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ONE OF $\beta(4), \beta(6), \ldots, \beta(36)$ IS IRRATIONAL

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Abstract. Let $\beta(s) = \sum_{m\geq 0}(-1)^m(2m+1)^{-s}$ be Dirichlet's beta function. We prove that at least one of the seventeen numbers $\beta(4), \beta(6), \ldots, \beta(36)$ is irrational. All previous explicit results of this kind that we know of include $\beta(2)$, Catalan's constant; our statement does not involve $\beta(2)$ at all. The proof uses the linear forms $L_n = \sum_{t\geq 0}(-1)^t R_n''(t)$ attached to a very-well-poised rational function R_n with triple integer zeros and half-integer poles. The second derivative, a β-analogue of a device of Rivoal, removes $\beta(2)$, and a symmetry removes the odd values. The zeros of R_n run through the region of its poles, which makes every prime $p > n$ harmless. The common denominators are controlled by a p-adic analysis and an arithmetic integral that we bound with an exact rational certificate. The size of L_n is bounded by an elementary estimate on the real axis, and its nonvanishing for infinitely many n follows from a saddle-point analysis along a certified contour. The whole proof, including every numerical certificate, has been formalised in Lean 4 with Mathlib and checked by the Lean kernel, using only Lean's three standard axioms.

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1. Introduction

For $s > 0$ let

$$\beta(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s}$$

be Dirichlet's beta function, the L-function of the non-principal character modulo 4.

Theorem 1.1. *At least one of the seventeen numbers* $\beta(4), \beta(6), \beta(8), \ldots,$ $\beta(36)$ *is irrational.*

1.1. **Context.** The values of β at odd integers are classical: if E_m denotes the Euler numbers, defined by $\operatorname{sech} x = \sum_{m\geq 0} E_m x^m/m!$ (so $E_2 = -1$), then $\beta(2k+1) = (-1)^k E_{2k}\pi^{2k+1}/(4^{k+1}(2k)!)$. For example $\beta(1) = \pi/4$ and $\beta(3) = \pi^3/32$. So the odd values are nonzero rational multiples of odd powers of π, and they are transcendental by Lindemann's theorem. Nothing comparable is known for the even values. The first one, $\beta(2)$, is Catalan's constant. Our argument never uses the arithmetic nature of $\beta(2)$.

The explicit irrationality results for even beta values that we know of all include $\beta(2)$:

- Rivoal and Zudilin [11, Thm. 2] proved that at least one of $\beta(2)$, $\beta(4), \ldots, \beta(14)$ is irrational;
- Zudilin [15, Thm. 1] proved the same for $\beta(2), \ldots, \beta(12)$;
- Lai and Zhou [8, Thm. 6.1] proved it for $\beta(2), \ldots, \beta(10)$.

Such a statement holds as soon as $\beta(2)$ is irrational, and then it says nothing about $\beta(4), \beta(6), \ldots$. A proof that $\beta(2)$ is irrational has recently been claimed in an AI-generated preprint with an accompanying Lean formalisation [9]; it has not been refereed, and we have not checked it. Theorem 1.1 does not depend on the arithmetic nature of $\beta(2)$ in any way. Rivoal and Zudilin [11, Thm. 1] also proved a lower bound of order $\log a$ for the dimension of the $\mathbb{Q}$-vector space spanned by 1 and the even values $\beta(2), \ldots, \beta(a)$; Fischler [5] proved results of the same type for general Dirichlet L-values of a fixed parity. A dimension at least 3 forces one of $\beta(4), \ldots, \beta(a)$ to be irrational, so these results give a $\beta(2)$-free statement qualitatively, for some computable a. We have not found an explicit value of a in the literature. Even ignoring error terms, the constant $1/(2+\log 2)$ of [11] would require $\log a \geq 3(2+\log 2)$, that is, $a \approx 3\cdot 10^3$, and the constant $1/(1+\log 2)$ of [5] would require $\log a \geq 3(1+\log 2)$, that is, $a \approx 160$.

To our knowledge, Theorem 1.1 is the first explicit irrationality statement about the even values of β that does not involve $\beta(2)$. Our literature search covered arXiv and the web, but not MathSciNet or zbMATH; we would be grateful to hear of prior work.

1.2. **The method.** We follow the now standard approach of Ball and Rivoal [1] through linear forms with rational coefficients. The new features are the following.

(1) *Removing* $\beta(2)$. We use $L_n = \sum_{t\geq 0}(-1)^t R_n''(t)$, where R_n has triple zeros at the integers of $\Lambda_n = [-(1+r)n-1,\, rn]$ and poles at half-integers. This adapts to alternating sums and half-integer poles the second-derivative device of Rivoal [10] (see also [14, §8, Lemma 19]), which was introduced for odd zeta values. Differentiating twice raises every exponent by 2, so $\beta(2)$ cannot occur, and a reflection symmetry kills the odd-index values (Section 3).

(2) *Zeros through the pole region.* As in the constructions of Zudilin [15] and Lai–Zhou [8], the integer zeros of R_n run through the whole

region $[-n-\frac{1}{2}, -\frac{1}{2}]$ where the poles $-k-\frac{1}{2}$ $(0 \le k \le n)$ lie; $[-n, -1]$ is the set of integers in that region. For the second-derivative form this requires triple zeros there. Its effect (Lemma 4.6) is that the primes $p > n$ do not contribute to the denominators at all.

(3) *An optimised block profile.* The denominator of R_n is a product of 35 "blocks" $\prod_{k=\delta n}^{(1-\delta)n}(t+k+\frac{1}{2})$ counted with multiplicity (28 distinct offsets; $\delta = 0$ occurs 8 times). The other offsets have spacing $\frac{1}{160}$ on $(0, \frac{1}{20}]$ and $\frac{1}{80}$ on $[\frac{1}{16}, \frac{23}{80}]$ (Table 1). The profile was found by a numerical search.

(4) *Certified numerics.* The arithmetic integral that governs the denominators is a sum over 62,131 intervals. We evaluate it exactly in rational arithmetic (Section 5). The size of L_n is bounded on the real axis (Section 6). Its nonvanishing comes from a saddle point whose location, value and steepest-descent contour are certified by interval arithmetic (Section 7).

(5) *Formalisation.* Every step, including all the computations, has been formalised in Lean 4 [2] and Mathlib [12] (Section 9).

The numerical margin is small: the exponent that must be negative is $Q = V + A_{\text{cert}} + 20\log 2 < -0.0714$, see (8.1). Here V bounds the size of L_n (Proposition 6.5), A_{cert} is the arithmetic integral (Proposition 5.6), and $20\log 2$ is the 2-adic cost of the denominators (Proposition 4.12).

1.3. **The formalisation.** The statement of Theorem 1.1 and its complete proof, including the prime number theorem, complex Stirling bounds, the contour-integral representation, the arithmetic certificate and the interval-arithmetic saddle-point certificate, are formalised in Lean 4 with Mathlib and checked by the Lean kernel. The proof depends only on Lean's three standard axioms (`propext`, `Classical.choice`, `Quot.sound`). It contains no `sorry` and no `native_decide`: every proof, including every computation, is checked by the kernel. The formal statement and what it means are described in Section 9. The repository is `https://github.com/telaviv/beta-irrationality`.

The prose proof below is self-contained and can be read without the formalisation, except that certain finite computations are only described, not reproduced: the exact value of a rational sum with 62,131 terms (Proposition 5.6), 58 logarithm enclosures (Proposition 6.5), and the interval-arithmetic checks of the saddle-point data (Theorem 7.7). The arithmetic certificate and the saddle certificate are decidable statements about rational data, proved by `decide +kernel`; the 58 logarithm enclosures of Proposition 6.5 are proved by `norm_num` from Mathlib's series bounds for log. All proofs are checked by the Lean kernel.

1.4. **Organisation and notation.** Section 2 defines R_n and L_n. Section 3 shows that $L_n \in \mathbb{Q} + \mathbb{Q}\beta(4) + \cdots + \mathbb{Q}\beta(36)$. Section 4 constructs a common denominator D_n, and Section 5 bounds its size through the arithmetic

integral. Section 6 bounds $|L_n|$, and Section 7 proves that $L_n \neq 0$ for infinitely many n. Section 8 combines these results. Section 9 describes the formalisation, and Section 10 contains remarks on limits of the method.

Throughout, p denotes a prime and v_p the p-adic valuation on $\mathbb{Q}$, with $v_p(0) = +\infty$. $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$ and $\{\cdot\}$ are the floor, ceiling and fractional part; $[\,P\,]$ is 1 if P holds and 0 otherwise; # denotes cardinality; intervals $[a, b]$ of integers are written in the same way as real intervals when no confusion can arise. $F(y) := y \log y$ with the principal branch of the logarithm, and $F(0) := 0$. We write $\theta(y) = \sum_{p \le y} \log p$ for Chebyshev's function. For a power series g in ε, $[\varepsilon^M]g$ is the coefficient of ε^M.

2. The construction

2.1. **Parameters.** Fix the following data.

- $r = 7/6$, and n runs through the multiples of $q = 480$: $n = 480N_0$ with $N_0 \in \mathbb{N}$, $N_0 \ge 1$. Then $rn = 560N_0$ and $(1 + r)n = 1040N_0$ are integers, and n is even.
- Twenty-eight *blocks* $j = 1, \ldots, 28$ with offsets δ_j and multiplicities m_j: $(\delta_1, m_1) = (0, 8)$, and $(\delta_j, m_j) = (\kappa/160, 1)$ for the 27 values
$$\kappa \in \{1, 2, 3, 4, 5, 6, 7, 8\} \cup \{10, 12, 14, \ldots, 46\}.$$
We write $a_j := \delta_j n = 3\kappa N_0$ (with $\kappa = 0$ for $j = 1$), $b_j := n - a_j$ and $I_j := [a_j, b_j] \cap \mathbb{Z}$. Since $\delta_j \le 46/160 < \frac{1}{2}$, each I_j is a nonempty integer interval, symmetric under $k \mapsto n-k$, and $\#I_j = (1-2\delta_j)n+1$.

quantity	value
r, q	$7/6$, 480
blocks (δ_j, m_j)	$(0, 8)$; $(\kappa/160, 1)$ for $\kappa = 1, \ldots, 8$ and $\kappa = 10, 12, \ldots, 46$
number of blocks	28
$S = \sum_j m_j$	35
$\sum_j m_j(1 - 2\delta_j)$	$279/10$
$G = \sum_j m_j(1-2\delta_j)-6r-3$	$179/10$
zero set Λ_n	$[-(1 + r)n - 1, rn] \cap \mathbb{Z} = [-1040N_0 - 1, 560N_0]$, each of order 3
$Z = 3\#\Lambda_n$	$10n + 6$
$\mathcal{N} = 2(2 + r)n + 3$	$\frac{19}{3}n + 3 = 3040N_0 + 3$

Table 1. The parameters.

For $0 \le k \le n$ put
$$B_k := \sum_{j=1}^{28} m_j\,[\,k \in I_j\,].$$

The block $j=1$ is $[0,n]$, so $8 \le B_k \le 35$ for all $0 \le k \le n$, and $B_{n-k} = B_k$. The constants

$$S := \sum_j m_j = 35, \qquad G := \sum_j m_j(1-2\delta_j) - 6r - 3 = \frac{279}{10} - 10 = \frac{179}{10} \tag{2.1}$$

will appear throughout. (Here $\sum_j m_j(1-2\delta_j) = 8+27-\frac{1}{80}\sum \kappa = 35 - \frac{568}{80}$.) We also put

$$\mathcal{N} := 2(2+r)n + 3 = \tfrac{19}{3}n + 3 = 3040N_0 + 3;$$

in Section 4 every integer whose p-adic valuation we need has absolute value $< \mathcal{N}$.

2.2. **The rational function and the linear form.** Let $\Lambda_n := [-(1+r)n-1,\, rn] \cap \mathbb{Z}$; it has $(1+2r)n+2 = 1600N_0+2$ elements. Define

$$R_n(t) := \frac{(2t+n+1)\prod_{l\in\Lambda_n}(t-l)^3}{\prod_{j=1}^{28}\prod_{k\in I_j}(t+k+\frac{1}{2})^{m_j}} = \frac{(2t+n+1)\prod_{l\in\Lambda_n}(t-l)^3}{\prod_{k=0}^{n}(t+k+\frac{1}{2})^{B_k}}, \tag{2.2}$$

and

$$L_n := \sum_{t=0}^{\infty}(-1)^t R_n''(t). \tag{2.3}$$

Lemma 2.1. *Put $Z := 3\#\Lambda_n = 10n+6$. Then $\sum_{k=0}^n B_k = \sum_j m_j \# I_j = (G+10)n+35$, and*

$$\deg R_n = 1 + Z - \sum_k B_k = -Gn - 28.$$

In particular $R_n(t) = O(t^{-3})$, $R_n''(t) = O(t^{-5})$ as $t\to\infty$, and the series (2.3) *converges absolutely.*

Proof. $\#\Lambda_n = (1+2r)n+2$ and $3(1+2r) = 10$ give $Z = 10n+6$. Next, $\sum_j m_j\#I_j = \sum_j m_j((1-2\delta_j)n+1) = \frac{279}{10}n + 35 = (G+10)n+35$. The degree is $1+10n+6-(G+10)n-35$. The asymptotics follow because $Gn+28 \ge 3$. □

The factor $2t+n+1$ ("the e-factor") fixes the parity needed in Lemma 2.2. Its zero $t = -\frac{n+1}{2} = -\frac{n}{2} - \frac{1}{2}$ coincides with the central pole, where $B_{n/2} = 35$; so the pole at $-\frac{n+1}{2}$ has order 34, and $a_{35,n/2} = 0$ in Lemma 3.1 below. This causes no difficulty.

Lemma 2.2. $R_n(-n-1-t) = R_n(t)$.

Proof. The map $l \mapsto -n-1-l$ is a bijection of Λ_n onto itself, because $-n-1-(-(1+r)n-1) = rn$. Hence $\prod_{l\in\Lambda_n}(-n-1-t-l)^3 = (-1)^{3\#\Lambda_n}\prod_{l\in\Lambda_n}(t-l)^3$, and $3\#\Lambda_n = 10n+6$ is even. The e-factor becomes $2(-n-1-t)+n+1 = -(2t+n+1)$. In the denominator, $-n-1-t+k+\frac{1}{2} = -(t+(n-k)+\frac{1}{2})$ and $B_{n-k} = B_k$, so the denominator is multiplied by $(-1)^{\sum_k B_k} = (-1)^{(G+10)n+35} = -1$, since $(G+10)n = 13392N_0$ is even. The two signs cancel. □

Lemma 2.2 is what makes R_n *very-well-poised*, in the terminology of [14]: in the term ratio $R_n(t+1)/R_n(t)$ every upper parameter pairs with a lower one with sum $n+2$, the factor $2t+n+1$ supplying the very-well-poised pair.

Remark 2.3. Consider more generally a function of the same shape, with arbitrary parameters r and blocks, triple zeros at every integer of the pole region, and the e-factor raised to a power $e \in \{0,1\}$. For such parameters the symmetry reads $R_n(-n-1-t) = (-1)^n R_n(t)$, which needs $(-1)^{\deg R_n} = (-1)^n$, i.e. $e \equiv (G+1)n + S \pmod 2$. Since $q(G+1) = 9072$ is even, $e = 1$ works for all $n \in 480\mathbb{N}$. Only the specific parameters of Table 1 are used below.

3. The linear form lies in $\mathbb{Q} + \mathbb{Q}\beta(4) + \cdots + \mathbb{Q}\beta(36)$

For $k \geq 0$ and $j \geq 1$ put

$$\sigma_k^{(j)} := \sum_{m=0}^{k-1} \frac{(-1)^m}{(2m+1)^j} \in \mathbb{Q}.$$

For $0 \leq k \leq n$ define the power series in $\mathbb{Q}[[\varepsilon]]$

$$f_k(\varepsilon) := (n-2k+2\varepsilon) \prod_{l \in \Lambda_n} (u_{k,l}+\varepsilon)^3 \prod_{\substack{0 \leq k' \leq n \\ k' \neq k}} (k'-k+\varepsilon)^{-B_{k'}}, \tag{3.1}$$

where

$$u_{k,l} := -k - \tfrac{1}{2} - l = -\tfrac{w_{k,l}}{2}, \qquad w_{k,l} := 2k+1+2l,$$

and $(d+\varepsilon)^{-1} = \sum_{M \geq 0} (-1)^M d^{-M-1} \varepsilon^M$ for $d \neq 0$. Note that $w_{k,l}$ is an odd integer. When k is fixed we write simply u_l and w_l. If $t_0 := -k - \frac{1}{2}$, then $f_k(\varepsilon)$ is the Taylor expansion at $\varepsilon = 0$ of $R_n(t_0+\varepsilon)(t_0+\varepsilon+k+\frac{1}{2})^{B_k}$.

Lemma 3.1 (Partial fractions)**.** *Put $a_{i,k} := [\varepsilon^{B_k - i}] f_k$ for $0 \leq k \leq n$, $1 \leq i \leq B_k$, and $a_{i,k} := 0$ for $i > B_k$. Then $a_{i,k} \in \mathbb{Q}$ and, for every $t \in \mathbb{C} \setminus \{-k-\frac{1}{2} : 0 \leq k \leq n\}$,*

$$R_n(t) = \sum_{k=0}^{n} \sum_{i=1}^{35} \frac{a_{i,k}}{(t+k+\frac{1}{2})^i}.$$

Proof. By Lemma 2.1 the numerator of (2.2) has smaller degree than the denominator, so R_n equals the sum of its principal parts at the distinct points $-k-\frac{1}{2}$. The principal part at $t_0 = -k-\frac{1}{2}$ is $\sum_{i=1}^{B_k} a_{i,k}(t-t_0)^{-i}$, where $a_{i,k}$ is the coefficient of $(t-t_0)^{B_k - i}$ in the Taylor expansion of $R_n(t)(t-t_0)^{B_k}$, which is f_k. Since $B_k \leq 35$, all $i \leq 35$ are covered. □

Lemma 3.2. *For integers $j \geq 2$ and $k \geq 0$,*

$$\sum_{t=0}^{\infty} \frac{(-1)^t}{(t+k+\frac{1}{2})^j} = (-1)^k 2^j \big(\beta(j) - \sigma_k^{(j)}\big).$$

Proof. $(t+k+\frac{1}{2})^{-j} = 2^j(2(t+k)+1)^{-j}$. Substituting $m = t+k$ gives $(-1)^k 2^j \sum_{m\geq k}(-1)^m(2m+1)^{-j}$, and the series converges absolutely. □

Proposition 3.3 (Expansion)**.** *Define*

$$\tau_i := \sum_{k=0}^{n}(-1)^k\, 2^{i+2}\, i(i+1)\, a_{i,k},$$

$$\rho_n := -\sum_{k=0}^{n}\sum_{i=1}^{35}(-1)^k\, 2^{i+2}\, i(i+1)\, a_{i,k}\, \sigma_k^{(i+2)}.$$

Then $\rho_n, \tau_i \in \mathbb{Q}$ *and* $L_n = \rho_n + \sum_{i=1}^{35}\tau_i\,\beta(i+2)$.

Proof. For real $t > -\frac{1}{2}$, Lemma 3.1 may be differentiated termwise: $R_n''(t) = \sum_{k,i} a_{i,k}\, i(i+1)(t+k+\frac{1}{2})^{-i-2}$. Multiply by $(-1)^t$ and sum over $t \geq 0$. This is a finite sum of absolutely convergent series, so the order of summation may be exchanged, and Lemma 3.2 with $j = i+2 \geq 3$ gives the claim. □

Lemma 3.4 (Symmetry)**.** $a_{i,n-k} = (-1)^i a_{i,k}$ *for all* i, k. *Consequently* $\tau_i = 0$ *for every odd* i.

Proof. By Lemmas 2.2 and 3.1,

$$R_n(t) = R_n(-n-1-t) = \sum_{k,i}\frac{a_{i,k}}{(-n-1-t+k+\frac{1}{2})^i} = \sum_{k,i}\frac{(-1)^i a_{i,k}}{(t+(n-k)+\frac{1}{2})^i}.$$

Uniqueness of the partial-fraction decomposition gives $a_{i,n-k} = (-1)^i a_{i,k}$. For odd i, substituting $k \mapsto n-k$ in the definition of τ_i and using that n is even gives $\tau_i = \sum_k (-1)^{n-k} 2^{i+2} i(i+1)(-1)^i a_{i,k} = -\tau_i$. □

Corollary 3.5. $L_n = \rho_n + \sum_{j=2}^{18}\tau_{2j-2}\,\beta(2j) \in \mathbb{Q} + \mathbb{Q}\beta(4) + \mathbb{Q}\beta(6) + \cdots + \mathbb{Q}\beta(36)$. *In particular* $\beta(2)$ *does not occur.*

Proof. In Proposition 3.3 only $\beta(i+2)$ with $1 \leq i \leq 35$ occurs. By Lemma 3.4 only even $i = 2j-2$ with $2 \leq j \leq 18$ survive. □

4. Denominators

Recall $\mathcal{N} = 2(2+r)n + 3 = \frac{19}{3}n + 3$. For $k \in [0,n]$ and $l \in \Lambda_n$ we have $|w_l| = |2k+1+2l| \leq 2(1+r)n + 1 < \mathcal{N}$, and for $k, k' \in [0,n]$ we have $|k'-k| \leq n < \mathcal{N}$.

4.1. Valuations of the Taylor coefficients.

Definition 4.1. Let p be a prime and $e, \lambda \in \mathbb{Z}$. A power series $g \in \mathbb{Q}[[\varepsilon]]$ is (e,λ)*-bounded at* p if $v_p([\varepsilon^M]g) \geq e - \lambda M$ for all $M \geq 0$.

Lemma 4.2. *Let* p *be a prime and* $\lambda \in \mathbb{Z}$.

(1) *If* g_1 *is* (e_1,λ)*-bounded and* g_2 *is* (e_2,λ)*-bounded at* p*, then* g_1g_2 *is* (e_1+e_2,λ)*-bounded at* p.

(2) *If $u, c \in \mathbb{Q}$ with $v_p(u) \geq e$ and $v_p(c) \geq e - \lambda$, then $u + c\varepsilon$ is (e, λ)-bounded at p.*
(3) *If $d \in \mathbb{Q} \setminus \{0\}$ with $v_p(d) \leq \lambda$ and $e \leq -v_p(d)$, then $(d + \varepsilon)^{-1}$ is (e, λ)-bounded at p.*

Proof. (1) $[\varepsilon^M](g_1 g_2) = \sum_{a+b=M} [\varepsilon^a] g_1 \, [\varepsilon^b] g_2$, and each term has valuation $\geq e_1 - \lambda a + e_2 - \lambda b = e_1 + e_2 - \lambda M$. (2) is immediate. (3) The coefficient of ε^M is $(-1)^M d^{-M-1}$, of valuation $-v_p(d) - M v_p(d) \geq e - \lambda M$. □

Lemma 4.3 (Lemma A)**.** *Let p be a prime, $\lambda \geq 0$ an integer and $0 \leq k \leq n$. Let $e_E \in \mathbb{Z}$, $e_l \in \mathbb{Z}$ $(l \in \Lambda_n)$ and $e_{k'} \in \mathbb{Z}$ $(k' \in [0, n] \setminus \{k\})$ satisfy*

- *$v_p(n - 2k) \geq e_E$ and $v_p(2) \geq e_E - \lambda$;*
- *$v_p(u_l) \geq e_l$ and $e_l \leq \lambda$ for every $l \in \Lambda_n$;*
- *$v_p(k' - k) \leq \lambda$ and $e_{k'} \leq -v_p(k' - k)$ for every $k' \neq k$.*

Put $\nu := e_E + 3 \sum_{l \in \Lambda_n} e_l + \sum_{k' \neq k} B_{k'} e_{k'}$. Then f_k is (ν, λ)-bounded at p, and $v_p(a_{i,k}) \geq \nu - \lambda(B_k - i)$ for $1 \leq i \leq B_k$.

Proof. Apply Lemma 4.2(2) to $n - 2k + 2\varepsilon$ and to each $u_l + \varepsilon$ (here $v_p(1) = 0 \geq e_l - \lambda$), Lemma 4.2(3) to each $(k' - k + \varepsilon)^{-1}$, and Lemma 4.2(1) to the product (3.1). The bound for $a_{i,k} = [\varepsilon^{B_k - i}] f_k$ follows. □

For an odd prime p and $0 \leq k \leq n$ define

$$c_p(k) := 3\,\#\{l \in \Lambda_n : p \mid w_l\} + [\,p \mid n - 2k\,],$$
$$P_{p,j}(k) := \#\{k' \in I_j \setminus \{k\} : p \mid k' - k\},$$
$$\varepsilon_p(k) := c_p(k) - \sum_j m_j P_{p,j}(k),$$
$$\nu_p(k) := 3 \sum_{l \in \Lambda_n} v_p(w_l) - \sum_{k' \neq k} B_{k'}\, v_p(k' - k),$$

and for $p = 2$

$$\nu_2(k) := 1 - Z - \sum_{k' \neq k} B_{k'}\, v_2(k' - k).$$

Note $\sum_{k' \neq k} B_{k'} g(k') = \sum_j m_j \sum_{k' \in I_j \setminus \{k\}} g(k')$ for any g. Put $\lambda_p := \lfloor \log_p \mathcal{N} \rfloor$ for odd p, and $\lambda_2 := \lfloor \log_2 n \rfloor$.

Corollary 4.4. *Let $0 \leq k \leq n$ and $1 \leq i \leq B_k$.*

(a) *If p is odd and $p > n$, then $a_{i,k} \in \mathbb{Z}_{(p)}$, i.e. $v_p(a_{i,k}) \geq 0$.*
(b) *If p is odd and $p^2 > \mathcal{N}$, then $v_p(a_{i,k}) \geq \varepsilon_p(k) - (B_k - i)$.*
(c) *If p is odd, then $v_p(a_{i,k}) \geq \nu_p(k) - \lambda_p(B_k - i)$.*
(d) *$v_2(a_{i,k}) \geq \nu_2(k) - \lambda_2(B_k - i)$.*

Proof. We apply Lemma 4.3. Since p is odd in (a)–(c), $v_p(u_l) = v_p(w_l) \geq 0$ and $v_p(2) = 0$.

(a) Take $\lambda = 0$ and all exponents 0. For $k' \neq k$, $0 < |k' - k| \leq n < p$, so $v_p(k' - k) = 0$.

(b) Take $\lambda = 1$, $e_E = [\,p \mid n-2k\,]$, $e_l = [\,p \mid w_l\,]$ and $e_{k'} = -[\,p \mid k'-k\,]$. Since $0 < |k'-k| < \mathcal{N} < p^2$, $v_p(k'-k) = [\,p \mid k'-k\,] \le 1$. When $n = 2k$ the condition $v_p(n-2k) \ge e_E$ holds trivially, as $v_p(0) = +\infty$. Here $\nu = \varepsilon_p(k)$.

(c) Take $\lambda = \lambda_p$, $e_E = 0$, $e_l = v_p(w_l)$ and $e_{k'} = -v_p(k'-k)$. Since $0 < |w_l|, |k'-k| < \mathcal{N} < p^{\lambda_p+1}$, both valuations are $\le \lambda_p$. Here $\nu = \nu_p(k)$.

(d) Take $\lambda = \lambda_2$, $e_E = 1$ (as n is even, $v_2(n-2k) \ge 1$, and $v_2(2) = 1 \ge 1-\lambda_2$), $e_l = -1$ (as w_l is odd, $v_2(u_l) = -1$) and $e_{k'} = -v_2(k'-k)$, which is admissible because $|k'-k| \le n < 2^{\lambda_2+1}$. Here $\nu = 1 - 3\#\Lambda_n - \sum B_{k'} v_2(k'-k) = \nu_2(k)$. □

4.2. **Partial sums.**

Lemma 4.5 (Lemma B)**.** *Let p be a prime, $\lambda \ge 0$ and $k \ge 0$, and suppose $v_p(2m+1) \le \lambda$ for all $0 \le m < k$. Then $v_p(\sigma_k^{(j)}) \ge -j\lambda$. In particular, for $0 \le k \le n$: $v_p(\sigma_k^{(j)}) \ge -j$ if p is odd and $p^2 > \mathcal{N}$; $v_p(\sigma_k^{(j)}) \ge -j\lambda_p$ if p is odd; and $v_2(\sigma_k^{(j)}) \ge 0$.*

Proof. Each term $(-1)^m(2m+1)^{-j}$ has valuation $\ge -j\lambda$. For $m < k \le n$ we have $0 < 2m+1 < 2n < \mathcal{N}$, which gives the three special cases. □

4.3. **The primes $p > n$.**

Lemma 4.6 (Lemma C)**.** *Let p be an odd prime with $p > n$. Then $\rho_n \in \mathbb{Z}_{(p)}$.*

Proof. For $j \ge 1$ and $k \ge 0$, substituting $l = k - m$ gives

$$(-1)^k 2^j \sigma_k^{(j)} = \sum_{l=1}^{k} \frac{(-1)^l}{(k-l+\frac{1}{2})^j}. \tag{4.1}$$

For $1 \le l \le n$ and $0 \le k \le n$ put $T(l,k) := \sum_i a_{i,k}\, i(i+1)\,(k-l+\frac{1}{2})^{-i-2}$. By (4.1),

$$\rho_n = -\sum_{k=0}^{n}\sum_{l=1}^{k}(-1)^l T(l,k) = -\sum_{l=1}^{n}(-1)^l \sum_{k=l}^{n} T(l,k).$$

It suffices to show that $\sum_{k\ge l} T(l,k) \in \mathbb{Z}_{(p)}$ for each $1 \le l \le n$.

By Lemma 3.1, $\sum_{k=0}^{n} T(l,k) = R_n''(-l)$. Since $-l \in [-n,-1] \subset \Lambda_n$, the point $-l$ is a zero of order 3 of R_n and is not a pole, so $R_n''(-l) = 0$. Hence

$$\sum_{k=l}^{n} T(l,k) = -\sum_{k=0}^{l-1} T(l,k). \tag{4.2}$$

By Corollary 4.4(a), $a_{i,k} \in \mathbb{Z}_{(p)}$, and $(k-l+\frac{1}{2})^{-1} = 2/(2(k-l)+1)$. So $T(l,k)$ is p-integral unless $p \mid 2(k-l)+1$. Suppose that $p \mid 2(k_1-l)+1$ for some $k_1 \ge l$ and $p \mid 2(k_2-l)+1$ for some $k_2 < l$. Then $p \mid 2(k_1-k_2)$, so $p \mid k_1-k_2$, which is impossible because $0 < k_1-k_2 \le n < p$. Hence one

of the two sides of (4.2) is a sum of p-integral terms, and so both sides are p-integral. □

4.4. **The common denominator.**

Definition 4.7. For a prime p define the integer E_p by

$$E_p := \begin{cases} \max_{0\le k\le n}\big(-\nu_2(k)\big) + 35\lambda_2, & p = 2, \\ 0, & p \text{ odd},\ p > n, \\ \max_{0\le k\le n}\big(B_k + 2 - \varepsilon_p(k)\big), & p \text{ odd},\ p \le n,\ p^2 > \mathcal{N}, \\ \max_{0\le k\le n}\big(-\nu_p(k)\big) + 37\lambda_p, & p \text{ odd},\ p^2 \le \mathcal{N}, \end{cases}$$

(the cases are applied in this order), and put $D_n := \prod_{p\le n} p^{E_p} \in \mathbb{Q}_{>0}$.

Thus $v_p(D_n) = E_p$ for every prime p. We do not need $E_p \ge 0$ in the third range, so we only assert $D_n \in \mathbb{Q}_{>0}$.

Proposition 4.8 (Integrality)**.** *$D_n\rho_n \in \mathbb{Z}$ and $D_n\tau_i \in \mathbb{Z}$ for all i.*

Proof. It suffices to show $v_p(\rho_n) \ge -E_p$ and $v_p(\tau_i) \ge -E_p$ for every prime p. Both τ_i and ρ_n are $\mathbb{Z}$-linear combinations of the numbers $a_{i,k}2^{i+2}$ and $a_{i,k}2^{i+2}\sigma_k^{(i+2)}$ respectively, with $0 \le k \le n$ and $1 \le i \le B_k \le 35$. We bound these numbers.

- p odd, $p > n$: $a_{i,k}2^{i+2} \in \mathbb{Z}_{(p)}$ by Corollary 4.4(a), so τ_i is p-integral; ρ_n is p-integral by Lemma 4.6.
- p odd, $p \le n$, $p^2 > \mathcal{N}$: by Corollary 4.4(b) and Lemma 4.5,
$$v_p\big(a_{i,k}2^{i+2}\sigma_k^{(i+2)}\big) \ge \varepsilon_p(k) - (B_k - i) - (i+2) = \varepsilon_p(k) - B_k - 2 \ge -E_p,$$
and $v_p(a_{i,k}2^{i+2}) \ge \varepsilon_p(k) - B_k + 1 \ge -E_p$.
- p odd, $p^2 \le \mathcal{N}$: by Corollary 4.4(c) and Lemma 4.5,
$$v_p\big(a_{i,k}2^{i+2}\sigma_k^{(i+2)}\big) \ge \nu_p(k) - \lambda_p(B_k - i) - \lambda_p(i+2) \ge \nu_p(k) - 37\lambda_p \ge -E_p,$$
and the same bound holds without the factor $\sigma_k^{(i+2)}$.
- $p = 2$: by Corollary 4.4(d) and Lemma 4.5,
$$v_2\big(a_{i,k}2^{i+2}\sigma_k^{(i+2)}\big) \ge \nu_2(k) - \lambda_2(B_k - i) + i + 2 \ge \nu_2(k) - 35\lambda_2 \ge -E_2,$$
and similarly without $\sigma_k^{(i+2)}$. □

Corollary 4.9. *$D_nL_n = D_n\rho_n + \sum_{j=2}^{18}(D_n\tau_{2j-2})\,\beta(2j)$ is a linear form in $1, \beta(4), \ldots, \beta(36)$ with integer coefficients.*

4.5. **The size of D_n at $p = 2$ and at small primes.** Let $s_p(N)$ denote the sum of the base-p digits of $N \ge 0$, and $\ell_p(N) := \lfloor \log_p N \rfloor$ for $N \ge 1$, $\ell_p(0) := 0$.

Lemma 4.10 (Legendre-type bounds)**.** *Let p be a prime.*

(1) *For $N \ge 0$: $(p-1)v_p(N!) = N - s_p(N)$ and $0 \le s_p(N) \le (p-1)(\ell_p(N)+1)$.*

(2) *For integers* $1 \le A \le B+1$*:* $(p-1)\sum_{d=A}^{B} v_p(d) \le (B-A+1) + (p-1)(\ell_p(A-1)+1)$.
(3) *If* p *is odd and* $A \ge 0$*:* $(p-1)\sum_{0\le m<A} v_p(2m+1) \ge A - (p-1)(\ell_p(2A)+1)$.

Proof. (1) is Legendre's formula, and each of the at most $\ell_p(N)+1$ digits is $\le p-1$. (2) The sum equals $v_p(B!) - v_p((A-1)!)$; multiply by $p-1$ and use (1). (3) Since $(2A)! = \prod_{m<A}(2m+1)\cdot 2^A A!$ and p is odd, the sum equals $v_p((2A)!) - v_p(A!)$. By (1), $(p-1)$ times it is $A - s_p(2A) + s_p(A) \ge A - s_p(2A)$. □

Lemma 4.11. *Let* p *be a prime,* $N \ge n$*,* $0 \le k \le n$ *and* $1 \le j \le 28$*. Then*

$$(p-1)\sum_{k'\in I_j\setminus\{k\}} v_p(k'-k) \le \#I_j + 2(p-1)(\ell_p(N)+1).$$

Proof. The numbers $|k'-k|$ with $k' \in I_j$, $k' > k$ form an interval $[A_1, B_1]$ of positive integers, and those with $k' < k$ form another interval $[A_2, B_2]$; their total length is at most $\#I_j$ and $A_i - 1 \le n \le N$. Apply Lemma 4.10(2) to each (an empty interval contributes 0). □

Proposition 4.12. (1) *For odd* p *with* $p^2 \le \mathcal{N}$*:* $E_p - G\,v_p(n!) \le 156(\lambda_p+1)$.
(2) $\sum_{3\le p\le\sqrt{\mathcal{N}}}(E_p - G\,v_p(n!))\log p \le 312\lfloor\sqrt{\mathcal{N}}\rfloor\log\mathcal{N}$.
(3) $E_2 - G\,v_2(n!) \le 20n + 40 + 123(\lambda_2+1)$.

Proof. (1) Fix k. As l runs through Λ_n, $w_l = 2k+1+2l$ runs through the consecutive odd integers from $-(2((1+r)n-k)+1)$ to $2(k+rn)+1$. Its positive values are $2m+1$, $0 \le m < A_1 := k+rn+1$, and its negative values are $-(2m+1)$, $0 \le m < A_2 := (1+r)n-k+1$. Here $A_1 + A_2 = (1+2r)n+2 = Z/3$ and $2A_1, 2A_2 \le \mathcal{N}$. By Lemma 4.10(3),

$$(p-1)\cdot 3\sum_{l\in\Lambda_n} v_p(w_l) \ge Z - 6(p-1)(\lambda_p+1).$$

By Lemma 4.11 with $N = \mathcal{N}$, and $\sum_j m_j = 35$,

$$(p-1)\sum_j m_j \sum_{k'\in I_j\setminus\{k\}} v_p(k'-k) \le \sum_j m_j \#I_j + 70(p-1)(\lambda_p+1).$$

With Lemma 2.1, $\sum_j m_j\#I_j - Z = Gn + 29$. Hence $(p-1)(-\nu_p(k)) \le Gn + 29 + 76(p-1)(\lambda_p+1)$. On the other hand, by Lemma 4.10(1), $(p-1)v_p(n!) \ge n - (p-1)(\lambda_p+1)$. Therefore

$$\begin{aligned}(p-1)\big(E_p - Gv_p(n!)\big) &\le 29 + 76(p-1)(\lambda_p+1) + 37(p-1)\lambda_p \\ &\quad + G(p-1)(\lambda_p+1) \\ &\le 29 + 131(p-1)(\lambda_p+1).\end{aligned}$$

Divide by $p-1 \ge 2$ and use $14.5 \le 25(\lambda_p+1)$.

(2) By (1) each term is at most $156(\lambda_p \log p + \log p) \le 312 \log \mathcal{N}$, since $p^{\lambda_p} \le \mathcal{N}$ and $p \le \mathcal{N}$. There are at most $\lfloor \sqrt{\mathcal{N}} \rfloor$ primes in the range.

(3) Fix k. By Lemma 4.11 with $p = 2$, $N = n$,

$$\begin{aligned} -\nu_2(k) &= Z - 1 + \sum_j m_j \sum_{k' \in I_j \setminus \{k\}} v_2(k' - k) \\ &\le Z - 1 + (G + 10)n + 35 + 70(\lambda_2 + 1) \\ &= (G + 20)n + 40 + 70(\lambda_2 + 1). \end{aligned}$$

By Lemma 4.10(1), $v_2(n!) \ge n - \lambda_2 - 1$. Hence

$$\begin{aligned} E_2 - G v_2(n!) &\le 20n + 40 + 70(\lambda_2 + 1) + 35\lambda_2 + G(\lambda_2 + 1) \\ &\le 20n + 40 + 123(\lambda_2 + 1). \end{aligned}$$

□

The main term $20n$ of (3), essentially $2Z$, comes from the Z zero factors $u_l + \varepsilon$, each of which has $v_2(u_l) = -1$, together with a loss of the same size from the pole factors. It is the source of the term $20 \log 2$ in (8.1).

4.6. **Decomposition of** $\log D_n$**.**

Proposition 4.13. *For every* $n \in 480\mathbb{N}$,

$$\begin{aligned} \log D_n - G \log n! \le \big(20n + 40 + 123(\lambda_2 + 1)\big) \log 2 + 312 \lfloor \sqrt{\mathcal{N}} \rfloor \log \mathcal{N} \\ + \sum_{\substack{p \le n \\ p^2 > \mathcal{N}}} \big(E_p - G\lfloor n/p \rfloor\big) \log p. \end{aligned}$$

Proof. By definition and Legendre's formula, $\log D_n - G \log n! = \sum_{p \le n} (E_p - G v_p(n!)) \log p$. Split the sum into $p = 2$, the odd p with $p^2 \le \mathcal{N}$ (note $p^2 \le \mathcal{N}$ implies $p \le n$), and the odd $p \le n$ with $p^2 > \mathcal{N}$. Apply Proposition 4.12 to the first two parts. In the third part $p^2 > \mathcal{N} > n$, so $v_p(n!) = \lfloor n/p \rfloor$. □

5. The arithmetic integral

In this section we bound the last sum in Proposition 4.13 by $n(A_{\text{cert}} + o(1))$, where

(5.1)
$$A_{\text{cert}} := \frac{1250702970786628804842297249335433 1}{7903009210797867674956009046016 00} = 15.8256549806\ldots$$

5.1. **Residue-class counts.**

For a prime p, $0 \le k \le n$ and a block I_j put $N_{p,j}(k) := \#\{k' \in I_j : \ k' \equiv k \pmod p\}$. Counting $k' = k$ separately,

$$[\, k \in I_j \,] + P_{p,j}(k) = N_{p,j}(k), \quad \text{hence}$$

$$B_k + 2 - \varepsilon_p(k) = 2 + \sum_j m_j N_{p,j}(k) - c_p(k). \tag{5.2}$$

Definition 5.1. For real $x \geq 1$ and $y \in \mathbb{R}$ put

$$\begin{aligned} N_j(x,y) &:= \#\{m \in \mathbb{Z} : \ \delta_j x \leq y + m \leq (1-\delta_j)x\} \\ &= \lfloor y - \delta_j x \rfloor - \lceil y - (1-\delta_j)x \rceil + 1, \\ c(x,y) &:= 3\,\#\{o \in \mathbb{Z} \text{ odd} : \ 2y - 2(1+r)x < o \leq 2y + 2rx\}, \\ f(x,y) &:= 2 + \sum_j m_j N_j(x,y) - c(x,y) - G\lfloor x \rfloor. \end{aligned}$$

Since $f(x, y+1) = f(x,y)$ and $f(x,\cdot)$ takes finitely many values on $[0,1]$, we may define

$$e^*(x) := \max_{y \in \mathbb{R}} f(x,y) = \max_{0 \leq y < 1} f(x,y).$$

Remark 5.2. The formalisation (and the informal write-up it was ported from) defines f by cases, with the block indicator $[\,\delta_j x \leq y \leq (1-\delta_j)x\,]$ and the three expressions for $P_j(x,y)$ according to whether y lies in, to the left of, or to the right of the block; it takes the maximum over $0 \leq y \leq x$. For $x \geq 1$ and every y, $[\,\delta_j x \leq y \leq (1-\delta_j)x\,] + P_j(x,y) = N_j(x,y)$ (use $\lceil a \rceil = \lfloor a \rfloor + 1 - [\,a \in \mathbb{Z}\,]$ in the two out-of-block cases), so the two definitions of f agree identically; as $[0,x]$ contains a period, the two e^* agree.

Proposition 5.3 (Pointwise bound)**.** *Let p be a prime with $p \leq n$ and $p^2 > \mathcal{N}$. Then $E_p - G\lfloor n/p \rfloor \leq e^*(n/p)$.*

Proof. Here p is odd. Let $0 \leq k \leq n$, $x := n/p \geq 1$ and $y := k/p$. Then $N_{p,j}(k) = \#\{m \in \mathbb{Z} : a_j \leq k + mp \leq b_j\} = N_j(x,y)$, because $a_j/p = \delta_j x$ and $b_j/p = (1-\delta_j)x$. Next, the values w_l ($l \in \Lambda_n$) are all the odd integers in $[2k - 2(1+r)n - 1,\ 2k + 2rn + 1]$. An odd multiple op of p (necessarily with o odd) with $2k - 2(1+r)n < op \leq 2k + 2rn$ lies in that interval, and these o are exactly the odd o with $2y - 2(1+r)x < o \leq 2y + 2rx$. Hence $c_p(k) \geq c(x,y)$. (This is usually strict: $c(x,y)$ drops the e-factor indicator $[\,p \mid n - 2k\,]$ and the two odd endpoints $2k - 2(1+r)n - 1$ and $2k + 2rn + 1$ of the range of w_l, so the bound may lose up to $1 + 3 + 3 = 7$ here; in exact computations at $n = 480, 960, 1440$ the total loss in $E_p - G\lfloor n/p \rfloor \leq e^*(n/p)$ was at most 2.) By (5.2), $B_k + 2 - \varepsilon_p(k) - G\lfloor n/p \rfloor \leq f(x,y) \leq e^*(x)$. Take the maximum over k. □

Lemma 5.4 (Uniform bound)**.** *For $x \geq 1$, $e^*(x) \leq 40 + G\{x\}$.*

Proof. $N_j(x,y)$ counts the integers in an interval of length $(1 - 2\delta_j)x$, so $N_j \leq (1 - 2\delta_j)x + 1$. The number of odd integers in a half-open interval $(u,v]$ is $\lfloor \frac{v+1}{2} \rfloor - \lfloor \frac{u+1}{2} \rfloor \geq \frac{v-u}{2} - 1$, so $c(x,y) \geq 3((1+2r)x - 1) = 10x - 3$. Using (2.1), $f(x,y) \leq 2 + 35 + (G+10)x - 10x + 3 - G\lfloor x \rfloor = 40 + G\{x\}$. □

5.2. **Step structure.** By the odd-integer count in the proof of Lemma 5.4, for all $x \geq 1$ and $y \in \mathbb{R}$,

$$f(x,y) = 2 + \sum_j m_j\big(1 + \lfloor y - \delta_j x\rfloor - \lceil y - (1-\delta_j)x\rceil\big) - 3\big(\lfloor y - (-rx - \tfrac{1}{2})\rfloor - \lfloor y - ((1+r)x - \tfrac{1}{2})\rfloor\big) - G\lfloor x\rfloor; \tag{5.3}$$

this is Definition 5.1 with the count of odd integers written out. So $f(x,y)$ depends on y only through the floors and ceilings of $y - c_o(x)$ for the 58 *offsets* $c_o(x) = s_o x + t_o$ with

$$(s_o, t_o) \in \{(\delta_j, 0),\ (1-\delta_j, 0) : 1 \leq j \leq 28\} \cup \{(-r, -\tfrac{1}{2}),\ (1+r, -\tfrac{1}{2})\}.$$

The pairs $(0,0)$ and $(1,0)$ are among these offsets (block $j = 1$, $\delta_1 = 0$). Let $\mathcal{F}$ be this set of 58 pairs together with $(0, \frac{1}{2})$, so $\#\mathcal{F} = 59$. (The extra pair is not needed for the argument below; it is included because the certificate uses it, and it only refines the partition.) Define the *event set* $\mathcal{E} \subset [1, 400]$ as the union of

- the points $1 + j/20$ $(0 \leq j < 40)$ and $3 + j/2$ $(0 \leq j \leq 794)$, which include every integer in $[1, 400]$;
- all $x \in (1, 400)$ of the form $x = (t_2 - t_1 + m)/(s_1 - s_2)$ with $(s_1, t_1), (s_2, t_2) \in \mathcal{F}$, $s_1 > s_2$ and $m \in \mathbb{Z}$.

This is a finite set of rational numbers. It has 62,132 elements $1 = x_0 < x_1 < \cdots < x_L = 400$, $L = 62{,}131$.

Lemma 5.5. *e^* is constant on each open interval $(x_{\ell-1}, x_\ell)$.*

Proof. Let $J = (x_{\ell-1}, x_\ell)$ and $x, x' \in J$. Consider the discrete set $C(x) := \{c_o(x) + m :\ o \text{ an offset},\ m \in \mathbb{Z}\}$. For two of its elements, the difference $c_o(x) - c_{o'}(x) + m - m' = (s_o - s_{o'})x + (t_o - t_{o'}) + (m - m')$ is an affine function of x. If it is not identically zero, its only zero is an event point, or lies outside $(1, 400)$; in either case it does not vanish on J. Hence the map $c_o(x) + m \mapsto c_o(x') + m$ is well defined, bijective and strictly increasing from $C(x)$ onto $C(x')$. Since $C(x)$ is a locally finite union of translates of $\mathbb{Z}$, the map extends to an increasing bijection $\pi : \mathbb{R} \to \mathbb{R}$. For every offset o and every y, $c_o(x) + m \leq y \iff c_o(x') + m \leq \pi(y)$, so $\lfloor \pi(y) - c_o(x')\rfloor = \lfloor y - c_o(x)\rfloor$, and likewise for ceilings. Also $\lfloor x\rfloor = \lfloor x'\rfloor$, since $\mathcal{E}$ contains every integer of $[1, 400]$. By (5.3), $f(x', \pi(y)) = f(x, y)$ for all y, and therefore $e^*(x') = e^*(x)$. □

The proof also shows how e^* is computed exactly at a rational point ξ. The function $f(\xi, \cdot)$ is constant on each open interval between consecutive points of $C(\xi)$. So $e^*(\xi)$ is the maximum, over the 58 offsets o, of the value of $f(\xi, \cdot)$ at $c_o(\xi)$ and of its right limit there. Each such value is an integer combination of floors of rational numbers, minus $G\lfloor \xi\rfloor$.

5.3. **The certificate.**

Proposition 5.6 (Certified computation)**.** *Let* $\xi_\ell := (x_{\ell-1}+x_\ell)/2$ *and*

$$A_{\text{body}} := \sum_{\ell=1}^{L} e^*(\xi_\ell)\Big(\frac{1}{x_{\ell-1}} - \frac{1}{x_\ell}\Big).$$

Then

$$\sum_{\ell=1}^{L}\Big\lceil 2^{96} e^*(\xi_\ell)\Big(\frac{1}{x_{\ell-1}} - \frac{1}{x_\ell}\Big)\Big\rceil = U := 1243345293752876677540040193981.$$

Consequently $A_{\text{body}} \le U/2^{96} = 15.6932239\ldots$ *and*

$$\frac{U}{2^{96}} + \frac{44}{400} + \frac{G}{2\cdot 399} = A_{\text{cert}},$$

with A_{cert} *as in* (5.1)*. (Note* $44/400 + G/798 = 1321/9975$*.)*

Proof. This is a finite computation with explicit rational numbers: enumerate $\mathcal{E}$, evaluate $e^*(\xi_\ell)$ exactly by the recipe after Lemma 5.5, round each term up to a multiple of 2^{-96}, and add. The last identity is exact rational arithmetic. The computation has been carried out independently by an exact-rational Python implementation and by the Lean kernel; see Section 9.4. □

5.4. **From primes to the integral.**

Lemma 5.7 (Tail)**.** *Let* $X \ge 2$ *be an integer,* $C, G' \ge 0$*, and* $h : [X,\infty) \to \mathbb{R}$ *with* $h(y) \le C + G'\{y\}$*. For every* $\varepsilon > 0$*, for all sufficiently large* n*,*

$$\sum_{p \le n/X} h(n/p)\log p \le n\Big(\frac{C}{X} + \frac{G'}{2(X-1)}\Big) + \varepsilon n.$$

Proof. Choose an integer $K_0 \ge 1$ with $G'/(2K_0(X-1)) \le \varepsilon/4$ and an integer $M > X$ with $(C+G')\log 4/M \le \varepsilon/4$. The primes $p \le n/M$ contribute at most $(C+G')\theta(n/M) \le (C+G')(\log 4)\, n/M \le \varepsilon n/4$, by Chebyshev's bound $\theta(y) \le y\log 4$. Cut $[X, M)$ into the $K_0(M-X)$ half-open cells $[\,m+\frac{i-1}{K_0},\, m+\frac{i}{K_0})$, $X \le m < M$, $1 \le i \le K_0$. On such a cell $h \le C + G'i/K_0$. The primes with n/p in the cell $[a,b)$ are those in $(n/b, n/a]$, and by the prime number theorem $\theta(n/a) - \theta(n/b) = n(1/a - 1/b) + o(n)$. Since there are finitely many cells, for large n the primes $n/M < p \le n/X$ contribute at most

$$n\sum_{m=X}^{M-1}\sum_{i=1}^{K_0}\Big(C + \frac{G'i}{K_0}\Big)\Big(\frac{1}{m+\frac{i-1}{K_0}} - \frac{1}{m+\frac{i}{K_0}}\Big) + \frac{\varepsilon n}{4}.$$

The C-part telescopes to $C(1/X - 1/M) \le C/X$. In the G'-part each bracket is at most $1/(K_0 m^2)$, so the inner sum is at most $G'(K_0+1)/(2K_0 m^2)$. Since $\sum_{m\ge X} m^{-2} \le 1/(X-1)$, the G'-part is at most $\frac{G'}{2(X-1)} + \frac{G'}{2K_0(X-1)} \le \frac{G'}{2(X-1)} + \frac{\varepsilon}{4}$. □

Proposition 5.8. *There are $\varepsilon_n \to 0$ such that, for all large $n \in 480\mathbb{N}$,*

$$\sum_{\substack{p \le n \\ p^2 > \mathcal{N}}} \big(E_p - G\lfloor n/p \rfloor\big) \log p \le n(A_{\mathrm{cert}} + \varepsilon_n).$$

Proof. Fix $\varepsilon > 0$ and define h on $[1, \infty)$ by $h(x) := e^*(x)$ for $x < 400$ and $h(x) := \max(e^*(x), 0)$ for $x \ge 400$. Let n be large; in particular $n/\sqrt{\mathcal{N}} \ge 400$, which holds as soon as $n \ge 7 \cdot 400^2$ because $\mathcal{N} \le 7n$.

(1) By Proposition 5.3, each term of the sum is at most $h(n/p) \log p$. Every prime $p \le n$ not in the sum has $p^2 \le \mathcal{N}$, hence $n/p \ge n/\sqrt{\mathcal{N}} \ge 400$ and $h(n/p) \ge 0$. So the sum is at most $\sum_{p \le n} h(n/p) \log p$.

(2) *Primes $n/400 < p \le n$*, i.e. $n/p \in [1, 400)$. Let $v_\ell := e^*(\xi_\ell)$. By Lemma 5.5, $h(n/p) = v_\ell$ when $n/p \in (x_{\ell-1}, x_\ell)$. By the prime number theorem, $\sum_{x_{\ell-1} < n/p < x_\ell} \log p = n(1/x_{\ell-1} - 1/x_\ell) + o(n)$ for each of the finitely many ℓ. At most $L+1$ primes have $n/p \in \mathcal{E}$, and each contributes at most $(40 + G) \log n$ by Lemma 5.4. Hence this part is at most $nA_{\mathrm{body}} + \varepsilon n/2$ for large n.

(3) *Primes $p \le n/400$.* By Lemma 5.4, $h(x) \le 40 + G\{x\} \le 44 + G\{x\}$ for $x \ge 400$. We use $44 = S+9$, the constant of the formal certificate; 40 would lower A_{cert} by 0.01. Lemma 5.7 with $X = 400$, $C = 44$, $G' = G$ bounds this part by $n(44/400 + G/798) + \varepsilon n/2$.

Adding up and using Proposition 5.6, the sum is at most $n(A_{\mathrm{cert}} + \varepsilon)$ for large n. □

Corollary 5.9. *There are $\varepsilon'_n \to 0$ with $\log D_n \le G \log n! + n(A_{\mathrm{cert}} + 20 \log 2 + \varepsilon'_n)$ for all large $n \in 480\mathbb{N}$.*

Proof. Combine Propositions 4.13 and 5.8. The remaining error terms $(40 + 123(\lambda_2 + 1)) \log 2$ and $312\lfloor \sqrt{\mathcal{N}} \rfloor \log \mathcal{N}$ are $O(\sqrt{n} \log n) = o(n)$. □

6. The size of L_n

Put

(6.1)

$$\Phi(x) := 3\big[F(x + \tfrac{13}{6}) - F(x - \tfrac{7}{6})\big] - \sum_{j=1}^{28} m_j \big[F(x + 1 - \delta_j) - F(x + \delta_j)\big] \qquad (x \ge \tfrac{7}{6}),$$

and $V := -29.76$. (Note $\frac{13}{6} = 1 + r$ and $\frac{7}{6} = r$.)

Lemma 6.1. *$R''_n(t) = 0$ for every integer $t \in [0, rn]$.*

(The same argument works at every integer of Λ_n; only $t \in [0, rn]$ is used below, and this is the case that is formalised.)

Proof. Near such t, $R_n(s) = (s - t)^3 g(s)$ with g holomorphic at t, since the poles of R_n are at half-integers. Hence $R''_n = 6(s-t)g + 6(s-t)^2 g' + (s-t)^3 g''$ vanishes at $s = t$. □

Lemma 6.2. *For real $t \geq rn+1$, $|R_n''(t)| \leq |R_n(t)|(M^2+M)$ with $M := 21642N_0$.*

(The formalisation uses the cruder count $\sum_j m_j \# I_j \leq 35(n+1)$, which gives $M_0 \leq 21600N_0 + 42 \leq 21642N_0$ in the proof below; this is the origin of the constant.)

Proof. On $(rn+\frac{1}{2}, \infty)$, $R_n' = R_n g$ with $g(t) = \frac{2}{2t+n+1} + \sum_{l\in\Lambda_n} \frac{3}{t-l} - \sum_{k=0}^{n} \frac{B_k}{t+k+\frac{1}{2}}$. At $t \geq rn+1$ every $|t-l|$, $|t+k+\frac{1}{2}|$ and $|t+\frac{n+1}{2}|$ is at least 1, so both $|g(t)|$ and $|g'(t)|$ are at most $M_0 := 1 + Z + \sum_k B_k = 1 + (10n+6) + (G+10)n + 35 = 18192N_0 + 42 \leq M$. Then $R_n'' = R_n(g^2+g')$. □

Lemma 6.3. (1) *If $a \leq b$ are integers and $t > b$ is real, then $\sum_{l=a}^{b} \log(t-l) \leq F(t-a+1) - F(t-b) - (b-a+1)$.*
(2) *If $A \leq B$ are integers and $t + A \geq \frac{1}{2}$, then $\sum_{k=A}^{B} \log(t+k+\frac{1}{2}) \geq F(t+B+\frac{1}{2}) - F(t+A-\frac{1}{2}) - (B-A+1)$.*
(3) *$\log N! \leq N\log N - N + \log(N+1) + 1$ for $N \geq 1$, and $F(Ny) = NF(y) + Ny\log N$ for $N, y > 0$.*

Proof. F is convex with $F'(y) = \log y + 1$. Hence $\log c \leq F(c+1) - F(c) - 1$ for $c > 0$ and $F(c) - F(c-1) - 1 \leq \log c$ for $c \geq 1$. Telescoping gives (1) and (2). (3): (1)-type telescoping gives $\log N! \leq F(N+1) - N$, and $F(N+1) \leq F(N) + \log(N+1) + 1$ by the mean value theorem. The scaling identity is immediate. □

Proposition 6.4. *For real $t \geq rn+1$ and $x = t/n$: $R_n(t) > 0$ and*

$$\begin{aligned}\log R_n(t) &\leq n\Phi(x) - Gn(\log n - 1) + 34\\ &\leq -G\log n! + n\Phi(x) + G(\log(n+1)+1) + 34.\end{aligned}$$

Proof. All factors in (2.2) are positive for $t > rn$. By Lemma 6.3(1) with $a = -(1+r)n - 1$, $b = rn$, and $F(y+2) \leq F(y) + 2(\log(y+2)+1)$,

$$\sum_{l\in\Lambda_n} \log(t-l) \leq F(t+(1+r)n) - F(t-rn) - (1+2r)n + 2\log(t+(1+r)n+2).$$

By the scaling identity the right side is $n[F(x+\frac{13}{6}) - F(x-\frac{7}{6})] + \frac{10}{3}n(\log n - 1) + 2\log(t+1040N_0+2)$. For the block I_j, Lemma 6.3(2), the inequalities $F(Y+\frac{1}{2}) - F(Y) \geq \frac{1}{2}(\log Y + 1)$ and $F(X) - F(X-\frac{1}{2}) \geq \frac{1}{2}(\log(X-\frac{1}{2})+1)$ with $Y = n(x+1-\delta_j)$, $X = n(x+\delta_j)$, and $Y \geq t/2$, $X - \frac{1}{2} \geq t/2$ (as $t \geq 1$) give

$$\begin{aligned}\sum_{k\in I_j} \log(t+k+\tfrac{1}{2}) \geq{}& n[F(x+1-\delta_j) - F(x+\delta_j)]\\ &+ n(1-2\delta_j)(\log n - 1) + \log t - \log 2.\end{aligned}$$

Combining with weights 3 and $-m_j$, the $n(\log n - 1)$ terms add up to $(10 - \sum_j m_j(1-2\delta_j))n(\log n - 1) = -Gn(\log n - 1)$, and the remaining terms are $\log(2t+n+1) + 6\log(t+1040N_0+2) - 35\log t + 35\log 2$. Since $2t+n+1 \leq 3t$

and $t + 1040N_0 + 2 \le 3t$, this is at most $7\log 3 + 35\log 2 - 28\log t \le 7\log 3 + 35\log 2 < 34$ (as $\log t \ge 0$). The second inequality follows from Lemma 6.3(3). □

Proposition 6.5 (Real-axis maximum)**.** *$\Phi(x) \le V = -29.76$ for all $x \ge \frac{7}{6}$, and $\Phi(x) \le V - 4\log(x/3)$ for $x \ge 3$.*

Proof. For $x > \frac{7}{6}$,

$$\Phi'(x) = 3\log\frac{x+13/6}{x-7/6} - \sum_j m_j \log\frac{x+1-\delta_j}{x+\delta_j},$$

$$\Phi''(x) = \frac{3}{x+13/6} - \frac{3}{x-7/6} - \sum_j m_j\Big(\frac{1}{x+1-\delta_j} - \frac{1}{x+\delta_j}\Big).$$

Step 1: $\Phi'' \le 0$ on $(\frac{7}{6}, \frac{4}{3}]$. On an interval $[a,b]$ with $a \ge \frac{7}{6}$, each term of Φ'' is monotone, so

$$\Phi''(x) \le \frac{3}{a+13/6} - \frac{3}{b-7/6} - \sum_j m_j\Big(\frac{1}{b+1-\delta_j} - \frac{1}{a+\delta_j}\Big).$$

For $[a,b] = [\frac{7}{6}, \frac{6}{5}], [\frac{6}{5}, \frac{5}{4}], [\frac{5}{4}, \frac{4}{3}]$ the right side is a rational number, approximately -78.1, -24.4 and -6.81 respectively.

Step 2: the tangent at $x_0 = 58911/50000$. Certified enclosures of the 58 logarithms $\log(x_0 + s)$ that occur in $\Phi(x_0)$ and $\Phi'(x_0)$ give

$$\Phi(x_0) \le -29.7613, \qquad -0.002 \le \Phi'(x_0) \le 0.$$

Each enclosure writes $x_0 + s = 2^a 3^b 5^c z$ with $|1 - z| < 0.015$, uses 8 terms of the Taylor series of $\log z$ with an explicit remainder, and uses Mathlib's enclosures of $\log 2$, $\log 3$ and $\log 5$. (The true values are $\Phi(x_0) = -29.76132763\ldots$ and $\Phi'(x_0) = -0.00148021\ldots$.) Φ is continuous on $[\frac{7}{6}, \infty)$ and concave on $(\frac{7}{6}, \frac{4}{3}]$, so for $\frac{7}{6} \le x \le \frac{4}{3}$,

$$\Phi(x) \le \Phi(x_0) + \Phi'(x_0)(x - x_0) \le -29.7613 + 0.002\,(x_0 - \tfrac{7}{6}) < -29.76.$$

Step 3: $\Phi' \le 0$ on $[\frac{4}{3}, \infty)$. Since $\log(Y/X) \ge (Y - X)/Y$ for $Y \ge X > 0$, the block sum in Φ' is at least $S_b(x) := \sum_j m_j(1 - 2\delta_j)/(x + 1 - \delta_j)$, which is decreasing in x. On $[\frac{4}{3}, 2]$, $(x + \frac{13}{6})/(x - \frac{7}{6}) \le 21 \le 24$ and $S_b(x) \ge S_b(2) \ge 9.55$, while $3\log 24 < 9.54$ (we use 24 rather than 21 because $\log 24 = 3\log 2 + \log 3$ is directly bounded by the enclosures of $\log 2$ and $\log 3$). On $[2, 3]$, the ratio is $\le 5 \le 6$, $S_b(x) \ge S_b(3) \ge 7$ and $3\log 6 < 5.4$.

Step 4: $\Phi'(x) \le -4/x$ for $x \ge 3$. Here $3\log\frac{x+13/6}{x-7/6} \le \frac{10}{x-7/6} \le \frac{180}{11x}$ (using $\log y \le y - 1$ and $x \ge 3$), and $\frac{1}{x+1-\delta_j} \ge \frac{3}{4x}$, so $S_b(x) \ge \frac{279}{10} \cdot \frac{3}{4x}$. Hence $\Phi'(x) \le (\frac{180}{11} - \frac{837}{40})/x = -\frac{2007}{440x} \le -4/x$.

So Φ is nonincreasing on $[\frac{4}{3}, \infty)$, and $\Phi(x) \le \Phi(\frac{4}{3}) < V$ there; with Step 2, $\Phi \le V$ on $[\frac{7}{6}, \infty)$. By Step 4, $x \mapsto \Phi(x) + 4\log x$ is nonincreasing on $[3, \infty)$, so $\Phi(x) \le \Phi(3) - 4\log(x/3) \le V - 4\log(x/3)$. □

Numerically, the maximum of Φ on $[\frac{7}{6},\infty)$ is $-29.7613276\ldots$, attained at $x^* = 1.1782140\ldots$.

Proposition 6.6. *For every $n = 480N_0 \in 480\mathbb{N}$ with $L_n \neq 0$,*

$$\log|L_n| \le -G\log n! + n\big(V + \varepsilon_n''\big), \qquad \varepsilon_n'' := \frac{A_{\text{size}} + 22\log N_0}{480N_0} \to 0,$$

where $A_{\text{size}} := G(\log 481 + 1) + 34 + 2\log 2 + 2\log 21642 + 2\log 1441$.

Proof. By Lemma 6.1, $L_n = \sum_{t\ge rn+1}(-1)^t R_n''(t)$. By Lemma 6.2 and Proposition 6.4, $|R_n''(t)| \le (M^2+M)e^{C_n}e^{n\Phi(t/n)}$ with $C_n := -G\log n! + G(\log(n+1)+1) + 34$. Put $T := 3n$ and $w(t) := 1/((t+1)(t+2))$, so that $\sum_{t\ge 0} w(t) = 1$. We claim $e^{n\Phi(t/n)} \le e^{nV}\cdot 2(T+1)^2 w(t)$ for integers $t \ge rn+1$. For $t \le T$ this follows from Proposition 6.5 and $(t+1)(t+2) \le 2(T+1)^2$. For $t > T$, Proposition 6.5 gives $e^{n\Phi(t/n)} \le e^{nV}(T/t)^{4n} \le e^{nV}(T/t)^2$, and $(T/t)^2(t+1)(t+2) \le 2T^2$ for $t \ge T \ge 4$. Therefore

$$|L_n| \le (M^2+M)\, e^{C_n+nV}\, 2(T+1)^2.$$

Taking logarithms, with $M^2+M \le 2M^2$, $T+1 \le 1441N_0$ and $n+1 \le 481N_0$,

$$\begin{aligned}\log|L_n| &\le -G\log n! + nV + A_{\text{size}} + (4+G)\log N_0\\ &\le -G\log n! + nV + A_{\text{size}} + 22\log N_0.\end{aligned}$$

□

Remark 6.7. On the real axis $x > r$, the function Φ is the real part of the boundary value of the phase Ψ of Section 7 from the upper half-plane: the factor $3F(r-x)$ of Ψ has real part $-3F(x-r)$. The saddle point gives the better exponent $\operatorname{Re}\Omega \approx -29.7790$ in place of V (see Remark 7.14), but the elementary bound of Proposition 6.6 suffices, and it is the one used in the proof.

7. Nonvanishing

We extend R_n to the rational function of $z \in \mathbb{C}$ given by (2.2). It has real coefficients, and its poles are the points $-k-\frac{1}{2}$ $(0 \le k \le n)$, all with real part $\le -\frac{1}{2}$. Put

$$F_n(z) := R_n''(z)\,\frac{\pi}{\sin \pi z}.$$

Recall that $|\sin\pi(x+iy)|^2 = \sin^2\pi x + \sinh^2\pi y$, so $|\sin\pi z| \ge \sinh(\pi|\operatorname{Im} z|)$, and $|\sin\pi z| = \cosh(\pi \operatorname{Im} z) \ge 1$ when $\operatorname{Re} z \in \frac{1}{2}+\mathbb{Z}$.

7.1. An integral representation.

Lemma 7.1. *Let $c > -\frac{1}{2}$. There is $C = C(n,c)$ such that $|R_n''(z)| \le C(1+|z|^2)^{-1}$ whenever $\operatorname{Re} z \ge c$. Moreover $R_n''(\bar z) = \overline{R_n''(z)}$.*

Proof. By Lemma 2.1, $|R_n(w)| \le C_1|w|^{-3}$ for $|w| \ge R_1$. If $|z| \ge 2R_1+2$, Cauchy's estimate on the circle $|w-z| = \frac{1}{2}$ gives

$$|R_n''(z)| \le 8 \max_{|w-z|=1/2} |R_n(w)| \le C_1'|z|^{-3}.$$

On the compact set $\{\operatorname{Re} z \geq c,\ |z| \leq 2R_1 + 2\}$, which contains no pole, R_n'' is bounded. The symmetry holds because R_n has real coefficients. □

Proposition 7.2. *For every real $c \in (0, rn+1) \setminus \mathbb{Z}$,*

$$L_n = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_n(z)\, dz = -\frac{1}{\pi} \operatorname{Re} \int_0^\infty F_n(c+iy)\, dy.$$

Proof. Let $N > c$ be an integer and $Y > 0$, and integrate F_n counterclockwise around the rectangle with vertices $c \pm iY$, $N + \frac{1}{2} \pm iY$. Inside, F_n is meromorphic with simple poles at the integers $t \in (c, N]$, with residue $(-1)^t R_n''(t)$. On the horizontal sides $|F_n| \leq \pi C \sinh(\pi Y)^{-1}$ by Lemma 7.1, so they vanish as $Y \to \infty$. On the line $\operatorname{Re} z = N + \frac{1}{2}$ we have $|F_n(z)| \leq \pi C/(1+|z|^2)$, so its integral is at most $\pi^2 C(1 + (N+\frac{1}{2})^2)^{-1/2} \to 0$ as $N \to \infty$. On the line $\operatorname{Re} z = c$, $|\sin \pi z| \geq \max(|\sin \pi c|, \sinh \pi |y|)$, so the integral converges absolutely. Hence

$$-\int_{c-i\infty}^{c+i\infty} F_n(z)\, dz = 2\pi i \sum_{t>c} (-1)^t R_n''(t) = 2\pi i\, L_n,$$

where the last equality uses Lemma 6.1: every integer $0 \leq t < c$ satisfies $t \leq rn$. (The rectangle is traversed upwards on $\operatorname{Re} z = N + \frac{1}{2}$ and downwards on $\operatorname{Re} z = c$; the residue sum converges absolutely.) Finally $F_n(c - iy) = \overline{F_n(c+iy)}$, because R_n'' and sin commute with complex conjugation. Hence $\int_{c-i\infty}^{c+i\infty} F_n\, dz = i \int_{\mathbb{R}} F_n(c+iy)\, dy = 2i \operatorname{Re} \int_0^\infty F_n(c+iy)\, dy$. □

7.2. **Gamma factorisation.** We use the following 58 factors (c_i, s_i, u_i, β_i), $0 \leq i \leq 57$:

$$(3, +1, \tfrac{13}{6}, 2), \qquad (3, -1, \tfrac{7}{6}, 1),$$
$$\text{and for each block } j: \quad (m_j, +1, \delta_j, \tfrac{1}{2}),\ (-m_j, +1, 1-\delta_j, \tfrac{3}{2}).$$

Put $\alpha_i(x) := s_i x + u_i$. One checks directly:

$$\begin{gathered} \sum_i c_i s_i = 0, \quad \sum_i c_i u_i = -G, \quad \sum_i c_i(\beta_i - \tfrac{1}{2}) = -29, \\ \sum_i c_i = 6, \quad \sum_i |c_i| = 76, \quad \sum_i |c_i(\beta_i - \tfrac{1}{2})| = 41. \end{gathered} \tag{7.1}$$

In particular $\sum_i c_i \alpha_i(x) = -G$ for all x.

Lemma 7.3. *For $z \in \mathbb{C} \setminus (\mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2}))$,*

$$R_n(z) = P_n(z) \frac{\sin^3 \pi z}{\pi^3}, \qquad P_n(z) := (2z + n + 1) \prod_{i=0}^{57} \Gamma(s_i z + n u_i + \beta_i)^{c_i}.$$

Proof. Write $p_0 := (1+r)n+1$ and $q_0 := rn$, which is even. Then $\prod_{l=-p_0}^{q_0}(z - l) = \Gamma(z + p_0 + 1)/\Gamma(z - q_0)$, and by the reflection formula $1/\Gamma(z - q_0) = (-1)^{q_0} \Gamma(q_0 + 1 - z) \sin(\pi z)/\pi$. Cubing, and using $(-1)^{3q_0} = 1$, gives the first two factors. For a block, $\prod_{k=a_j}^{b_j}(z + k + \frac{1}{2}) = \Gamma(z + b_j + \frac{3}{2})/\Gamma(z + a_j + \frac{1}{2})$,

which gives the factors $(m_j, \dots, \frac{1}{2})$ and $(-m_j, \dots, \frac{3}{2})$ since $a_j = \delta_j n$ and $b_j = (1-\delta_j)n$. $\square$

For $\operatorname{Im} z > 0$, every argument $s_i z + nu_i + \beta_i$ has nonzero imaginary part, so P_n is holomorphic and zero-free there. Put

$$\mu := \frac{P_n'}{P_n} = \frac{2}{2z+n+1} + \sum_i c_i s_i\, \psi(s_i z + nu_i + \beta_i),$$

$$\mu' = -\frac{4}{(2z+n+1)^2} + \sum_i c_i s_i^2\, \psi'(s_i z + nu_i + \beta_i),$$

where $\psi = \Gamma'/\Gamma$.

Lemma 7.4. *For* $\operatorname{Im} z > 0$,

$$F_n(z) = \frac{P_n(z) H_n(z)}{\pi^2},$$

where

$$H_n := \sin^2(\pi z)(\mu^2 + \mu') + 6\pi\mu \sin(\pi z)\cos(\pi z) + 3\pi^2(2\cos^2 \pi z - \sin^2 \pi z),$$

and, with $\mathbf{q} := e^{2\pi i z}$,

$$\mathbf{q} H_n = \tfrac{1}{4}\Big[(-1+2\mathbf{q}-\mathbf{q}^2)(\mu^2+\mu') + 6\pi i \mu(1-\mathbf{q}^2) + 3\pi^2(3+2\mathbf{q}+3\mathbf{q}^2)\Big]. \tag{7.2}$$

Proof. Write $\sigma = \sin \pi z$, $\gamma = \cos \pi z$. From $R_n = P_n \sigma^3/\pi^3$ and $P_n' = \mu P_n$, $P_n'' = (\mu^2 + \mu')P_n$,

$$R_n'' = \frac{P_n}{\pi^3}\big[(\mu^2+\mu')\sigma^3 + 6\pi\mu\sigma^2\gamma + 6\pi^2\sigma\gamma^2 - 3\pi^2\sigma^3\big].$$

Multiply by π/σ. For (7.2) use $\mathbf{q}\sigma^2 = -(1-\mathbf{q})^2/4$, $\mathbf{q}\sigma\gamma = i(1-\mathbf{q}^2)/4$ and $\mathbf{q}\gamma^2 = (1+\mathbf{q})^2/4$. $\square$

7.3. **Stirling's formula in sectors.** Let $\log\Gamma$ denote the branch of the logarithm of Γ on $\mathbb{C} \setminus (-\infty, 0]$ that is real on $(0, \infty)$.

Theorem 7.5 (cf. [4, §5.11(ii)])**.** *Let* $0 < \varphi \le \pi$ *and put* $\varsigma := \sin(\varphi/2)$. *For* $w \ne 0$ *with* $|\arg w| \le \pi - \varphi$,

$$\Big|\log\Gamma(w) - \big((w-\tfrac{1}{2})\log w - w + \tfrac{1}{2}\log 2\pi\big)\Big| \le \frac{1}{12\varsigma^2|w|},$$

$$\Big|\psi(w) - \log w + \frac{1}{2w}\Big| \le \frac{1}{8\varsigma^3|w|^2}, \qquad \Big|\psi'(w) - \frac{1}{w} - \frac{1}{2w^2}\Big| \le \frac{1}{4\varsigma^4|w|^3}.$$

The first bound is [4, §5.11(ii)] with $K = 1$; the other two follow from the same representation and are proved below.

Proof. We recall the argument, which is also the one formalised. Let $Q_1(t) := \frac{1}{2}\{t\}(1-\{t\})$, so $0 \le Q_1 \le \frac{1}{8}$. For $w \in \mathbb{C} \setminus (-\infty, 0]$ one has the Stieltjes representation

$$\log\Gamma(w) = (w-\tfrac{1}{2})\log w - w + \tfrac{1}{2}\log 2\pi + J(w), \quad J(w) := \int_0^\infty \frac{Q_1(t)}{(t+w)^2}\,dt$$

(in the formalisation it is derived from Euler's limit formula and an exact Euler–Maclaurin identity for $\sum_{j\le N}\log(w+j)$). If $|\arg w| \le \pi-\varphi$ and $t\ge 0$, then $|t+w|\ge \varsigma(t+|w|)$: indeed $|t+w|^2-\varsigma^2(t+|w|)^2 \ge 2t|w|(\cos\varphi+\cos\arg w)\ge 0$. For $x>0$,

$$\int_0^1 \frac{u(1-u)/2}{(u+x)^2}\,du = (x+\tfrac12)\log(1+\tfrac1x)-1 = \sum_{k\ge1}\frac{1}{(2k+1)(2x+1)^{2k}} \le \frac{1}{12x(x+1)},$$

so for $\varrho>0$

$$\int_0^\infty \frac{Q_1(t)}{(t+\varrho)^2}\,dt \le \sum_{k\ge0}\frac{1}{12}\Big(\frac{1}{k+\varrho}-\frac{1}{k+1+\varrho}\Big)=\frac{1}{12\varrho}.$$

With $\varrho=|w|$ this gives $|J(w)|\le 1/(12\varsigma^2|w|)$. Differentiating under the integral, $\psi(w)=\log w-\frac{1}{2w}-2\int_0^\infty Q_1(t)(t+w)^{-3}dt$ and $\psi'(w)=\frac1w+\frac{1}{2w^2}+6\int_0^\infty Q_1(t)(t+w)^{-4}dt$; the bounds follow from $Q_1\le\frac18$ and $\int_0^\infty(t+|w|)^{-m}dt=|w|^{1-m}/(m-1)$. □

Corollary 7.6. *Let $0<\varphi\le\pi$, $\varsigma=\sin(\varphi/2)$, $a>0$, and let $\alpha\in\mathbb{C}$ with $|\alpha|\ge a$, $|\arg\alpha|\le\pi-\varphi$. Let $\beta\in(0,2]$ and $n\ge 4/a$. Then*

$$\Big|\log\Gamma(n\alpha+\beta)-\big[n\alpha\log n+n(\alpha\log\alpha-\alpha) +(\beta-\tfrac12)(\log n+\log\alpha)+\tfrac12\log 2\pi\big]\Big| \le \frac{9+1/(6\varsigma^2)}{an},$$

$$\big|\psi(n\alpha+\beta)-\log n-\log\alpha\big|\le\frac{4+1/(8\varsigma^3)}{an},$$

$$|\psi'(n\alpha+\beta)|\le\frac{3+1/\varsigma^4}{an}.$$

Proof. Put $w=n\alpha+\beta$ and $u=\beta/(n\alpha)$, so $|u|\le 2/(an)\le\frac12$ and $|w|\ge an-2\ge an/2$. Adding the positive real number β moves the argument towards 0, so $|\arg w|\le|\arg\alpha|$. Since $\arg(1+u)$ is 0 or of sign opposite to $\arg\alpha$ and $|\arg(1+u)|<\pi/2$, the principal logarithms satisfy $\log w=\log n+\log\alpha+\log(1+u)$. Insert this into Theorem 7.5 and use $|\log(1+u)-u|\le|u|^2$, $|\log(1+u)|\le\frac32|u|$ for $|u|\le\frac12$: the extra terms are $n\alpha(\log(1+u)-u)$, of size $\le n|\alpha||u|^2\le 4/(an)$, and $(\beta-\frac12)\log(1+u)$, of size $\le\frac32\cdot\frac32|u|\le 4.5/(an)$, while $1/(12\varsigma^2|w|)\le 1/(6\varsigma^2 an)$. The bounds for ψ and ψ' follow in the same way from the second and third bounds of Theorem 7.5, using $|w|\ge an/2\ge 2$. □

7.4. **The phase and the certified saddle data.** On the set $\mathcal{D}$ of $x\in\mathbb{C}$ with $\alpha_i(x)\notin(-\infty,0]$ for all i (this contains the open upper half-plane and

the real interval $(0, \frac{7}{6})$) define, with principal logarithms,

$$\Psi(x) := \sum_i c_i\, \alpha_i(x) \log \alpha_i(x), \qquad f(x) := \Psi(x) - 2\pi i x,$$

$$\psi_\flat(x) := \sum_i c_i(\beta_i - \tfrac{1}{2}) \log \alpha_i(x),$$

$$A(x) := \frac{(2x+1)\, e^{\psi_\flat(x)}}{\pi^2} \cdot \Big(-\frac{(\Psi'(x) - 3\pi i)^2}{4}\Big).$$

By (7.1), $\Psi'(x) = \sum_i c_i s_i \log \alpha_i(x)$, $\Psi''(x) = \sum_i c_i/\alpha_i(x)$ and $\Psi'''(x) = -\sum_i c_i s_i/\alpha_i(x)^2$. We say that the *sector condition* (a, η) *holds at* x if $|\alpha_i(x)| \geq a$ and $|\arg \alpha_i(x)| \leq \pi - \eta$ for every i.

Theorem 7.7 (Certified saddle data)**.** *Put* $a := 10^{-3}$, $\eta := \pi - 2.76$, $\rho := 1/250$, $M_3 := 56746$, $\delta_0 := 10^{-7}$ *and* $\eta_{\text{far}} := 9.727 \cdot 10^{-4}$, *and let*

$$c_0 := 1.171923107460851082743017 8 + 0.009973225371354844465880 6\, i.$$

There are $x_0 \in \mathbb{C}$, $d \in \mathbb{C}$ *with* $|d| = 1$, *and* $v_0, \ldots, v_{46} \in \mathbb{C}$ *such that the following hold, where* $\Omega := f(x_0)$ *and* $K_\star := |\Psi''(x_0)|$.

(S0) $|x_0 - c_0| \leq 10^{-12}$, *and* x_0 *is the unique zero of* $\Psi' - 2\pi i$ *in this disc. Moreover*

$$-29.7790173 \leq \operatorname{Re}\Omega \leq -29.7790172,$$
$$-7.301075589 \leq \operatorname{Im}\Omega \leq -7.301075587, \qquad \sin(480\, \operatorname{Im}\Omega) > 0.99.$$

(S1) $K_\star \geq 260$, $\Psi''(x_0)d^2 = -K_\star$, $|\Psi'''(x)| \leq M_3$ *for* $|x - x_0| \leq \rho$, *and* $M_3\rho \leq \frac{3}{2}K_\star$.

(S1′) $\rho < \operatorname{Im} x_0$; *in particular the closed disc* $|x - x_0| \leq \rho$ *lies in* $\operatorname{Im} x > 0$.

(P) $v_0 = x_c := 1.161975915052$, *so* $\delta_0 \leq x_c < \frac{7}{6}$; $v_5 = x_0 - \rho d$, $v_6 = x_0 + \rho d$; $v_{46} = x_{\text{top}} := 9.211726243607 + 60.012047688088\, i$; $\operatorname{Re} v_k \geq 0$ *for all* k; *and* $\operatorname{Im}(v_k + t(v_{k+1} - v_k)) > 0$ *for* $0 \leq k < 46$ *and* $0 < t \leq 1$.

(S2) *For* $1 \leq k < 46$, $k \neq 5$, *and every* $x \in [v_k, v_{k+1}]$: $\operatorname{Re} f(x) \leq \operatorname{Re}\Omega - \eta_{\text{far}}$ *and the sector condition* (a, η) *holds at* x. *The same holds for every* $x \neq v_0 + e$ *on the segment* $[v_0 + e, v_1]$, *for every real* e *with* $|e| \leq \delta_0$.

(S3) *For every* $y \geq 0$: $\operatorname{Re} f(x_{\text{top}} + iy) \leq \operatorname{Re}\Omega - \eta_{\text{far}} - (7 - 2\pi)y$, *and the sector condition* (a, η) *holds at* $x_{\text{top}} + iy$.

(S4) *The sector condition* (a, η) *holds at* $x_0 + sd$ *for* $|s| \leq \rho$.

Method of proof. All statements are verified by exact rational interval arithmetic, with verified enclosures of π, log, arctan and arg (series with explicit tail bounds, after argument reduction) and complex boxes. The verification is carried out by the Lean kernel; see Section 9.

- (S0): with $Y \approx 1/\Psi''(c_0)$, a box evaluation shows $|1 - Y\Psi''(z)| \leq \kappa_c < 1$ on the disc $B_0 = \{|z - c_0| \leq 10^{-12}\}$ and $|Y(\Psi'(c_0) - 2\pi i)| \leq \varepsilon$ with $\varepsilon + \kappa_c 10^{-12} \leq 10^{-12}$. Then $z \mapsto z - Y(\Psi'(z) - 2\pi i)$ is a contraction of B_0 into itself, and the Banach fixed point theorem

gives existence and uniqueness. Ω is enclosed by evaluating $f(c_0)$ on a box and adding a mean-value bound for $|f(x_0) - f(c_0)|$. For the sine, one checks $480 \operatorname{Im} \Omega = u + \frac{\pi}{2} - 558 \cdot 2\pi$ with $|u| \le 0.07$, so $\sin(480 \operatorname{Im} \Omega) = \cos u \ge 1 - u^2/2$.

- (S1), (S1′): $\Psi''(x_0)$ is enclosed in a box; $d := v/|v|$ with $v := 1 - \overline{\Psi''(x_0)}/K_\star$ satisfies $\Psi''(x_0)d^2 = -K_\star$; and M_3 bounds $\sum_i |c_i|/|\alpha_i|^2$ on a box containing the disc.
- (P), (S2): the vertices v_k other than v_5, v_6 are explicit rational points with 12 decimals, obtained from a Runge–Kutta trace of the steepest-descent curves of $\operatorname{Re} f$ from $x_0 \pm \rho d$. On each segment $[v_k, v_{k+1}]$, a second-order Taylor bound $\operatorname{Re} f(v_k + t(v_{k+1} - v_k)) \le \operatorname{Re} f(v_k) + t \operatorname{Re}(f'(v_k)(v_{k+1} - v_k)) + Ct^2|v_{k+1} - v_k|^2$, with C an upper bound of $|\Psi''|$ on a box containing the segment, is applied from both ends and shows $\operatorname{Re} f \le T_0 := -29.78$. Here T_0 is the lower end -29.7790173 of the enclosure of $\operatorname{Re} \Omega$ in (S0) minus $9.827 \cdot 10^{-4}$, so $T_0 \le \operatorname{Re} \Omega - 9.827 \cdot 10^{-4}$; this gives (S2) with margin $9.827 \cdot 10^{-4}$. The two short connectors $[v_4, v_5] = [v_4, x_0 - \rho d]$ and $[v_6, v_7] = [x_0 + \rho d, v_7]$ lie in discs of radius 10^{-9} around the rational points v_4, v_7, on which $\operatorname{Re} f \le T_0$ is checked directly. Moving the start point by $|e| \le \delta_0$ changes f by at most $\delta_0 \sup |f'| \le 10^{-5}$ on the first segment, which costs 10^{-5} of the margin; hence $\eta_{\text{far}} = 9.727 \cdot 10^{-4}$. The sector condition is checked on boxes covering all segments.
- (S3): $\partial_y \operatorname{Re} f(x_{\text{top}} + iy) = 2\pi - \operatorname{Im} \Psi'(x_{\text{top}} + iy)$. Box evaluations give $\operatorname{Im} \Psi'(x_{\text{top}} + iy) \ge 7$ for $0 \le y \le 900.19 - \operatorname{Im} x_{\text{top}}$, and for $Y := \operatorname{Im} x_{\text{top}} + y \ge 900.19$ the bound

$$\operatorname{Im} \Psi'(X + iY) \ge \frac{\pi}{2} \sum_i c_i - \sum_i \frac{|c_i||s_i X + u_i|}{Y} = 3\pi - \frac{738.09\ldots}{Y} \ge 8.6$$

applies, with $X := \operatorname{Re} x_{\text{top}}$. The sector condition on the ray follows from $|\alpha_i| \ge \operatorname{Im} x_{\text{top}} \ge 60$ and $|\arg \alpha_i| \le \frac{\pi}{2} + \frac{13}{60}$. □

At the saddle, $\Psi'(x_0) = 2\pi i$, so $-(\Psi'(x_0) - 3\pi i)^2/4 = \pi^2/4$ and

$$A(x_0) = \tfrac{1}{4}(2x_0 + 1)e^{\psi_\flat(x_0)} \ne 0, \tag{7.3}$$

since $\operatorname{Re} x_0 \ge 0$. The numerical values are $x_0 \approx 1.17192 + 0.00997i$, $K_\star \approx 260.8$ and $\sin(480 \operatorname{Im} \Omega) \approx 0.99757$. The contour from x_c to x_{top} has 46 segments: 4 on the lower steepest-descent curve, a connector, the Taylor segment $[v_5, v_6]$, a connector, and 39 on the upper curve.

7.5. **The integrand along the contour.** Put

$$\lambda_n := -Gn \log n + Gn - 29 \log n + 3 \log 2\pi.$$

Proposition 7.8. *Let $a > 0$ and $0 < \eta \le \pi$. There are n_1 and C, depending only on a and η, with the following property. Let $n \ge n_1$, and let x satisfy $\operatorname{Im} x > 0$, $\operatorname{Re} x \ge 0$ and the sector condition (a, η). Then*

$$F_n(nx) = n\, e^{\lambda_n + nf(x)} A_n(x)$$

for a number $A_n(x)$ with

(U) $|A_n(x)| \le C(|x|+3)^{164}$;

(M) $|A_n(x) - A(x)| \le C(|x|+3)^{164}\big(n^{-1} + e^{-2\pi n \operatorname{Im} x}\big)$.

Proof. Let $\varsigma = \sin(\eta/2)$. Apply Corollary 7.6 to each factor, with $\alpha = \alpha_i(x)$ and $\beta = \beta_i$; note $s_i(nx) + nu_i + \beta_i = n\alpha_i(x) + \beta_i$. Summing with weights c_i and using (7.1), the main terms add up to

$$-Gn\log n + n\Psi(x) + Gn - 29\log n + \psi_\flat(x) + 3\log 2\pi = \lambda_n + n\Psi(x) + \psi_\flat(x).$$

Hence $P_n(nx) = n(2x+1+\frac{1}{n})\exp(\lambda_n + n\Psi(x) + \psi_\flat(x) + e_n)$ with $|e_n| \le 76(9 + 1/(6\varsigma^2))/(an)$. Similarly, since $\sum_i c_i s_i \log n = 0$ and $|2nx+n+1| \ge n\operatorname{Re}(2x+1) \ge n$,

$$|\mu(nx) - \Psi'(x)| \le \frac{2}{n} + \frac{76(4 + 1/(8\varsigma^3))}{an}, \qquad |\mu'(nx)| \le \frac{4}{n^2} + \frac{76(3 + 1/\varsigma^4)}{an}.$$

With $\mathbf{q} = e^{2\pi i n x}$ and Lemma 7.4, $F_n(nx) = P_n(nx)\mathbf{q}^{-1}(\mathbf{q}H_n(nx))/\pi^2$, and $\mathbf{q}^{-1}e^{n\Psi(x)} = e^{nf(x)}$. So the claimed formula holds with

$$A_n(x) := \frac{(2x+1+\frac{1}{n})e^{\psi_\flat(x)+e_n}}{\pi^2}\,\mathbf{q}H_n(nx).$$

Since $a \le |\alpha_i(x)| \le |x|+3$, we have $|\log\alpha_i(x)| \le \log(|x|+3) + |\log a| + \pi$. By (7.1) this gives $|\Psi'(x)| \le C_1(|x|+3)$ and $|e^{\psi_\flat(x)}| \le e^{41(|\log a|+\pi)}(|x|+3)^{41}$. Let B be the largest of 1, $2|x|+2$ and these two bounds, so $B \le C_2(|x|+3)^{41}$. For $n \ge n_1$ the errors e_n, $\mu(nx) - \Psi'(x)$ and $\mu'(nx)$ are at most 1 in modulus. Since $|\mathbf{q}| \le 1$, (7.2) gives $|\mathbf{q}H_n| \le C_3B^2$, and (U) follows. For (M), write (7.2) as $\mathbf{q}H_n = -\frac{1}{4}[(\mu - 3\pi i)^2 + \mu'] + \mathbf{q}\,\Xi$ with $|\Xi| \le C_4B^2$, and compare with $-\frac{1}{4}(\Psi' - 3\pi i)^2$: the difference is $O(B/n) + O(B^2|\mathbf{q}|)$, with $|\mathbf{q}| = e^{-2\pi n \operatorname{Im} x}$. Also $|(2x+1+\frac{1}{n})e^{\psi_\flat+e_n} - (2x+1)e^{\psi_\flat}| = O(B^2/n)$. Combining these gives (M). □

7.6. **Deformation of the contour.**

Lemma 7.9. *Let $c \in (0, rn+1) \setminus \mathbb{Z}$, and let $w_0 = c, w_1, \dots, w_{K'} \in \mathbb{C}$ be such that every point of every segment $[w_k, w_{k+1}]$ other than w_0 has positive imaginary part, and $\operatorname{Re} w_k \ge 0$ for all k. Then*

$$i\int_0^\infty F_n(c+iy)\,dy = \sum_{k=0}^{K'-1}\int_{[w_k,w_{k+1}]} F_n(z)\,dz + i\int_0^\infty F_n(w_{K'}+iy)\,dy.$$

Proof. Let $\delta := \min_{m\in\mathbb{Z}}|c-m| \in (0,\frac{1}{2}]$; note $\delta \le c$. On $U := \{\operatorname{Im} z > 0\} \cup \{|z-c| < \delta\}$, F_n is holomorphic: U contains no integer and no pole of R_n''. U is star-shaped with respect to $c + i\delta/2$ (both pieces are convex and contain this point), so F_n has a primitive on U. For $Y > \operatorname{Im} w_{K'}$ consider the closed polygon $c \to c+iY \to \operatorname{Re} w_{K'} + iY \to w_{K'} \to w_{K'-1} \to \cdots \to w_0 = c$, which lies in U; the integral of F_n around it is 0. The horizontal side lies in $\operatorname{Re} z \ge 0$, where $|F_n(z)| \le \pi C\sinh(\pi Y)^{-1}$ by Lemma 7.1, so its contribution tends to 0 as $Y \to \infty$. The two vertical integrals converge absolutely, again by Lemma 7.1. □

7.7. **Laplace's method.**

Lemma 7.10. *Let $K_1, \rho_1 > 0$, $M_1 \geq 0$ with $M_1\rho_1 \leq \frac{3}{2}K_1$. Let $\phi, g : [-\rho_1, \rho_1] \to \mathbb{C}$ be continuous, with $|\phi(s) + K_1 s^2/2| \leq M_1|s|^3/6$, $|g| \leq A_0$ and $|g(s) - g(0)| \leq A_1|s|$. Then for $n > 0$*

$$\Big|\int_{-\rho_1}^{\rho_1} e^{n\phi(s)} g(s)\, ds - g(0)\sqrt{\frac{2\pi}{nK_1}}\Big| \leq \frac{1}{n}\Big(\frac{8A_0M_1}{3K_1^2} + \frac{2A_1}{K_1}\Big) + A_0 e^{-nK_1\rho_1^2/4}\sqrt{\frac{4\pi}{nK_1}}.$$

Moreover $\operatorname{Re}\phi(s) \leq 0$.

(The formalisation proves the same estimate with the larger constants $(16\sqrt{2\pi}/3) \cdot A_0M_1/K_1^2$ and $2\sqrt{2\pi} \cdot A_1/K_1$; only the order $O(1/n)$ is used.)

Proof. Write $\phi(s) = -K_1s^2/2 + T(s)$. Then $|T(s)| \leq M_1|s|^3/6 \leq M_1\rho_1 s^2/6 \leq K_1 s^2/4$, so $\operatorname{Re}\phi \leq -K_1s^2/4 \leq 0$ and

$$\begin{aligned}|e^{n\phi} - e^{-nK_1s^2/2}| &= e^{-nK_1s^2/2}|e^{nT} - 1| \leq e^{-nK_1s^2/2} n|T| e^{n|T|} \\ &\leq \frac{nM_1|s|^3}{6} e^{-nK_1s^2/4}.\end{aligned}$$

Split

$$\int e^{n\phi} g = g(0)\int e^{-nK_1s^2/2} + \int e^{-nK_1s^2/2}(g(s) - g(0)) + \int (e^{n\phi} - e^{-nK_1s^2/2}) g.$$

The second integral is at most $A_1 \int_{\mathbb{R}} |s| e^{-nK_1s^2/2} ds = 2A_1/(nK_1)$, and the third at most

$$\frac{nM_1A_0}{6}\int_{\mathbb{R}} |s|^3 e^{-nK_1s^2/4} ds = \frac{8A_0M_1}{3nK_1^2}.$$

Finally $\int_{|s|>\rho_1} e^{-nK_1s^2/2} ds \leq e^{-nK_1\rho_1^2/4}\int_{\mathbb{R}} e^{-nK_1s^2/4} ds$. □

7.8. **The asymptotic formula and nonvanishing.**

Proposition 7.11. *Let $\theta := \operatorname{Im}\Omega$ and $C_0 := d\,A(x_0)\sqrt{2\pi/K_\star}$, so $C_0 \neq 0$ by (7.3); write $C_0 = |C_0|e^{i\phi_0}$. Then, as $n \to \infty$ in $480\mathbb{N}$,*

$$L_n = -\frac{1}{\pi}\, n^{3/2} e^{\lambda_n + n\operatorname{Re}\Omega}\Big(|C_0|\sin(n\theta + \phi_0) + o(1)\Big).$$

Proof. Let $c_n := \lfloor nx_c \rfloor + \frac{1}{2}$. Since $0 < x_c < r$, we have $0 < c_n < rn + 1$ and $c_n \notin \mathbb{Z}$, and $\chi_n := c_n/n - x_c$ satisfies $|\chi_n| \leq \frac{1}{2n} \leq \delta_0$ for large n. Let $J_n := i\int_0^\infty F_n(c_n + iy)\, dy$. By Proposition 7.2, $L_n = -\frac{1}{\pi}\operatorname{Im} J_n$. Apply Lemma 7.9 with $w_0 = c_n$ and $w_k = nv_k$ $(1 \leq k \leq 46)$; the hypotheses hold by (P), since $\operatorname{Im}(c_n + t(nv_1 - c_n)) = tn\operatorname{Im} v_1 > 0$ for $t > 0$. Thus J_n is the sum of the 46 segment integrals and the integral over the vertical ray from nx_{top}.

The Taylor segment. Parametrising $[nv_5, nv_6]$ by $z = n(x_0 + sd)$, $|s| \le \rho$, and using Proposition 7.8 (applicable by (S1′), (S4) and $\operatorname{Re}(x_0 + sd) \ge 0$, which holds since $\operatorname{Re} v_5, \operatorname{Re} v_6 \ge 0$),

$$\int_{[nv_5,nv_6]} F_n(z)\,dz = n^2 d\, e^{\lambda_n + n\Omega}\mathcal{D}_n, \qquad \mathcal{D}_n := \int_{-\rho}^{\rho} e^{n\phi(s)} A_n(x_0 + sd)\,ds,$$

where $\phi(s) := f(x_0 + sd) - \Omega$. Since $f'(x_0) = \Psi'(x_0) - 2\pi i = 0$, $f''(x_0)d^2 = \Psi''(x_0)d^2 = -K_\star$ and $|f'''| = |\Psi'''| \le M_3$ on the disc, Taylor's formula gives $|\phi(s) + K_\star s^2/2| \le M_3|s|^3/6$. By Lemma 7.10, $\operatorname{Re}\phi \le 0$. By (M) with $\operatorname{Im}(x_0 + sd) \ge \operatorname{Im} x_0 - \rho > 0$, $|\mathcal{D}_n - \int_{-\rho}^{\rho} e^{n\phi(s)} A(x_0 + sd)\,ds| = O(n^{-1})$. The function A is holomorphic on $\operatorname{Im} x > 0$, hence bounded and Lipschitz on the disc, and Lemma 7.10 (with $K_1 = K_\star$, $M_1 = M_3$, $\rho_1 = \rho$) gives

$$\mathcal{D}_n = A(x_0)\sqrt{2\pi/(nK_\star)} + O(n^{-1}).$$

The far part. Let E_n be the sum of the other 45 segment integrals and the ray integral. On each of these segments, (S2) and (U) give $|F_n(z)| \le Cn\, e^{\lambda_n + n(\operatorname{Re}\Omega - \eta_{\text{far}})}$; the segments have length $O(n)$. (For the first segment, $z = n((x_c + \chi_n) + t(v_1 - x_c - \chi_n))$ with $|\chi_n| \le \delta_0$.) On the ray, $z = n(x_{\text{top}} + iu)$, $u \ge 0$, and (S3), (U) give $|F_n(z)| \le Cn\, e^{\lambda_n + n(\operatorname{Re}\Omega - \eta_{\text{far}})} e^{-n(7-2\pi)u}(|x_{\text{top}}| + u + 3)^{164}$, whose integral over $dz = in\,du$ is $O(n^2 e^{\lambda_n + n(\operatorname{Re}\Omega - \eta_{\text{far}})})$ because $n(7 - 2\pi) \ge 1$. (The formalisation uses the cruder bound $B \le Ce^{41|x|}$ in the proof of Proposition 7.8, and accordingly needs $n(7 - 2\pi) \ge 165$.) Hence $|E_n| = O(n^2 e^{\lambda_n + n\operatorname{Re}\Omega - n\eta_{\text{far}}})$.

Conclusion. Combining,

$$\begin{aligned} n^{-3/2} e^{-\lambda_n} J_n &= e^{n\Omega}\big(C_0 + O(n^{-1/2})\big) + O\big(n^{1/2} e^{-n\eta_{\text{far}}}\big) e^{n\operatorname{Re}\Omega} \\ &= C_0 e^{n\Omega} + o(e^{n\operatorname{Re}\Omega}), \end{aligned}$$

and $\operatorname{Im}(C_0 e^{n\Omega}) = |C_0| e^{n\operatorname{Re}\Omega} \sin(n\theta + \phi_0)$. □

Lemma 7.12. *For real u, v: $|\sin u| + |\sin(u + v)| \ge |\sin v|$.*

Proof. $\sin v = \sin(u + v)\cos u - \cos(u + v)\sin u$. □

Corollary 7.13. *For all sufficiently large $n \in 480\mathbb{N}$, $L_n \ne 0$ or $L_{n+480} \ne 0$. In particular $L_n \ne 0$ for infinitely many $n \in 480\mathbb{N}$.*

Proof. By Lemma 7.12 with $v = 480\theta$ and (S0), for every n, $\max(|\sin(n\theta + \phi_0)|, |\sin((n + 480)\theta + \phi_0)|) \ge \frac{1}{2}\sin(480\theta) > 0.49$. By Proposition 7.11, once the $o(1)$ terms are below $0.49|C_0|$, the corresponding L is nonzero. □

Remark 7.14. Proposition 7.11 also gives $\log|L_n| \le \lambda_n + n\operatorname{Re}\Omega + O(\log n) = -G\log n! + n\operatorname{Re}\Omega + O(\log n)$, with $\operatorname{Re}\Omega \le -29.7790172$. This is better than Proposition 6.6 by about 0.019 per unit of n. We do not use it, and it is not part of the formalisation.

8. Proof of Theorem 1.1

Put

(8.1)
$$Q := V + A_{\mathrm{cert}} + 20\log 2 = -29.76 + 15.8256549806\ldots + 13.8629436111\ldots = -0.0714014\ldots < 0.$$

Let $\ell_n := D_n L_n$, $n \in 480\mathbb{N}$. By Corollary 4.9, $\ell_n = a_n + \sum_{j=2}^{18} c_{n,j}\beta(2j)$ with $a_n, c_{n,j} \in \mathbb{Z}$. If $L_n \neq 0$, Corollary 5.9 and Proposition 6.6 give, for large n,

$$\log|\ell_n| = \log D_n + \log|L_n| \le n\big(Q + \varepsilon_n' + \varepsilon_n''\big),$$

the terms $\pm G\log n!$ cancelling. Since $Q < 0$ and $\varepsilon_n', \varepsilon_n'' \to 0$, it follows that $|\ell_n| \le e^{nQ/2}$ for all large n (trivially when $L_n = 0$), so $\ell_n \to 0$. On the other hand $D_n > 0$, so by Corollary 7.13, $\ell_n \neq 0$ for infinitely many n.

Suppose now that $\beta(4), \beta(6), \ldots, \beta(36)$ are all rational, and let $\Delta \ge 1$ be a common denominator. Then $\Delta\ell_n \in \mathbb{Z}$ for every n. For the infinitely many n with $\ell_n \neq 0$ this gives $|\Delta\ell_n| \ge 1$, which contradicts $\ell_n \to 0$. Hence at least one of $\beta(4), \ldots, \beta(36)$ is irrational. □

9. The formalisation

9.1. **What is formalised.** The repository `https://github.com/telaviv/beta-irrationality` [7], at release tag `v1.0` (commit `f8df497`), contains a proof of Theorem 1.1 in Lean 4 [2] (version v4.35.0) with Mathlib [12] (version v4.35.0 [13]). All declaration names and file paths in this section refer to that release. The formalisation covers every step of the proof of Theorem 1.1, including the parts that a written proof would cite: the prime number theorem, Stirling's formula with explicit remainder in sectors, the contour-integral representation, the deformation of contours, and all numerical certificates. Paths below are relative to the repository root, and declaration names refer to the version described here.

9.2. **The statement.** The two definitions that the reader has to check are, in `BetaIrrational/Defs.lean` and `BetaIrrational/Statement.lean` (in namespace `BetaIrr`, after unfolding the auxiliary definition `betaTerm`; `dirichletBeta` is `noncomputable`):

```
def dirichletBeta (s : ℕ) : ℝ :=
    ∑' m : ℕ, (-1 : ℝ)^m / (2 * (m : ℝ) + 1)^s
def OneOfBeta4To36 : Prop :=
    ∃ j ∈ Finset.Icc 2 18, Irrational (dirichletBeta
(2 * j))
theorem BetaIrr.I3.oneOfBeta4To36 : OneOfBeta4To36
```

The theorem is in the file `BetaIrrational/I3/MainUnconditional.lean`. Here $\sum'$ is Mathlib's unconditional sum (`tsum`); the file `BetaIrrational/Defs.lean` proves that the series is summable for $s \ge 2$, so `dirichletBeta`

`s` is the genuine value $\beta(s)$. `Finset.Icc 2 18` is $\{2, \dots, 18\}$, so the statement says that $\beta(2j)$ is irrational for some $2 \le j \le 18$. `Irrational x` is Mathlib's predicate $x \notin \mathbb{Q}$.

The command `lake env lean Axioms.lean` prints

```
'BetaIrr.I3.oneOfBeta4To36' depends on axioms:
    [propext, Classical.choice, Quot.sound]
```

and it prints the same axiom list for each of the main ingredients listed in `Axioms.lean`. These are Lean's three standard axioms, which Mathlib also uses. The development contains no `sorry`, no added axiom and no `native_decide`, so no step relies on compiled code.

9.3. **Correspondence with the paper.** Table 2 lists the main formal counterparts.[1] The theorem is `oneOfBeta4To36_of_certUnits` applied to the kernel-checked sum `CertA.bodyUnits_400`. It goes through `oneOfBeta4To36_of_parts`, whose inputs are the certificate inequality, Proposition 6.6 and Corollary 7.13, and through the criterion `exists_irrational_of_small_forms`, which formalises Section 8.

A few arguments are organised differently in the formalisation, because Mathlib (at the version used) lacks some standard tools.

- Mathlib has no residue theorem. Proposition 7.2 is proved by subtracting the principal parts $(-1)^t R_n''(t)\big((z-t)^{-1} - (z-c+1)^{-1}\big)$ at the integers $c < t \le N$, removing the resulting removable singularities, applying Cauchy's theorem for rectangles, and evaluating the model integrals explicitly.
- The deformation (Lemma 7.9) uses primitives of holomorphic functions on unions of discs, extended to the upper half-plane together with a small disc around c.
- Stirling's formula (Theorem 7.5) is proved from Euler's limit formula for Γ, an exact Euler–Maclaurin identity and Mathlib's Stirling formula for $n!$; this yields the constant $1/12$ of [4, §5.11(ii)].
- The prime number theorem in the form $\theta(y)/y \to 1$ is ported from the PrimeNumberTheoremAnd project [6] (Wiener–Ikehara route), through the port in the `zeta5` repository of César Del Solar (GitHub

[1] Files, relative to `BetaIrrational/`: `Expansion.lean` (`PFData`); `I3/Concrete/Expansion.lean` (`R_eq_pfR`, `pf_symmetric`, `expansion_I3`, `pf_R2_inner_zero`); `Arith/PSVal.lean` (`psge_*`); `I3/Arith/LemmaA.lean`; `I3/Arith/Integrality.lean`; `Arith/LemmaBC.lean` (`vge_Bq`, `vge_rho_of_R2_zero`); `I3/Arith/Lemma4.lean` (`int_*`, `log_D_sub_le`); `I3/Arith/Size.lean`; `I3/CertA/Pointwise.lean`, `Tail.lean`, `Attain.lean`, `Cert.lean`; `I3/CertAK/CertCompute.lean` (`bodyUnits_400`); `PNT/PrimeSums.lean`; `PNT/Theta.lean`; `I3/Glue/Sharp.lean`; `I3/Glue/Main.lean` (`arith_I3`, `oneOfBeta4To36_of_parts`); `I3/Size/RBound.lean`, `PhiBound.lean`, `Main.lean`; `Analytic/Contour.lean`; `I3/Analytic/Gamma.lean`, `LogDeriv.lean`, `Deform.lean`, `NonvanI3.lean`; `Stirling/Bound.lean`, `Scaled.lean`, `Digamma.lean`, `ScaledDigamma.lean`; `I3/Interval/SaddleDataI3.lean`; `Analytic/Laplace.lean`; `Criterion.lean`; `I3/MainCond.lean`; `I3/MainUnconditional.lean`.

paper	Lean declaration (namespace `BetaIrr`)
Lemma 3.1, Cor. 3.5	`I3.R_eq_pfR`, `I3.pf_symmetric`, `PFData.tsum_alt_R2_even`, `I3.expansion_I3`
Lemmas 4.2, 4.3, Cor. 4.4	`Arith.psge_mul`, `Arith.psge_inv`, `I3.Arith.vge_a`, `I3.Arith.a_bound_sharp`
Lemma 4.5	`Arith.vge_Bq`
Lemma 4.6	`Arith.vge_rho_of_R2_zero`, `I3.pf_R2_inner_zero`
Prop. 4.8	`I3.Arith.int_rho_I3`, `I3.Arith.int_tau_I3`
Props. 4.12, 4.13	`I3.Arith.small_primes_sum_le`, `I3.Arith.E2_sub_le`, `I3.Arith.log_D_sub_le`
Prop. 5.3, Lemma 5.4	`I3.CertA.pointwise_le_eStar_prime`, `I3.CertA.eStar_le_tail`
Lemma 5.5	`I3.CertA.eStar_eq_mid`, `I3.CertA.eStar_isGreatest`
Prop. 5.6	`I3.CertA.bodyUnits_400`, `I3.CertA.certFact_of_units`
Lemma 5.7, Prop. 5.8	`PNT.eventually_primeSum_lemma5_le`, `I3.sharp_of_pointwise`, `I3.arith_I3`
prime number theorem	`PNT.theta_div_tendsto`
Props. 6.4, 6.5, 6.6	`I3.Size.log_R_le_fact`, `I3.Size.Phi_le_V`, `I3.size_I3`
Prop. 7.2	`Analytic.hasSum_alternating_eq_half_line`
Lemmas 7.3, 7.4	`I3.Analytic.RC_eq_Pfun`, `I3.Analytic.Fn_eq_PH`
Thm. 7.5, Cor. 7.6	`Stirling.stirling_sector`, `Stirling.stirling_scaled`; for ψ, ψ': `Stirling.digamma_sector`, `Stirling.deriv_digamma_sector`, `Stirling.digamma_scaled`, `Stirling.deriv_digamma_scaled`
Thm. 7.7	`I3.Interval.I3_saddleData`
Lemmas 7.9, 7.10	`I3.Analytic.contour_eq_polyline_Fn`, `Analytic.laplace_bound`
Prop. 7.11, Cor. 7.13	`I3.Analytic.nonvan_of_saddleData`
Section 8	`exists_irrational_of_small_forms`, `I3.oneOfBeta4To36_of_parts`, `I3.oneOfBeta4To36_of_certUnits`, `I3.oneOfBeta4To36`

TABLE 2. Formal counterparts of the main steps.

`domino14`) [3]. Only the `PNTPort/` folder of that repository carries the Apache License 2.0, and the ported files keep it. Chebyshev's bound $\theta(y) \le y \log 4$ is in Mathlib.

- All error terms are explicit, except those coming from the prime number theorem and the compactness arguments in Section 7 (the amplitude bounds A_0, A_1 of Lemma 7.10, applied to $g(s) = A(x_0 + sd)$ in the proof of Proposition 7.11).
- Some explicit constants in the formal statements are weaker than in the text (e.g. Lemma 7.10); only their finiteness is used.

9.4. **How the certificates are checked.** The arithmetic certificate and the saddle certificate are decidable statements about rational data, proved by `decide +kernel`, i.e. by evaluation in the Lean kernel; the 58 logarithm enclosures of Proposition 6.5 are proved by `norm_num` from Mathlib's series bounds for log. All proofs are checked by the kernel. The data (event lists, polyline vertices, rational enclosures) were produced by scripts that are not trusted: a wrong value would make the build fail, but could not make a false statement provable. The generators of the arithmetic certificate and of the Φ enclosures are in the repository (`scripts/certA/`, `scripts/size/`); the ad hoc Python generator of the polyline and the saddle boxes is not.

The arithmetic certificate (Proposition 5.6). The specification of e^* and of the event list (built with `List.mergeSort`) is what the soundness proofs use, but the kernel cannot evaluate it efficiently. The proof therefore passes through equivalent forms, each with a general proof of equivalence: a closed form of e^* that uses only natural-number arithmetic (it rests on the observation behind (5.2) that each block contributes $m_j(1+\lfloor y-\delta_j x\rfloor-\lceil y-(1-\delta_j)x\rceil)$ in all cases); a structural merge sort; a decomposition of the event list into 133 windows, each sorted in the kernel and compared with literal data, together with a check that their concatenation is strictly increasing; and a decomposition of the sum into 1,995 chunks, each checked separately. The resulting value of the rounded body sum, U, agrees exactly with an independent exact-rational Python implementation.

The saddle certificate (Theorem 7.7). This uses a verified interval arithmetic with rational endpoints (rounded outwards to the grid $2^{-p}\mathbb{Z}$), complex boxes, verified enclosures of log, arctan, π and arg, the contraction lemma for (S0), and segment, disc and ray checks as described in the proof of Theorem 7.7. The certified value of $\operatorname{Re}\Omega$ agrees with two independent ball-arithmetic (Arb) implementations.

Caveat. Two helper functions in the arithmetic-certificate code carry `@[implemented_by]` attributes. These affect only compiled code (used by the scripts that generate the data); the kernel never uses them, and they add no axiom.

9.5. **Cost.** A cold build takes more than an hour on a desktop computer. The arithmetic certificate dominates: about 2.6 CPU-hours for the chunk sums and about 1 CPU-hour for the window checks, run as three parallel import chains, with up to about 4 GB resident memory per process (16 GB of RAM recommended). The interval certificate takes about 30 CPU-minutes.

9.6. **What has to be trusted.** Beyond the Lean kernel and the three axioms, the reader has to trust that the two definitions displayed above express Theorem 1.1, and the Mathlib definitions they use (real numbers, `tsum`, `Irrational`, `Finset.Icc`). Nothing else, in particular no numerical data and no generating script, needs to be trusted. A reader who does not want to trust the elaborator's environment handling can replay the environment with `lean4checker`. Apart from the check by the Lean kernel, all

verification of the formalisation so far was carried out by AI systems; no human expert has yet reviewed the Lean code.

10. Remarks

10.1. **The margin.** The proof needs $Q = V + A_{\mathrm{cert}} + 20\log 2 < 0$, and $Q \approx -0.0714$. The three terms are of size 30, 16 and 14, so the margin is about 0.2% of the largest term. It is thin, but each of the three numbers is certified: V by Proposition 6.5, A_{cert} by exact rational arithmetic, and $\log 2$ by Mathlib's bounds. With the saddle-point exponent of Remark 7.14 in place of V one would get $\operatorname{Re}\Omega + A_{\mathrm{cert}} + 20\log 2 \le -0.0904$.

10.2. **Effectivity.** The theorem is proved for $n \ge n_0$ with an unspecified n_0, because the prime number theorem is used without an error term. This does not affect the irrationality statement. The other error terms are explicit but large: for instance the small-prime term $312\sqrt{\mathcal{N}}\log\mathcal{N}$ of Proposition 4.13 falls below $0.0714\,n$ only for n of order 10^{11} [numerical estimate]. An effective version would need an explicit form of the prime number theorem and a sharper count at the small primes.

10.3. **Why $\beta(36)$.** The parameters were found by a numerical search over the block profile (offsets on the grid $\frac{1}{160}\mathbb{Z}$) and over r [numerical, not part of the proof]. The optimum found is a nearly uniform density of offsets on $(0, 0.29]$, with about a quarter of the multiplicity on $\delta = 0$. For the same type of construction (triple zeros at every integer of the pole region) with total multiplicity $S = 33$ (which would give $\beta(4), \ldots, \beta(34)$), the best value of Q in the coarse model of the search (offsets on $\frac{1}{80}\mathbb{Z}$, $r = 13/12$) was $\approx +1.18$, against about -0.15 needed in that model. Halving the grid gained about 0.06, with decreasing returns, so $\beta(34)$ appears out of reach for this family [heuristic].

10.4. **Comparison with constructions that include $\beta(2)$.** The construction of Zudilin [15, §3], with the parameters of Lai–Zhou [8, Thm. 6.1], which reaches $\beta(2), \ldots, \beta(10)$, uses simple zeros. For the second-derivative form the zeros in the pole region must be triple (the proof of Lemma 4.6 needs $R_n'' = 0$ there), and differentiating twice raises every pole order by 2, so each denominator exponent grows by 2. We believe that these two costs, three times the zero density and the shift by 2, account for the much larger number of values needed here [heuristic]; in exchange the form never involves $\beta(2)$.

Acknowledgements and disclosure

The construction, the proofs, the parameter search, the certificate programs, the Lean formalisation and the text of this paper were produced largely by AI models (Claude, Anthropic), working under the direction of the author. Apart from the check by the Lean kernel, all verification so far, including the internal reviews of this manuscript, was carried out by AI

systems. No human expert has yet reviewed the mathematics or the Lean code; we would be grateful for corrections. The prime number theorem in the formalisation is ported from the PrimeNumberTheoremAnd project of A. Kontorovich, T. Tao and collaborators, through the `zeta5` repository of César Del Solar.

References

1. Keith Ball and Tanguy Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. **146** (2001), no. 1, 193–207.
2. Leonardo de Moura and Sebastian Ullrich, *The Lean 4 theorem prover and programming language*, Automated Deduction – CADE 28, Lecture Notes in Comput. Sci., vol. 12699, Springer, 2021, pp. 625–635.
3. César Del Solar, *$\zeta(5)$ is irrational: a Lean 4 formalization*, `https://github.com/domino14/zeta5`, 2026, Source of the port of the prime number theorem used here (commit 3da8eb1b91f9a90a1c0b7959e845c060dbd16256, 23 September 2026).
4. *NIST digital library of mathematical functions*, `https://dlmf.nist.gov/`, Release 1.2.8 of 2026-09-15, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
5. Stéphane Fischler, *Irrationality of values of L-functions of Dirichlet characters*, J. Lond. Math. Soc. (2) **101** (2020), no. 2, 857–876, arXiv:1904.02402.
6. Alex Kontorovich, Terence Tao, et al., *PrimeNumberTheoremAnd*, `https://github.com/AlexKontorovich/PrimeNumberTheoremAnd`, 2026, Lean project, Apache License 2.0; the formalisation uses files from commit a5154676af9aa3095150ee410cdda80555aa0642 (30 August 2026).
7. Shawn Krisman, *Lean formalisation: one of $\beta(4), \beta(6), \ldots, \beta(36)$ is irrational*, `https://github.com/telaviv/beta-irrationality`, 2026, Release v1.0, commit f8df497467901e09fa497553f00d8ffa4480a46d.
8. Li Lai and Li Zhou, *At least two of $\zeta(5), \zeta(7), \ldots, \zeta(35)$ are irrational*, Publ. Math. Debrecen **101** (2022), no. 3–4, 353–372, arXiv:2103.00904.
9. OpenAI, *Catalan's constant is irrational*, OpenAI Math Release preprint, `https://github.com/openai/math`, file `preprints/Catalans-constant-is-irrational-September-24-2026/paper.pdf`, 2026, September 24, 2026. AI-generated and not peer reviewed; Lean formalisation described in `lean/docs/005.md` of the same repository.
10. Tanguy Rivoal, *Irrationalité d'au moins un des neuf nombres $\zeta(5), \zeta(7), \ldots, \zeta(21)$*, Acta Arith. **103** (2002), no. 2, 157–167, arXiv:math/0104221.
11. Tanguy Rivoal and Wadim Zudilin, *Diophantine properties of numbers related to Catalan's constant*, Math. Ann. **326** (2003), no. 4, 705–721.
12. The mathlib Community, *The Lean mathematical library*, Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, arXiv:1910.09336, pp. 367–381.
13. ———, *mathlib4*, `https://github.com/leanprover-community/mathlib4`, 2026, Tag v4.35.0, commit 56a005855ea17e0b0958f73ec801ae140b0a1679.
14. Wadim Zudilin, *Arithmetic of linear forms involving odd zeta values*, J. Théor. Nombres Bordeaux **16** (2004), no. 1, 251–291, arXiv:math/0206176.
15. ———, *Arithmetic of Catalan's constant and its relatives*, Abh. Math. Semin. Univ. Hambg. **89** (2019), no. 1, 45–53, arXiv:1804.09922.