < 1$ otherwise. By continuity there are $t_0 > 0$ and an open $V \ni \kappa^0$ in $(0, \infty)^n$ such that these strict inequalities and positivity hold at (κ, t_0) for all $\kappa \in V$. Then $h(\kappa) = H(\kappa, t_0)$ is C^1 , moderate, and satisfies $\mathcal{G} = 0$, $\mathcal{T} = 2\pi$. $\square$

4.3. Transport of anchors.

Proposition 4.5. *If $\kappa > 0$ admits a regular anchor, so do (a) $\lambda\kappa$ for $\lambda > 0$; (b) the shift $\kappa(\cdot + a)$ for $a \in \mathbb{Z}/n$; (c) the reflection $\kappa(-\cdot)$.*

Proof. (a) Keep the frame and take $\bar{L}/\lambda$; all β_j are unchanged and $\Phi^{\lambda\kappa}(L) = (\lambda^{-1}\Phi_1^\kappa(\lambda L), \Phi_2^\kappa(\lambda L))$. (b) Take $a = 1$. If $e_0 \geq 1$, the frame $(e_j - 1)_j$ has the same readings, and $\bar{L}$ works. If $e_0 = 0$, take the frame $(e_1 - 1, e_2 - 1, e_3 - 1, n - 1)$ and $L'_j = L_{j+1}$ (indices mod 4). Then $\beta'_j = \beta_{j+1}$ and, since $\sum_j \beta_j = \Phi_2 + 2\pi$,

$$\Phi'(L') = (e^{-i\beta_1}(\Phi_1(L) + L_0(e^{i\Phi_2(L)} - 1)), \Phi_2(L)).$$

(c) Edge e of κ is edge $-e - 1$ of $\kappa' = \kappa(-\cdot)$, traversed backwards. Take the frame $e'_j = n - 1 - e_{3-j}$, with $s'_j = f_{3-j}$, $f'_j = s_{3-j}$, and $L'_j = L_{3-j}$. Then $(\beta'_0, \dots, \beta'_3) = (\beta_0, \beta_3, \beta_2, \beta_1)$ and $\Phi'(L') = (e^{i(\beta_1 + \beta_2 + \beta_3)} \overline{\Phi_1(L)}, \Phi_2(L))$.

In each case Φ' vanishes at the new point, and since Φ_1 and Φ_2 vanish there, the derivatives of the prefactors drop out: $D\Phi'$ is $D\Phi(\bar{L})$ composed with a linear automorphism of $\mathbb{C} \times \mathbb{R}$ (in (b), $(z, s) \mapsto (e^{-i\beta_1}(z + i\bar{L}_0 s), s)$) and a permutation of $\mathbb{R}^4$. Hence it is onto. $\square$

The controller depends on κ only through the eight readings. So if a frame e for μ and a frame e' for κ have equal readings, a regular anchor (e, L) for μ gives the regular anchor (e', L) for κ , even if μ and κ have different lengths.

Lemma 4.6. *Let $\kappa \in \mathbb{R}^{m+1}$, $m \geq 1$, with $\kappa_a = \kappa_{a+1}$ for some $0 \leq a \leq m$, and let $\mu \in \mathbb{R}^m$ be obtained by deleting the entry a : $\mu_i = \kappa_{\tau(i)}$ with $\tau(i) = i$ for $i < a$ and $\tau(i) = i + 1$ for $i \geq a$. (i) If $\text{DFV}(\kappa, c)$, then $\text{DFV}(\mu, c)$. (ii) If (e, L) is a regular anchor for μ , then $(\tau(e), L)$ is a regular anchor for κ .*

Proof. (i) The map $\nu: \mathbb{N} \rightarrow \mathbb{N}$ with $\nu(p(m+1) + j) = pm + j$ for $0 \leq j \leq a$ and $\nu(p(m+1) + j) = pm + j - 1$ for $a < j \leq m$ is monotone, satisfies $\nu(i + m + 1) = \nu(i) + m$ and $\mu_{\nu(i)} = \kappa_i$. It maps a window $i_1 < \dots < i_4 < i_1 + m + 1$ of κ to indices $\nu(i_1) \leq \dots \leq \nu(i_4) \leq \nu(i_1) + m$, and each inequality is strict because the two sides carry readings on opposite sides of c . (ii) τ is strictly increasing, the start readings agree by definition, and the residues $\tau(e_j) + 1$ and $\tau(e_j + 1)$ either coincide or are a and $a + 1$. So the readings agree. $\square$

A word is *adjacent-unequal* if $\kappa_i \neq \kappa_{i+1}$ for all i .

Proposition 4.7. *If every positive adjacent-unequal word with a strict four-vertex window admits a regular anchor, then every positive profile with a strict four-vertex window does, and Theorem 3.8 holds.*

Proof. Induct on the length. If $\kappa_a = \kappa_{a+1}$, the shortened word of Lemma 4.6 is positive with a window at the same level, so it has a regular anchor, which lifts to κ . Then apply Theorem 4.4. $\square$

4.4. Peaks, flags and the case split. For $0 < l < 1 < h$ put

$$(9) \quad \sigma(h, l) = \sqrt{\frac{1 - l^2}{h^2 - l^2}}, \quad \varphi(p, t, h, l) = \arcsin p + \arcsin(t\sigma(h, l)) - \arcsin \sigma(h, l) - \frac{\pi}{2}$$

for $0 < p < 1$ and $0 < t \leq h$. Here $\sigma \in (0, 1)$, $\sigma^2(h^2 - l^2) = 1 - l^2$ and $h\sigma < 1$, because $(1 - h^2\sigma^2)(h^2 - l^2) = l^2(h^2 - 1) > 0$; so $t\sigma \leq h\sigma < 1$ and all arguments of $\arcsin$ lie in $(0, 1)$.

A vertex i is a *strict local maximum* if $\kappa_{i-1} < \kappa_i > \kappa_{i+1}$. For an index H , a vertex $K \neq H$ is a *highest secondary peak relative to H* if K is a strict local maximum and $\kappa_i \leq \kappa_K$ for every strict local maximum $i \neq H$.

Lemma 4.8. *Let κ be adjacent-unequal, let 0 be a highest secondary peak relative to H , and let $\kappa_0 < \kappa_H$. Represent H in $\{0, \dots, n - 1\}$.*

- (a) *H is the unique global maximum: $\kappa_i < \kappa_H$ for $i \neq H$.*
- (b) *$2 \leq H \leq n - 2$, and there is a with $2 \leq a \leq H$, $\kappa_{a-1} \leq \kappa_0 < \kappa_a$, and κ strictly increasing on $\{a, \dots, H\}$.*

Proof. In an adjacent-unequal word every global maximum is a strict local maximum.

(a) If $G \neq H$ is a global maximum, then $\kappa_G \leq \kappa_0 < \kappa_H \leq \kappa_G$.

(b) Since $\kappa_{\pm 1} < \kappa_0 < \kappa_H$, $H \notin \{1, n-1\}$. Let a be the least index in $\{1, \dots, H\}$ with $\kappa_j > \kappa_0$ for $a \leq j \leq H$; then $a \geq 2$ and $\kappa_{a-1} \leq \kappa_0$. If $\kappa_j > \kappa_{j+1}$ for some $a \leq j < H$, the maximum of κ on $\{a-1, \dots, j+1\}$ is at least $\kappa_j > \kappa_0$ and is not attained at the ends, so it is attained at a strict local maximum $i \neq H$ with $\kappa_i > \kappa_0$, which is impossible. $\square$

Let now μ be positive and adjacent-unequal with $\mu_0 = 1$, and let 0 be a highest secondary peak relative to H with $\mu_H > 1$. By Lemma 4.8, $0 < \mu_{\pm 1} < 1$ and $\mu_{H\pm 1} < \mu_H$, so the *forward* and *backward flags*

$$\varphi^+ = \varphi(\mu_1, \mu_{H-1}, \mu_H, \mu_{-1}), \quad \varphi^- = \varphi(\mu_{-1}, \mu_{H+1}, \mu_H, \mu_1)$$

are defined. The reflection $\mu'_i = \mu_{-i}$ has 0 as a highest secondary peak relative to $-H$, and its forward flag is φ^- .

Proposition 4.9. *Every positive adjacent-unequal word with a strict four-vertex window admits a regular anchor, provided the following hold for all positive adjacent-unequal words μ with $\mu_0 = 1$.*

(R): *If $\mu \leq 1$ and $\mu_J = 1$ for some $J \neq 0$, then μ admits a regular anchor.*

(F): *If 0 is a highest secondary peak relative to H , $\mu_H > 1$ and $\varphi^+ \geq 0$, then μ admits a regular anchor.*

(N): *If 0 is a highest secondary peak relative to H , $\mu_H > 1$, $\varphi^+ < 0$ and $\varphi^- < 0$, then μ admits a regular anchor.*

Proof. Let κ be such a word and H a global maximum. For $g \in \mathbb{Z}/n$ the word $\mu_i^{(g)} = \kappa_{i+g}/\kappa_g$ is positive and adjacent-unequal with $\mu_0^{(g)} = 1$, and by Proposition 4.5 it suffices to find an anchor for one of them. If the maximum is also attained at $J \neq H$, apply (R) to $\mu^{(H)}$. Otherwise, by Lemma 3.7 κ has two distinct plateau maxima, which are strict local maxima, so there is a strict local maximum other than H ; let K be one of largest value. Then 0 is a highest secondary peak of $\mu = \mu^{(K)}$ relative to $H - K$, and $\mu_{H-K} > 1$. If $\varphi^+ \geq 0$, apply (F). If $\varphi^+ < 0 \leq \varphi^-$, apply (F) to the reflection of μ and use Proposition 4.5(c). If both flags are negative, apply (N). $\square$

Statements (R), (F) and (N) are proved in Section 6 (Propositions 6.1, 6.2 and 6.7). Together with Propositions 4.7 and 4.9 this proves Theorem 3.8.

5. THE FOUR-LENGTH SYSTEM

All three constructions use a frame $(0, i-1, i, n-1)$ with $\kappa_0 = 1$. Writing $p = \kappa_1$, $t = \kappa_{i-1}$, $q = \kappa_i$, $h = \kappa_{i+1}$ and $l = \kappa_{-1}$, its readings are

$$(10) \quad (s_0, s_1, s_2, s_3) = (1, t, q, l), \quad (f_0, f_1, f_2, f_3) = (p, q, h, 1).$$

In this section $p, t, q, h, l > 0$ are arbitrary, and $\Phi = \Phi_{p,t,q,h,l}$ is the controller (7) with these readings. Its block turns are

$$\begin{aligned} \beta_0 &= \arcsin L_3 + \arcsin L_0, & \beta_1 &= \arcsin(pL_0) + \arcsin(tL_1), \\ \beta_2 &= \arcsin(qL_1) + \arcsin(qL_2), & \beta_3 &= \arcsin(hL_2) + \arcsin(lL_3). \end{aligned}$$

Corners 0 and 2 are *pure* (equal readings 1, resp. q , on both sides), corners 1 and 3 are *mixed*. Let Ω be the open set of $L \in \mathbb{R}^4$ with $L_j > 0$ and

$$(11) \quad L_0 < 1, \quad pL_0 < 1, \quad tL_1 < 1, \quad qL_1 < 1, \quad qL_2 < 1, \quad hL_2 < 1, \quad lL_3 < 1, \quad L_3 < 1.$$

This is condition (A1). So a zero $L \in \Omega$ of Φ at which $D\Phi(L)$ is onto is a regular anchor on every frame with readings (10).

5.1. Split coordinates. For $L \in \Omega$ the *split angles*

$$\alpha_0 = \arcsin L_0, \quad \alpha_3 = \arcsin L_3, \quad \gamma_1 = \arcsin(qL_1), \quad \gamma_2 = \arcsin(qL_2)$$

lie in $(0, \pi/2)$, and $L = (\sin \alpha_0, \sin \gamma_1/q, \sin \gamma_2/q, \sin \alpha_3)$. The pure turns are $S = \alpha_0 + \alpha_3 = \beta_0$ and $T = \gamma_1 + \gamma_2 = \beta_2$, and the mixed turns are $\beta_1 = \arcsin(p \sin \alpha_0) + \arcsin(\frac{t}{q} \sin \gamma_1)$ and $\beta_3 = \arcsin(\frac{h}{q} \sin \gamma_2) + \arcsin(l \sin \alpha_3)$. Put

$$\chi = \sin S - \frac{\sin T}{q}, \quad \varepsilon_1 = \alpha_3 + \gamma_2 + \beta_1 - \pi, \quad \varepsilon_3 = \alpha_0 + \gamma_1 + \beta_3 - \pi,$$

so that $\varepsilon_1 + \varepsilon_3 = \Phi_2$. Geometrically, $\chi = 0$ says that the diagonal between the mixed corners is a common chord of the circles of curvature 1 and q through the pure corners, and $\varepsilon_1 = \varepsilon_3 = 0$ says that the turns at the mixed corners are the inscribed-angle turns of these two circles.

Lemma 5.1 (Closure identity). *Let $q \neq 0$ and let $\alpha_0, \alpha_3, \gamma_1, \gamma_2, \beta_1, \beta_3$ be arbitrary real numbers. Define $S, T, \chi, \varepsilon_1, \varepsilon_3$ by the formulas above, $\tau = \varepsilon_1 + \varepsilon_3$, and $E = \sin \alpha_0 + \frac{\sin \gamma_1}{q} e^{i\beta_1} + \frac{\sin \gamma_2}{q} e^{i(\beta_1+T)} + \sin \alpha_3 e^{i(\beta_1+T+\beta_3)}$. Then*

$$(12) \quad E = e^{-i\alpha_3} \left(\sin S - \frac{\sin T}{q} e^{i\varepsilon_1} \right) + \sin \alpha_3 e^{-iS} (e^{i\tau} - 1).$$

If $L \in \Omega$ and the six numbers are its split angles and mixed turns, then $\Phi(L) = (E, \tau)$, and $\Phi(L) = 0$ if and only if $\chi = \varepsilon_1 = \varepsilon_3 = 0$.

Proof. We use $\sin x + \sin y e^{\pm i(x+y)} = \sin(x+y) e^{\pm iy}$. With $(x, y) = (\gamma_1, \gamma_2)$ and $\beta_1 + \gamma_2 = \varepsilon_1 - \alpha_3 + \pi$,

$$\frac{e^{i\beta_1}}{q} (\sin \gamma_1 + \sin \gamma_2 e^{iT}) = \frac{\sin T}{q} e^{i(\beta_1+\gamma_2)} = -\frac{\sin T}{q} e^{-i\alpha_3} e^{i\varepsilon_1}.$$

Since $\beta_1 + T + \beta_3 = \tau + 2\pi - S$, the last term of E is $\sin \alpha_3 e^{-iS} + \sin \alpha_3 e^{-iS} (e^{i\tau} - 1)$, and $\sin \alpha_0 + \sin \alpha_3 e^{-iS} = \sin S e^{-i\alpha_3}$. Adding up gives (12).

For $L \in \Omega$, $(E, \tau) = \Phi(L)$ by the definitions. If $\chi = \varepsilon_1 = \varepsilon_3 = 0$, then $\tau = 0$ and $E = e^{-i\alpha_3} \chi = 0$. Conversely, if $E = \tau = 0$, then $\sin S = \frac{\sin T}{q} e^{i\varepsilon_1}$. Here $\sin T > 0$, and $\varepsilon_1 \in (-\pi, \pi)$ because $\alpha_3, \gamma_2 \in (0, \pi/2)$ and $\beta_1 \in (0, \pi)$. The imaginary part gives $\varepsilon_1 = 0$, the real part $\chi = 0$, and $\varepsilon_3 = \tau - \varepsilon_1 = 0$. $\square$

5.2. Rank. For $a > 0$ and $z \in (0, \pi/2)$ with $a \sin z < 1$ let $\varsigma_a(z) = \frac{d}{dz} \arcsin(a \sin z) = a \cos z / \sqrt{1 - a^2 \sin^2 z} > 0$. Since $a^2 \cos^2 z - (1 - a^2 \sin^2 z) = a^2 - 1$,

$$(13) \quad \varsigma_a(z) < 1 \quad \text{if } a < 1, \quad \varsigma_a(z) > 1 \quad \text{if } a > 1.$$

For $L \in \Omega$ put

$$A = 1 - \varsigma_p(\alpha_0), \quad B = 1 - \varsigma_l(\alpha_3), \quad C = 1 - \varsigma_{t/q}(\gamma_1), \quad D = 1 - \varsigma_{h/q}(\gamma_2).$$

They are the derivatives of the defects $z - \arcsin(a \sin z)$, and $D < 1$.

Lemma 5.2. *Let $0 < p < 1$, $0 < l < 1$, $q \geq 1$, $0 < t < q$ and $h > 0$, and let $L \in \Omega$ with $\Phi(L) = 0$. Then $A, B, C \in (0, 1)$, $S + T > \pi$ and $\cos S < 0$. If $D\Phi(L)$ is not onto, then*

$$(14) \quad AD = BC \quad \text{and} \quad \cos S (A + B - AD) + \frac{\cos T}{q} (A + B - AB) = 0.$$

Proof. By (13), $A, B, C \in (0, 1)$. By Lemma 5.1, $\chi = \varepsilon_1 = \varepsilon_3 = 0$. Now $\varepsilon_1 = 0$ reads $S + T - \pi = (\alpha_0 - \arcsin(p \sin \alpha_0)) + (\gamma_1 - \arcsin(\frac{t}{q} \sin \gamma_1)) > 0$. If $S \leq \pi/2$, then $0 < \pi - T < S \leq \pi/2$ gives $\sin T < \sin S = \sin T/q \leq \sin T$. So $S \in (\pi/2, \pi)$.

We use two charts near the split angles of L : in chart I the coordinates are (S, α_0, γ_1) with T fixed ($\alpha_3 = S - \alpha_0$, $\gamma_2 = T - \gamma_1$), in chart II they are (S, α_0, T) with γ_1 fixed. In each chart, $x \mapsto L(x) = (\sin \alpha_0, \sin \gamma_1/q, \sin \gamma_2/q, \sin \alpha_3)$ is smooth with values in Ω near x , and if $D(\Phi \circ L)(x) = D\Phi(L) \circ DL(x)$ is onto, so is $D\Phi(L)$. Since $\chi = \varepsilon_1 = \tau = 0$ at x , differentiating (12) kills every term with an undifferentiated vanishing factor:

$$D(\Phi \circ L) = \Lambda \circ D(\chi, \varepsilon_1, \varepsilon_3),$$

$$\Lambda(x_0, x_1, x_2) = \left(e^{-i\alpha_3} (x_0 - i \frac{\sin T}{q} x_1) + i \sin \alpha_3 e^{-iS} (x_1 + x_2), x_1 + x_2 \right).$$

$\Lambda: \mathbb{R}^3 \rightarrow \mathbb{C} \times \mathbb{R}$ is injective (if it vanishes, $x_1 + x_2 = 0$, then $x_0 = x_1 = 0$), hence bijective. So $D\Phi(L)$ is onto as soon as the Jacobian of $(\chi, \varepsilon_1, \varepsilon_3)$ in either chart is invertible. Since $\varepsilon_1 = S + T - \pi - \text{def}_p(\alpha_0) - \text{def}_{t/q}(\gamma_1)$ and $\varepsilon_3 = S + T - \pi - \text{def}_l(\alpha_3) - \text{def}_{h/q}(\gamma_2)$ with $\text{def}_a(z) = z - \arcsin(a \sin z)$, these Jacobians are

$$J_I = \begin{pmatrix} \cos S & 0 & 0 \\ 1 & -A & -C \\ 1-B & B & D \end{pmatrix}, \quad J_{II} = \begin{pmatrix} \cos S & 0 & -\frac{\cos T}{q} \\ 1 & -A & 1 \\ 1-B & B & 1-D \end{pmatrix},$$

with $\det J_I = \cos S (BC - AD)$ and $\det J_{II} = -\cos S (A + B - AD) - \frac{\cos T}{q} (A + B - AB)$. As $\cos S \neq 0$, rank loss forces (14). $\square$

Corollary 5.3. *Under the hypotheses of Lemma 5.2, $D\Phi(L)$ is onto in each of the following cases: (a) $h > q$; (b) $T \geq \pi/2$; (c) $q = 1$.*

Proof. (a) By (13), $D < 0$, so $BC > 0 > AD$. (b) $A + B - AD = B + A(1 - D) > 0$, $A + B - AB = A + B(1 - A) > 0$, $\cos S < 0$ and $\cos T \leq 0$, so the second expression in (14) is negative. (c) Now $\sin S = \sin T$ and $S + T > \pi$. If $T < \pi/2$, then $0 < \pi - S < T < \pi/2$ gives $\sin S < \sin T$. So $T \geq \pi/2$, and (b) applies. $\square$

The remaining case, $h < q$ and $T < \pi/2$, is only needed at the orthogonal points of the next subsection, where it is settled by an elimination.

Proposition 5.4. *Let $0 < p, l < 1 < q$, $0 < t, h < q$, and $0 < u, v < 1$. If $L = \hat{L}(u, v)$ (see (15)) lies in Ω , $\Phi(L) = 0$ and $T < \pi/2$, then $D\Phi(L)$ is onto.*

The proof is in Appendix B.

5.3. The orthogonal core. For $q \geq 1$ and $u, v \in [0, 1]$ with $uv < q$ put $\Delta = q^2 - u^2v^2 > 0$ and

$$(15) \quad \hat{L}(u, v) = \Delta^{-1/2} \left(\sqrt{q^2 - v^2}, \sqrt{1 - v^2}, \sqrt{1 - u^2}, \sqrt{q^2 - u^2} \right).$$

Given p, t, h, l , the *residuals* are, where defined,

$$\begin{aligned} \rho_1(u, v) &= \arcsin(p\hat{L}_0) + \arcsin(t\hat{L}_1) - \arcsin(u\hat{L}_0) - \arcsin(v\hat{L}_1), \\ \rho_3(u, v) &= \arcsin(h\hat{L}_2) + \arcsin(l\hat{L}_3) - \arcsin(v\hat{L}_2) - \arcsin(u\hat{L}_3). \end{aligned}$$

Lemma 5.5 (Orthogonal core). *Let $q \geq 1$ and write $\hat{L} = \hat{L}(u, v)$.*

- (i) $\hat{L}_0^2 + (v\hat{L}_2)^2 = \hat{L}_3^2 + (u\hat{L}_1)^2 = (q\hat{L}_1)^2 + (v\hat{L}_3)^2 = (q\hat{L}_2)^2 + (u\hat{L}_0)^2 = 1$.
- (ii) Let $0 < u, v < 1$. The points $z_0 = 2\hat{L}_0\hat{L}_3$, $z_1 = 2iu\hat{L}_0\hat{L}_1$, $z_2 = -2q\hat{L}_1\hat{L}_2$, $z_3 = -2iv\hat{L}_2\hat{L}_3$ form a convex quadrilateral with perpendicular diagonals and sides $|z_{j+1} - z_j| = 2\hat{L}_j$, the circle through z_{j-1}, z_j, z_{j+1} has curvature k_j , where $(k_0, \dots, k_3) = (1, u, q, v)$, and the exterior angle at z_j is $\arcsin(k_j\hat{L}_{j-1}) + \arcsin(k_j\hat{L}_j)$. The controller with readings $(1, u, q, v)$, $(u, q, v, 1)$ vanishes at $\hat{L}$.
- (iii) If $0 < u, v < 1$, $\hat{L} \in \Omega$ and $\rho_1(u, v) = \rho_3(u, v) = 0$, then $\Phi(\hat{L}) = 0$.
- (iv) $\hat{L}(v, u)$ is $\hat{L}(u, v)$ in reverse order, so $\rho_3(u, v)$ equals $\rho_1(v, u)$ computed with (p, t) replaced by (l, h) .
- (v) Let $q > 1$ and $0 < p, l < 1$. Where defined, $\rho_1(0, v) = \arcsin(p\hat{L}_0) + \arcsin(t\hat{L}_1)$ with $\hat{L}_0 > 0$; on the face $v = 1$, $\hat{L}_1 = 0$, $\hat{L}_0 > 0$, and $\rho_1(u, 1)$ has the sign of $p - u$; and at the corner $(1, l)$, $\hat{L}_0 = 1$, $\hat{L}_1 = \sigma(q, l)$ and $\rho_1(1, l) = \varphi(p, t, q, l)$ if $t \leq q$. By (iv), $\rho_3(u, 0) = \arcsin(h\hat{L}_2) + \arcsin(l\hat{L}_3)$ with $\hat{L}_3 > 0$, $\rho_3(1, v)$ has the sign of $l - v$, $\rho_3(1, 1) = \arcsin l - \pi/2$, and $\rho_3(p, 1) = \arcsin l + \arcsin(h\sigma(q, p)) - \arcsin \sigma(q, p) - \pi/2$, which is $\varphi(l, h, q, p)$ if $h \leq q$.

Proof. (i) Each sum is Δ^{-1} times $q^2 - v^2 + v^2(1 - u^2)$, $q^2 - u^2 + u^2(1 - v^2)$, $q^2(1 - v^2) + v^2(q^2 - u^2)$ or $q^2(1 - u^2) + u^2(q^2 - v^2)$, all equal to Δ .

(ii) Here $\hat{L} > 0$. The z_j lie on the four half-axes in counterclockwise order, so they form a convex quadrilateral whose diagonals meet at 0. By Pythagoras and (i), $|z_1 - z_0|^2 = 4\hat{L}_0^2(\hat{L}_3^2 + u^2\hat{L}_1^2) =$

$4\hat{L}_0^2$, and likewise for the other sides. In the triangle $z_{j-1}z_jz_{j+1}$ the neighbours $z_{j\pm 1}$ lie on one axis on opposite sides of 0 and z_j on the other axis, so the foot of the altitude from z_j is 0, strictly between z_{j-1} and z_{j+1} , and both base angles are acute. The area is $\frac{1}{2}|z_{j+1} - z_{j-1}||z_j|$, so the circumradius is $|z_j - z_{j-1}||z_{j+1} - z_j|/(2|z_j|) = 1/k_j$ (for $j = 1$: $4\hat{L}_0\hat{L}_1/(4u\hat{L}_0\hat{L}_1)$). By the law of sines the base angle at z_{j+1} has $\sin \frac{k_j}{2}|z_j - z_{j-1}| = k_j\hat{L}_{j-1}$, so it equals $\arcsin(k_j\hat{L}_{j-1})$; likewise the base angle at z_{j-1} is $\arcsin(k_j\hat{L}_j)$. The exterior angle at z_j is π minus the interior angle, that is, the sum of the two base angles. In particular the moderate-arc inequalities $k_j\hat{L}_{j-1}, k_j\hat{L}_j < 1$ hold. Hence the turning law (3) for the profile k with half-lengths $\hat{L}$ returns the exterior angles, which sum to 2π , and the development reproduces the sides of the quadrilateral up to a rotation, so it closes. By Proposition 4.2 with $n = 4$ and no dormant edges, the controller vanishes at $\hat{L}$.

(iii) The pure turns β_0, β_2 of Φ at $\hat{L}$ are those of the controller in (ii), and ρ_1, ρ_3 are the differences of the mixed turns. So $\Phi(\hat{L})$ is the vanishing controller of (ii).

(iv) Both claims are immediate from (15), since Δ is symmetric.

(v) At $u = 0$ the subtracted terms vanish. At $v = 1$, $\hat{L} = (a, 0, \sqrt{1-u^2}/\sqrt{q^2-u^2}, 1)$ with $a = \sqrt{q^2-1}/\sqrt{q^2-u^2} > 0$, so $\rho_1(u, 1) = \arcsin(pa) - \arcsin(ua)$. At $(1, l)$, $\Delta = q^2 - l^2$, and the formula follows from (9). The rest follows from (iv) and $\hat{L}(1, 1) = (1, 0, 0, 1)$. $\square$

5.4. Crossed signs.

Lemma 5.6. *Let F, G be continuous on $Q = [u_1, u_2] \times [v_1, v_2]$, and suppose:*

- (1) $F(u_1, v) > 0$ and $G(u, v_1) > 0$ for all u, v ;
- (2) if $G(u_2, v) = 0$ then $F(u_2, v) < 0$, and if $F(u, v_2) = 0$ then $G(u, v_2) < 0$;
- (3) $G(u_2, v_2) < 0$.

Then F and G have a common zero in the interior of Q .

Proof. Let c_1 and c_2 be the clamps of $\mathbb{R}$ onto $[u_1, u_2]$ and $[v_1, v_2]$, and $R(u, v) = (u - c_1(u + F), v - c_2(v + G))$. Then $R_1 \leq 0$ on $u = u_1$, $R_1 \geq 0$ on $u = u_2$, $R_2 \leq 0$ on $v = v_1$ and $R_2 \geq 0$ on $v = v_2$, so by the Poincaré–Miranda theorem $R(z) = 0$ for some $z = (u, v) \in Q$. If $u = u_1$, then $F(z) \leq 0$, contradicting (1); so $u > u_1$, and likewise $v > v_1$. If $u = u_2$, then $F(z) \geq 0$; if also $v = v_2$, then $G(z) \geq 0$, contradicting (3), and if $v < v_2$, then $G(z) = 0$, contradicting (2). If $v = v_2$ and $u < u_2$, then $F(z) = 0$ and $G(z) \geq 0$, contradicting (2). So z is interior, and there $F(z) = G(z) = 0$. $\square$

6. EXISTENCE OF ANCHORS

We prove statements (R), (F) and (N) of Proposition 4.9. In each case μ is positive and adjacent-unequal with $\mu_0 = 1$, and we identify $\mathbb{Z}/n$ with $\{0, \dots, n-1\}$.

6.1. A repeated maximum.

Proposition 6.1 (Statement (R)). *If $\mu \leq 1$ and $\mu_J = 1$ for some $J \neq 0$, then μ has a regular anchor on the frame $(0, J-1, J, n-1)$.*

Proof. Adjacent inequality gives $2 \leq J \leq n-2$, so the frame is valid (Figure 5). Its readings are (10) with $q = 1$ and $p = \mu_1$, $t = \mu_{J-1}$, $h = \mu_{J+1}$, $l = \mu_{-1}$, which lie in $(0, 1)$ because they are at most 1 and adjacent to a reading equal to 1. For $u, v \in (0, 1)$ the orthogonal core with $q = 1$ is $\hat{L} = (\lambda, \lambda, \lambda', \lambda')$ with $\lambda = \sqrt{1-v^2}/\sqrt{1-u^2v^2}$ and $\lambda' = \sqrt{1-u^2}/\sqrt{1-u^2v^2}$ in $(0, 1)$, and $\rho_1 = \arcsin(p\lambda) + \arcsin(t\lambda) - 2\arcsin(u\lambda)$, $\rho_3 = \arcsin(h\lambda') + \arcsin(l\lambda') - 2\arcsin(v\lambda')$. Let $a = \frac{1}{2}\min(p, t, h, l)$ and $b = \frac{1}{2}(1 + \max(p, t, h, l))$, so $0 < a < p, t, h, l < b < 1$. Since $\arcsin$ is increasing, $\rho_1 > 0$ on $u = a$, $\rho_1 < 0$ on $u = b$, $\rho_3 > 0$ on $v = a$ and $\rho_3 < 0$ on $v = b$. By the Poincaré–Miranda theorem, applied to $(-\rho_1, -\rho_3)$ on $[a, b]^2$, there is a common zero $(u, v) \in (0, 1)^2$. All entries of $\hat{L}(u, v)$ and all of p, t, h, l are less than 1, so $\hat{L} \in \Omega$, and $\Phi(\hat{L}) = 0$ by Lemma 5.5(iii). Corollary 5.3(c) gives the rank. $\square$

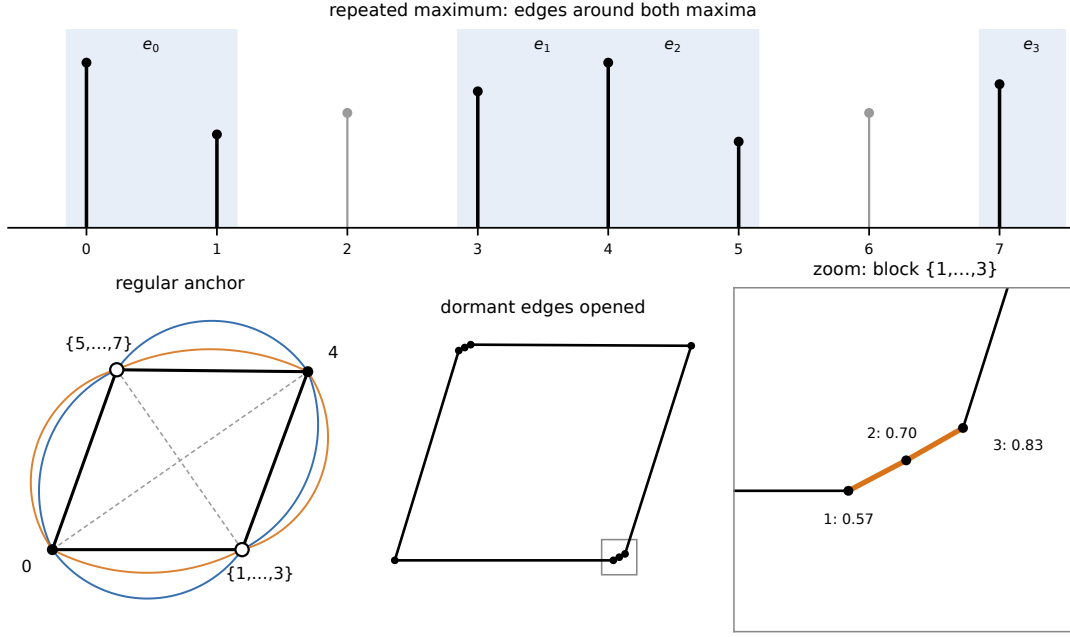


FIGURE 5. Construction (R) for $\mu = (1, 0.57, 0.70, 0.83, 1, 0.52, 0.70, 0.87)$, whose maximum 1 is attained at 0 and $J = 4$. The frame is $(0, J - 1, J, n - 1)$ and the anchor is an orthogonal quadrilateral (dashed diagonals; Lemma 5.5). Dormant readings are grey; the arcs are as in Figure 4. Bottom row: the anchor, the polygon after opening the dormant edges, and a zoom on the block $\{1, 2, 3\}$, which is a single point in the anchor and now has short dormant edges (orange).

6.2. Forward flag nonnegative.

Proposition 6.2 (Statement (F)). *Let 0 be a highest secondary peak relative to H , $\mu_H > 1$ and $\varphi^+ \geq 0$. Then μ has a regular anchor on a frame $(0, i - 1, i, n - 1)$ with $2 \leq i < H$.*

Choice of the frame. Let $p = \mu_1$ and $l = \mu_{-1}$, both in $(0, 1)$, and let a be as in Lemma 4.8(b). For $a - 1 \leq j \leq H - 1$ we have $\mu_j \leq \mu_{j+1}$ and $\mu_{j+1} > 1$, so $\varphi(p, \mu_j, \mu_{j+1}, l)$ is defined, and it is nonnegative for $j = H - 1$. Let i be the least such j with $\varphi(p, \mu_j, \mu_{j+1}, l) \geq 0$. For $j = a - 1$ the flag is negative: with $\sigma = \sigma(\mu_a, l)$, $\mu_{a-1} \leq 1$ gives $\arcsin(\mu_{a-1}\sigma) \leq \arcsin \sigma$, and $\arcsin p < \pi/2$. So $i \neq a - 1$. So $a \leq i < H$, and with

$$t = \mu_{i-1}, \quad q = \mu_i, \quad h = \mu_{i+1}$$

we have $0 < t < q$, $1 < q < h$ and

$$(16) \quad \varphi(p, q, h, l) \geq 0 > \varphi(p, t, q, l).$$

Since $2 \leq a \leq i$ and $i + 1 \leq H \leq n - 2$, $(0, i - 1, i, n - 1)$ is a frame with readings (10) (Figure 6). By Corollary 5.3(a), every zero of Φ in Ω is regular. If $t \leq p$, then $pt < 1$ and Proposition 6.6 below gives a zero; if $t > p$, Proposition 6.4 does. $\square$

Lemma 6.3. *Let $0 < p < 1$, $0 < l < 1 < q < h$, $\sigma = \sigma(h, l)$ and $\varphi(p, q, h, l) \geq 0$. Then (a) $\sigma < p$ and $\sqrt{1 - p^2}\sqrt{1 - \sigma^2} \leq (q - p)\sigma$; (b) $pq > 1$; (c) $(1 - p^2)(h^2 - 1) \leq (1 - l^2)(q - p)^2$.*

Proof. (a) As $q\sigma < h\sigma < 1$, $\arcsin(q\sigma) < \pi/2$, and the flag gives $\arcsin \sigma < \arcsin p$. The angle $\pi/2 - \arcsin p + \arcsin \sigma$ then lies in $(0, \pi/2)$ and is at most $\arcsin(q\sigma)$; take sines. (b) From $\sigma < p$, $\sqrt{1 - p^2}\sigma < p\sqrt{1 - \sigma^2}$, so by (a) $p(q - p)\sigma \geq p\sqrt{1 - p^2}\sqrt{1 - \sigma^2} > (1 - p^2)\sigma$. (c) Square (a) and multiply by $h^2 - l^2$, using $\sigma^2(h^2 - l^2) = 1 - l^2$ and $(1 - \sigma^2)(h^2 - l^2) = h^2 - 1$. $\square$

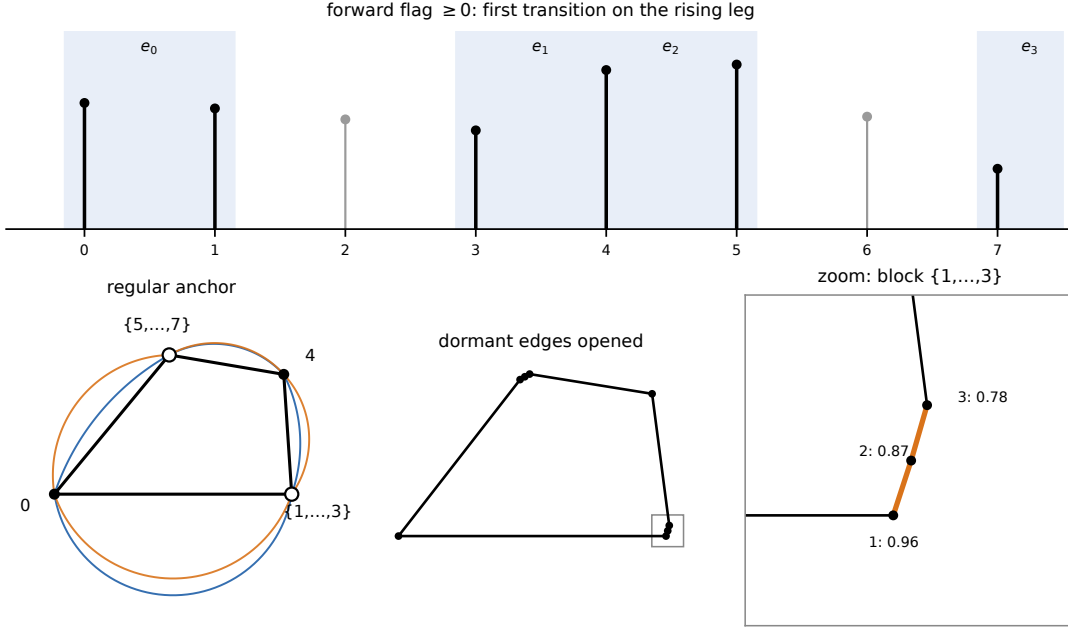


FIGURE 6. Construction (F) for $\mu = (1, 0.96, 0.87, 0.78, 1.26, 1.30, 0.89, 0.48)$, with maximum at $H = 5$. The frame $(0, i-1, i, n-1)$ sits at the first transition $i = 4$ on the rising leg towards H . Here $t = \mu_3 \leq p = \mu_1$, so the anchor comes from the chart used when $pt < 1$ and is not orthogonal. Bottom row: the anchor, the polygon after opening the dormant edges, and a zoom on the block $\{1, 2, 3\}$, which is a single point in the anchor and now has short dormant edges (orange).

The case $t > p$: an orthogonal core.

Proposition 6.4. *Let $0 < p < t < q$, $p < 1$, $0 < l < 1 < q < h$, and assume (16). Then Φ has a zero in Ω .*

Proof. Put $\bar{p} = \sqrt{1-p^2}$ and $a_* = \sqrt{(h^2-1)/(1-l^2)}$. Since $q > 1 > p$, Lemma 6.3(c) says $p + \bar{p}a_* \leq q$. As $\sigma(h, l) < p$ and $\sigma(h, l)^2(1+a_*^2) = 1$, we get $\bar{p} < pa_*$; also $a_*^2 \geq h^2 - 1$. Hence $(p + \bar{p}a_*)^2 - h^2\bar{p}^2 - p^2 = \bar{p}^2(a_*^2 - h^2) + 2p\bar{p}a_* > \bar{p}^2(a_*^2 - h^2 + 2) > 0$, that is,

$$(17) \quad q^2 - h^2 + p^2(h^2 - 1) > 0.$$

For $u, v \in [0, 1]$ we have the identities

$$\begin{aligned} q^2 - u^2v^2 - t^2(1-v^2) &= q^2 - t^2 + v^2(t^2 - u^2), \\ q^2 - u^2v^2 - h^2(1-u^2) &= (q^2 - h^2 + p^2(h^2 - 1)) + (u^2 - p^2)(h^2 - v^2) + p^2(1 - v^2). \end{aligned}$$

The first is positive (if $u > t$ it is at least $q^2 - u^2$), so $t\hat{L}_1 < 1$ on $[0, 1]^2$. By (17) there is $p' \in (0, p)$ with $(p^2 - p'^2)h^2 < q^2 - h^2 + p^2(h^2 - 1)$, and then the second is positive for $u \geq p'$, so $h\hat{L}_2 < 1$ on $[p', 1] \times [0, 1]$.

Fix $l' \in (0, l)$ and apply Lemma 5.6 to $F = \rho_1$, $G = \rho_3$ on $Q = [p', 1] \times [l', 1]$. Both are continuous on Q , since all arguments of arcsin lie in $[0, 1]$ by Lemma 5.5(i) and the two walls just proved. On $u = p'$, $\rho_1 = (\arcsin(p\hat{L}_0) - \arcsin(p'\hat{L}_0)) + (\arcsin(t\hat{L}_1) - \arcsin(p'\hat{L}_1)) > 0$ since $\hat{L}_0 > 0$ and $p' < p < t$. On $v = l'$, likewise $\rho_3 > 0$ since $\hat{L}_3 > 0$ and $l' < l < h$. By Lemma 5.5(v), $\rho_3(1, v) = 0$ only for $v = l$, where $\rho_1(1, l) = \varphi(p, t, q, l) < 0$ by (16); $\rho_1(u, 1) = 0$ only for $u = p$, where $\rho_3(p, 1) < 0$ by Lemma 6.5; and $\rho_3(1, 1) = \arcsin l - \pi/2 < 0$. So there is a common zero (u, v) in the interior of Q . There $\hat{L} > 0$, $\hat{L}_0, \hat{L}_3, q\hat{L}_1, q\hat{L}_2 < 1$ by Lemma 5.5(i), hence $p\hat{L}_0, l\hat{L}_3 < 1$, and $t\hat{L}_1, h\hat{L}_2 < 1$ were shown above. So $\hat{L} \in \Omega$ and $\Phi(\hat{L}) = 0$ by Lemma 5.5(iii). $\square$

Lemma 6.5. *Under the hypotheses of Proposition 6.4, with $\sigma_p = \sigma(q, p)$, we have $h\sigma_p < 1$ and $\rho_3(p, 1) = \arcsin l + \arcsin(h\sigma_p) - \arcsin \sigma_p - \pi/2 < 0$.*

Proof. The function $g(x) = \arcsin(hx) - \arcsin x$ is strictly increasing on $[0, 1/h]$, because $h/\sqrt{1-h^2x^2} > 1/\sqrt{1-x^2}$. By (17), $h^2\sigma_p^2(q^2 - p^2) = h^2(1 - p^2) < q^2 - p^2$, so $\sigma_p < 1/h$. If $hl \leq 1$, then $g(\sigma_p) < g(1/h) = \frac{\pi}{2} - \arcsin \frac{1}{h} \leq \frac{\pi}{2} - \arcsin l$. If $hl > 1$, put $d_0 = h^2 + 1 - 2hl > 0$ and $x_0 = \sqrt{(1-l^2)/d_0}$. Then $d_0(1 - h^2x_0^2) = (hl - 1)^2$ and $d_0(1 - x_0^2) = (h - l)^2$, so $x_0 < 1/h$ and $\cos g(x_0) = ((hl - 1)(h - l) + h(1 - l^2))/d_0 = l$, that is, $g(x_0) = \frac{\pi}{2} - \arcsin l$. Finally $\sigma_p < x_0$: using $(1 - l^2)a_*^2 = h^2 - 1$,

$$(1 - l^2)((p + \bar{p}a_*)^2 - p^2) - \bar{p}^2d_0 = 2(1 - l^2)p\bar{p}a_* + 2\bar{p}^2(hl - 1) > 0,$$

and $(p + \bar{p}a_*)^2 \leq q^2$ gives $\bar{p}^2d_0 < (1 - l^2)(q^2 - p^2)$, that is, $\sigma_p^2 < x_0^2$. $\square$

The case $pt < 1$: a chart.

Proposition 6.6. *Let $0 < p < 1$, $0 < l < 1 < q < h$, $t > 0$, $pt < 1$ and $\varphi(p, q, h, l) \geq 0$. Then Φ has a zero in Ω .*

By Lemma 6.3(b), $pq > 1 > pt$, so $t < q$. The chart solves $\chi = 0$ and $\varepsilon_1 = 0$ for (α_3, γ_2) given (α_0, γ_1) . For $(\alpha_0, \gamma_1) \in \mathbb{R}^2$ let $\beta_1 = \arcsin(p \sin \alpha_0) + \arcsin(\frac{t}{q} \sin \gamma_1)$ and

$$X = q \cos(\beta_1 - \alpha_0) - \cos \gamma_1, \quad Y = \sin \gamma_1 - q \sin(\beta_1 - \alpha_0).$$

For every γ_2 , putting $\alpha_3 = \pi - \gamma_2 - \beta_1$ (that is, $\varepsilon_1 = 0$) gives $q\chi = X \sin \gamma_2 - Y \cos \gamma_2$, by expanding $\sin(\gamma_2 + \beta_1 - \alpha_0)$. Where $X > 0$ let $\hat{\gamma}_2 = \arctan(Y/X)$ and $\hat{\alpha}_3 = \pi - \hat{\gamma}_2 - \beta_1$, so that $\chi = \varepsilon_1 = 0$. Let $\mathcal{W}$ be the open set where $X > 0$, $Y > 0$, $0 < \hat{\alpha}_3 < \pi/2$ and $\frac{h}{q} \sin \hat{\gamma}_2 < 1$, and on $\mathcal{W}$ let

$$\mathcal{E}(\alpha_0, \gamma_1) = \alpha_0 + \gamma_1 + \arcsin\left(\frac{h}{q} \sin \hat{\gamma}_2\right) + \arcsin(l \sin \hat{\alpha}_3) - \pi$$

be the value of ε_3 . It is continuous, and by Lemma 5.1 every zero of $\mathcal{E}$ in $\mathcal{W} \cap (0, \pi/2)^2$ gives the zero $(\sin \alpha_0, \sin \gamma_1/q, \sin \hat{\gamma}_2/q, \sin \hat{\alpha}_3) \in \Omega$ of Φ (here $\hat{\gamma}_2 \in (0, \pi/2)$, and $t < q$).

The wall $\alpha_0 = \pi/2$. There $\beta_1 = \arcsin p + \arcsin(\frac{t}{q} \sin \gamma_1)$, and for $\gamma_1 \in [0, \pi/2]$

$$X = p\sqrt{q^2 - t^2 \sin^2 \gamma_1} + t\sqrt{1 - p^2} \sin \gamma_1 - \cos \gamma_1,$$

$$Y = \sqrt{1 - p^2} \sqrt{q^2 - t^2 \sin^2 \gamma_1} + (1 - pt) \sin \gamma_1.$$

So $Y > 0$, and $X > 0$ because $p^2(q^2 - t^2 \sin^2 \gamma_1) - \cos^2 \gamma_1 = (p^2q^2 - 1) + (1 - p^2t^2) \sin^2 \gamma_1 > 0$. On the wall $\chi = 0$ reads $\cos \hat{\alpha}_3 = \sin(\gamma_1 + \hat{\gamma}_2)/q > 0$. At $\gamma_1 = 0$, $\hat{\alpha}_3 = \pi - \hat{\gamma}_2 - \arcsin p > 0$, and $\hat{\alpha}_3 = 0$ would force $q = \sin(\gamma_1 + \hat{\gamma}_2) \leq 1$; so $0 < \hat{\alpha}_3 < \pi/2$ on the wall. At the corner $\gamma_1 = 0$ we have $Y = q\sqrt{1 - p^2}$, $X = pq - 1$, and with $Z = \sqrt{q^2 - 2pq + 1}$, $\sin \hat{\gamma}_2 = q\sqrt{1 - p^2}/Z$ and $\sin \hat{\alpha}_3 = (q - p)/Z$. For $x = \frac{h}{q} \sin \hat{\gamma}_2$ and $y = l \sin \hat{\alpha}_3$,

$$Z^2(1 - x^2 - y^2) = (1 - l^2)(q - p)^2 - (1 - p^2)(h^2 - 1) \geq 0,$$

$$(1 - l^2)Z^2(1 - x^2) = [(1 - l^2)(q - p)^2 - (1 - p^2)(h^2 - 1)] + l^2(1 - p^2)(h^2 - 1) > 0$$

by Lemma 6.3(c). So $x < 1$ and $\arcsin x + \arcsin y \leq \pi/2$, that is,

$$(18) \quad \left(\frac{\pi}{2}, 0\right) \in \mathcal{W} \quad \text{and} \quad \mathcal{E}\left(\frac{\pi}{2}, 0\right) = \arcsin x + \arcsin y - \frac{\pi}{2} \leq 0.$$

Exit. Let $f(\gamma_1) = \frac{h}{q} \sin \hat{\gamma}_2(\frac{\pi}{2}, \gamma_1)$ and $g(\gamma_1) = \mathcal{E}(\frac{\pi}{2}, \gamma_1)$ on $[0, \pi/2]$; g is continuous where $f \leq 1$, and $f(0) < 1$. If $f = 1$ somewhere, let e be the first such point; then $g(e) = e + \arcsin(l \sin \hat{\alpha}_3) > 0$. Otherwise let $e = \pi/2$; then $g(e) = \arcsin(\frac{h}{q} \sin \hat{\gamma}_2) + \arcsin(l \sin \hat{\alpha}_3) > 0$. Choose $\gamma_{hi} < e$ with $g(\gamma_{hi}) > 0$. Then $\{\pi/2\} \times [0, \gamma_{hi}] \subset \mathcal{W}$.

Inward derivative. At $\gamma_1 = 0$, $\beta_1 = \arcsin(p \sin \alpha_0)$ has derivative 0 at $\alpha_0 = \pi/2$. There $\partial_{\alpha_0} X = -q\sqrt{1 - p^2}$ and $\partial_{\alpha_0} Y = qp$, so $\partial_{\alpha_0} \hat{\gamma}_2 = q(pX + \sqrt{1 - p^2}Y)/(X^2 + Y^2) > 0$, $\partial_{\alpha_0} \hat{\alpha}_3 = -\partial_{\alpha_0} \hat{\gamma}_2$, and $\partial_{\alpha_0} \mathcal{E}(\frac{\pi}{2}, 0) = 1 + (\varsigma_{h/q}(\hat{\gamma}_2) - \varsigma_l(\hat{\alpha}_3)) \partial_{\alpha_0} \hat{\gamma}_2 > 1$ by (13).

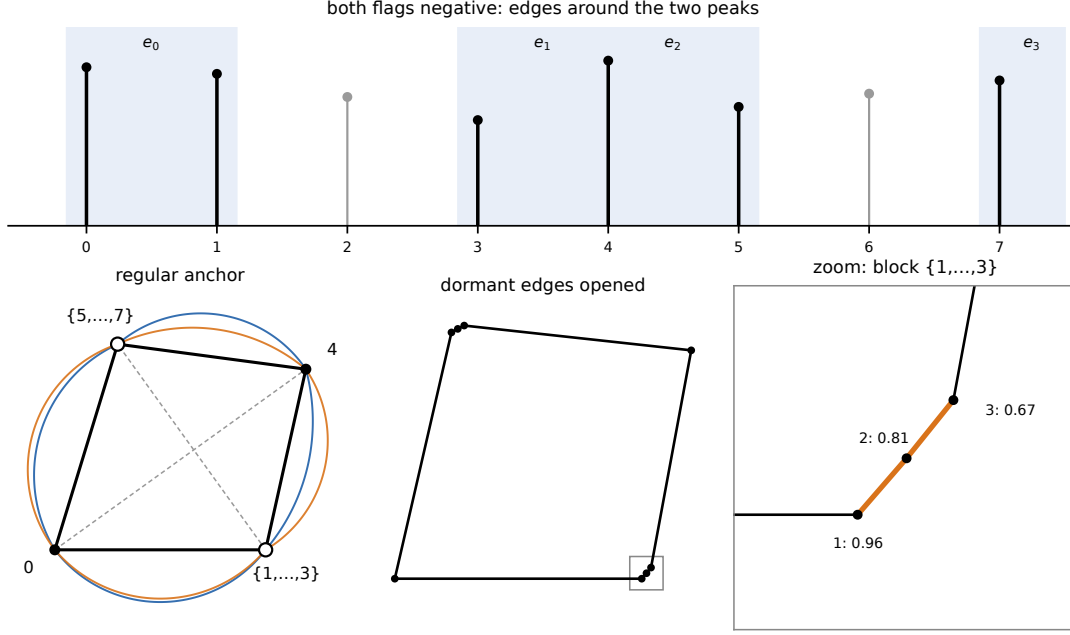


FIGURE 7. Construction (N) for $\mu = (1, 0.96, 0.81, 0.67, 1.04, 0.75, 0.83, 0.92)$: the secondary peak is 0 and the maximum is $H = 4$, and both flags are negative. The frame is $(0, H - 1, H, n - 1)$ and the anchor is an orthogonal quadrilateral (dashed diagonals; Lemma 5.5). Bottom row: the anchor, the polygon after opening the dormant edges, and a zoom on the block $\{1, 2, 3\}$, which is a single point in the anchor and now has short dormant edges (orange).

Proof of Proposition 6.6. Since $\{\pi/2\} \times [0, \gamma_{hi}]$ is compact, the segments $\{\alpha_0\} \times [0, \gamma_{hi}]$ lie in $\mathcal{W}$ and $\mathcal{E}(\alpha_0, \gamma_{hi}) > 0$ for α_0 near $\pi/2$. By (18) and the inward derivative (needed when $\mathcal{E}(\frac{\pi}{2}, 0) = 0$), $\mathcal{E}(\alpha_0, 0) < 0$ for $\alpha_0 < \pi/2$ near $\pi/2$. For such an $\alpha_0 \in (0, \pi/2)$ the intermediate value theorem gives $\gamma_1 \in (0, \gamma_{hi})$ with $\mathcal{E}(\alpha_0, \gamma_1) = 0$. $\square$

6.3. Both flags negative.

Proposition 6.7 (Statement (N)). *Let 0 be a highest secondary peak relative to H , $\mu_H > 1$, $\varphi^+ < 0$ and $\varphi^- < 0$. Then μ has a regular anchor on the frame $(0, H - 1, H, n - 1)$.*

Proof. By Lemma 4.8, $2 \leq H \leq n - 2$ and $\mu_{H \pm 1} < \mu_H$. The frame (Figure 7) has readings (10) with $i = H$: $p = \mu_1$, $t = \mu_{H-1}$, $q = \mu_H$, $h = \mu_{H+1}$, $l = \mu_{-1}$, where $0 < p, l < 1 < q$ and $t, h < q$; moreover $\varphi^+ = \varphi(p, t, q, l)$ and $\varphi^- = \varphi(l, h, q, p)$. Apply Lemma 5.6 to $F = \rho_1$, $G = \rho_3$ on $[0, 1]^2$; they are continuous there, because all arguments of arcsin lie in $[0, 1]$ by Lemma 5.5(i). By Lemma 5.5(v): $\rho_1(0, v) > 0$ and $\rho_3(u, 0) > 0$; $\rho_3(1, v) = 0$ forces $v = l$, where $\rho_1(1, l) = \varphi^+ < 0$; $\rho_1(u, 1) = 0$ forces $u = p$, where $\rho_3(p, 1) = \varphi^- < 0$; and $\rho_3(1, 1) < 0$. So there is a common zero $(u, v) \in (0, 1)^2$. By Lemma 5.5(i), $\hat{L}_0, \hat{L}_3, q\hat{L}_1, q\hat{L}_2 < 1$, and with $p, l < 1$ and $t, h < q$ all eight walls hold, so $\hat{L} \in \Omega$ and $\Phi(\hat{L}) = 0$ by Lemma 5.5(iii). If $T \geq \pi/2$, Corollary 5.3(b) gives the rank, and otherwise Proposition 5.4. $\square$

7. CONTINUITY

We identify a curvature profile with a vector $\kappa \in \mathbb{R}^n$ and a labelled polygon with a vector $(P_i)_{i \in \mathbb{Z}/n} \in \mathbb{C}^n$, with the product topologies.

7.1. Local continuity.

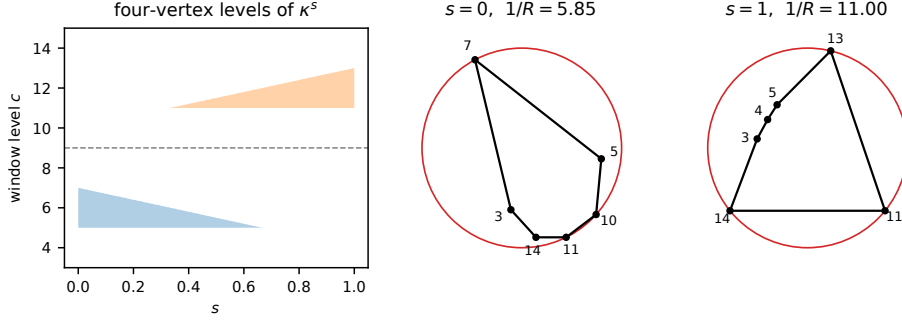


FIGURE 8. Left: the levels of strict four-vertex windows of κ^s . Right: realizing hexagons at $s = 0$ and $s = 1$ with their smallest enclosing circles. The reciprocal enclosing radius is at most 7 at $s = 0$, at least 11 at $s = 1$, and never 9, so it cannot vary continuously.

Theorem 7.1. *Let $n \geq 4$ and let $\kappa^0 \in \mathbb{R}^n$ be positive with a strict four-vertex window (1). There are an open neighbourhood V of κ^0 and C^1 maps $h: V \rightarrow \mathbb{R}^n$ and $P: V \rightarrow \mathbb{C}^n$ such that for every $\kappa \in V$ the half-lengths $h(\kappa)$ realize κ and $P(\kappa)$ is the polygon developed from them by (4), normalized by $P_0 = 0$. In particular $P(\kappa)$ is simple, strictly convex and coherent, with curvature κ_i at every vertex.*

Proof. By Propositions 4.7 and 4.9 and Section 6, κ^0 has a regular anchor, and Theorem 4.4 gives V and h . The vertices (4) are real-analytic in (κ, h) , so P is C^1 . The remaining properties are Section 3. $\square$

7.2. No global continuous choice. For $0 \leq s \leq 1$ consider the hexagon profile

$$(19) \quad \kappa^s = (14, 11, 10 + 3s, 5, 7 - 3s, 3).$$

Every κ^s is positive and nonconstant. A direct check of all quadruples of indices shows that the levels c of strict four-vertex windows of κ^s form the set

$$(5, 7 - 3s) \cup (11, 10 + 3s),$$

with an empty interval when the endpoints are reversed. The lower band is nonempty for $s < 2/3$ (window indices 0, 3, 4, 5), the upper band for $s > 1/3$ (indices 0, 1, 2, 5). Hence every κ^s is admissible and, by Theorem 1.1, realized by a coherent convex hexagon.

A polygon $(P_i)_{i \in \mathbb{Z}/n}$, not necessarily developed, *realizes* κ if for every i the circle through P_{i-1}, P_i, P_{i+1} has radius $1/\kappa_i$.

Theorem 7.2. *There is no continuous map $s \mapsto P^s \in \mathbb{C}^6$, $s \in [0, 1]$, such that each P^s is a coherent convex hexagon realizing κ^s .*

The invariant is the smallest enclosing circle. For a finite set X of points let $D(X)$ be its smallest enclosing closed disk and $R(X)$ its radius; $D(X)$ exists and is unique, and R is 1-Lipschitz for the maximum distance between labelled tuples, since enlarging a disk enclosing one tuple by that distance encloses the other. The *contacts* are the points of X on $\partial D(X)$.

Lemma 7.3. *$D(X)$ is also the smallest enclosing disk of its set of contacts.*

Proof. Let $D(X)$ have center O and radius R , and suppose a disk with center O' and radius $R' < R$ contains every contact. For $O_t = (1 - t)O + tO'$ and every contact x ,

$$|O_t - x|^2 = (1 - t)|O - x|^2 + t|O' - x|^2 - t(1 - t)|O - O'|^2 \leq (1 - t)R^2 + tR'^2 < R^2,$$

and non-contacts satisfy $|O - x| < R$, hence $|O_t - x| < R$ for small $t > 0$. Then a disk of radius smaller than R encloses X , a contradiction. $\square$

In the next two lemmas P is a coherent convex polygon with curvatures κ_i , $D = D(P)$ has radius R , and ω_i is the circle through P_{i-1}, P_i, P_{i+1} , of radius $1/\kappa_i$.

Lemma 7.4. *If P_i is a contact, then $\kappa_i \geq 1/R$. If all contacts lie in $\{P_{i-1}, P_i, P_{i+1}\}$, then $\kappa_i \leq 1/R$.*

Proof. Place P_i at the origin with $D = \{z : |z|^2 \leq 2\langle O, z \rangle\}$, $|O| = R$. The circle ω_i is $\{|z|^2 = 2\langle C, z \rangle\}$ with $|C| = 1/\kappa_i$. By coherence $C = aQ_- + bQ_+$ with $a, b \geq 0$, where $Q_\pm = P_{i\pm 1}$. Since $Q_\pm \in \omega_i \cap D$,

$$|C|^2 = a\langle C, Q_- \rangle + b\langle C, Q_+ \rangle = \frac{1}{2}(a|Q_-|^2 + b|Q_+|^2) \leq a\langle O, Q_- \rangle + b\langle O, Q_+ \rangle = \langle O, C \rangle \leq R|C|,$$

so $1/\kappa_i = |C| \leq R$. For the second claim, the closed disk bounded by ω_i contains all contacts, so its radius is at least R by Lemma 7.3. $\square$

Lemma 7.5. *If the disk bounded by ω_j contains every vertex, then $\kappa_j \leq \kappa_{j-1}$ and $\kappa_j \leq \kappa_{j+1}$. If P is not inscribed in a circle, there are two vertices $j \neq j'$ with $\omega_j \neq \omega_{j'}$ whose disks contain every vertex.*

Proof. The circles ω_j and ω_{j+1} both pass through $A = P_j$ and $B = P_{j+1}$, and by coherence both centers lie in the open half-plane H bounded by the line AB that contains the polygon. For circles through A and B with centers in H , the parts of their disks in H increase strictly with the radius. The vertex P_{j+2} lies in H , on ω_{j+1} and in the disk of ω_j ; if ω_{j+1} had the larger radius, P_{j+2} would lie in the open disk of ω_{j+1} . Hence $1/\kappa_{j+1} \leq 1/\kappa_j$, and symmetrically for $j-1$. The second statement is Dahlberg's containing-circle lemma [3], which applies to strictly convex polygons not inscribed in a circle. $\square$

Proof of Theorem 7.2. Suppose P^s is such a family and put $c(s) = 1/R(P^s)$, a continuous function. Index the vertices $0, \dots, 5$ as in (19).

$c(s) \neq 9$. If $c(s) = 9$, the contacts have $\kappa_i \geq 9$ by Lemma 7.4, so they lie in $\{0, 1, 2\}$. The same lemma at vertex 1 gives $11 = \kappa_1 \leq 9$, which is absurd.

$c(0) \leq 7$. If $c(0) > 7$, the contacts lie in $\{0, 1, 2\}$, so $c(0) \geq \kappa_1 = 11$. Then vertex 2, with $\kappa_2 = 10$, is not a contact, the contacts lie in $\{0, 1\} \subset \{5, 0, 1\}$, and $c(0) \geq \kappa_0 = 14$. Now the only possible contact is vertex 0, and by Lemma 7.3 the enclosing radius would be 0.

$c(1) \geq 11$. Here $\kappa^1 = (14, 11, 13, 5, 4, 3)$. The hexagon is not inscribed in a circle, since its profile is not constant. By Lemma 7.5 there are two distinct vertices whose curvature disks contain the polygon, and each is a weak local minimum of κ^1 . The only weak local minima are vertices 1 and 5, so vertex 1 qualifies, and $R \leq 1/\kappa_1 = 1/11$.

By the intermediate value theorem $c(s) = 9$ for some s , a contradiction. $\square$

The argument uses only continuity of the labelled vertices. Whether a global continuous choice exists for $n = 4$ or $n = 5$ is left open.

8. CIRCLES TANGENT TO CONSECUTIVE SIDES

There is a second discrete curvature attached to a convex polygon: the reciprocal radius of the inward circle tangent to three consecutive supporting lines. Wegner's four-edge theorem concerns these radii [13, Theorem 1]; see also [9, Corollary 21.11]. We give a converse as a companion to Theorem 1.1. The construction below uses a tangential polygon and positive edge scaling; it is independent of the four-length system used for the main theorem.

Definition 8.1. Let P_i be the vertices of a counterclockwise strictly convex polygon, with outward unit normal N_i on edge $[P_i, P_{i+1}]$. The *inward three-line circle at edge i* has center C_i and radius $r_i > 0$ satisfying

$$(20) \quad \langle N_{i-1}, P_i - C_i \rangle = \langle N_i, P_i - C_i \rangle = \langle N_{i+1}, P_{i+1} - C_i \rangle = r_i.$$

Its central contact is required to lie in the open edge (P_i, P_{i+1}) . The contacts on the two neighbouring supporting lines may lie on their extensions. The circle need not be contained in the entire polygon.

Put $c_i = r_{i-1} - r_i$. By four cyclic sign changes we mean that there are $i_1 < i_2 < i_3 < i_4 < i_1 + n$ with $c_{i_1} > 0$, $c_{i_2} < 0$, $c_{i_3} > 0$ and $c_{i_4} < 0$; zero differences are retained but do not count as a sign. This formulation includes repeated radii without a genericity assumption.

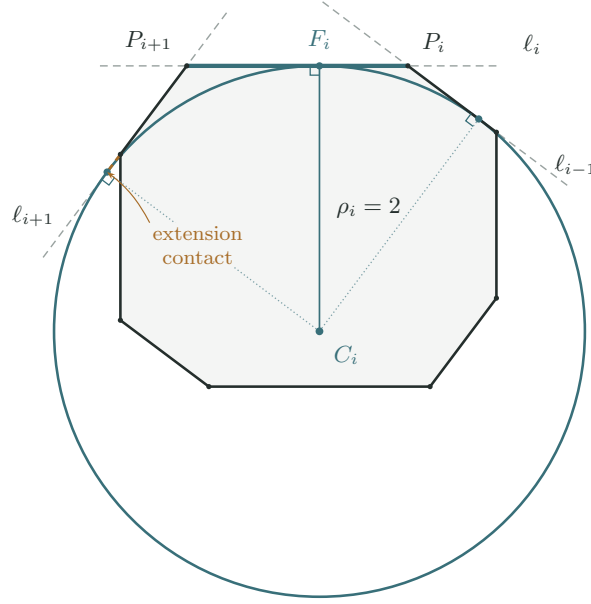


FIGURE 9. The selected circle is tangent to three supporting lines, with its central contact in the open edge. Here one neighbouring contact lies beyond its edge. All coordinates, circle equations and contact parameters in this example are checked with exact rational arithmetic.

Theorem 8.2. *Let $n \geq 4$ and let $r_i > 0$ be a cyclic sequence. Suppose that r is constant or that $c_i = r_{i-1} - r_i$ has four cyclic sign changes. Then a simple strictly convex polygon realizes r_i as the radii of its inward three-line circles. Each center and radius are uniquely determined by the three equations (20), and every central contact lies in its open edge.*

The main ingredient is a balanced polygon circumscribed about the unit circle. Its vertices, rather than the contact points, carry the coefficients.

Lemma 8.3. *Suppose $\sum_i c_i = 0$ and c has four cyclic sign changes. There is a strictly convex polygon Y_i , circumscribed about the unit circle with each contact in its open edge, such that $\sum_i c_i Y_i = 0$.*

Proof. Choose four alternating nonzero coefficients, and place their labels at $(1, 1), (-1, 1), (-1, -1), (1, -1)$ in that order. Collapse all intervening labels to the corresponding side contact of the square. All labels and their coefficients remain present. Denote this limiting configuration by X_i , and consider the finite rational field

$$G(a) = \sum_i \frac{c_i X_i}{1 - \langle a, X_i \rangle}.$$

For sufficiently small $\varepsilon > 0$, all denominators are positive on $D_\varepsilon = \{(x, y) : |x| + |y| \leq 1 - \varepsilon\}$. On its upper boundary arc $y \geq |x|$, one of the two upper corner denominators equals ε . If the alternating corner coefficients are $\alpha, -\beta, \gamma, -\delta$ with $\alpha, \beta, \gamma, \delta > 0$, and the left and right contact coefficients are u, v , then

$$\operatorname{Re} G(x, y) \geq \frac{\min(\alpha, \beta)}{\varepsilon} - \gamma - \delta - 2|u| - 2|v| > 0.$$

The upper and lower contact labels have zero horizontal contribution, including near their poles. The corresponding bounds on the other three arcs give, successively, $\operatorname{Im} G < 0$ on the left, $\operatorname{Re} G < 0$ on the bottom and $\operatorname{Im} G > 0$ on the right. Thus the boundary field has winding number -1 .

Now separate the repeated normals into small, strictly ordered clusters. This produces a genuine tangential polygon X_i^η , with every edge positive and every contact interior, converging

to the labelled square as $\eta \downarrow 0$. Positivity of all denominators and the strict boundary margins persist uniformly on the compact diamond. The perturbed field has the same nonzero winding number and hence a zero a in D_ε . In particular $|a| < 1$.

Set $\sigma = \sqrt{1 - |a|^2}$ and

$$A = \sigma I + \frac{aa^\top}{1 + \sigma}, \quad B_a(x) = \frac{Ax - a}{1 - \langle a, x \rangle}.$$

This projective transformation preserves the unit disk, its tangent lines, and the cyclic order and open contacts of our polygon. All relevant denominators are positive. Since $\sum_i c_i = 0$,

$$\sum_i \frac{c_i}{1 - \langle a, X_i^\eta \rangle} = \langle a, G(a) \rangle = 0, \quad \sum_i c_i B_a(X_i^\eta) = AG(a) - a \langle a, G(a) \rangle = 0.$$

The vertices $Y_i = B_a(X_i^\eta)$ have the required properties. $\square$

Lemma 8.4. *Let Y_i be a counterclockwise strictly convex polygon circumscribed about the unit circle, with its contacts in its open edges. If $r_i > 0$ and $\sum_i (r_{i-1} - r_i)Y_i = 0$, the cyclic edge equations*

$$(21) \quad P_{i+1} - P_i = r_i(Y_{i+1} - Y_i)$$

define a closed simple strictly convex polygon. Its inward three-line circle at edge i has center $C_i = P_i - r_i Y_i$ and radius r_i .

Proof. Cyclic summation by parts gives $\sum_i r_i(Y_{i+1} - Y_i) = \sum_i (r_{i-1} - r_i)Y_i = 0$, so the edges close. Positive scaling preserves their ordered directions and their total turn of 2π . For any fixed edge, the projections of the subsequent edges on its left normal are first positive and then negative. Every intermediate partial sum is positive, using closure for the second part. Hence every nonincident vertex is strictly to its left. This gives strict support and simplicity.

The outward normals satisfy $\langle N_{i-1}, Y_i \rangle = \langle N_i, Y_i \rangle = 1$ and $\langle N_{i+1}, Y_{i+1} \rangle = 1$. Substituting $C_i = P_i - r_i Y_i = P_{i+1} - r_i Y_{i+1}$ gives all three signed-distance equations. The central contact is the image of N_i under the positive homothety $x \mapsto C_i + r_i x$, and therefore lies in its open edge. Adjacent normals are linearly independent. The first two equations force any other center with radius s to equal $P_i - sY_i$; the third then gives $s = r_i$, since $\langle N_{i+1}, Y_i \rangle < 1$. Finally, unit normals turn the signed-distance equations into actual circle tangency. $\square$

Proof of Theorem 8.2. For nonconstant data, apply Lemma 8.3 to $c_i = r_{i-1} - r_i$, whose sum is zero, and then Lemma 8.4. For constant $r_i = r$, take a regular polygon circumscribed about the unit circle and scale its edges by r . $\square$

The forward implication is Wegner's four-edge theorem [13, Theorem 1], whose convention already includes plateau extrema. For the supporting-line circles of Definition 8.1, we give a proof using the signed affine-balance argument of [9, Theorem 21.15], allowing zero coefficients.

Lemma 8.5. *Let Y_i be the cyclic vertices of a counterclockwise strictly convex polygon. If $\sum_i c_i = 0$ and $\sum_i c_i Y_i = 0$, then either every $c_i = 0$ or the cyclic coefficient sequence has at least four sign changes.*

Proof. A nonzero zero-sum sequence has both signs. If it has no four alternating signs, it has one nonpositive block and one nonnegative block, with zeros allowed in either. Choose a negative entry c_a beginning the first block and a positive entry c_b beginning the second, using cyclic lifts $a < b < a + n$. The chord $Y_a Y_b$ has strictly negative signed area against vertices in (a, b) and strictly positive signed area against those in $(b, a + n)$. Thus

$$c_i[Y_a, Y_b, Y_i] \geq 0 \quad \text{for every } i, \quad \sum_i c_i[Y_a, Y_b, Y_i] = 0,$$

where $[A, B, C]$ is the signed determinant. The equality follows from the zero mass and zero vector sum, since the determinant is affine in its third argument. Every summand vanishes. Strict chord separation forces all coefficients except c_a, c_b to vanish. The two balance equations then give $c_a(Y_a - Y_b) = 0$, contradicting $c_a < 0$ and distinctness of the vertices. $\square$

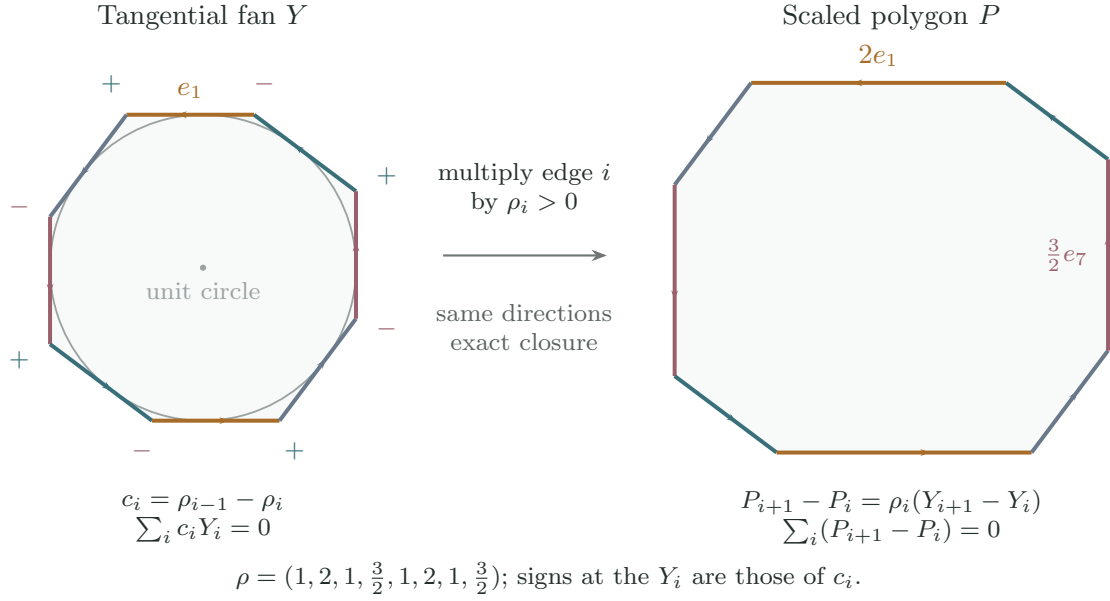


FIGURE 10. A balanced polygon circumscribed about the unit circle (left) and its positive edge scaling (right). The prescribed cyclic radii are $(1, 2, 1, 3/2, 1, 2, 1, 3/2)$. Closure, strict support and all eight circles are verified exactly. The highlighted upper edge is the central edge of the circle in Figure 9.

Corollary 8.6. *Let $n \geq 4$ and $r_i > 0$. The sequence r is realized by the inward three-line circles of a simple strictly convex polygon, with their central contacts in the open edges, if and only if r is constant or its cyclic differences $r_{i-1} - r_i$ have at least four sign changes.*

Proof. Sufficiency is Theorem 8.2. For necessity, start from the actual polygon and its circles, and set $Y_i = (P_i - C_i)/r_i$. The two incident normal equations give $\langle N_{i-1}, Y_i \rangle = \langle N_i, Y_i \rangle = 1$. These normals are linearly independent, because the polygon is strictly convex. The circle at the preceding edge satisfies the same two normalized equations at P_i , so it gives the same Y_i . Consequently

$$Y_{i+1} - Y_i = \frac{P_{i+1} - P_i}{r_i}.$$

This is a closed positive edge scaling of the original polygon; the strict support argument in Lemma 8.4 therefore applies to Y . Summation by parts gives $\sum_i (r_{i-1} - r_i)Y_i = 0$, and the differences themselves sum to zero. Lemma 8.5 supplies four sign changes unless every difference vanishes. In the latter case the cyclic radius sequence is constant. $\square$

9. TURNING-ANGLE CURVATURE AND THE MIXED-SIGN PROBLEM

The choice of discrete curvature matters. Two elementary examples explain why the results above neither extend to turning angles without further hypotheses nor give a converse for arbitrary signed Menger profiles.

9.1. Turning angles need not have four extrema. If curvature is defined simply as the exterior turning angle, a strictly convex polygon can have only one local maximum and one local minimum.

Example 9.1. Take the quadrilateral with counterclockwise vertices

$$A = (0, 0), \quad B = (3, 0), \quad C = (4, 1), \quad D = (0, 2).$$

It is strictly convex. Its exterior turns, in the same order, are

$$\theta_A = \frac{\pi}{2}, \quad \theta_B = \frac{\pi}{4}, \quad \theta_C = \frac{\pi}{2} + \arctan \frac{3}{5}, \quad \theta_D = \frac{\pi}{2} + \arctan \frac{1}{4}.$$

Thus $\theta_B < \theta_A < \theta_D < \theta_C$. Around the cycle A, B, C, D , only B is a local minimum and only C is a local maximum. All four readings are distinct, so plateau conventions do not affect the example.

The construction is just a quadrilateral with unequal side lengths; there is no limiting or degenerate configuration. It disproves the unrestricted turning-angle version of the forward theorem. Additional assumptions can restore such a statement: generic equilateral convex polygons with at least four vertices satisfy an angle version of the discrete four-vertex theorem [9, Corollary 21.6].

9.2. Four extrema do not suffice for mixed-sign Menger curvature. For signed Menger curvature, the analogous difficulty concerns the *converse*. The strict four-vertex window alone does not guarantee realizability once negative values are allowed.

Example 9.2. For any $M, \varepsilon > 0$, consider the cyclic profile

$$(22) \quad \kappa = (M, -\varepsilon, M, -\varepsilon).$$

It has two strict maxima and two strict minima, and satisfies $\text{DFV}(\kappa, 0)$: the two negative readings lie below zero and the two positive readings above it. Nevertheless, no oriented simple quadrilateral has this signed Menger profile.

Proof. For any counterclockwise simple polygon, let $\theta_i \in (-\pi, \pi)$ be its principal exterior turns. They sum to 2π . By (2),

$$\kappa_i = \frac{2 \sin \theta_i}{|P_{i+1} - P_{i-1}|},$$

so κ_i and θ_i have the same sign. There must therefore be at least three positive curvature readings: with at most two positive turns,

$$2\pi = \sum_i \theta_i \leq \sum_{\theta_i > 0} \theta_i < 2\pi,$$

a contradiction. For a clockwise simple polygon the same argument with signs reversed requires at least three negative readings. The profile in (22) has only two of each sign, so neither orientation is possible. $\square$

This obstruction is independent of coherence or local regularity. Making M arbitrarily large or ε arbitrarily small does not remove it: each individual positive turn is still less than π . Its moderate-arc form is proved in `Gluck/Discrete/Budget.lean`, by `three_le_card_pos_of_realizes` and `killerProfile_not_realizes`. These Lean statements concern the `RealizesR2` development model; the general polygon argument above and Example 9.1 are elementary arguments supplied here, not separately formalized.

What additional conditions on a mixed-sign cyclic profile, beyond a strict four-vertex window, characterize its realization as the signed Menger curvature of a counterclockwise simple, locally regular polygon in the sense of Dahlberg [3]? The positive-support condition just proved is necessary, but we do not claim it is sufficient. Thus the direct mixed-sign extension is false, while finding the right strengthened converse remains an open problem left by this work. Theorem 1.1 settles the positive case; a mixed-sign characterization would be a further discrete counterpart of Dahlberg’s smooth converse.

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APPENDIX A. FORMAL STATEMENTS

The main existence, continuity, and characterization results are proved in Lean 4 (v4.35.0-rc3) with Mathlib in <https://github.com/kejace/cd4vt> at version bfd433640794, in the library `Gluck`. Each displayed theorem depends only on the standard axioms `propext`, `Classical.choice` and `Quot.sound`. After `lake build`, the command `lake env lean`

scripts/axioms.lean prints these axioms for every theorem below, using `#print axioms`, for example

```
#print axioms Gluck.Discrete.tabachnikov_converse_geometric
```

The paper sources name the Lean declarations behind the mathematical statements.

Theorem 1.1. File `Gluck/Discrete/TabachnikovConverse.lean`. Here `vertexCircumradius v i` is the distance from v_i to the circumcenter of $v_{(i-1)}, v_i, v_{(i+1)}$, and `IsCoherentConvexPolygon` is simplicity, strict convexity and coherence.

```
theorem tabachnikov_converse_geometric [NeZero n]
  (hn : 4 ≤ n) {r : ZMod n → ℝ}
  (hr : ∀ i, 0 < r i) (hext : IsPlateauTwoMaxTwoMin r) :
  ∃ v : ZMod n → ℂ, IsCoherentConvexPolygon v ∧
    ∀ i, vertexCircumradius v i = r i
```

For this same polygon, the lemmas `IsCoherentConvexPolygon.signedMengerR2_eq_inv_of_radii` and `IsCoherentConvexPolygon.not_concyclic4_of_radius_ne`, in `Gluck/Discrete/TabachnikovCharacterization.lean`, give curvature $1/r_i$ and exclude four consecutive concyclic vertices at each individual pair with $r_i \neq r_{i+1}$. The original theorem `tabachnikov_converse`, with the combined conclusions, is retained.

Corollary 1.2. File `Gluck/Discrete/TabachnikovCharacterization.lean`. The hypotheses include positivity and nonconstancy, and the conclusion uses the same distinct plateau runs as Theorem 1.1.

```
theorem tabachnikov_radius_characterization [NeZero n]
  (hn : 4 ≤ n) {r : ZMod n → ℝ}
  (hr : ∀ i, 0 < r i) (hnc : ¬ ∃ c, ∀ i, r i = c) :
  (∃ v : ZMod n → ℂ, IsCoherentConvexPolygon v ∧
    ∀ i, vertexCircumradius v i = r i) ↔ IsPlateauTwoMaxTwoMin r
```

Theorem 8.2. File `Gluck/Discrete/SideTangentConverse.lean`, namespace `Gluck.Discrete.SideTangent`. The predicate `SideTangentAdmissible r` means positive radii which are constant or whose incoming-minus-outgoing differences have a strict four-sign window. The predicate `RealizesSideTangent r` includes simplicity, strict support, outward unit normals, all three signed-distance equations, uniqueness of center and radius, and the open central contact.

```
theorem sideTangentConverse [NeZero n] (hn : 4 ≤ n)
  (r : ZMod n → ℝ) (h : SideTangentAdmissible r) :
  RealizesSideTangent r
```

The same file proves `IsSideTangentRealization.tangent_sphere`: the sphere meets each of the three supporting lines in exactly its stated contact point. Thus the linear equations imply metric circle tangency.

Corollary 8.6. File `Gluck/Discrete/SideTangentForward.lean`, in the same namespace and proof branch. The forward implication starts from an actual `IsSideTangentRealization`; the balanced polygon and the necessary sign condition are proved from it.

```
theorem realizesSideTangent_iff_admissible [NeZero n]
  (hn : 4 ≤ n) (r : ZMod n → ℝ) :
  RealizesSideTangent r ↔ SideTangentAdmissible r
```

For positive nonconstant radii, `realizesSideTangent_iff_dfv` removes the constant alternative and concludes DFV (`radiusDifference r`) 0 exactly.

Theorem 7.1. File `Gluck/Discrete/LocalContinuity.lean`. A `LocalDiscreteRealizationFamily  $\kappa_0$` is an open neighbourhood of κ_0 with a C^1 length map whose values are realizations and whose vertex map is C^1 . The strict window already implies nonconstancy and $n \geq 4$; neither needs to be a separate hypothesis.

```

theorem localContinuousRealization_of_dfv [NeZero n]
  {κ₀ : ZMod n → ℝ} {c : ℝ}
  (hκ₀ : ∀ i, 0 < κ₀ i) (hdfv : DFV κ₀ c) :
  Nonempty (LocalDiscreteRealizationFamily κ₀)

```

Theorem 7.2. File

Gluck/Discrete/GlobalContinuity/EnclosingCircleObstruction.lean, where
bandProfile s is κ^s of (19).

```

theorem no_continuous_regular_hexagon_path :
  ¬ ∃ p : ℝ → (ZMod 6 → ℂ), ContinuousOn p (Set.Icc 0 1) ∧
    ∀ s ∈ Set.Icc (0 : ℝ) 1,
      IsSimplePolygon (p s) ∧ DahlbergRegular (p s) ∧
      SignedMengerProfile (p s) = bandProfile s

```

Here **DahlbergRegular** asks that at every vertex the circumcenter lie in the closed vertex cone, with a collinear alternative when the middle point lies between the others. The prescribed positive profile excludes that alternative and already forces every consecutive triple to be positively oriented. The function **SignedMengerProfile** is $i \mapsto M(P_{i-1}, P_i, P_{i+1})$. The orientation of a continuous path of convex hexagons is constant, and complex conjugation reverses it; after choosing counterclockwise orientation, a coherent convex realization of κ^s satisfies the three displayed conditions. Thus the formal statement implies Theorem 7.2.

APPENDIX B. THE ACUTE ORTHOGONAL CASE

We prove Proposition 5.4. Let $L = \hat{L}(u, v)$ be as there and suppose that $D\Phi(L)$ is not onto. All polynomial identities below were verified by computer algebra and are checked in Lean.

Projective coordinates. By Lemma 5.5(i), $\cos \alpha_0 = v\hat{L}_2$, $\cos \alpha_3 = u\hat{L}_1$, $\cos \gamma_1 = v\hat{L}_3$ and $\cos \gamma_2 = u\hat{L}_0$, hence

$$(23) \quad \tan \alpha_0 \tan \gamma_2 = \tan \alpha_3 \tan \gamma_1 = \frac{q}{uv} > 1.$$

Put $x = \tan \alpha_0 / \tan \gamma_1$, $y = 1/(\tan \gamma_1 \tan \gamma_2)$, $w = 1/\tan \gamma_2$ and $m = w^2$. By (23), $\tan \alpha_0 = xw/y$, $\tan \alpha_3 = x/w$, $\tan \gamma_1 = w/y$ and $\tan \gamma_2 = 1/w$. Since $T < \pi/2$, $\tan \gamma_1 \tan \gamma_2 < 1$, so $y > 1$; and $x/y = \tan \alpha_0 \tan \gamma_2 > 1$, so $x > y > 1$. With $\lambda_* = \cos \alpha_0 \cos \alpha_3 / y > 0$ one finds

$$(24) \quad \cos S = \lambda_* (y - x^2), \quad \frac{\cos T}{q} = \lambda_* x (y - 1);$$

the first is $\cos S = \cos \alpha_0 \cos \alpha_3 (1 - \tan \alpha_0 \tan \alpha_3)$, and the second uses $q \sin \alpha_3 \cos \alpha_0 = \cos \gamma_1 \sin \gamma_2$, which follows from the cosine relations above.

The equations. By Lemma 5.2, $AD = BC$. Put $a = A/2$, $b = C/2$ and $\tau = A/B$. Then $B = 2a/\tau$ and $D = 2b/\tau$, and since $0 < p, l < 1$ and $t, h < q$, all of A, B, C, D lie in $(0, 1)$:

$$(25) \quad 0 < a < \frac{1}{2}, \quad 0 < b < \frac{1}{2}, \quad 2a < \tau, \quad 2b < \tau.$$

Substituting (24) into the second equation of (14) and dividing by $\lambda_* B$ gives

$$(26) \quad \mathcal{A} := -(x+1)(x-y)(1+\tau) + 2(x^2-y)b - 2x(y-1)a = 0.$$

Since $\tan \arcsin(c \sin z) = \varsigma_c(z) \tan z$, the four mixed arcsines have tangents $(1-2a)xw/y$, $(1-2b)w/y$, $(1-2b/\tau)/w$ and $(1-2a/\tau)x/w$. The equations $\varepsilon_1 = 0$ and $\varepsilon_3 = 0$ say that four angles with nonzero cosines sum to π , and for such angles $(\tan z_1 \tan z_2 - 1)(\tan z_3 + \tan z_4) = (\tan z_1 + \tan z_2)(1 - \tan z_3 \tan z_4)$. Clearing denominators gives

$$(27) \quad \mathcal{E}_1 := y(x-m)(x(1-2a) + (1-2b)) - (x+1)(y^2 - xm(1-2a)(1-2b)) = 0,$$

$$(28) \quad \mathcal{E}_2 := (xm - y^2)\tau(\tau - 2b + x(\tau - 2a)) - y(x+1)(m\tau^2 - x(\tau - 2b)(\tau - 2a)) = 0.$$

So Proposition 5.4 follows from the next lemma.

Lemma B.1. *There are no reals x, y, m, a, b, τ with $1 < y < x$, $m > 0$, $a > 0$, $0 < b < \frac{1}{2}$, $2a < \tau$ and $2b < \tau$ satisfying (26), (27) and (28).*

Proof. Put

$$D_0 = x(x+1)a - (x^2 - y), \quad N_0 = (x+1)(x-y) + 2x(y-1)a, \quad \mathcal{H} = 2bD_0 + N_0.$$

Step 1: $\mathcal{H} = 0$. Write $\mathcal{E}_1 = c_1m + d_1$ and $\mathcal{E}_2 = c_2m + d_2$; then $c_1d_2 - d_1c_2 = 0$. Here $c_1 = -y(x(1-2a) + 1 - 2b) + x(x+1)(1-2a)(1-2b)$, $d_1 = xy(x(1-2a) + 1 - 2b) - (x+1)y^2$, $c_2 = x\tau(\tau-2b+x(\tau-2a)) - y(x+1)\tau^2$ and $d_2 = -y^2\tau(\tau-2b+x(\tau-2a)) + xy(x+1)(\tau-2b)(\tau-2a)$. Solving (26) for τ (note $(x+1)(x-y) \neq 0$) and substituting, one has the identity of rational functions

$$(29) \quad (x+1)(x-y)(c_1d_2 - d_1c_2)|_{\tau=\tau(a,b)} = 8xy(a-b)(b(x^2-y) + x^2a(y-1))\mathcal{H}.$$

If $a = b$, then (26) reads $(x+1)(x-y)(2a-1-\tau) = 0$, so $\tau = 2a-1 < 2a$, contradicting $\tau > 2a$. Also $b(x^2-y) + x^2a(y-1) > 0$. Hence $\mathcal{H} = 0$.

Step 2: two consequences. The identities

$$(x+1)((x-y)\tau - 2bxa) + \mathcal{H} + \mathcal{A} = 0, \quad D_0\mathcal{E}_1 + (x+1)\mathcal{C} = c_H\mathcal{H},$$

with $c_H = 2mx^2a - mx^2 + 2mxa - mx + my - xy$ and $\mathcal{C} = mx\mathcal{D}(a) + y\mathcal{P}(a)$, where

$$\mathcal{P}(a) = 2x^3a^2 + (-3x^3 + x^2y + x^2 + xy)a + (x^3 - x^2y - x^2 + y^2),$$

$$\mathcal{D}(a) = 2x(x+y-1)a^2 + (-x^2 - xy + 3x - y)a + (y - x),$$

give $(x-y)\tau = 2bxa$ and $\mathcal{C} = 0$.

Step 3: a forced interval. $N_0 > 0$, and $2bD_0 = -N_0$ gives $D_0 < 0$. From $(x-y)\tau = 2bxa$ and $\tau > 2b$ we get $a > (x-y)/x$. From $\mathcal{H} = 0$, $N_0 + D_0 = (1-2b)D_0 < 0$, and $N_0 + D_0 = x(a(x+2y-1) - (y-1))$. Thus

$$(30) \quad a_- := \frac{x-y}{x} < a < a_+ := \frac{y-1}{x+2y-1},$$

and in particular $a_- < a_+$, which is equivalent to $x^2 - 2y^2 + y < 0$.

Step 4: a sign obstruction. $\mathcal{P}$ and $\mathcal{D}$ are convex quadratics in a , so it suffices that each is negative at two points bracketing (30). Direct evaluation gives

$$\begin{aligned} \mathcal{P}(a_-) &= -xy(x-y), & \mathcal{P}\left(\frac{x^2-y}{x(x+1)}\right) &= -\frac{x(x+y)(x^2-2y+1)}{(x+1)^2}, \\ \mathcal{D}(a_-) &= \frac{(x-y)(x^2-xy-2y^2+y)}{x}, & \mathcal{D}(a_+) &= -\frac{y(x+y)(x^2-2y+1)}{(x+2y-1)^2}. \end{aligned}$$

Here $x^2 - 2y + 1 > 2x - 2y > 0$, $x^2 - xy - 2y^2 + y < x^2 - 2y^2 + y < 0$, and $a_+ < \frac{x^2-y}{x(x+1)}$, which after cross-multiplying is $(x+y)(x^2-2y+1) > 0$. So $\mathcal{P}(a) < 0$ and $\mathcal{D}(a) < 0$, hence $\mathcal{C} < 0$, contradicting $\mathcal{C} = 0$. $\square$

REFERENCES

1. Stanko Bilinski, *Die primitivste Form des Vierscheitelsatzes*, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 85–93.
2. Björn E. J. Dahlberg, *The converse of the four vertex theorem*, Proc. Amer. Math. Soc. **133** (2005), no. 7, 2131–2135, <https://doi.org/10.1090/S0002-9939-05-07788-9>.
3. ———, *A discrete four vertex theorem*, Geom. Dedicata **133** (2008), 111–128, <https://doi.org/10.1007/s10711-008-9238-0>.
4. Dennis DeTurck, Herman Gluck, Daniel Pomerleano, and David Shea Vick, *The four vertex theorem and its converse*, Notices Amer. Math. Soc. **54** (2007), no. 2, 192–207, <https://arxiv.org/abs/math/0609268>.
5. Herman Gluck, *The converse to the four vertex theorem*, Enseign. Math. (2) **17** (1971), 295–309, <https://doi.org/10.5169/seals-44584>.
6. Kristoffer Josefsson, *Lean formalization of converse four-vertex theorems*, 2026, <https://github.com/kejace/cd4vt>.
7. Oleg R. Musin, *Curvature extrema and four-vertex theorems for polygons and polyhedra*, J. Math. Sci. (N.Y.) **119** (2004), no. 2, 268–277, <https://doi.org/10.1023/B:JOTH.0000008769.58818.6d>; Revised version: [arXiv:0805.0629v3](https://arxiv.org/abs/0805.0629v3).

8. Valentin Ovsienko and Serge Tabachnikov, *Projective geometry of polygons and discrete 4-vertex and 6-vertex theorems*, Enseign. Math. (2) **47** (2001), no. 1–2, 3–19, <https://doi.org/10.5169/seals-65426>; [arXiv:math/9909150](https://arxiv.org/abs/math/9909150).
9. Igor Pak, *Lectures on discrete and polyhedral geometry*, 2010, Draft, April 20, 2010; <https://www.math.ucla.edu/~pak/geomp018.pdf>.
10. Vyacheslav D. Sedykh, *Discrete versions of the four-vertex theorem*, Topics in Singularity Theory, Amer. Math. Soc. Transl. Ser. 2, vol. 180, Amer. Math. Soc., 1997, <https://doi.org/10.1090/trans2/180/17>, pp. 197–207.
11. Serge Tabachnikov, *A four vertex theorem for polygons*, Amer. Math. Monthly **107** (2000), no. 9, 830–833, <https://doi.org/10.1080/00029890.2000.12005277>.
12. ———, *Converse Sturm–Hurwitz–Kellogg theorem and related results*, J. Fixed Point Theory Appl. **3** (2008), no. 1, 121–130, <https://doi.org/10.1007/s11784-007-0049-y>; [arXiv:0710.5902](https://arxiv.org/abs/0710.5902).
13. Bernd Wegner, *On the evolutes of piecewise linear curves in the plane*, Rad Hrvatske Akad. Znan. Umjet. Mat. Znan. **11** (467) (1994), 1–16, <https://dizbi.hazu.hr/?pr=i&id=172424>.