
Noncommutative Maccone-Pati Uncertainty Principles

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Date: October 16, 2026

Abstract: Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ be possibly unbounded self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we show that

$$(1) \quad d_x(A)^2 + d_x(B)^2 \geq \mp \langle \{A, B\}x, x \rangle + \langle (A \pm B)x, y \rangle \langle y, (A \pm B)x \rangle \pm \{ \langle Ax, x \rangle, \langle Bx, x \rangle \},$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0$$

$$(2) \quad d_x(A)^2 + d_x(B)^2 \geq \mp i \langle [A, B]x, x \rangle + \langle (A \pm iB)x, y \rangle \langle y, (A \pm iB)x \rangle \mp i[\langle Ax, x \rangle, \langle Bx, x \rangle],$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

where

$$d_x(A) := \sqrt{\langle Ax, Ax \rangle - \langle Ax, x \rangle^2}, \quad [A, B] := AB - BA,$$

$$[\langle Ax, x \rangle, \langle Bx, x \rangle] := \langle Ax, x \rangle \langle Bx, x \rangle - \langle Bx, x \rangle \langle Ax, x \rangle.$$

We call Inequalities (1) and (2) as noncommutative Maccone-Pati uncertainty principles. They reduce to uncertainty principles derived by Maccone and Pati [*Phys. Rev. Lett.*, 2014] whenever $\mathcal{A} = \mathbb{C}$.

Keywords: Uncertainty Principle, Hilbert C^* -module.

Mathematics Subject Classification (2020): 46L08.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space and A be a possibly unbounded self-adjoint linear operator defined on domain $\mathcal{D}(A) \subseteq \mathcal{H}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, define the **uncertainty** of A at the point h as

$$\Delta_h(A) := \|Ah - \langle Ah, h \rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h \rangle^2}.$$

In 1929, Robertson [14] derived the following mathematical form of the uncertainty principle of Heisenberg derived in 1927 [6]. Recall that for two linear operators $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$, we define the commutator $[A, B] := AB - BA$ and the anti-commutator $\{A, B\} := AB + BA$.

Theorem 1.1. [4, 6, 14, 17] (**Heisenberg-Robertson Uncertainty Principle**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$(1) \quad \frac{1}{2} (\Delta_h(A)^2 + \Delta_h(B)^2) \geq \frac{1}{4} (\Delta_h(A) + \Delta_h(B))^2 \geq \Delta_h(A) \Delta_h(B) \geq \frac{1}{2} |\langle [A, B]h, h \rangle|.$$

In 1930, Schrodinger improved Inequality (1) [15].

Theorem 1.2. [3, 15] (**Heisenberg-Robertson-Schrodinger Uncertainty Principle**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$\begin{aligned} \Delta_h(A)\Delta_h(B) &\geq |\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle| = \frac{\sqrt{|\langle [A, B]h, h \rangle|^2 + |\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|^2}}{2} \\ &= \frac{\sqrt{(\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle)^2 - \langle [A, B]h, h \rangle^2}}{2}. \end{aligned}$$

In 2014, Maccone and Pati derived the following uncertainty principles which work for orthogonal vectors [10].

Theorem 1.3. [10] (**Maccone-Pati Uncertainty Principles**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

(i)

$$\begin{aligned} \Delta_h(A)^2 + \Delta_h(B)^2 &\geq |\langle (A+B)h, k \rangle|^2 - \langle \{A, B\}h, h \rangle + 2\langle Ah, h \rangle \langle Bh, h \rangle, \\ &\forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0. \end{aligned}$$

(ii)

$$\begin{aligned} \Delta_h(A)^2 + \Delta_h(B)^2 &\geq |\langle (A-B)h, k \rangle|^2 + \langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle, \\ &\forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0. \end{aligned}$$

(iii)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq \frac{|\langle (A+B)h, k \rangle|^2 + |\langle (A-B)h, k \rangle|^2}{2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

(iv)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq -i\langle [A, B]h, h \rangle + |\langle (A+iB)h, k \rangle|^2, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

(v)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq i\langle [A, B]h, h \rangle + |\langle (A-iB)h, k \rangle|^2, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

(vi)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq \frac{|\langle (A+iB)h, k \rangle|^2 + |\langle (A-iB)h, k \rangle|^2}{2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

(vii)

$$\Delta_h(A)\Delta_h(B) \geq \frac{-i\langle [A, B]h, h \rangle}{2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} + i \frac{B}{\Delta_h(B)} \right) h, k \right\rangle \right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

(viii)

$$\Delta_h(A)\Delta_h(B) \geq \frac{i\langle [A, B]h, h \rangle}{2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} - i \frac{B}{\Delta_h(B)} \right) h, k \right\rangle \right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

(ix)

$$\Delta_h(A)\Delta_h(B) \geq \frac{2\langle Ah, h \rangle \langle Bh, h \rangle - \langle \{A, B\}h, h \rangle}{2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} + \frac{B}{\Delta_h(B)} \right) h, k \right\rangle \right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

(x)

$$\Delta_h(A)\Delta_h(B) \geq \frac{-2\langle Ah, h \rangle \langle Bh, h \rangle + \langle \{A, B\}h, h \rangle}{2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} - \frac{B}{\Delta_h(B)} \right) h, k \right\rangle \right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

We ask: What is noncommutative analogues of Theorems 1.3? For this, we need the notion of noncommutative Hilbert spaces known as Hilbert C^* -modules which are first introduced by Kaplansky [8] for modules over commutative C^* -algebras and later developed for modules over arbitrary C^* -algebras by Paschke [12] and Rieffel [13]. Idea behind Hilbert C^* -modules is simple: replace the field of complex numbers in the theory of Hilbert spaces by C^* -algebras. Now the positive numbers are replaced by positive elements in C^* -algebras. Hilbert C^* -modules play an essential role in operator algebras, K-theory, KK-theory, operator spaces, quantum group theory, Morita equivalence, AW^* -algebras, operator K-theory, group representation theory etc. We refer monographs [9, 11, 16, 18] for the beautiful use of Hilbert C^* -modules in these areas.

Definition 1.4. [8, 12, 13] Let $\mathcal{A}$ be a unital C^* -algebra. A left module $\mathcal{E}$ over $\mathcal{A}$ is said to be a (left) **Hilbert C^* -module** if there exists a map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ such that the following conditions hold.

- (i) $\langle x, x \rangle \geq 0, \forall x \in \mathcal{E}$. If $x \in \mathcal{E}$ satisfies $\langle x, x \rangle = 0$, then $x = 0$.
- (ii) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle, \forall x, y, z \in \mathcal{E}, \forall a \in \mathcal{A}$.
- (iii) $\langle x, y \rangle = \langle y, x \rangle^*, \forall x, y \in \mathcal{E}$.
- (iv) $\mathcal{E}$ is complete w.r.t. the norm $\|x\| := \sqrt{\|\langle x, x \rangle\|}, \forall x \in \mathcal{E}$.

In the paper, we use the following noncommutative Cauchy-Schwarz inequality.

Theorem 1.5. [12, 13] Let $\mathcal{E}$ be a Hilbert C^* -module. Then

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle y, y \rangle\| \langle x, x \rangle, \quad \forall x, y \in \mathcal{E}.$$

Note that, eventhough we use Theorem 1.5, there are other variations of noncommutative Cauchy-Schwarz inequality, see [1, 2, 5, 7]. We derive noncommutative analogues of Theorem 1.3 in Theorem 2.1.

2. NONCOMMUTATIVE MACCONE-PATI UNCERTAINTY PRINCIPLES

Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$ and A be a possibly unbounded self-adjoint morphism defined on domain $\mathcal{D}(A) \subseteq \mathcal{E}$. For $x \in \mathcal{D}(A)$ with $\langle x, x \rangle = 1$, we define the noncommutative uncertainty of A at the point x as

$$\Delta_x(A) := \|Ax - \langle Ax, x \rangle x\| = \sqrt{\|\langle Ax, Ax \rangle - \langle Ax, x \rangle^2\|}.$$

We define

$$d_x(A) := \sqrt{\langle Ax, Ax \rangle - \langle Ax, x \rangle^2} = \sqrt{\langle Ax - \langle Ax, x \rangle x, Ax - \langle Ax, x \rangle x \rangle}.$$

Then $d_x(A) \in \mathcal{A}$ and

$$\Delta_x(A) = \|d_x(A)\|.$$

Given two elements $a, b \in \mathcal{A}$, we define $[a, b] := ab - ba$ and $\{a, b\} := ab + ba$. Similarly, for two morphisms $A : \mathcal{D}(A) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \rightarrow \mathcal{E}$, we define $[A, B] := AB - BA$ and $\{A, B\} := AB + BA$.

Theorem 2.1. (Noncommutative Maccone-Pati Uncertainty Principles) *Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we have*

(i)

$$d_x(A)^2 + d_x(B)^2 \geq -\langle \{A, B\}x, x \rangle + \langle (A+B)x, y \rangle \langle y, (A+B)x \rangle + \{\langle Ax, x \rangle, \langle Bx, x \rangle\},$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

(ii)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \| -\langle \{A, B\}x, x \rangle + \langle (A+B)x, y \rangle \langle y, (A+B)x \rangle + \{\langle Ax, x \rangle, \langle Bx, x \rangle\} \|,$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

(iii)

$$d_x(A)^2 + d_x(B)^2 \geq \langle \{A, B\}x, x \rangle + \langle (A-B)x, y \rangle \langle y, (A-B)x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\},$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

(iv)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \| \langle \{A, B\}x, x \rangle + \langle (A-B)x, y \rangle \langle y, (A-B)x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\} \|,$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

(v)

$$d_x(A)^2 + d_x(B)^2 \geq \frac{\langle (A+B)x, y \rangle \langle y, (A+B)x \rangle + \langle (A-B)x, y \rangle \langle y, (A-B)x \rangle}{2},$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

(vi)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \frac{\| \langle (A+B)x, y \rangle \langle y, (A+B)x \rangle + \langle (A-B)x, y \rangle \langle y, (A-B)x \rangle \|}{2},$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

(vii)

$$d_x(A)^2 + d_x(B)^2 \geq -i\langle [A, B]x, x \rangle + \langle (A+iB)x, y \rangle \langle y, (A+iB)x \rangle - i[\langle Ax, x \rangle, \langle Bx, x \rangle],$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

(viii)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \| -i\langle [A, B]x, x \rangle + \langle (A+iB)x, y \rangle \langle y, (A+iB)x \rangle - i[\langle Ax, x \rangle, \langle Bx, x \rangle] \|,$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

(ix)

$$d_x(A)^2 + d_x(B)^2 \geq i\langle [A, B]x, x \rangle + \langle (A-iB)x, y \rangle \langle y, (A-iB)x \rangle + i[\langle Ax, x \rangle, \langle Bx, x \rangle],$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.$$

(x)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \|i\langle[A, B]x, x\rangle + \langle(A - iB)x, y\rangle\langle y, (A - iB)x\rangle + i[\langle Ax, x\rangle, \langle Bx, x\rangle]\|,$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(xi)

$$d_x(A)^2 + d_x(B)^2 \geq \frac{\langle(A + iB)x, y\rangle\langle y, (A + iB)x\rangle + \langle(A - iB)x, y\rangle\langle y, (A - iB)x\rangle}{2},$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(xii)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \frac{\|\langle(A + iB)x, y\rangle\langle y, (A + iB)x\rangle + \langle(A - iB)x, y\rangle\langle y, (A - iB)x\rangle\|}{2},$$

$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

Proof. Let $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x\rangle = 1$. Let $y \in \mathcal{E}$ be such that $\|y\| \leq 1$ and $\langle x, y\rangle = 0$.

(i)

$$\begin{aligned} & \langle(A + B)x, y\rangle\langle y, (A + B)x\rangle \\ &= \langle Ax - \langle Ax, x\rangle x + Bx - \langle Bx, x\rangle x, y\rangle\langle y, Ax - \langle Ax, x\rangle x + Bx - \langle Bx, x\rangle x\rangle \\ &\leq \|\langle y, y\rangle\| \langle(Ax - \langle Ax, x\rangle x) + (Bx - \langle Bx, x\rangle x), (Ax - \langle Ax, x\rangle x) + (Bx - \langle Bx, x\rangle x)\rangle \\ &\leq \langle Ax - \langle Ax, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + \langle Bx - \langle Bx, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\ &\quad + \langle Bx - \langle Bx, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + \langle Ax - \langle Ax, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\ &= d_x(A)^2 + d_x(B)^2 + \langle Bx, Ax\rangle - \langle Bx, x\rangle\langle Ax, x\rangle + \langle Ax, Bx\rangle - \langle Ax, x\rangle\langle Bx, x\rangle \\ &= d_x(A)^2 + d_x(B)^2 + \langle\{A, B\}x, x\rangle - \{\langle Ax, x\rangle, \langle Bx, x\rangle\}. \end{aligned}$$

Therefore

$$d_x(A)^2 + d_x(B)^2 \geq -\langle\{A, B\}x, x\rangle + \langle(A + B)x, y\rangle\langle y, (A + B)x\rangle + \{\langle Ax, x\rangle, \langle Bx, x\rangle\}.$$

(ii) Take norm on (i).

(iii) Similar to (i).

(iv) Take norm on (iii).

(v)

$$\begin{aligned} & \langle(A + B)x, y\rangle\langle y, (A + B)x\rangle + \langle(A - B)x, y\rangle\langle y, (A - B)x\rangle = 2\langle Ax, y\rangle\langle y, Ax\rangle + 2\langle Bx, y\rangle\langle y, Bx\rangle \\ &= 2\langle Ax - \langle Ax, x\rangle x, y\rangle\langle y, Ax - \langle Ax, x\rangle x\rangle + 2\langle Bx - \langle Bx, x\rangle x, y\rangle\langle y, Bx - \langle Bx, x\rangle x\rangle \\ &\leq 2\|\langle y, y\rangle\| \langle Ax - \langle Ax, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + 2\|\langle y, y\rangle\| \langle Bx - \langle Bx, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\ &\leq 2d_x(A)^2 + 2d_x(B)^2. \end{aligned}$$

(vi) Take norm on (v).

(vii)

$$\begin{aligned}
 \langle (A + iB)x, y \rangle \langle y, (A + iB)x \rangle &= \langle Ax + iBx, y \rangle \langle y, Ax + iBx \rangle \\
 &= \langle Ax - \langle Ax, x \rangle x + i(Bx - \langle Bx, x \rangle x), y \rangle \langle y, Ax - \langle Ax, x \rangle x + i(Bx - \langle Bx, x \rangle x) \rangle \\
 &\leq \| \langle y, y \rangle \| \langle (Ax - \langle Ax, x \rangle x) + i(Bx - \langle Bx, x \rangle x), (Ax - \langle Ax, x \rangle x) + i(Bx - \langle Bx, x \rangle x) \rangle \\
 &\leq \langle Ax - \langle Ax, x \rangle x, Ax - \langle Ax, x \rangle x \rangle + \langle Bx - \langle Bx, x \rangle x, Bx - \langle Bx, x \rangle x \rangle \\
 &\quad + i \langle Bx - \langle Bx, x \rangle x, Ax - \langle Ax, x \rangle x \rangle - i \langle Ax - \langle Ax, x \rangle x, Bx - \langle Bx, x \rangle x \rangle \\
 &= d_x(A)^2 + d_x(B)^2 + i(\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle) - i(\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle) \\
 &= d_x(A)^2 + d_x(B)^2 + i\langle [A, B]x, x \rangle + i[\langle Ax, x \rangle, \langle Bx, x \rangle].
 \end{aligned}$$

Therefore

$$d_x(A)^2 + d_x(B)^2 \geq -i\langle [A, B]x, x \rangle + \langle (A + iB)x, y \rangle \langle y, (A + iB)x \rangle - i[\langle Ax, x \rangle, \langle Bx, x \rangle].$$

(viii) Take norm on (vii).

(ix) Similar to (vii).

(x) Take norm on (ix).

(xi) Add (vii) and (ix).

(xii) Add (viii) and (ix), then take norm.

□

Corollary 2.2. *Let $\mathcal{E}$ be a Hilbert C^* -module over a unital commutative C^* -algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we have*

(i) *If $\mathcal{A}$ is commutative, then*

$$\begin{aligned}
 d_x(A)^2 + d_x(B)^2 &\geq -i\langle [A, B]x, x \rangle + \langle (A + iB)x, y \rangle \langle y, (A + iB)x \rangle, \\
 &\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.
 \end{aligned}$$

(ii) *If $\mathcal{A}$ is commutative, then*

$$\begin{aligned}
 \Delta_x(A)^2 + \Delta_x(B)^2 &\geq \| -i\langle [A, B]x, x \rangle + \langle (A + iB)x, y \rangle \langle y, (A + iB)x \rangle \|, \\
 &\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.
 \end{aligned}$$

We also have following uncertainty principles.

Theorem 2.3. (Noncommutative Maccone-Pati Uncertainty Principles) *Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we have*

(i)

$$\begin{aligned}
 \Delta_x(A)^2 + \Delta_x(B)^2 &\geq \langle y, (A + B)x \rangle \langle (A + B)x, y \rangle - \| \langle \{A, B\}x, x \rangle - \{ \langle Ax, x \rangle, \langle Bx, x \rangle \} \|, \\
 &\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.
 \end{aligned}$$

(ii)

$$\begin{aligned}
 \Delta_x(A)^2 + \Delta_x(B)^2 &\geq \langle y, (A - B)x \rangle \langle (A - B)x, y \rangle - \| \langle \{A, B\}x, x \rangle - \{ \langle Ax, x \rangle, \langle Bx, x \rangle \} \| \\
 &\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.
 \end{aligned}$$

(iii)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \frac{\|\langle y, (A+B)x \rangle \langle (A+B)x, y \rangle + \langle (A-B)x, y \rangle \langle y, (A-B)x \rangle\|}{2},$$

$\forall y \in \mathcal{E}$ satisfying $\|y\| \leq 1$ and $\langle x, y \rangle = 0$.

(iv)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \langle y, (A+iB)x \rangle \langle (A+iB)x, y \rangle - \|\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle]\|,$$

$\forall y \in \mathcal{E}$ satisfying $\|y\| \leq 1$ and $\langle x, y \rangle = 0$.

(v)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \langle y, (A-iB)x \rangle \langle (A-iB)x, y \rangle - \|\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle]\|,$$

$\forall y \in \mathcal{E}$ satisfying $\|y\| \leq 1$ and $\langle x, y \rangle = 0$.

Proof. Let $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$. Let $y \in \mathcal{E}$ be such that $\|y\| \leq 1$ and $\langle x, y \rangle = 0$.

(i)

$$\begin{aligned} & \langle y, (A+B)x \rangle \langle (A+B)x, y \rangle \\ &= \langle y, Ax - \langle Ax, x \rangle x + (Bx - \langle Bx, x \rangle x) \rangle \langle Ax - \langle Ax, x \rangle x + (Bx - \langle Bx, x \rangle x), y \rangle \\ &\leq \|\langle Ax - \langle Ax, x \rangle x + (Bx - \langle Bx, x \rangle x), (Ax - \langle Ax, x \rangle x) + (Bx - \langle Bx, x \rangle x) \rangle\| \langle y, y \rangle \\ &\leq \|\langle Ax - \langle Ax, x \rangle x + (Bx - \langle Bx, x \rangle x), (Ax - \langle Ax, x \rangle x) + (Bx - \langle Bx, x \rangle x) \rangle\| \|\langle y, y \rangle\| \\ &\leq \|\langle Ax - \langle Ax, x \rangle x, Ax - \langle Ax, x \rangle x \rangle + \langle Bx - \langle Bx, x \rangle x, Bx - \langle Bx, x \rangle x \rangle \\ &\quad + \langle Bx - \langle Bx, x \rangle x, Ax - \langle Ax, x \rangle x \rangle + \langle Ax - \langle Ax, x \rangle x, Bx - \langle Bx, x \rangle x \rangle\| \\ &= \|d_x(A)^2 + d_x(B)^2 + (\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle) + \langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle\| \\ &\leq \|d_x(A)^2 + d_x(B)^2\| + \|\langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\}\|. \end{aligned}$$

Therefore

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \|d_x(A)^2 + d_x(B)^2\| \geq \langle y, (A+B)x \rangle \langle (A+B)x, y \rangle - \|\langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\}\|.$$

(ii) Similar to (i).

(iii)

$$\begin{aligned} & \langle y, (A+B)x \rangle \langle (A+B)x, y \rangle + \langle y, (A-B)x \rangle \langle (A-B)x, y \rangle = 2\langle y, Ax \rangle \langle Ax, y \rangle + 2\langle y, Bx \rangle \langle Bx, y \rangle \\ &= 2\langle y, Ax - \langle Ax, x \rangle x \rangle \langle Ax - \langle Ax, x \rangle x, y \rangle + 2\langle y, Bx - \langle Bx, x \rangle x \rangle \langle Bx - \langle Bx, x \rangle x, y \rangle \\ &\leq 2\|\langle Ax - \langle Ax, x \rangle x, Ax - \langle Ax, x \rangle x \rangle\| \langle y, y \rangle + 2\|\langle Bx - \langle Bx, x \rangle x, Bx - \langle Bx, x \rangle x \rangle\| \langle y, y \rangle \\ &\leq 2\Delta_x(A)^2 + 2\Delta_x(B)^2. \end{aligned}$$

(iv)

$$\begin{aligned}
& \langle y, (A + iB)x \rangle \langle (A + iB)x, y \rangle = \langle y, Ax + iBx \rangle \langle Ax + iBx, y \rangle \\
& = \langle y, Ax - \langle Ax, x \rangle x + i(Bx - \langle Bx, x \rangle x) \rangle \langle Ax - \langle Ax, x \rangle x + i(Bx - \langle Bx, x \rangle x), y \rangle \\
& \leq \| \langle Ax - \langle Ax, x \rangle x + i(Bx - \langle Bx, x \rangle x), (Ax - \langle Ax, x \rangle x) + i(Bx - \langle Bx, x \rangle x) \rangle \| \langle y, y \rangle \\
& \leq \| \langle Ax - \langle Ax, x \rangle x, Ax - \langle Ax, x \rangle x \rangle + \langle Bx - \langle Bx, x \rangle x, Bx - \langle Bx, x \rangle x \rangle \\
& \quad + i \langle Bx - \langle Bx, x \rangle x, Ax - \langle Ax, x \rangle x \rangle - i \langle Ax - \langle Ax, x \rangle x, Bx - \langle Bx, x \rangle x \rangle \| \\
& = \| d_x(A)^2 + d_x(B)^2 + i(\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle) - i(\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle) \| \\
& = \| d_x(A)^2 + d_x(B)^2 \| + \| (\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle) - (\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle) \| \\
& = \| d_x(A)^2 + d_x(B)^2 \| + \| \langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle] \|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta_x(A)^2 + \Delta_x(B)^2 & \geq \| d_x(A)^2 + d_x(B)^2 \| \\
& \geq \langle y, (A + iB)x \rangle \langle (A + iB)x, y \rangle - \| \langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle] \|.
\end{aligned}$$

(v) Similar to (iv).

□

Theorem 2.4. Let $\mathcal{E}$ be a Hilbert C^* -module over a unital commutative C^* -algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we have

(i) If $d_x(A)$, $d_x(B)$ are invertible, then

$$\begin{aligned}
d_x(A)d_x(B) & \geq \frac{-i\langle [A, B]x, x \rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)} \right) x \right\rangle}, \\
& \forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.
\end{aligned}$$

(ii)

$$\begin{aligned}
\Delta_x(A)\Delta_x(B) & \geq \frac{-i\langle [A, B]x, x \rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)} \right) x \right\rangle}, \\
& \forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.
\end{aligned}$$

(iii) If $d_x(A)$, $d_x(B)$ are invertible, then

$$\begin{aligned}
d_x(A)d_x(B) & \geq \frac{i\langle [A, B]x, x \rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} - \frac{iB}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} - \frac{iB}{d_x(B)} \right) x \right\rangle}, \\
& \forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.
\end{aligned}$$

(iv)

$$\begin{aligned}
\Delta_x(A)\Delta_x(B) & \geq \frac{i\langle [A, B]x, x \rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} - \frac{iB}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} - \frac{iB}{\Delta_x(B)} \right) x \right\rangle}, \\
& \forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y \rangle = 0.
\end{aligned}$$

(v) If $d_x(A)$, $d_x(B)$ are invertible, then

$$d_x(A)d_x(B) \geq \frac{2\langle Ax, x \rangle \langle Bx, x \rangle - \langle \{A, B\}x, x \rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)} \right) x \right\rangle},$$

$\forall y \in \mathcal{E}$ satisfying $\|y\| \leq 1$ and $\langle x, y \rangle = 0$.

(vi)

$$\Delta_x(A)\Delta_x(B) \geq \frac{2\langle Ax, x \rangle \langle Bx, x \rangle - \langle \{A, B\}x, x \rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)} \right) x \right\rangle},$$

$\forall y \in \mathcal{E}$ satisfying $\|y\| \leq 1$ and $\langle x, y \rangle = 0$.

(vii) If $d_x(A)$, $d_x(B)$ are invertible, then

$$d_x(A)d_x(B) \geq \frac{-2\langle Ax, x \rangle \langle Bx, x \rangle + \langle \{A, B\}x, x \rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} - \frac{B}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} - \frac{B}{d_x(B)} \right) x \right\rangle},$$

$\forall y \in \mathcal{E}$ satisfying $\|y\| \leq 1$ and $\langle x, y \rangle = 0$.

(viii)

$$\Delta_x(A)\Delta_x(B) \geq \frac{-2\langle Ax, x \rangle \langle Bx, x \rangle + \langle \{A, B\}x, x \rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} - \frac{B}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} - \frac{B}{\Delta_x(B)} \right) x \right\rangle},$$

$\forall y \in \mathcal{E}$ satisfying $\|y\| \leq 1$ and $\langle x, y \rangle = 0$.

Proof. Let $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$. Let $y \in \mathcal{E}$ be such that $\|y\| \leq 1$ and $\langle x, y \rangle = 0$.

(i) Since $\mathcal{A}$ is commutative and $d_x(A)$, $d_x(B)$ are invertible,

$$\begin{aligned} & \left\langle \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)} \right) x \right\rangle \\ &= \left\langle \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} + \frac{i(Bx - \langle Bx, x \rangle x)}{d_x(B)}, y \right\rangle \left\langle y, \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} + \frac{i(Bx - \langle Bx, x \rangle x)}{d_x(B)} \right\rangle \\ &\leq \|y\| \left\langle \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} + \frac{i(Bx - \langle Bx, x \rangle x)}{d_x(B)}, \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} + \frac{i(Bx - \langle Bx, x \rangle x)}{d_x(B)} \right\rangle \\ &\leq \left\langle \frac{Ax - \langle Ax, x \rangle x}{d_x(A)}, \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} \right\rangle + \left\langle \frac{Bx - \langle Bx, x \rangle x}{d_x(B)}, \frac{Bx - \langle Bx, x \rangle x}{d_x(B)} \right\rangle \\ &\quad + i \left\langle \frac{Bx - \langle Bx, x \rangle x}{d_x(B)}, \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} \right\rangle - i \left\langle \frac{Ax - \langle Ax, x \rangle x}{d_x(A)}, \frac{Bx - \langle Bx, x \rangle x}{d_x(B)} \right\rangle \\ &= 1 + 1 + \frac{i\langle Bx, Ax \rangle - i\langle Bx, x \rangle \langle Ax, x \rangle}{d_x(A)d_x(B)} + \frac{-i\langle Ax, Bx \rangle + i\langle Ax, x \rangle \langle Bx, x \rangle}{d_x(A)d_x(B)} \\ &= 2 + \frac{i\langle [A, B]x, x \rangle}{d_x(A)d_x(B)}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{-i\langle [A, B]x, x \rangle}{d_x(A)d_x(B)} &\leq 2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)} \right) x \right\rangle \\ &\implies \frac{-i\langle [A, B]x, x \rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)} \right) x \right\rangle} \leq d_x(A)d_x(B). \end{aligned}$$

(ii) Since $\mathcal{A}$ is commutative,

$$\begin{aligned}
& \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)} \right) x \right\rangle \\
&= \left\langle \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} + \frac{i(Bx - \langle Bx, x \rangle x)}{\Delta_x(B)}, y \right\rangle \left\langle y, \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} + \frac{i(Bx - \langle Bx, x \rangle x)}{\Delta_x(B)} \right\rangle \\
&\leq \|y\| \left\langle \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} + \frac{i(Bx - \langle Bx, x \rangle x)}{\Delta_x(B)}, \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} + \frac{i(Bx - \langle Bx, x \rangle x)}{\Delta_x(B)} \right\rangle \\
&\leq \left\langle \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)}, \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} \right\rangle + \left\langle \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)}, \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)} \right\rangle \\
&\quad + i \left\langle \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)}, \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} \right\rangle - i \left\langle \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)}, \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)} \right\rangle \\
&= 1 + 1 + \frac{i\langle Bx, Ax \rangle - i\langle Bx, x \rangle \langle Ax, x \rangle}{\Delta_x(A)\Delta_x(B)} + \frac{-i\langle Ax, Bx \rangle + i\langle Ax, x \rangle \langle Bx, x \rangle}{\Delta_x(A)\Delta_x(B)} \\
&= 2 + \frac{i\langle [A, B]x, x \rangle}{\Delta_x(A)\Delta_x(B)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \frac{-i\langle [A, B]x, x \rangle}{\Delta_x(A)\Delta_x(B)} \leq 2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)} \right) x \right\rangle \\
&\quad - i\langle [A, B]x, x \rangle \\
&\implies \frac{-i\langle [A, B]x, x \rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)} \right) x \right\rangle} \leq \Delta_x(A)\Delta_x(B).
\end{aligned}$$

(iii) Similar to (i).

(iv) Similar to (ii).

(v) Since $\mathcal{A}$ is commutative and $d_x(A)$, $d_x(B)$ are invertible,

$$\begin{aligned}
& \left\langle \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)} \right) x \right\rangle \\
&= \left\langle \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} + \frac{Bx - \langle Bx, x \rangle x}{d_x(B)}, y \right\rangle \left\langle y, \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} + \frac{Bx - \langle Bx, x \rangle x}{d_x(B)} \right\rangle \\
&\leq \|y\| \left\langle \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} + \frac{Bx - \langle Bx, x \rangle x}{d_x(B)}, \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} + \frac{Bx - \langle Bx, x \rangle x}{d_x(B)} \right\rangle \\
&\leq \left\langle \frac{Ax - \langle Ax, x \rangle x}{d_x(A)}, \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} \right\rangle + \left\langle \frac{Bx - \langle Bx, x \rangle x}{d_x(B)}, \frac{Bx - \langle Bx, x \rangle x}{d_x(B)} \right\rangle \\
&\quad + \left\langle \frac{Bx - \langle Bx, x \rangle x}{d_x(B)}, \frac{Ax - \langle Ax, x \rangle x}{d_x(A)} \right\rangle + \left\langle \frac{Ax - \langle Ax, x \rangle x}{d_x(A)}, \frac{Bx - \langle Bx, x \rangle x}{d_x(B)} \right\rangle \\
&= 1 + 1 + \frac{\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle}{d_x(A)d_x(B)} + \frac{\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle}{d_x(A)d_x(B)} \\
&= 2 + \frac{-2\langle Ax, x \rangle \langle Bx, x \rangle + \langle \{A, B\}x, x \rangle}{d_x(A)d_x(B)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& -\frac{2\langle Ax, x \rangle \langle Bx, x \rangle + \langle \{A, B\}x, x \rangle}{d_x(A)d_x(B)} \leq 2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)} \right) x \right\rangle \\
&\implies \frac{2\langle Ax, x \rangle \langle Bx, x \rangle - \langle \{A, B\}x, x \rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)} \right) x \right\rangle} \leq d_x(A)d_x(B).
\end{aligned}$$

(vi) Since $\mathcal{A}$ is commutative,

$$\begin{aligned}
 & \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)} \right) x \right\rangle \\
 &= \left\langle \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} + \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)}, y \right\rangle \left\langle y, \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} + \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)} \right\rangle \\
 &\leq \|y\| \left\| \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} + \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)} \right\| \\
 &\leq \left\langle \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)}, \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} \right\rangle + \left\langle \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)}, \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)} \right\rangle \\
 &\quad + \left\langle \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)}, \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)} \right\rangle + \left\langle \frac{Ax - \langle Ax, x \rangle x}{\Delta_x(A)}, \frac{Bx - \langle Bx, x \rangle x}{\Delta_x(B)} \right\rangle \\
 &= 1 + 1 + \frac{\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle}{\Delta_x(A)\Delta_x(B)} + \frac{\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle}{\Delta_x(A)\Delta_x(B)} \\
 &= 2 + \frac{2\langle Ax, x \rangle \langle Bx, x \rangle + \langle \{A, B\}x, x \rangle}{\Delta_x(A)\Delta_x(B)}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & -\frac{2\langle Ax, x \rangle \langle Bx, x \rangle + \langle \{A, B\}x, x \rangle}{\Delta_x(A)\Delta_x(B)} \leq 2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)} \right) x \right\rangle \\
 & \implies \frac{2\langle Ax, x \rangle \langle Bx, x \rangle - \langle \{A, B\}x, x \rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)} \right) x, y \right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)} \right) x \right\rangle} \leq \Delta_x(A)\Delta_x(B).
 \end{aligned}$$

(vii) Similar to (v).

(viii) Similar to (vi).

□

3. ACKNOWLEDGMENTS

The author thanks anonymous reviewer for his/her constructive feedback, which significantly improved the final manuscript.

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