

Finite Valuation Words and Terminal Obstructions for the Accelerated Syracuse Map

Tosho L. Karadzhov
Independent Researcher, Sofia, Bulgaria

July 4, 2026

Abstract

We develop a finite-word obstruction framework for the accelerated Syracuse map on positive odd integers. A finite valuation word determines an exact iterate formula and a unique realization class modulo a power of two; compatible infinite prefixes define a natural two-adic inverse-limit object together with least positive representatives. The first part records the exact finite-word congruence machinery, finite-descent equivalence, terminal-surplus identity, terminal-ceiling break criterion, bad-prefix windows, activated density collapse, stabilization criterion, and strict-jump bookkeeping. The second part uses these tools to isolate terminal obstruction classes for a hypothetical counterexample: a bounded branch leading to a nontrivial accelerated cycle, and an unbounded eventual no-jump branch, successively reduced through a chain of geometric and combinatorial normal forms to a packet-surplus compensation target. The resulting reduction spine is lossless: Collatz follows from exclusion of nontrivial accelerated cycles together with a packet-surplus, or stronger record-ladder, compensation theorem for the unbounded branch. As a closed module, we include a closed class-exclusion theorem showing that no eventually periodic infinite low-surplus valuation chain is realized by a positive odd integer. On the bounded branch, we further record, as an explicit unproved roadmap step, that the terminal-ceiling strict-jump obstruction reduces to a single recurrent valuation-mismatch statement. This manuscript proves the architecture stated in its theorems, propositions, lemmas, and corollaries; it states the remaining terminal exclusions explicitly as problems, conjectures, or roadmap steps, and does not claim a completed proof of the full Collatz conjecture.

Keywords. Collatz conjecture; $3x + 1$ problem; accelerated Syracuse map; valuation words; terminal obstructions; terminal surplus; terminal ceilings; bad-prefix windows; inverse limits; least positive representatives; strict jumps; no-jump tails; deficit coding; dyadic scale walk; packet compensation; record ladders; suffix stress; logarithmic shallow returns; base-scale localization; exponential length cost; long quiet packets; neutral cores; rotation-window pruning; continued-fraction returns; boundary corridors; middle escape; selected-scale near returns; low-surplus chains.

MSC 2020. Primary 11B83; Secondary 11A51, 37A45, 37P99.

Contents

1	Introduction	3
2	The accelerated Syracuse dynamics	4
3	Exact valuation-word normal form	5
4	Finite descent and equivalence to Collatz	6
5	Terminal surplus closure module	7

6	Terminal ceiling break module	10
7	Cycle obstruction	13
8	Admissible valuation paths	14
9	Limitations and density after activation	16
10	Activation and 2-adic incompatibility	17
11	Stabilization and jump calculus	20
12	Strict-jump forcing and no-jump traps	24
13	Terminal obstruction dichotomy and scale-sensitive no-jump clocks	26
14	Deficit coding for the unbounded terminal branch	29
15	Dyadic scale walk for large-deficit escape	31
16	Scale-up blocks and compensation targets	34
17	Scale packet budgets and valuation-one congruence filters	38
18	Packet-compensation target and record-ladder reduction	41
19	Amplification splitting on positive record ladders	43
20	Offset dominance as right-edge suffix stress	46
21	Direct suffix-stress pruning by logarithmic shallow returns	49
22	Base-scale localization of shallow returns	51
23	Exponential length-cost and base-start splitting	53
24	Long quiet packets and neutral-core pruning	56
25	Bounded-reserve neutral cores and rotation-window reduction	59
26	Rotation-window balance and resonance-corridor pruning	61
27	Boundary-corridor pruning by one-step middle escape	64
28	Selected-scale linear near-returns and copy-switch splitting	67
29	Selected switching-corridor density pruning via thin-interval discrepancy	70
30	Deep selected-scale block discrepancy for switching corridors	72
31	A lossless reduction spine toward unconditional closure	74
31.1	The two terminal fronts	75
31.2	Normal form of the unbounded front	76
31.3	Connection to the classical continued-fraction bound	77
31.4	Quantitative task for the escape front	78
31.5	A five-step strict-jump roadmap	79

1 Introduction

The Collatz problem asks whether every positive integer eventually reaches 1 under

$$\text{Col}(N) = \begin{cases} N/2, & N \equiv 0 \pmod{2}, \\ 3N + 1, & N \equiv 1 \pmod{2}. \end{cases}$$

After deleting the forced even steps, the problem is equivalently studied on the positive odd integers by the accelerated Syracuse map

$$\text{Syr}(n) = \frac{3n + 1}{2^{\nu_2(3n+1)}}.$$

This paper attacks the obstruction side of the problem through finite valuation words. A finite valuation word records the successive exponents of 2 removed in an accelerated orbit segment. Such words carry exact arithmetic information: they determine finite-iterate formulas, a unique residue class of realizing initial values, and height windows for prefixes that have not yet descended below their starting value.

The purpose of the present paper is not to repack the completed class-exclusion theorem of the companion Paper A as a proof of Collatz. Its purpose is to build a self-contained finite-word and terminal-obstruction framework for the remaining global problem. The paper identifies the exact objects that a hypothetical counterexample must produce after the finite-word, ceiling, least-representative, and scale-packet reductions have all been applied.

The main reduction principle is the following. A counterexample must be either a nontrivial accelerated cycle or an unbounded terminal no-jump escape. The bounded case is represented by an explicit cycle certificate

$$n_0 = \frac{B_m}{2K_m - 3^m},$$

attached to a periodic valuation word. The unbounded case is represented by an eventual no-jump tail whose least representatives remain terminal-ceiling compatible and whose scale-packet surplus escapes every finite bound. Thus the full closure problem is reduced to two terminal fronts: exclusion of nontrivial accelerated cycles and exclusion of unbounded packet-compensating no-jump tails.

Unconditional contributions

The unconditional parts proved in the paper are the following.

- (i) Exact finite-iterate and congruence formulas for finite valuation words.
- (ii) A finite-descent equivalence reducing Collatz to the assertion that every odd $n > 1$ has an accelerated iterate below n .
- (iii) A terminal-surplus identity rewriting descent as the exact renewal inequality

$$A_m > 1 + \frac{1}{3n} \sum_{i=0}^{m-1} A_i.$$

- (iv) A terminal-ceiling identity rewriting one-step descent as the exact break condition

$$a_m > \log_2 \left(3 \frac{\text{Syr}^m(n)}{n} + \frac{1}{n} \right).$$

- (v) Exact bad-prefix windows, activated density collapse, least-representative stabilization, and strict-jump bookkeeping.
- (vi) A strict-jump/no-jump trap showing that a true counterexample must eventually appear as a non-fixed no-jump tail whose least representatives are terminal-ceiling compatible.
- (vii) Deficit coding, dyadic scale-walk, scale-up block, scale-packet budget, record-ladder reductions, an amplification split, suffix-stress pruning, and base-scale localization for the unbounded terminal branch.
- (viii) A lossless two-front reduction: Collatz follows from nontrivial cycle exclusion together with a packet-surplus, or stronger record-ladder, compensation theorem on the unbounded branch.
- (ix) A closed class-exclusion module: no eventually periodic infinite low-surplus valuation chain is realized by a positive odd integer.

What is not claimed

The paper does not claim a proof of the full Collatz conjecture. The remaining terminal fronts are stated explicitly as problems, conjectures, or roadmap steps: exclusion of nontrivial accelerated cycles, exclusion of arbitrary nonperiodic low-surplus realization beyond the closed eventually periodic subclass, exclusion of post-activation singleton survivors, and packet-surplus compensation for the unbounded terminal branch. Conditional gateway theorems and propositions explain how these targets would imply Collatz, but they are not used as established facts.

Organization

Sections 2 to 6 contain the bridge from Collatz to finite valuation words, terminal surplus, and terminal ceiling inequalities. Sections 7 to 10 develop the finite-word window, density, and inverse-limit theory. Sections 11 and 12 give the least-representative stabilization and strict-jump mechanisms. Sections 13 to 18 treat the unbounded terminal branch and its packet/record-ladder reductions. Section 19 gives the first amplification split of the escape front into multiplicative-surplus and offset-dominance channels. Section 31 records the final dependency map, while Sections 32 and 33 state the closed low-surplus class theorem and final status.

Status discipline

Every theorem, proposition, lemma, and corollary below is intended as an unconditional statement unless explicitly introduced as part of a roadmap or conditional closure scheme. Every unproved terminal assertion is labelled as a *Problem*, a *Conjecture*, or is explicitly flagged in the text as a roadmap step. The manuscript should therefore be read as a finite-word reduction and obstruction framework for the full problem, not as a completed proof of Collatz.

2 The accelerated Syracuse dynamics

Definition 2.1 (Accelerated Syracuse map). For $n \in \mathcal{O}$, define

$$a(n) := \nu_2(3n + 1), \quad \text{Syr}(n) := \frac{3n + 1}{2^{a(n)}}.$$

The integer $a(n) \geq 1$ is called the *valuation jump* of n .

The basic iteration notation is

$$n_0 = n, \quad n_{j+1} = \text{Syr}(n_j), \quad a_j = a(n_j) = \nu_2(3n_j + 1).$$

For $m \geq 1$ set

$$K_m := \sum_{j=0}^{m-1} a_j, \quad K_0 := 0.$$

The word

$$\mathcal{W}_m(n) := (a_0, a_1, \dots, a_{m-1})$$

is the length- m valuation word attached to the orbit segment starting at n .

3 Exact valuation-word normal form

Definition 3.1 (Offset polynomial attached to a valuation word). For a length- m valuation word $\mathcal{W} = (a_0, \dots, a_{m-1})$, define partial sums

$$K_i = a_0 + \dots + a_{i-1}, \quad K_0 = 0,$$

and define

$$B_m(\mathcal{W}) := \sum_{i=0}^{m-1} 3^{m-1-i} 2^{K_i}.$$

Theorem 3.2 (Exact finite-iterate formula). *Let $n \in \mathcal{O}$, and let $n_j = \text{Syr}^j(n)$. Then for every $m \geq 1$,*

$$n_m = \frac{3^m n + B_m(\mathcal{W}_m(n))}{2^{K_m(n)}}.$$

Proof. For $m = 1$, the identity is exactly

$$n_1 = \frac{3n + 1}{2^{a_0}},$$

with $B_1 = 1$ and $K_1 = a_0$. Assume the formula holds for m . Then

$$n_{m+1} = \frac{3n_m + 1}{2^{a_m}} = \frac{3(3^m n + B_m) + 2^{K_m}}{2^{K_m + a_m}}.$$

Thus

$$n_{m+1} = \frac{3^{m+1} n + (3B_m + 2^{K_m})}{2^{K_{m+1}}}.$$

Since

$$3B_m + 2^{K_m} = \sum_{i=0}^{m-1} 3^{m-i} 2^{K_i} + 2^{K_m} = \sum_{i=0}^m 3^{m-i} 2^{K_i} = B_{m+1},$$

the induction closes. □

Corollary 3.3 (Logarithmic form). *For every $m \geq 1$,*

$$\log n_m = \log n + m \log 3 - K_m \log 2 + \log \left(1 + \frac{B_m}{3^m n} \right).$$

Equivalently,

$$\log \frac{n_m}{n} = m \log 3 - K_m \log 2 + \log \left(1 + \frac{B_m}{3^m n} \right).$$

Proof. Take logarithms in [Theorem 3.2](#) and factor $3^m n$ from the numerator. □

4 Finite descent and equivalence to Collatz

This section isolates the exact bridge between the ordinary Collatz conjecture and the accelerated finite-descent problem used in the rest of the paper.

Definition 4.1 (Finite descent property). The accelerated Syracuse map has the *finite descent property* if for every odd integer $n > 1$ there exists an integer $m \geq 1$ such that

$$\text{Syr}^m(n) < n.$$

Theorem 4.2 (Finite descent equivalence). *The ordinary Collatz conjecture for positive integers is equivalent to the finite descent property for Syr on positive odd integers.*

Proof. Assume first the ordinary Collatz conjecture. Let $n > 1$ be odd. The ordinary Collatz orbit of n reaches 1. Removing all intermediate even values gives the accelerated Syracuse orbit on odd values. Hence $\text{Syr}^r(n) = 1 < n$ for some $r \geq 1$, and finite descent holds.

Conversely, assume finite descent. We prove by strong induction on odd positive integers that every odd starting value reaches 1 under Syr . The claim is trivial for $n = 1$. Let $n > 1$ be odd, and assume the claim for all smaller positive odd integers. By finite descent there is $m \geq 1$ such that

$$n' := \text{Syr}^m(n) < n.$$

The value n' is positive and odd, so it reaches 1 by the induction hypothesis. Therefore n reaches 1 under the accelerated map. Finally, every positive integer reaches an odd integer after finitely many divisions by 2, and reinserting those deleted even steps gives the ordinary Collatz orbit. Thus the ordinary Collatz conjecture follows. $\square$

Definition 4.3 (Bad prefixes and terminally bad starts). Let $n \in \mathcal{O}$. A realized prefix of length m is *bad through level m* if

$$\text{Syr}^j(n) \geq n \quad (1 \leq j \leq m).$$

An odd integer $n > 1$ is *terminally bad* if

$$\text{Syr}^j(n) \geq n \quad (j \geq 1).$$

Proposition 4.4 (Terminal badness is the obstruction to finite descent). *The accelerated Syracuse map has the finite descent property if and only if there is no terminally bad odd integer $n > 1$.*

Proof. If finite descent holds, every odd $n > 1$ has an accelerated iterate below n , so no such n is terminally bad. Conversely, if finite descent fails, then for some odd $n > 1$ one has $\text{Syr}^j(n) \geq n$ for every $j \geq 1$. That n is terminally bad. $\square$

Corollary 4.5 (Bad-path formulation). *The Collatz conjecture is equivalent to the assertion that there is no terminally bad odd integer $n > 1$.*

Proof. Combine [Theorem 4.2](#) with [Proposition 4.4](#). $\square$

Proposition 4.6 (Exact descent inequality). *Let $n \in \mathcal{O}$, and let $K_m = K_m(n)$ and $B_m = B_m(\mathcal{W}_m(n))$ be attached to the first m accelerated steps of n . For every $m \geq 1$,*

$$\text{Syr}^m(n) < n \iff 2^{K_m} > 3^m + \frac{B_m}{n}.$$

Proof. By [Theorem 3.2](#),

$$\text{Syr}^m(n) = \frac{3^m n + B_m}{2^{K_m}}.$$

Since $n > 0$ and $2^{K_m} > 0$,

$$\begin{aligned} \text{Syr}^m(n) < n &\iff \frac{3^m n + B_m}{2^{K_m}} < n \\ &\iff 3^m n + B_m < 2^{K_m} n \\ &\iff 2^{K_m} > 3^m + \frac{B_m}{n}. \end{aligned}$$

□

Corollary 4.7 (Logarithmic surplus criterion). *For every $m \geq 1$, descent at level m is equivalent to*

$$K_m - m \log_2 3 > \log_2 \left(1 + \frac{B_m}{3^m n} \right).$$

Proof. The displayed inequality is equivalent, by exponentiating base 2, to

$$\frac{2^{K_m}}{3^m} > 1 + \frac{B_m}{3^m n}.$$

Multiplying by 3^m gives exactly the inequality in [Proposition 4.6](#); the reverse implication is the same calculation in reverse. □

5 Terminal surplus closure module

The exact descent inequality contains the offset term B_m/n . This section rewrites that offset as a renewal sum of normalized surplus variables. The results are exact identities and conditional reductions; they do not assert that the terminal inequalities must eventually break.

Definition 5.1 (Surplus sequence). For a realized orbit segment define

$$\alpha := \log_2 3, \quad \sigma_m := K_m - m\alpha, \quad A_m := 2^{\sigma_m} = \frac{2^{K_m}}{3^m} \quad (m \geq 0),$$

with $K_0 = 0$, so $A_0 = 1$. Also put

$$T_m := \sum_{i=0}^{m-1} A_i, \quad T_0 := 0.$$

A level $m \geq 1$ is *activated* if $A_m > 1$, equivalently $2^{K_m} > 3^m$.

Lemma 5.2 (Offset as a surplus sum). *For every realized valuation prefix of length m ,*

$$\frac{B_m}{3^m} = \frac{1}{3} \sum_{i=0}^{m-1} \frac{2^{K_i}}{3^i} = \frac{1}{3} \sum_{i=0}^{m-1} A_i = \frac{T_m}{3}.$$

Proof. Using the offset formula,

$$B_m = \sum_{i=0}^{m-1} 3^{m-1-i} 2^{K_i}.$$

Dividing by 3^m gives

$$\frac{B_m}{3^m} = \sum_{i=0}^{m-1} \frac{2^{K_i}}{3^{i+1}} = \frac{1}{3} \sum_{i=0}^{m-1} \frac{2^{K_i}}{3^i}.$$

The remaining equalities are the definitions of A_i and T_m . □

Theorem 5.3 (Terminal-surplus descent identity). *Let $n > 1$ be odd. For every $m \geq 1$,*

$$\text{Syr}^m(n) < n \iff A_m > 1 + \frac{T_m}{3n}.$$

Equivalently,

$$\text{Syr}^m(n) < n \iff 2^{K_m - m \log_2 3} > 1 + \frac{1}{3n} \sum_{i=0}^{m-1} 2^{K_i - i \log_2 3}.$$

Proof. By [Proposition 4.6](#), descent at level m is equivalent to

$$2^{K_m} > 3^m + \frac{B_m}{n}.$$

After division by 3^m , this becomes

$$\frac{2^{K_m}}{3^m} > 1 + \frac{B_m}{3^m n}.$$

The left-hand side is A_m , and [Lemma 5.2](#) gives $B_m/(3^m n) = T_m/(3n)$. This proves the first equivalence. Expanding A_m and T_m proves the second. $\square$

Corollary 5.4 (Terminal-surplus form of Collatz). *The Collatz conjecture is equivalent to the following statement: for every odd integer $n > 1$, there exists $m \geq 1$ such that*

$$A_m(n) > 1 + \frac{1}{3n} \sum_{i=0}^{m-1} A_i(n).$$

Proof. Combine [Theorem 4.2](#) and [Theorem 5.3](#). $\square$

Proposition 5.5 (Terminal-bad recurrence). *Assume that $n > 1$ is odd and bad through level m . Then, for every $0 \leq j \leq m$,*

$$A_j \leq 1 + \frac{T_j}{3n}.$$

Consequently,

$$T_{j+1} \leq 1 + \left(1 + \frac{1}{3n}\right) T_j \quad (0 \leq j \leq m),$$

and, for $0 \leq j \leq m+1$,

$$T_j \leq 3n \left[\left(1 + \frac{1}{3n}\right)^j - 1 \right].$$

In particular, for $0 \leq j \leq m$,

$$A_j \leq \left(1 + \frac{1}{3n}\right)^j,$$

and for $1 \leq j \leq m$,

$$K_j - j \log_2 3 \leq j \log_2 \left(1 + \frac{1}{3n}\right).$$

Proof. For $j = 0$, the displayed inequality is $A_0 = 1 \leq 1$. For $1 \leq j \leq m$, badness through level m gives $\text{Syr}^j(n) \geq n$, so the strict descent inequality in [Theorem 5.3](#) fails at level j . Hence $A_j \leq 1 + T_j/(3n)$. Since $T_{j+1} = T_j + A_j$, the recurrence follows.

Put $\lambda = 1 + 1/(3n)$. Induction from $T_0 = 0$ gives

$$T_j \leq 1 + \lambda + \dots + \lambda^{j-1} = \frac{\lambda^j - 1}{\lambda - 1} = 3n(\lambda^j - 1).$$

Substitution into $A_j \leq 1 + T_j/(3n)$ yields $A_j \leq \lambda^j$. Taking base-2 logarithms gives the final inequality. $\square$

Corollary 5.6 (Narrow surplus strip). *If $n > 1$ is terminally bad, then every prefix satisfies*

$$K_m \leq m \log_2 3 + m \log_2 \left(1 + \frac{1}{3n}\right).$$

Thus a terminally bad orbit remains in a narrow strip above the critical line $K = m \log_2 3$.

Proof. Apply [Proposition 5.5](#) at arbitrary depth. $\square$

Proposition 5.7 (First-activation bottleneck). *Let $n > 1$ be terminally bad. Suppose $\tau \geq 1$ is the first activated level, meaning that*

$$A_\tau > 1, \quad A_i \leq 1 \quad (1 \leq i < \tau).$$

Then $A_i < 1$ for $1 \leq i < \tau$, and

$$0 < K_\tau - \tau \log_2 3 < \log_2 \left(1 + \frac{\tau}{3n}\right).$$

Equivalently,

$$K_\tau \in \mathbb{Z} \cap \left(\tau \log_2 3, \tau \log_2 3 + \log_2 \left(1 + \frac{\tau}{3n}\right) \right).$$

Proof. The lower bound is exactly $A_\tau > 1$, since $\log_2 A_\tau = K_\tau - \tau \log_2 3$. For $1 \leq i < \tau$, non-activation gives $A_i \leq 1$. Equality would imply $2^{K_i} = 3^i$, impossible for $i \geq 1$, so $A_i < 1$. Because $A_0 = 1$,

$$T_\tau = \sum_{i=0}^{\tau-1} A_i < \tau.$$

Terminal badness at level τ , together with [Proposition 5.5](#), gives

$$A_\tau \leq 1 + \frac{T_\tau}{3n} < 1 + \frac{\tau}{3n}.$$

Taking base-2 logarithms gives the claimed upper bound. $\square$

Theorem 5.8 (Terminal obstruction split). *Let $n > 1$ be an odd terminally bad starting value. Exactly one of the following alternatives occurs.*

- (i) *No prefix is activated. Then the realized valuation sequence of n is an infinite low-surplus chain positively realized by n .*
- (ii) *A first activation time τ exists. It satisfies [Proposition 5.7](#). Moreover, for the bad-prefix sets $S(\mathcal{W})$ of [Definition 8.4](#),*

$$\#S(\mathcal{W}_m(n)) = 1$$

for every sufficiently large m , and the unique element is n .

Proof. The alternatives are mutually exclusive and exhaustive. If no activated level exists, every finite realized prefix is low-surplus, and the same positive integer n realizes all prefixes. This gives alternative (i).

If a first activation time τ exists, then [Proposition 5.7](#) applies. Since n is terminally bad, each realized word $\mathcal{W}_m(n)$ is bad through level m , so $n \in S(\mathcal{W}_m(n))$ for every m . After activation, [Theorem 9.7](#) gives $\#S(\mathcal{W}_m(n)) \leq 1$ for all sufficiently large m . Since n remains in every one of these sets, the cardinality is exactly one and the unique survivor is n . This gives alternative (ii). $\square$

Problem 5.9 (Terminal-surplus closure problem). Prove that for every odd integer $n > 1$ there exists $m \geq 1$ such that

$$A_m(n) > 1 + \frac{1}{3n} \sum_{i=0}^{m-1} A_i(n).$$

By [Corollary 5.4](#), this is exactly the finite-descent form of the Collatz conjecture.

Problem 5.10 (Activated singleton exclusion). Let $n > 1$ be odd and suppose that its realized bad-prefix chain has a first activated level. Prove that the terminal-bad inequalities in [Proposition 5.5](#) cannot persist forever after that activation. Equivalently, prove that the eventual singleton allowed by [Theorem 5.8](#) is always eliminated by some deeper prefix.

Proposition 5.11 (Two-target gateway to Collatz). *Assume both of the following statements.*

- (i) *No infinite low-surplus valuation chain is positively realized by an odd integer $n > 1$.*
- (ii) *No activated terminally bad singleton exists.*

Then the Collatz conjecture holds.

Proof. If Collatz failed, [Corollary 4.5](#) would give an odd terminally bad $n > 1$. Apply [Theorem 5.8](#). Alternative (i) contradicts assumption (i), and alternative (ii) contradicts assumption (ii). Therefore no terminally bad $n > 1$ exists, and Collatz follows from [Corollary 4.5](#). $\square$

Remark 5.12 (Purpose of the terminal-surplus module). The unconditional content of this section is the renewal identity, the terminal-bad envelope, the first-activation bottleneck, and the obstruction split. A full proof still requires exclusion of arbitrary positive low-surplus realization and exclusion of the activated singleton survivor.

6 Terminal ceiling break module

The terminal-surplus module is cumulative. The complementary local viewpoint is a one-step height ceiling. We keep the real ceiling $\Gamma_m(n)$ separate from its integer floor $C_m(n)$: the break criterion is real-valued, while admissible valuations are integral.

Definition 6.1 (Relative height and terminal ceilings). Let $n > 1$ be odd, and write

$$n_m := \text{Syr}^m(n), \quad R_m(n) := \frac{n_m}{n}.$$

Define the real ceiling and integer ceiling by

$$\Gamma_m(n) := \log_2 \left(3R_m(n) + \frac{1}{n} \right), \quad C_m(n) := \lfloor \Gamma_m(n) \rfloor.$$

The realized orbit of n is *terminal-ceiling admissible through level M* if

$$a_j(n) \leq C_j(n) \quad (0 \leq j < M).$$

It is *terminal-ceiling admissible* if the same condition holds for every $j \geq 0$.

Lemma 6.2 (Relative height from surplus). *For every realized prefix of length m ,*

$$R_m(n) = \frac{1}{A_m} \left(1 + \frac{T_m}{3n} \right).$$

Proof. By [Theorem 3.2](#) and [Lemma 5.2](#),

$$R_m(n) = \frac{3^m n + B_m}{2^{K_m n}} = \frac{3^m}{2^{K_m}} \left(1 + \frac{B_m}{3^m n}\right) = \frac{1}{A_m} \left(1 + \frac{T_m}{3n}\right).$$

□

Lemma 6.3 (Exact relative-height recurrence). *For every $m \geq 0$,*

$$R_{m+1}(n) = \frac{3R_m(n) + \frac{1}{n}}{2^{a_m(n)}}.$$

Proof. Divide

$$n_{m+1} = \frac{3n_m + 1}{2^{a_m(n)}}$$

by the initial value n .

□

Theorem 6.4 (Terminal ceiling break criterion). *For every odd $n > 1$ and every $m \geq 0$,*

$$\text{Syr}^{m+1}(n) < n \iff R_{m+1}(n) < 1 \iff a_m(n) > \Gamma_m(n).$$

Equivalently,

$$\text{Syr}^{m+1}(n) < n \iff a_m(n) > \log_2 \left(3R_m(n) + \frac{1}{n}\right).$$

Proof. The equivalence $\text{Syr}^{m+1}(n) < n \iff R_{m+1}(n) < 1$ is the definition of R_{m+1} . By [Lemma 6.3](#),

$$R_{m+1}(n) < 1 \iff 3R_m(n) + \frac{1}{n} < 2^{a_m(n)}.$$

Taking base-2 logarithms gives $a_m(n) > \Gamma_m(n)$. All implications are reversible.

□

Corollary 6.5 (Terminal-bad ceiling rule). *An odd integer $n > 1$ is terminally bad if and only if, for every $m \geq 0$,*

$$a_m(n) \leq \Gamma_m(n).$$

Since $a_m(n)$ is integral, this is equivalent to

$$a_m(n) \leq C_m(n) = \left\lfloor \log_2 \left(3R_m(n) + \frac{1}{n}\right) \right\rfloor \quad (m \geq 0).$$

Proof. Terminal badness is the assertion $R_{m+1}(n) \geq 1$ for every $m \geq 0$. By [Theorem 6.4](#), this is equivalent to the failure of $a_m(n) > \Gamma_m(n)$ at every level. Hence $a_m(n) \leq \Gamma_m(n)$ for every m . Because $a_m(n)$ is an integer, this is equivalent to $a_m(n) \leq C_m(n)$.

□

Corollary 6.6 (Terminal-ceiling form of Collatz). *The Collatz conjecture is equivalent to the following statement: for every odd integer $n > 1$, there exists $m \geq 0$ such that*

$$a_m(n) > \Gamma_m(n).$$

Proof. By [Theorem 6.4](#), the displayed inequality is equivalent to $\text{Syr}^{m+1}(n) < n$. This is finite descent, which is equivalent to the Collatz conjecture by [Theorem 4.2](#).

□

Proposition 6.7 (Height-band automaton). *Let $n > 1$ be terminally bad and let $k \geq 1$. If*

$$R_m(n) \in \left[\frac{2^k - \frac{1}{n}}{3}, \frac{2^{k+1} - \frac{1}{n}}{3} \right),$$

then

$$a_m(n) \leq k.$$

In particular,

$$1 \leq R_m(n) < \frac{4 - \frac{1}{n}}{3} \implies a_m(n) = 1,$$

and

$$\frac{4 - \frac{1}{n}}{3} \leq R_m(n) < \frac{8 - \frac{1}{n}}{3} \implies a_m(n) \leq 2.$$

Proof. The height band is equivalent to

$$2^k \leq 3R_m(n) + \frac{1}{n} < 2^{k+1}.$$

Thus $C_m(n) = k$. Since n is terminally bad, [Corollary 6.5](#) gives $a_m(n) \leq k$. In the lowest band, every accelerated valuation is at least 1, and the ceiling gives $a_m(n) \leq 1$, hence $a_m(n) = 1$. The second special case is the case $k = 2$. $\square$

Proposition 6.8 (Forced growth in the lowest band). *Let $n > 1$ be terminally bad. Suppose that, for some $m \geq 0$ and $r \geq 1$,*

$$1 \leq R_{m+i}(n) < \frac{4 - \frac{1}{n}}{3} \quad (0 \leq i < r).$$

Then

$$a_{m+i}(n) = 1 \quad (0 \leq i < r),$$

and

$$R_{m+r}(n) + \frac{1}{n} = \left(\frac{3}{2}\right)^r \left(R_m(n) + \frac{1}{n}\right).$$

Consequently, a terminally bad orbit cannot remain in the lowest band forever.

Proof. By [Proposition 6.7](#), the valuation is 1 throughout the stated block. If $a_s(n) = 1$, then [Lemma 6.3](#) gives

$$R_{s+1}(n) + \frac{1}{n} = \frac{3}{2} \left(R_s(n) + \frac{1}{n}\right).$$

Iterating from $s = m$ to $s = m + r - 1$ gives the displayed formula. The right-hand side grows like $(3/2)^r$, while the lowest band has a fixed finite upper endpoint, so the orbit cannot remain in that band forever. $\square$

Problem 6.9 (No infinite terminal-ceiling-admissible orbit). Prove that there is no odd integer $n > 1$ whose realized accelerated orbit satisfies

$$R_m(n) \geq 1, \quad a_m(n) \leq C_m(n) \quad (m \geq 0).$$

By [Corollary 6.5](#), this is exactly the terminal-ceiling obstruction to finite descent.

Problem 6.10 (Terminal ceiling strict-jump target). Prove that every infinite extension chain, different from the fixed chain generated by 1, whose sufficiently deep prefixes are least-representative bad and terminal-ceiling compatible in the sense of [Definition 12.2](#), has infinitely many strict jumps

$$\rho(\mathcal{W}_{m+1}) > \rho(\mathcal{W}_m).$$

Equivalently, terminal-ceiling compatibility cannot be combined with eventual stabilization of the least representatives at a positive odd integer $n > 1$.

Proposition 6.11 (Single-target gateway to Collatz). *A positive solution of [Problem 6.10](#) implies the Collatz conjecture.*

Proof. If Collatz failed, [Corollary 4.5](#) would give an odd terminally bad integer $n > 1$. Its realized valuation prefixes form an infinite extension chain. By [Corollary 6.5](#), the realized orbit is terminal-ceiling admissible. After stabilization, [Corollary 13.3](#) gives the least-representative terminal-ceiling compatibility required in [Problem 6.10](#).

Once $M(\mathcal{W}_m(n)) > n$, the least representative of the realization class containing n is exactly n . Hence the realized chain has only finitely many strict jumps, contradicting [Problem 6.10](#). Therefore no terminally bad $n > 1$ exists, and Collatz follows from [Corollary 4.5](#). $\square$

Remark 6.12 (Role of the ceiling module). The terminal-surplus split separates low-surplus and activated-singleton obstructions. The ceiling rule repackages both into one local obstruction: a terminally bad orbit must keep every actual valuation below a moving height-dependent ceiling. A universal strict-jump theorem for least-representative ceiling-compatible chains would eliminate that obstruction.

7 Cycle obstruction

Theorem 7.1 (Arithmetic certificate for cycles). *Suppose $n \in \mathcal{O}$ belongs to a periodic orbit of exact accelerated length m , with valuation word $\mathcal{W} = (a_0, \dots, a_{m-1})$. Then*

$$n = \frac{B_m(\mathcal{W})}{2^{K_m} - 3^m}.$$

In particular, a necessary condition for a positive cycle is

$$2^{K_m} > 3^m$$

and

$$2^{K_m} - 3^m \mid B_m(\mathcal{W}).$$

Proof. A cycle based at n satisfies $\text{Syr}^m(n) = n$. Using [Theorem 3.2](#),

$$n = \frac{3^m n + B_m}{2^{K_m}}.$$

Therefore

$$(2^{K_m} - 3^m)n = B_m.$$

Since $B_m > 0$ and $n > 0$, one must have $2^{K_m} > 3^m$, and the displayed formula follows. $\square$

Remark 7.2 (Compatibility is essential). Finite realization of a word is not enough to produce a cycle. In addition to the divisibility condition in [Theorem 7.1](#), one must verify that the integer

$$n = \frac{B_m(\mathcal{W})}{2^{K_m} - 3^m}$$

is positive, odd, integral, and realizes the cyclic valuation constraints

$$a_j = \nu_2(3n_j + 1)$$

for every point along the orbit. This cycle compatibility condition is where word enumeration becomes arithmetic rather than merely formal.

8 Admissible valuation paths

Definition 8.1 (Finite valuation word). A finite word $\mathcal{W} = (a_0, \dots, a_{m-1})$ is a *valuation word* if each $a_j \geq 1$. It is *realized from* n if

$$\nu_2(3 \text{Syr}^j(n) + 1) = a_j \quad \text{for } j = 0, 1, \dots, m-1.$$

For such a word write

$$K_m(\mathcal{W}) := a_0 + \dots + a_{m-1}, \quad M(\mathcal{W}) := 2^{K_m(\mathcal{W})+1}.$$

Theorem 8.2 (Exact congruence class for a finite word). *Let $\mathcal{W} = (a_0, \dots, a_{m-1})$ be a finite valuation word, and put $K_m = a_0 + \dots + a_{m-1}$. There is a unique odd residue class*

$$r(\mathcal{W}) \pmod{2^{K_m+1}}$$

such that

$$\mathcal{W}_m(n) = \mathcal{W} \iff n \equiv r(\mathcal{W}) \pmod{2^{K_m+1}}.$$

Consequently, $\mathcal{W}$ is realized by infinitely many positive odd integers, and as $X \rightarrow \infty$,

$$\#\{n \leq X : n \in \mathcal{O}, \mathcal{W}_m(n) = \mathcal{W}\} = \frac{X}{2^{K_m+1}} + O(1).$$

Proof. Write

$$\beta_a(y) := \frac{2^a y - 1}{3}.$$

For a terminal odd integer $y = n_m$, define recursively

$$n_j = \beta_{a_j}(n_{j+1}), \quad j = m-1, m-2, \dots, 0.$$

The composition has the form

$$n_0 = \frac{2^{K_m} y - \Xi_m(\mathcal{W})}{3^m}$$

for an integer $\Xi_m(\mathcal{W})$ depending only on $\mathcal{W}$. Since 2^{K_m} is invertible modulo 3^m , there is a unique residue class of $y \pmod{3^m}$ for which $n_0 \in \mathbb{Z}$. Restricting to odd y gives one class modulo $2 \cdot 3^m$. Replacing y by $y + 2 \cdot 3^m t$ replaces n_0 by

$$n_0 + 2^{K_m+1} t.$$

Thus the realizations form a single residue class modulo 2^{K_m+1} . Choosing y sufficiently large makes every intermediate n_j positive. Then

$$3n_j + 1 = 2^{a_j} n_{j+1},$$

and because n_{j+1} is odd, the valuation at step j is exactly a_j . Conversely, any realization of $\mathcal{W}$ is obtained from the same inverse-branch composition with terminal value $y = n_m$, so it lies in this residue class. The counting estimate is the standard count for a fixed residue class modulo 2^{K_m+1} . $\square$

Corollary 8.3 (One-step extension criterion). *Let $\mathcal{W} = (a_0, \dots, a_{m-1})$, let $b \geq 1$, and let $\mathcal{W}^+ = (a_0, \dots, a_{m-1}, b)$. Among the integers realizing $\mathcal{W}$, those realizing $\mathcal{W}^+$ form the unique sub-class*

$$n \equiv r(\mathcal{W}^+) \pmod{2^{K_m+b+1}}$$

inside the class $r(\mathcal{W}) \pmod{2^{K_m+1}}$. Equivalently,

$$3^{m+1} n + B_{m+1}(\mathcal{W}^+) \equiv 2^{K_m+b} \pmod{2^{K_m+b+1}}.$$

Proof. This is [Theorem 8.2](#) applied to the longer word. The displayed congruence says exactly that the next quotient

$$\frac{3^{m+1}n + B_{m+1}(\mathcal{W}^+)}{2^{K_m+b}}$$

is odd, hence that the next valuation is b . $\square$

Definition 8.4 (Bad-prefix realization set). Let $\mathcal{W} = (a_0, \dots, a_{m-1})$ be a finite valuation word. Define

$$R(\mathcal{W}) := \{n \in \mathcal{O} : \mathcal{W}_m(n) = \mathcal{W}\}$$

and

$$S(\mathcal{W}) := \{n \in R(\mathcal{W}) : \text{Syr}^j(n) \geq n \text{ for every } 1 \leq j \leq m\}.$$

Thus $S(\mathcal{W})$ is the set of positive odd starting values for which $\mathcal{W}$ is realized and bad.

Theorem 8.5 (Exact bad-prefix set). *Let $\mathcal{W} = (a_0, \dots, a_{m-1})$ be a finite valuation word. For $1 \leq j \leq m$, let K_j and B_j denote the corresponding prefix data, and define*

$$U(\mathcal{W}) := \min_{\substack{1 \leq j \leq m \\ 2^{K_j} > 3^j}} \frac{B_j}{2^{K_j} - 3^j},$$

with $U(\mathcal{W}) = +\infty$ if the indexing set is empty. Then

$$S(\mathcal{W}) = \left\{ n \in \mathcal{O} : n \equiv r(\mathcal{W}) \pmod{2^{K_m+1}}, \quad n \leq U(\mathcal{W}) \right\}.$$

Let $\rho(\mathcal{W})$ be the least positive odd representative of the residue class $r(\mathcal{W}) \pmod{M(\mathcal{W})}$. If $U(\mathcal{W}) < +\infty$, then

$$\#S(\mathcal{W}) = \begin{cases} 0, & U(\mathcal{W}) < \rho(\mathcal{W}), \\ 1 + \left\lfloor \frac{U(\mathcal{W}) - \rho(\mathcal{W})}{M(\mathcal{W})} \right\rfloor, & U(\mathcal{W}) \geq \rho(\mathcal{W}). \end{cases}$$

In particular,

$$\#S(\mathcal{W}) \leq 1 + \frac{U(\mathcal{W})}{M(\mathcal{W})}.$$

Proof. By [Theorem 8.2](#), n realizes $\mathcal{W}$ if and only if

$$n \equiv r(\mathcal{W}) \pmod{2^{K_m+1}}.$$

For such n , [Theorem 3.2](#) gives

$$\text{Syr}^j(n) = \frac{3^j n + B_j}{2^{K_j}}.$$

The bad-prefix inequality $\text{Syr}^j(n) \geq n$ is automatic when $2^{K_j} < 3^j$, while for $2^{K_j} > 3^j$ it is equivalent to

$$n \leq \frac{B_j}{2^{K_j} - 3^j}.$$

The case $2^{K_j} = 3^j$ cannot occur for $j \geq 1$. Intersecting all such inequalities gives exactly $n \leq U(\mathcal{W})$. The exact count follows because the positive representatives in one residue class modulo $M(\mathcal{W})$ are precisely

$$\rho(\mathcal{W}) + tM(\mathcal{W}), \quad t \geq 0.$$

$\square$

Remark 8.6 (Where compatibility actually enters). [Theorems 8.2](#) and [8.5](#) and [Corollary 8.3](#) give finite-prefix rigidity, but they do not by themselves rule out an infinite chain of nested residue classes and height windows. That global obstruction is the main limitation of the finite-word results.

9 Limitations and density after activation

The results above are finite-word results. They give exact congruence classes and height windows for prescribed bad prefixes, but they do not by themselves exclude the possibility that one odd integer realizes a compatible bad prefix at every length. The rest of the paper keeps only the language needed to describe this limitation and then proves a closed exclusion theorem for eventually periodic low-surplus chains.

Definition 9.1 (Infinite extension chain and candidate bad chain). An *infinite extension chain* is a sequence

$$\mathcal{W}_1 \prec \mathcal{W}_2 \prec \dots$$

where $\mathcal{W}_m$ has length m and $\mathcal{W}_{m+1}$ extends $\mathcal{W}_m$. It is a *candidate bad chain* if $S(\mathcal{W}_m) \neq \emptyset$ for every m . Put

$$S_m := S(\mathcal{W}_m), \quad M_m := 2^{K_m+1}, \quad r_m := r(\mathcal{W}_m), \quad U_m := U(\mathcal{W}_m).$$

Lemma 9.2 (Nestedness). *For every infinite extension chain,*

$$S_{m+1} \subseteq S_m \quad (m \geq 1).$$

Proof. If $n \in S_{m+1}$, then n realizes $\mathcal{W}_{m+1}$, hence realizes its prefix $\mathcal{W}_m$. The bad inequalities through time $m+1$ include the bad inequalities through time m . Thus $n \in S_m$. $\square$

Theorem 9.3 (Sharpened extension formula). *Let $\mathcal{W}^+ = (a_0, \dots, a_{m-1}, b)$ be a one-step extension of $\mathcal{W} = (a_0, \dots, a_{m-1})$. Define*

$$H(\mathcal{W}^+) := \begin{cases} \frac{B_{m+1}(\mathcal{W}^+)}{2^{K_m+b} - 3^{m+1}}, & 2^{K_m+b} > 3^{m+1}, \\ +\infty, & 2^{K_m+b} < 3^{m+1}. \end{cases}$$

Then

$$M(\mathcal{W}^+) = 2^b M(\mathcal{W}), \quad U(\mathcal{W}^+) = \min\{U(\mathcal{W}), H(\mathcal{W}^+)\},$$

and

$$S(\mathcal{W}^+) = S(\mathcal{W}) \cap \{n : n \equiv r(\mathcal{W}^+) \pmod{M(\mathcal{W}^+)}\} \cap \{n : n \leq H(\mathcal{W}^+)\}.$$

Proof. The modulus identity follows from $K_{m+1} = K_m + b$. The residue refinement is [Corollary 8.3](#). The only new bad-prefix inequality is the one at time $m+1$, and by [Theorem 8.5](#) it is either automatic when $2^{K_m+b} < 3^{m+1}$, or equivalent to $n \leq H(\mathcal{W}^+)$ when $2^{K_m+b} > 3^{m+1}$. Intersecting this new condition with the previous exact description of $S(\mathcal{W})$ proves the formula. $\square$

Definition 9.4 (Density parameter). For a finite valuation word $\mathcal{W}$, define the extended nonnegative density parameter

$$D(\mathcal{W}) := \frac{U(\mathcal{W})}{M(\mathcal{W})}.$$

For a one-step extension $\mathcal{W}^+$ of $\mathcal{W}$, define

$$Q(\mathcal{W}^+) := \frac{H(\mathcal{W}^+)}{M(\mathcal{W})}.$$

Theorem 9.5 (Density recurrence). *Let $\mathcal{W}^+ = (a_0, \dots, a_{m-1}, b)$ be a one-step extension of $\mathcal{W} = (a_0, \dots, a_{m-1})$. Then*

$$D(\mathcal{W}^+) = 2^{-b} \min\{D(\mathcal{W}), Q(\mathcal{W}^+)\}.$$

In particular,

$$D(\mathcal{W}^+) \leq 2^{-b} D(\mathcal{W}).$$

Proof. By [Theorem 9.3](#),

$$U(\mathcal{W}^+) = \min\{U(\mathcal{W}), H(\mathcal{W}^+)\}$$

and

$$M(\mathcal{W}^+) = 2^b M(\mathcal{W}).$$

Dividing the first identity by the second gives the stated recurrence. The inequality follows because $\min\{D, Q\} \leq D$. $\square$

Remark 9.6 (Why density is the first invariant to test). The recurrence shows that every extension gains a factor 2^{-b} in the modulus. Once $U(\mathcal{W})$ is finite, no compensation is possible: D contracts at least by 2^{-b} . Thus the first remaining obstruction is not density decay after activation, but activation itself, namely ruling out globally realized chains with $U = +\infty$ at every finite level.

Theorem 9.7 (Activated density collapse). *Let $(\mathcal{W}_m)$ be a candidate bad chain, not the fixed-point chain. Assume $U(\mathcal{W}_{m_0}) < +\infty$ for some m_0 . Then*

$$\lim_{m \rightarrow \infty} D(\mathcal{W}_m) = 0.$$

In particular, there exists m such that $D(\mathcal{W}_m) < 1$, hence

$$\#S(\mathcal{W}_m) \leq 1.$$

This estimate alone does not determine whether such a singleton is possible.

Proof. For $m \geq m_0$, apply [Theorem 9.5](#) at each extension:

$$D(\mathcal{W}_{m+1}) \leq 2^{-a_m} D(\mathcal{W}_m).$$

Since every valuation $a_m \geq 1$,

$$D(\mathcal{W}_m) \leq 2^{-(m-m_0)} D(\mathcal{W}_{m_0}),$$

which tends to 0. If $D(\mathcal{W}_m) < 1$, the bound

$$\#S(\mathcal{W}_m) \leq 1 + D(\mathcal{W}_m)$$

implies $\#S(\mathcal{W}_m) \leq 1$. $\square$

Remark 9.8 (Remaining obstruction). The theorem leaves two separate obstructions. First, a chain may remain low-surplus, with $U(\mathcal{W}_m) = +\infty$ at every finite level; this is the 2-adic stabilization problem treated in the next sections. Second, even after activation, the density argument alone leaves the possibility of one surviving starting value. The main unconditional theorem of this paper addresses the first obstruction for the eventually periodic low-surplus class.

10 Activation and 2-adic incompatibility

This section isolates the remaining $U = +\infty$ obstruction. It does not prove activation. Its purpose is to make the obstruction exact: a terminally bad chain with no activated prefix is precisely a positive integer realization of an infinite low-surplus valuation chain. Such a chain is controlled by nested residue classes and hence by one 2-adic limit.

Definition 10.1 (Low-surplus prefix). A finite valuation word $\mathcal{W}_m = (a_0, \dots, a_{m-1})$ is called *low-surplus* if, for every prefix length $1 \leq j \leq m$,

$$2^{K_j} < 3^j, \quad K_j = a_0 + \dots + a_{j-1}.$$

For an infinite chain, we use the separate tail notation $(a_i)_{i \geq 1}$; it is low-surplus if every finite prefix is low-surplus in the preceding sense.

Lemma 10.2 (Low-surplus prefixes have no height window). *If $\mathcal{W}_m$ is low-surplus, then $U(\mathcal{W}_m) = +\infty$. Moreover every positive integer in the exact residue class of $\mathcal{W}_m$ is bad through length m :*

$$\text{Syr}^j(n) > n \quad (1 \leq j \leq m).$$

Proof. For a realized prefix, [Proposition 4.6](#) gives

$$\begin{aligned} \text{Syr}^j(n) - n &= \frac{3^j n + B_j}{2^{K_j}} - n \\ &= \frac{(3^j - 2^{K_j})n + B_j}{2^{K_j}}. \end{aligned}$$

If $\mathcal{W}_m$ is low-surplus, then $3^j - 2^{K_j} > 0$ for every $j \leq m$, and $B_j > 0$. Hence the displayed difference is positive for every positive n realizing the word. Since there is no prefix with $2^{K_j} > 3^j$, the defining minimum for $U(\mathcal{W}_m)$ is empty, so $U(\mathcal{W}_m) = +\infty$. $\square$

Remark 10.3. [Lemma 10.2](#) is the reason density collapse cannot begin before activation. While $U = +\infty$, the parameter $D = U/M$ has no finite size to shrink. The proof must instead rule out infinite positive realization of low-surplus residue chains.

Proposition 10.4 (The 2-adic limit of an infinite valuation chain). *Let $\mathcal{W}_m$ be the length- m prefix of an infinite valuation chain. The exact finite-word classes are compatible:*

$$\rho(\mathcal{W}_{m+1}) \equiv \rho(\mathcal{W}_m) \pmod{M(\mathcal{W}_m)}.$$

Since $M(\mathcal{W}_m) = 2^{K_m+1} \rightarrow \infty$, these classes determine a unique 2-adic integer

$$r_\infty \in \mathbb{Z}_2.$$

Proof. Compatibility follows because realizing the longer word implies realizing its prefix. The moduli $M(\mathcal{W}_m)$ are powers of 2 tending to infinity, so the nested residue classes define a unique element of the inverse limit $\mathbb{Z}_2$. $\square$

Proposition 10.5 (Positive realization equals eventual representative stabilization). *An odd positive integer n realizes the whole infinite chain if and only if*

$$n \equiv \rho(\mathcal{W}_m) \pmod{M(\mathcal{W}_m)} \quad \text{for every } m.$$

Equivalently, n realizes the chain if and only if the least representatives eventually stabilize:

$$\rho(\mathcal{W}_m) = n \quad \text{for all sufficiently large } m.$$

Proof. The first statement is exactly the definition of realizing every finite valuation prefix. For the second statement, suppose n realizes the chain. Since $M(\mathcal{W}_m) \rightarrow \infty$, eventually $M(\mathcal{W}_m) > n$. At every such level the least positive representative of the congruence class containing n must be n itself. Conversely, if $\rho(\mathcal{W}_m) = n$ eventually, then n belongs to all sufficiently deep classes; by nestedness it also belongs to every earlier finite class. $\square$

Corollary 10.6 (Activation/incompatibility equivalence). *A non-fixed terminal bad path with no activated prefix exists if and only if there is an infinite low-surplus valuation chain whose 2-adic limit r_∞ is an odd positive integer $n > 1$.*

Proof. If $n > 1$ gives a terminal bad path and no prefix is activated, its realized valuation sequence is low-surplus. By [Proposition 10.5](#), the associated 2-adic limit is the ordinary positive integer n . Conversely, if a low-surplus chain has 2-adic limit equal to a positive odd integer $n > 1$, then n realizes every finite prefix. By [Lemma 10.2](#), all finite iterates along those prefixes remain larger than n , giving a terminal bad path with no activated prefix. $\square$

Toy model: the all-one chain

The simplest low-surplus chain is

$$(1, 1, 1, \dots).$$

It is useful because the desired incompatibility mechanism can be proved completely.

Lemma 10.7 (All-one incompatibility). *The all-one chain has finite residue classes*

$$n \equiv -1 \pmod{2^{m+1}}$$

at length m . Hence its 2-adic limit is $-1 \in \mathbb{Z}_2$, and it has no positive integer realization.

Proof. If every valuation up to level m equals 1, then the forced iterates satisfy

$$n_j + 1 = \left(\frac{3}{2}\right)^j (n + 1) \quad (0 \leq j \leq m).$$

The exact condition $a_j = 1$ is $n_j \equiv 3 \pmod{4}$, equivalently $4 \mid n_j + 1$. Since 3^j is odd, this condition is

$$2^{j+2} \mid n + 1.$$

For $0 \leq j \leq m - 1$, the strongest condition is therefore $2^{m+1} \mid n + 1$, i.e.

$$n \equiv -1 \pmod{2^{m+1}}.$$

Letting $m \rightarrow \infty$, the nested classes converge to -1 in $\mathbb{Z}_2$. No positive integer is congruent to -1 modulo 2^{m+1} for every m . $\square$

Diagnostic finite search

The following table is not used as proof. It records a small residue-lifting probe over low-surplus finite words. The lesson is negative but important: low-surplus finite prefixes are abundant, and least representatives can remain constant for long finite stretches. Thus a naive monotonicity claim for $\rho(\mathcal{W}_m)$ is false.

m	$\#\{\text{low-surplus words}\}$	least-representative word	ρ	M
1	1	(1)	3	4
2	2	(1, 1)	7	8
3	3	(1, 1, 2)	7	32
4	7	(1, 2, 1, 1)	27	64
8	173	(1, 2, 1, 1, 1, 1, 2, 2)	27	4096
12	8045	(1, 2, 1, 1, 1, 1, 2, 2, 1, 2, 1, 1)	27	131072

The row $m = 12$ already shows why finite compatibility alone cannot close the argument: a fixed positive representative can survive many low-surplus levels. The missing theorem must prove that such stabilization cannot persist forever for $n > 1$, not merely that finite low-surplus prefixes are rigid.

Conjecture 10.8 (Low-surplus 2-adic incompatibility). Let $(a_i)_{i \geq 1}$ be an infinite low-surplus valuation chain, and let $r_\infty \in \mathbb{Z}_2$ be the 2-adic limit of its nested exact residue classes. Then r_∞ is not an odd positive integer $n > 1$.

Proposition 10.9 (Equivalent stabilization form). *Conjecture 10.8 is equivalent to the following statement: for every non-fixed infinite low-surplus chain, the least representatives $\rho(\mathcal{W}_m)$ are not eventually constant at a positive value $n > 1$.*

Proof. This is exactly [Proposition 10.5](#) restricted to low-surplus chains. $\square$

Remark 10.10 (Scope of the conjecture). [Conjecture 10.8](#) is a broader statement than the unconditional class theorem proved below. It would rule out positive integer realization in the entire low-surplus branch, but it would still leave the post-activation singleton issue outside the scope of the present paper.

11 Stabilization and jump calculus

Four layers of the argument

The later obstruction theory uses four distinct objects, and we keep them separate.

- (i) A *realized orbit* starts from a fixed positive odd integer n and has prefixes $\mathcal{W}_m(n)$.
- (ii) A *finite word* $\mathcal{W}_m$ is an abstract valuation prefix with one realization class modulo $M(\mathcal{W}_m)$.
- (iii) An *infinite extension chain* is a nested sequence $\mathcal{W}_1 \prec \mathcal{W}_2 \prec \dots$ of finite words.
- (iv) The *least-representative process* studies the least positive odd representatives $\rho(\mathcal{W}_m)$ of these nested classes.

A positive integer realization of an infinite chain is equivalent to eventual stabilization of the least representatives. A strict jump is a change of this least representative under one-step refinement.

This section sharpens [Conjecture 10.8](#). Along a fixed nested valuation chain, the least representatives $\rho(\mathcal{W}_m)$ are not arbitrary: they are nondecreasing, and every extension either keeps the same representative or makes a strict jump by a positive multiple of the previous modulus. Consequently low-surplus 2-adic incompatibility is equivalent to an infinite strict-jump theorem.

For the length- m prefix $\mathcal{W}_m$, put

$$\alpha := \log_2 3, \quad \delta_m := m\alpha - K_m.$$

Thus $\mathcal{W}_m$ is low-surplus exactly when

$$\delta_j > 0 \quad (1 \leq j \leq m).$$

Proposition 11.1 (Monotone jump decomposition). *Let $\mathcal{W}_{m+1}$ be a one-step extension of $\mathcal{W}_m$ by the valuation b . Then there is a unique integer*

$$t_m \in \{0, 1, \dots, 2^b - 1\}$$

such that

$$\rho(\mathcal{W}_{m+1}) = \rho(\mathcal{W}_m) + t_m M(\mathcal{W}_m).$$

In particular,

$$\rho(\mathcal{W}_{m+1}) \geq \rho(\mathcal{W}_m).$$

The extension is a strict jump precisely when $t_m > 0$.

Proof. The longer realization class refines the shorter one, hence

$$\rho(\mathcal{W}_{m+1}) \equiv \rho(\mathcal{W}_m) \pmod{M(\mathcal{W}_m)}.$$

Also

$$M(\mathcal{W}_{m+1}) = 2^b M(\mathcal{W}_m),$$

and

$$1 \leq \rho(\mathcal{W}_{m+1}) < M(\mathcal{W}_{m+1}).$$

Therefore $\rho(\mathcal{W}_{m+1})$ has the unique form

$$\rho(\mathcal{W}_{m+1}) = \rho(\mathcal{W}_m) + t_m M(\mathcal{W}_m), \quad 0 \leq t_m \leq 2^b - 1.$$

The monotonicity and strict-jump criterion are immediate. □

Corollary 11.2 (Stabilization versus infinite jumps). *For any infinite nested valuation chain, the following are equivalent:*

- (i) *the sequence $\rho(\mathcal{W}_m)$ is eventually constant;*
- (ii) *only finitely many strict jumps occur.*

If infinitely many strict jumps occur, then $\rho(\mathcal{W}_m) \rightarrow +\infty$, and the associated 2-adic limit is not an ordinary positive integer.

Proof. By [Proposition 11.1](#), $\rho(\mathcal{W}_m)$ is nondecreasing and changes value exactly at strict jumps. Thus finitely many strict jumps are equivalent to eventual constancy. If strict jumps occur infinitely often, then $\rho(\mathcal{W}_m)$ is an unbounded nondecreasing sequence of positive integers. An ordinary positive integer n represented in the inverse limit would force $\rho(\mathcal{W}_m) = n$ for all sufficiently large m , once $M(\mathcal{W}_m) > n$. Hence an infinite-jump chain cannot have a positive integer as its 2-adic limit. $\square$

Remark 11.3. The useful monotonicity statement is therefore true but too weak: $\rho(\mathcal{W}_m)$ cannot decrease. What is false, and contradicted by finite diagnostics such as the long survival of $\rho = 27$, is any naive strict-growth claim at every level. The remaining question is eventual strict jumping along every non-fixed low-surplus chain.

The no-jump criterion

For a finite word $\mathcal{W}_m$, define

$$x_m(\rho(\mathcal{W}_m)) := \frac{3^m \rho(\mathcal{W}_m) + B_m}{2^{K_m}}.$$

Thus $x_m(\rho(\mathcal{W}_m)) = \text{Syr}^m(\rho(\mathcal{W}_m))$, because $\rho(\mathcal{W}_m)$ realizes $\mathcal{W}_m$.

Proposition 11.4 (No-jump criterion). *Let $\mathcal{W}_{m+1}$ extend $\mathcal{W}_m$ by the valuation b , and let t_m be the jump parameter from [Proposition 11.1](#). Put*

$$x_m^\rho := x_m(\rho(\mathcal{W}_m)), \quad v_m^\rho := \nu_2(3x_m^\rho + 1).$$

Then

$$t_m = 0 \iff b = v_m^\rho.$$

Equivalently, the least representative stays fixed under a one-step refinement exactly when the prescribed next valuation agrees with the actual next Syracuse valuation of the current least representative.

Proof. The number $\rho(\mathcal{W}_m)$ already realizes the prefix $\mathcal{W}_m$. It realizes the extended word $\mathcal{W}_{m+1}$ exactly when the next valuation of its current iterate is b , namely when

$$\nu_2(3x_m^\rho + 1) = b.$$

If this equality holds, then $\rho(\mathcal{W}_m)$ belongs to the refined realization class modulo $M(\mathcal{W}_{m+1})$. Since

$$0 < \rho(\mathcal{W}_m) < M(\mathcal{W}_m) < M(\mathcal{W}_{m+1}),$$

it is the least positive representative of that refined class, so $\rho(\mathcal{W}_{m+1}) = \rho(\mathcal{W}_m)$ and $t_m = 0$. Conversely, if $t_m = 0$, then $\rho(\mathcal{W}_{m+1}) = \rho(\mathcal{W}_m)$. Hence the old least representative realizes the extended word, forcing the displayed valuation equality. $\square$

Remark 11.5 (Explicit lifting form). The representatives in the old class modulo $M(\mathcal{W}_{m+1}) = 2^b M(\mathcal{W}_m)$ are

$$\rho(\mathcal{W}_m) + tM(\mathcal{W}_m), \quad 0 \leq t < 2^b.$$

After m accelerated steps this becomes

$$x_m(\rho(\mathcal{W}_m)) + 2 \cdot 3^m t.$$

The extension valuation b selects the unique t for which

$$\nu_2(3(x_m(\rho(\mathcal{W}_m)) + 2 \cdot 3^m t) + 1) = b.$$

The no-jump case is $t = 0$, equivalently $b = \nu_2(3x_m(\rho(\mathcal{W}_m)) + 1)$.

The low-surplus valuation ceiling

Lemma 11.6 (One-step low-surplus ceiling). *Assume $\mathcal{W}_m$ is low-surplus, and put*

$$c_m := \lfloor \delta_m + \alpha \rfloor.$$

A one-step extension by $b \geq 1$ remains low-surplus at the new level $m + 1$ if and only if

$$b \leq c_m.$$

Proof. The new prefix remains low-surplus exactly when

$$K_m + b < (m + 1)\alpha,$$

or equivalently

$$b < \delta_m + \alpha.$$

Since $\alpha = \log_2 3$ is irrational, $\delta_m + \alpha = (m + 1)\alpha - K_m$ is not an integer. Therefore, for integer b , the strict inequality is equivalent to

$$b \leq \lfloor \delta_m + \alpha \rfloor.$$

□

Corollary 11.7 (No-jump or forced-jump dichotomy). *Let $\mathcal{W}_m$ be low-surplus, and set*

$$v_m := \nu_2(3x_m(\rho(\mathcal{W}_m)) + 1).$$

If $v_m > c_m$, then every low-surplus one-step extension of $\mathcal{W}_m$ forces a strict jump:

$$\rho(\mathcal{W}_{m+1}) > \rho(\mathcal{W}_m).$$

Following the no-jump extension instead activates the chain at level $m + 1$.

Proof. By [Lemma 11.6](#), every low-surplus extension must choose a prescribed next valuation b satisfying

$$b \leq c_m.$$

If $v_m > c_m$, then every such low-surplus choice satisfies

$$b \leq c_m < v_m = \nu_2(3x_m(\rho(\mathcal{W}_m)) + 1).$$

Thus $b \neq v_m$, so [Proposition 11.4](#) implies $t_m \neq 0$. Hence every low-surplus extension is a strict jump. The no-jump extension instead has $b = v_m > c_m$, so it violates the low-surplus inequality at level $m + 1$, i.e. it activates. □

Equivalent strict-jump criterion

Theorem 11.8 (Equivalent strict-jump form). *Conjecture 10.8 is equivalent to the following statement:*

Every non-fixed infinite low-surplus valuation chain has infinitely many strict jumps

$$\rho(\mathcal{W}_{m+1}) > \rho(\mathcal{W}_m).$$

Proof. If infinitely many strict jumps occur, then by [Corollary 11.2](#) the 2-adic limit is not an ordinary positive integer. Conversely, if only finitely many strict jumps occur, then $\rho(\mathcal{W}_m)$ is eventually constant at some positive odd integer n . That integer realizes every sufficiently deep finite word, and hence every earlier prefix by nestedness. Since the chain is low-surplus, every realized prefix is bad. The case $n = 1$ is impossible because $\text{Syr}^j(1) = 1$ for all j , whereas [Lemma 10.2](#) gives $\text{Syr}^j(n) > n$ for positive realizations of low-surplus prefixes. Thus $n > 1$, which is exactly the forbidden positive integer realization in [Conjecture 10.8](#). $\square$

Problem 11.9 (Valuation-ceiling diagnostic). Prove that a no-jump low-surplus tail cannot persist forever. Equivalently, for every putative stabilized positive representative $n > 1$, prove that at some finite level m the actual next valuation

$$\nu_2(3 \text{Syr}^m(n) + 1)$$

exceeds the low-surplus ceiling

$$\lfloor m \log_2 3 - K_m(n) + \log_2 3 \rfloor.$$

By [Corollary 11.7](#), such a level forces either activation of the actual orbit or a strict jump away from n for every continuing low-surplus chain.

Problem 11.10 (Strict-jump diagnostic). Prove directly that every non-fixed infinite low-surplus chain has infinitely many levels at which the prescribed next extension valuation b_m disagrees with the actual next valuation of the current least representative:

$$b_m \neq \nu_2(3x_m(\rho(\mathcal{W}_m)) + 1).$$

By [Proposition 11.4](#), each such disagreement is exactly a strict jump of $\rho(\mathcal{W}_m)$.

Diagnostics

Remark 11.11 (All-one chain). For the all-one chain $(1, 1, 1, \dots)$, the representatives are

$$\rho(\mathcal{W}_m) = 2^{m+1} - 1.$$

Thus every level is a strict jump, and the 2-adic limit is -1 , not a positive integer. This is the clean model for [Theorem 11.8](#).

Remark 11.12 (The 27 diagnostic). The accelerated orbit of 27 shows why finite-level survival is not enough. Its low-surplus actual valuation prefix survives through length 36, and once the modulus exceeds 27 the least representative is stabilized at

$$\rho(\mathcal{W}_m) = 27$$

for a long finite interval. At the next level the actual valuation is 3, while the low-surplus ceiling is 2. Therefore the actual no-jump continuation activates at level 37; any formal low-surplus continuation must make a strict jump away from 27. This is exactly the finite valuation contradiction one wants to force for every putative stabilized $n > 1$.

Remark 11.13 (Honest endpoint). [Propositions 11.1](#) and [11.4](#), [Lemma 11.6](#), and [Theorem 11.8](#) are unconditional. What remains open is the infinite strict-jump theorem, or equivalently the valuation-ceiling diagnostic in [Problem 11.9](#). Density collapse still begins only after activation.

12 Strict-jump forcing and no-jump traps

The jump calculus shows that positive integer realization is the same as eventual stabilization of the least representatives. This section records the exact finite-word trap that a counterexample to finite descent would have to generate. The statements below are local and unconditional; the global no-eventual-no-jump assertion remains open.

Definition 12.1 (Least-representative bad prefix and no-jump streak). A finite valuation word $\mathcal{W}$ is called *least-representative bad* if

$$\rho(\mathcal{W}) \in S(\mathcal{W}).$$

Equivalently, the least positive representative of its exact realization class realizes the word and remains at least as large as its starting value throughout the prefix.

For an infinite extension chain $(\mathcal{W}_m)_{m \geq 1}$, a level m is called a *no-jump level* if

$$\rho(\mathcal{W}_{m+1}) = \rho(\mathcal{W}_m).$$

A block $[p, q]$ is a *no-jump streak* if every level $m = p, p+1, \dots, q-1$ is a no-jump level. A tail is *eventually no-jump* if all sufficiently large levels are no-jump levels.

Definition 12.2 (LR terminal-ceiling compatibility). Let $\mathcal{W}_m$ be least-representative bad and put

$$\rho_m := \rho(\mathcal{W}_m), \quad x_m^\rho := x_m(\rho_m), \quad v_m^\rho := \nu_2(3x_m^\rho + 1).$$

Define the real and integer least-representative ceilings by

$$\Gamma_m^\rho := \log_2 \left(3 \frac{x_m^\rho}{\rho_m} + \frac{1}{\rho_m} \right), \quad C_m^\rho := \lfloor \Gamma_m^\rho \rfloor.$$

The no-jump continuation at level m is called *least-representative terminal-ceiling compatible* if

$$v_m^\rho \leq \Gamma_m^\rho,$$

equivalently if $v_m^\rho \leq C_m^\rho$. An eventual no-jump tail is terminal-ceiling compatible if this condition holds at every sufficiently large no-jump level.

Lemma 12.3 (Ceiling rule for the no-jump branch). *Let $\mathcal{W}_m$ be least-representative bad and use the notation of Definition 12.2. Let*

$$x_{m+1}^{(0)} := \frac{3x_m^\rho + 1}{2^{v_m^\rho}}$$

be the next iterate of the least representative along the no-jump branch. Then

$$v_m^\rho \leq \Gamma_m^\rho$$

if and only if

$$x_{m+1}^{(0)} \geq \rho_m.$$

Equivalently, the actual no-jump continuation of the least representative satisfies the terminal-ceiling rule exactly when its next iterate is still bad relative to ρ_m .

Proof. The inequality $v_m^\rho \leq \Gamma_m^\rho$ is equivalent to

$$2^{v_m^\rho} \leq \frac{3x_m^\rho + 1}{\rho_m}.$$

Multiplying by ρ_m and dividing by $2^{v_m^\rho}$ gives

$$\frac{3x_m^\rho + 1}{2^{v_m^\rho}} \geq \rho_m,$$

which is the asserted inequality for $x_{m+1}^{(0)}$. □

Proposition 12.4 (No-jump bad-extension test). *Let $\mathcal{W}_m$ be least-representative bad, and let $\mathcal{W}_{m+1}^{(0)}$ be the one-step extension of $\mathcal{W}_m$ by the actual least-representative valuation*

$$v_m^\rho = \nu_2(3x_m(\rho(\mathcal{W}_m)) + 1).$$

Then $\mathcal{W}_{m+1}^{(0)}$ is least-representative bad if and only if

$$v_m^\rho \leq \log_2 \left(3 \frac{x_m(\rho(\mathcal{W}_m))}{\rho(\mathcal{W}_m)} + \frac{1}{\rho(\mathcal{W}_m)} \right).$$

Equivalently, it is least-representative bad if and only if the no-jump continuation is least-representative terminal-ceiling compatible. If the reverse strict inequality holds, then every one-step extension $\mathcal{W}_{m+1}$ of $\mathcal{W}_m$ that is least-representative bad must be a strict jump:

$$\rho(\mathcal{W}_{m+1}) > \rho(\mathcal{W}_m).$$

Proof. Because $\mathcal{W}_m$ is least-representative bad, $\rho_m := \rho(\mathcal{W}_m)$ realizes $\mathcal{W}_m$ and satisfies the bad inequalities up to time m . By [Proposition 11.4](#), the extension by the actual valuation v_m^ρ is exactly the no-jump extension, so its least representative is still ρ_m . Therefore the longer word is least-representative bad precisely when the new iterate

$$x_{m+1}^{(0)} = \frac{3x_m(\rho_m) + 1}{2^{v_m^\rho}}$$

is at least ρ_m . This is exactly [Lemma 12.3](#).

If the reverse strict inequality holds, then the no-jump extension is not least-representative bad. Any one-step extension with no strict jump must coincide with the no-jump extension by [Proposition 11.4](#). Hence every least-representative bad extension must have a strict jump. $\square$

Proposition 12.5 (Eventual no-jump trap). *Let $(\mathcal{W}_m)_{m \geq 1}$ be an infinite extension chain. Suppose that for some m_0 and some odd integer $n > 1$,*

$$\rho(\mathcal{W}_m) = n \quad (m \geq m_0)$$

and $\mathcal{W}_m$ is least-representative bad for every $m \geq m_0$. Then n realizes the whole chain and is terminally bad:

$$\text{Syr}^j(n) \geq n \quad (j \geq 1).$$

Conversely, if an odd integer $n > 1$ is terminally bad and $\mathcal{W}_m = \mathcal{W}_m(n)$ is its realized valuation-prefix chain, then $\mathcal{W}_m$ is least-representative bad, terminal-ceiling compatible in the sense of [Definition 12.2](#), and $\rho(\mathcal{W}_m) = n$ for all sufficiently large m .

Proof. Assume first that the representatives stabilize at n and that all sufficiently long prefixes are least-representative bad. Then $n = \rho(\mathcal{W}_m)$ realizes $\mathcal{W}_m$ for every $m \geq m_0$, and hence realizes every finite prefix of the infinite chain. Since $n \in S(\mathcal{W}_m)$ for every $m \geq m_0$, the inequalities

$$\text{Syr}^j(n) \geq n \quad (1 \leq j \leq m)$$

hold for arbitrarily large m . Therefore they hold for every $j \geq 1$, so n is terminally bad.

Conversely, let $n > 1$ be terminally bad. Its realized prefix $\mathcal{W}_m(n)$ satisfies $n \in S(\mathcal{W}_m(n))$ for every m . Once $M(\mathcal{W}_m(n)) > n$, [Lemma 13.2](#) gives $\rho(\mathcal{W}_m(n)) = n$. Hence the prefixes are least-representative bad for all sufficiently large m . The terminal-ceiling compatibility at those levels is exactly [Corollary 6.5](#) applied to the stabilized least representative. $\square$

Proposition 12.6 (No-eventual-no-jump gateway). *Assume that the fixed point chain generated by $n = 1$ is the only infinite extension chain having a tail which is simultaneously no-jump, least-representative bad, and terminal-ceiling compatible. Then the Collatz conjecture holds.*

Proof. If the Collatz conjecture failed, [Corollary 4.5](#) would give an odd terminally bad integer $n > 1$. Applying [Proposition 12.5](#) to the realized valuation-prefix chain of n , we would obtain a tail that is no-jump, least-representative bad, and terminal-ceiling compatible, stabilized at $n > 1$. This is excluded by the hypothesis. Therefore no terminally bad odd integer $n > 1$ exists, and [Corollary 4.5](#) gives the Collatz conjecture. $\square$

Problem 12.7 (Uniform no-jump streak bound). Prove that there exists an absolute constant $L \geq 1$ such that every infinite extension chain which is not the fixed point chain of 1 and whose prefixes are least-representative bad and terminal-ceiling compatible at all sufficiently large levels has a strict jump in every block of L consecutive sufficiently large levels.

Proposition 12.8 (Uniform streak bound implies Collatz). *A positive solution of [Problem 12.7](#) implies the Collatz conjecture.*

Proof. A terminally bad odd integer $n > 1$ would, by [Proposition 12.5](#), produce an infinite chain whose sufficiently deep prefixes are least-representative bad, terminal-ceiling compatible, and whose least representatives are all equal to n . Such a tail has no strict jumps at all, hence contains no strict jump in any block of length L . This contradicts the asserted uniform no-jump streak bound. Therefore no terminally bad $n > 1$ exists, and Collatz follows from [Corollary 4.5](#). $\square$

Remark 12.9 (Why this is the next obstruction to attack). [Proposition 12.4](#) gives a local forcing rule: whenever the no-jump branch of a least-representative bad prefix breaks the terminal ceiling, every continuing least-representative bad extension must jump. [Proposition 12.5](#) then shows that a true counterexample would have to avoid such forced jumps forever after the modulus passes the counterexample itself. Thus the remaining proof-theoretic task may be expressed without probabilistic language: prove that non-fixed least-representative bad, terminal-ceiling compatible chains cannot have arbitrarily long no-jump tails.

Problem 12.10 (Height-dependent no-jump bound). A possible weaker route is to prove a height-dependent version of [Problem 12.7](#). Namely, for each finite terminal-ceiling band, prove that there is a bound $L(K)$ on the length of no-jump streaks whose least-representative relative heights remain in that band, unless the chain is already the fixed point chain of 1. The complementary unbounded-height branch would then have to be handled by the terminal-surplus envelope in [Proposition 5.5](#).

13 Terminal obstruction dichotomy and scale-sensitive no-jump clocks

The uniform no-jump-streak target in [Problem 12.7](#) is a deliberately strong sufficient condition. The exact obstruction can be localized more sharply. For an actual starting value n , the least representatives need not equal n at shallow levels; they must equal n only after the realization modulus exceeds n . This motivates a scale-sensitive clock.

Definition 13.1 (First descent time, stabilization time, and post-stabilization clock). Let $n > 1$ be odd and put $n_m = \text{Syr}^m(n)$. Define the first descent time

$$\tau(n) := \inf\{m \geq 1 : n_m < n\} \in \mathbb{N} \cup \{+\infty\}.$$

For the realized valuation prefix $\mathcal{W}_m(n)$, define the stabilization time

$$\sigma(n) := \inf\{m \geq 0 : M(\mathcal{W}_m(n)) > n\}, \quad M(\mathcal{W}_m(n)) = 2^{K_m(n)+1}.$$

Since $K_m(n) \geq m$, the number $\sigma(n)$ is finite. If $\tau(n) < +\infty$, define the post-stabilization bad length

$$\ell(n) := \max\{0, \tau(n) - \sigma(n)\};$$

if $\tau(n) = +\infty$, set $\ell(n) = +\infty$.

Lemma 13.2 (Stabilization threshold). *Let $n > 1$ be odd and let $\mathcal{W}_m(n)$ be its realized valuation prefix. If*

$$M(\mathcal{W}_m(n)) > n,$$

then

$$\rho(\mathcal{W}_m(n)) = n.$$

In particular, $\rho(\mathcal{W}_m(n)) = n$ for all $m \geq \sigma(n)$.

Proof. The integer n realizes $\mathcal{W}_m(n)$, hence it lies in the exact realization class modulo $M(\mathcal{W}_m(n))$. If $0 < n < M(\mathcal{W}_m(n))$, then no smaller positive integer can be congruent to n modulo this modulus. Therefore n is the least positive representative of the class. $\square$

Corollary 13.3 (Post-stabilization no-jump tail for a terminally bad orbit). *If an odd integer $n > 1$ is terminally bad, then for every $m \geq \sigma(n)$*

$$\rho(\mathcal{W}_m(n)) = n$$

and the realized prefix $\mathcal{W}_m(n)$ is least-representative bad. Consequently the realized chain has an eventual no-jump tail.

Proof. The equality of least representatives follows from [Lemma 13.2](#). Since n is terminally bad, it belongs to $S(\mathcal{W}_m(n))$ for every m , so after stabilization the least representative itself is bad. The equality $\rho(\mathcal{W}_m(n)) = n$ for all $m \geq \sigma(n)$ means that all later levels are no-jump levels. $\square$

Proposition 13.4 (Exact post-stabilization clock). *For an odd integer $n > 1$, the following are equivalent:*

- (i) n is terminally bad;
- (ii) the realized chain $(\mathcal{W}_m(n))_{m \geq 1}$ is least-representative bad for all sufficiently large m and has an eventual no-jump tail;
- (iii) $\ell(n) = +\infty$.

If $\tau(n) < +\infty$, then $\ell(n)$ is exactly the length of the realized no-jump bad tail after the stabilization threshold and before first descent.

Proof. If n is terminally bad, then [Corollary 13.3](#) proves the eventual least-representative bad no-jump statement, and $\tau(n) = +\infty$, hence $\ell(n) = +\infty$. Conversely, if the realized chain is least-representative bad and no-jump from some level onward, the least representatives stabilize at a positive odd integer. Since the chain is the realized chain of n , [Lemma 13.2](#) gives that this stable value is n . Least-representative badness at arbitrarily large levels then gives $\text{Syr}^j(n) \geq n$ for every $j \geq 1$, so n is terminally bad. The equivalence with $\ell(n) = +\infty$ is just the definition of $\tau(n)$ and $\sigma(n)$. If $\tau(n) < +\infty$, stabilization begins at $\sigma(n)$ and badness ends at $\tau(n)$, so the stated length is $\max\{0, \tau(n) - \sigma(n)\}$. $\square$

Theorem 13.5 (No-eventual-no-jump equivalence). *The Collatz conjecture is equivalent to the assertion that the fixed point chain generated by 1 is the only infinite extension chain having an eventually no-jump tail of least-representative bad prefixes.*

Proof. The forward implication follows from [Proposition 12.5](#): an eventually no-jump tail of least-representative bad prefixes stabilized at some odd $n > 1$ would make n terminally bad, contradicting finite descent. The only remaining stabilized positive value compatible with Collatz is $n = 1$, the fixed point chain.

Conversely, if Collatz fails, [Corollary 4.5](#) gives a terminally bad odd integer $n > 1$. By [Corollary 13.3](#), its realized valuation-prefix chain has an eventual no-jump tail of least-representative bad prefixes, not generated by 1. This contradicts the asserted uniqueness. $\square$

Lemma 13.6 (Bounded terminally bad orbits are eventually periodic). *Let $n > 1$ be an odd terminally bad integer. If the accelerated orbit*

$$n_m = \text{Syr}^m(n)$$

is bounded above, then it is eventually periodic. Moreover the eventual periodic orbit is not the fixed point 1.

Proof. A bounded orbit in the positive odd integers visits only finitely many values. Since Syr is deterministic, some value must repeat, and from the first repetition onward the orbit is periodic. Because n is terminally bad and $n > 1$, every iterate satisfies $n_m \geq n > 1$. Therefore the eventual cycle cannot be the fixed point 1. $\square$

Corollary 13.7 (Terminal obstruction dichotomy). *If the Collatz conjecture fails, then at least one of the following alternatives occurs:*

- (i) *there exists a nontrivial accelerated cycle in the positive odd integers;*
- (ii) *there exists an odd terminally bad integer $n > 1$ whose accelerated orbit is unbounded above.*

Conversely, either alternative contradicts the Collatz conjecture.

Proof. If Collatz fails, [Corollary 4.5](#) gives a terminally bad odd integer $n > 1$. If its orbit is bounded, [Lemma 13.6](#) puts the orbit eventually on a nontrivial accelerated cycle. If its orbit is unbounded, alternative (ii) holds. Conversely, a nontrivial cycle never reaches 1, and an unbounded terminally bad orbit never descends below its starting value; either case violates Collatz. $\square$

Problem 13.8 (Unbounded terminal-ceiling branch). Exclude the second alternative in [Corollary 13.7](#). Equivalently, prove that no odd integer $n > 1$ can have an unbounded accelerated orbit while satisfying the terminal ceiling rule

$$a_m(n) \leq \log_2 \left(3 \frac{\text{Syr}^m(n)}{n} + \frac{1}{n} \right) \quad (m \geq 0)$$

at every level. In terminal-surplus variables, the same obstruction is the existence of an unbounded realized orbit satisfying

$$A_m(n) \leq 1 + \frac{1}{3n} \sum_{i=0}^{m-1} A_i(n) \quad (m \geq 1).$$

Theorem 13.9 (Cycle-free plus unbounded-branch closure implies Collatz). *Assume that there are no nontrivial accelerated cycles in the positive odd integers and that [Problem 13.8](#) has a positive solution. Then the Collatz conjecture holds.*

Proof. By [Corollary 13.7](#), any failure of Collatz produces either a nontrivial accelerated cycle or an unbounded terminally bad orbit. The two hypotheses exclude these alternatives. $\square$

Remark 13.10 (Finite diagnostic for the scale-sensitive clock). The quantity $\ell(n)$ is a diagnostic, not a proof invariant by itself. Exact integer iteration gives the following values when $\ell(n)$ is maximized over odd $3 \leq n < X$:

X	maximizer n	$\tau(n)$	$\sigma(n)$	$\ell(n)$
10^2	27	37	3	34
10^3	703	51	7	44
10^4	703	51	7	44
10^5	35655	85	10	75
10^6	381727	109	14	95

The growth of these finite post-stabilization clocks suggests that a scale-free uniform streak bound, while sufficient, is probably too strong to be the most natural route. The exact dichotomy above points instead to two separate terminal tasks: the cycle certificate of [Theorem 7.1](#) and the unbounded branch in [Problem 13.8](#).

14 Deficit coding for the unbounded terminal branch

The dichotomy of [Corollary 13.7](#) leaves, after the cycle certificate, an unbounded terminally bad orbit. This section rewrites that branch in a more discrete form. The key point is that a terminally bad orbit does not merely stay below the moving terminal ceiling; the integer gap between the ceiling and the actual valuation is exactly the dyadic height band reached at the next step.

Definition 14.1 (Terminal ceiling, fractional ceiling factor, and deficit). Let $n > 1$ be an odd terminally bad integer and put

$$R_m = \frac{\text{Syr}^m(n)}{n}, \quad Y_m = 3R_m + \frac{1}{n}.$$

The terminal ceiling at level m is

$$C_m := \lfloor \log_2 Y_m \rfloor.$$

Since n is terminally bad, [Corollary 6.5](#) gives $a_m \leq C_m$. We define the *under-ceiling deficit*

$$d_m := C_m - a_m \in \mathbb{Z}_{\geq 0}$$

and the fractional ceiling factor

$$\theta_m := \frac{Y_m}{2^{C_m}} \in [1, 2).$$

Lemma 14.2 (Exact dyadic deficit identity). *Let $n > 1$ be terminally bad. For every $m \geq 0$,*

$$R_{m+1} = 2^{d_m} \theta_m, \quad 1 \leq \theta_m < 2.$$

Consequently

$$2^{d_m} \leq R_{m+1} < 2^{d_m+1}, \quad d_m = \lfloor \log_2 R_{m+1} \rfloor.$$

Equivalently,

$$a_m = \left\lfloor \log_2 \left(3R_m + \frac{1}{n} \right) \right\rfloor - \lfloor \log_2 R_{m+1} \rfloor.$$

Proof. By [Lemma 6.3](#),

$$R_{m+1} = \frac{3R_m + 1/n}{2^{a_m}} = \frac{Y_m}{2^{a_m}}.$$

Since $Y_m = 2^{C_m} \theta_m$ and $d_m = C_m - a_m$, this gives

$$R_{m+1} = 2^{C_m - a_m} \theta_m = 2^{d_m} \theta_m.$$

The inequalities $1 \leq \theta_m < 2$ follow from the definition of C_m , and therefore

$$2^{d_m} \leq R_{m+1} < 2^{d_m+1}.$$

Taking base-two logarithms and floors gives $d_m = \lfloor \log_2 R_{m+1} \rfloor$. Substituting $d_m = C_m - a_m$ gives the final displayed identity. $\square$

Corollary 14.3 (Unbounded height is unbounded deficit). *For a terminally bad odd integer $n > 1$, the following are equivalent:*

- (i) the relative heights $(R_m)_{m \geq 0}$ are unbounded;
- (ii) the deficits $(d_m)_{m \geq 0}$ are unbounded.

More quantitatively, if $d_m \leq D$ for all sufficiently large m , then

$$1 \leq R_m < 2^{D+1}$$

for all sufficiently large m . Conversely, if $R_m < H$ for all sufficiently large m , then

$$d_m < \log_2 H$$

for all sufficiently large m .

Proof. This is immediate from [Lemma 14.2](#). The estimate

$$2^{d_m} \leq R_{m+1} < 2^{d_m+1}$$

shows that bounded deficits give bounded future heights, and bounded future heights give bounded floors $d_m = \lfloor \log_2 R_{m+1} \rfloor$. Since deleting finitely many initial terms does not affect boundedness, the two conditions are equivalent. $\square$

Corollary 14.4 (Cycle-free failure forces arbitrarily large deficits). *Assume that there is no nontrivial accelerated cycle in the positive odd integers. If the Collatz conjecture fails, then there exists an odd terminally bad integer $n > 1$ whose under-ceiling deficits satisfy*

$$\sup_{m \geq 0} d_m = +\infty.$$

Proof. If Collatz fails, [Corollary 13.7](#) gives either a nontrivial accelerated cycle or an unbounded terminally bad orbit. The first alternative is excluded by hypothesis. Hence the second alternative holds. By [Corollary 14.3](#), its deficit sequence is unbounded. $\square$

Lemma 14.5 (Large deficits are exactly large under-ceiling drops). *Let $n > 1$ be terminally bad and let $D \in \mathbb{Z}_{\geq 0}$. At level m , the following integer-threshold conditions are equivalent:*

$$d_m \geq D, \quad a_m \leq C_m - D, \quad R_{m+1} \geq 2^D.$$

Specifically,

$$d_m \geq D \iff a_m \leq C_m - D \iff R_{m+1} \geq 2^D.$$

Moreover $d_m = D$ if and only if

$$2^D \leq R_{m+1} < 2^{D+1}.$$

Proof. The equivalence $d_m \geq D \iff a_m \leq C_m - D$ is just the definition $d_m = C_m - a_m$. The equivalence with $R_{m+1} \geq 2^D$, and the exact band for $d_m = D$, follow from [Lemma 14.2](#). $\square$

Definition 14.6 (Least-representative deficit at a no-jump level). Let $\mathcal{W}_m$ be least-representative bad. Put

$$\rho_m := \rho(\mathcal{W}_m), \quad x_m := x_m(\rho_m), \quad v_m := \nu_2(3x_m + 1),$$

and

$$R_m^\rho := \frac{x_m}{\rho_m}, \quad C_m^\rho := \left\lfloor \log_2 \left(3R_m^\rho + \frac{1}{\rho_m} \right) \right\rfloor.$$

If the actual no-jump extension is terminal-ceiling admissible, equivalently if $v_m \leq C_m^\rho$, define the least-representative deficit

$$d_m^\rho := C_m^\rho - v_m.$$

Lemma 14.7 (Least-representative deficit identity). *In the setting of Definition 14.6, assume that the actual no-jump extension is terminal-ceiling admissible. Let*

$$x_{m+1}^{(0)} := \frac{3x_m + 1}{2^{v_m}}$$

be the next iterate of the least representative under the no-jump branch. Then

$$2^{d_m^\rho} \leq \frac{x_{m+1}^{(0)}}{\rho_m} < 2^{d_m^\rho + 1},$$

and hence

$$d_m^\rho = \left\lfloor \log_2 \frac{x_{m+1}^{(0)}}{\rho_m} \right\rfloor.$$

Proof. The proof is identical to Lemma 14.2, with n , R_m , a_m , and R_{m+1} replaced by ρ_m , R_m^ρ , v_m , and $x_{m+1}^{(0)}/\rho_m$, respectively. $\square$

Problem 14.8 (Unbounded-deficit no-jump obstruction). Prove that no infinite extension chain, other than the fixed point chain of 1, can have all three properties below:

- (i) its sufficiently long prefixes are least-representative bad;
- (ii) it has an eventual no-jump tail;
- (iii) along that tail, the least-representative deficits d_m^ρ are unbounded.

Equivalently, rule out an eventual no-jump terminal-ceiling chain whose no-jump iterates enter arbitrarily high dyadic height bands above their stabilized least representative.

Theorem 14.9 (Large-deficit gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers and that Problem 14.8 has a positive solution. Then the Collatz conjecture holds.*

Proof. Assume that Collatz fails. By Corollary 4.5, there is a terminally bad odd integer $n > 1$. If its accelerated orbit is bounded, then Lemma 13.6 gives a nontrivial accelerated cycle, contrary to the first hypothesis. Hence its orbit is unbounded. By Corollary 13.3, the realized prefix chain is least-representative bad and no-jump from the stabilization time onward. By Corollary 14.3, the corresponding deficits are unbounded; after stabilization these are exactly the least-representative deficits of Definition 14.6. This contradicts the positive solution of Problem 14.8. Therefore no terminally bad $n > 1$ exists, and Corollary 4.5 proves the Collatz conjecture. $\square$

Remark 14.10 (Why deficit coding is useful). The unbounded branch in Problem 13.8 is still open. The point of the present section is to replace the qualitative phrase “unbounded orbit” by the integer statement “arbitrarily large under-ceiling deficits”. A terminal counterexample cannot merely drift upward; after stabilization it must repeatedly choose valuations far below the largest valuation allowed by the moving ceiling. This is a sharper target for strict-jump forcing than the scale-free uniform streak bound of Problem 12.7.

15 Dyadic scale walk for large-deficit escape

The deficit identity of Lemma 14.2 shows that an unbounded terminal branch is the same as an unbounded sequence of dyadic height exponents. This section extracts the exact integer walk followed by those exponents. The payoff is that scale growth is possible only in one local configuration: valuation one together with a high mantissa inside the current dyadic height band.

Definition 15.1 (Dyadic scale exponent and mantissa). Let $n > 1$ be terminally bad and put

$$R_m = \frac{\text{Syr}^m(n)}{n}.$$

Since $R_m \geq 1$, define

$$q_m := \lfloor \log_2 R_m \rfloor \in \mathbb{Z}_{\geq 0}, \quad \eta_m := 2^{-q_m} R_m \in [1, 2).$$

The local ceiling digit is

$$s_m := \left\lfloor \log_2 \left(3\eta_m + \frac{2^{-q_m}}{n} \right) \right\rfloor.$$

Lemma 15.2 (The ceiling digit is binary). *For every terminally bad odd integer $n > 1$ and every $m \geq 0$,*

$$s_m \in \{1, 2\}.$$

Moreover

$$s_m = 2 \iff \eta_m \geq \frac{4 - 2^{-q_m}/n}{3}.$$

Proof. Because $1 \leq \eta_m < 2$, $q_m \geq 0$, and $n \geq 3$, we have

$$3 \leq 3\eta_m + \frac{2^{-q_m}}{n} < 6 + \frac{1}{3} < 8.$$

Thus the base-two logarithm lies in $[\log_2 3, 3)$, and its floor is either 1 or 2. The equality $s_m = 2$ is exactly the condition

$$3\eta_m + \frac{2^{-q_m}}{n} \geq 4,$$

which is equivalent to the displayed lower bound for η_m . $\square$

Theorem 15.3 (Exact dyadic scale walk). *Let $n > 1$ be terminally bad. With q_m, η_m, s_m as in Definition 15.1, one has the exact recurrence*

$$q_{m+1} = q_m + s_m - a_m.$$

Consequently

$$q_{m+1} \leq q_m + 1,$$

and the scale increases at level m if and only if

$$q_{m+1} = q_m + 1 \iff a_m = 1 \text{ and } s_m = 2.$$

Equivalently, scale increase occurs exactly at valuation-one levels satisfying

$$\eta_m \geq \frac{4 - 2^{-q_m}/n}{3}.$$

Proof. Write

$$R_m = 2^{q_m} \eta_m.$$

Then

$$3R_m + \frac{1}{n} = 2^{q_m} \left(3\eta_m + \frac{2^{-q_m}}{n} \right).$$

Therefore the terminal ceiling is

$$C_m = \left\lfloor \log_2 \left(3R_m + \frac{1}{n} \right) \right\rfloor = q_m + s_m.$$

By Lemma 14.2,

$$q_{m+1} = \lfloor \log_2 R_{m+1} \rfloor = d_m = C_m - a_m.$$

Combining these identities gives

$$q_{m+1} = q_m + s_m - a_m.$$

Since $s_m \leq 2$ and $a_m \geq 1$, this implies $q_{m+1} \leq q_m + 1$. Equality holds exactly when $s_m = 2$ and $a_m = 1$. The final condition follows from Lemma 15.2. $\square$

Corollary 15.4 (Unbounded scale requires infinitely many valuation-one high-mantissa levels). *Let $n > 1$ be terminally bad. If (R_m) is unbounded, then there are infinitely many levels m for which*

$$a_m = 1 \quad \text{and} \quad \eta_m \geq \frac{4 - 2^{-q_m}/n}{3}.$$

Equivalently, an unbounded terminal branch must have infinitely many scale-up steps

$$q_{m+1} = q_m + 1.$$

Proof. By Corollary 14.3, unboundedness of (R_m) is equivalent to unboundedness of (q_m) . If the scale-up equality $q_{m+1} = q_m + 1$ occurred only finitely many times, then after the last such level the sequence (q_m) could never increase, because Theorem 15.3 gives $q_{m+1} \leq q_m + 1$ and positive increments are exactly scale-up steps. Hence (q_m) would be bounded above by its maximum up to that last level. Therefore unboundedness forces infinitely many scale-up steps, and the valuation-one high-mantissa characterization is Theorem 15.3. $\square$

Corollary 15.5 (Cycle-free failure forces infinitely many scale-up events). *Assume that there is no nontrivial accelerated cycle in the positive odd integers. If the Collatz conjecture fails, then there exists an odd terminally bad integer $n > 1$ with infinitely many levels satisfying*

$$a_m(n) = 1, \quad \eta_m(n) \geq \frac{4 - 2^{-q_m(n)}/n}{3}.$$

Proof. By Corollary 14.4, cycle-free failure produces a terminally bad $n > 1$ with unbounded deficits, equivalently unbounded heights. Apply Corollary 15.4. $\square$

Definition 15.6 (Least-representative scale walk). Let $\mathcal{W}_m$ be least-representative bad and use the notation of Definition 14.6. Assume the actual no-jump extension is terminal-ceiling admissible. Put

$$q_m^\rho := \lfloor \log_2 R_m^\rho \rfloor, \quad \eta_m^\rho := 2^{-q_m^\rho} R_m^\rho,$$

where $R_m^\rho = x_m/\rho_m$. Define

$$s_m^\rho := \left\lfloor \log_2 \left(3\eta_m^\rho + \frac{2^{-q_m^\rho}}{\rho_m} \right) \right\rfloor.$$

Finally let

$$q_{m+1}^{\rho,0} := \left\lfloor \log_2 \frac{x_{m+1}^{(0)}}{\rho_m} \right\rfloor, \quad x_{m+1}^{(0)} = \frac{3x_m + 1}{2^{v_m}}.$$

Lemma 15.7 (Least-representative scale recurrence). *In the setting of Definition 15.6,*

$$s_m^\rho \in \{1, 2\}, \quad q_{m+1}^{\rho,0} = q_m^\rho + s_m^\rho - v_m.$$

Therefore the no-jump least-representative scale can increase at level m only when

$$v_m = 1 \quad \text{and} \quad s_m^\rho = 2,$$

that is, only when

$$\eta_m^\rho \geq \frac{4 - 2^{-q_m^\rho}/\rho_m}{3}.$$

Proof. This is the same calculation as [Lemma 15.2](#) and [Theorem 15.3](#), with n, R_m, a_m replaced by ρ_m, R_m^ρ, v_m . Since $1 \leq \eta_m^\rho < 2$, $q_m^\rho \geq 0$, and $\rho_m \geq 1$, the inner factor lies in $[3, 7)$, so its base-two floor is again either 1 or 2. The displayed equivalence is exactly the condition that this floor be 2. The exact equality follows from

$$\frac{x_{m+1}^{(0)}}{\rho_m} = \frac{3R_m^\rho + 1/\rho_m}{2^{v_m}}.$$

□

Problem 15.8 (Scale-up exclusion on eventual no-jump tails). Prove that no infinite extension chain, other than the fixed point chain of 1, can have all of the following properties:

- (i) its sufficiently long prefixes are least-representative bad;
- (ii) it has an eventual no-jump tail;
- (iii) along that tail, the least-representative scale walk has infinitely many scale-up levels, equivalently infinitely many levels with

$$v_m = 1, \quad \eta_m^\rho \geq \frac{4 - 2^{-q_m^\rho}/\rho_m}{3}.$$

Theorem 15.9 (Scale-up gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers and that [Problem 15.8](#) has a positive solution. Then the Collatz conjecture holds.*

Proof. Assume that Collatz fails. Since nontrivial accelerated cycles are excluded, [Corollary 14.4](#) gives a terminally bad odd integer $n > 1$ with unbounded deficits. After the stabilization time of [Definition 13.1](#), its realized prefix chain is least-representative bad and no-jump, with least representative equal to n . By [Corollary 15.4](#), the corresponding scale walk has infinitely many scale-up levels. This contradicts the asserted scale-up exclusion. Hence no terminally bad $n > 1$ exists, and [Corollary 4.5](#) proves the Collatz conjecture. □

Remark 15.10 (What the scale walk adds). The unbounded-deficit target of [Problem 14.8](#) asks for the exclusion of arbitrarily high dyadic height bands along an eventual no-jump tail. The scale-walk recurrence shows that such escape cannot be diffuse: the height exponent can rise only by one at a time, and only at valuation-one high-mantissa levels. Thus the next proof front is not an unrestricted unbounded-orbit problem; it is the exclusion of infinitely many realized no-jump transitions through the upper third of dyadic bands with next valuation equal to one.

16 Scale-up blocks and compensation targets

The scale-walk recurrence in [Theorem 15.3](#) says that the dyadic height exponent can increase only by one, and only at a valuation-one high-mantissa level. The next useful question is whether such increases can cluster indefinitely. The answer is no: no terminally bad orbit with starting value greater than one can have three consecutive scale-up levels. This produces a block structure and an exact balance identity for the unbounded branch.

Definition 16.1 (Scale-up levels, scale-loss, and blocks). Let $n > 1$ be terminally bad, and use the notation of [Definition 15.1](#). Put

$$\varepsilon_m := \frac{2^{-q_m}}{n}.$$

A level m is called a *scale-up level* if

$$q_{m+1} = q_m + 1.$$

By [Theorem 15.3](#), this is equivalent to

$$a_m = 1, \quad s_m = 2, \quad \eta_m \geq \frac{4 - \varepsilon_m}{3}.$$

A maximal consecutive interval of scale-up levels is called a *scale-up block*. For an interval $[M, N)$, define

$$U_{M,N} := \#\{m \in [M, N) : q_{m+1} = q_m + 1\},$$

and define the total scale-loss by

$$L_{M,N} := \sum_{m=M}^{N-1} (a_m - s_m)_+, \quad (x)_+ := \max\{x, 0\}.$$

Lemma 16.2 (Exact mantissa map). *Let $n > 1$ be terminally bad. Then, for every $m \geq 0$,*

$$\eta_{m+1} = 2^{-s_m}(3\eta_m + \varepsilon_m), \quad \varepsilon_m = \frac{2^{-q_m}}{n}.$$

Thus

$$s_m = 1 \implies \eta_{m+1} = \frac{3\eta_m + \varepsilon_m}{2},$$

and

$$s_m = 2 \implies \eta_{m+1} = \frac{3\eta_m + \varepsilon_m}{4}.$$

Proof. From $R_m = 2^{q_m}\eta_m$,

$$3R_m + \frac{1}{n} = 2^{q_m}(3\eta_m + \varepsilon_m).$$

The relative-height recurrence gives

$$R_{m+1} = \frac{2^{q_m}(3\eta_m + \varepsilon_m)}{2^{a_m}}.$$

By [Theorem 15.3](#), $q_{m+1} = q_m + s_m - a_m$. Hence

$$\eta_{m+1} = 2^{-q_{m+1}}R_{m+1} = 2^{-(q_m+s_m-a_m)} \frac{2^{q_m}(3\eta_m + \varepsilon_m)}{2^{a_m}} = 2^{-s_m}(3\eta_m + \varepsilon_m).$$

□

Lemma 16.3 (Double scale-up window). *Let $n > 1$ be terminally bad and put $\varepsilon_m = 2^{-q_m}/n$. If both m and $m+1$ are scale-up levels, then*

$$\eta_m \geq \frac{16 - 5\varepsilon_m}{9}.$$

Conversely, if

$$a_m = a_{m+1} = 1 \quad \text{and} \quad \eta_m \geq \frac{16 - 5\varepsilon_m}{9},$$

then m and $m+1$ are scale-up levels.

Proof. If m is a scale-up level, then $s_m = 2$, $a_m = 1$, and $q_{m+1} = q_m + 1$. Therefore

$$\varepsilon_{m+1} = \frac{2^{-q_{m+1}}}{n} = \frac{\varepsilon_m}{2}$$

and, by [Lemma 16.2](#),

$$\eta_{m+1} = \frac{3\eta_m + \varepsilon_m}{4}.$$

The next level $m + 1$ is a scale-up level only if $s_{m+1} = 2$, which is equivalent to

$$\eta_{m+1} \geq \frac{4 - \varepsilon_{m+1}}{3} = \frac{4 - \varepsilon_m/2}{3}.$$

Substituting the formula for η_{m+1} gives

$$\frac{3\eta_m + \varepsilon_m}{4} \geq \frac{4 - \varepsilon_m/2}{3},$$

which is equivalent to

$$\eta_m \geq \frac{16 - 5\varepsilon_m}{9}.$$

This proves the necessary condition. Conversely, the displayed inequality implies in particular

$$\eta_m \geq \frac{4 - \varepsilon_m}{3},$$

because $0 < \varepsilon_m \leq 1/3$. Thus, if $a_m = 1$, level m is a scale-up level. The same calculation then gives $\eta_{m+1} \geq (4 - \varepsilon_m/2)/3$, and if $a_{m+1} = 1$, level $m + 1$ is also a scale-up level. $\square$

Theorem 16.4 (No three consecutive scale-up levels). *Let $n > 1$ be terminally bad. Then no scale-up block has length greater than two. More precisely, if m and $m + 1$ are scale-up levels, then*

$$s_{m+2} = 1,$$

and therefore

$$q_{m+3} \leq q_{m+2}.$$

Proof. Assume that m and $m + 1$ are scale-up levels. Put $\varepsilon = \varepsilon_m$. Then $0 < \varepsilon \leq 1/3$, $q_{m+1} = q_m + 1$, and $q_{m+2} = q_m + 2$. Applying Lemma 16.2 twice gives

$$\eta_{m+2} = \frac{3\eta_{m+1} + \varepsilon/2}{4} = \frac{9\eta_m + 5\varepsilon}{16}.$$

Since $\eta_m < 2$,

$$\eta_{m+2} < \frac{18 + 5\varepsilon}{16} \leq \frac{18 + 5/3}{16} = \frac{59}{48}.$$

On the other hand the high-mantissa threshold at level $m + 2$ is

$$\frac{4 - \varepsilon_{m+2}}{3} = \frac{4 - \varepsilon/4}{3} \geq \frac{4 - 1/12}{3} = \frac{47}{36}.$$

Since $59/48 < 47/36$, level $m + 2$ is below the high-mantissa threshold. Hence $s_{m+2} = 1$. The scale recurrence then gives

$$q_{m+3} = q_{m+2} + s_{m+2} - a_{m+2} = q_{m+2} + 1 - a_{m+2} \leq q_{m+2},$$

because $a_{m+2} \geq 1$. In particular, $m + 2$ cannot be a scale-up level. $\square$

Corollary 16.5 (Scale-up block decomposition). *Every scale-up block in a terminally bad orbit with $n > 1$ has length either one or two. If the block has length two, then the immediately following level is a non-increasing scale level.*

Proof. This is exactly Theorem 16.4. A block has positive length by definition, cannot have length three or more, and the final assertion is the inequality $q_{m+3} \leq q_{m+2}$ after a double block. $\square$

Proposition 16.6 (Exact scale-balance identity). *Let $n > 1$ be terminally bad. For every interval $[M, N)$,*

$$q_N - q_M = U_{M,N} - L_{M,N},$$

where $U_{M,N}$ and $L_{M,N}$ are defined in Definition 16.1. Consequently, if (q_m) is unbounded, then for every M

$$\sup_{N > M} (U_{M,N} - L_{M,N}) = +\infty.$$

Proof. By Theorem 15.3,

$$q_N - q_M = \sum_{m=M}^{N-1} (s_m - a_m).$$

Since $s_m \in \{1, 2\}$ and $a_m \geq 1$, the quantity $s_m - a_m$ is positive exactly when $s_m = 2$ and $a_m = 1$, in which case it equals 1. This is precisely the scale-up case. If $s_m - a_m < 0$, then its negative contribution is $(a_m - s_m)_+$. Zero contributions occur when $a_m = s_m$. Hence the sum decomposes as

$$\sum_{m=M}^{N-1} (s_m - a_m) = U_{M,N} - L_{M,N}.$$

If (q_m) is unbounded, then $q_N - q_M$ is unbounded above as $N \rightarrow \infty$, and the final statement follows from the identity. $\square$

Problem 16.7 (Scale-compensation on terminal no-jump tails). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail satisfies a scale-compensation bound: for some tail index M and some constant $C = C(M)$, one has

$$U_{M,N} - L_{M,N} \leq C \quad \text{for all } N > M.$$

Equivalently, along a non-fixed eventual no-jump tail, scale-loss events must compensate scale-up blocks up to a bounded error.

Theorem 16.8 (Scale-compensation gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers and that Problem 16.7 has a positive solution. Then the Collatz conjecture holds.*

Proof. Assume that Collatz fails. Since nontrivial accelerated cycles are excluded, Corollary 14.4 gives a terminally bad odd integer $n > 1$ with unbounded relative height, equivalently unbounded (q_m) . After the stabilization time of Definition 13.1, its realized prefix chain is an eventual no-jump tail of least-representative bad prefixes. By Proposition 16.6, for every tail index M the quantity $U_{M,N} - L_{M,N}$ is unbounded above as $N \rightarrow \infty$. This contradicts the asserted scale-compensation bound. Therefore no terminally bad $n > 1$ exists, and Corollary 4.5 proves the Collatz conjecture. $\square$

Remark 16.9 (What remains after block arithmetic). The scale-up block theorem rules out arbitrary clustering of the only events that can increase dyadic height. Therefore the remaining unbounded branch is not merely an orbit that grows: it must contain infinitely many isolated or double scale-up blocks whose cumulative contribution is not cancelled by scale-loss levels. The next genuinely decisive step is to prove Problem 16.7, or a stronger local theorem forcing a scale-loss event after every bounded number of scale-up blocks on a non-fixed no-jump tail.

17 Scale packet budgets and valuation-one congruence filters

The scale-up block theorem leaves the following precise obstruction: a cycle-free counterexample must have infinitely many isolated or double scale-up blocks whose positive contribution is not cancelled by scale-loss levels. This section refines that statement into a signed budget and a packet decomposition. The point is not yet to prove compensation, but to isolate the exact arithmetic that any uncompensated unbounded branch must obey.

Definition 17.1 (High levels, low levels, and signed scale budget). Let $n > 1$ be terminally bad, and use the notation of [Definition 15.1](#). A level m is called *high* if $s_m = 2$, and *low* if $s_m = 1$. For an interval $[M, N)$, define

$$P_{M,N} := \#\{m \in [M, N) : s_m = 2, a_m = 1\},$$

$$L_{M,N}^{\text{low}} := \sum_{\substack{M \leq m < N \\ s_m = 1}} (a_m - 1), \quad L_{M,N}^{\text{high}} := \sum_{\substack{M \leq m < N \\ s_m = 2}} (a_m - 2)_+.$$

Thus $P_{M,N}$ counts valuation-one high-mantissa scale-up levels, while $L_{M,N}^{\text{low}}$ and $L_{M,N}^{\text{high}}$ measure the loss below the neutral values $a_m = s_m$ in the low and high bands.

Proposition 17.2 (Signed scale budget). *Let $n > 1$ be terminally bad. For every interval $[M, N)$,*

$$q_N - q_M = P_{M,N} - L_{M,N}^{\text{low}} - L_{M,N}^{\text{high}}.$$

Consequently the scale-balance identity of [Proposition 16.6](#) is equivalently the statement that every positive scale contribution is a high valuation-one level and every negative contribution is a low-band or high-band loss.

Proof. By [Theorem 15.3](#),

$$q_N - q_M = \sum_{m=M}^{N-1} (s_m - a_m).$$

If $s_m = 1$, then $s_m - a_m = 1 - a_m = -(a_m - 1)$, which contributes to $L_{M,N}^{\text{low}}$. If $s_m = 2$, then $s_m - a_m = 1$ for $a_m = 1$, is zero for $a_m = 2$, and equals $-(a_m - 2)$ for $a_m \geq 3$. These three alternatives give respectively the count $P_{M,N}$, no contribution, and the loss $L_{M,N}^{\text{high}}$. Summing over $[M, N)$ gives the formula. $\square$

Corollary 17.3 (Cycle-free failure has unbounded signed surplus). *Assume that there is no nontrivial accelerated cycle in the positive odd integers. If Collatz fails, then there exists an odd terminally bad integer $n > 1$ such that, for every tail index M ,*

$$\sup_{N > M} (P_{M,N} - L_{M,N}^{\text{low}} - L_{M,N}^{\text{high}}) = +\infty.$$

Proof. Under the hypotheses, [Corollary 14.4](#) gives a terminally bad $n > 1$ with unbounded dyadic scale (q_m) . Applying [Proposition 17.2](#) on $[M, N)$ gives

$$P_{M,N} - L_{M,N}^{\text{low}} - L_{M,N}^{\text{high}} = q_N - q_M,$$

which is unbounded above as $N \rightarrow \infty$. $\square$

Lemma 17.4 (Consecutive valuation-one congruence). *Let x be a positive odd integer and define*

$$T_1(y) := \frac{3y + 1}{2}.$$

For an integer $r \geq 1$, the following are equivalent:

- (i) $\nu_2(3T_1^j(x) + 1) = 1$ for every $0 \leq j < r$;
- (ii) $x \equiv -1 \pmod{2^{r+1}}$.

When these equivalent conditions hold,

$$T_1^r(x) = \frac{3^r(x+1)}{2^r} - 1.$$

Proof. First observe that $\nu_2(3y+1) = 1$ for odd y if and only if $y \equiv -1 \pmod{4}$. If $x \equiv -1 \pmod{2^{r+1}}$, then an induction using $T_1(y) + 1 = 3(y+1)/2$ gives

$$T_1^j(x) + 1 = \frac{3^j(x+1)}{2^j}$$

for $0 \leq j \leq r$. For $j < r$, the right side is divisible by 4, so $T_1^j(x) \equiv -1 \pmod{4}$ and the valuation at that level is exactly one.

Conversely, assume the valuations are equal to one for $0 \leq j < r$. Then each step is given by T_1 , so the displayed formula for $T_1^j(x) + 1$ holds up to $j = r$. At $j = r - 1$, the condition $\nu_2(3T_1^{r-1}(x) + 1) = 1$ implies $T_1^{r-1}(x) \equiv -1 \pmod{4}$, hence $T_1^{r-1}(x) + 1$ is divisible by 4. Since

$$T_1^{r-1}(x) + 1 = \frac{3^{r-1}(x+1)}{2^{r-1}}$$

and 3^{r-1} is odd, it follows that $x+1$ is divisible by 2^{r+1} . This is the desired congruence. The formula for $T_1^r(x)$ follows from the same induction. $\square$

Corollary 17.5 (Congruence filter for scale-up blocks). *Let $n > 1$ be terminally bad, and let a scale-up block begin at level m with length r . Then*

$$\text{Syr}^m(n) \equiv -1 \pmod{2^{r+1}}.$$

Since [Theorem 16.4](#) gives $r \in \{1, 2\}$, every single scale-up block begins with

$$\text{Syr}^m(n) \equiv 3 \pmod{4},$$

while every double scale-up block begins with

$$\text{Syr}^m(n) \equiv 7 \pmod{8}.$$

Proof. A scale-up level has $a_m = 1$ by [Theorem 15.3](#). Hence a block of length r gives r consecutive valuation-one levels starting at $x = \text{Syr}^m(n)$. Apply [Lemma 17.4](#). The restriction $r \leq 2$ is [Theorem 16.4](#). $\square$

Definition 17.6 (Scale packets). Let $n > 1$ be terminally bad. Suppose that scale-up blocks begin at levels

$$b_1 < b_2 < b_3 < \cdots,$$

with block lengths $r_j \in \{1, 2\}$. The j -th scale packet is the interval

$$\Pi_j := [b_j, b_{j+1})$$

when b_{j+1} exists. Its packet loss and packet surplus are

$$\Lambda_j := L_{b_j, b_{j+1}}^{\text{low}} + L_{b_j, b_{j+1}}^{\text{high}}, \quad \Omega_j := r_j - \Lambda_j.$$

For a finite sequence of packets set

$$\Omega_{J,K} := \sum_{j=J}^{K-1} \Omega_j.$$

Proposition 17.7 (Exact packet budget). *Let $n > 1$ be terminally bad and suppose that the scale-up blocks are infinite. For all $J < K$,*

$$q_{b_K} - q_{b_J} = \Omega_{J,K}.$$

In particular, if (q_m) is unbounded, then for every J ,

$$\sup_{K > J} \Omega_{J,K} = +\infty.$$

Proof. The intervals $[b_j, b_{j+1})$, $J \leq j < K$, partition $[b_J, b_K)$. On each packet, the only positive scale contributions are the r_j scale-up levels in the initial block; all negative contributions are exactly the low-band and high-band losses entering Λ_j . Thus Proposition 17.2 gives

$$q_{b_{j+1}} - q_{b_j} = r_j - \Lambda_j = \Omega_j.$$

Summing over $J \leq j < K$ proves the identity. If (q_m) is unbounded, then the scale at the later block starts is unbounded, since scale can increase only at scale-up blocks. Hence $q_{b_K} - q_{b_J}$, and therefore $\Omega_{J,K}$, is unbounded above. $\square$

Problem 17.8 (Packet-surplus compensation). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail satisfies a packet-surplus bound: after some packet index J , there is a constant $C = C(J)$ such that

$$\Omega_{J,K} \leq C \quad (K > J).$$

Equivalently, prove that the positive contribution of valuation-one high-mantissa scale-up blocks is compensated, up to bounded error, by low-band and high-band scale losses on every non-fixed eventual no-jump tail.

Theorem 17.9 (Packet-compensation gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers and that Problem 17.8 has a positive solution. Then the Collatz conjecture holds.*

Proof. Assume that Collatz fails. Since nontrivial accelerated cycles are excluded, Corollary 15.5 gives a terminally bad odd integer $n > 1$ with infinitely many scale-up levels and unbounded scale. After the stabilization time of Definition 13.1, the realized prefix chain is an eventual no-jump tail of least-representative bad prefixes. By Proposition 17.7, every tail of scale packets has unbounded partial packet surplus. This contradicts the asserted packet-surplus compensation bound. Therefore no terminally bad positive odd integer exists, and Corollary 4.5 proves the Collatz conjecture. $\square$

Remark 17.10 (Why packet budgets are sharper than raw scale balance). Proposition 16.6 already records the identity $q_N - q_M = U_{M,N} - L_{M,N}$. The packet budget keeps the same information but aligns it with the geometry of the only possible upward events: blocks of one or two valuation-one high-mantissa levels. The congruence filter Corollary 17.5 further shows that these upward blocks must start in very specific residue classes, namely $-1 \pmod{4}$ or $-1 \pmod{8}$. The remaining terminal task is therefore not arbitrary growth control; it is the much narrower problem of proving that an eventual no-jump tail cannot accumulate unbounded packet surplus from these filtered scale-up blocks.

18 Packet-compensation target and record-ladder reduction

The previous section reduces the unbounded escape front to packet surplus. This section sharpens that target without claiming its solution. The goal is to replace the qualitative phrase “unbounded packet surplus” by finite, checkable ladder intervals made only from explicitly classified positive packets. All results in this section are unconditional; the last problem is the remaining packet-compensation theorem in its most local form.

Definition 18.1 (Packet signs and dangerous packets). Let a terminally bad orbit have scale packets Π_j as in [Definition 17.6](#). A packet is called *positive*, *neutral*, or *negative* according as

$$\Omega_j > 0, \quad \Omega_j = 0, \quad \Omega_j < 0.$$

A positive packet is also called *dangerous*. It is called *strongly dangerous* if $\Omega_j \geq 2$.

Proposition 18.2 (Positive-packet classification). *For every terminally bad orbit with $n > 1$, a packet Π_j is positive if and only if*

$$\Lambda_j < r_j.$$

Since $r_j \in \{1, 2\}$, the only positive-packet possibilities are

$$(r_j, \Lambda_j) = (1, 0), \quad (2, 0), \quad (2, 1).$$

Moreover, a packet is strongly dangerous if and only if

$$(r_j, \Lambda_j) = (2, 0).$$

Proof. By [Definition 17.6](#),

$$\Omega_j = r_j - \Lambda_j.$$

Thus $\Omega_j > 0$ is exactly $\Lambda_j < r_j$. By [Corollary 16.5](#), the block length satisfies $r_j \in \{1, 2\}$, and by definition Λ_j is a nonnegative integer. The listed triples are therefore the only possibilities with $r_j - \Lambda_j > 0$. Finally, $\Omega_j \geq 2$ can occur only when $r_j = 2$ and $\Lambda_j = 0$, and in that case $\Omega_j = 2$. $\square$

Corollary 18.3 (Loss-sparse form of positive packets). *Every positive packet is loss-sparse in the following precise sense.*

- (i) *If $r_j = 1$, then positivity forces $\Lambda_j = 0$. Hence every low level in Π_j has $a_m = 1$, and every high non-scale-up level in Π_j has $a_m \leq 2$.*
- (ii) *If $r_j = 2$, then positivity forces $\Lambda_j \in \{0, 1\}$. Thus the whole packet contains at most one unit of scale loss.*

Proof. This is [Proposition 18.2](#) together with the definition

$$\Lambda_j = L_{b_j, b_{j+1}}^{\text{low}} + L_{b_j, b_{j+1}}^{\text{high}}.$$

The loss L^{low} is a sum of the nonnegative terms $a_m - 1$ over low levels $s_m = 1$, while L^{high} is a sum of the nonnegative terms $(a_m - 2)_+$ over high levels $s_m = 2$. Therefore $\Lambda_j = 0$ is exactly the absence of all these loss units, and $\Lambda_j \leq 1$ allows at most one such unit. $\square$

Definition 18.4 (Packet partial sums and ladder intervals). Fix a packet tail starting at index J . For $K \geq J$, set

$$S_J(K) := \Omega_{J,K}, \quad S_J(J) := 0.$$

A packet interval $[U, V)$, with $J \leq U < V$, is called a *positive ladder interval* if

$$\Omega_{U,T} > 0 \quad (U < T \leq V).$$

Its height is $\Omega_{U,V}$.

Lemma 18.5 (Record-ladder extraction). *Let $(\Omega_j)_{j \geq J}$ be any integer packet-surplus sequence. If*

$$\sup_{K > J} \Omega_{J,K} = +\infty,$$

then for every integer $H \geq 1$ there exists a positive ladder interval $[U, V)$ with

$$\Omega_{U,V} \geq H.$$

Proof. Write $S(K) = \Omega_{J,K}$. Since $S(K)$ is unbounded above, choose $V > J$ such that

$$S(V) > H + \min_{J \leq T \leq V} S(T).$$

Let U be the last index in $[J, V)$ at which the minimum of $S(T)$ over $J \leq T \leq V$ is attained. Then $S(T) > S(U)$ for every $U < T \leq V$, by the choice of the last minimum. Hence

$$\Omega_{U,T} = S(T) - S(U) > 0 \quad (U < T \leq V),$$

so $[U, V)$ is a positive ladder interval. Also

$$\Omega_{U,V} = S(V) - S(U) \geq S(V) - \min_{J \leq T \leq V} S(T) > H.$$

Since the left side is an integer, it is at least H . □

Corollary 18.6 (Unbounded escape contains arbitrarily high ladders). *Let $n > 1$ be an unbounded terminal no-jump escape. Then on every sufficiently late packet tail and for every $H \geq 1$, there is a positive ladder interval of packet height at least H .*

Proof. Since the escape is unbounded, the dyadic scales $q_m = \lfloor \log_2 R_m \rfloor$ are unbounded. Scale can increase only at scale-up blocks by [Theorem 15.3](#), so the scale-up blocks are infinite. By the exact packet budget [Proposition 17.7](#), every sufficiently late packet tail has unbounded partial packet surplus. Apply [Lemma 18.5](#) to that tail. □

Proposition 18.7 (Ladder intervals are built from dangerous packets). *Every positive ladder interval contains at least one dangerous packet. More quantitatively, if $[U, V)$ is a positive ladder interval of height H , then*

$$\sum_{j=U}^{V-1} \max(\Omega_j, 0) \geq H,$$

and each positive summand is one of the three cases listed in [Proposition 18.2](#).

Proof. The identity

$$\Omega_{U,V} = \sum_{j=U}^{V-1} \Omega_j$$

implies

$$H \leq \Omega_{U,V} \leq \sum_{j=U}^{V-1} \max(\Omega_j, 0).$$

Thus at least one positive summand occurs whenever $H \geq 1$. The classification of each positive summand is [Proposition 18.2](#). □

Problem 18.8 (Record-ladder compensation). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail has a uniform ladder-height bound: after some packet index J , there exists a constant $C = C(J)$ such that every positive ladder interval $[U, V)$ with $U \geq J$ satisfies

$$\Omega_{U,V} \leq C.$$

Equivalently, prove that arbitrarily high positive record-packet ladders force either a strict jump, a one-step ceiling break, or entry into the fixed tail of 1.

Theorem 18.9 (Record-ladder compensation implies packet compensation). *For a fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, a positive solution of Problem 18.8 implies the packet-surplus bound in Problem 17.8.*

Proof. Assume record-ladder compensation. Then there are a packet index J and a constant C such that every positive ladder interval $[U, V)$ with $U \geq J$ has height at most C . Suppose, for contradiction, that the packet-surplus bound fails on the tail starting at J . Then

$$\sup_{K > J} \Omega_{J,K} = +\infty.$$

By Lemma 18.5, for every $H \geq 1$ there is a positive ladder interval $[U, V)$ with $U \geq J$ and height at least H . Choosing $H > C$ contradicts the record-ladder bound. Therefore $\Omega_{J,K}$ is bounded above on the tail, which is the desired packet-surplus compensation. $\square$

Corollary 18.10 (Failure of packet compensation produces arbitrarily high ladders). *If packet-surplus compensation fails on a packet tail J , then that tail contains positive ladder intervals of arbitrarily large height.*

Proof. Failure on the tail J means

$$\sup_{K > J} \Omega_{J,K} = +\infty.$$

The conclusion is exactly Lemma 18.5. $\square$

Theorem 18.11 (Record-ladder closure gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers and that Problem 18.8 has a positive solution. Then the Collatz conjecture holds.*

Proof. By Theorem 18.9, record-ladder compensation implies packet-surplus compensation on every non-fixed eventual no-jump tail. Then Theorem 17.9 applies, and gives the Collatz conjecture. $\square$

Remark 18.12 (Why this is the next reduction target). The packet problem is now reduced to a finite local obstruction: exclude arbitrarily high positive ladder intervals made from loss-sparse packets of the three types

$$(r, \Lambda) = (1, 0), \quad (2, 0), \quad (2, 1).$$

This is strictly sharper than controlling all packets at once. A future proof can focus on these record ladders and try to show that their repeated occurrence is incompatible with eventual no-jump least representatives, terminal ceilings, and the congruence filters $-1 \pmod{4}$ and $-1 \pmod{8}$ from Corollary 17.5.

19 Amplification splitting on positive record ladders

The packet and record-ladder reductions isolate the unbounded escape front, but Problem 18.8 is still a compensation problem. This section makes the next attack more explicit. Every high positive record ladder must amplify the orbit by a large factor. The exact finite-word formula then shows that this amplification has only two possible sources: either the ladder word has large multiplicative surplus, or the finite-word offset dominates the base point at the start of the ladder. This does not close the escape front, but it splits the remaining task into two precise channels.

Definition 19.1 (Ladder block data). Let $n > 1$ be an unbounded terminal no-jump escape, and let $[U, V)$ be a packet interval with scale-up block starts $b_U < b_V$. Put

$$\ell_{U,V} := b_V - b_U, \quad K_{U,V} := \sum_{m=b_U}^{b_V-1} a_m,$$

and let

$$\mathcal{L}_{U,V} := (a_{b_U}, a_{b_U+1}, \dots, a_{b_V-1})$$

be the finite valuation word read along the ladder. Define

$$B_{U,V} := B_{\ell_{U,V}}(\mathcal{L}_{U,V}).$$

Finally, define the multiplicative surplus and the normalized offset dominance of the ladder by

$$M_{U,V} := \ell_{U,V} \log_2 3 - K_{U,V}, \quad O_{U,V} := \log_2 \left(\frac{B_{U,V}}{2^{K_{U,V}} n_{b_U}} \right).$$

Lemma 19.2 (Exact amplification identity for a ladder). *Let $n > 1$ be an unbounded terminal no-jump escape and let $[U, V)$ be a positive ladder interval of packet height*

$$H := \Omega_{U,V} = q_{b_V} - q_{b_U} > 0.$$

Then

$$2^{H-1} < \frac{n_{b_V}}{n_{b_U}} < 2^{H+1},$$

and, with the notation of [Definition 19.1](#),

$$\frac{n_{b_V}}{n_{b_U}} = \frac{3^{\ell_{U,V}}}{2^{K_{U,V}}} + \frac{B_{U,V}}{2^{K_{U,V}} n_{b_U}}.$$

Proof. By [Definition 15.1](#),

$$R_m = \frac{n_m}{n} = 2^{q_m} \eta_m, \quad 1 \leq \eta_m < 2.$$

Therefore

$$\frac{n_{b_V}}{n_{b_U}} = \frac{R_{b_V}}{R_{b_U}} = 2^{q_{b_V} - q_{b_U}} \frac{\eta_{b_V}}{\eta_{b_U}} = 2^H \frac{\eta_{b_V}}{\eta_{b_U}}.$$

Since $1 \leq \eta_{b_V} < 2$ and $1 \leq \eta_{b_U} < 2$, the displayed two-sided bound follows.

For the second identity, apply the exact finite-iterate formula [Theorem 3.2](#) to the orbit segment starting at n_{b_U} and having valuation word $\mathcal{L}_{U,V}$. This gives

$$n_{b_V} = \frac{3^{\ell_{U,V}} n_{b_U} + B_{U,V}}{2^{K_{U,V}}}.$$

Dividing by n_{b_U} proves the identity. □

Corollary 19.3 (High-ladder dichotomy). *Let $[U, V)$ be a positive ladder interval of height H . Then at least one of the following two alternatives holds:*

$$M_{U,V} > H - 2,$$

or

$$O_{U,V} > H - 2.$$

Equivalently,

$$H < 2 + \max\{M_{U,V}, O_{U,V}\}.$$

Proof. By Lemma 19.2,

$$\frac{3^{\ell_{U,V}}}{2^{K_{U,V}}} + \frac{B_{U,V}}{2^{K_{U,V}n_{b_U}}} > 2^{H-1}.$$

If both summands were at most 2^{H-2} , their sum would be at most 2^{H-1} , a contradiction. Hence at least one summand is strictly larger than 2^{H-2} . Taking base-two logarithms gives the two alternatives and the final equivalent bound. $\square$

Definition 19.4 (Two escape channels). A positive ladder interval is called *multiplicative-dominant* if

$$M_{U,V} > H - 2,$$

and *offset-dominant* if

$$O_{U,V} > H - 2.$$

Thus Corollary 19.3 says that every high positive ladder lies in at least one of these two channels.

Proposition 19.5 (Unbounded escape forces an unbounded channel). *If an unbounded terminal no-jump escape exists, then for every integer $H_0 \geq 1$ there is a positive ladder interval $[U, V)$ of height $H \geq H_0$ for which at least one of the two inequalities*

$$M_{U,V} > H - 2, \quad O_{U,V} > H - 2$$

holds. In particular, along such an escape, the quantity

$$\max\{M_{U,V}, O_{U,V}\}$$

is unbounded on positive ladder intervals.

Proof. By Corollary 18.6, every sufficiently late packet tail contains positive ladder intervals of arbitrarily large height. Applying Corollary 19.3 to each of those ladders gives the asserted channel alternative. Since their heights are unbounded and $H < 2 + \max\{M_{U,V}, O_{U,V}\}$, the maximum is also unbounded. $\square$

Problem 19.6 (Multiplicative ladder bound). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail has an eventual upper bound on multiplicative surplus over positive ladder intervals: there are a packet index J and a constant C_M such that every positive ladder interval $[U, V)$ with $U \geq J$ satisfies

$$M_{U,V} \leq C_M.$$

Problem 19.7 (Offset-dominance ladder bound). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail has an eventual upper bound on normalized offset dominance over positive ladder intervals: there are a packet index J and a constant C_O such that every positive ladder interval $[U, V)$ with $U \geq J$ satisfies

$$O_{U,V} \leq C_O.$$

Theorem 19.8 (Two-channel compensation criterion). *For a fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, positive solutions of Problem 19.6 and Problem 19.7 imply record-ladder compensation Problem 18.8 for that tail, and hence packet-surplus compensation for that tail.*

Proof. Let J be chosen after both channel bounds hold, and put $C := 2 + \max\{C_M, C_O\}$. For any positive ladder interval $[U, V)$ with $U \geq J$, Corollary 19.3 gives

$$H < 2 + \max\{M_{U,V}, O_{U,V}\} \leq C.$$

Thus all positive ladder intervals on the tail have uniformly bounded height, which is exactly the record-ladder compensation statement. Packet-surplus compensation follows from Theorem 18.9. $\square$

Theorem 19.9 (Two-channel escape-front gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers and that [Problems 19.6](#) and [19.7](#) have positive solutions for every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail. Then the Collatz conjecture holds.*

Proof. The two channel bounds imply record-ladder compensation on every non-fixed eventual no-jump tail by [Theorem 19.8](#). Hence [Theorem 18.11](#) applies and gives the Collatz conjecture. $\square$

Remark 19.10 (What the split achieves). The unbounded branch is no longer a single opaque packet-compensation problem. A hypothetical escape must generate arbitrarily high positive record ladders, and each such ladder must be explained by either a large multiplicative surplus of its finite valuation word or a large offset term relative to its initial value. Thus the next exact targets are [Problems 19.6](#) and [19.7](#).

20 Offset dominance as right-edge suffix stress

The previous section splits every sufficiently high positive record ladder into one of two channels. This section attacks the offset channel. The point is that $B_{U,V}$ is not an opaque error term: after normalization by $2^{K_{U,V}} n_{b_U}$, it is an exact soft maximum of multiplicative surpluses of suffixes ending at the right edge of the ladder. Thus offset dominance can occur only if a hidden right-edge suffix has sufficiently large multiplicative surplus relative to the starting value of the ladder.

Definition 20.1 (Right-edge suffix surplus). Let $[U, V)$ be a positive ladder interval with ladder word

$$\mathcal{L}_{U,V} = (a_{b_U}, a_{b_U+1}, \dots, a_{b_V-1})$$

of length $\ell = \ell_{U,V}$. For $0 \leq j \leq \ell$, set

$$K_j^{U,V} := \sum_{t=0}^{j-1} a_{b_U+t}, \quad K_0^{U,V} := 0,$$

so that $K_\ell^{U,V} = K_{U,V}$. For $0 \leq i \leq \ell - 1$, define the *right-edge suffix surplus*

$$\Sigma_{i;U,V}^{\text{sup}} := (\ell - i) \log_2 3 - (K_\ell^{U,V} - K_i^{U,V}).$$

Finally put

$$\Sigma_{U,V}^* := \max_{0 \leq i \leq \ell-1} \Sigma_{i;U,V}^{\text{sup}}.$$

Lemma 20.2 (Exact suffix expansion of the normalized offset). *For every positive ladder interval $[U, V)$,*

$$\frac{B_{U,V}}{2^{K_{U,V}} n_{b_U}} = \frac{1}{3n_{b_U}} \sum_{i=0}^{\ell_{U,V}-1} 2^{\Sigma_{i;U,V}^{\text{sup}}}.$$

Equivalently,

$$O_{U,V} = \log_2 \left(\sum_{i=0}^{\ell_{U,V}-1} 2^{\Sigma_{i;U,V}^{\text{sup}}} \right) - \log_2(3n_{b_U}).$$

Proof. By the definition of the finite-word offset,

$$B_{U,V} = \sum_{i=0}^{\ell-1} 3^{\ell-1-i} 2^{K_i^{U,V}}.$$

Dividing by $2^{K_\ell^{U,V}} n_{b_U}$ gives

$$\frac{B_{U,V}}{2^{K_\ell^{U,V}} n_{b_U}} = \frac{1}{n_{b_U}} \sum_{i=0}^{\ell-1} 3^{\ell-1-i} 2^{-(K_\ell^{U,V} - K_i^{U,V})}.$$

Since

$$3^{\ell-1-i} 2^{-(K_\ell^{U,V} - K_i^{U,V})} = \frac{1}{3} 2^{(\ell-i) \log_2 3 - (K_\ell^{U,V} - K_i^{U,V})},$$

the first formula follows from [Definition 20.1](#). Taking base-two logarithms gives the formula for $O_{U,V}$. $\square$

Corollary 20.3 (Soft-maximum offset bounds). *For every positive ladder interval $[U, V]$,*

$$\Sigma_{U,V}^* - \log_2(3n_{b_U}) \leq O_{U,V} \leq \Sigma_{U,V}^* + \log_2 \ell_{U,V} - \log_2(3n_{b_U}).$$

Proof. For any finite set of real numbers s_i ,

$$\max_i s_i \leq \log_2 \left(\sum_i 2^{s_i} \right) \leq \max_i s_i + \log_2 N,$$

where N is the number of terms. Apply this with $s_i = \Sigma_{i;U,V}^{\text{sup}}$ and $N = \ell_{U,V}$, then use [Lemma 20.2](#). $\square$

Corollary 20.4 (Offset dominance forces hidden suffix surplus). *Let $[U, V]$ be an offset-dominant positive ladder interval of height $H = q_{b_V} - q_{b_U}$. Then there exists $0 \leq i \leq \ell_{U,V} - 1$ such that*

$$\Sigma_{i;U,V}^{\text{sup}} > H - 2 + \log_2(3n_{b_U}) - \log_2 \ell_{U,V}.$$

In particular, since $n_{b_U} \geq 2^{q_{b_U}}$, there exists such a suffix with

$$\Sigma_{i;U,V}^{\text{sup}} > q_{b_V} - 2 + \log_2 3 - \log_2 \ell_{U,V}.$$

Proof. Offset dominance means $O_{U,V} > H - 2$. The upper bound in [Corollary 20.3](#) gives

$$H - 2 < \Sigma_{U,V}^* + \log_2 \ell_{U,V} - \log_2(3n_{b_U}),$$

which proves the first assertion by the definition of $\Sigma_{U,V}^*$. For the second assertion, use $n_{b_U} \geq 2^{q_{b_U}}$ and $H + q_{b_U} = q_{b_V}$. $\square$

Definition 20.5 (Offset suffix-stress). The *upper offset suffix-stress* of a positive ladder interval is

$$\mathfrak{S}_{U,V}^{\text{off}} := \Sigma_{U,V}^* + \log_2 \ell_{U,V} - \log_2(3n_{b_U}).$$

Thus [Corollary 20.3](#) gives the unconditional estimate

$$O_{U,V} \leq \mathfrak{S}_{U,V}^{\text{off}}.$$

Problem 20.6 (Right-edge suffix-stress bound). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail has an eventual upper bound on offset suffix-stress over positive ladder intervals: there are a packet index J and a constant C_S such that every positive ladder interval $[U, V]$ with $U \geq J$ satisfies

$$\mathfrak{S}_{U,V}^{\text{off}} \leq C_S.$$

Equivalently, one must prevent the right-edge suffix surplus maximum from keeping pace with $\log_2 n_{b_U}$ up to the logarithmic length allowance $\log_2 \ell_{U,V}$.

Proposition 20.7 (Suffix-stress bound implies offset-dominance bound). *For any fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, a positive solution of Problem 20.6 implies the offset-dominance ladder bound Problem 19.7 for that tail.*

Proof. If $\mathfrak{S}_{U,V}^{\text{off}} \leq C_S$ eventually on positive ladders, then Definition 20.5 gives

$$O_{U,V} \leq \mathfrak{S}_{U,V}^{\text{off}} \leq C_S$$

for all sufficiently late positive ladder intervals. This is exactly Problem 19.7, with $C_O = C_S$. $\square$

Proposition 20.8 (Length-moderate suffix control kills offset dominance). *Let $[U, V)$ range over sufficiently late positive ladder intervals of a fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail. Assume that there are constants $A \in \mathbb{R}$, $C_L \geq 1$, and $0 \leq \theta < 1$ such that*

$$\ell_{U,V} \leq C_L n_{b_U}^\theta$$

and

$$\Sigma_{U,V}^* \leq (1 - \theta) \log_2 n_{b_U} + A$$

for all such ladders. Then the offset-dominance bound holds, quantitatively,

$$O_{U,V} \leq A + \log_2 C_L - \log_2 3.$$

Proof. By Corollary 20.3,

$$O_{U,V} \leq \Sigma_{U,V}^* + \log_2 \ell_{U,V} - \log_2(3n_{b_U}).$$

Using the two assumptions gives

$$O_{U,V} \leq (1 - \theta) \log_2 n_{b_U} + A + \log_2 C_L + \theta \log_2 n_{b_U} - \log_2 3 - \log_2 n_{b_U},$$

which simplifies to the asserted bound. $\square$

Proposition 20.9 (Suffix-stress escape-front gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers, that Problem 19.6 has a positive solution for every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, and that Problem 20.6 has a positive solution for every such tail. Then the Collatz conjecture holds.*

Proof. By Proposition 20.7, the right-edge suffix-stress bound gives the offset-dominance ladder bound. The multiplicative ladder bound and the offset-dominance ladder bound then imply the two-channel compensation criterion Theorem 19.8. The two-channel escape-front gateway Theorem 19.9 gives the Collatz conjecture under cycle exclusion. $\square$

Remark 20.10 (What the offset attack achieves). The offset channel has been converted into a precise suffix-stress problem. A large normalized offset is possible only when a right-edge suffix of the same ladder carries enough multiplicative surplus to overcome the current starting size n_{b_U} , up to the harmless soft-maximum allowance $\log_2 \ell_{U,V}$. Thus offset dominance is not a third independent mechanism; it is hidden multiplicative surplus occurring inside the right edge of the ladder.

21 Direct suffix-stress pruning by logarithmic shallow returns

The suffix-stress formulation still contains a maximum over all right-edge suffixes. This section attacks that maximum directly. A large right-edge suffix surplus cannot be hidden at arbitrary scale: the same suffix produces a large ordinary amplification from its own starting point to the right edge of the ladder. Comparing this amplification with dyadic height bands forces the suffix to begin at logarithmically small scale, relative to the length of the ladder, whenever the ladder is offset-dominant. Thus the offset channel is reduced to a new and more geometric obstruction: high positive record ladders with an internal logarithmically shallow return.

Definition 21.1 (Right-edge suffix scale gap). Let $[U, V)$ be a positive ladder interval of length $\ell = \ell_{U,V}$. For $0 \leq i \leq \ell - 1$, put

$$t_i := b_U + i$$

and define the scale gap from the suffix start to the right edge by

$$G_{i;U,V} := q_{b_V} - q_{t_i}.$$

We say that the suffix start t_i is *logarithmically shallow* for the ladder $[U, V)$ if

$$q_{t_i} < \log_2 \ell_{U,V} + 3 - \log_2 3.$$

Lemma 21.2 (Suffix surplus is bounded by suffix scale gain). *For every positive ladder interval $[U, V)$ and every right-edge suffix start $t_i = b_U + i$,*

$$\Sigma_{i;U,V}^{\text{sup}} < G_{i;U,V} + 1.$$

Proof. Let $r = \ell_{U,V} - i$ and

$$K_{i;U,V}^{\text{sup}} := K_{\ell}^{U,V} - K_i^{U,V}.$$

The exact finite-word formula applied to the suffix from t_i to b_V gives

$$\frac{n_{b_V}}{n_{t_i}} = \frac{3^r}{2^{K_{i;U,V}^{\text{sup}}}} + \frac{B_{i;U,V}^{\text{sup}}}{2^{K_{i;U,V}^{\text{sup}} n_{t_i}}},$$

where $B_{i;U,V}^{\text{sup}} > 0$ is the offset of the suffix word. Hence

$$\frac{n_{b_V}}{n_{t_i}} > \frac{3^r}{2^{K_{i;U,V}^{\text{sup}}}} = 2^{\Sigma_{i;U,V}^{\text{sup}}}.$$

On the other hand, by the dyadic scale decomposition $n_m/n = 2^{q_m} \eta_m$ with $1 \leq \eta_m < 2$,

$$\frac{n_{b_V}}{n_{t_i}} = 2^{q_{b_V} - q_{t_i}} \frac{\eta_{b_V}}{\eta_{t_i}} < 2^{G_{i;U,V} + 1}.$$

Combining the two strict inequalities gives $2^{\Sigma_{i;U,V}^{\text{sup}}} < 2^{G_{i;U,V} + 1}$, hence the claim. $\square$

Theorem 21.3 (Offset dominance forces a logarithmically shallow suffix). *Let $[U, V)$ be an offset-dominant positive ladder interval of height*

$$H = q_{b_V} - q_{b_U}.$$

Then $[U, V)$ contains a logarithmically shallow right-edge suffix start. More precisely, there exists $0 \leq i \leq \ell_{U,V} - 1$ such that

$$q_{t_i} < \log_2 \ell_{U,V} + 3 - \log_2 3.$$

Proof. By [Corollary 20.4](#), there exists $0 \leq i \leq \ell_{U,V} - 1$ such that

$$\Sigma_{i;U,V}^{\text{suf}} > H - 2 + \log_2(3n_{b_U}) - \log_2 \ell_{U,V}.$$

Since $n_{b_U} \geq 2^{q_{b_U}}$, this implies

$$\Sigma_{i;U,V}^{\text{suf}} > H - 2 + \log_2 3 + q_{b_U} - \log_2 \ell_{U,V}.$$

By [Lemma 21.2](#),

$$\Sigma_{i;U,V}^{\text{suf}} < q_{b_V} - q_{t_i} + 1 = H + q_{b_U} - q_{t_i} + 1.$$

Combining the last two inequalities and cancelling $H + q_{b_U}$ gives

$$-2 + \log_2 3 - \log_2 \ell_{U,V} < 1 - q_{t_i},$$

which is equivalent to

$$q_{t_i} < \log_2 \ell_{U,V} + 3 - \log_2 3.$$

Thus the suffix start is logarithmically shallow. $\square$

Corollary 21.4 (Deep ladders are not offset-dominant). *Let $[U, V)$ be a positive ladder interval. If every right-edge suffix start $t_i = b_U + i$, $0 \leq i \leq \ell_{U,V} - 1$, satisfies*

$$q_{t_i} \geq \log_2 \ell_{U,V} + 3 - \log_2 3,$$

then $[U, V)$ is not offset-dominant.

Proof. This is the contrapositive of [Theorem 21.3](#). $\square$

Problem 21.5 (Logarithmic shallow-return obstruction). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail has an eventual height bound for positive ladder intervals which contain a logarithmically shallow right-edge suffix start. Equivalently, prove that there are a packet index J and a constant C_{sh} such that every positive ladder interval $[U, V)$ with $U \geq J$ and

$$q_{b_U+i} < \log_2 \ell_{U,V} + 3 - \log_2 3$$

for some $0 \leq i \leq \ell_{U,V} - 1$ satisfies

$$q_{b_V} - q_{b_U} \leq C_{\text{sh}}.$$

Proposition 21.6 (Multiplicative bound plus shallow-return obstruction closes the unbounded front). *For a fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, assume the multiplicative ladder bound [Problem 19.6](#) and the logarithmic shallow-return obstruction [Problem 21.5](#). Then record-ladder compensation [Problem 18.8](#) holds for that tail, and hence packet-surplus compensation holds for that tail.*

Proof. Let J be chosen so that both assumptions hold after packet index J , and let C_M and C_{sh} be the corresponding constants. Consider a positive ladder interval $[U, V)$ with $U \geq J$, and put $H = q_{b_V} - q_{b_U}$. If

$$H > 2 + C_M,$$

then the multiplicative alternative in [Corollary 19.3](#) is impossible, because $M_{U,V} \leq C_M$. Therefore $[U, V)$ must be offset-dominant. By [Theorem 21.3](#), the ladder contains a logarithmically shallow right-edge suffix start. The shallow-return obstruction then gives $H \leq C_{\text{sh}}$. Hence every sufficiently late positive ladder has

$$H \leq \max\{2 + C_M, C_{\text{sh}}\}.$$

This is exactly record-ladder compensation. Packet-surplus compensation follows from [Theorem 18.9](#). $\square$

Proposition 21.7 (Shallow-return escape-front gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers, that [Problem 19.6](#) has a positive solution for every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, and that [Problem 21.5](#) has a positive solution for every such tail. Then the Collatz conjecture holds.*

Proof. By [Proposition 21.6](#), the two unbounded-front assumptions imply packet-surplus compensation on every non-fixed eventual no-jump tail. The two-front closure theorem [Theorem 31.12](#) then applies under cycle exclusion and gives the Collatz conjecture. $\square$

Remark 21.8 (What this pruning step achieves). The right-edge suffix-stress target is sharpened into a scale-geometric target. Offset dominance cannot persist on high record ladders whose interior stays deep in dyadic scale. A high offset-dominant ladder must contain a suffix start at scale only logarithmic in the ladder length. Therefore the offset channel has been reduced to a shallow-return obstruction: repeated high record gains would have to be accompanied by recurrent logarithmic resets inside the same ladders.

22 Base-scale localization of shallow returns

The logarithmic shallow-return obstruction in [Section 21](#) can be sharpened once the packet-ladder geometry is used. A positive packet ladder is not merely a pair of endpoint scales: by definition every intermediate packet endpoint is above the ladder base. Since scale-up events occur only at packet beginnings and the remainder of a packet is non-increasing in scale, the whole interior of a positive ladder stays above its base scale. Consequently the shallow suffix forced by offset dominance cannot be a hidden deep internal dip. It forces the base scale of the ladder itself to be logarithmic in the ladder length. Equivalently, an offset-dominant ladder beginning at scale q_{b_U} must have length exponential in q_{b_U} .

Lemma 22.1 (Packet floor between scale-up blocks). *Let $n > 1$ be terminally bad, and let $[b_j, b_{j+1})$ be one scale packet. Then for every level $m \in [b_j, b_{j+1}]$,*

$$q_m \geq \min\{q_{b_j}, q_{b_{j+1}}\}.$$

Proof. Let the initial scale-up block of the packet have length $r_j \in \{1, 2\}$. On this block the scale increases by one at each level, so

$$q_m \geq q_{b_j} \quad (b_j \leq m \leq b_j + r_j).$$

After the initial block and before the next packet begins there are no scale-up levels, by the definition of b_{j+1} as the next scale-up block start. Hence [Theorem 15.3](#) gives

$$q_{m+1} \leq q_m$$

throughout the remaining part of the packet. Therefore, for $b_j + r_j \leq m \leq b_{j+1}$, the scale is bounded below by its value at the right endpoint b_{j+1} . Combining the two subintervals gives the asserted minimum bound. $\square$

Corollary 22.2 (Interior floor of a positive ladder). *Let $[U, V)$ be a positive ladder interval. Then every level in its underlying orbit interval satisfies*

$$q_m \geq q_{b_U} \quad (b_U \leq m \leq b_V).$$

Proof. For each packet $[b_j, b_{j+1})$ with $U \leq j < V$, positivity of the ladder means

$$q_{b_T} - q_{b_U} = \Omega_{U,T} > 0 \quad (U < T \leq V).$$

Thus $q_{b_j} \geq q_{b_U}$ and $q_{b_{j+1}} > q_{b_U}$ for every $U \leq j < V$. Applying [Lemma 22.1](#) on each packet therefore gives $q_m \geq q_{b_U}$ throughout $[b_U, b_V]$. $\square$

Theorem 22.3 (Offset dominance localizes at the ladder base). *Let $[U, V)$ be an offset-dominant positive ladder interval of length $\ell_{U,V} = b_V - b_U$. Then*

$$q_{b_U} < \log_2 \ell_{U,V} + 3 - \log_2 3.$$

Equivalently,

$$\ell_{U,V} > \frac{3}{8} 2^{q_{b_U}}.$$

Proof. By [Theorem 21.3](#), offset dominance gives a right-edge suffix start $t_i = b_U + i$ such that

$$q_{t_i} < \log_2 \ell_{U,V} + 3 - \log_2 3.$$

By the interior floor of a positive ladder, [Corollary 22.2](#), we also have

$$q_{t_i} \geq q_{b_U}.$$

The first displayed inequality therefore holds with q_{b_U} in place of q_{t_i} . Exponentiating gives

$$2^{q_{b_U}} < 2^{3-\log_2 3} \ell_{U,V} = \frac{8}{3} \ell_{U,V},$$

which is the equivalent length-cost form. □

Corollary 22.4 (High-base ladders are not offset-dominant unless exponentially long). *Let $[U, V)$ be a positive ladder interval. If*

$$\ell_{U,V} \leq \frac{3}{8} 2^{q_{b_U}},$$

then $[U, V)$ is not offset-dominant.

Proof. This is the contrapositive of [Theorem 22.3](#). □

Problem 22.5 (Base-start obstruction). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail has an eventual height bound for positive ladder intervals whose base scale is logarithmic in their length. Equivalently, prove that there are a packet index J and a constant C_{base} such that every positive ladder interval $[U, V)$ with $U \geq J$ and

$$q_{b_U} < \log_2 \ell_{U,V} + 3 - \log_2 3$$

satisfies

$$q_{b_V} - q_{b_U} \leq C_{\text{base}}.$$

Proposition 22.6 (Base-start obstruction implies shallow-return obstruction). *For a fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, a positive solution of [Problem 22.5](#) implies the logarithmic shallow-return obstruction [Problem 21.5](#).*

Proof. Let $[U, V)$ be a positive ladder interval containing a logarithmically shallow right-edge suffix start t_i . Then

$$q_{t_i} < \log_2 \ell_{U,V} + 3 - \log_2 3.$$

By [Corollary 22.2](#), $q_{b_U} \leq q_{t_i}$. Hence

$$q_{b_U} < \log_2 \ell_{U,V} + 3 - \log_2 3.$$

The base-start obstruction therefore bounds the height of every sufficiently late shallow-return ladder, which is exactly [Problem 21.5](#). □

Proposition 22.7 (Multiplicative bound plus base-start obstruction closes the unbounded front). *For a fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, assume the multiplicative ladder bound [Problem 19.6](#) and the base-start obstruction [Problem 22.5](#). Then record-ladder compensation [Problem 18.8](#) holds for that tail, and hence packet-surplus compensation holds for that tail.*

Proof. Let J be chosen so that both assumptions hold after packet index J , and let C_M and C_{base} be the corresponding constants. Let $[U, V)$ be a positive ladder interval with $U \geq J$, and put

$$H = q_{b_V} - q_{b_U}.$$

If $H > 2 + C_M$, the multiplicative alternative in [Corollary 19.3](#) is impossible. Hence $[U, V)$ is offset-dominant. By [Theorem 22.3](#), it satisfies the base-start condition

$$q_{b_U} < \log_2 \ell_{U,V} + 3 - \log_2 3.$$

The base-start obstruction then gives $H \leq C_{\text{base}}$. Thus every sufficiently late positive ladder satisfies

$$H \leq \max\{2 + C_M, C_{\text{base}}\}.$$

This is record-ladder compensation, and packet-surplus compensation follows from [Theorem 18.9](#). $\square$

Proposition 22.8 (Base-start escape-front gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers, that [Problem 19.6](#) has a positive solution for every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, and that [Problem 22.5](#) has a positive solution for every such tail. Then the Collatz conjecture holds.*

Proof. By [Proposition 22.7](#), the two unbounded-front assumptions imply packet-surplus compensation on every non-fixed eventual no-jump tail. The two-front closure theorem [Theorem 31.12](#) then applies under cycle exclusion and gives the Collatz conjecture. $\square$

Remark 22.9 (What the base-scale localization achieves). The logarithmic shallow-return target no longer allows a hidden deep dip inside a positive record ladder. Positivity of the ladder and the packet structure force the whole interior to stay above the ladder base. Therefore an offset-dominant ladder must begin at scale only logarithmic in its own length, or equivalently it must pay an exponential length cost in its base scale. The unbounded escape front is thereby sharpened from an internal shallow-return problem to a base-start length-cost problem.

23 Exponential length-cost and base-start splitting

The base-scale localization theorem turns offset dominance into an exponential length cost: an offset-dominant positive ladder beginning at dyadic base scale q_{b_U} must have length at least a fixed multiple of $2^{q_{b_U}}$. This section attacks the resulting base-start obstruction. The new point is that such an exponential length cost cannot remain amorphous. It must appear in one of two concrete ways: either the ladder contains very many packet beginnings and hence very many filtered scale-up blocks, or one of its packets is itself a very long scale-up-free interval. Thus the base-start front splits into a “many filtered starts” channel and a “long quiet packet” channel.

Definition 23.1 (Base-length ratio and packet lengths). Let $[U, V)$ be a positive ladder interval. Its *base-length ratio* is

$$\Lambda_{U,V} := \ell_{U,V} 2^{-q_{b_U}}.$$

For $U \leq j < V$, write

$$\lambda_j := b_{j+1} - b_j$$

for the length of the j -th scale packet. Thus

$$\ell_{U,V} = \sum_{j=U}^{V-1} \lambda_j.$$

We call $[U, V)$ *base-long* if

$$\Lambda_{U,V} > \frac{3}{8},$$

or equivalently $\ell_{U,V} > (3/8)2^{q_{b_U}}$.

Proposition 23.2 (Base-start condition equals base-long condition). *For a positive ladder interval $[U, V)$, the base-start inequality*

$$q_{b_U} < \log_2 \ell_{U,V} + 3 - \log_2 3$$

is equivalent to the base-long condition

$$\ell_{U,V} > \frac{3}{8} 2^{q_{b_U}}.$$

Proof. The displayed base-start inequality is equivalent to

$$2^{q_{b_U}} < 2^{3-\log_2 3} \ell_{U,V} = \frac{8}{3} \ell_{U,V},$$

which is exactly $\ell_{U,V} > (3/8)2^{q_{b_U}}$. □

Lemma 23.3 (Packet-start base scales diverge on unbounded escapes). *Let $n > 1$ be an unbounded terminal no-jump escape. Then*

$$q_{b_j} \longrightarrow \infty$$

along the packet starts.

Proof. Suppose not. Then for some integer B there are infinitely many packet starts b_j with $q_{b_j} \leq B$. Since

$$n_{b_j} = n 2^{q_{b_j}} \eta_{b_j}, \quad 1 \leq \eta_{b_j} < 2,$$

all these values lie in the finite set of positive odd integers below $2^{B+1}n$. Two of them must therefore be equal. The accelerated Syracuse map is deterministic, so equality at two different times makes the subsequent orbit periodic and hence bounded. This contradicts the assumed unboundedness. □

Theorem 23.4 (Failure of base-start obstruction produces high-base base-long ladders). *Let a fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail be an unbounded terminal no-jump escape. If the base-start obstruction [Problem 22.5](#) fails for this tail, then there are positive ladder intervals $[U_r, V_r)$, with $U_r \rightarrow \infty$, such that*

$$q_{b_{U_r}} \rightarrow \infty, \quad q_{b_{V_r}} - q_{b_{U_r}} \rightarrow \infty,$$

and

$$\ell_{U_r, V_r} > \frac{3}{8} 2^{q_{b_{U_r}}}.$$

Proof. Failure of [Problem 22.5](#) gives, for arbitrarily large height bounds and arbitrarily late packet indices, positive ladder intervals satisfying the base-start inequality and having unbounded height. By [Proposition 23.2](#), these intervals are base-long. Since the tail is an unbounded terminal no-jump escape, [Lemma 23.3](#) shows that, after passing to a later subsequence, their base scales q_{b_U} tend to infinity. □

Lemma 23.5 (Square-root splitting of the exponential length cost). *Let $[U, V)$ be a base-long positive ladder interval and put*

$$Q := q_{b_U}.$$

Then at least one of the following two alternatives holds:

$$V - U > 2^{Q/2},$$

or

$$\max_{U \leq j < V} \lambda_j > \frac{3}{8} 2^{Q/2}.$$

Proof. If both alternatives failed, then $V - U \leq 2^{Q/2}$ and $\lambda_j \leq (3/8)2^{Q/2}$ for every $U \leq j < V$. Hence

$$\ell_{U,V} = \sum_{j=U}^{V-1} \lambda_j \leq (V - U) \max_{U \leq j < V} \lambda_j \leq \frac{3}{8} 2^Q,$$

contradicting base-longness. $\square$

Problem 23.6 (Many-packet base-long obstruction). Prove that every non-fixed eventual no-jump tail satisfying the least-representative and terminal-ceiling conditions has an eventual height bound for base-long positive ladders with many packet starts. Equivalently, prove that there are a packet index J and a constant C_{many} such that every positive ladder interval $[U, V)$ with $U \geq J$,

$$\ell_{U,V} > \frac{3}{8} 2^{q_{b_U}} \quad \text{and} \quad V - U > 2^{q_{b_U}/2},$$

satisfies

$$q_{b_V} - q_{b_U} \leq C_{\text{many}}.$$

Problem 23.7 (Long-packet base-long obstruction). Prove that every non-fixed eventual no-jump tail satisfying the least-representative and terminal-ceiling conditions has an eventual height bound for base-long positive ladders containing a very long quiet packet. Equivalently, prove that there are a packet index J and a constant C_{quiet} such that every positive ladder interval $[U, V)$ with $U \geq J$,

$$\ell_{U,V} > \frac{3}{8} 2^{q_{b_U}} \quad \text{and} \quad \max_{U \leq j < V} \lambda_j > \frac{3}{8} 2^{q_{b_U}/2},$$

satisfies

$$q_{b_V} - q_{b_U} \leq C_{\text{quiet}}.$$

Proposition 23.8 (The two length-cost channels imply base-start obstruction). *For a fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, positive solutions of Problems 23.6 and 23.7 imply the base-start obstruction Problem 22.5 for that tail.*

Proof. Choose J large enough so that both asserted channel bounds hold after packet index J , and put

$$C := \max\{C_{\text{many}}, C_{\text{quiet}}\}.$$

Let $[U, V)$ be a positive ladder interval with $U \geq J$ satisfying the base-start inequality. By Proposition 23.2, it is base-long. Applying Lemma 23.5, either it has many packet starts or it contains a very long quiet packet. The corresponding channel bound gives

$$q_{b_V} - q_{b_U} \leq C.$$

This is precisely the base-start obstruction. $\square$

Theorem 23.9 (Length-cost splitting closes the base-start front). *For a fixed non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, assume the multiplicative ladder bound Problem 19.6, the many-packet base-long obstruction Problem 23.6, and the long-packet base-long obstruction Problem 23.7. Then record-ladder compensation Problem 18.8 holds for that tail, and hence packet-surplus compensation holds for that tail.*

Proof. By Proposition 23.8, the two length-cost channel bounds imply the base-start obstruction. The result then follows from Proposition 22.7. $\square$

Proposition 23.10 (Length-splitting escape-front gateway). *Assume that there are no nontrivial accelerated cycles in the positive odd integers, that Problem 19.6 has a positive solution for every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, and that Problems 23.6 and 23.7 have positive solutions for every such tail. Then the Collatz conjecture holds.*

Proof. By Theorem 23.9, the unbounded terminal front satisfies packet-surplus compensation on every non-fixed eventual no-jump tail. Under cycle exclusion, the two-front closure theorem Theorem 31.12 gives the Collatz conjecture. $\square$

Remark 23.11 (What the exponential length-cost attack achieves). The base-start obstruction has been split into two visibly narrower mechanisms. A high offset-dominant ladder must be base-long. A base-long ladder must either contain exponentially many packet starts, each beginning at one of the filtered scale-up configurations, or contain an exponentially long quiet packet with no new scale-up start. Thus the remaining base-start front is no longer a single length inequality; it is the disjunction of a many-filtered-starts obstruction and a long-quiet-packet obstruction.

24 Long quiet packets and neutral-core pruning

The previous section splits the base-start front into a many-packet channel and a long-quiet-packet channel. The latter is more local: inside one scale packet, after the initial scale-up block, no further scale-up level occurs until the next packet begins. Hence the dyadic scale walk on the quiet part has non-positive drift. This section records the exact consequence. A long quiet packet can survive inside a positive ladder only in one of two ways: either it consumes a pre-existing positive height reserve, or it contains a long loss-free neutral core. Thus the long-packet obstruction is reduced to a bounded-reserve neutral core problem.

Definition 24.1 (Quiet core, reserve, and core losses). Let $\Pi_j = [b_j, b_{j+1})$ be a scale packet with initial scale-up block length $r_j \in \{1, 2\}$. Its *quiet core* is

$$\mathcal{Q}_j := [c_j, d_j), \quad c_j := b_j + r_j, \quad d_j := b_{j+1},$$

and its quiet-core length is

$$\mu_j := d_j - c_j = \lambda_j - r_j.$$

For a packet ladder interval $[U, V)$ and $U \leq j < V$, define the *height reserve before packet j* by

$$h_{U,j} := \Omega_{U,j} = q_{b_j} - q_{b_U}.$$

On a quiet core put

$$\delta_m := a_m - s_m \quad (c_j \leq m < d_j).$$

A subinterval $I \subseteq \mathcal{Q}_j$ is called a *loss-free neutral core* if $\delta_m = 0$ for every $m \in I$.

Lemma 24.2 (Non-positive drift on a quiet core). *Let $n > 1$ be terminally bad and let Π_j be one of its scale packets. For every $m \in \mathcal{Q}_j$,*

$$q_{m+1} - q_m = -\delta_m \leq 0.$$

Moreover

$$\Lambda_j = \sum_{m=c_j}^{d_j-1} \delta_m,$$

and for $c_j \leq t \leq d_j$,

$$q_t = q_{c_j} - \sum_{m=c_j}^{t-1} \delta_m.$$

In particular

$$q_{b_{j+1}} = q_{b_j} + r_j - \Lambda_j.$$

Proof. The levels $b_j, b_j + 1, \dots, b_j + r_j - 1$ are precisely the initial scale-up block, so each of them raises the scale by one. Hence

$$q_{c_j} = q_{b_j} + r_j.$$

By definition of b_{j+1} , no level in $[c_j, d_j)$ is a scale-up level. The exact scale-walk formula [Theorem 15.3](#) gives

$$q_{m+1} - q_m = s_m - a_m.$$

Since no positive increment occurs on the quiet core, one has $a_m \geq s_m$, so $q_{m+1} - q_m = -(a_m - s_m) = -\delta_m \leq 0$. Summing gives the displayed formula for q_t . The packet loss Λ_j is the sum of the low-band losses $a_m - 1$ on levels with $s_m = 1$ and the high-band losses $a_m - 2$ on levels with $s_m = 2$. On the quiet core this is exactly $\sum \delta_m$, and the initial scale-up block contributes no loss. Taking $t = d_j = b_{j+1}$ gives the final identity. $\square$

Proposition 24.3 (Reserve-loss alternative for a long quiet packet). *Let $[U, V)$ be a positive ladder interval and let $j \in [U, V)$. Fix an integer $A \geq 0$. Then one of the following alternatives holds:*

- (i) $h_{U,j} > A$, in which case $[U, j)$ is already a positive ladder prefix of height greater than A whenever $j > U$;
- (ii) $h_{U,j} \leq A$, in which case

$$\Lambda_j \leq A + 1.$$

Consequently, in the second case the quiet core $\mathcal{Q}_j$ contains a loss-free neutral subinterval of length at least

$$\frac{\mu_j - (A + 1)}{A + 2}.$$

Proof. If $h_{U,j} > A$, then for every $U < T \leq j$, positivity of $[U, V)$ gives $\Omega_{U,T} > 0$. Thus $[U, j)$ is a positive ladder prefix, with height $h_{U,j} > A$, provided $j > U$.

Assume now that $h_{U,j} \leq A$. Positivity at the next packet endpoint gives

$$0 < \Omega_{U,j+1} = h_{U,j} + \Omega_j = h_{U,j} + r_j - \Lambda_j.$$

All terms are integers, and $r_j \leq 2$. Hence

$$\Lambda_j \leq h_{U,j} + r_j - 1 \leq A + 1.$$

The number of loss levels in the quiet core is at most Λ_j , since each loss level contributes at least one unit to $\sum \delta_m = \Lambda_j$. Removing at most $A + 1$ loss levels from an interval of length μ_j leaves at most $A + 2$ loss-free neutral gaps. One of those gaps has length at least $(\mu_j - (A + 1))/(A + 2)$. $\square$

Lemma 24.4 (The finite neutral automaton). *On a loss-free neutral core one has*

$$q_{m+1} = q_m, \quad a_m = s_m \in \{1, 2\}.$$

Write L for a low-neutral level $(s_m, a_m) = (1, 1)$, and H for a high-neutral level $(s_m, a_m) = (2, 2)$. Then a loss-free neutral core contains no word LL and no word $HHHH$.

Proof. Loss-free neutrality means $\delta_m = 0$, hence $a_m = s_m$. By Lemma 15.2, $s_m \in \{1, 2\}$, and the scale is constant. If level m is low-neutral, then

$$\eta_{m+1} = \frac{3\eta_m + \varepsilon_m}{2} \geq \frac{3}{2}.$$

Since the next high threshold is at most $4/3$, the next ceiling digit is $s_{m+1} = 2$. Thus LL is impossible.

For high-neutral levels one has

$$\eta_{m+1} = \frac{3\eta_m + \varepsilon_m}{4}, \quad 0 < \varepsilon_m = \frac{2^{-q_m}}{n} \leq \frac{1}{3}.$$

Starting from $\eta_m < 2$, three consecutive high-neutral steps give

$$\eta_{m+3} \leq \frac{199}{192} < \frac{11}{9}.$$

But the high threshold is always at least $(4 - 1/3)/3 = 11/9$. Therefore after three consecutive high-neutral levels the next ceiling digit must be 1, and $HHHH$ is impossible. $\square$

Definition 24.5 (Bounded-reserve neutral-core obstruction). A non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail is said to satisfy the *bounded-reserve neutral-core obstruction* if, for every integer $A \geq 0$, there are a packet index J_A , a length bound N_A , and a height bound C_A such that the following never occurs after J_A : a positive ladder interval $[U, V)$ with $U \geq J_A$, height $\Omega_{U,V} > C_A$, and a packet $j \in [U, V)$ satisfying

$$h_{U,j} \leq A$$

whose quiet core contains a loss-free neutral subinterval of length at least N_A .

Problem 24.6 (Neutral-core pruning). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail satisfies the bounded-reserve neutral-core obstruction of Definition 24.5.

Theorem 24.7 (Long-packet pruning reduction). *Fix a non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail and assume Problem 24.6 for this tail. Then a failure of the long-packet base-long obstruction can occur only through arbitrarily high positive ladders whose first very long packet is preceded by an arbitrarily large positive height reserve.*

Proof. Suppose the long-packet base-long obstruction fails, and choose positive ladder intervals of unbounded height containing packets with $\mu_j \rightarrow \infty$. Fix $A \geq 0$. If infinitely many of these long packets satisfy $h_{U,j} \leq A$, then Proposition 24.3 extracts loss-free neutral cores of length tending to infinity inside positive ladders of unbounded height and bounded reserve. This contradicts Problem 24.6. Hence, for each fixed A , all sufficiently long packets occurring in sufficiently high counter-ladders must satisfy $h_{U,j} > A$. Equivalently, the first very long packet in such a ladder is preceded by a positive height reserve which tends to infinity along the counterexample sequence. $\square$

Remark 24.8 (What the long-packet attack achieves). A long quiet packet cannot itself create height: after its initial scale-up block its core has non-positive drift. If it has substantial loss, it must spend height already accumulated before the packet begins. If it does not have substantial loss, then it contains a long loss-free neutral core, and such cores are forced into the finite symbolic automaton excluding LL and $HHHH$. Thus the long-packet channel is reduced to a sharper local problem: exclude long bounded-reserve neutral cores, or else prove that every long packet appears only after a large earlier positive ladder has already been formed.

25 Bounded-reserve neutral cores and rotation-window reduction

The preceding section reduces the long-quiet-packet channel to the local problem of excluding arbitrarily long loss-free neutral cores with bounded ladder reserve. This section makes that target more rigid. On a loss-free neutral core the dyadic scale exponent is constant, so the mantissa dynamics is an exact small perturbation of the rigid rotation by $\log_2 3$ on the unit logarithmic circle. Moreover, under cycle exclusion, any such neutral core has uniformly bounded total perturbation budget. Thus the bounded-reserve neutral-core front is no longer an unrestricted finite-automaton problem; it is a cycle-free rotation-window problem.

Definition 25.1 (Neutral logarithmic coordinate and perturbation budget). Let $I = [r, r + N)$ be a loss-free neutral subinterval of a quiet core. Since $a_m = s_m$ on I , the scale walk gives $q_{m+1} = q_m$. We write this constant scale as

$$Q_I := q_m \quad (r \leq m \leq r + N),$$

and put

$$y_m := \log_2 \eta_m \in [0, 1), \quad \varepsilon_I := \frac{2^{-Q_I}}{n}.$$

The *neutral perturbation budget* of I is

$$\mathcal{E}(I) := N\varepsilon_I = \frac{N2^{-Q_I}}{n}.$$

Lemma 25.2 (Exact logarithmic recurrence on a neutral core). *Let $I = [r, r + N)$ be a loss-free neutral subinterval. For every $r \leq m < r + N$, define*

$$\xi_m := \log_2 \left(1 + \frac{\varepsilon_I}{3\eta_m} \right).$$

Then

$$0 < \xi_m \leq \frac{\varepsilon_I}{3 \ln 2},$$

and

$$y_{m+1} = y_m + \log_2 3 - s_m + \xi_m.$$

Equivalently, if

$$E_t := \sum_{m=r}^{r+t-1} \xi_m, \quad S_t := \sum_{m=r}^{r+t-1} s_m,$$

then for $0 \leq t \leq N$,

$$y_{r+t} = y_r + t \log_2 3 - S_t + E_t, \quad 0 < E_t \leq \frac{\mathcal{E}(I)}{3 \ln 2}.$$

Proof. On a loss-free neutral core one has $q_m = Q_I$, hence $\varepsilon_m = 2^{-q_m}/n = \varepsilon_I$. The exact mantissa map [Lemma 16.2](#) gives

$$\eta_{m+1} = 2^{-s_m}(3\eta_m + \varepsilon_I) = 2^{-s_m}3\eta_m \left(1 + \frac{\varepsilon_I}{3\eta_m} \right).$$

Taking base-two logarithms gives the displayed recurrence. Since $\eta_m \geq 1$, the elementary bound $\log_2(1+u) \leq u/\ln 2$ for $u \geq 0$ gives

$$\xi_m \leq \frac{\varepsilon_I}{3\eta_m \ln 2} \leq \frac{\varepsilon_I}{3 \ln 2}.$$

Summing over $[r, r + t)$ gives the formula for y_{r+t} and the cumulative perturbation bound. $\square$

Proposition 25.3 (Finite-state bound for a cycle-free neutral core). *Assume that the positive odd orbit under consideration contains no nontrivial accelerated cycle. Let $I = [r, r + N)$ be a loss-free neutral subinterval with constant scale Q_I . Then*

$$N \leq n2^{Q_I}.$$

Consequently

$$\mathcal{E}(I) \leq 1.$$

Proof. For every $r \leq m \leq r + N$ one has $q_m = Q_I$, hence

$$n_m = nR_m = n2^{Q_I}\eta_m$$

with $1 \leq \eta_m < 2$. Therefore all states n_m in the neutral core lie among the positive odd integers in the interval

$$[n2^{Q_I}, n2^{Q_I+1}).$$

There are at most $n2^{Q_I}$ such odd integers. If $N > n2^{Q_I}$, then the $N + 1$ states

$$n_r, n_{r+1}, \dots, n_{r+N}$$

would contain a repetition. The accelerated Syracuse map is deterministic, so such a repetition gives an accelerated cycle. Since the whole orbit is terminally bad and starts above 1, this cycle is nontrivial, contradicting the hypothesis. Hence $N \leq n2^{Q_I}$. Multiplying by $2^{-Q_I}/n$ gives $\mathcal{E}(I) \leq 1$. $\square$

Corollary 25.4 (Uniform rotation-window shadowing under cycle exclusion). *Assume that the orbit contains no nontrivial accelerated cycle. Every loss-free neutral subinterval I satisfies*

$$0 < E_t \leq \frac{1}{3 \ln 2} \quad (0 \leq t \leq |I|),$$

with E_t as in [Lemma 25.2](#). Thus every cycle-free loss-free neutral core is a uniformly bounded positive-error shadow of the rigid logarithmic rotation

$$y \mapsto y + \log_2 3 \pmod{1}.$$

Proof. This follows immediately from [Lemma 25.2](#) and [Proposition 25.3](#). $\square$

Definition 25.5 (Cycle-free rotation-window obstruction). A non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail is said to satisfy the *cycle-free rotation-window obstruction* if, for every integer $A \geq 0$, there are a packet index J_A , a length bound N_A , and a height bound C_A such that the following never occurs after J_A : a positive ladder interval $[U, V)$ of height $\Omega_{U,V} > C_A$ and a packet $j \in [U, V)$ with $h_{U,j} \leq A$, whose quiet core contains a loss-free neutral subinterval I of length at least N_A and satisfying

$$\mathcal{E}(I) \leq 1.$$

Problem 25.6 (Rotation-window pruning). Prove that every non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail with no nontrivial accelerated cycle satisfies the cycle-free rotation-window obstruction of [Definition 25.5](#).

Theorem 25.7 (Rotation-window pruning implies bounded-reserve neutral-core pruning). *Fix a non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail and assume that its positive odd orbit contains no nontrivial accelerated cycle. If [Problem 25.6](#) holds for this tail, then the bounded-reserve neutral-core obstruction [Definition 24.5](#) holds for this tail.*

Proof. Suppose the bounded-reserve neutral-core obstruction fails for some reserve bound A . Then there are positive ladders of unbounded height, occurring arbitrarily far out in the tail, with packets whose reserve satisfies $h_{U,j} \leq A$ and whose quiet cores contain loss-free neutral subintervals of arbitrarily large length. By [Proposition 25.3](#), every such neutral subinterval satisfies $\mathcal{E}(I) \leq 1$. This is exactly a failure of the cycle-free rotation-window obstruction, contradicting [Problem 25.6](#). $\square$

Remark 25.8 (Neutral-core reduction). The finite neutral automaton of [Lemma 24.4](#) alone cannot be expected to bound neutral cores: long aperiodic words are not excluded merely by forbidding LL and $HHHH$. The additional information is dynamical. A loss-free neutral core is an exact finite window of the logarithmic rotation by $\log_2 3$, with a one-sided perturbation whose total size is bounded under cycle exclusion. Therefore the next terminal object is not an arbitrary binary word, but a bounded-error rotation window embedded inside a bounded-reserve positive ladder.

Problem 25.9 (Large-reserve recycling). Close the complementary branch left by [Theorem 24.7](#): prove that, in a non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail, arbitrarily high positive ladders cannot have their first very long quiet packet preceded by arbitrarily large positive height reserve without forcing either a record-ladder compensation bound or a strict jump.

Remark 25.10 (Current status of the long-packet front). The long-packet channel is now split into two sharper pieces. Bounded-reserve long neutral cores are reduced by [Theorems 25.7, 27.6](#) and [28.10](#) to the rotation-window pruning problem. The remaining complementary case is large-reserve recycling: if every very long packet begins only after a large positive ladder reserve has already been accumulated, then that reserve must be shown to recycle into compensation, descent, or a strict jump. No final closure of this complementary branch is claimed here.

26 Rotation-window balance and resonance-corridor pruning

The preceding section reduces bounded-reserve neutral cores to cycle-free rotation windows. This section extracts the next layer of structure. A loss-free neutral core is not merely a bounded-error orbit of an irrational rotation; its binary neutral word has a rigid prefix-balance law. Moreover, continued-fraction denominators of the rotation number force a dichotomy: either a long neutral window contains a genuine near-return subwindow, or the corresponding endpoints are trapped in narrow boundary corridors of the logarithmic circle. Thus rotation-window pruning is reduced to two sharper fronts: linear near-return pruning and boundary-corridor pruning.

Throughout this section we put

$$\alpha := \log_2 3 - 1 \in (0, 1).$$

Thus the neutral logarithmic recurrence of [Lemma 25.2](#) is the rotation by α on $\mathbb{R}/\mathbb{Z}$, with a positive perturbation.

Definition 26.1 (Neutral counts and rotation discrepancy). Let $I = [r, r + N)$ be a loss-free neutral subinterval. For $0 \leq t \leq N$, define

$$H_I(t) := \#\{r \leq m < r + t : s_m = 2\}, \quad L_I(t) := \#\{r \leq m < r + t : s_m = 1\}.$$

Thus $L_I(t) + H_I(t) = t$, and

$$S_t := \sum_{m=r}^{r+t-1} s_m = t + H_I(t).$$

The *rotation discrepancy* of the first t neutral steps is

$$\Delta_I(t) := H_I(t) - \alpha t.$$

For a subwindow $J = [a, a + t) \subseteq I$, we analogously write $H_J(t)$ and $\Delta_J(t)$.

Lemma 26.2 (Exact prefix-balance identity). *Let $I = [r, r + N)$ be a loss-free neutral subinterval. With E_t as in Lemma 25.2, one has*

$$H_I(t) - \alpha t = y_r - y_{r+t} + E_t \quad (0 \leq t \leq N).$$

Consequently, if the orbit has no nontrivial accelerated cycle, then

$$|H_I(t) - \alpha t| \leq 1 + \frac{1}{3 \ln 2} \quad (0 \leq t \leq N).$$

The same estimate holds for every subwindow $J \subseteq I$.

Proof. By Lemma 25.2,

$$y_{r+t} = y_r + t \log_2 3 - S_t + E_t.$$

Since $\log_2 3 = 1 + \alpha$ and $S_t = t + H_I(t)$, this becomes

$$y_{r+t} = y_r + \alpha t - H_I(t) + E_t.$$

Rearranging gives the identity. Since $0 \leq y_r, y_{r+t} < 1$, the first term on the right is bounded in absolute value by 1. Under cycle exclusion, Corollary 25.4 gives $0 < E_t \leq 1/(3 \ln 2)$. Applying the same argument after restarting the neutral recurrence at any subwindow gives the final assertion. $\square$

Corollary 26.3 (Uniform two-sided balance of neutral words). *Assume cycle exclusion. There is an absolute constant*

$$D_0 := 1 + \frac{1}{3 \ln 2}$$

such that every loss-free neutral subwindow J satisfies

$$|H_J(t) - \alpha t| \leq D_0 \quad (0 \leq t \leq |J|).$$

Equivalently, the number of high neutral letters H in every neutral subwindow is forced to follow the irrational slope α with uniformly bounded discrepancy.

Proof. This is Lemma 26.2 applied to all subwindows. $\square$

Definition 26.4 (Continued-fraction return data). Let

$$\frac{p_k}{d_k} \quad (k \geq 0)$$

be the convergents of the continued fraction of $\alpha = \log_2 3 - 1$, with $d_k \geq 1$, and put

$$\delta_k := \|d_k \alpha\|_{\mathbb{R}/\mathbb{Z}}.$$

If $I = [r, r + N)$ is a loss-free neutral subinterval of constant scale Q_I , define the *effective return width*

$$\Theta_k(I) := \delta_k + \frac{d_k 2^{-Q_I}}{3 \ln 2}.$$

Lemma 26.5 (Continued-fraction return or boundary corridor). *Let $I = [r, r + N)$ be a loss-free neutral subinterval, and let k be such that $d_k \leq N$ and $\Theta_k(I) < 1/4$. For every start a with $r \leq a \leq r + N - d_k$, at least one of the following alternatives holds:*

(i) linear near-return:

$$|y_{a+d_k} - y_a| \leq \Theta_k(I);$$

(ii) upper-to-lower boundary crossing:

$$y_a \geq 1 - \Theta_k(I), \quad y_{a+d_k} \leq \Theta_k(I);$$

(iii) lower-to-upper boundary crossing:

$$y_a \leq \Theta_k(I), \quad y_{a+d_k} \geq 1 - \Theta_k(I).$$

Proof. Apply [Lemma 25.2](#) to the subwindow $[a, a + d_k]$. There is an integer M_a such that

$$y_{a+d_k} - y_a = d_k \alpha - M_a + E_{a,k},$$

where

$$0 < E_{a,k} \leq \frac{d_k 2^{-Q_I}}{3n \ln 2}.$$

Since $\delta_k = \|d_k \alpha\|_{\mathbb{R}/\mathbb{Z}}$, the right-hand side differs from an integer by at most $\Theta_k(I)$. But $y_{a+d_k} - y_a \in (-1, 1)$. Because $\Theta_k(I) < 1/4$, the only possibilities are that this difference lies within $\Theta_k(I)$ of 0, 1, or -1 . Being within $\Theta_k(I)$ of 0 gives (i). Being within $\Theta_k(I)$ of -1 forces $y_a \geq 1 - \Theta_k(I)$ and $y_{a+d_k} \leq \Theta_k(I)$, giving (ii). Being within $\Theta_k(I)$ of 1 gives (iii). $\square$

Definition 26.6 (Resonant and boundary-corridor neutral windows). Let $I = [r, r + N]$ be a loss-free neutral subinterval, and let k satisfy $d_k \leq N$ and $\Theta_k(I) < 1/4$. A start $a \in [r, r + N - d_k]$ is called *k-resonant* if alternative (i) of [Lemma 26.5](#) holds. It is called *k-boundary* if alternative (ii) or (iii) holds.

The window I is called *k-resonance-dominated* if it contains a *k-resonant* start. It is called *k-boundary-dominated* if every admissible start $a \in [r, r + N - d_k]$ is *k-boundary*.

Problem 26.7 (Linear near-return pruning). Prove that, in a non-fixed least-representative terminal-ceiling-compatible cycle-free eventual no-jump tail, arbitrarily high bounded-reserve positive ladders cannot contain arbitrarily long loss-free neutral cores with effective continued-fraction denominators d_k for which the neutral core is *k-resonance-dominated*.

Problem 26.8 (Boundary-corridor pruning). Prove that, in a non-fixed least-representative terminal-ceiling-compatible cycle-free eventual no-jump tail, arbitrarily high bounded-reserve positive ladders cannot contain arbitrarily long loss-free neutral cores with effective continued-fraction denominators d_k for which the neutral core is *k-boundary-dominated*.

Theorem 26.9 (Rotation-window pruning via resonance-corridor splitting). *Fix a non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail with no nontrivial accelerated cycle. Suppose that the linear near-return pruning problem [Problem 26.7](#) and the boundary-corridor pruning problem [Problem 26.8](#) hold for this tail. Then the cycle-free rotation-window obstruction [Definition 25.5](#) holds for this tail.*

Proof. Assume that the cycle-free rotation-window obstruction fails. Then there is a fixed reserve bound A and arbitrarily late positive ladders of unbounded height containing packets of reserve at most A , whose quiet cores contain loss-free neutral subintervals $I = [r, r + N]$ of arbitrarily large length with $\mathcal{E}(I) \leq 1$.

Choose any continued-fraction denominator $d_k \leq N$ with $\Theta_k(I) < 1/4$. For all sufficiently long such windows, at least one such k exists unless the window lies entirely below the finite initial range of continued-fraction denominators; that exceptional case is absorbed by increasing the length bound in [Definition 25.5](#). By [Lemma 26.5](#), every admissible d_k -start is resonant

or boundary. If at least one start is resonant, then the windows violate [Problem 26.7](#). If no start is resonant, then every admissible start is boundary, so the windows violate [Problem 26.8](#). Either alternative contradicts the two assumed pruning statements. Hence the rotation-window obstruction holds. $\square$

Remark 26.10 (What the rotation front has become). The rotation-window front is now split into two visibly sharper objects. The linear near-return branch asks one to rule out long neutral cores containing a continued-fraction return of the logarithmic rotation. The boundary-corridor branch asks one to rule out the complementary possibility that all such returns are absorbed by repeated visits to the endpoints of the logarithmic band. Thus a long bounded-reserve neutral core is no longer treated as an arbitrary long binary word or even as a generic rotation shadow; it must be either resonant or boundary-dominated at a continued-fraction scale.

27 Boundary-corridor pruning by one-step middle escape

The resonance-corridor split of the preceding section leaves the boundary branch as the more rigid alternative. This section proves a selected-scale form of boundary-corridor pruning. The point is local: a neutral point lying near either endpoint of the logarithmic band cannot remain near an endpoint for one more neutral step. It is immediately pushed into the middle of the band. Consequently a long neutral window cannot be boundary-dominated at any continued-fraction scale whose effective corridor width is sufficiently small.

Throughout this section we keep the notation

$$\alpha := \log_2 3 - 1 \in (0, 1)$$

from [Section 26](#), and we define the absolute corridor constant

$$\kappa_\alpha := \frac{1}{4} \min\{\alpha, 1 - \alpha\}.$$

For a width $0 < \theta < 1/2$, write

$$\mathcal{B}_\theta := [0, \theta] \cup [1 - \theta, 1]$$

for the boundary corridor and

$$\mathcal{M}_\theta := (\theta, 1 - \theta)$$

for the middle band.

Lemma 27.1 (One-step escape from a boundary corridor). *Let $I = [r, r + N)$ be a loss-free neutral subinterval and let $0 < \theta < \kappa_\alpha$. Suppose that, for some $m \in [r, r + N)$,*

$$\xi_m \leq \theta$$

in the logarithmic recurrence of [Lemma 25.2](#). If

$$y_m \in \mathcal{B}_\theta,$$

then

$$y_{m+1} \in \mathcal{M}_\theta.$$

In particular, two consecutive neutral vertices cannot both lie in $\mathcal{B}_\theta$.

Proof. On a loss-free neutral core the only neutral letters are

$$L = (s_m, a_m) = (1, 1), \quad H = (s_m, a_m) = (2, 2),$$

and

$$y_{m+1} = y_m + \log_2 3 - s_m + \xi_m.$$

First assume $0 \leq y_m \leq \theta$. If $s_m = 2$, then

$$y_{m+1} = y_m + \alpha - 1 + \xi_m \leq \alpha - 1 + 2\theta < 0,$$

because $2\theta < 1 - \alpha$. This is impossible since $y_{m+1} \in [0, 1)$. Hence $s_m = 1$, and then

$$y_{m+1} = y_m + \alpha + \xi_m.$$

Thus

$$y_{m+1} \geq \alpha > \theta,$$

and also

$$y_{m+1} \leq \alpha + 2\theta < 1 - \theta,$$

since $3\theta < 1 - \alpha$. Hence $y_{m+1} \in \mathcal{M}_\theta$.

Now assume $1 - \theta \leq y_m < 1$. If $s_m = 1$, then

$$y_{m+1} = y_m + 1 + \alpha - 1 + \xi_m \geq 1 + \alpha - \theta > 1,$$

because $\theta < \alpha$. This is impossible. Hence $s_m = 2$, and

$$y_{m+1} = y_m + \alpha - 1 + \xi_m.$$

Therefore

$$y_{m+1} \geq \alpha - \theta > \theta,$$

because $2\theta < \alpha$, while

$$y_{m+1} < \alpha + \theta < 1 - \theta,$$

because $2\theta < 1 - \alpha$. Again $y_{m+1} \in \mathcal{M}_\theta$. The final statement follows immediately. $\square$

Lemma 27.2 (Small effective continued-fraction scales). *There are constants $0 < \eta_\alpha < 1/2$ and $N_\alpha \geq 1$, depending only on α , with the following property. Let $I = [r, r + N)$ be a loss-free neutral subinterval with*

$$\mathcal{E}(I) \leq 1,$$

and suppose $N \geq N_\alpha$. Then there is a continued-fraction denominator d_k for α such that

$$d_k \leq \eta_\alpha N, \quad \Theta_k(I) < \kappa_\alpha.$$

In particular, $d_k < N - 1$ for all sufficiently large N .

Proof. Choose

$$0 < \eta_\alpha < \min \left\{ \frac{1}{2}, \frac{3 \ln 2}{4} \kappa_\alpha \right\}.$$

For each sufficiently large N , let d_k be the largest continued-fraction denominator with $d_k \leq \eta_\alpha N$. Since the denominators tend to infinity, this choice is possible and $k \rightarrow \infty$ as $N \rightarrow \infty$. Hence

$$\delta_k = \|d_k \alpha\|_{\mathbb{R}/\mathbb{Z}} \rightarrow 0.$$

For all sufficiently large N , we therefore have

$$\delta_k < \frac{1}{4} \kappa_\alpha.$$

Using $\mathcal{E}(I) = N2^{-Q_I}/n \leq 1$, we get

$$\frac{d_k 2^{-Q_I}}{3n \ln 2} \leq \frac{\eta_\alpha}{3 \ln 2} < \frac{1}{4} \kappa_\alpha.$$

Thus

$$\Theta_k(I) = \delta_k + \frac{d_k 2^{-Q_I}}{3n \ln 2} < \frac{1}{2} \kappa_\alpha < \kappa_\alpha.$$

The inequality $d_k < N - 1$ follows from $d_k \leq \eta_\alpha N$ and $\eta_\alpha < 1/2$, after increasing N_α if necessary. $\square$

Theorem 27.3 (Small-width boundary-corridor pruning). *Let $I = [r, r + N)$ be a loss-free neutral subinterval, and let k satisfy*

$$d_k < N, \quad \Theta_k(I) < \kappa_\alpha.$$

Then I is not k -boundary-dominated.

Proof. Put $\theta = \Theta_k(I)$. Since $d_k < N$, the starts r and $r + 1$ are both admissible for the d_k -windows in I . Suppose, for contradiction, that I is k -boundary-dominated. Then every admissible start is k -boundary. In particular, the start r is boundary, so

$$y_r \in \mathcal{B}_\theta.$$

The one-step perturbation satisfies

$$\xi_r \leq \frac{2^{-Q_I}}{3n \ln 2} \leq \Theta_k(I) = \theta,$$

because $d_k \geq 1$. By [Lemma 27.1](#),

$$y_{r+1} \in \mathcal{M}_\theta.$$

But $r + 1$ is also an admissible start and the window is assumed k -boundary-dominated, so y_{r+1} must lie in $\mathcal{B}_\theta$. This contradiction proves the theorem. $\square$

Corollary 27.4 (Boundary branch removed at selected effective scales). *Let $I_j = [r_j, r_j + N_j)$ be any sequence of loss-free neutral subintervals with*

$$N_j \rightarrow \infty, \quad \mathcal{E}(I_j) \leq 1.$$

After passing to sufficiently large j , there are continued-fraction denominators $d_{k_j} \leq \eta_\alpha N_j$ such that

$$\Theta_{k_j}(I_j) < \kappa_\alpha,$$

and none of the windows I_j is k_j -boundary-dominated.

Proof. The existence of d_{k_j} is [Lemma 27.2](#). The exclusion of k_j -boundary domination is [Theorem 27.3](#). $\square$

Problem 27.5 (Selected-scale linear near-return pruning). Prove that, in a non-fixed least-representative terminal-ceiling-compatible cycle-free eventual no-jump tail, arbitrarily high bounded-reserve positive ladders cannot contain arbitrarily long loss-free neutral cores which are k_j -resonance-dominated at the selected effective scales supplied by [Lemma 27.2](#).

Theorem 27.6 (Rotation-window pruning after boundary-corridor pruning). *Fix a non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail with no nontrivial accelerated cycle. If the selected-scale linear near-return pruning problem [Problem 27.5](#) holds for this tail, then the cycle-free rotation-window obstruction [Definition 25.5](#) holds for this tail.*

Proof. Assume that the cycle-free rotation-window obstruction fails. Then, as in the proof of [Theorem 26.9](#), there are arbitrarily late positive ladders of unbounded height containing bounded-reserve packets whose quiet cores contain loss-free neutral subintervals

$$I_j = [r_j, r_j + N_j), \quad N_j \rightarrow \infty, \quad \mathcal{E}(I_j) \leq 1.$$

Choose the selected denominators d_{k_j} from [Lemma 27.2](#). By [Lemma 26.5](#), every admissible d_{k_j} -start is either resonant or boundary. By [Corollary 27.4](#), the windows cannot be k_j -boundary-dominated. Hence each sufficiently long I_j contains a k_j -resonant start. This violates [Problem 27.5](#). Therefore the rotation-window obstruction holds. $\square$

Remark 27.7 (What has been gained). The boundary-corridor branch is not merely renamed. At the selected effective continued-fraction scales it is eliminated by a one-step local fact: boundary points are immediately ejected into the middle band. Thus the rotation-window front is sharpened again. After cycle exclusion and bounded perturbation, a long bounded-reserve neutral core must now produce a selected-scale linear near-return; the boundary-corridor alternative no longer survives at the scale needed for the reduction.

28 Selected-scale linear near-returns and copy-switch splitting

The previous section removes the boundary-corridor branch at selected effective continued-fraction scales. Consequently a sufficiently long cycle-free neutral window must contain many selected-scale linear near-returns. This section makes that remaining front more concrete. A selected near-return almost preserves the logarithmic coordinate. Therefore, except when the starting point lies close to the neutral switching threshold, it copies the neutral letter after one selected return length. The selected-scale linear near-return problem is thereby split into two sharper alternatives: frequent visits to a thin switching corridor, or a large periodic-copy mesh at the selected denominator.

Throughout the section, let $I = [r, r + N)$ be a loss-free neutral subinterval of constant scale Q_I , perturbation parameter $\varepsilon_I = 2^{-Q_I}/n$, and logarithmic coordinates y_m as in [Definition 25.1](#). For a continued-fraction denominator $d_k < N$, put

$$\theta_k(I) := \Theta_k(I).$$

We write

$$\mathcal{A}_k^\circ(I) := \{a : r \leq a \leq r + N - d_k - 1\}$$

for the interior selected starts, so that both neutral letters s_a and s_{a+d_k} are defined.

Definition 28.1 (Neutral switching threshold). On a loss-free neutral subinterval I , define

$$\tau_I := \log_2 \left(\frac{4 - \varepsilon_I}{3} \right).$$

For a width $\theta > 0$, the associated switching corridor is

$$\mathcal{C}_I(\theta) := \{y \in [0, 1) : |y - \tau_I| \leq \theta\}.$$

Lemma 28.2 (Exact neutral switching rule). *Let $I = [r, r + N)$ be loss-free neutral. For every $r \leq m < r + N$,*

$$s_m = 1 \iff y_m < \tau_I, \quad s_m = 2 \iff y_m \geq \tau_I.$$

Proof. By the exact mantissa map, the scale digit s_m is determined by whether

$$3\eta_m + \varepsilon_I$$

lies in $[2, 4)$ or in $[4, 8)$. Since $1 \leq \eta_m < 2$ and $\varepsilon_I > 0$, only the values $s_m = 1$ and $s_m = 2$ can occur on a neutral step. Thus

$$s_m = 2 \iff 3\eta_m + \varepsilon_I \geq 4 \iff \eta_m \geq \frac{4 - \varepsilon_I}{3}.$$

Taking base-two logarithms gives the asserted threshold rule. $\square$

Lemma 28.3 (Boundary starts are isolated). *Let $I = [r, r + N)$ be loss-free neutral, and let k satisfy*

$$d_k < N - 1, \quad \Theta_k(I) < \kappa_\alpha.$$

Then two consecutive starts in $\mathcal{A}_k^\circ(I)$ cannot both be k -boundary starts.

Proof. Put $\theta = \Theta_k(I)$. If $a \in \mathcal{A}_k^\circ(I)$ is k -boundary, then by [Definition 26.6](#) and [Lemma 26.5](#) one has

$$y_a \in \mathcal{B}_\theta.$$

Moreover

$$\xi_a \leq \frac{2^{-Q_I}}{3n \ln 2} \leq \Theta_k(I) = \theta.$$

Since $\theta < \kappa_\alpha$, [Lemma 27.1](#) gives

$$y_{a+1} \in \mathcal{M}_\theta.$$

But a k -boundary start must begin in $\mathcal{B}_\theta$. Hence $a + 1$ cannot be k -boundary. $\square$

Corollary 28.4 (Selected-scale resonance density). *Let $I = [r, r + N)$ be loss-free neutral and let k satisfy*

$$d_k < N - 1, \quad \Theta_k(I) < \kappa_\alpha.$$

Then at least one half, up to an endpoint error, of the interior selected starts are k -resonant. More precisely, if

$$M_k(I) := |\mathcal{A}_k^\circ(I)| = N - d_k,$$

then the number of k -resonant starts in $\mathcal{A}_k^\circ(I)$ is at least

$$\frac{1}{2}M_k(I) - 1.$$

Proof. For every admissible selected start, [Lemma 26.5](#) says that the start is either k -resonant or k -boundary. By [Lemma 28.3](#), the boundary starts in the ordered set $\mathcal{A}_k^\circ(I)$ are isolated, and therefore there are at most $M_k(I)/2 + 1$ of them. The displayed lower bound for resonant starts follows. $\square$

Lemma 28.5 (A selected near-return copies the neutral letter off the switching corridor). *Let $I = [r, r + N)$ be loss-free neutral. Let $a \in \mathcal{A}_k^\circ(I)$ be a k -resonant start and put $\theta = \Theta_k(I)$. If*

$$y_a \notin \mathcal{C}_I(\theta),$$

then

$$s_{a+d_k} = s_a.$$

Proof. Since a is k -resonant,

$$|y_{a+d_k} - y_a| \leq \theta.$$

If $y_a < \tau_I - \theta$, then $y_{a+d_k} < \tau_I$. By [Lemma 28.2](#), both letters are L -letters, so $s_a = s_{a+d_k} = 1$. If $y_a > \tau_I + \theta$, then $y_{a+d_k} > \tau_I$, and both letters are H -letters, so $s_a = s_{a+d_k} = 2$. These are the only possibilities when $y_a \notin \mathcal{C}_I(\theta)$. $\square$

Definition 28.6 (Switching hits and selected copy starts). For a loss-free neutral subinterval I and selected denominator d_k , put

$$\text{Sw}_k(I) := \#\{a \in \mathcal{A}_k^\circ(I) : y_a \in \mathcal{C}_I(\Theta_k(I))\},$$

and

$$\text{Copy}_k(I) := \#\{a \in \mathcal{A}_k^\circ(I) : s_{a+d_k} = s_a\}.$$

Theorem 28.7 (Selected near-return copy-switch splitting). *Let $I = [r, r + N)$ be a loss-free neutral subinterval, and let k satisfy*

$$d_k < N - 1, \quad \Theta_k(I) < \kappa_\alpha.$$

Set $M_k(I) = N - d_k$. Then at least one of the following alternatives holds:

$$\text{Sw}_k(I) \geq \frac{1}{4}M_k(I) - 1,$$

or

$$\text{Copy}_k(I) \geq \frac{1}{4}M_k(I) - 1.$$

Proof. By [Corollary 28.4](#), at least $M_k(I)/2 - 1$ starts in $\mathcal{A}_k^\circ(I)$ are k -resonant. Split these resonant starts according to whether they lie in the switching corridor $\mathcal{C}_I(\Theta_k(I))$. If at least $M_k(I)/4 - 1$ of them lie in the switching corridor, the first alternative holds. Otherwise at least $M_k(I)/4 - 1$ resonant starts lie outside the switching corridor, and each of those is a copy start by [Lemma 28.5](#). This is the second alternative. $\square$

Problem 28.8 (Selected switching-corridor density pruning). Prove that, in a non-fixed least-representative terminal-ceiling-compatible cycle-free eventual no-jump tail, arbitrarily high bounded-reserve positive ladders cannot contain arbitrarily long loss-free neutral cores I_j with selected denominators $d_{k_j} \leq \eta_\alpha |I_j|$ such that

$$\text{Sw}_{k_j}(I_j) \geq \frac{1}{4}(|I_j| - d_{k_j}) - 1.$$

Problem 28.9 (Selected copy-mesh pruning). Prove that, in a non-fixed least-representative terminal-ceiling-compatible cycle-free eventual no-jump tail, arbitrarily high bounded-reserve positive ladders cannot contain arbitrarily long loss-free neutral cores I_j with selected denominators $d_{k_j} \leq \eta_\alpha |I_j|$ such that

$$\text{Copy}_{k_j}(I_j) \geq \frac{1}{4}(|I_j| - d_{k_j}) - 1.$$

Equivalently, a long bounded-reserve neutral core cannot contain a positive density of selected denominator shifts which copy the neutral letter.

Theorem 28.10 (Selected-scale linear near-return pruning via copy-switch splitting). *Fix a non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail with no nontrivial accelerated cycle. If the selected switching-corridor density pruning problem [Problem 28.8](#) and the selected copy-mesh pruning problem [Problem 28.9](#) hold for this tail, then the selected-scale linear near-return pruning problem [Problem 27.5](#) holds for this tail.*

Proof. Assume that [Problem 27.5](#) fails. Then there are arbitrarily high bounded-reserve positive ladders containing loss-free neutral cores $I_j = [r_j, r_j + N_j)$ with $N_j \rightarrow \infty$, together with the selected effective denominators supplied by [Lemma 27.2](#). For all sufficiently large j , these denominators satisfy

$$d_{k_j} < N_j - 1, \quad \Theta_{k_j}(I_j) < \kappa_\alpha.$$

Applying [Theorem 28.7](#) to each I_j , one obtains either switching-corridor density along an infinite subsequence or selected copy-mesh density along an infinite subsequence. The first alternative contradicts [Problem 28.8](#); the second contradicts [Problem 28.9](#). Hence the selected-scale linear near-return pruning problem holds. $\square$

Remark 28.11 (What the selected-scale front has become). The selected-scale linear near-return front is no longer a black-box resonance condition. At the selected effective denominators, boundary starts are isolated; therefore resonant starts have positive density. Each resonant start either lies near the switching threshold or copies the neutral letter after one selected return length. Thus the remaining rotation front is reduced to two very concrete symbolic targets: prune dense threshold visits, or prune dense selected-period copy meshes.

29 Selected switching-corridor density pruning via thin-interval discrepancy

The previous section reduces the selected-scale linear near-return front to two symbolic alternatives: dense selected switching-threshold visits, or a dense selected copy mesh. This section sharpens the switching branch. A switching corridor is simply a thin interval around the neutral threshold τ_I . Thus the remaining switching problem is an interval-occupancy problem for the perturbed irrational rotation governing a loss-free neutral core. We isolate that interval-occupancy statement as a separate pruning target and show that it is already strong enough to eliminate the selected switching-density branch, because the selected corridor width is uniformly smaller than one quarter.

Throughout the section, let $I = [r, r + N)$ be a loss-free neutral subinterval with selected denominator $d_k < N - 1$, selected effective width $\Theta_k(I) < \kappa_\alpha$, and interior selected starts $\mathcal{A}_k^\circ(I)$ as in [Section 28](#). Put

$$M_k(I) := |\mathcal{A}_k^\circ(I)| = N - d_k.$$

Recall that

$$\text{Sw}_k(I) = \#\{a \in \mathcal{A}_k^\circ(I) : y_a \in \mathcal{C}_I(\Theta_k(I))\}$$

is exactly the selected switching-corridor occupancy count.

Definition 29.1 (Selected thin-interval discrepancy pruning). We say that *selected thin-interval discrepancy pruning* holds if there is an absolute constant $C_\alpha > 0$, depending only on $\alpha = \log_2 3 - 1$, such that for every bounded-reserve loss-free neutral subinterval I appearing inside a positive ladder of a non-fixed least-representative terminal-ceiling-compatible cycle-free eventual no-jump tail, and for every selected denominator d_k with $\Theta_k(I) < \kappa_\alpha$, one has

$$\text{Sw}_k(I) \leq 2\Theta_k(I) M_k(I) + C_\alpha.$$

In words: selected visits to the switching corridor are bounded by the interval length $2\Theta_k(I)$ times the selected sample size, up to a uniform additive discrepancy.

Lemma 29.2 (Uniform one-quarter gap at selected scales). *Let*

$$\delta_\alpha := \frac{1}{4} - 2\kappa_\alpha = \frac{1}{4} - \frac{1}{2} \min\{\alpha, 1 - \alpha\}.$$

Then $\delta_\alpha > 0$. In particular, whenever $\Theta_k(I) < \kappa_\alpha$,

$$2\Theta_k(I) \leq 2\kappa_\alpha = \frac{1}{4} - \delta_\alpha < \frac{1}{4}.$$

Proof. Since

$$\alpha = \log_2 3 - 1 \in (0, 1)$$

and numerically $\alpha \approx 0.5849625$, one has

$$\min\{\alpha, 1 - \alpha\} = 1 - \alpha \approx 0.4150375.$$

Hence

$$2\kappa_\alpha = \frac{1}{2} \min\{\alpha, 1 - \alpha\} \approx 0.2075187 < \frac{1}{4}.$$

Thus $\delta_\alpha = \frac{1}{4} - 2\kappa_\alpha > 0$, and the displayed bound follows immediately from $\Theta_k(I) < \kappa_\alpha$. $\square$

Theorem 29.3 (Thin-interval discrepancy implies selected switching pruning). *Fix a non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail with no nontrivial accelerated cycle. If the selected thin-interval discrepancy pruning problem Definition 29.1 holds for this tail, then the selected switching-corridor density pruning problem Problem 28.8 also holds for this tail.*

Proof. Assume that Problem 28.8 fails. Then there are arbitrarily high bounded-reserve positive ladders containing loss-free neutral cores I_j with selected denominators $d_{k_j} \leq \eta_\alpha |I_j|$ such that

$$\text{Sw}_{k_j}(I_j) \geq \frac{1}{4}(|I_j| - d_{k_j}) - 1 = \frac{1}{4}M_{k_j}(I_j) - 1.$$

If selected thin-interval discrepancy pruning holds, then for each j ,

$$\text{Sw}_{k_j}(I_j) \leq 2\Theta_{k_j}(I_j)M_{k_j}(I_j) + C_\alpha.$$

By Lemma 29.2,

$$2\Theta_{k_j}(I_j) \leq \frac{1}{4} - \delta_\alpha$$

for some fixed $\delta_\alpha > 0$. Therefore

$$\text{Sw}_{k_j}(I_j) \leq \left(\frac{1}{4} - \delta_\alpha\right) M_{k_j}(I_j) + C_\alpha.$$

Comparing with the lower bound gives

$$\delta_\alpha M_{k_j}(I_j) \leq C_\alpha + 1.$$

Hence the selected sample sizes $M_{k_j}(I_j) = |I_j| - d_{k_j}$ are uniformly bounded. Since $d_{k_j} \leq \eta_\alpha |I_j|$ with $0 < \eta_\alpha < 1$, this forces $|I_j|$ itself to be uniformly bounded, contradicting the assumption that the selected switching-density front appears along arbitrarily long loss-free neutral cores. Thus the selected switching-corridor density pruning problem holds. $\square$

Problem 29.4 (Selected switching-corridor discrepancy bound). Prove the selected thin-interval discrepancy pruning estimate of Definition 29.1. Equivalently, show that on every bounded-reserve loss-free neutral core, the perturbed logarithmic rotation visits the selected switching corridor with frequency at most the corridor length $2\Theta_k(I)$, up to a uniform additive error.

Theorem 29.5 (Selected switching branch pruned by discrepancy plus copy pruning). *Fix a non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail with no nontrivial accelerated cycle. If the selected switching-corridor discrepancy bound Problem 29.4 and the selected copy-mesh pruning problem Problem 28.9 hold for this tail, then the selected-scale linear near-return pruning problem Problem 27.5 holds for this tail.*

Proof. By [Theorem 29.3](#), the selected switching-corridor discrepancy bound implies the selected switching-corridor density pruning problem. Therefore the hypotheses of [Theorem 28.10](#) are satisfied, and that theorem yields selected-scale linear near-return pruning. $\square$

Remark 29.6 (What remains of the switching branch). The switching branch is no longer a free-standing symbolic obstruction. It has been reduced to a thin-interval discrepancy problem for the perturbed rotation $y_{m+1} = y_m + \alpha + \xi_m \pmod{1}$. Because the selected corridor width is strictly less than one quarter, any uniform discrepancy estimate of the form

$$\#\{m < M : y_m \in J\} \leq |J|M + O_\alpha(1)$$

for the selected threshold interval $J = \mathcal{C}_I(\Theta_k(I))$ is already strong enough to prune the entire switching-density front.

30 Deep selected-scale block discrepancy for switching corridors

The preceding section reduces dense switching-corridor visits to a thin-interval discrepancy statement. This section attacks that statement directly at continued-fraction block scales. The point is that a selected denominator d_k is not merely a return length: over each block of d_k consecutive neutral starts, the unperturbed rotation by α has Denjoy–Koksma interval discrepancy $O(1)$. The perturbation is one-sided and has a controlled block budget. Therefore a sufficiently deep selected denominator forces switching-corridor occupancy strictly below the bad density threshold $1/4$.

We keep the notation of [Section 29](#). Put

$$\Delta_k(I) := \frac{d_k 2^{-Q_I}}{3n \ln 2}.$$

Thus

$$\Theta_k(I) = \|d_k \alpha\|_{\mathbb{R}/\mathbb{Z}} + \Delta_k(I).$$

Definition 30.1 (Deep selected denominator). Let

$$\theta_\alpha^\# := \min \left\{ \frac{\kappa_\alpha}{2}, \frac{1}{64} \right\}.$$

A selected denominator $d_k < N$ for a loss-free neutral window I is called *deep* if

$$\Theta_k(I) \leq \theta_\alpha^\#.$$

Lemma 30.2 (Block perturbation envelope). *Let $I = [r, r + N)$ be loss-free neutral, and let $B = [u, u + d_k) \subset I$. For $0 \leq t \leq d_k$,*

$$y_{u+t} = y_u + t\alpha + e_t \pmod{1}, \quad 0 \leq e_t \leq \Delta_k(I).$$

Consequently, if $J \subset \mathbb{R}/\mathbb{Z}$ is an interval and $y_{u+t} \in J$, then the unperturbed point $y_u + t\alpha$ lies in the $\Delta_k(I)$ -neighbourhood $J + \Delta_k(I)$.

Proof. On a loss-free neutral window the logarithmic coordinate satisfies

$$y_{m+1} = y_m + \alpha + \xi_m \pmod{1}, \quad 0 \leq \xi_m \leq \frac{2^{-Q_I}}{3n \ln 2}.$$

Summing from u to $u + t - 1$ gives

$$e_t := \sum_{m=u}^{u+t-1} \xi_m \leq t \frac{2^{-Q_I}}{3n \ln 2} \leq d_k \frac{2^{-Q_I}}{3n \ln 2} = \Delta_k(I).$$

The interval-neighbourhood conclusion is immediate. $\square$

Lemma 30.3 (Denjoy–Koksma block bound for intervals). *Let d_k be a continued-fraction denominator for $\alpha = \log_2 3 - 1$. For any interval $J \subset \mathbb{R}/\mathbb{Z}$, any starting point z , and any block of length d_k ,*

$$\#\{0 \leq t < d_k : z + t\alpha \in J\} \leq d_k|J| + 2.$$

Proof. Apply the Denjoy–Koksma inequality to the bounded-variation function

$$f = \mathbf{1}_J - |J|$$

over the denominator d_k . Since the total variation of $\mathbf{1}_J$ on the circle is at most 2, one has

$$\left| \sum_{t=0}^{d_k-1} \mathbf{1}_J(z + t\alpha) - d_k|J| \right| \leq 2.$$

The displayed upper bound follows. $\square$

Proposition 30.4 (Deep selected switching block bound). *Let $I = [r, r + N)$ be loss-free neutral and let $d_k < N$ be a deep selected denominator. Let $J = \mathcal{C}_I(\Theta_k(I))$. For every complete block $B = [u, u + d_k) \subset I$,*

$$\#\{m \in B : y_m \in J\} \leq 4\Theta_k(I)d_k + 2.$$

In particular, since d_k is deep,

$$\#\{m \in B : y_m \in J\} \leq \frac{1}{16}d_k + 2.$$

Proof. The interval J has length $2\Theta_k(I)$. By Lemma 30.2, each perturbed visit to J gives an unperturbed visit to $J^{+\Delta_k(I)}$, whose length is at most

$$2\Theta_k(I) + 2\Delta_k(I) \leq 4\Theta_k(I),$$

because $\Delta_k(I) \leq \Theta_k(I)$. Applying Lemma 30.3 to $J^{+\Delta_k(I)}$ gives

$$\#\{m \in B : y_m \in J\} \leq 4\Theta_k(I)d_k + 2.$$

If d_k is deep, then $\Theta_k(I) \leq 1/64$, and hence $4\Theta_k(I) \leq 1/16$. $\square$

Theorem 30.5 (Deep selected scales prune switching density). *Let $I = [r, r + N)$ be loss-free neutral, let $d_k \leq \eta_\alpha N$ be a deep selected denominator with $0 < \eta_\alpha < 1$, and set*

$$M_k(I) = N - d_k.$$

Then

$$\text{Sw}_k(I) \leq \frac{1}{16}M_k(I) + O_\alpha\left(\frac{M_k(I)}{d_k} + d_k\right).$$

Consequently, along any sequence of loss-free neutral cores for which

$$N_j \rightarrow \infty, \quad d_{k_j} \rightarrow \infty, \quad d_{k_j} \leq \eta_\alpha N_j, \quad \frac{d_{k_j}}{N_j} \rightarrow 0,$$

deep selected denominators are incompatible with the bad switching-density lower bound

$$\text{Sw}_{k_j}(I_j) \geq \frac{1}{4}M_{k_j}(I_j) - 1.$$

Proof. Tile the selected-start interval by complete blocks of length d_k , leaving at most two incomplete end pieces of total length $O(d_k)$. On each complete block, [Proposition 30.4](#) gives at most

$$\frac{1}{16}d_k + 2$$

switching hits. With $O(M_k(I)/d_k)$ complete blocks, this yields

$$\text{Sw}_k(I) \leq \frac{1}{16}M_k(I) + O\left(\frac{M_k(I)}{d_k} + d_k\right).$$

The constants are absolute, hence depend at most on α through the fixed choice of selected scales.

Under the displayed asymptotic assumptions, the error term is $o(M_{k_j}(I_j))$. Thus the upper density tends to at most $1/16$, whereas the bad switching-density alternative requires density at least $1/4$, up to a negligible endpoint error. This is a contradiction for all sufficiently large j . $\square$

Problem 30.6 (Deep selected-scale availability). Prove that every counterexample sequence to selected switching-corridor density pruning contains, after passing to a subsequence, deep selected denominators d_{k_j} satisfying

$$d_{k_j} \rightarrow \infty, \quad d_{k_j} \leq \eta_\alpha N_j, \quad \frac{d_{k_j}}{N_j} \rightarrow 0.$$

Theorem 30.7 (Switching-density pruning from deep-scale availability). *Fix a non-fixed least-representative terminal-ceiling-compatible eventual no-jump tail with no nontrivial accelerated cycle. If the deep selected-scale availability problem [Problem 30.6](#) holds for this tail, then the selected switching-corridor density pruning problem [Problem 28.8](#) holds for this tail.*

Proof. Assume selected switching-corridor density pruning fails. By [Problem 30.6](#), pass to a subsequence with deep selected denominators satisfying the asymptotic hypotheses of [Theorem 30.5](#). That theorem rules out the required switching density along the subsequence, a contradiction. $\square$

Remark 30.8 (What has been proved at the discrepancy front). The selected switching-corridor discrepancy problem is now reduced to one specific scale-selection issue. Once a genuinely deep selected denominator is available with $d_k \rightarrow \infty$ and $d_k = o(N)$, the Denjoy–Koksma block estimate and the one-sided perturbation envelope force switching density below the threshold $1/4$. Thus the remaining obstruction is not interval discrepancy itself, but the possibility that all effective selected denominators stay too shallow or too close to the full window length.

31 A lossless reduction spine toward unconditional closure

The preceding sections produce several equivalent or sufficient terminal formulations. This section consolidates them into a single reduction spine. No new completion of the Collatz conjecture is claimed here. The purpose is to state the shortest remaining route in theorem-level form: one bounded/cyclic front and one unbounded/packet front, now sharpened by record-ladder, two-channel, suffix-stress, logarithmic shallow-return, base-start length-cost, exponential length-splitting, long-quiet-packet neutral-core, cycle-free rotation-window, resonance-corridor, selected-scale boundary-pruning, selected copy-switch targets, thin-interval switching-discrepancy targets, and deep selected-scale block-discrepancy targets.

31.1 The two terminal fronts

Definition 31.1 (Unbounded terminal no-jump escape). An *unbounded terminal no-jump escape* is a positive odd integer $n > 1$ such that, with $n_m = \text{Syr}^m(n)$,

$$n_m \geq n \quad (m \geq 0),$$

$$\sup_m n_m = +\infty,$$

and, after the stabilization time $\sigma(n)$ of [Definition 13.1](#), the associated least representatives satisfy

$$\rho(\mathcal{W}_m(n)) = n \quad (m \geq \sigma(n)).$$

Equivalently, it is an unbounded terminally bad orbit whose finite valuation prefixes form an eventual no-jump tail of least-representative bad prefixes.

Theorem 31.2 (Lossless two-front reduction). *The finite-descent form of the Collatz conjecture is equivalent to the conjunction of the following two assertions:*

- (i) *there are no nontrivial accelerated cycles in the positive odd integers;*
- (ii) *there are no unbounded terminal no-jump escapes.*

Proof. Assume first the finite-descent form of Collatz. Then every positive odd $n > 1$ has some iterate below n . A nontrivial accelerated cycle is impossible, since the minimum element of such a cycle would never descend below itself. An unbounded terminal no-jump escape is also impossible by definition.

Conversely, suppose that finite descent fails. Then [Corollary 4.5](#) gives a terminally bad positive odd integer $n > 1$, so $n_m \geq n$ for all m . If the orbit (n_m) is bounded, then it takes only finitely many positive odd values and hence is eventually periodic. Since it never falls below $n > 1$, its eventual period is not the fixed point 1. This gives a nontrivial accelerated cycle.

If the orbit is unbounded, then the modulus $M(\mathcal{W}_m(n))$ tends to infinity because $K_m \geq m$. Hence there is a stabilization time $\sigma(n)$ with $M(\mathcal{W}_m(n)) > n$ for all $m \geq \sigma(n)$. By [Lemma 13.2](#), the least representative of the realized prefix is then exactly n , so the prefix chain is eventual no-jump. Therefore n is an unbounded terminal no-jump escape. Thus failure of finite descent implies one of the two listed obstructions. This proves the equivalence. $\square$

Remark 31.3 (Why this is the manuscript's actual reduction spine). [Theorem 31.2](#) is the point at which the reduction becomes lossless. The paper no longer asks vaguely for a proof that all orbits reach 1. It asks for two explicit exclusions: the finite cycle certificate of [Section 7](#), and the unbounded terminal no-jump escape governed by the ceiling, deficit, scale-walk, scale-up, and packet-budget modules.

Proposition 31.4 (Dependency map for the two-front reduction). *The two-front reduction uses only the following unconditional inputs.*

- (i) [Theorem 4.2](#) and [Proposition 4.4](#): *failure of Collatz is equivalent to a terminally bad odd value $n > 1$.*
- (ii) [Theorem 7.1](#): *a bounded terminally bad orbit is eventually periodic and therefore contains a nontrivial accelerated cycle.*
- (iii) [Lemma 13.2](#): *an unbounded terminally bad orbit has an eventual no-jump least-representative tail after the modulus exceeds n .*
- (iv) [Corollaries 6.5 and 14.3](#), [Theorems 15.3 and 16.4](#), and [Proposition 17.7](#): *the unbounded no-jump tail has unbounded deficits, infinitely many valuation-one high-mantissa scale-up events, scale-up blocks of length at most two, and unbounded packet surplus.*

Thus the only open inputs needed for a complete closure through this spine are the cycle front [Problem 31.6](#) and the packet compensation front [Problem 31.11](#), with the stronger record-ladder front [Problem 18.8](#) giving a sufficient local form.

Proof. Items (i)–(iv) are exactly the cited unconditional results, listed in the order in which they are used in [Theorem 31.2](#) and [Proposition 31.5](#). The final sentence follows from [Theorem 31.12](#). $\square$

31.2 Normal form of the unbounded front

Proposition 31.5 (Unbounded escape normal form). *Let $n > 1$ be an unbounded terminal no-jump escape. Then, on its eventual no-jump tail, the following properties hold.*

(i) *The terminal ceiling rule holds at every level:*

$$a_m \leq \left\lfloor \log_2 \left(3R_m + \frac{1}{n} \right) \right\rfloor.$$

(ii) *The under-ceiling deficits are unbounded:*

$$\sup_m d_m = +\infty.$$

(iii) *The dyadic scales are unbounded:*

$$\sup_m q_m = +\infty.$$

(iv) *There are infinitely many valuation-one high-mantissa scale-up levels:*

$$a_m = 1, \quad s_m = 2.$$

(v) *No scale-up block has length greater than two.*

(vi) *For every packet tail index J ,*

$$\sup_{K > J} \Omega_{J,K} = +\infty.$$

Proof. The terminal ceiling rule is [Corollary 6.5](#). Since the orbit is unbounded, the relative heights $R_m = n_m/n$ are unbounded. Hence [Corollary 14.3](#) gives unbounded deficits, while the definition $q_m = \lfloor \log_2 R_m \rfloor$ in [Definition 15.1](#) gives unbounded dyadic scale. The scale-walk identity of [Theorem 15.3](#) shows that scale can increase only when $a_m = 1$ and $s_m = 2$, so unbounded scale forces infinitely many such levels. The no-three-scale-ups theorem [Theorem 16.4](#) gives the block-length bound. Finally, [Proposition 17.7](#) identifies packet surplus with scale increase at packet starts, and therefore every packet tail has unbounded positive partial surplus. $\square$

Problem 31.6 (Cycle front). Prove that no nontrivial finite valuation word $\mathcal{W} = (a_0, \dots, a_{m-1})$ gives a positive odd accelerated cycle. Equivalently, prove that the cycle certificate

$$n_0 = \frac{B_m(\mathcal{W})}{2^{K_m} - 3^m}$$

never produces a positive odd integer realizing $\mathcal{W}$ as its own periodic valuation word, except for the fixed point 1.

31.3 Connection to the classical continued-fraction bound

The cycle certificate of [Theorem 7.1](#) identifies any hypothetical nontrivial cycle with a finite valuation word $\mathcal{W} = (a_0, \dots, a_{m-1})$ for which

$$n_0 = \frac{B_m(\mathcal{W})}{2^{K_m} - 3^m}$$

is a positive odd integer, where $K_m = a_0 + \dots + a_{m-1}$ and n_0 is (without loss of generality) the least element of the cycle. This is exactly the situation studied by the classical continued-fraction approach to Collatz cycles, which we now import as a cited external result rather than re-derive.

Lemma 31.7 (Eliahou–Simons–de Weger–Hercher criterion; cited). *Let $\alpha := \log_2 3$, and for a positive integer R let $p(R)/q(R)$ denote the fraction of least denominator among the convergents (and intermediate fractions) of the continued fraction expansion of α satisfying*

$$\alpha < \frac{p(R)}{q(R)} \leq \log_2 \left(3 + \frac{1}{R} \right).$$

If a nontrivial positive Collatz cycle exists with least element $n_0 \geq R$, accelerated length m , and total valuation $K_m = a_0 + \dots + a_{m-1}$, then

$$K_m \geq p(R), \quad m \geq q(R).$$

Equivalently, the pair (K_m, m) attached to any nontrivial cycle in [Theorem 7.1](#) cannot be a “smaller” rational approximation to α than $p(R)/q(R)$.

Source. This is the sharp form of Eliahou’s diophantine-approximation argument [14], in the least-denominator formulation of Simons and de Weger [15] and Hungerbühler [17] (Theorem 2 there), and sharpened further via Baker’s theorem on linear forms in logarithms by Hercher [16]. We do not re-derive the inequality here; we only translate its conclusion into the notation of [Theorem 7.1](#). $\square$

Corollary 31.8 (Quantitative obstruction at the current verified range). *Take $R = 2^{71}$, the largest range for which the Collatz conjecture has been computationally verified with no cycle found, established by Bařina [12]. Computing the continued fraction expansion of $\alpha = \log_2 3$ to 200 decimal digits of precision and searching convergents for the least-denominator criterion of [Lemma 31.7](#) gives, as the unique minimal solution,*

$$p(2^{71}) = 217\,976\,794\,617, \quad q(2^{71}) = 137\,528\,045\,312.$$

(The previous convergent, with denominator 6 586 818 670, fails the criterion at $R = 2^{71}$; this was checked by exact rational arithmetic on the continued fraction convergents, not by floating point.) Hence, by [Lemma 31.7](#), no nontrivial cycle compatible with [Theorem 7.1](#) can have

$$K_m < 217\,976\,794\,617 \quad \text{or} \quad m < 137\,528\,045\,312,$$

and the total unaccelerated Collatz length of any such cycle, $m + K_m$, is at least 355 504 839 929. These values agree exactly with the current literature record [18], confirming that [Lemma 31.7](#) together with the Bařina range [12] reproduces the sharpest bound presently known on any solution of [Problem 31.6](#).

Remark 31.9 (No new record available yet). Angeltveit [13] describes an algorithm that would make verifying $n < 2^{72}$ or beyond computationally feasible, and states explicitly that a concrete new verified range $N \geq 72$ is left as “future work.” It reports no completed verification beyond Bařina’s 2^{71} [12]. Consequently [Corollary 31.8](#) currently represents the sharpest quantitative

form of the obstruction to [Problem 31.6](#) available from the verified-range route; it should be revisited if Angeltveit or others publish a concrete new N . (For reference, the next convergent above α after $p(2^{71})/q(2^{71})$ has denominator 5 409 303 924 479 and numerator 8 573 543 875 303; it would become the operative bound only once a verified range R large enough to defeat it is established.)

Remark 31.10 (What this does, and does not, resolve). [Lemma 31.7](#) and [Corollary 31.8](#) do not solve [Problem 31.6](#): they are a conditional lower bound on (K_m, m) , not a proof that no valid $(\mathcal{W}, n_0)$ pair exists. [Problem 31.6](#) remains open, and closing it unconditionally requires either (i) an unconditional lower bound on partial quotients of the continued fraction of $\log_2 3$ (currently unavailable; would follow from an effective irrationality measure sharper than what is known, cf. Baker–Wüstholz-type bounds used by Hercher [16]), or (ii) an entirely different argument that rules out the divisibility condition $2^{K_m} - 3^m \mid B_m(\mathcal{W})$ of [Theorem 7.1](#) directly, independent of the size of m . What this subsection adds is a citation-complete link between [Problem 31.6](#) and the established 30-year chain of results on this exact problem [14, 15, 17, 16], which the present manuscript did not previously reference.

Problem 31.11 (Escape-front compensation). Prove that every least-representative terminal-ceiling-compatible eventual no-jump tail, different from the fixed tail of 1, has bounded packet surplus on each sufficiently late tail: for some packet index J and constant $C = C(J)$,

$$\Omega_{J,K} \leq C \quad (K > J).$$

Equivalently, prove that valuation-one high-mantissa scale-up packets cannot accumulate unbounded positive surplus without forcing either a strict jump or a descent.

Theorem 31.12 (Two-front closure theorem). *A positive solution of [Problem 31.6](#) and [Problem 31.11](#) implies the Collatz conjecture.*

Proof. By [Problem 31.6](#), the first obstruction in [Theorem 31.2](#) is absent. If the second obstruction existed, [Proposition 31.5](#) would give unbounded packet surplus on every sufficiently late packet tail. This contradicts [Problem 31.11](#). Hence both obstructions in [Theorem 31.2](#) are absent, so the finite-descent form of Collatz holds. By [Theorem 4.2](#), this is equivalent to the Collatz conjecture. $\square$

31.4 Quantitative task for the escape front

The escape front is now a finite-word compensation problem. The following sharper statement is the most useful next target because it converts an unbounded qualitative obstruction into a quantitative inequality.

Problem 31.13 (Packet loss lower bound). For every least-representative terminal-ceiling-compatible eventual no-jump tail different from the fixed tail of 1, prove a bound of the following form. There exist a packet index J , a constant C , and a nonnegative correction functional $E_{J,K}$ such that

$$\sum_{j=J}^{K-1} \Lambda_j \geq \sum_{j=J}^{K-1} r_j - C + E_{J,K} \quad (K > J).$$

In particular, it would suffice to prove

$$\Omega_{J,K} = \sum_{j=J}^{K-1} (r_j - \Lambda_j) \leq C \quad (K > J).$$

Remark 31.14 (Research protocol after the reduction-spine theorem). The remaining work should not introduce new terminology unless it proves force toward [Problem 31.6](#) or toward

one of the unbounded-front targets. On the cycle front, the exact certificate $n_0 = B_m/(2^{K_m} - 3^m)$ is the object to attack. On the escape front, the hierarchy of targets is now: packet compensation [Problem 31.11](#), record-ladder compensation [Problem 18.8](#), the two-channel bounds [Problems 19.6](#) and [19.7](#), the offset suffix-stress target [Problem 20.6](#), the logarithmic shallow-return target [Problem 21.5](#), the base-start length target [Problem 22.5](#), the length-cost split targets [Problems 23.6](#) and [23.7](#), and the neutral rotation-window target [Problem 25.6](#), its resonance-corridor split [Problems 26.7](#) and [26.8](#), the selected-scale boundary-pruning reduction [Theorem 27.6](#), and large-reserve recycling [Problem 25.9](#). A successful proof of either a direct strict-jump theorem for unbounded deficits or one complete chain through these unbounded-front bounds would remove the unbounded terminal branch; a successful cycle exclusion would remove the bounded branch.

The reduction architecture developed above still leaves several open fronts. Among them, the terminal-ceiling strict-jump front admits a particularly short closure chain. We record that chain here in order to isolate the single decisive unresolved statement.

31.5 A five-step strict-jump roadmap

The steps below are recorded as a roadmap, not as new theorems of this manuscript; only the deductions explicitly marked *unconditional* are proved earlier in the paper and merely re-cited here for self-containedness. The one step that is not proved anywhere in this manuscript is stated as an open **problem**, in the same status as [Problems 6.10](#) and [31.6](#) elsewhere in this paper, precisely so that it cannot be mistaken for an established lemma. Every other step is either a direct citation of an already-proved result or an immediate corollary of one. No statement in this subsection is used later as an unproved input to any unconditional theorem of the manuscript.

Lemma 31.15 (Ceiling-to-no-jump normalization; unconditional). *Assume that the Collatz conjecture fails. Then there exists a non-fixed infinite extension chain*

$$\mathcal{W}_1 \prec \mathcal{W}_2 \prec \dots$$

such that, for all sufficiently large m , the prefix $\mathcal{W}_m$ is least-representative bad and terminal-ceiling compatible in the sense of [Definition 12.2](#). Moreover, the least positive representatives $\rho(\mathcal{W}_m)$ are eventually constant if and only if the chain has only finitely many strict jumps.

Proof. This is exactly the forward direction already carried out inside the proof of [Proposition 6.11](#): a terminally bad starting value ([Corollary 4.5](#)) gives a realized infinite bad chain; [Corollary 6.5](#) transfers badness into ceiling compatibility; and [Corollary 13.3](#) identifies positive realization with eventual constancy of the least representatives. The finite-versus-infinite strict-jump equivalence is the monotone jump decomposition of [Proposition 11.1](#). This lemma adds no content beyond [Proposition 6.11](#)'s own proof; it is isolated here only so the present chain can be read independently of that proof. $\square$

Lemma 31.16 (Low-band recurrence obstruction; unconditional). *Let*

$$\mathcal{W}_1 \prec \mathcal{W}_2 \prec \dots$$

be a non-fixed infinite extension chain whose sufficiently deep prefixes are least-representative bad and terminal-ceiling compatible. If the associated least-representative tail is eventually no-jump, then the relative-height process cannot remain forever in the lowest terminal-ceiling band. In particular, higher ceiling bands must be visited infinitely often.

Proof. This is exactly [Proposition 6.8](#) (forced growth in the lowest band), already proved above: repeated valuation 1 forces multiplicative growth of the relative height, so the orbit cannot remain in the lowest band forever. This lemma is a restatement of [Proposition 6.8](#) in the eventual-no-jump vocabulary used by the rest of this subsection; it adds no new content. $\square$

Problem 31.17 (Recurrent ceiling mismatch). Let

$$\mathcal{W}_1 \prec \mathcal{W}_2 \prec \dots$$

be a non-fixed infinite extension chain whose sufficiently deep prefixes are least-representative bad and terminal-ceiling compatible. Assume that the least-representative tail is eventually no-jump. Prove that there exist infinitely many indices m such that the actual next accelerated valuation of the current least representative is strictly larger than the maximal valuation compatible with continuing the same no-jump obstruction regime.

Remark 31.18. This is the one step in the present roadmap that is not proved anywhere in this manuscript. A proof would need to show that eventual no-jump transport is incompatible with the recurrent terminal-ceiling constraints forced by [Lemma 31.16](#) on a non-fixed bad chain; no such argument is given here.

Lemma 31.19 (Mismatch forces strict jump; unconditional). *At every index m for which the valuation mismatch of [Problem 31.17](#) occurs, the extension*

$$\mathcal{W}_m \prec \mathcal{W}_{m+1}$$

is a strict jump. Equivalently,

$$\rho(\mathcal{W}_{m+1}) > \rho(\mathcal{W}_m).$$

Proof. This is the direct contrapositive of [Proposition 11.4](#): the no-jump extension is exactly the case $b = v_m^\rho$ (prescribed valuation equals actual valuation); a mismatch means $b \neq v_m^\rho$, and [Proposition 11.4](#) then forces $t_m \neq 0$, i.e. a strict jump. $\square$

Remark 31.20 (What a positive solution of [Problem 31.17](#) would give). If [Problem 31.17](#) were proved, then by [Lemma 31.19](#) every chain covered by [Problem 31.17](#) would have infinitely many strict jumps, and so could not be eventually no-jump. Combined with [Lemma 31.15](#), this is verbatim a positive solution of [Problem 6.10](#) stated earlier in this paper ([Section 7](#)) – we record no separate “lemma” for this step, since it would only restate [Problem 6.10](#) itself under a new name.

Proposition 31.21 (Conditional closure via the ceiling-mismatch problem). *A positive solution of [Problem 31.17](#) would resolve [Problem 6.10](#), and hence, by the already-established [Proposition 6.11](#), would establish the Collatz conjecture.*

Proof. By [Lemma 31.15](#), any failure of the Collatz conjecture would produce a non-fixed infinite extension chain that is least-representative bad, terminal-ceiling compatible, and, after stabilization, eventually no-jump. [Remark 31.20](#) shows that a positive solution of [Problem 31.17](#) rules out exactly this possibility, which is a positive solution of [Problem 6.10](#). The conclusion then follows from [Proposition 6.11](#), already proved above, without any further argument. $\square$

Remark 31.22 (Decisive closure point). The only genuinely unresolved step in this subsection is [Problem 31.17](#), stated deliberately as a problem rather than a lemma. [Lemmas 31.15](#) and [31.16](#) restate content already proved earlier in this manuscript (respectively, part of the proof of [Proposition 6.11](#) and [Proposition 6.8](#)) and add nothing beyond that; [Lemma 31.19](#) is an immediate corollary of [Proposition 11.4](#); and [Remark 31.20](#) is a direct restatement of [Problem 6.10](#) itself. The only new mathematical object introduced in this subsection is [Problem 31.17](#). [Proposition 31.21](#) is a conditional proposition, not a new theorem: it introduces no claim toward Collatz beyond what [Proposition 6.11](#) already establishes, given a solution of [Problem 31.17](#). Thus this subsection’s sole contribution is to isolate [Problem 31.17](#) as the precise, self-contained target that would close the terminal-ceiling strict-jump front.

32 Eventually periodic low-surplus class theorem

The theorem in this section is the closed module isolated separately as Paper A [1]. It is included here to keep the reduction framework self-contained.

This section gives a completed class-exclusion theorem inside the low-surplus branch. The class is defined purely by valuation pattern: eventually periodic infinite low-surplus chains. The proof does not use a conjectural terminal step.

Why infinitely many ones are not immediate

A tempting question is whether strict-jump incompatibility holds for every non-fixed infinite low-surplus chain containing infinitely many valuations equal to 1. This remains natural, but it does not follow directly from the current local machinery.

Indeed, if

$$\alpha = \log_2 3, \quad \delta_m = m\alpha - K_m,$$

then by Lemma 11.6 a low-surplus one-step extension by b must satisfy

$$b \leq \lfloor \delta_m + \alpha \rfloor.$$

For every low-surplus prefix, $\delta_m > 0$, hence

$$\lfloor \delta_m + \alpha \rfloor \geq 1.$$

Thus the valuation $b = 1$ never violates the low-surplus ceiling. Recurrent 1's force strict jumps only when they disagree with the actual next valuation of the current least representative. This is why the all-one chain is a useful model but does not, by itself, prove the full infinitely-many-ones theorem.

Definition 32.1 (Eventually periodic valuation chain). An infinite valuation chain $(a_i)_{i \geq 1}$ is *eventually periodic* if there are integers $m_0 \geq 0$ and $p \geq 1$, and a block

$$P = (b_1, \dots, b_p),$$

such that

$$a_{m_0+kp+r} = b_r \quad (k \geq 0, 1 \leq r \leq p).$$

Write

$$L := b_1 + \dots + b_p.$$

For the period block P , define its offset

$$B_P := \sum_{i=1}^p 3^{p-i} 2^{L_{i-1}}, \quad L_i := b_1 + \dots + b_i, \quad L_0 := 0.$$

The period map attached to P is

$$F_P(x) = \frac{3^p x + B_P}{2^L}.$$

Lemma 32.2 (A low-surplus periodic tail has expanding real multiplier). *If an eventually periodic chain is low-surplus, then its period block satisfies*

$$2^L < 3^p.$$

Proof. Along the periodic tail, the deficit

$$\delta_m = m\alpha - K_m$$

changes by

$$p\alpha - L$$

after each full period. Since the whole infinite chain is low-surplus, $\delta_m > 0$ for all sufficiently large m . If $p\alpha - L < 0$, repeated periods would eventually make δ_m negative. If $p\alpha - L = 0$, then $\alpha = L/p$ would be rational, impossible because $\log_2 3$ is irrational. Hence $p\alpha - L > 0$, which is equivalent to $2^L < 3^p$. $\square$

Lemma 32.3 (The periodic tail fixed point is negative). *Assume $2^L < 3^p$. The affine period map*

$$F_P(x) = \frac{3^p x + B_P}{2^L}$$

has the unique fixed point

$$x_P = \frac{B_P}{2^L - 3^p},$$

and this number is negative.

Proof. Solving $F_P(x) = x$ gives

$$3^p x + B_P = 2^L x,$$

hence

$$x = \frac{B_P}{2^L - 3^p}.$$

Every summand in B_P is positive, so $B_P > 0$, while $2^L - 3^p < 0$ by assumption. Therefore $x_P < 0$. $\square$

Lemma 32.4 (Tail realization forces an ordinary period-map fixed point). *Let P^∞ be a purely periodic valuation chain with period block P . If a positive odd integer y realizes P^∞ , then*

$$F_P(y) = y$$

as ordinary integers. Equivalently,

$$(2^L - 3^p)y = B_P.$$

Proof. Let $\mathcal{P}_m$ be the length- m prefix of P^∞ , and let $K_m(\mathcal{P}_m)$ be its total valuation sum. Since y realizes P^∞ , it realizes every finite prefix $\mathcal{P}_m$. In particular,

$$y \equiv \rho(\mathcal{P}_m) \pmod{2^{K_m(\mathcal{P}_m)+1}} \quad (m \geq 1).$$

Since y realizes the first full period block P , the exact finite-iterate formula gives

$$\text{Syr}^p(y) = F_P(y) = \frac{3^p y + B_P}{2^L},$$

and this is an ordinary positive odd integer. After removing the first period block, the remaining valuation tail is again P^∞ . Therefore $F_P(y) = \text{Syr}^p(y)$ also realizes every finite prefix $\mathcal{P}_m$, so

$$F_P(y) \equiv \rho(\mathcal{P}_m) \pmod{2^{K_m(\mathcal{P}_m)+1}} \quad (m \geq 1).$$

Consequently

$$F_P(y) \equiv y \pmod{2^{K_m(\mathcal{P}_m)+1}} \quad (m \geq 1).$$

Because every valuation in P^∞ is at least 1, one has $K_m(\mathcal{P}_m) \geq m$, and hence $K_m(\mathcal{P}_m) \rightarrow \infty$. Therefore the ordinary integer $F_P(y) - y$ is divisible by arbitrarily large powers of 2. Hence $F_P(y) - y = 0$ in $\mathbb{Z}$. Substituting the formula for F_P gives

$$3^p y + B_P = 2^L y,$$

or equivalently

$$(2^L - 3^p)y = B_P.$$

□

Theorem 32.5 (Eventually periodic low-surplus exclusion). *No eventually periodic infinite low-surplus valuation chain is realized by a positive odd integer.*

Proof. Suppose, for contradiction, that a positive odd integer n realizes such a chain. Let m_0 be the start of the periodic tail, and put

$$y = \text{Syr}^{m_0}(n).$$

Since the original chain is realized from n and periodicity starts at m_0 , the shifted value y is a positive odd integer realizing the purely periodic tail P^∞ .

By Lemma 32.4, y is an ordinary fixed point of the period map:

$$(2^L - 3^p)y = B_P.$$

By Lemmas 32.2 and 32.3,

$$y = \frac{B_P}{2^L - 3^p} < 0,$$

contradicting $y > 0$. Thus no positive odd integer realizes the chain. □

Corollary 32.6 (Strict jumps for eventually periodic low-surplus chains). *Every eventually periodic infinite low-surplus valuation chain has infinitely many strict jumps of its least representatives $\rho(\mathcal{W}_m)$. Consequently its 2-adic limit is not a positive odd integer.*

Proof. By Theorem 32.5, the chain has no positive integer realization. By Proposition 10.5, eventual constancy of $\rho(\mathcal{W}_m)$ is equivalent to positive realization of the chain. Therefore $\rho(\mathcal{W}_m)$ cannot be eventually constant. Since the representatives are nondecreasing and change exactly at strict jumps by Proposition 11.1, strict jumps must occur infinitely often. □

Remark 32.7 (Relation to the all-one toy model). The all-one chain is the special case $P = (1)$. Then $p = 1$, $L = 1$, $B_P = 1$, and

$$x_P = \frac{1}{2 - 3} = -1.$$

Thus the earlier all-one incompatibility is exactly the period-one instance of Theorem 32.5.

Remark 32.8 (Why this is a completed theorem). The class is defined purely by valuation pattern; the proof uses finite-word congruence, the 2-adic inverse limit, and the low-surplus inequality; and there is no exceptional set or conjectural terminal step. The theorem does not prove the infinitely-many-ones question, but it gives a nontrivial closed class-exclusion result inside the same mechanism.

33 Proved results and remaining questions

This final section records the status of the manuscript as a reduction paper.

Unconditional results proved here

- (i) The exact finite-iterate formula, exact descent inequality, and finite-descent equivalence to Collatz.
- (ii) The exact realization class and bad-prefix window for every finite valuation word.
- (iii) Activated density collapse: once a finite ceiling appears, continuing bad-prefix windows have normalized density tending to zero and eventually contain at most one candidate.
- (iv) The inverse-limit and least-representative calculus: positive realization is equivalent to eventual stabilization, while infinitely many strict jumps exclude positive realization.
- (v) The terminal-surplus identity, terminal-bad recurrence, narrow surplus strip, and first-activation bottleneck.
- (vi) The terminal-ceiling break criterion, integer ceiling rule, and height-band automaton.
- (vii) The no-jump trap: an eventual no-jump tail of least-representative bad prefixes is exactly the finite-word form of a terminally bad positive realization.
- (viii) The terminal obstruction dichotomy: a counterexample must be either a nontrivial accelerated cycle or an unbounded terminally bad orbit.
- (ix) Deficit coding and scale mechanics for the unbounded branch: unbounded height is equivalent to unbounded under-ceiling deficits; the dyadic scale walk increases only at valuation-one high-mantissa levels; no three scale-up levels are consecutive; and packet budgets and record-ladder extraction encode the remaining positive scale surplus.
- (x) The reduction-spine theorem: Collatz follows from cycle exclusion together with packet-surplus compensation.
- (xi) The amplification split of high record ladders into multiplicative and offset-dominance channels, the suffix-stress expansion reducing offset dominance to right-edge suffix surplus, and the logarithmic shallow-return pruning theorem for offset-dominant ladders.
- (xii) The base-scale localization, exponential length-cost split, long-quiet-packet neutral-core reduction, rotation-window balance theorem, resonance-corridor split, selected-scale boundary-corridor pruning by one-step middle escape, and selected near-return copy-switch splitting.
- (xiii) The closed class theorem: no eventually periodic infinite low-surplus valuation chain is realized by a positive odd integer.

Open fronts not proved here

- (i) **Cycle front.** Exclude all nontrivial accelerated cycles satisfying the certificate in [Theorem 7.1](#).
- (ii) **Low-surplus front.** Exclude arbitrary nonperiodic positive realizations of infinite low-surplus chains; the eventually periodic subclass is closed in [Theorem 32.5](#).
- (iii) **Post-activation singleton front.** After activation, density collapse leaves at most one candidate; excluding that candidate remains open.
- (iv) **Unbounded escape front.** Prove the packet-surplus compensation target [Problem 31.11](#), the stronger record-ladder target [Problem 18.8](#), the two-channel targets [Problems 19.6](#) and [19.7](#), the suffix-stress target [Problem 20.6](#), the logarithmic shallow-return target

Problem 21.5, the selected-scale linear-return target Problem 27.5, the selected switching-corridor target Problem 28.8, the selected copy-mesh target Problem 28.9, or the large-reserve recycling target Problem 25.9; alternatively prove any theorem forcing strict jumps or descent along unbounded terminal no-jump tails.

Final dependency

The clean final dependency is

$$\begin{aligned}
 & \text{cycle exclusion} \\
 & + \text{one of the packet, record-ladder, two-channel,} \\
 & \text{suffix-stress, shallow-return, selected-scale rotation, or copy-switch targets} \\
 & \implies \text{finite descent for every odd } n > 1 \\
 & \implies \text{Collatz.}
 \end{aligned}$$

The first implication is given by Theorems 19.9, 25.7, 27.6, 28.10 and 31.12 and Propositions 20.9, 21.7, 22.8 and 23.10, depending on which sufficient unbounded-front target is used; the second is Theorem 4.2. The present manuscript proves the reduction architecture, the amplification, suffix-stress, length-cost, neutral-core, rotation-window, resonance-corridor, selected-scale boundary-pruning, and selected copy-switch reductions, and the eventually periodic low-surplus class theorem, but it does not prove the cycle front, the large-reserve recycling front, the selected switching-corridor density front, the selected copy-mesh front, or the unbounded-front terminal bound and therefore does not claim a full proof of the Collatz conjecture.

References

- [1] T. L. Karadzhov, *Eventually periodic low-surplus valuation chains for the accelerated Syracuse map*, Zenodo preprint, 2026, doi:10.5281/zenodo.21169719.
- [2] J. C. Lagarias, *The $3x + 1$ problem: an overview*, arXiv:2111.02635, 2021.
- [3] J. C. Lagarias, *The $3x + 1$ problem: an annotated bibliography (1963–1999)*, arXiv:math/0309224, 2003.
- [4] J. C. Lagarias, *The $3x + 1$ problem and its generalizations*, Amer. Math. Monthly **92** (1985), no. 1, 3–23.
- [5] R. Terras, *A stopping time problem on the positive integers*, Acta Arith. **30** (1976), 241–252.
- [6] R. E. Crandall, *On the “ $3x + 1$ ” problem*, Math. Comp. **32** (1978), no. 144, 1281–1292.
- [7] L. E. Garner, *On the Collatz $3n + 1$ algorithm*, Proc. Amer. Math. Soc. **82** (1981), no. 1, 19–22.
- [8] G. J. Wirsching, *The Dynamical System Generated by the $3n + 1$ Function*, Lecture Notes in Mathematics, vol. 1681, Springer, Berlin, 1998.
- [9] I. Krasikov and J. C. Lagarias, *Bounds for the $3x + 1$ problem using difference inequalities*, Acta Arith. **109** (2003), no. 3, 237–258.
- [10] T. Tao, *Almost all orbits of the Collatz map attain almost bounded values*, Forum Math. Pi **10** (2022), e12; arXiv:1909.03562.
- [11] D. Bařina, *Convergence verification of the Collatz problem*, J. Supercomput. **77** (2021), 2681–2688.

- [12] D. Bařina, *Improved verification limit for the convergence of the Collatz conjecture*, J. Supercomput. **81** (2025), article 810.
- [13] V. Angeltveit, *An improved algorithm for checking the Collatz conjecture for all $n < 2^N$* , arXiv:2602.10466, 2026.
- [14] S. Eliahou, *The $3x+1$ problem: new lower bounds on nontrivial cycle lengths*, Discrete Math. **118** (1993), no. 1–3, 45–56.
- [15] J. Simons and B. de Weger, *Theoretical and computational bounds for m -cycles of the $3n + 1$ -problem*, Acta Arith. **117** (2005), no. 1, 51–70.
- [16] C. Hercher, *There are no Collatz m -cycles with $m \leq 91$* , J. Integer Sequences **26** (2023), Article 23.2.8.
- [17] N. Hungerbühler, *Optimal bounds for the length of rational Collatz cycles*, preprint, ETH Zürich, available at https://www.math.ch/norbert.hungerbuehler/publications/Optimal_bounds_for_the_length_of_rational_Collatz_cycles.pdf.
- [18] Wikipedia contributors, *Collatz conjecture*, Wikipedia, The Free Encyclopedia, accessed July 2026, https://en.wikipedia.org/wiki/Collatz_conjecture.