

A fourteen-dimensional biperfect Lie bialgebra

ChatGPT 6 Pro

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Abstract

We construct a fourteen-dimensional Lie bialgebra for which both the underlying Lie algebra and its dual are perfect. The construction is defined over $\mathbb{Q}$ and is self-dual over $\mathbb{C}$. It gives an affirmative answer to Question 4.1(3) of Etingof–Gelaki–Guralnick–Saxl. Its twenty-eight-dimensional Drinfeld double is quasitriangular with injective cobracket, also answering Question 4.1(4).

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1 Statement and construction

A Lie algebra $\mathfrak{g}$ is *perfect* if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. A finite-dimensional Lie bialgebra is *biperfect* if both $\mathfrak{g}$ and its dual Lie algebra $\mathfrak{g}^*$ are perfect. The existence of a nonzero example over $\mathbb{C}$ was asked in [2, Question 4.1(3)].

Theorem 1. *There exists a fourteen-dimensional Lie bialgebra over $\mathbb{Q}$ such that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are perfect. After extension of scalars to $\mathbb{C}$, it is isomorphic to its dual as a Lie bialgebra. Both underlying Lie algebras have Levi factor $\mathfrak{sl}_2$ and an eleven-dimensional two-step nilpotent radical with three-dimensional center.*

We construct two complementary perfect Lagrangian subalgebras of a quadratic Lie algebra. The Manin-triple correspondence then supplies the required Lie bialgebra [1, §3, pp. 802–803]. All vector spaces below are over $\mathbb{Q}$ until the discussion of self-duality.

1.1 The ambient quadratic Lie algebra

Let A and B be two-dimensional vector spaces with bases a_0, a_1 and b_0, b_1 , respectively. Equip them with alternating forms $\varepsilon_A, \varepsilon_B$ normalized by

$$\varepsilon_A(a_0, a_1) = \varepsilon_B(b_0, b_1) = 1.$$

Let $M = \mathbb{Q}^4$ with the symmetric form $(u, v) = u^t v$, and put

$$\mathfrak{s} = \mathfrak{sl}(A), \quad \mathfrak{t} = \mathfrak{sl}(B), \quad \mathfrak{k} = \mathfrak{s} \oplus \mathfrak{t}, \quad V = A \otimes B \otimes M.$$

The natural action of $\mathfrak{k}$ on V preserves the nondegenerate *symmetric* form

$$q(a \otimes b \otimes u, a' \otimes b' \otimes v) = \varepsilon_A(a, a') \varepsilon_B(b, b') u^t v. \quad (1)$$

Define $\mu : \wedge^2 V \longrightarrow \mathfrak{k}^*$ by

$$\mu(v, w)(x) = q(xv, w), \quad x \in \mathfrak{k}. \quad (2)$$

This is alternating because the action preserves q .

On $\mathfrak{d} = \mathfrak{k} \oplus V \oplus \mathfrak{k}^*$ define

$$[(x, v, \alpha), (y, w, \beta)] = ([x, y], xw - yv, \text{ad}_x^* \beta - \text{ad}_y^* \alpha + \mu(v, w)), \quad (3)$$

where $(\text{ad}_x^* \alpha)(y) = -\alpha([x, y])$. Thus $V \oplus \mathfrak{k}^*$ is two-step nilpotent, with $[V, V] \subseteq \mathfrak{k}^*$ and $[V, \mathfrak{k}^*] = [\mathfrak{k}^*, \mathfrak{k}^*] = 0$. The form

$$\langle (x, v, \alpha), (y, w, \beta) \rangle = q(v, w) + \alpha(y) + \beta(x) \quad (4)$$

is nondegenerate, symmetric, and invariant.

These formulas are the double extension of the abelian quadratic Lie algebra V by $\mathfrak{k}$ [3, §2.2]. For completeness, the Jacobi identity follows from the representation property of the actions of $\mathfrak{k}$ and the equivariance

$$\text{ad}_x^* \mu(v, w) = \mu(xv, w) + \mu(v, xw).$$

Indeed, evaluation on $y \in \mathfrak{k}$ gives on the right $q(yxv, w) + q(yv, xw) = q([y, x]v, w)$, which is the left-hand side. Jacobi identities involving three elements of $V \oplus \mathfrak{k}^*$ are immediate. Invariance of (4) follows from invariance of q , the definition of the coadjoint action, and $\langle [v, w], x \rangle = q(xv, w)$.

1.2 Two complementary Lagrangian subspaces

Choose

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}. \quad (5)$$

Both matrices are symmetric and invertible, and

$$ST = -TS, \quad \text{Spec}(S) = \text{Spec}(T) = \{1, -1, 2, -2\}. \quad (6)$$

In particular, each of the triples I, S, S^2 and I, T, T^2 is linearly independent.

Define linear embeddings

$$\iota_T : A \otimes M \longrightarrow V, \quad \iota_T(a, u) = a \otimes b_0 \otimes u + a \otimes b_1 \otimes Tu, \quad (7)$$

$$j_S : B \otimes M \longrightarrow V, \quad j_S(b, u) = a_0 \otimes b \otimes u + a_1 \otimes b \otimes Su, \quad (8)$$

where the notation on pure tensors is extended linearly. Set $U = \text{im } \iota_T$ and $W = \text{im } j_S$. Then U is $\mathfrak{s}$ -stable and W is $\mathfrak{t}$ -stable.

Symmetry of T gives

$$q(\iota_T(a, u), \iota_T(a', v)) = \varepsilon_A(a, a')(u^t T v - (T u)^t v) = 0.$$

Likewise $q(W, W) = 0$. Since $\dim U = \dim W = 8$ and $\dim V = 16$, both are Lagrangian.

To see that they are complementary, write $z \in U$ uniquely as

$$z = a_0 \otimes b_0 \otimes u + a_0 \otimes b_1 \otimes Tu + a_1 \otimes b_0 \otimes v + a_1 \otimes b_1 \otimes Tv.$$

Membership in W requires $v = Su$ and $Tv = STu$, hence $(TS - ST)u = 0$. But $TS - ST = 2TS$ is invertible by (6). Thus $u = v = 0$, proving

$$V = U \oplus W. \quad (9)$$

1.3 The Manin triple

Identify $\mathfrak{k}^* = \mathfrak{s}^* \oplus \mathfrak{t}^*$ in the natural way, and set

$$\mathfrak{g} = \mathfrak{s} \oplus U \oplus \mathfrak{t}^*, \quad \mathfrak{h} = \mathfrak{t} \oplus W \oplus \mathfrak{s}^*. \quad (10)$$

These are Lie subalgebras of $\mathfrak{d}$. For example, if $u, u' \in U$ and $x \in \mathfrak{s}$, then

$$\mu(u, u')(x) = q(xu, u') = 0,$$

because U is $\mathfrak{s}$ -stable and isotropic. Consequently $[U, U] \subseteq \mathfrak{t}^*$. Also $[\mathfrak{s}, U] \subseteq U$ and $\mathfrak{t}^*$ is central in $\mathfrak{g}$. The argument for $\mathfrak{h}$ is identical.

The form (4) vanishes on each of $\mathfrak{g}$ and $\mathfrak{h}$: the only pairings between $\mathfrak{k}$ and $\mathfrak{k}^*$ are $\mathfrak{s}$ with $\mathfrak{s}^*$ and $\mathfrak{t}$ with $\mathfrak{t}^*$. Equation (9) gives

$$\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}, \quad \dim \mathfrak{g} = \dim \mathfrak{h} = 3 + 8 + 3 = 14.$$

Thus $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple.

2 Perfectness and the Lie bialgebra structure

Since $U \cong A \otimes M$ as an $\mathfrak{s}$ -module, we have

$$[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}, \quad [\mathfrak{s}, U] = U. \quad (11)$$

It remains to prove $[U, U] = \mathfrak{t}^*$. Let

$$x = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathfrak{t}.$$

Directly from (1) and (7),

$$\mu(\iota_T(a, u), \iota_T(a', v))(x) = \varepsilon_A(a, a') u^t (2\alpha T + \beta T^2 - \gamma I) v. \quad (12)$$

If x annihilates $[U, U]$, taking $a = a_0$, $a' = a_1$ and arbitrary $u, v \in M$ gives

$$2\alpha T + \beta T^2 - \gamma I = 0.$$

The independence of I, T, T^2 implies $x = 0$. Hence the annihilator of $[U, U] \subseteq \mathfrak{t}^*$ is zero, so

$$[U, U] = \mathfrak{t}^*. \quad (13)$$

Together with (11), this proves $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Interchanging A, B and replacing T by S gives

$$[\mathfrak{t}, \mathfrak{t}] = \mathfrak{t}, \quad [\mathfrak{t}, W] = W, \quad [W, W] = \mathfrak{s}^*.$$

Thus $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$ as well.

To specify the cobracket explicitly, identify $\mathfrak{h}$ with $\mathfrak{g}^*$ by $\xi(x) = \langle x, \xi \rangle$, and define

$$\langle \delta(x), \xi \otimes \eta \rangle = \langle x, [\xi, \eta] \rangle, \quad x \in \mathfrak{g}, \quad \xi, \eta \in \mathfrak{h}. \quad (14)$$

Here $\wedge^2 \mathfrak{g}$ is identified with the skew tensors in $\mathfrak{g} \otimes \mathfrak{g}$. The Manin-triple correspondence guarantees that δ satisfies the co-Jacobi identity and the Lie bialgebra cocycle identity. Its dual Lie algebra is precisely $\mathfrak{h}$, which we have proved perfect. Moreover,

$$\ker \delta = [\mathfrak{h}, \mathfrak{h}]^\perp \cap \mathfrak{g} = 0.$$

Every formula and every subspace used above is defined over $\mathbb{Q}$; in particular the nondegenerate pairing between $\mathfrak{g}$ and $\mathfrak{h}$ defines (14) over $\mathbb{Q}$.

Put $\mathfrak{n} = U \oplus \mathfrak{t}^*$. Then

$$\text{rad}(\mathfrak{g}) = \mathfrak{n}, \quad \mathfrak{g}/\mathfrak{n} \cong \mathfrak{s}, \quad Z(\mathfrak{n}) = Z(\mathfrak{g}) = \mathfrak{t}^*.$$

Indeed, $\mathfrak{n}$ is a two-step nilpotent ideal with semisimple quotient, so it is the radical of $\mathfrak{g}$. The component of (12) corresponding to γ is the nondegenerate alternating form $-\varepsilon_A \otimes (u^t v)$ on $A \otimes M$, so no nonzero element of U is central in $\mathfrak{n}$. Thus $Z(\mathfrak{n}) = \mathfrak{t}^*$. Since $\mathfrak{s}$ has zero center, every central element of $\mathfrak{g}$ lies in $\mathfrak{n}$; as $\mathfrak{t}^*$ is central in $\mathfrak{g}$, this also gives $Z(\mathfrak{g}) = \mathfrak{t}^*$. The same statements hold for $\mathfrak{h}$ with $\mathfrak{s}$ and $\mathfrak{t}$ interchanged.

3 Self-duality and the quasitriangular double

Over $\mathbb{C}$, put

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Then $P^t P = P^2 = I$ and

$$PSP = T, \quad PTP = S.$$

Identify A and B through their chosen bases. Swapping their tensor factors and applying P to M gives an isometry

$$\Phi_V(a \otimes b \otimes u) = b \otimes a \otimes Pu$$

of V . Together with the swaps $\mathfrak{s} \leftrightarrow \mathfrak{t}$ and $\mathfrak{s}^* \leftrightarrow \mathfrak{t}^*$, it extends to an involutive isometric Lie algebra automorphism Φ of $\mathfrak{d}$. The identities $PT = SP$ and $PS = TP$ show that $\Phi(U) = W$ and $\Phi(W) = U$. Hence $\Phi(\mathfrak{g}) = \mathfrak{h}$.

Reversing the two Lagrangian subalgebras gives the dual Lie bialgebra: $\mathfrak{h}$ carries $\delta_{\mathfrak{h}}$, the transpose of the bracket on $\mathfrak{g}$. For $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned} \langle \delta_{\mathfrak{h}}(\Phi x), y \otimes z \rangle &= \langle \Phi x, [y, z] \rangle \\ &= \langle x, [\Phi y, \Phi z] \rangle \\ &= \langle \delta(x), \Phi y \otimes \Phi z \rangle \\ &= \langle (\Phi \otimes \Phi) \delta(x), y \otimes z \rangle. \end{aligned}$$

Here we used $\Phi^2 = 1$ and preservation of the bracket and pairing. Thus $\delta_{\mathfrak{h}} \Phi = (\Phi \otimes \Phi) \delta$, so $\Phi|_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{h} \cong \mathfrak{g}^*$ is a Lie bialgebra isomorphism. This proves the self-duality asserted in Theorem 1 and completes its proof. The canonical double cobracket restricts to $-\delta_{\mathfrak{h}}$ on $\mathfrak{h}$, as in Corollary 3 below.

Remark 2. The fourteen-dimensional example is not coboundary. Its central subspace $\mathfrak{t}^*$ has a nonzero, indeed injective, cobracket: the restriction of (14) to $\mathfrak{t}^*$ is the transpose of the bracket on $\mathfrak{t}$. A coboundary cobracket vanishes on the center. The same observation applies to the dual example.

Corollary 3. *There exists a twenty-eight-dimensional quasitriangular Lie bialgebra over $\mathbb{Q}$ with injective cobracket.*

Proof. Equip $\mathfrak{d} = D(\mathfrak{g})$ with its canonical quasitriangular structure. If $e_i \in \mathfrak{g}$ and $f_i \in \mathfrak{h}$ are dual bases for the pairing, its classical r -matrix is $r = \sum_i e_i \otimes f_i$. The cobracket $\delta_{\mathfrak{d}}(z) = [z \otimes 1 + 1 \otimes z, r]$ restricts to δ on $\mathfrak{g}$ and to $-\delta_{\mathfrak{h}}$ on $\mathfrak{h}$, where $\delta_{\mathfrak{h}}$ is the transpose of the bracket on $\mathfrak{g}$. Both restrictions are injective because $\mathfrak{g}$ and $\mathfrak{h}$ are perfect, and their images lie in the disjoint subspaces $\wedge^2 \mathfrak{g}$ and $\wedge^2 \mathfrak{h}$ of $\wedge^2 \mathfrak{d}$. Therefore $\delta_{\mathfrak{d}}$ is injective. $\square$

This is the consequence anticipated in [2, Remark 4.2(3)] and answers its Question 4.1(4).

Acknowledgements

This paper was written in response to the following prompt by Pavel Etingof:

Is there a nonzero finite-dimensional Lie bialgebra $\mathfrak{g}$ over $\mathbb{C}$ for which both $\mathfrak{g}$ and $\mathfrak{g}^*$ are perfect?

References

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