

INTEGRAL TAIL SIGNATURES AND THE HILBERT–SMITH CONJECTURE

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ABSTRACT. The obstruction to an effective p -adic manifold action is an integer signature tail. On asymptotically controlled rational modules over one fixed free C_p -space, finite-cover Mayer–Vietoris makes the difference between p copies and quotient-permutation tensoring nilpotent. Every degree-four class therefore has eventually p -divisible signatures. An effective action gives a class of signature 1: coordinate averaging provides degree-one sphere control, free sheets turn quotient representatives into a compatible homotopy idempotent, and its normalized Poincaré corner forgets to a scalar manifold germ. Successive localization boundaries recover one copy of $\mathbb{CP}^2$, up to sign. The resulting contradiction proves the Hilbert–Smith conjecture.

1. INTRODUCTION

The Hilbert–Smith conjecture extends Hilbert’s fifth problem from locally Euclidean groups to groups acting on manifolds. The cases of one- and two-manifolds are classical [Pa19]; Pardon proved the three-dimensional case [Pa13]. In arbitrary dimension, the conjecture holds for Lipschitz actions [RS97] and quasiconformal actions [Mar99]. We prove the following.

Theorem 1.1 (p -adic exclusion). *For every prime p , a continuous action of $\mathbb{Z}_p$ on a connected finite-dimensional manifold has nontrivial kernel.*

Corollary 1.2 (Hilbert–Smith). *A locally compact Hausdorff group acting continuously and effectively on a connected finite-dimensional manifold is a Lie group with its given topology.*

Manifolds are topological and Hausdorff, and may have boundary. Lie groups are allowed arbitrarily many connected components. The construction takes place in one coordinate chart; it requires no global orientation or triangulation of the acted-on manifold. The standard reduction to p -adic exclusion uses locally compact group structure and Newman’s theorem [Pa19].

The obstruction is an integer signature tail. For cyclic groups $G_i = C_{p^{a_i}}$ with index- p subgroups P_i , and a fixed compact free C_p -space T , tensoring with the positive permutation form $\tau_i = \mathbb{Q}[G_i/P_i]$ is locally p copies of the identity. Controlled Mayer–Vietoris therefore makes $p \text{id} - \tau \otimes (-)$ nilpotent, and equivariant signature characters force every class in $L_4(\mathcal{A}_G(T))$ to have eventually p -divisible ordinary signatures.

An effective $\mathbb{Z}_p$ -action, however, would supply a class with signature 1. Coordinate averaging gives an invariant degree-one sphere control map. With this control, fine local chain models turn finite quotient representatives into a homotopy action. Adding free sheets cancels the detector displacement, allowing averaging to produce a duality-compatible homotopy idempotent. Its normalized corner forgets to the scalar manifold germ of $S^n \times \mathbb{CP}^2$. Successive localization boundaries then recover one copy of $\mathbb{CP}^2$, up to sign. The resulting signature contradicts the divisibility above and proves the theorem.

The construction and boundary evaluation use Karoubi localization [CP95], controlled Poincaré–Lefschetz duality [Ran99], and algebraic Poincaré pairs and unions [Ran80I, Ran80II].

Idempotent splitting uses [BS01]; the local carrier and normalization calculations are given below.

Conventions. All actions are jointly continuous. Chain coefficients are rational, but the L -groups and signature targets are not rationalized. Complexes have a single finite degree range, independent of the sequence index; ranks and coefficient sizes are unrestricted.

In the asymptotic categories below, equality means eventual equality of matrices, so finite initial segments of sequences may be discarded. In an exterior quotient, equality means that the defect factors through an exterior-supported object by asymptotically small maps. We use maximum product metrics and take empty suprema to be zero.

We use ultimate lower quadratic L -theory, $L_k(\mathcal{A}) = \pi_k \mathbf{L}^{<-\infty}(\mathcal{A})$ [CP95, Definition 4.16], [Ran92, §17]. The properties used below are functoriality and additivity, Karoubi localization with natural connecting maps [CP95, Theorem 4.2], projective-to-ultimate decoration maps, and invariance under idempotent completion [Ran92, Theorem 17.2 and Definition 17.7]. For localization boundaries we use the pair description in [CP95, proof of Theorem 4.1]. Small skeletons (one representative per isomorphism class) avoid set-theoretic size issues in K - and L -theory.

2. ASYMPTOTIC CATEGORIES AND SUPPORT

Fix $G_i = C_{p^{a_i}}$, $a_i \geq 1$, index- p subgroups P_i , and $G_i/P_i = C_p$. Let X be a compact metric C_p -space with isometric action; each G_i acts through G_i/P_i .

Definition 2.1. An object of $\mathcal{A}_{\mathbf{G}}(X)$ is a sequence of finite free based modules $M_i = \mathbb{Q}[B_i]$, with B_i a free G_i -set and equivariant labels $\ell_i : B_i \rightarrow X$. Morphisms are sequences of equivariant rational matrices, modulo eventual equality, satisfying

$$(2.1) \quad \text{prop}(u_i) = \sup_{(u_i)_{b' \neq 0}} d(\ell'_i(b'), \ell_i(b)) \longrightarrow 0.$$

Duality fixes labeled bases and transposes matrices. Trivial groups give $\mathcal{A}_1(X)$. Ranks and coefficients are unrestricted.

The propagation bounds

$$(2.2) \quad \text{prop}(vu) \leq \text{prop}(v) + \text{prop}(u), \quad \text{prop}(u + v) \leq \max\{\text{prop}(u), \text{prop}(v)\}$$

show that composition and addition are well defined. Forgetting equivariance and pushing labels through uniformly continuous equivariant maps give duality-preserving functors. For a closed invariant $S \subset X$, let $\mathcal{A}_S = \mathcal{A}_{\mathbf{G}, S}(X)$ consist of objects whose labels approach S uniformly, with $\mathcal{A}_{\emptyset} = 0$. Write I_S for the ideal of maps factoring through $\mathcal{A}_S$. For a closed invariant exterior $Z \subset X$, put

$$(2.3) \quad \mathcal{B}_{\mathbf{G}, Z}(X) = \mathcal{A}_{\mathbf{G}}(X)/\mathcal{A}_Z.$$

A map belongs to I_S exactly when both endpoints of every nonzero entry approach S uniformly: necessity follows from propagation, and sufficiency by restricting the source to its nonzero columns.

Lemma 2.2 (Support calculus). *The inclusion $\mathcal{A}_S \subset \mathcal{A}_{\mathbf{G}}(X)$ is a duality-invariant Karoubi filtration [CP95, Definition 1.27]. For a closed invariant cover $X = A \cup B$, with $Y = A \cap B$, there are involution-preserving equivalences*

$$(2.4) \quad \mathcal{A}_A \cap \mathcal{A}_B = \mathcal{A}_Y, \quad \mathcal{A}_A/\mathcal{A}_Y \simeq \mathcal{A}_{\mathbf{G}}(X)/\mathcal{A}_B.$$

For nonempty closed invariant S , $\mathcal{A}_{\mathbf{G}}(S) \simeq \mathcal{A}_S$. If $d(S, Z) > 0$, $\mathcal{A}_S \rightarrow \mathcal{B}_{\mathbf{G}, Z}(X)$ is fully faithful.

Proof. For a null sequence $r_i \geq 0$, split each basis at distance r_i from S . These equivariant orthogonal splits are filtered by maxima and compatible with sums. A map to or from an object supported within distance a_i of S , with propagation ϵ_i , factors through the near summand for $r = a + \epsilon$. This verifies the filtration axioms.

By compactness, points approaching both A and B approach Y . Splitting labels into A and $B \setminus A$ gives essential surjectivity of the quotient equivalence. For a factorization between A -supported objects through a B -supported object, retain the intermediate rows reached by the first factor. They approach both sets, so the factorization is through $\mathcal{A}_Y$. This proves full faithfulness, including $Y = \emptyset$ by compact separation.

Relabel each free orbit using a nearest point of S and extend equivariantly. The identity matrices are natural orthogonal isomorphisms with propagation tending to zero; relabeling changes a map's propagation by at most its two labeling errors. Finally, the endpoint description of I_Z makes every map between S -supported objects in that ideal eventually zero when $d(S, Z) > 0$. $\square$

Splits precede idempotent completion, denoted Kar ; an object (M, e) carries the support of M .

3. INTEGER SIGNATURES AND TAILS

For a finite group G , the correspondence between invariant rational forms b and group-ring forms uses the unnormalized trace:

$$(3.1) \quad \lambda(v, w) = \sum_{g \in G} b(v, gw)g, \quad \text{tr}_1 \lambda(v, w) = b(v, w).$$

Thus the coefficient-of-identity trace tr_1 makes forgetting equivariance duality-preserving, including on projective summands. The half-unit comparison and [Ran89, Theorem 5.2] identify the projectively decorated group $L_4^p(\mathbb{Q}[G])$ with the Witt group of nonsingular invariant symmetric rational forms. Such a form has signature representation

$$(3.2) \quad \text{Sign}_G(V, b) = [V_{\mathbb{C}}^+] - [V_{\mathbb{C}}^-] \in R_{\mathbb{C}}(G),$$

where $R_{\mathbb{C}}(G)$ is the complex representation ring and $V^{\pm}$ are the positive and negative spectral subspaces on $V \otimes_{\mathbb{Q}} \mathbb{R}$ for an invariant positive inner product. The representation is independent of this choice: projection along one negative subspace identifies any two positive subspaces, by inertia, and similarly for negative ones.

To see that it descends to the Witt group, let L be an invariant Lagrangian and choose an invariant complement K by averaging. Replacing $u \in K$ by $u - c(u)/2$, where $c : K \rightarrow L$ satisfies $b(cu, v) = b(u, v)$ for $u, v \in K$, makes K Lagrangian. Hence metabolic forms have zero signature representation. The character of $\text{Sign}_G(V, b)$ at the identity is the ordinary integer signature $\sigma(V, b)$; tensoring with a positive form multiplies characters.

Lemma 3.1 (Cyclic arithmetic). *For $G = C_{p^a} = \langle g \rangle$, if $\text{Sign}_G(V, b)(g) = 0$, then $p \mid \sigma(V, b)$.*

Proof. Write the signature character as $\sum_{j=0}^{p^a-1} m_j \chi_j$, $m_j \in \mathbb{Z}$, with $\chi_j(g) = \zeta_{p^a}^j$, where ζ_{p^a} is a primitive p^a -th root of unity. Then $F(t) = \sum_j m_j t^j$ is divisible in $\mathbb{Z}[t]$ by the p^a -th cyclotomic polynomial $\Phi_{p^a}(t)$, so $F(1) = \sigma(V, b)$ is divisible by $\Phi_{p^a}(1) = p$. $\square$

Write $\text{Free}(R)$ for the category of finite free based R -modules. Put $\mathcal{P} = \prod_i \text{Free}(\mathbb{Q}[G_i])$ and let $\mathcal{S}$ be its finite-support subcategory. Forgetting labels gives a functor to

$$(3.3) \quad \mathcal{V}_G = \mathcal{P}/\mathcal{S},$$

whose morphisms are sequences modulo eventual equality.

Proposition 3.2 (Tail extraction). *There is a natural homomorphism*

$$(3.4) \quad L_4(\mathcal{V}_{\mathbf{G}}) \longrightarrow \prod_i L_4^p(\mathbb{Q}[G_i]) / \bigoplus_i L_4^p(\mathbb{Q}[G_i]),$$

given by the coordinate classes of any lift to $L_4(\mathcal{P})$. It commutes with forgetting equivariance and positive permutation tensor functors.

Proof. The semisimple rings $\mathbb{Q}[G_i]$ have vanishing negative K -groups [Wei13, III, §4], also as free-module categories [Ran92, §11, p. 101]. Bass's polynomial definition commutes with $\mathcal{S} = \bigcup_{F \text{ finite}} \prod_{i \in F} \text{Free}(\mathbb{Q}[G_i])$: every finite list of objects, idempotents, polynomial matrices and relations lies in one finite block. Consequently $K_{-k}(\mathcal{S}) = 0$ for $k \geq 1$. Rothenberg comparison then gives $L_j(\mathcal{S}) = \bigoplus_i L_j^p(\mathbb{Q}[G_i])$ and $L_j(\text{Free}(\mathbb{Q}[G_i])) = L_j^p(\mathbb{Q}[G_i])$.

Splitting at finite index sets is an invariant Karoubi filtration. Coordinate projections detect $L_3(\mathcal{S})$, so its map to $L_3(\mathcal{P})$ is injective. Localization therefore lifts every $L_4(\mathcal{V}_{\mathbf{G}})$ class to $L_4(\mathcal{P})$; two lifts differ by the image of $L_4(\mathcal{S})$. Their coordinate classes agree eventually. These constructions are natural for the stated functors. $\square$

Applying ordinary signature after extraction defines

$$(3.5) \quad \sigma_{\text{tail}} : L_4(\mathcal{A}_{\mathbf{G}}(X)) \longrightarrow \mathbb{Z}^{\mathbb{N}} / \mathbb{Z}^{(\mathbb{N})}.$$

Here $\mathbb{Z}^{(\mathbb{N})}$ denotes the finitely supported integer sequences. We use this map only after localization has removed the exterior.

4. DIVISIBILITY OVER A FREE C_p -SPACE

Let T be a fixed nonempty compact free C_p -space with a compatible invariant metric. Tensoring by $\tau_i = \mathbb{Q}[G_i/P_i]$, with its orthonormal form, diagonal action and unchanged spatial labels, defines

$$(4.1) \quad \nu = p \text{ id} - \tau \otimes (-) \quad \text{on } L_*(\mathcal{A}_{\mathbf{G}}(T)).$$

Lemma 4.1 (Local triviality). *For compact $U \subset T/C_p$ contained in an open covering trivialization, $\nu = 0$ on $L_*(\mathcal{A}_{\mathbf{G}, T_U}(T))$, where T_U is its inverse image.*

Proof. Choose a closed invariant neighborhood L of T_U inside that trivialization, with sheet function $c : L \rightarrow C_p$ satisfying $c(hx) = \bar{h}c(x)$, where $\bar{h}$ is the image of h in C_p . Labels of supported objects eventually lie in L . The orthogonal equivariant map

$$(4.2) \quad e_a \otimes e_b \longmapsto e_b \quad \text{in copy } c(\ell_i(b))^{-1}a \in C_p$$

has propagation zero. The sheet pieces of L are compact and positively separated, so controlled morphisms eventually preserve them. Hence this is a natural isometry $\tau \otimes (-) \cong \text{id}^{\oplus p}$, inducing equality in ultimate L -theory. $\square$

Choose a finite compact cover $U_1, \dots, U_s$ of T/C_p inside trivializations, fixed independently of i , and put $T_a = T_{U_a}$.

Lemma 4.2 (Nilpotence). *One has $\nu^s = 0$ on $L_*(\mathcal{A}_{\mathbf{G}}(T))$.*

Proof. Induct on s over all such compact free spaces. Set $A = T_1 \cup \dots \cup T_{s-1}$ and $B = T_s$. The natural support Mayer–Vietoris sequence follows from lemma 2.2. Since $\nu = 0$ on $A \cap B$, $\partial(\nu x) = 0$. Thus νx comes from $L_*(\mathcal{A}_A) \oplus L_*(\mathcal{A}_B)$. Relabeling and induction kill the first summand by ν^{s-1} ; local triviality kills the second by ν . For $s > 1$, applying ν^{s-1} proves the claim; $s = 1$ is local triviality. $\square$

Theorem 4.3 (Integer tail divisibility). *Every $\alpha \in L_4(\mathcal{A}_{\mathbf{G}}(T))$ has signature tail eventually divisible by p .*

Proof. Extract forms (V_i, b_i) . Naturality and nilpotence imply

$$(4.3) \quad (p - \tau_i)^s [V_i, b_i] = 0 \quad \text{in } L_4^p(\mathbb{Q}[G_i]) \quad \text{for all sufficiently large } i.$$

For a generator g_i of G_i , $\chi_{\tau_i}(g_i) = 0$. Taking signature characters gives $p^s \text{Sign}_{G_i}(V_i, b_i)(g_i) = 0$. Apply lemma 3.1. $\square$

5. RATIONAL POINCARÉ CORNERS

5.1. Signs and rational structures. Complexes are homologically graded. We use

$$(5.1) \quad (C^{N-*})_r = C_{N-r}^*, \quad d_{C^{N-*}, r} = (-1)^r d_C^*, \quad (T\phi)_r = (-1)^{r(N-r)} \phi_{N-r}^*.$$

The double shifted dual has differential $(-1)^{N+1} d_C$; its identification with C is $(-1)^{r(N-r)} = (-1)^{r(N+1)}$ in degree r . Right slant sends $x_r \otimes y_{N-r}$ to $\alpha \mapsto x_r \alpha(y_{N-r})$, without an extra sign. For a bounded complex in a rational additive category with involution put

$$M(C)_k = \bigoplus_r \text{Hom}(C_{k-r}^*, C_r), \quad (\partial f)_r = d_C f_{r+1} + (-1)^r f_r d_C^*, \quad (Tf)_r = (-1)^{r(k-r)} f_{k-r}^*.$$

Thus $T^2 = 1$, $T\partial = \partial T$, and conjugation by chain maps commutes with T . For finite scalar complexes this is $C \otimes C$ under right slant. Exactness of rational invariants and coinvariants gives

$$(5.2) \quad Q^N(C) = H_N(M(C)^{C_2}), \quad Q_N(C) = H_N(M(C)_{C_2}).$$

The norm $1 + T$ is an isomorphism with inverse $t \mapsto [t/2]$. We write $\text{Sym} = (1 + T)/2$. Consequently a symmetric structure has a strict symmetric representative ϕ , and its quadratic lift is $\psi_0 = \phi/2$, $\psi_s = 0$ for $s > 0$. If symmetric cycles differ by ∂H , replace H by $(H + TH)/2$; halving that primitive identifies their quadratic classes. The same argument applies to the relative cone complexes, since their maps commute with T [Ran80I, Proposition 3.3], [Ran89, Section 3]. All these operations preserve propagation and exterior factorizations.

5.2. Compatible idempotents and normalization. Let a finite cyclic group G , of order q , act up to homotopy on (C, ϕ) , with

$$(5.3) \quad A_g A_h \simeq A_{gh}, \quad A_1 = 1, \quad A_g \phi \simeq \phi A_{g^{-1}}^*.$$

Only an action in the chain-homotopy category is assumed; no higher coherence data are used. On $\mathcal{C} = C \otimes \mathbb{Q}[G]$, with strict left sheet action, set

$$(5.4) \quad \Phi = \phi \otimes 1, \quad E = q^{-1} \sum_g A_g \otimes R_{g^{-1}},$$

where $R_{g^{-1}}$ sends sheet b to bg^{-1} . Then

$$(5.5) \quad E^2 \simeq E, \quad E\Phi \simeq \Phi E^*, \quad E_{ab} = q^{-1} A_{a^{-1}b}.$$

In the controlled application these equations, including their homotopies, hold in the exterior quotient, as checked in §11.

Lemma 5.1 (Compatible splitting). *Let $\mathcal{Q}$ be rational additive with involution, $\mathcal{C} \in [0, N]$, and $\Phi : \mathcal{C}^{N-*} \rightarrow \mathcal{C}$ a strict symmetric Poincaré equivalence. For an invertible self-adjoint central coefficient q and a chain map E with homotopies $E^2 \simeq E$, $E\Phi \simeq \Phi E^*$ in $\mathcal{Q}$, there is a splitting in the bounded chain-homotopy category $K^b(\text{Kar } \mathcal{Q})$,*

$$(5.6) \quad D \xrightarrow{\iota} \mathcal{C} \xrightarrow{r} D, \quad \iota r \simeq E, \quad r\iota \simeq 1, \quad D \in [0, N].$$

The strict symmetric Poincaré duality and a homotopy inverse are

$$(5.7) \quad \phi_D = q^{-1}r\Phi r^*, \quad b_D = q\iota^*\Phi^{-1}\iota.$$

Here Φ^{-1} denotes a chosen chain-homotopy inverse. The resulting quadratic Poincaré complex determines a natural class in $L_N(\mathcal{Q})$.

Proof. With its split exact structure, $\text{Kar } \mathcal{Q}$ has bounded derived category $K^b(\text{Kar } \mathcal{Q})$, which is idempotent complete [BS01, Theorem 2.8]. Hence $[E]$ splits. Outside $[0, N]$, the identity homotopy $r\iota \simeq 1$ makes the extremal differential split: surjective at the bottom and injective at the top. Removing the resulting contractible two-term summands puts D in $[0, N]$. Compatibility gives

$$\phi_D b_D \simeq r\Phi E^* \Phi^{-1} \iota \simeq rE\iota \simeq 1, \quad b_D \phi_D \simeq \iota^* \Phi^{-1} E \Phi r^* \simeq \iota^* E^* r^* \simeq 1.$$

Conjugation preserves strict symmetry, so $\phi_D/2$ is its complete quadratic structure. Two splittings are related by $r_2\iota_1$, an equivalence carrying their symmetric structures to one another up to homotopy. The primitive argument above and the Poincaré mapping-cylinder cobordism [Ran89, Definition 3.10] identify their ordinary L -classes in $\text{Kar } \mathcal{Q}$. Finally pass to ultimate decoration and use the natural isomorphism $L_N(\mathcal{Q}) \cong L_N(\text{Kar } \mathcal{Q})$ [Ran92, Definition 17.7]. This also proves functoriality. $\square$

For an asymptotic exterior category the splitting takes place once in the sequence category. Its finite collection of maps and homotopies therefore has asymptotically small representatives, with equations factoring through exterior-supported objects.

Lemma 5.2 (Scalar normalization). *For (5.4), suppose a duality-preserving forgetting functor retains a coordinate in which all A_g and the homotopies in (5.3) are uniformly small. Its image of (5.7) is homotopy-isometric to (C, ϕ) .*

Proof. Only after forgetting, let j_1, π_1 include and project the identity sheet, and put $v = Ej_1$, $u = q\pi_1 E$. Thus $v_a = q^{-1}A_{a-1}$, $u_b = A_b$, and

$$(5.8) \quad uv \simeq 1, \quad vu \simeq E, \quad uE \simeq u, \quad Ev \simeq v, \quad u\Phi u^* = \sum_b A_b \phi A_b^* \simeq q\phi.$$

Hence $w = u\iota$ and $z = rv$ are inverse equivalences and

$$(5.9) \quad w\phi_D w^* \simeq q^{-1}uE\Phi E^* u^* \simeq q^{-1}u\Phi u^* \simeq \phi.$$

The rational primitive argument identifies the quadratic classes; the mapping-cylinder cobordism and ultimate completion invariance then identify their L -classes. If $q = q_i$ varies, multiplication by $q_i^{\pm 1}$ has propagation zero, and the uniform entry and homotopy bounds make u, v, w, z valid sequence morphisms. $\square$

6. LOCALIZATION BOUNDARIES

For a closed invariant cover $X = A \cup B$, put $Y = A \cap B$. Support filtration, excision, and [CP95, Theorem 4.2] give

$$(6.1) \quad \partial_{A,B} : L_k(\mathcal{A}(X)) \longrightarrow L_{k-1}(\mathcal{A}_Y(X)).$$

Here $\mathcal{A} = \mathcal{A}_{\mathbf{G}}$; the group sequence is unchanged.

Proposition 6.1. *If $Z \subset A$ is closed invariant and $d(Z, B) > 0$, there is a boundary*

$$(6.2) \quad \partial_{A,B} : L_k(\mathcal{A}(X)/\mathcal{A}_Z(X)) \longrightarrow L_{k-1}(\mathcal{A}_Y(X)),$$

natural for duality-preserving functors preserving these support categories.

Proof. Put $\mathcal{Q} = \mathcal{A}(X)/\mathcal{A}_Z(X)$ and $\mathcal{U} = \mathcal{A}_A(X)/\mathcal{A}_Z(X)$. Separation embeds $\mathcal{A}_B$ and $\mathcal{A}_Y$ fully faithfully in $\mathcal{Q}$. Choose controlled representatives for finite families of quotient maps and perform the based support decompositions before quotienting; this verifies the support filtrations there. The same based split as in lemma 2.2 gives the duality-preserving equivalence

$$(6.3) \quad \mathcal{U}/\mathcal{A}_Y \simeq \mathcal{Q}/\mathcal{A}_B.$$

Indeed, on A -supported objects I_B restricts to I_Y , whereas I_Z already factors through A -supported objects; thus the respective ideals are $I_Z + I_Y$ and $I_Z + I_B$. The based inclusion and projection are adjoints and inverse in the quotient. The boundary is induced by

$$\mathbf{L}^{(-\infty)}(\mathcal{Q}) \longrightarrow \mathbf{L}^{(-\infty)}(\mathcal{Q}/\mathcal{A}_B) \simeq \mathbf{L}^{(-\infty)}(\mathcal{U}/\mathcal{A}_Y) \longrightarrow \Sigma \mathbf{L}^{(-\infty)}(\mathcal{A}_Y).$$

Localization for the two filtrations gives Mayer–Vietoris and naturality. The ordinary case is $Z = \emptyset$. $\square$

Lemma 6.2. *Let $V_A = (P \rightarrow D_A)$ and $V_B = (P \rightarrow D_B)$ be k -dimensional quadratic Poincaré pairs supported on A, B , with oppositely oriented common boundary P supported on Y . Orient P as the boundary of V_A . For $V = V_A \cup_P V_B$, in the ordinary or exterior quotient setting,*

$$(6.4) \quad \partial_{A,B}[V] = [P].$$

Proof. Use $\mathcal{Q}, \mathcal{U}$ above, with $Z = \emptyset$ in the ordinary case. The relative-pair representative $P \rightarrow D_A$ in the cofiber of $\mathbf{L}^{(-\infty)}(\mathcal{A}_Y) \rightarrow \mathbf{L}^{(-\infty)}(\mathcal{U})$ has boundary $[P]$. The relative-to-quotient comparison [CP95, proof of Theorem 4.1], followed by ultimate decoration, sends it to D_A with its full induced quadratic structure. Modulo $\mathcal{A}_B, D_B, P$, and the suspended interface of the union vanish degree-wise, along with every interface structure correction. The surviving complex and structure are exactly those on D_A . Excision thus identifies the two representatives, whose connecting image is $[P]$. Completion and the passage to ultimate decoration preserve this pair comparison. $\square$

7. LOCAL CHAIN CARRIERS

Write ε for geometric augmentation of zero chains. We use the acyclic-carrier method to prove the required shrinking-mesh estimates directly. Its application to dual cells is the elementary PL form of combinatorially controlled Poincaré–Lefschetz duality; compare [Ran99, Proposition 5.3, Theorem 6.7 and Corollary 6.8].

Lemma 7.1 (Finite acyclic carriers). *Let E be a finite based chain complex in nonnegative degrees. Assign to each basis element e an augmented acyclic subcomplex $\Gamma(e)$ of a rational chain complex F , with augmentations agreeing on inclusions. Assume $\Gamma(e') \subset \Gamma(e)$ whenever e' occurs in de . Two carried chain maps $u, v : E \rightarrow F$ agreeing in augmentation on degree-zero basis elements admit a carried homotopy*

$$u - v = dH + Hd, \quad H(e) \in \Gamma(e).$$

Carried, augmentation-compatible prescribed degree-zero images likewise extend to a carried chain map from an augmented source.

Proof. Induct on degree. The residual $u(e) - v(e) - H(de)$ lies in $\Gamma(e)$, is a cycle, and in degree zero has augmentation zero. Fill it in that carrier and use the filling as $H(e)$. The map-existence induction fills the cycle given by the already defined image of de . $\square$

Corollary 7.2. *If labels in $\Gamma(e)$ are within a of the label of e , the constructed map or homotopy has propagation at most a . Relabeling by a fixed continuous map c on a compact metric space replaces this bound by $\omega_c(a)$, where ω_c is a modulus of continuity for c . Signed cellular bases use geometric augmentation.*

Proof. The construction introduces no basis elements outside the assigned carriers. $\square$

Lemma 7.3 (Filling on a fixed triangulation). *Let K triangulate a compact m -manifold W with mesh at most η . Let C be convex in one coordinate chart. A simplicial r -cycle γ whose full simplices lie in C bounds a chain q on the same K , with full-simplex support in the closed η -neighborhood of C ; for $r = 0$ require augmentation zero. For $r = m$ this forces $\gamma = 0$.*

Proof. The zero chain needs no filling. Otherwise let A be the subcomplex generated by its support and cone its inclusion to a point in C . This gives $F : |\text{cone } A| \rightarrow C$ restricting to the inclusion on A . Subdivide only this cone and choose a simplicial approximation $t : L \rightarrow K$ satisfying $F(\text{st}_L^\circ v) \subset \text{st}_K^\circ t(v)$ [Hat02, Theorem 2C.1 and its proof]; write S for subdivision. On the base, tS is carried in each original simplex and fixes vertices, so $tS|_{C_*(A)} = 1$: induction reduces the difference on an r -simplex to a top-dimensional cycle in that simplex, which is zero for $r > 0$. Thus $q = tS(c\gamma)$ satisfies $dq = \gamma$, where c is the augmented cone operator. For z in the interior of a simplex λ of L , the approximation property says that every vertex of $t\lambda$ belongs to the smallest target simplex containing $F(z)$. That simplex meets C and has diameter at most η , proving the support claim. In top degree $q = 0$ because $C_{m+1}(K) = 0$. $\square$

Lemma 7.4 (Relative dual-block carrier maps). *Let K be a finite combinatorial triangulation of a compact oriented m -manifold A with ∂A a subcomplex and mesh η . For nonboundary simplices choose dual-block orientations with*

$$\partial D\sigma = (-1)^{\dim \sigma + 1} \sum_{\substack{\tau \supset \sigma \\ \dim \tau = \dim \sigma + 1}} [\tau : \sigma] D\tau, \quad D\rho = \epsilon_\rho [b_\rho]$$

on top simplices, where ϵ_ρ is the fundamental coefficient and b_ρ the barycenter. Let D_{rel} be the cellular subcomplex generated by these blocks and put $J_\sigma = \bigcup_{\tau \supset \sigma, \tau \not\subset \partial A} D\tau$. Every J_σ is augmented acyclic. There are maps $S_{\text{rel}} : D_{\text{rel}} \rightarrow C_*(\text{sd } K)$ and R_{rel} in the reverse direction, with $R_{\text{rel}}S_{\text{rel}} = 1$ and $S_{\text{rel}}R_{\text{rel}} \simeq 1$, all having propagation $O_m(\eta)$ when cells are labeled by points of their geometric supports.

Proof. Use the dual orientations induced by the manifold and simplex orientations; the displayed incidence formula follows. On an interior codimension-one simplex, the augmentation of its dual boundary is $(-1)^m \sum_{\rho \supset \sigma} \epsilon_\rho [\rho : \sigma] = 0$, by the relative fundamental-chain equation. Thus geometric augmentation is a chain map, including the signed zero-cell convention. All cofaces of a nonboundary simplex are nonboundary, giving the subcomplex. If σ is interior, $J_\sigma = D\sigma$ is a ball. If σ is boundary, its link is an l -ball, $l = m - \dim \sigma - 1$, with boundary the link in ∂A . Nonboundary cofaces correspond to nonboundary faces λ of this link. With join orientations the map

$$\lambda^* \mapsto (-1)^{r(l+1)} D(\sigma * \lambda)$$

identifies its relative shifted cochain complex in degree r with $C_r(J_\sigma)$. Hence $H_r(J_\sigma) = \mathbb{Q}$ for $r = 0$ and zero otherwise; augmentation is nonzero on a top-coface dual vertex and identifies H_0 with $\mathbb{Q}$. For a barycentric flag use J_{σ_0} , its minimum-face carrier. These shrink on deleting the minimum and are unchanged on deleting other faces. The carrier lemma constructs R_{rel} ; take S_{rel} to be cellular subdivision. Augmentation fixes the composite on zero cells; on a positive-dimensional $D\sigma$ the composite lies in that dual ball and equals the identity by induction and absence of top-dimensional cycles there. The other composite and the identity are carried in $\text{sd}(\text{st}_K \sigma_0)$, so the carrier lemma supplies their homotopy. Closed stars are contractible and have diameter at most 2η , giving every asserted bound independently of valence and rank. $\square$

Lemma 7.5 (Controlled cap-block comparison). *In the setting of Lemma 7.4, put $R = C_*(K, \partial K; \mathbb{Q})$ and write δ for its shifted-dual differential $(-1)^r d^*$. Let $F : R^{m-*} \rightarrow C_*(K)$ be a*

chain map carried, on σ^* , in the closed star $\text{st}_K \sigma$. Suppose $\varepsilon F(\rho^*) = \epsilon_\rho$ on each top simplex. Then F has a chain homotopy inverse and both inverse homotopies of propagation $O_m(\eta)$. More precisely, it is carried-homotopic to F_0 , with homotopy inverse B , where

$$F_0 = R_K S_{\text{rel}} \mathcal{J}, \quad B = \mathcal{J}^{-1} R_{\text{rel}} S_K, \quad \mathcal{J}_r(\sigma^*) = (-1)^{r(m+1)} D\sigma,$$

where S_K is subdivision and R_K a simplex-carried inverse. Fixed continuous relabelings give bounds tending to zero with η .

Proof. The incidence formula makes $\mathcal{J}$ a chain isomorphism. The carrier lemma on maximum-face carriers constructs R_K , with $R_K S_K = 1$ by top-chain uniqueness in a simplex; it also gives $S_K R_K \simeq 1$ in carriers $C_*(\text{sd } \sigma_r)$ on flags with maximum σ_r . Together with the preceding lemma these identities prove $F_0 B \simeq 1$ and $B F_0 \simeq 1$ by a fixed number of compositions. Both F, F_0 have the same top augmentations and closed-star carriers. The dual differential uses cofaces, and $\text{st}_K \tau \subset \text{st}_K \sigma$ for $\tau \supset \sigma$, so the carrier lemma gives $F - F_0 = dH + H\delta$. Add HB and BH to the inverse homotopies for F_0 . All carriers have diameter at most 2η ; fixed compositions give $C_m \eta$ and relabeling gives $\omega_c(C_m \eta)$. $\square$

Corollary 7.6 (A common simplicial diagonal). *In the setting of Lemma 7.4, let j be the boundary inclusion. Use one ordered Alexander–Whitney diagonal Δ on K and its subcomplexes. Choose oriented fundamental chains satisfying $dz_A = jz_{\partial A}$, and put $t_A = \frac{1}{2}(1+T)\Delta z_A$ and $t_{\partial A} = \frac{1}{2}(1+T)\Delta z_{\partial A}$. Then $dt_A = j^{\otimes 2} t_{\partial A}$ exactly. The right slant of t_A on relative cochains is a chain equivalence with inverse data of propagation $O_m(\eta)$, also when the boundary is empty.*

Proof. The diagonal restricts to subcomplexes. Its boundary equation gives

$$d\phi_{A,r} - (-1)^r \phi_{A,r-1} d^* = j\phi_{\partial A,r-1} j^*.$$

The right side vanishes on relative cochains. Each diagonal term consists of faces of one simplex, so slant on σ^* has closed coface-star support. On a top simplex it is $\epsilon_\rho([v_0] + [v_m])/2$, of augmentation ϵ_ρ . Apply the cap–block comparison. $\square$

8. GEOMETRIC DATA AND QUOTIENT REPRESENTATIVES

Suppose that $\mathbb{Z}_p$ acts effectively on a connected boundaryless n -manifold, $n \geq 1$, and put $H_j = p^j \mathbb{Z}_p$. First note that, for any compact group K acting on a connected manifold, every countable union of compact sets lies in a K -invariant connected second-countable open submanifold. Indeed, every compact set lies in a relatively compact connected open set covered by finitely many charts: connect a finite chart cover by finitely many paths. Choose such sets V_j inductively with $K\overline{V_j} \subset V_{j+1}$, including one prescribed compact set at each stage; their union has the required properties. For $K = \mathbb{Z}_p$, include a point moved by p^j for every $j \geq 0$. The restriction remains effective, so we henceforth work on this second-countable open submanifold.

Lemma 8.1. *There is a point with trivial stabilizer.*

Proof. Every nonzero closed subgroup of $\mathbb{Z}_p$ is an H_j : an element of minimum valuation j has integer multiples dense in H_j . If all stabilizers were nonzero, the discrete subgroup $\mathbb{Z}$ would have finite point orbits. By [Rob75, Theorem 2] its image would be finite. Some $m > 0$ would act trivially; the closed kernel would then contain $\overline{m\mathbb{Z}} = H_{v_p(m)}$, contradicting effectivity. $\square$

Center a chart at this point, with image containing $\overline{B(0,4)}$. Joint continuity gives $H = H_j$ and $0 < D < 1/64$ such that

$$(8.1) \quad |hx - x| \leq D \quad (h \in H, x \in \overline{B(0,2)}).$$

The invariant saturation $\Omega = HB(0, 2)$ lies in $B(0, 2 + D)$ and has H -displacements at most $2D$. Its coordinate average with respect to normalized Haar measure μ

$$(8.2) \quad F_0(x) = \int_H hx \, d\mu(h)$$

is continuous, invariant, and satisfies

$$(8.3) \quad |F_0(x) - x| \leq 2D, \quad F_0(x) = x \text{ on } \text{Fix}(H).$$

Choose $0 < \rho < \rho_+ < 1/4$ with $\overline{B(0, \rho_+)} \cap \text{Fix}(H) = \emptyset$. Then

$$(8.4) \quad Z^+ = F_0^{-1}(\overline{B(0, \rho_+)})$$

is compact, invariant, and disjoint from $\text{Fix}(H)$: it is a closed preimage contained in $B(0, \rho_+ + 2D) \Subset \Omega$. Choose $s_\rho : \mathbb{R}^n \rightarrow S^n$ restricting to an orientation-preserving homeomorphism from $B(0, \rho)$ to $S^n \setminus \{\infty\}$, and constant ∞ elsewhere. Define

$$(8.5) \quad f = s_\rho F_0 \text{ on } \Omega, \quad f = \infty \text{ elsewhere.}$$

Its nonconstant locus lies in $B(0, \rho + 2D)$, so it extends to the compactified coordinate sphere. The homotopy

$$(8.6) \quad s_\rho((1 - v)F_0(x) + vx), \quad 0 \leq v \leq 1,$$

extended by ∞ , has the same compact support bound. Hence f has degree one and Z^+ is nonempty.

Put $H' = H_{j+1}$. Every proper closed subgroup of H lies in H' , so $C_p = H/H'$ acts freely on the compact metric space Z^+/H' . Its orbit map is a covering: a neighborhood can be shrunk to have disjoint translates. A finite trivializing cover and a subordinate partition of unity give, by multiplying sheet coordinates by the square roots of the partition functions, an equivariant detector

$$(8.7) \quad \tilde{\rho} : Z^+ \longrightarrow T = S^{2m-1} \subset \mathbb{C}^m, \quad \tilde{\rho}(hx) = \chi(h)\tilde{\rho}(x),$$

where $\chi : H \rightarrow H/H' = C_p \subset S^1$ is the quotient character. Scalar multiplication makes T a fixed compact free isometric C_p -space. Choose a round target metric with $y = s_\rho(0)$ antipodal to ∞ , and fix

$$(8.8) \quad 0 < r < R < t < u < d(y, \infty), \quad Z_0 = \{z : d(y, z) \geq R\}, \quad Z = T \times Z_0.$$

For every $v < d(y, \infty)$, the compact invariant set $\mathcal{K}_v = f^{-1}(\overline{B(y, v)})$ lies in $\text{int } Z^+$. Only now choose

$$(8.9) \quad K_i = H_{\ell_i} \subset H', \quad \ell_i \rightarrow \infty, \quad G_i = H/K_i, \quad P_i = H'/K_i.$$

Write χ also for the induced character on each G_i . The K_i -displacements on $\overline{HB(0, 2)}$ tend uniformly to zero; T , the detector, and all buffers remain fixed.

9. COMMON CHAIN MODELS

Put $W = S^n \times \mathbb{CP}^2$, $N = n + 4$. The sphere factor is the compactified coordinate space. Fix a product metric, a finite smooth atlas with uniform chart buffers and Lipschitz constants, and a common continuity modulus ω for the compact map families below. Choose a combinatorial triangulation compatible with the later finite PL cutting flag and barycentric refinements Y_i of mesh $\eta_i \rightarrow 0$. Write $D_i = C_*(Y_i; \mathbb{Q})$ and let z_i be its oriented fundamental cycle; bases are simplices labeled at their barycenters. All control maps ignore $\mathbb{CP}^2$, and support estimates concern whole closed simplices.

Lemma 9.1. *The D_i have strict symmetric cap equivalences $\phi_i : D_i^{N-*} \rightarrow D_i$, inverses $b_i : D_i \rightarrow D_i^{N-*}$, and both inverse homotopies of propagation $O_N(\eta_i)$. An equicontinuous family of self-maps has chain approximations on these unchanged complexes, with uniform full-simplex graph error*

$$(9.1) \quad \mu_i = O(\eta_i + \omega(C\eta_i)) \longrightarrow 0.$$

Proof. The cap assertion is [corollary 7.6](#) with empty boundary. Subdivide the source enough for the target open-star cover and the common modulus, and take $a_h = t_h S$ by simplicial approximation [[Hat02](#), Theorem 2C.1]. The identity approximation is exactly 1 using the last-vertex map. Each image simplex lies within one target mesh of h of a point of its original source simplex; equicontinuity gives [\(9.1\)](#). $\square$

Choose a continuous cutoff $0 \leq \theta \leq 1$, equal to 1 on $B(0, 1)$ and 0 off $B(0, 3/2)$, and complete each $h \in H$ to a sphere map

$$(9.2) \quad \hat{h}(x) = x + \theta(x)(hx - x),$$

using the identity elsewhere. These maps are equicontinuous and degree one, by scaling θ , and

$$(9.3) \quad f\hat{h} = f :$$

inside $B(0, 1)$ use invariance; outside it both points miss the nonconstant locus. Choose representatives h_g of G_i , $h_1 = 1$, and write $h_g h_h = h_{gh} k_{g,h}$ with $k_{g,h} \in K_i$. The tracks

$$(9.4) \quad L_{g,h}(x, v) = (1 - v)\hat{h}_g \hat{h}_h(x) + v\hat{h}_{gh}(x)$$

are constant at infinity. For $|x| \leq 1/2$ they lie within $\beta_i \rightarrow 0$ of $h_{gh}x$, uniformly; for $|x| \geq 1/2$ they lie within $2D$ of x and have f -value ∞ .

A sequence of matrices has graph type g if its global f -propagation tends to zero and, for every fixed $v < d(y, \infty)$, its entries $x \rightarrow x'$ with $x \in \mathcal{K}_v$ satisfy $d_T(\tilde{\rho}(x'), \chi(g)\tilde{\rho}(x)) \rightarrow 0$ uniformly. Their target labels are defined eventually. A larger fixed $v' > v$ gives the same estimates for transposes and products of bounded length: the transpose has type g^{-1} , and graph types multiply. Apply [lemma 9.1](#) to $\hat{h}_g \times 1$ to obtain $a_{g,i}$, $a_{1,i} = 1$. Equation [\(9.3\)](#), detector equivariance, and the graph error give type g . Degree one and the absence of $(N + 1)$ -chains give

$$(9.5) \quad a_{g,i} z_i = z_i.$$

We now suppress the index i .

Lemma 9.2. *On the same complexes there are homotopies*

$$(9.6) \quad a_g a_h - a_{gh} = dB_{g,h} + B_{g,h}d$$

of graph type gh , uniformly in i, g, h .

Proof. Induct over simplex dimension, filling the residual cycle $\gamma_\sigma = a_g a_h \sigma - a_{gh} \sigma - B(\partial\sigma)$ using [lemma 7.3](#); in degree zero it has augmentation zero. Take first-coordinate convex hulls of $L_{g,h}(|\sigma| \times [0, 1])$, and small coordinate-convex carriers for the second projection. In a product chart, take convex hulls including full endpoint chains and already fixed face fillings. With $\delta_i = \eta_i + \mu_i + \omega(C\eta_i) + \omega(C\mu_i) + \beta_i \rightarrow 0$, the full supports stay within widths w_r of the nested track hulls and second-coordinate projections, where $w_{-1} = 0$, $w_r \leq C(w_{r-1} + \delta_i)$: coordinate changes cost a fixed factor and filling adds one target mesh. Thus $w_r \rightarrow 0$ uniformly for $r \leq N$, independently of all ranks and auxiliary subdivisions. These convex carriers lie in one product chart; near infinity use small charts in the identity region. The top residual vanishes by the same filling lemma, so no extra top chain is required.

For a source barycenter with $|x| \leq 3/8$, every filling lies within $o(1)$ of $h_{gh}x$. Equivariance of the detector on each $\mathcal{K}_v$ gives type gh . For $|x| \geq 3/8$ its support is within $2D + o(1)$ of x ; the fixed margin $3/8 - \rho - 4D > 1/16$ makes its f -value ∞ . Uniform continuity finishes both control estimates. $\square$

The same filling recursion, now on the based source D^{N-*} , gives a local primitive U_g with

$$(9.7) \quad a_g \phi a_g^* - \phi = dU_g + U_g d_{D^{N-*}}.$$

Indeed, an entry of a_g^* selects a source cell mapping near its given target cell; ϕ moves within $O(\eta_i)$, and a_g maps back near that target cell. Thus both endpoint maps are local with error $O(\mu_i + \omega(C\eta_i) + \eta_i) \rightarrow 0$. Their degree-zero augmentations agree, since simplicial approximations preserve geometric augmentation ($\varepsilon a_g = \varepsilon$): for $\alpha \in D_N^*$, $\varepsilon \phi(\alpha) = \alpha(z_i)$, while $\varepsilon a_g \phi a_g^*(\alpha) = \alpha(a_g z_i) = \alpha(z_i)$. For σ^* take a chart-convex hull of its endpoint supports and the carriers and fillings already assigned to cofaces occurring in the shifted dual differential, enlarging by one mesh after filling. These geometric carriers are nested under cofaces. Every coface meets σ , so the same width induction keeps them within $o(1)$ of σ . The top residual again vanishes. Put $a = a_g$, $b' = a_{g-1}$. For $b'a - 1 = dT + Td$ set $K_r = (-1)^{r+1} T_{N-r-1}^*$. The identity $a\phi - \phi b'^* = a\phi(1 - a^* b'^*) + (a\phi a^* - \phi) b'^*$ yields

$$(9.8) \quad V_g = -a\phi K + U_g b'^*, \quad dV_g + V_g d_{D^{N-*}} = a_g \phi - \phi a_{g-1}^*.$$

Since T, K, U_g have type 1 and b'^* type g , V_g has type g . Fix one $\epsilon_i \rightarrow 0$ bounding all these f and detector errors, cell oscillations, and local duality propagations on the fixed buffers.

10. COMPRESSION AND EXTERIOR CONTROL

Keep the fixed detector neighborhood and $R < t < u$. All primitive local and graph errors below are bounded uniformly by $\epsilon_i \rightarrow 0$; cell oscillations of f are bounded by $\kappa_i \rightarrow 0$. Let $F_i \subset D_i$ be spanned by simplices whose closed simplices miss $f^{-1}(\overline{B(y, t)})$. This is a subcomplex, since faces of a missing simplex also miss that set. Put

$$(10.1) \quad C = D/F, \quad \pi : D \rightarrow C, \quad \sigma : C \rightarrow D, \quad Q = 1 - \sigma\pi, \quad \hat{d} = \pi d\sigma,$$

where σ is the degreewise based section. Thus $\pi dQ = 0$. Call the simplices surviving in C retained and those in F far. Retained labels have sphere radius $d(y, f(x))$ at most $t + \kappa_i$; far labels have radius greater than t . Only retained matrices and the restricted witnesses below need detector labels; the following computations on D_i are raw matrix computations.

Lemma 10.1 (Exterior band). *Consider words in the preceding primitive maps and homotopies, their shifted duals, and the based projections and sections. A retained-to-retained word of fixed length at most L containing Q or Q^* factors through a based module labeled in*

$$(10.2) \quad t - L\epsilon_i \leq d(y, f(x)) \leq t + \kappa_i + L\epsilon_i.$$

Local words have controlled scalar factors; graph words have controlled equivariant factors after compensating sheet translations. Their cut objects are eventually supported in the fixed exterior Z .

Proof. At a far projection restrict to basis elements reached from retained sources and contributing to retained targets, and cut the matrix product there. This is an exact factorization. The far label has radius greater than t , and a retained endpoint and at most L sphere errors give the upper bound in (10.2). Every contributing intermediate label also has radius at most this upper bound, eventually less than u ; all required detector labels therefore exist in the fixed buffered domain. The cut lies outside R . Local propagation is at most $L\epsilon_i$. For graph words use all sheets, label (x, s) by $(\chi(s)\tilde{\rho}(x), f(x))$, and accompany a graph- g primitive by $s \mapsto sg^{-1}$.

Its detector displacement cancels because $\chi(sg^{-1})\chi(g) = \chi(s)$. Apply this separately to the prefix and suffix. Their intermediate basis is a free G_i -set and both factors are controlled. Sums factor through direct sums of the cut objects; their propagation is bounded by the maximum summand bound. $\square$

Let $b_i : D_i \rightarrow D_i^{N-*}$ be the local cap inverse supplied by lemma 9.1, and put

$$(10.3) \quad A_g = \pi a_g \sigma, \quad \widehat{\phi} = \pi \phi \pi^*, \quad \widehat{b} = \sigma^* b \sigma.$$

For a continuous $c : W \rightarrow S^n$, denote by $\mathfrak{m}_c \in L_N(\mathcal{B}_{1,Z_0}(S^n))$ the scalar exterior class of $(D_i, \phi_i/2)$ with labels c .

Proposition 10.2. *The compressed complex is Poincaré in $\mathcal{B}_{1,Z}(T \times S^n)$, with quadratic structure $\widehat{\phi}/2$. Its group and adjoint homotopies have the stated uniform graph control modulo controlled exterior witnesses, and $A_1 = 1$. After forgetting T , its class is $\mathfrak{m}_f$.*

Proof. For a degree-compatible raw operator put $\widehat{\alpha} = \pi \alpha \sigma$. Insertion of $\sigma \pi = 1 - Q$ gives

$$(10.4) \quad \widehat{\alpha} \widehat{\beta} - \widehat{\alpha} \widehat{\beta} = -\pi \alpha Q \beta \sigma,$$

$$(10.5) \quad \widehat{d} A_g - A_g \widehat{d} = \pi a_g Q d \sigma,$$

$$(10.6) \quad A_g A_h - A_{gh} - \widehat{d} \widehat{B} - \widehat{B} \widehat{d} = -\pi a_g Q a_h \sigma + \pi B_{g,h} Q d \sigma, \quad \widehat{B} = \pi B_{g,h} \sigma.$$

The band lemma kills all defects after compensation. The map $\widehat{\phi}$ is a strict symmetric chain map because π is a chain map. For its inverse, let $\delta, \widehat{\delta}$ be the shifted-dual differentials of D, C , and put $e = d\sigma - \sigma \widehat{d} = Q d \sigma$ and $e^\vee = \widehat{\delta} \sigma^* - \sigma^* \delta$, so $(e^\vee)_r = (-1)^{r+1} e_{N-r+1}^*$ factors through Q^* . Then

$$\widehat{\delta} \widehat{b} - \widehat{b} \widehat{d} = e^\vee b \sigma + \sigma^* b e.$$

Compress the local inverse homotopies $\phi b - 1 = dH + Hd$ and $b\phi - 1 = \delta H' + H'\delta$:

$$\widehat{\phi} \widehat{b} - 1 = \widehat{d}(\pi H \sigma) + (\pi H \sigma) \widehat{d} + \pi H e - \pi \phi Q^* b \sigma,$$

$$\widehat{b} \widehat{\phi} - 1 = \widehat{\delta}(\sigma^* H' \pi^*) + (\sigma^* H' \pi^*) \widehat{\delta} - e^\vee H' \pi^* - \sigma^* b Q \phi \pi^*.$$

All residuals are local bounded-length words containing Q or Q^* . Similarly, for $\widehat{V}_g = \pi V_g \pi^*$,

$$A_g \widehat{\phi} - \widehat{\phi} A_{g-1}^* - \widehat{d} \widehat{V}_g - \widehat{V}_g \widehat{d} = -\pi a_g Q \phi \pi^* + \pi \phi Q^* a_{g-1}^* \pi^*.$$

These residuals have graph type g and vanish after compensation by the same lemma. Finally $A_1 = \pi \sigma = 1$ and

$$(10.7) \quad A_g \pi z_i - \pi z_i = -\pi a_g Q z_i,$$

whose output misses $\overline{B(y, R)}$ eventually. After forgetting T , the entire D_i is labeled by f and F_i is Z_0 -supported. Consequently Q and e vanish in the scalar exterior quotient, so π, σ are inverse chain equivalences. Transport of $\phi/2$ along π is exactly $\widehat{\phi}/2$. $\square$

11. THE GRAPH CORNER AND ITS SCALAR IMAGE

Give $\mathcal{C}_i = C_i \otimes \mathbb{Q}[G_i]$ the free left sheet action and labels as follows. Let x_c be the first coordinate of the barycenter of a retained simplex c . Then

$$(11.1) \quad \ell(c \otimes s) = (\chi(s) \tilde{\rho}(x_c), f(x_c)).$$

Copying local scalar matrices sheetwise preserves duality, exterior support, and propagation. Set

$$(11.2) \quad \Phi = \hat{\phi} \otimes 1, \quad E_i = q_i^{-1} \sum_{g \in G_i} A_{g,i} \otimes R_{g^{-1}}, \quad q_i = |G_i|, \quad q = (q_i).$$

The detector cancellation in the band lemma applies to the maps and all product and adjoint homotopies. In particular $q_i^{-2} \sum_{g,h} \hat{B}_{g,h} \otimes R_{(gh)^{-1}}$ is a controlled product homotopy modulo the direct sum of the exterior witnesses. Uniform control, rather than a bound on the number of summands, gives

$$(11.3) \quad E^2 \simeq E, \quad E\Phi \simeq \Phi E^* \quad \text{in } K^b(\mathcal{B}_{\mathbf{G},Z}(T \times S^n)).$$

By lemma 5.1, split E in the Karoubi-completed homotopy category:

$$(11.4) \quad D \xrightarrow{L} \mathcal{C} \xrightarrow{r} D, \quad \phi_D = q^{-1} r \Phi r^*.$$

Its quadratic structure is $\phi_D/2$. Ultimate completion invariance gives

$$(11.5) \quad \mathfrak{d} \in L_{n+4}(\mathcal{B}_{\mathbf{G},Z}(T \times S^n)).$$

Let U forget equivariance and T , retaining sphere control. Every A_g and every relation homotopy is uniformly small in this coordinate. Scalar normalization (lemma 5.2) and the last assertion of proposition 10.2 give

$$(11.6) \quad U_* \mathfrak{d} = \mathfrak{m}_f \quad \text{in } L_{n+4}(\mathcal{B}_{\mathbf{1},Z_0}(S^n)).$$

12. EVALUATION OF THE ITERATED BOUNDARY

Put $N = n + 4$ and $W = S^n \times \mathbb{CP}^2$, with product orientation; all sphere control ignores $\mathbb{CP}^2$. Fix

$$A = T \times (S^n \setminus B(y, r)), \quad B = T \times \overline{B(y, r)}, \quad Y = T \times S_r(y).$$

Here $S_r(y) = \partial B(y, r)$. The exterior $Z \subset A$ is positively separated from B . Apply proposition 6.1, relabel its target onto Y , and successively cut the sphere interfaces into two fixed hemispheres. After n boundaries the support is $T \times S^0$; project onto one component. This gives

$$(12.1) \quad \Delta_T : L_N(\mathcal{B}_{\mathbf{G},Z}(T \times S^n)) \longrightarrow L_4(\mathcal{A}_{\mathbf{G}}(T)).$$

Every cut, relabeling, orientation, and component is fixed before the index varies. The same choices give Δ_1 without T and equivariance, with target $L_4(\mathcal{V}_1)$. Forgetting equivariance and T , while retaining sphere control, gives

$$(12.2) \quad U_* \Delta_T = \Delta_1 U_*.$$

These boundaries also commute with $\tau \otimes (-)$, by localization naturality.

Lemma 12.1 (Controlled relative duality). *Let A be a compact oriented combinatorial m -manifold with boundary. Use a finite combinatorial triangulation, with boundary a subcomplex, and one ordered diagonal. Write $C = C_*(\partial A; \mathbb{Q})$, $D = C_*(A; \mathbb{Q})$, $j : C \rightarrow D$, and $J = \text{Cone}(j)$. Fundamental chains $dz_A = jz_{\partial A}$ give invariant tensors*

$$t_D = \text{Sym } \Delta z_A, \quad t_C = \text{Sym } \Delta z_{\partial A}, \quad dt_D = j^{\otimes 2} t_C, \quad dt_C = 0.$$

Write ϕ_D, ϕ_C for their right slants, which give the relative duality map

$$(12.3) \quad \Psi_r(\alpha, \beta) = \phi_{D,r} \alpha + (-1)^{r+1} j \phi_{C,r} \beta : J_r^{m-*} \longrightarrow D_r.$$

For mesh η , this map, an inverse, and both inverse homotopies have geometric propagation $O_m(\eta)$. Any fixed continuous labeling therefore gives an asymptotically controlled Poincaré pair, whose full quadratic structure is $(t_D, t_C)/2$.

Proof. The tensor equation and

$$d_{J^{m-*},r} = (-1)^r \begin{pmatrix} d_D^* & 0 \\ j^* & -d_C^* \end{pmatrix}$$

make (12.3) a chain map: its two relative-boundary terms cancel. Let $\mathcal{R} = D/C$ and $q : J \rightarrow \mathcal{R}$ be the quotient. For the based section $s : \mathcal{R} \rightarrow D$, write $je = d_D s - s\bar{d}$. Then $\tau = (s, -e)$ is a chain section of q ; $\ker q = \text{Cone}(1_C)$ has contraction $k(x, c) = (0, x)$. Thus

$$(12.4) \quad dh + hd = 1 - \tau q, \quad h = k(1 - \tau q).$$

All these maps use only a simplex and its faces. By corollary 7.6, $F = \Psi q^*$ has a local cap inverse b and local inverse homotopies. Consequently q^*b is an inverse of Ψ : use $\Psi \simeq F\tau^*$ and $q^*\tau^* \simeq 1$, obtained by shifted dualizing (12.4). Only finitely many local compositions are used, so their propagation is $O_m(\eta)$, independently of rank and valence. Apply the labeling's continuity modulus. Taking halves gives the full relative quadratic structure by §5.1. $\square$

Lemma 12.2 (Controlled unions). *A cut of a closed oriented combinatorial manifold W into two manifolds along a common boundary, using one triangulation and diagonal, gives controlled Poincaré pairs whose union has the closed manifold's full rational quadratic structure.*

Proof. Write the chains as A, B, P , with inclusions j_A, j_B , and choose $z_W = z_A + z_B$, $dz_A = j_A z_P$, $dz_B = -j_B z_P$. On $V = \text{Cone}((j_A, -j_B))$, let e_A, e_B include the two tops, put $a = e_A j_A$, $b = e_B j_B$, and let s include the suspended interface. Then $ds + sd = a - b$. With tensor Koszul signs, set

$$K = s \otimes a + b \otimes s, \quad K_{\text{sym}} = (K + T_V K T_P)/2, \\ t_V = e_A^{\otimes 2} t_A + e_B^{\otimes 2} t_B - K_{\text{sym}} t_P.$$

The identity $dK + Kd = a^{\otimes 2} - b^{\otimes 2}$ proves $dt_V = 0$. This is the Poincaré union of [Ran80I, §3]. The collapse $p : V \rightarrow C_*(W; \mathbb{Q})$ has kernel $\text{Cone}(1_P)$ and satisfies

$$(12.5) \quad (p \otimes p)t_V = t_W.$$

Use the kernel's standard contraction k . For a based section r of p , $\ell = r - k(dr - rd)$ is a chain section and $k(1 - \ell p)$ contracts $1 - \ell p$. These maps, the interface correction, and the cap inverse of W are local. Hence p is a controlled chain equivalence identifying the full symmetric structures; their halves identify the quadratic structures. The propagation bounds are $O_m(\eta)$ before labeling and tend to zero after any fixed continuous labeling. $\square$

Lemma 12.3 (Scalar control homotopies). *Let Y_i be the fixed ordered combinatorial triangulations of a compact closed oriented PL m -manifold W , of mesh $\eta_i \rightarrow 0$, and let $c : W \times [0, 1] \rightarrow X$ be a fixed continuous map to a compact metric space. The complexes $C_*(Y_i; \mathbb{Q})$ with their prescribed fundamental structures and endpoint labels c_0, c_1 are rational quadratic Poincaré cobordant. Their L -classes remain equal in every fixed exterior quotient.*

Proof. Triangulate the product prisms with time mesh $\lambda_i \rightarrow 0$, keeping both endpoint triangulations, vertex orders, and fundamental structures exactly those of Y_i . Choose one vertex order restricting to the prescribed endpoint orders and use its ordered diagonal. The cylinder fundamental chain

$$Z_i = (-1)^m z_i \times [0, 1], \quad dZ_i = j_1 z_i - j_0 z_i$$

gives the exact relative tensor equation with boundary $t_1 - t_0$. Label simplex barycenters by c . By lemma 12.1, the pair's duality, inverse, and inverse homotopies have propagation at most $\omega_c(C_m(\eta_i + \lambda_i)) \rightarrow 0$ and degree range $[0, m + 1]$. Taking halves gives the endpoint quadratic structures. Apply the exterior quotient functor to this completed cylinder; no condition on the

homotopy's behavior on the exterior is required. See [Ran80II, Propositions 6.1–6.3] for the relative symmetric construction. $\square$

Proposition 12.4 (Boundary evaluation). *For one constant $\varepsilon \in \{1, -1\}$ determined by the fixed orientation and component choices,*

$$(12.6) \quad \Delta_1 \mathbf{m}_f = \varepsilon[C_*(\mathbb{CP}^2; \mathbb{Q}), \phi_{\mathbb{CP}^2}/2], \quad \sigma_{\text{tail}}(\Delta_1 \mathbf{m}_f) = [(\varepsilon, \varepsilon, \dots)].$$

Proof. The fixed scalar homotopy from f to s_ρ and lemma 12.3 give

$$(12.7) \quad \mathbf{m}_f = \mathbf{m}_{s_\rho}.$$

The degree-one sphere map s_ρ is homotopic to an orientation-preserving homeomorphism g carrying the fixed source PL disk D and hemisphere flag, chosen before the Y_i , to the target ball and flag; degree classifies sphere maps for $n \geq 1$ [Hat02, Corollary 4.25]. Keep every target cut fixed and apply lemma 12.3 again. The fixed Y_i already respect $D \times \mathbb{CP}^2$, its complement, and the finite flag; retain these same complexes and ordered diagonals. By lemmas 12.1 and 12.2, these give compatible controlled quadratic pairs supported on the designated target sides. Their interface is $\partial D \times \mathbb{CP}^2$; it is separated from Z_0 . By lemma 6.2, the first boundary is this interface, and each subsequent boundary is its next equator times $\mathbb{CP}^2$. After n boundaries, projecting onto one component retains exactly one copy of $\mathbb{CP}^2$, with sign ε independent of the index. At point control, subdivision identifies it with the constant model. The complex orientation gives middle form $\langle 1 \rangle$; its half-unit quadratic lift has the same symmetrized form and signature 1. This proves (12.6). $\square$

13. THE CONTRADICTION

Proposition 13.1 (Signature-one realization). *An effective continuous $\mathbb{Z}_p$ -action on a connected positive-dimensional manifold without boundary gives a fixed compact free C_p -space T , finite cyclic groups G_i with index- p subgroups P_i , and $\alpha \in L_4(\mathcal{A}_G(T))$ with*

$$(13.1) \quad \sigma_{\text{tail}}(\alpha) = [(1, 1, \dots)].$$

Proof. Use the fixed geometric data and the compensated normalized corner $\mathfrak{d}$. Naturality of the iterated boundary and the scalar comparison give

$$U_* \Delta_T \mathfrak{d} = \Delta_1 \mathbf{m}_f = \varepsilon[C_*(\mathbb{CP}^2; \mathbb{Q}), \phi_{\mathbb{CP}^2}/2], \quad \varepsilon \in \{1, -1\}.$$

Set $\alpha = \varepsilon \Delta_T \mathfrak{d}$. The coefficient-of-identity trace identifies the ordinary signature of each equivariant form with that of its scalar image, so its tail signature is the constant sequence 1. $\square$

Proof of theorem 1.1. A connected zero-dimensional manifold is a point. For a connected manifold with boundary, its interior is dense and connected: each boundary half-ball has connected interior and meets just one interior component, so different components would separate the manifold. Homeomorphisms preserve the interior by local homology, and fixing it fixes its closure. Restriction to the interior therefore preserves effectivity.

If the boundaryless action were effective, proposition 13.1 would give a class whose signatures are eventually 1. Theorem 4.3 makes the same signatures eventually divisible by p . A common tail gives $1 \in p\mathbb{Z}$, a contradiction. $\square$

Proof of corollary 1.2. The point case is immediate; otherwise restrict to the interior, and write G for the acting group. Let K be a compact subgroup of G . The invariant-neighborhood construction in §8 gives a nonempty K -invariant connected second-countable open submanifold U . Put $N_U = \ker(K \curvearrowright U)$. The compact quotient K/N_U embeds in the metrizable group $\text{Homeo}(U)$, so it is second countable. The standard compact p -adic reduction [Pa19] and

[theorem 1.1](#) make K/N_U a Lie group. If $U \subset V$ is another such invariant submanifold, the compact Lie group $N_U/N_V \subset K/N_V$ fixes U , so Newman’s theorem on V [[Pa19](#), Theorem (Newman)] gives $N_U = N_V$. Every point lies in such a V containing U , by the same construction. Thus N_U fixes the whole manifold and is trivial. Hence every compact subgroup of G is Lie.

Gleason–Yamabe gives an open subgroup $G_0 \subset G$ with a compact normal subgroup K and Lie quotient G_0/K [[Gle52](#), [Yam53](#)]. Both K and G_0/K have no small subgroups (NSS). Intersect the inverse image of an NSS neighborhood in G_0/K with an identity neighborhood whose intersection with K contains no nontrivial subgroup. The intersection contains no nontrivial subgroup, so G_0 , and hence G , has no small subgroups. Therefore G is Lie with its original topology [[Gle52](#), [Yam53](#)]. $\square$

AI DISCLOSURE

Various large language models (LLMs), including from Anthropic, OpenAI, and open-source projects, were used under human guidance at every stage and for nearly all tasks, from understanding, planning, and ideation to research, preparation, and writing. A custom LLM orchestration framework was used for research. Critical construction steps were developed jointly by Opus 5.5 and GPT 6.1. A formalization of [theorem 1.1](#) and [corollary 1.2](#) under second countability assumptions, completed in Lean 4 by Opus 5.5 and verified with Comparator and SafeVerify, is available at https://github.com/adbrw/HS_proof. This work was carried out as an independent research project.

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