

FIXED-LEVEL TQFT TRANSPORT OF MARKED HANDLESIDE COHERENCE: DISJOINT CUBES AND A THREE-HANDLE PENTAGON

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ABSTRACT. Fix an ordinary level $K \geq 1$ and a fully specified family of pairwise disjoint positive handleslides on a compression body. The main question is how a literal geometric relation becomes a literal finite-level operator relation. We construct a finite reduced skein fibre and one normalized isomorphism

$$Q_{K,d} : \mathcal{M}_{K,d} \xrightarrow{\sim} \mathcal{V}_{K,d}^+$$

to the corresponding scalar-extended Walker–TQFT state space. Reusing this map at every vertex sends the supplied weight-zero slide cylinders to E_K -linear automorphisms: geometric squares become operator squares, finite slide orders give the same operator, and forward/reverse maps telescope for all declared phases and integer weights. The construction is fibrewise and makes no comparison between levels. These operator relations transport the geometric disjoint-cube and marked-pentagon relations established in companion works; they do not constitute new geometric coherence theorems.

We then treat a fully specified positive three-handle pentagon. Its vertex-zero compression body admits a compatible fixed-level comparison core: a rational disk-kernel cap, a normalized complementary handlebody, a reduced spine model, and one universal-construction isomorphism $Q_{K,d}^q : \mathcal{M}_{K,d}^q \xrightarrow{\sim} \mathcal{V}_{K,d}^+$. The geometric weight-zero Walker pentagon consequently has a literal operator image over E_K , with explicit inverse formulas and a fixed-basis matrix corollary. Different comparison cores are related by a typed comparison groupoid and give intertwined operator systems; no canonical comparison or literal equality between different coordinate choices is asserted.

1. INTRODUCTION

How does a marked handleslide relation become a literal finite-level operator relation? One normalized skein-to-TQFT comparison provides a common coordinate model and avoids carrying vertex-dependent conjugations. A projective mapping-class action would retain the relation only up to an anomaly scalar. We construct one normalized comparison isomorphism, reuse it at every vertex, and retain the central factors by working in Walker’s extended cobordism category.

The geometric cube and pentagon are supplied by Papers 1 and 2. Their literal Walker equalities become literal E_K -linear operator equalities under this fixed comparison. The contribution here is its construction and exact normalization, coefficient and anomaly control, and the typed groupoid comparing auxiliary choices. The chosen isomorphism, called a comparison map below, relates two specified models; it is not a canonical identification or a choice that may vary from vertex to vertex.

1.1. From marked topology to the finite TQFT fibre. The source relation lives in Walker’s extended cobordism category. Its marked representatives determine the geometric composites, its Lagrangian data determine the Maslov corrections, and its integer weights retain the central anomaly. At a fixed root, the handlebody skein module is reduced by the negligible-color relations and compared with the finite TQFT state space through the universal construction. At fixed K , the universal handlebody map factors through the L -reduced skein quotient. The exact weight-zero S^3 pairing, normalized by $\mathcal{D}_K^{-1}$, proves both injectivity and surjectivity, giving the

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isomorphism $Q_{K,d}$. Strict functoriality then transports the already proved geometric relation without changing factor order or discarding its weight.

Cyclotomic arithmetic enters only to fix the root, coefficient field, and anomaly branch. Completion theory and disk q -series belong to separate all-level questions and are not proof dependencies here.

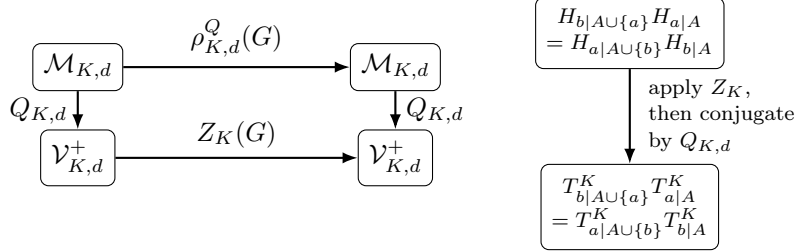


FIGURE 1. The fixed comparison square and the transported coherence relation. The source equality is a weight-zero Walker equality proved geometrically. The same $Q_{K,d}$ is used on every edge; the diagram has no reverse implication and makes no comparison between levels.

Theorem A (Fixed-level transport of the disjoint handleslide cube). *Fix $K \geq 1$ and a supplied marked disjoint handleslide cube from Paper 1. There is one normalized comparison isomorphism $Q_{K,d}$, reused at every vertex, such that each geometric edge is sent to an E_K -linear automorphism and every weight-zero geometric face is sent to a literal operator square. The same transport preserves the finite route equalities and the declared forward/reverse phase and weight telescoping identities. It is a fixed-level image of the geometric cube and makes no comparison between different levels.*

The complete normalization and edge statements are in Theorem 5.5. Its geometric input is [Sun26b, Theorem 1.2].

The pentagon reuses three handles, so its overlapping directions cannot be treated as a disjoint cube. Its geometric source is [Sun26a, Theorem 4.1, Proposition 12.3, Theorems 13.5 and 14.1, and Section 15]. Every selected vertex-zero datum admits a rational disk-kernel cap, a weight-zero S^3 complement, a reduced spine model, and a single comparison isomorphism. Two such models compare through their common TQFT fibre and intertwine every Walker endocylinder. Applying this comparison core and the strict-functor theorem directly to the geometric Walker pentagon gives the second result.

Theorem B (Fixed-level transport of the marked handleslide pentagon). *Fix $K \geq 1$ and a selected marked three-handle pentagon from Paper 2. A compatible comparison core transports its weight-zero Walker relation to the literal E_K -linear operator equality*

$$\mathsf{T}_{B\star}^{K,q} \mathsf{T}_{0B}^{K,q} = \mathsf{T}_{AC\star}^{K,q} \mathsf{T}_{AAC}^{K,q} \mathsf{T}_{0A}^{K,q},$$

with no projective scalar. Comparison cores for the same TQFT fibre give intertwined operator systems and satisfy the corresponding comparison groupoid law; no canonical coordinate choice is asserted.

Theorem 6.7 gives the operator equality, and Theorem 6.8 gives the change of comparison data. These complement the finite-order relations for independent supports without inferring overlapping coherence from disjoint squares.

1.2. Established inputs and scope. Przytycki’s skein-module framework provides the algebraic starting point [Prz99, Definition 2.1], while BHMV supplies the Kauffman-bracket TQFT, universal construction, and handlebody presentations [BHMV95, Theorem 1.4 and Proposition 1.9]. At the chosen root, admissibly colored banded trivalent graphs give finite state-space coordinates in the BHMV theory [BHMV95, Theorem 4.11]; the evaluation and recoupling conventions are those of Masbaum–Vogel [MV94, Theorems 1–3 and Section 2]. Walker supplies the

integral-weight and Lagrangian extension [Wal91, Definitions 1.1–1.6], with the Maslov-corrected mapping-cylinder convention of Gilmer–Masbaum [GM13, Equations (1)–(2) and Sections 9 and 11]. The paper does not claim a new TQFT, skein module, recoupling theory, central extension, or geometric cube or pentagon.

1.3. Organization. Section 2 fixes the geometric, Walker, skein, and TQFT interfaces. Sections 3–5 construct the fixed comparison map and prove the ordinary operator theorem. Section 6 constructs the compatible pentagon comparison package, proves the five-edge operator relation, and compares different auxiliary choices. The final section separates this fixed-level bridge from higher coherence and from the all-level program.

2. THE DISJOINT-CUBE INPUT AND FIXED-LEVEL QUANTUM CONVENTIONS

The operator construction begins from the marked cube, its common homological Lagrangian, and fixed Walker/TQFT conventions. These data are transported, not reproved. We call the subset-zero datum the *reference* datum; the word “old” is retained in formulas inherited from the source notation and has the same meaning.

2.1. Specified disjoint handleslide cubes with complete markings. For $n \geq 1$, write $[n] = \{1, \dots, n\}$. Fix integers $g \geq 1$, $s \geq 1$, and $k \geq 2s$. Let

$$X = \Sigma_g^{\text{out}}, \quad Y = \Sigma_{g+k}^{\text{out}}, \quad d = g + k \geq 3,$$

and let $W : X \rightarrow Y_{\text{geom}}$ be a completed compression body with k ordered product handles. Fix an injective assignment $a \mapsto (i_a, j_a)$ from $[s]$ to ordered handle pairs such that all $2s$ active labels are distinct. Assume that this family satisfies all protected-sector hypotheses of the geometric cube theorem cited below. In particular, each of the corresponding s protected standard sectors carries the complete local data and the prescribed positive product-zero-framed slide, and their closures, including every protected product neighborhood and full interface collar, are pairwise disjoint. The one-holed-torus phase-support collars of all k old marked rows are pairwise disjoint and disjoint from the old-region marking.

Let $\lambda_g \subset H_1(X; \mathbb{Q})$ be the old disk-kernel Lagrangian. The old-region marking fixes a symplectic inclusion

$$\iota_X : H_1(X; \mathbb{Q}) \hookrightarrow H_1(Y; \mathbb{Q}), \quad \lambda_g^Y := \iota_X(\lambda_g).$$

Let m_t, ℓ_t be the meridian–longitude row of the t -th product handle. The complete old marking supplies pairwise-disjoint one-holed-torus phase-support collars and the symplectic splitting

$$(1) \quad H_1(Y; \mathbb{Q}) = \iota_X(H_1(X; \mathbb{Q})) \oplus \bigoplus_{t=1}^k \text{span}_{\mathbb{Q}}\{[m_t], [\ell_t]\}.$$

Put

$$(2) \quad \Lambda = \lambda_g^Y \oplus \bigoplus_{t=1}^k \mathbb{Q}[m_t] \subset H_1(Y; \mathbb{Q}).$$

We take [Sun26b, Theorem 1.2] as the following independent geometric input.

Theorem 2.1 (Disjoint positive handleslide cube with complete markings). *For every subset $A \subseteq [s]$, there is a completed marked vertex $\mathfrak{C}_A$ and a parameterization $p_A : \partial_+ W \rightarrow Y$. For every $a \notin A$, there is a supported orientation-preserving map $F_{a|A} : (W, \mathfrak{C}_A) \rightarrow (W, \mathfrak{C}_{A \cup \{a\}})$ and an outgoing map*

$$(3) \quad h_{a|A} = p_{A \cup \{a\}}(F_{a|A}|_{\partial_+ W})p_A^{-1}.$$

The maps have the identity germ on the full interface collar and fix the incoming boundary, old region, other protected sectors, and unused handles pointwise. They preserve Λ in both directions. Hence

$$(4) \quad H_{a|A} = C_Y(h_{a|A}, 0) : (Y, \Lambda) \longrightarrow (Y, \Lambda)$$

is a well-typed weight-zero Walker cylinder.

For distinct $a, b \notin A$, the supported maps form a literal complete-marked square, and the corresponding cylinders satisfy

$$(5) \quad H_{b|A \cup \{a\}} H_{a|A} = H_{a|A \cup \{b\}} H_{b|A}$$

with both composite weights zero. The same package supplies the completed endpoint phase cylinders, the independently typed forward and reverse cobordisms, their edge covariances, finite order independence, and telescoping, for arbitrary label-indexed integer weights.

Its content is geometric: the square is established before any quantum operator is formed. In particular, the operator equalities below are one-way images of these source equalities.

2.2. Walker cylinders and Maslov weights. The boundary surface is equipped with the rational Lagrangian Λ . For mapping classes f, g and integer weights m, n , our convention is

$$(6) \quad C_Y(g, n) C_Y(f, m) = C_Y(gf, n + m - \mu_Y(f_*\Lambda, \Lambda, (g^{-1})_*\Lambda)),$$

with the right factor acting first; see [GM13, Equation (2) and Section 9] for this exact sign and order convention, and [Wal91, Definitions 1.1–1.6] for the extended-cobordism framework. Repeated Lagrangians give zero Maslov index, but that vanishing is checked at each seam needed below rather than silently assumed.

2.3. Fixed-level TQFT, reduction, and universal construction. We use the connected-surface Kauffman-bracket TQFT and handlebody presentations of BHMV [BHMV95, Theorem 1.4 and Proposition 1.9], in the integral-weight/Lagrangian formulation of [GM13, Section 11, especially Remark 11.3]. The input is the coefficient ring at the selected even root, finite state modules, nondegenerate duality, surjective handlebody skein maps, and strict functoriality on extended cobordisms.

We also use the accompanying positive rank, weight-zero S^3 normalization, and the selected anomaly branch. Their exact formulas are derived in Proposition 3.1; the branch is not reconstructed from its square alone.

At a primitive $4L$ -th root, reduction uses the local f_{L-1} -relations, the finite admissible colored-spine basis, and the nondegenerate complementary-handlebody bracket [BHMV95, Proposition 3.3(i) and Theorem 4.11]; trivalent evaluations follow [MV94]. The declared boundary Lagrangian remains distinct from the actual disk kernel.

The universal-construction input is only the radical mechanism: a handlebody skein is zero in the state module exactly when it pairs trivially with every opposite handlebody state. No odd-prime lattice or all-level completion is imported.

3. ROOT NORMALIZATION, REDUCED FIBRES, AND CUBE COMPARISON

The first algebraic obstruction is finiteness: the geometric surface has a natural skein module, but the operator theorem needs a finite fibre at one fixed level. This section constructs the normalized reduced source and one comparison isomorphism from the reduced skein model to the fixed-level TQFT state space. That map is fixed once and reused at every vertex rather than rebuilt along the cube.

We retain the quantifier order and notation fixed in Section 2. In particular, the ordinary level K is fixed before a surface, a handlebody, or a state space is constructed. The coefficient ring and even- p fixed-level theory are kept together with their handlebody skein surjections and duality. The positive rank, weight-zero S^3 value, and anomaly branch fix how the complementary pairing is normalized. The geometric cube contributes only the reference compression body. On its skein module, the L -reduced quotient has a finite admissible spine basis and a nondegenerate complementary bracket; these are the ingredients that make the normalized comparison invertible. The Walker boundary-Lagrangian and gluing-weight conventions remain in force. Surjectivity and TQFT duality identify the universal kernel with the pairing radical. The local reduced relations then make the map descend, and nondegeneracy of the reduced complementary bracket proves injectivity.

3.1. Root, ring, field, and the raw-field obstruction.

Proposition 3.1 (Exact root and coefficient normalization). *Fix $K \geq 1$, and put*

$$(7) \quad L := K + 2, \quad p := 2L, \quad \zeta_K := \exp\left(\frac{\pi i}{2L}\right), \quad A_K := -\zeta_K.$$

Then A_K is a primitive $4L$ -th root, the even- p palette is $\{0, \dots, K\}$, and the anomaly data are

$$(8) \quad \begin{aligned} \mathcal{D}_K &= \frac{\sqrt{L/2}}{\sin(\pi/L)} > 0, \\ \Delta_K &= \mathcal{D}_K \exp\left(-\frac{3\pi i K}{4L}\right), \\ \kappa_K &= \mathcal{D}_K \Delta_K^{-1} = \exp\left(\frac{3\pi i K}{4L}\right). \end{aligned}$$

Moreover,

$$(9) \quad \kappa_K^2 = A_K^{-6-p(p+1)/2},$$

where the positive-exponent formula in (8), rather than the square alone, fixes the branch. With

$$(10) \quad F_K := \mathbb{Q}(A_K), \quad E_K := \mathbb{Q}(A_K, \kappa_K), \quad k_{p,K} := \mathbb{Z}[1/p, A_K, \kappa_K] \subset E_K,$$

complex conjugation restricts to the involution $\iota_K(A_K) = A_K^{-1}$, $\iota_K(\kappa_K) = \kappa_K^{-1}$, and

$$(11) \quad \boxed{Z_p(S^3, w=0) = \mathcal{D}_K^{-1} \in k_{p,K} \cap E_K^\times.}$$

For every connected extended surface (Y, Λ) , the central cylinder $W_Y = C_Y(\text{id}_Y, 1)$ satisfies

$$(12) \quad (E_K \otimes_{k_{p,K}} Z_p)(W_Y) = \kappa_K \text{id}_{E_K \otimes_{k_{p,K}} V_p(Y, \Lambda)}.$$

Finally, at $K = 1$ one has $\kappa_1 \notin F_1$; hence F_K cannot uniformly replace E_K .

After these data have been fixed, take $g \geq 1$, $s \geq 1$, $k \geq 2s$, and $d = g + k \geq 3$, and import one datum supplied by Input Theorem 2.1 with

$$X = \Sigma_g^{\text{out}}, \quad Y = \Sigma_d^{\text{out}}, \quad \Lambda = \lambda_g^Y \oplus \bigoplus_{t=1}^k \langle [m_t] \rangle_{\mathbb{Q}}.$$

Here k counts handle sites and is unrelated to the coefficient ring $k_{p,K}$, while s counts slides and $[s]$ is the set of slide indices. Retain all source vertices $\mathfrak{C}_A$, $A \subseteq [s]$. For every supplied vertex $A \subseteq [s]$ and every $a \notin A$, retain, exactly as source data,

$$H_{a|A} = C_Y(h_{a|A}, 0),$$

together with all endpoint phase cylinders, all completed forward and reverse traces, all literal Walker squares, and the supplied common total cylinder $H_{[s]}$. This section makes no assertion about their linear images. Fix also the low cap B_g with its fixed boundary parameterization, regarded as the weight-zero extended cobordism $B_g^{(0)} : \emptyset \rightarrow (X, \lambda_g)$, with disk kernel λ_g , and distinguish

$$(13) \quad \begin{aligned} V_{p,K}^+ &:= V_p(Y, \Lambda), & \mathcal{V}_{K,d}^+ &:= E_K \otimes_{k_{p,K}} V_{p,K}^+, \\ V_{p,K}^- &:= V_p(X, \lambda_g), & \mathcal{V}_{K,g}^- &:= E_K \otimes_{k_{p,K}} V_{p,K}^-. \end{aligned}$$

Proof. Anomaly branch. The normalization is BHMV's η and unknot formula in [BHMV95, Section 2, pp. 897–898, formula (*) and the definition of η], with $p = 2L$, palette $0, \dots, K$, and $A = A_K$. BHMV's anomaly parameter has cube κ_K , the weight-one character in [GM13, Section 11]; thus $\eta = \mathcal{D}_K^{-1}$ and $\langle U_{-1}(\Omega_p) \rangle = \Delta_K$. With the displayed positive branch, the anomaly rank formula is the first line of (8). Since $L \geq 3$, $\sin(\pi/L) > 0$, so this is the positive rank. The same calibration gives

$$\Delta_K \mathcal{D}_K^{-1} = \exp\left(-\frac{3\pi i(L-2)}{4L}\right).$$

As $K = L - 2$, this proves the second line and, after inversion, the third line of (8). In particular it selects κ_K , not its negative.

Root-square identity. For the square convention, note first that $-1 = \zeta_K^{2L}$, hence $A_K = \zeta_K^{2L+1}$. With $p = 2L$ and $K = L - 2$, direct integer arithmetic gives

$$(14) \quad (2L + 1)(-6 - L(2L + 1)) - 3(L - 2) = -4L(L^2 + L + 4).$$

The two exponents are therefore congruent modulo the order $4L$ of ζ_K , and

$$A_K^{-6-p(p+1)/2} = \zeta_K^{3K} = \exp\left(\frac{3\pi i K}{2L}\right) = \kappa_K^2.$$

This proves (9) without weakening (14) to an unspecified congruence representative.

Coefficient field and involution. The ring inclusion in (10) is literal: $1/p \in \mathbb{Q} \subset E_K$, and both displayed generators belong to E_K . Complex conjugation in $\mathbb{C}$ inverts the two roots of unity A_K and κ_K , so it preserves E_K and has the asserted values.

Vacuum normalization and central weight. The BHMV normalization [BHMV95, Section 2, p. 898] identifies the invariant of the empty, weight-zero S^3 with $\mathcal{D}_K^{-1}$. On the other hand the fixed-level TQFT is defined over $k_{p,K}$ and has $V_p(\emptyset) = k_{p,K}$. The same closed cobordism therefore has its value in $k_{p,K}$. Equality with the nonzero number $\mathcal{D}_K^{-1}$ proves both the source-ring membership and invertibility in (11); the membership is not inferred merely from later scalar extension. The weight convention of [GM13, Section 11] says that increasing a cobordism weight by one multiplies its map by κ_K . Applying this to the identity cylinder and then extending scalars proves (12); integer weights, including negative ones, act by the corresponding powers because $\kappa_K \in E_K^\times$.

Root order and reduced palette. Also,

$$\gcd(2L + 1, 4L) = \gcd(2L + 1, 2) = 1,$$

so $A_K = \zeta_K^{2L+1}$ is primitive of order $4L$. The fixed-level TQFT even- $p = 2K + 4$ palette is consequently $\{0, \dots, K\}$. This proves the root-order assertion and fixes the reduced convention used below.

The raw-field obstruction. It remains to check the raw-field obstruction. For $K = 1$, $L = 3$,

$$A_1 = -\zeta_{12} = \zeta_{12}^7, \quad F_1 = \mathbb{Q}(\zeta_{12}) = \mathbb{Q}(i, \sqrt{3}), \quad \kappa_1 = e^{\pi i/4} = \zeta_8.$$

If ζ_8 belonged to F_1 , then $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$ would belong to the maximal real subfield

$$F_1^+ = \mathbb{Q}(\zeta_{12} + \zeta_{12}^{-1}) = \mathbb{Q}(\sqrt{3}).$$

But if $\sqrt{2} = a + b\sqrt{3}$ with $a, b \in \mathbb{Q}$, squaring forces $2ab = 0$. The alternatives would make either $a^2 = 2$ or $b^2 = 2/3$, neither of which has a rational solution. Thus $\sqrt{2} \notin \mathbb{Q}(\sqrt{3})$, a contradiction. Hence $\kappa_1 \notin F_1$.

Geometric and module types. Finally, $d = g + k \geq 1 + 2 = 3$. The surfaces, Λ , and the weight-zero source cylinders are the supplied geometric-cube data, while the cap and its disk kernel are fixed before the high handlebody is formed. Equation (13) is then simply the typed TQFT source-module/scalar-extension distinction. This completes the root, ring, field, anomaly, involution, and coefficient-field claims stated above. $\square$

3.2. The old handlebody and its reduced fibre. The old vertex is the anchor for all later TQFT transport. Topologically, the fixed low cap is attached to the reference compression body first; algebraically, its disk kernel is identified and root-of-unity reduction is then imposed. Only after the reduced fibre and its complementary pairing are available is the common scalar extension used to define the comparison map. This order is why later vertices reuse one fixed comparison rather than acquiring separate vertex-dependent maps.

Proposition 3.2 (Old cap, disk kernel, reduction, and spine basis). *Glue the fixed low cap, through its fixed boundary parameterization, to the common geometric-cube compression body in the reference chart $\mathfrak{C}_\emptyset$, and use the fixed old incoming and outgoing parameterizations:*

$$(15) \quad H_{\emptyset,d} := B_g \cup_X (W, \mathfrak{C}_\emptyset), \quad \partial H_{\emptyset,d} \xrightarrow{p_\emptyset} Y.$$

Then $H_{\emptyset,d}$ is a genus- d handlebody, and, when assigned weight zero,

$$H_{\emptyset,d}^{(0)} : \emptyset \longrightarrow (Y, \Lambda)$$

has

$$(16) \quad \ker(H_1(Y; \mathbb{Q}) \longrightarrow H_1(H_{\emptyset,d}; \mathbb{Q})) = \Lambda.$$

There is a fixed parameterized complementary handlebody $H_{\emptyset,d}^c$ such that $H_{\emptyset,d} \cup_Y H_{\emptyset,d}^c = S^3$. It is used with weight zero as

$$(H_{\emptyset,d}^c)^{(0)} : \emptyset \longrightarrow (\bar{Y}, \Lambda).$$

Here Λ is its declared boundary Lagrangian; its actual disk kernel is a different, complementary Lagrangian.

For one fixed banded trivalent spine $\Gamma_{\emptyset,d} \subset H_{\emptyset,d}$, let

$$(17) \quad \mathcal{S}_{K,d}^{\text{red}} := \mathcal{S}_{A_K}(H_{\emptyset,d}) / \langle f_{L-1}\text{-relations} \rangle.$$

Then

$$(18) \quad \mathcal{B}_{K,d} = \{(\Gamma_{\emptyset,d})_c : c \text{ is } L\text{-admissible}\}$$

is a finite basis: every edge color is in $\{0, \dots, K\}$, every vertex sum is even, the triangle inequalities hold, and the sum is at most $2K$. The ordinary old fibre

$$(19) \quad \mathcal{M}_{K,d} := E_K \otimes_{F_K} \mathcal{S}_{K,d}^{\text{red}}$$

is finite-dimensional, its displayed basis has nonzero double-handlebody norms after scalar extension, and the reduced pairing with the complementary handlebody is nondegenerate.

For $U = B_g, H_{\emptyset,d}, H_{\emptyset,d}^c$, there is the canonical base-change identification

$$(20) \quad E_K \otimes_{k_{p,K}} \mathcal{S}_{k_{p,K}}(U) \cong E_K \otimes_{F_K} \mathcal{S}_{A_K}(U).$$

Proof. Topologically, the reference compression body is obtained from $X \times [0, 1]$ by the k fixed handle sites, whose co-cores have old meridians $m_1, \dots, m_k$. Gluing B_g to the incoming boundary therefore attaches those k one-handles to the genus- g handlebody. The result is a genus- $g+k=d$ handlebody, proving the first assertion of (15).

Every class in λ_g^Y is the old-region image of a class in λ_g and dies in B_g , while every m_t bounds the corresponding co-core disk. Thus Λ is contained in the kernel in (16). The inclusion of the boundary of a genus- d handlebody onto the handlebody is surjective on rational first homology and has a d -dimensional kernel. The direct sum defining Λ has dimension $g+k=d$. The inclusion is therefore an equality. Assigning weight zero is part of the fixed extended cobordism and introduces no scalar.

Choose the fixed standard complementary handlebody used by the reduced pairing. Walker's extended-cobordism formalism permits the boundary Lagrangian of an extended three-manifold to be declared independently of its disk kernel, so its boundary is declared to be $(\bar{Y}, \Lambda)$ and its weight is set to zero [GM13, Section 2]. Throughout, complementary disk kernels are transported through the identity-on-chains vector-space identification

$$\text{or}_Y : H_1(\bar{Y}; \mathbb{Q}) \xrightarrow{\cong} H_1(Y; \mathbb{Q})$$

before they are compared with subspaces of $H_1(Y; \mathbb{Q})$ or entered in a Maslov index; this identification reverses the intersection pairing. To keep the two notions visibly distinct, write

$$\Lambda^c := \text{or}_Y(\ker(H_1(\bar{Y}; \mathbb{Q}) \longrightarrow H_1(H_{\emptyset,d}^c; \mathbb{Q}))) \subset H_1(Y; \mathbb{Q}).$$

The Mayer–Vietoris sequence for $S^3 = H_{\emptyset,d} \cup_Y H_{\emptyset,d}^c$ identifies $H_1(Y; \mathbb{Q})$ with the direct sum of the first homologies of the two handlebodies. Hence $\Lambda \cap \Lambda^c = 0$, and both summands have dimension d . In particular $\Lambda^c \neq \Lambda$; only the declared boundary Lagrangian of the complement is Λ .

Because A_K is primitive of order $4L$, the reduction root hypothesis holds with its source parameter $r = L$. Its negligible relation is therefore exactly f_{L-1} , and its allowed colors are $0, \dots, L-2 = K$. Root-of-unity reduction gives the admissible colored-spine basis. Its

admissibility inequality is a strict vertex-sum inequality $< 2L - 2$; the sum is even, so this is equivalent to the bound $\leq 2L - 4 = 2K$ stated above. A finite graph and a finite palette give only finitely many such colorings.

The reduced-spine orthogonality formula expresses every diagonal norm as a quotient of quantum dimensions and theta evaluations. All quantum integers occurring for an L -admissible coloring are nonzero at a primitive $4L$ -th root, and the theta formula is nonzero. Thus every norm is nonzero. Its complementary-handlebody linking proposition makes the bilinear bracket pairing nondegenerate. Extending the corresponding matrices from F_K to the field E_K preserves their nonzero determinants, proving all assertions about $\mathcal{M}_{K,d}$.

Finally, a Kauffman-bracket skein module is presented by isotopy classes of banded links and the local bracket relations, whose coefficients use only $A_K^{\pm 1}$. Extending either the $k_{p,K}$ -presentation or the F_K -presentation to their common field E_K therefore produces the same E_K -module presentation. The identity on link classes gives (20). No spine or basis at any other vertex has been introduced. $\square$

3.3. Quotient-first finiteness of the low state space. The low state space is deliberately treated before coordinates. The cap first gives a surjection onto the TQFT fibre, while the reduced relations produce a finite source; this establishes finiteness without selecting a low spine basis or claiming injectivity. The next factorization isolates the universal-radical and complementary-pairing argument that will later justify the fixed comparison map.

Proposition 3.3 (Low-fibre finiteness before coordinates). *The weight-zero cap induces the typed surjection*

$$(21) \quad \begin{aligned} q_{K,g}^- : E_K \otimes_{k_{p,K}} \mathcal{S}_{k_{p,K}}(B_g) &\rightarrow \mathcal{V}_{K,g}^-, \\ a \otimes x &\longmapsto a \otimes Z_p(B_g^{(0)}, x)(1). \end{aligned}$$

It kills every local f_{L-1} -relation and consequently factors as a surjection

$$(22) \quad E_K \otimes_{F_K} \mathcal{S}_{A_K}^{\text{red}}(B_g) \rightarrow \mathcal{V}_{K,g}^-, \quad \boxed{\dim_{E_K} \mathcal{V}_{K,g}^- < \infty}.$$

This conclusion chooses no low spine basis and asserts no injectivity.

Proof. Handlebody surjectivity first presents the low state space as an image of the skein module. Complementary-handlebody testing makes the local reduced relation vanish after gluing, so the map factors through the reduced quotient; a finite admissible spanning family then proves finite-dimensionality of the image without choosing a basis or claiming injectivity.

The fixed-level TQFT handlebody presentation is surjective for every handlebody whose boundary is the connected extended surface in question; it does not require the declared boundary Lagrangian to equal the handlebody disk kernel [GM13, Remark 11.3]. Apply it to $B_g^{(0)}$. Extension of scalars is right exact, so its scalar extension is precisely the surjection (21).

Let n be a local relation containing f_{L-1} . Pair its TQFT state with an arbitrary state of the oppositely oriented surface. The latter state has a handlebody-skein representative by the same TQFT surjectivity. After gluing, the resulting closed skein still contains the local f_{L-1} -relation. The reduced vanishing trace says that its evaluation is zero. Since the TQFT duality pairing is nondegenerate, the state of n is zero. Thus (21) kills the entire negligible submodule and factors through the reduced quotient. Using (20) gives the source displayed in (22).

The root-of-unity reduction argument supplies a finite spanning family indexed by the L -admissible colorings of any finite handlebody spine. Therefore the source of the factored map is finite-dimensional, and its surjective image is finite-dimensional. Only a spanning family was used; no low basis, comparison map, or injectivity statement has been made. $\square$

3.4. The normalized universal pairing and one comparison map. Set

$$\begin{aligned} \mathcal{U}_{K,d} &:= E_K \otimes_{k_{p,K}} \mathcal{S}_{k_{p,K}}(H_{\emptyset,d}), \\ \mathcal{U}_{K,d}^c &:= E_K \otimes_{k_{p,K}} \mathcal{S}_{k_{p,K}}(H_{\emptyset,d}^c), \\ \mathcal{V}_{K,d}^c &:= E_K \otimes_{k_{p,K}} V_p(\bar{Y}, \Lambda). \end{aligned}$$

Proposition 3.4 (Exact pairing, factorization, and both kernels). *The two weight-zero handlebodies give maps*

$$(23) \quad \begin{aligned} q_{K,d}^{\text{univ}} : \mathcal{U}_{K,d} &\cong E_K \otimes_{F_K} \mathcal{S}_{A_K}(H_{\emptyset,d}) \rightarrow \mathcal{V}_{K,d}^+, & a \otimes x &\mapsto a \otimes Z_p(H_{\emptyset,d}^{(0)}, x)(1), \\ q_{K,d}^{\text{c,univ}} : \mathcal{U}_{K,d}^c &\cong E_K \otimes_{F_K} \mathcal{S}_{A_K}(H_{\emptyset,d}^c) \rightarrow \mathcal{V}_{K,d}^c, & b \otimes y &\mapsto b \otimes Z_p((H_{\emptyset,d}^c)^{(0)}, y)(1). \end{aligned}$$

For $a, b \in E_K$ and coefficient-ring skeins x, y in the displayed handlebodies, the scalar-extended evaluation pairing is exactly

$$(24) \quad \boxed{\text{ev}_{Y,K}(q_{K,d}^{\text{univ}}(a \otimes x), q_{K,d}^{\text{c,univ}}(b \otimes y)) = ab \mathcal{D}_K^{-1} \langle x \cup y \rangle_{A_K}.$$

If $\mathcal{N}_{K,d} \subset \mathcal{U}_{K,d}$ denotes the E_K -span of the local f_{L-1} -relations, then

$$(25) \quad \mathcal{N}_{K,d} \subseteq \ker(q_{K,d}^{\text{univ}}) = \text{Rad}_{\text{left}}(\beta_{K,d}^{\text{univ}}),$$

where $\beta_{K,d}^{\text{univ}}(u, v) = \text{ev}_{Y,K}(q_{K,d}^{\text{univ}}(u), q_{K,d}^{\text{c,univ}}(v))$. Consequently the first map in (23) descends to a surjection

$$(26) \quad \bar{q}_{K,d} : \mathcal{M}_{K,d} \rightarrow \mathcal{V}_{K,d}^+.$$

The two required kernel inclusions are, with every scalar extension understood as displayed above,

$$(27) \quad \begin{aligned} \ker(q_{K,d}^{\text{univ}}) &\supseteq \mathcal{N}_{K,d}, \\ \ker(\bar{q}_{K,d}) &\subseteq \text{Rad}_{\text{left}} \langle -, - \rangle_{\text{red}, E_K} = 0. \end{aligned}$$

Proof. The two kernel inclusions serve different purposes. Vanishing of the local reduced relations makes the universal map factor through the reduced quotient; nondegeneracy of the reduced pairing proves that the descended map is injective. The exact gluing scalar connects the TQFT pairing to the bracket used in both steps.

Surjectivity and the formulas in (23) are the fixed-level TQFT handlebody presentation applied separately to $H_{\emptyset,d}^{(0)}$ and to $(H_{\emptyset,d}^c)^{(0)}$, followed by scalar extension. This is why the source ring, the weight, the value on 1, and the target scalar extension all appear explicitly.

We next compute the gluing normalization. Let Λ^c be the actual disk kernel of the complementary handlebody, as in the proof of Proposition 3.2. In the Walker gluing formula the three Lagrangians are

$$\lambda_{H_{\emptyset,d}}(Y) = \Lambda, \quad \lambda(Y) = \Lambda, \quad \lambda_{H_{\emptyset,d}^c}(\bar{Y}) = \Lambda^c.$$

Thus the glued weight is

$$0 + 0 - \mu_Y(\Lambda, \Lambda, \Lambda^c) = 0,$$

because a Maslov index with two equal entries vanishes. Notice that this calculation uses the equality between the *old* disk kernel and the declared boundary Lagrangian; it does not identify the complementary disk kernel with either one.

The glued closed manifold is consequently the weight-zero S^3 , with skein $x \cup y$. The Kauffman-bracket skein module of S^3 is generated by the empty skein [Prz99, Theorem 2.3(a)], and the class of $x \cup y$ is $\langle x \cup y \rangle_{A_K}$ times that generator. By (11), its TQFT value is therefore

$$\mathcal{D}_K^{-1} \langle x \cup y \rangle_{A_K}.$$

Multiplying the two scalar coefficients gives (24). In particular the normalizing scalar is the named element $\mathcal{D}_K^{-1} \in k_{p,K} \cap E_K^\times$, not an unspecified unit.

For completeness, we now reconstruct the universal radical statement in the present even- $p = 2K + 4$ theory; compare [BHMV95, the universal construction following Proposition 1.1]. If $u \in \ker(q_{K,d}^{\text{univ}})$, then it pairs to zero with every $v \in \mathcal{U}_{K,d}^c$, so it lies in the left radical of $\beta_{K,d}^{\text{univ}}$. Conversely, suppose u lies in that radical. The complementary map in (23) is surjective, so $q_{K,d}^{\text{univ}}(u)$ evaluates to zero against every vector of $\mathcal{V}_{K,d}^c$. Nondegeneracy of the scalar-extended TQFT duality pairing forces $q_{K,d}^{\text{univ}}(u) = 0$. Hence

$$\ker(q_{K,d}^{\text{univ}}) = \text{Rad}_{\text{left}}(\beta_{K,d}^{\text{univ}}).$$

This is a purely universal-construction argument; no odd-prime lattice, integrality, basis, or disconnected-surface theorem has entered it.

Now take $n \in \mathcal{N}_{K,d}$. For every complementary skein y , the closed bracket $\langle n \cup y \rangle_{A_K}$ vanishes by the local reduced trace identity for f_{L-1} . Equation (24) therefore puts n in the universal left radical, which is the kernel just proved. This gives the first inclusion in (27) and the well-defined surjection (26). Indeed, (20) and right exactness of scalar extension identify its source quotient explicitly as

$$\mathcal{U}_{K,d}/\mathcal{N}_{K,d} \cong E_K \otimes_{F_K} \mathcal{S}_{K,d}^{\text{red}} = \mathcal{M}_{K,d}.$$

The same local argument on the complementary side makes the bracket descend to the two reduced quotients.

Finally let $v \in \ker(\bar{q}_{K,d})$ and pair a representative with every reduced complementary skein w . Equation (24) and the nonzero scalar $\mathcal{D}_K^{-1}$ show that $\langle v, w \rangle_{\text{red}, E_K} = 0$ for every w . The reduced complementary pairing is nondegenerate over F_K , and remains so after extension to E_K . Hence $v = 0$. This proves the second inclusion in (27) independently of the first; no equality of dimensions has been used. $\square$

Proposition 3.5 (The comparison isomorphism at the old vertex). *The descended universal map is the isomorphism*

$$(28) \quad \boxed{Q_{K,d} := \bar{q}_{K,d} : \mathcal{M}_{K,d} \xrightarrow{\sim} \mathcal{V}_{K,d}^+}.$$

It is tied to the fixed low cap, old parameterization, coefficient embedding, old weight-zero handlebody, old spine, and weight-zero complementary testing handlebody. For every source vertex and every source edge, and for every later phase word, order word, and trace in which the common state space occurs, the literal maps used are

$$(29) \quad Q_{K,d} \quad \text{and} \quad Q_{K,d}^{-1};$$

no vertex-indexed or route-indexed comparison map is introduced.

Proof. The descended map is surjective by the quotient construction and injective by the separately established complementary pairing. Because its cap, parameterization, coefficient embedding, spine, and testing handlebody are all fixed at the old vertex, the same comparison map can be reused at every source vertex and edge. This is one fixed comparison, not a claim of canonicity under changing those choices.

The map $\bar{q}_{K,d}$ is surjective by (26). Its kernel is zero by the second inclusion in (27). It is therefore the isomorphism (28), proving surjectivity and injectivity by separate arguments.

Every object used to define this map was fixed at the old vertex and depends on none of the vertex subsets. The geometric cube theorem identifies every vertex with the same abstract extended surface (Y, Λ) , so the codomain is the same $\mathcal{V}_{K,d}^+$ throughout. Declaring (29) to be the sole comparison map therefore requires no compatibility or cancellation argument and creates no new choice. No canonicity under a change of cap, parameterization, coefficient embedding, spine, or handlebody weight is asserted. $\square$

4. STRICT TRANSPORT OF CUBE EDGES, PHASES, AND TRACES

The fixed comparison identifies the finite skein fibre with the TQFT state space, so a geometric edge now acts on the skein fibre by conjugation. Retaining the Walker integer records its central scalar, and retaining the composition order lets the phase changes and forward/reverse traces obey the same identities as their source cobordisms.

Fix all data of Sections 2 and 3. In particular, $K \geq 1$, $d = g + k \geq 3$, and every surface, Lagrangian, marking, cylinder, phase, and completed trace is the corresponding fixed object in the supplied geometric cube input. Write

$$Q := Q_{K,d} : \mathcal{M}_{K,d} \xrightarrow{\sim} \mathcal{V}_{K,d}^+$$

for the single old-vertex comparison isomorphism of Proposition 3.5. It is never replaced by a vertex-dependent map. Products of source cobordisms and of their images are written

right-factor first. Every source equality used below is a literal Walker equality supplied by the geometric input; none is inferred from a target operator equality.

First fix the strict scalar-extended Walker representation and the one comparison map that identifies the finite source fibre. Next transport each weight-zero slide edge, including its twist-word correction, without replacing the geometric edge by a skein presentation. Then add the completed endpoint phases and their central weights and edge covariance. Finally factor the forward and reverse traces separately, since their high boundaries have opposite categorical roles. The arguments below follow this order: strict functoriality first, supplied source equalities second, and matrix or scalar consequences only after the typed transport is complete.

For an oriented product-framed simple closed curve $\alpha \subset Y$, let T_α denote the positive (right) Dehn twist supported in its specified one-holed-torus collar. For a signed twist word $\mathfrak{w} = \alpha_1^{\varepsilon_1} \cdots \alpha_q^{\varepsilon_q}$, write

$$\mathrm{Tw}(\mathfrak{w}) := T_{\alpha_1}^{\varepsilon_1} \cdots T_{\alpha_q}^{\varepsilon_q}.$$

4.1. The strict scalar-extended Walker representation. Let $Z_K = E_K \otimes_{k_{p,K}} Z_p$. Scalar extension preserves the composition and identity axioms of the fixed-level extended TQFT, so for composable extended cobordisms

$$(30) \quad Z_K(M_2 M_1) = Z_K(M_2) Z_K(M_1), \quad Z_K(C_Y(\mathrm{id}_Y, 0)) = \mathrm{id}.$$

The Walker weight of the composite on the left is already the Maslov-corrected weight. Thus this is a strict functor on extended cobordisms, not the unextended projective mapping-class action.

Proposition 4.1 (Strict representation in the fixed comparison model). *For every high-surface extended cylinder G , define*

$$(31) \quad \rho_{K,d}^Q(G) := Q^{-1} Z_K(G) Q.$$

Then

$$(32) \quad \rho_{K,d}^Q(G_2 G_1) = \rho_{K,d}^Q(G_2) \rho_{K,d}^Q(G_1),$$

$$(33) \quad \rho_{K,d}^Q(G^{-1}) = \rho_{K,d}^Q(G)^{-1}.$$

If $W_Y = C_Y(\mathrm{id}_Y, 1)$, then for every $n \in \mathbb{Z}$,

$$(34) \quad \rho_{K,d}^Q(W_Y^n G) = \rho_{K,d}^Q(G W_Y^n) = \kappa_K^n \rho_{K,d}^Q(G).$$

Finally, a supplied literal source equality

$$(35) \quad G_r \cdots G_1 = G'_r \cdots G'_1$$

implies the literal operator equality

$$(36) \quad \rho_{K,d}^Q(G_r) \cdots \rho_{K,d}^Q(G_1) = \rho_{K,d}^Q(G'_r) \cdots \rho_{K,d}^Q(G'_1).$$

There is no converse implication.

Proof. Strict functoriality preserves the composition of extended cobordisms. When the fixed comparison map changes the source fibre, its adjacent inverse factors cancel in each product. The central cylinder records the weight character, so this conjugation transports a supplied Walker equality with its integer weight intact.

Insert $Q Q^{-1} = \mathrm{id}_{\mathcal{V}_{K,d}^+}$ between the two factors in (30):

$$Q^{-1} Z_K(G_2 G_1) Q = Q^{-1} Z_K(G_2) Q Q^{-1} Z_K(G_1) Q.$$

This proves (32). Applying it to $G^{-1} G = G G^{-1} = C_Y(\mathrm{id}_Y, 0)$ proves both sides of (33).

Increasing the Walker weight by one multiplies Z_p , and hence its scalar extension, by κ_K . The same statement for arbitrary positive and negative n , together with centrality of W_Y , gives (34). Repeated use of (32) on the two sides of the already supplied equality (35) proves (36). This last argument only applies a function to a source equality and therefore creates no target-to-source inference. $\square$

We use the right-twist convention of [GM13, Section 4, Theorem 4.2 and Definition 4.3; Section 11, Theorem 11.2]. For a word $\mathfrak{w}$, its surgery link has the curve α_i with framing $-\varepsilon_i$ relative to the surface, ordered with α_q nearest $0 \times Y$ and α_1 nearest $1 \times Y$. Cabling every component by the source's ω gives $s(\mathfrak{w})$. The corrected link is formed in the source's Heegaard embedding, with Λ the first handlebody disk kernel, by adjoining the zero-framed unlink of pushed-off meridians in the complement. Here $L_\Lambda^0(\mathfrak{w})$ denotes its linking matrix; the source uses that notation for the framed link itself.

Proposition 4.2 (Exact word-signature correction). *Let $f \in \Gamma(Y)$, $n \in \mathbb{Z}$, and let*

$$\mathfrak{w} = \alpha_1^{\varepsilon_1} \cdots \alpha_q^{\varepsilon_q}, \quad \text{Tw}(\mathfrak{w}) = f.$$

For the prescribed surgery skein $s(\mathfrak{w})$ and corrected linking-matrix signature

$$n_\Lambda^0(\mathfrak{w}) := \text{sign}(L_\Lambda^0(\mathfrak{w})),$$

one has

$$(37) \quad Z_K(C_Y(f, n)) = \kappa_K^{n-n_\Lambda^0(\mathfrak{w})} Z_K(I \times Y, s(\mathfrak{w})).$$

In particular,

$$(38) \quad Z_K(C_Y(f, 0)) = \kappa_K^{-n_\Lambda^0(\mathfrak{w})} Z_K(I \times Y, s(\mathfrak{w})).$$

The corrected expression is independent of the chosen Dehn-twist word.

Proof. The twist word is used as a presentation of an already fixed extended cylinder. Surgery supplies its corrected Walker weight, and changing the weight to the requested integer contributes the central character. Word independence follows only because both corrected expressions represent the same typed extended cylinder, not because a bare skein word has been declared canonical.

Gilmer–Masbaum’s surgery and TQFT word formulas [GM13, Theorems 4.2 and 11.2, equation (21)] say that extended surgery on the weight-zero identity cylinder along the framed link for $\mathfrak{w}$ produces $C_Y(f, n_\Lambda^0(\mathfrak{w}))$. The surgery axiom therefore identifies the bare surgery operator with

$$Z_K(C_Y(f, n_\Lambda^0(\mathfrak{w}))).$$

Changing that Walker weight to n multiplies by $\kappa_K^{n-n_\Lambda^0(\mathfrak{w})}$, which proves (37). Setting $n = 0$ gives (38); the minus sign is forced by the direction of the weight change. If two words represent f , both corrected expressions equal the same already typed extended cylinder $C_Y(f, n)$, so word independence is a consequence rather than a bare skein relation. $\square$

4.2. Weight-zero slide edges. For $A \subseteq [s]$ and $a \notin A$, set

$$(39) \quad T_{a|A}^K := \rho_{K,d}^Q(H_{a|A}) = Q^{-1} Z_K(C_Y(h_{a|A}, 0)) Q.$$

Proposition 4.3 (Typed edge inverse and word formula). *Every $T_{a|A}^K$ is an E_K -linear automorphism of $\mathcal{M}_{K,d}$, with*

$$(40) \quad (T_{a|A}^K)^{-1} = Q^{-1} Z_K(C_Y(h_{a|A}^{-1}, 0)) Q.$$

For every Dehn-twist word $\mathfrak{w}_{a|A}$ representing $h_{a|A}$,

$$(41) \quad T_{a|A}^K = Q^{-1} \left(\kappa_K^{-n_\Lambda^0(\mathfrak{w}_{a|A})} Z_K(I \times Y, s(\mathfrak{w}_{a|A})) \right) Q.$$

Proof. Invertibility is established in the source category first: each product with the proposed inverse has identity underlying map and zero Maslov correction because the edge preserves the declared Lagrangian in both directions. The twist-word formula is then just the one-way word correction conjugated by the same old comparison map, so it does not introduce a new edge or a new comparison map.

The geometric cube theorem supplies both $(h_{a|A})_*\Lambda = \Lambda$ and $(h_{a|A}^{-1})_*\Lambda = \Lambda$. Consequently both products of the proposed inverse cylinder with $H_{a|A}$ have underlying map the identity, and every Lagrangian in the ordered Walker seam is Λ . Their correction is therefore $\mu_Y(\Lambda, \Lambda, \Lambda) = 0$, so

$$C_Y(h_{a|A}^{-1}, 0)C_Y(h_{a|A}, 0) = C_Y(\text{id}_Y, 0) = C_Y(h_{a|A}, 0)C_Y(h_{a|A}^{-1}, 0).$$

Apply (33) to obtain (40). Equation (41) is (38) conjugated by the one fixed map Q . It computes the image of the already constructed geometric edge; it does not manufacture a slide from a skein word. $\square$

4.3. Completed endpoint phases. At each completed vertex $A \subseteq [s]$, let $\bar{m}_t^A, \bar{\ell}_t^A$ be the actual oriented product-framed row on $\partial_+(W, \mathfrak{C}_A)$, and let

$$m_t^A := p_A(\bar{m}_t^A), \quad \ell_t^A := p_A(\bar{\ell}_t^A) \subset Y$$

be its oriented counterpart in the fixed abstract surface, with its actual one-holed-torus support collar. For $t \in [k]$ and $u \in \mathbb{Z}$, retain the geometric input definitions

$$\sigma_{t,A} = T_{m_t^A} T_{\ell_t^A} T_{m_t^A}, \quad \Theta_{t,A}(u) = C_Y(\sigma_{t,A}, u),$$

and define

$$(42) \quad P_{t,A}^K(u) := \rho_{K,d}^Q(\Theta_{t,A}(u)).$$

Fix $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{Z}^k$. For $\epsilon \in \{0, 1\}^k$, let $I_\epsilon = \{t : \epsilon_t = 1\}$ and $U^+(\epsilon) = \sum_{t \in I_\epsilon} u_t^+$. If $\varrho = (\varrho_1, \dots, \varrho_q)$ orders I_ϵ , put

$$(43) \quad P_{A,\varrho}^K(\epsilon, \mathbf{u}^+) := P_{\varrho_q,A}^K(u_{\varrho_q}^+) \cdots P_{\varrho_1,A}^K(u_{\varrho_1}^+).$$

The corresponding common source data are

$$(44) \quad \sigma_A^\epsilon := \prod_{t \in I_\epsilon} \sigma_{t,A}, \quad S_A^\epsilon(\mathbf{u}^+) := C_Y(\sigma_A^\epsilon, U^+(\epsilon)),$$

where the mapping-class product is the supplied order-independent product. For the reverse assignment, set $I_\eta = \{t : \eta_t = 1\}$, $U^-(\eta) = \sum_{t \in I_\eta} u_t^-$, and define $S_A^\eta(\mathbf{u}^-)$ by the identical construction.

Proposition 4.4 (Common phase, central weight, and edge conjugacy). *For every ordering ϱ ,*

$$(45) \quad P_{A,\varrho}^K(\epsilon, \mathbf{u}^+) = P_A^K(\epsilon, \mathbf{u}^+) := \rho_{K,d}^Q(S_A^\epsilon(\mathbf{u}^+)).$$

This is the identity when $I_\epsilon = \emptyset$. For arbitrary integer weights,

$$(46) \quad P_{t,A}^K(u) = \kappa_K^u P_{t,A}^K(0),$$

$$(47) \quad P_A^K(\epsilon, \mathbf{u}^+) = \kappa_K^{U^+(\epsilon)} P_A^K(\epsilon, \mathbf{0}).$$

If $A' = A \cup \{a\}$, then

$$(48) \quad T_{a|A}^K P_A^K(\epsilon, \mathbf{u}^+) = P_{A'}^K(\epsilon, \mathbf{u}^+) T_{a|A}^K.$$

The same statements hold for the reverse assignment $\eta \in \{0, 1\}^k$ and weights $\mathbf{u}^-$.

Transporting phase data. Phase transport has two independent parts. Source-level disjoint support gives a common ordered cylinder and source-level covariance moves that cylinder across an edge; the strict representation then transports both facts to the finite fibre. Central weights are added through the one central cylinder, so the labels on the phase parameters remain visible throughout the argument.

Proof. The geometric cube theorem first supplies, at the source, the literal equality between every ordered product of the cylinders $\Theta_{t,A}(u_t^+)$ and the common cylinder

$$S_A^\epsilon(\mathbf{u}^+) = C_Y(\sigma_A^\epsilon, U^+(\epsilon)).$$

Apply the one-way shadow (36); this proves (45) while keeping each label t attached to its integer u_t^+ . The empty source word is the Walker identity, so its image is the identity.

The source cylinders satisfy

$$\Theta_{t,A}(u) = W_Y^u \Theta_{t,A}(0), \quad S_A^\epsilon(\mathbf{u}^+) = W_Y^{U^+(\epsilon)} S_A^\epsilon(\mathbf{0}).$$

Equation (34) proves (46) and (47), including negative and zero weights.

Finally the geometric cube theorem supplies the literal Walker equality

$$H_{a|A} S_A^\epsilon(\mathbf{u}^+) = S_{A'}^\epsilon(\mathbf{u}^+) H_{a|A}.$$

Its image under the same Q on every factor is exactly (48). The reverse phase statement is identical after replacing $(\epsilon, \mathbf{u}^+)$ by $(\eta, \mathbf{u}^-)$. $\square$

4.4. Forward and reverse trace maps. For label-indexed vectors $\mathbf{c}, \mathbf{r} \in \mathbb{Z}^k$, write

$$c_\Sigma = \sum_{t=1}^k c_t, \quad r_\Sigma = \sum_{t=1}^k r_t.$$

For the supplied completed source cobordisms $J_A^\epsilon(\mathbf{c}, \mathbf{u}^+) : X \rightarrow Y$ and $R_A^\eta(\mathbf{r}, \mathbf{u}^-) : Y \rightarrow X$, define

$$(49) \quad J_A^K(\epsilon; \mathbf{c}, \mathbf{u}^+) := Q^{-1} Z_K(J_A^\epsilon(\mathbf{c}, \mathbf{u}^+)) : \mathcal{V}_{K,g}^- \longrightarrow \mathcal{M}_{K,d},$$

$$(50) \quad R_A^K(\eta; \mathbf{r}, \mathbf{u}^-) := Z_K(R_A^\eta(\mathbf{r}, \mathbf{u}^-)) Q : \mathcal{M}_{K,d} \longrightarrow \mathcal{V}_{K,g}^-.$$

Proposition 4.5 (Phase factorization, weights, and both covariances). *The completed maps factor as*

$$(51) \quad J_A^K(\epsilon; \mathbf{c}, \mathbf{u}^+) = P_A^K(\epsilon, \mathbf{u}^+) J_A^K(\mathbf{0}; \mathbf{c}, \mathbf{0}),$$

$$(52) \quad R_A^K(\eta; \mathbf{r}, \mathbf{u}^-) = R_A^K(\mathbf{0}; \mathbf{r}, \mathbf{0}) P_A^K(\eta, \mathbf{u}^-).$$

Their exact central-weight dependence is

$$(53) \quad J_A^K(\epsilon; \mathbf{c}, \mathbf{u}^+) = \kappa_K^{c_\Sigma + U^+(\epsilon)} J_A^K(\epsilon; \mathbf{0}, \mathbf{0}),$$

$$(54) \quad R_A^K(\eta; \mathbf{r}, \mathbf{u}^-) = \kappa_K^{r_\Sigma + U^-(\eta)} R_A^K(\eta; \mathbf{0}, \mathbf{0}).$$

Here only the integer weights are zeroed on the right; the phase assignment remains fixed. If $A' = A \cup \{a\}$, then

$$(55) \quad T_{a|A}^K J_A^K(\epsilon; \mathbf{c}, \mathbf{u}^+) = J_{A'}^K(\epsilon; \mathbf{c}, \mathbf{u}^+),$$

$$(56) \quad R_{A'}^K(\eta; \mathbf{r}, \mathbf{u}^-) T_{a|A}^K = R_A^K(\eta; \mathbf{r}, \mathbf{u}^-).$$

Forward and reverse traces. The forward and reverse traces are factored separately because their high boundary has opposite categorical roles. Strict functoriality supplies the two factorizations, the central character supplies their exact weights, and the already established phase and edge covariances supply the two transport identities. No identification of the forward and reverse maps is made.

Proof. The supplied endpoint definitions are literal source factorizations

$$J_A^\epsilon(\mathbf{c}, \mathbf{u}^+) = S_A^\epsilon(\mathbf{u}^+) J_A^0(\mathbf{c}, \mathbf{0}),$$

and

$$R_A^\eta(\mathbf{r}, \mathbf{u}^-) = R_A^0(\mathbf{r}, \mathbf{0}) S_A^\eta(\mathbf{u}^-).$$

Apply strict functoriality and insert QQ^{-1} on the high side. The resulting two typed compositions are precisely (51) and (52). Notice that their orders differ because the high boundary is the target of J but the source of R .

The geometric cube theorem also identifies the completed forward and reverse weights as $c_\Sigma + U^+(\epsilon)$ and $r_\Sigma + U^-(\eta)$, respectively, relative to the same phase assignments with every integer weight zero. Applying the central character (34) on the high side, or equivalently the fixed-level weight axiom before inserting Q , gives (53) and (54).

For covariance, the supplied source equalities are independently typed as

$$H_{a|A} J_A^\epsilon(\mathbf{c}, \mathbf{u}^+) = J_{A'}^\epsilon(\mathbf{c}, \mathbf{u}^+) : X \longrightarrow Y,$$

and

$$R_{A'}^\eta(\mathbf{r}, \mathbf{u}^-) H_{a|A} = R_A^\eta(\mathbf{r}, \mathbf{u}^-) : Y \longrightarrow X.$$

For the first, apply $Q^{-1}Z_K(-)$ and insert QQ^{-1} between the two source factors. For the second, apply $Z_K(-)Q$ and make the same insertion. These calculations give (55) and (56) separately. No adjoint, dagger, or unitarity statement connects the two arguments. $\square$

The results of this section establish strict functoriality, the corrected word formula, the edge and phase operators, and the two trace families. They use the old comparison map only through Proposition 3.5; they do not prove a source relation, introduce a slid-vertex basis, construct a low-side gauge, or assert any finite-level dagger structure.

5. CUBE SQUARES, ORDER INDEPENDENCE, AND FINITE MATRICES

The supplied geometric square is already an equality of Walker cobordisms. Transporting it through the strict functor and the same comparison map at every vertex gives an operator square. Finite matrix identities then record that square in chosen coordinates.

All notation and conventions are those of Sections 4 and the standing assumptions. In particular, every operator is formed with the one map $Q_{K,d}$, every source product and target product is written right-factor first, and source topology always precedes its operator image.

Here $H_{a|A} = C(h_{a|A}, 0)$ is the marked source handleslide cylinder, while $T_{a|A}^K$ is its image on the fixed-level operator fibre. The set tag A belongs to both symbols: it records the source vertex of the geometric edge before that edge is transported to an operator.

First transport each literal geometric square through the one fixed comparison map and strict finite-level functor. Then use the resulting operator squares to compare adjacent orders, keeping the intermediate-set tag attached to every factor. The forward and reverse trace families are telescoped only after that order-independence step. Finally choose bases to record the established operator identities as finite square and rectangular matrices. Thus the matrix formulas are

coordinates on fixed-level transport, not a new source of coherence or a comparison between levels.

5.1. Literal operator squares.

Proposition 5.1 (Every disjoint source face has a literal image). *Let $A \subseteq [s]$ and let $a, b \notin A$ be distinct. Then*

$$(57) \quad T_{b|A \cup \{a\}}^K T_{a|A}^K = T_{a|A \cup \{b\}}^K T_{b|A}^K$$

as a literal equality in $\text{End}_{E_K}(\mathcal{M}_{K,d})$.

Proof. The geometric cube theorem supplies the Walker equality

$$(58) \quad H_{b|A \cup \{a\}} H_{a|A} = H_{a|A \cup \{b\}} H_{b|A}.$$

Every cylinder in (58) has geometric weight zero, and the source seam calculation shows that both composites also have weight zero. Apply (36) and the edge definition (39); the result is exactly (57). No matrix or operator equality is used in the source direction. $\square$

Smallest square. The first nontrivial instance has $s = 2$, $A = \emptyset$, and $a = 1, b = 2$. Its four vertices are $\emptyset, \{1\}, \{2\}, \{1, 2\}$, and the proposition says that the two routes from $\emptyset$ to $\{1, 2\}$ give the same operator: $T_{2|\{1\}}^K T_{1|\emptyset}^K = T_{1|\{2\}}^K T_{2|\emptyset}^K$. This is the smallest picture in which the intermediate-set tag matters. It also shows what the fixed comparison map accomplishes: all four edge maps act on the same finite fibre $\mathcal{M}_{K,d}$, so the equality is obtained before any basis is chosen. The finite matrix statement below is only the coordinate form of this already established operator square.

5.2. Order independence and equality-only Coxeter closure.

For $\pi = (\pi_1, \dots, \pi_s) \in S_s$, set

$$(59) \quad A_q^\pi := \{\pi_1, \dots, \pi_q\}, \quad A_0^\pi := \emptyset,$$

and define

$$(60) \quad T_\pi^K := T_{\pi_s|A_{s-1}^\pi}^K \cdots T_{\pi_1|\emptyset}^K.$$

The set tag records slides already applied to the right of the indicated factor.

Proposition 5.2 (Finite order closure). *For every $\pi \in S_s$,*

$$(61) \quad T_\pi^K = T_{[s]}^K := \rho_{K,d}^Q(H_{[s]}).$$

Adjacent applications of (57) give equality replacements τ_t , and these replacements close around

$$(62) \quad \tau_t^2 = 1, \quad \tau_t \tau_u = \tau_u \tau_t \quad (|t - u| > 1), \quad \tau_t \tau_{t+1} \tau_t = \tau_{t+1} \tau_t \tau_{t+1}.$$

The first two families use $1 \leq t, u \leq s - 1$; the third uses $1 \leq t \leq s - 2$. These are equalities between replacement paths, not operators and not an action of S_s .

Proof. The geometric cube theorem proves, for each π , the literal source identity

$$H_{\pi_s|A_{s-1}^\pi} \cdots H_{\pi_1|\emptyset} = H_{[s]}.$$

Apply the strict representation and use (60). This proves (61) directly, including the fact that every product has total Walker weight zero.

An adjacent transposition changes precisely the two factors spanning one square at the correct intermediate set tag; Proposition 5.1 replaces those two factors by the other side. Repeating

the same equality replacement returns to the original word, which gives the first family in (62). Replacements in disjoint word positions commute, giving the second family. The two length-three replacement paths in the last family have the same final order word; each intermediate word equals $T_{[s]}^K$ by (61), so the two paths close. This proves equality-only closure without assigning a linear map to any τ_t . When $s = 1$, no index range is nonempty. When $s = 2$, only the involution range is nonempty. $\square$

5.3. Forward and reverse telescoping.

Proposition 5.3 (All-phase and all-weight telescoping). *For every phase assignment, every integer base-weight vector, and every integer phase-weight vector,*

$$(63) \quad T_{[s]}^K J_{\emptyset}^K(\epsilon; \mathbf{c}, \mathbf{u}^+) = J_{[s]}^K(\epsilon; \mathbf{c}, \mathbf{u}^+),$$

$$(64) \quad R_{[s]}^K(\eta; \mathbf{r}, \mathbf{u}^-) T_{[s]}^K = R_{\emptyset}^K(\eta; \mathbf{r}, \mathbf{u}^-).$$

Proof. Fix an order π . Iterating (55) in chronological order gives, for every $1 \leq q \leq s$,

$$T_{\pi_q | A_{q-1}^\pi}^K \cdots T_{\pi_1 | \emptyset}^K J_{\emptyset}^K(\epsilon; \mathbf{c}, \mathbf{u}^+) = J_{A_q^\pi}^K(\epsilon; \mathbf{c}, \mathbf{u}^+).$$

At $q = s$, the operator product is $T_\pi^K = T_{[s]}^K$ and $A_s^\pi = [s]$, proving (63). Starting at the terminal vertex and iterating (56) from the left instead gives

$$R_{A_s^\pi}^K(\eta; \mathbf{r}, \mathbf{u}^-) T_{\pi_s | A_{s-1}^\pi}^K \cdots T_{\pi_1 | \emptyset}^K = R_{\emptyset}^K(\eta; \mathbf{r}, \mathbf{u}^-),$$

which is (64). The argument never changes a phase assignment or any integer weight, so it includes empty phase words, zero weights, and arbitrary negative weights. Their scalar content remains exactly (53) and (54). $\square$

5.4. Fibrewise finite matrices. At this point the operator identities have already been proved on the fixed finite-level fibres. Choosing the old-spine basis now produces square arrays for endomorphisms and rectangular arrays for the forward and reverse maps between the low and high fibres. These matrices record the transported relations in coordinates; they do not define the comparison map, remove its choice dependence, or introduce a comparison between different levels.

An exact $K = 1$ matrix. The central weight-one cylinder gives a small-level check with no unspecified entries. At $K = 1$, the normalization in proposition 3.1 has

$$\kappa_1 = \exp(\pi i/4) = \zeta_8.$$

Hence, in any ordered old-spine basis $\mathcal{B}_{1,d}$,

$$[\rho_{1,d}^Q(W_Y)]_{\mathcal{B}_{1,d}} = \zeta_8 I_{r_{1,d}}, \quad r_{1,d} = \dim_{E_1} \mathcal{M}_{1,d}.$$

This exact scalar matrix is already enough to expose the coefficient-field issue: $\kappa_1 \notin F_1$, so the matrix belongs naturally over E_1 , not uniformly over the raw field F_1 . It is a normalization example, not a computation of a handleslide-edge matrix.

Proposition 5.4 (Fixed-level matrix realization). *Order the finite old-spine basis $\mathcal{B}_{K,d}$. For $\epsilon, \eta \in \{0, 1\}^k$ and $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{Z}^k$, every edge, phase, and total-order operator has a finite matrix over E_K :*

$$(65) \quad [T_{a|A}^K]_{\mathcal{B}_{K,d}}, \quad [P_A^K(\epsilon, \mathbf{u}^+)]_{\mathcal{B}_{K,d}}, \quad [P_A^K(\eta, \mathbf{u}^-)]_{\mathcal{B}_{K,d}}, \quad [T_{[s]}^K]_{\mathcal{B}_{K,d}}.$$

After Proposition 3.3 has established $\dim_{E_K} \mathcal{V}_{K,g}^- < \infty$, choose any basis of the low module. The maps J_A^K and R_A^K then have finite rectangular matrices. In these bases, every equality proved above remains a literal matrix equality.

Proof. The reduced spine basis is finite by Proposition 3.2, and every high operator is an E_K -linear endomorphism of its span. This proves (65). The separate quotient-first argument in Proposition 3.3 proves low finiteness before a low basis is chosen, so the forward and reverse maps have finite rectangular coordinate arrays afterward. Passing an already proved equality to coordinates preserves equality. A change of the high basis simultaneously conjugates all high endomorphisms and gives the corresponding one-sided base change for the rectangular maps. Nothing in this argument relates different values of K . $\square$

5.5. Main theorem. The theorem below is the algebraic payoff of keeping one comparison isomorphism in view. The geometric cube has already supplied literal Walker equalities; the remaining task is to show that the same normalized map carries every vertex to the same finite-level fibre. Once that is done, squares, route products, and telescoping identities survive as literal operator equalities rather than only as projective or basis-dependent statements.

Reusing the same comparison map lets its inverse cancel between successive edge images. Strict functoriality therefore carries each geometric two-face to an operator square, and adjacent replacements give finite order independence. The endpoint covariance identities then make the phase-weight traces telescope. Bases are chosen afterward to express these operator identities as finite square and rectangular matrix identities at the fixed level.

Theorem 5.5 (Fixed-level disjoint-cube operator transport). *Uniformly for every datum quantified in the standing assumptions, the following nine statements hold.*

- (1) *The root, coefficient-ring, field, rank, anomaly, involution, and weight-zero S^3 identities hold, including the exact positive square-root branch and the $K = 1$ obstruction to the raw field.*
- (2) *The weight-zero old handlebody, its complementary pairing witness, the finite reduced old fibre, its admissible spine basis, and the old disk kernel are as constructed in Section 3; the low module is finite by Proposition 3.3.*
- (3) *The universal map factors through the reduced quotient, its pairing is exactly $\mathcal{D}_K^{-1}$ times the reduced complementary pairing, both kernel inclusions hold, and the result is the single isomorphism $Q_{K,d}$ used at every vertex.*
- (4) *Equations (32)–(36) and (37)–(38) give the strict corrected extended functor with its central character.*
- (5) *Equations (39)–(41) and (45)–(48) give every edge, its exact inverse, all completed phases, their labelled orders, and their weights.*
- (6) *Equations (49)–(56) give both independently typed forward and reverse families, phase factorizations, total-weight scalars, and covariances.*
- (7) *Every supplied disjoint source face has the literal operator square (57), only in the source-to-target direction.*
- (8) *Equations (61)–(64) give finite order independence, equality-only Coxeter closure with the exact vacuities, and forward/reverse telescoping for all phases and weights.*
- (9) *Proposition 5.4 gives only the claimed fixed- K , E_K -linear finite matrices and no cross-level, refined, topological, or closed-manifold promotion.*

Proof. Item one is Proposition 3.1. Item two combines Propositions 3.2 and 3.3; item three combines Propositions 3.4 and 3.5. Items four through six are Propositions 4.1, 4.2, 4.3, 4.4, and 4.5. The last three items are Propositions 5.1, 5.2, 5.3, and 5.4. Every use of the geometric cube theorem in those proofs applies a supplied source equality; every target conclusion is then obtained by the strict fixed-level functor. $\square$

5.6. Boundary of the cube-branch theorem. Theorem 5.5 includes the degenerate edge cases needed for a uniform statement: $K = 1$, genus $g = 1$, one or two active slides, spectator handles, empty phase words, and arbitrary integral Walker weights. The theorem is nevertheless deliberately fixed-level and choice-dependent. More precisely:

- it is proved at one level K and supplies no comparison between different levels;

- every operator uses one fixed comparison map $Q_{K,d}$, with no claim that another admissible comparison map gives the identical matrix;
- the source relations are the supplied relations among disjoint slide directions; source equality implies operator equality, but no converse is asserted;
- the five overlapping directions of the pentagon with complete markings in Section 6 are not instances of these disjoint directions; and
- no Cerf-completeness, stabilization, closed-manifold invariance, refined theory, WRT–Habiro comparison, or $\widehat{Z}$ statement is deduced here.

6. THE GEOMETRIC PENTAGON INPUT AND ITS FIXED-LEVEL OPERATOR IMAGE

The pentagon reuses three handles along routes of different lengths, so it requires a geometric input separate from the disjoint cube. We start from one selected pentagon with complete markings and transport it using the already proved root normalization, comparison construction, and strict functoriality. The geometric relation and its operator image retain their separate owners.

From the geometric pentagon to its operator image. The construction has four stages. First, the selected geometric witness is placed on one extended surface with one common Lagrangian, so the two routes are literal Walker composites. Second, a comparison core is built once at vertex zero from the cap, handlebody, pairing, and spine data. Third, the same core transports all five edge cylinders, allowing the intermediate comparison maps to cancel along both routes. Finally, changing the auxiliary core is handled by a typed intertwiner rather than by identifying the resulting coordinate matrices. The operator pentagon is therefore a transported geometric relation, not a new relation reconstructed from matrices.

The route notation remains fixed throughout the section. The short source route is $H_{B\star}H_{0B}$, and the long source route is $H_{AC\star}H_{AAC}H_{0A}$, with the right factor applied first. Their operator images use the same edge names with T^K in place of H . This parallel notation is the type check connecting the geometric pentagon to its transported coherence relation.

6.1. The selected geometric input. Fix $K \geq 1$ and the root and coefficient data of Proposition 3.1. Fix a tuple

$$(66) \quad \mathcal{T} = (g, N, W : X \rightarrow Y_{\text{geom}}, i, j, \ell, \mathfrak{C}_{\text{ext}}, \lambda_g, \mathcal{L}_{ij\ell}),$$

where X is a closed oriented surface of genus $g \geq 1$, and W is the compact oriented compression body obtained from $X \times [0, 1]$ by attaching $N \geq 3$ ordered product 1-handles at the top. The indices $i, j, \ell \in \{1, \dots, N\}$ are pairwise distinct; $\mathfrak{C}_{\text{ext}}$ is the fixed ordered exterior completed row, including the common exterior parameterization and old-summand identification; $\lambda_g \subset H_1(X; \mathbb{Q})$ is its rational old disk-kernel Lagrangian; and $\mathcal{L}_{ij\ell}$ is a supplied fully protected standard three-handle sector. The symbol ℓ avoids a collision with the ordinary handle-count symbol k . Choose one five-vertex witness in its vertex-first order and set $d = g + N \geq 4$. Its vertex and edge sets are

$$(67) \quad V_5 = \{0, B, A, AC, \star\}, \quad E_5 = \{0B, B\star, 0A, AAC, AC\star\}.$$

The selected witness supplies, for every $v \in V_5$, a complete marking $\mathfrak{C}_v$ on the same compression body W and an orientation-preserving parameterization $p_v : \partial_+ W \rightarrow Y$ onto one fixed oriented surface $Y = \Sigma_d^{\text{out}}$. The exterior marking identifies the old summand $\lambda_g^Y \subset H_1(Y; \mathbb{Q})$ and the ordered product-handle meridians $m_1, \dots, m_N$. The directed edges are

$$0B : 0 \rightarrow B, \quad B\star : B \rightarrow \star, \quad 0A : 0 \rightarrow A, \quad AAC : A \rightarrow AC, \quad AC\star : AC \rightarrow \star.$$

For $e : v \rightarrow w$, let F_e be the selected global map and put

$$(68) \quad h_e = p_w(F_e|_{\partial_+ W})p_v^{-1} : Y \longrightarrow Y.$$

The common rational Lagrangian is

$$(69) \quad \Lambda = \lambda_g^Y \oplus \bigoplus_{t \notin \{i, j, \ell\}} \mathbb{Q}[m_t] \oplus \text{span}_{\mathbb{Q}}\{[m_i], [m_j], [m_\ell]\},$$

where the middle sum is zero when $N = 3$.

Theorem 6.1 (Selected Walker pentagon with complete markings). *Every h_e preserves Λ in both directions. Thus*

$$(70) \quad H_e = C_Y([h_e], 0) : (Y, \Lambda) \longrightarrow (Y, \Lambda)$$

is a weight-zero Walker endocylinder. With right factor acting first,

$$(71) \quad \boxed{H_{B\star}H_{0B} = H_{AC\star}H_{AAC}H_{0A} = C_Y([h_{B\star}h_{0B}], 0).}$$

All five constituent weights, all three seams, and both route weights are zero.

This is the selected-system theorem, two-sided Lagrangian proposition, relative pentagon, Walker lift, and exact theorem boundary proved in [Sun26a, Theorem 4.1, Proposition 12.3, Theorems 13.5 and 14.1, and Section 15]. In particular, the route equality comes from a relative isotopy fixed on the protected collar, not from equality of homology matrices.

6.2. Existence of a compatible comparison core. The proof of the ordinary-cube map $Q_{K,d}$ in Section 3 uses the geometric cube only to obtain its reference compression body. No cube edge, phase, trace, square, order, or telescoping relation enters the cap, pairing, reduction, or kernel arguments. We now apply the same reference-data argument to vertex zero of the pentagon. Because its cap, high handlebody, complement, and spine are chosen independently, its source module and comparison map receive a core index and are not identified with the ordinary-cube objects.

Lemma 6.2 (Rational Lagrangians are handlebody disk kernels). *Let X be a closed oriented surface of genus g , and let $\lambda \subset H_1(X; \mathbb{Q})$ be a rational Lagrangian. There is a parameterized genus- g handlebody B such that*

$$(72) \quad \ker(H_1(X; \mathbb{Q}) \longrightarrow H_1(B; \mathbb{Q})) = \lambda.$$

Proof. The rational disk-kernel is first made integral by intersecting it with the surface lattice. A symplectic-basis correction then realizes that primitive isotropic lattice as the meridian system of a standard handlebody, so the desired cap is obtained by a mapping-class change of parameterization.

Put $H = H_1(X; \mathbb{Z})$, with its unimodular symplectic intersection form ω , and $L_\lambda = H \cap \lambda$. Rationality implies that L_λ has rank g and spans λ . It is primitive: if $nx \in L_\lambda$ for $0 \neq n \in \mathbb{Z}$ and $x \in H$, then $x = n^{-1}(nx) \in \lambda$, so $x \in L_\lambda$. It is isotropic because λ is.

Choose a basis $a_1, \dots, a_g$ of L_λ . Primitivity lets every homomorphism $L_\lambda \rightarrow \mathbb{Z}$ extend to H , while unimodularity identifies H with H^* . Hence there are $b_1, \dots, b_g \in H$ with $\omega(a_i, b_j) = \delta_{ij}$. If $A_{ij} = \omega(b_i, b_j)$, choose an integral matrix $C = (c_{ij})$ satisfying $C^T - C = -A$, and replace

$$(73) \quad b_i \quad \text{by} \quad b'_i = b_i + \sum_{r=1}^g c_{ri} a_r.$$

Then $\omega(b'_i, b'_j) = A_{ij} + c_{ji} - c_{ij} = 0$, so $(a_1, \dots, a_g, b'_1, \dots, b'_g)$ is an integral symplectic basis. An orientation-preserving mapping class realizes its change from a standard symplectic basis: elementary Dehn-twist transvections generate the integral symplectic group. Parameterize a standard handlebody so that its meridians map to the a_i . Its rational disk kernel is $\text{span}_{\mathbb{Q}}\{a_1, \dots, a_g\} = \lambda$, proving (72). $\square$

The five edge operators can be compared only after they have been transported through one shared source-to-TQFT identification. The next definition packages the cap, spine, coefficient extension, and reduced comparison map that must be chosen once and then held fixed on every edge and both routes.

Definition 6.3 (Compatible fixed comparison data). A COMP- Q core, denoted $\mathfrak{q}$, for the selected pentagon package consists of the following literal choices and identifications.

- (1) The oriented extended surface is the same named object (Y, Λ) , with $d = g + N$.
- (2) The reference marking and parameterization are literally $(\mathfrak{C}_0, p_0)$ on the same W .

- (3) The low cap $B_g^{(0)} : \emptyset \rightarrow (X, \lambda_g)$ has weight zero and disk kernel λ_g .
- (4) The reference high handlebody is literally

$$H_{0,d} = B_g \cup_X (W, \mathfrak{C}_0),$$

with outgoing parameterization p_0 , weight zero, and disk kernel Λ .

- (5) The complementary high handlebody has weight zero, declared boundary Lagrangian Λ , glues to $H_{0,d}$ to give S^3 , and uses the exact weight-zero normalization $\mathcal{D}_K^{-1} \in k_{p,K} \cap E_K^\times$.
- (6) One admissible spine, the exact source-ring inclusion $k_{p,K} \subset E_K$, the core-indexed reduced source

$$\mathcal{M}_{K,d}^q := E_K \otimes_{F_K} \mathcal{S}_{A_K}^{\text{red}}(H_{0,d}),$$

the two universal-map kernel inclusions—the negligible ideal is killed and the descended kernel lies in the zero reduced radical—and the fixed isomorphism

$$Q_{K,d}^q : \mathcal{M}_{K,d}^q \xrightarrow{\sim} \mathcal{V}_{K,d}^+ = E_K \otimes_{k_{p,K}} V_p(Y, \Lambda)$$

determined by those choices.

- (7) The same literal $Q_{K,d}^q$ and $(Q_{K,d}^q)^{-1}$ are used on all five edges and both routes. There is no vertex-, edge-, or route-dependent comparison map.

Abstract isomorphism, equality of dimensions, or equality of genus does not satisfy this definition. These are auxiliary choices; their existence is proved next. No uniqueness or canonicity is included in the definition.

Proposition 6.4 (Existence of compatible comparison data). *For every fixed $K \geq 1$, every tuple $\mathcal{T}$ in the stated range, and every selected five-vertex witness package furnished by Input Theorem 6.1, there exists a COMP- Q core $\mathfrak{q}$.*

Building one comparison core. Build the core once at vertex zero: cap the old summand, attach the ordered product handles, choose a complementary Heegaard side, and use the reduced pairing to identify the resulting source with the fixed TQFT fibre. Because every edge cylinder has the same boundary object (Y, Λ) , that one core can then be reused on both routes without introducing edge-dependent comparison maps.

Proof. Apply Lemma 6.2 to λ_g , and give the resulting cap Walker weight zero. Capping the incoming boundary of W attaches its N product 1-handles to B_g , so $H_{0,d} = B_g \cup_X W$ is a genus- $d = g + N$ handlebody. Its disk kernel contains λ_g^Y and every product-handle meridian. The sum in (69) is direct and has dimension d , which is the dimension of a genus- d handlebody disk kernel; hence the kernel is exactly Λ .

Identify $H_{0,d}$ with one side of a standard genus- d Heegaard splitting of S^3 , and pull back the other side as $H_{0,d}^c$. Give the complement weight zero and declare its boundary Lagrangian to be Λ . Its actual disk kernel Λ^c is complementary to Λ and is not being identified with it. The Walker seam is

$$\mu_Y(\Lambda, \Lambda, \Lambda^c) = 0,$$

so the glued manifold is the weight-zero S^3 . Proposition 3.1 gives its exact value $\mathcal{D}_K^{-1} \in k_{p,K} \cap E_K^\times$.

Choose a banded trivalent spine in $H_{0,d}$, and denote the resulting collection of cap, complement, normalization, and spine choices by $\mathfrak{q}$. The reduced handlebody theorem supplies its finite admissible-coloring basis and a nondegenerate pairing with the complementary reduced skein module. Put

$$\mathcal{W}_{K,d}^0 := E_K \otimes_{k_{p,K}} \mathcal{S}_{k_{p,K}}(H_{0,d}), \quad \mathcal{W}_{K,d}^{c,0} := E_K \otimes_{k_{p,K}} \mathcal{S}_{k_{p,K}}(H_{0,d}^c),$$

and let $\mathcal{N}_{K,d}^0 \subset \mathcal{W}_{K,d}^0$ be the span of the local f_{L-1} -relations. The two weight-zero handlebodies give surjective universal maps q^0 and $q^{c,0}$, and gluing gives the exact pairing

$$(74) \quad \text{ev}_{Y,K}(q^0(a \otimes x), q^{c,0}(b \otimes y)) = ab \mathcal{D}_K^{-1} \langle x \cup y \rangle_{A_K}.$$

A local f_{L-1} -relation pairs trivially with every complementary skein; surjectivity on the complementary side and TQFT duality therefore put it in $\ker q^0$. The canonical base change and right-exact quotient identify

$$(75) \quad \mathcal{U}_{K,d}^0 / \mathcal{N}_{K,d}^0 \cong E_K \otimes_{F_K} \mathcal{S}_{A_K}^{\text{red}}(H_{0,d}) = \mathcal{M}_{K,d}^q.$$

The same local argument makes the complementary pairing descend. If a class in $\mathcal{M}_{K,d}^q$ maps to zero, then (74) makes it pair trivially with every reduced complementary skein. Nondegeneracy of that reduced pairing makes the class zero. Thus the descended map $\bar{q}^0$ is independently surjective and injective, and is the required $Q_{K,d}^q := \bar{q}^0 : \mathcal{M}_{K,d}^q \xrightarrow{\sim} \mathcal{V}_{K,d}^+$.

Every choice in this construction is made at vertex zero and is independent of every pentagon edge. Since the five cylinders have the same source and target (Y, Λ) , declaring this one $Q_{K,d}^q$ to be the comparison map on all five edges completes all seven clauses. $\square$

6.3. Transport of the pentagon. Fix a compatible core $\mathfrak{q}$. Its source module and comparison map are the indexed objects $\mathcal{M}_{K,d}^q$ and $Q_{K,d}^q$, not the independently chosen ordinary-cube objects $\mathcal{M}_{K,d}$ and $Q_{K,d}$. For $e \in E_5$, define

$$(76) \quad \mathsf{T}_e^{K,q} = (Q_{K,d}^q)^{-1} Z_K(H_e) Q_{K,d}^q \in \text{Aut}_{E_K}(\mathcal{M}_{K,d}^q).$$

Lemma 6.5 (Zero-weight inverse cylinders). *For every $e \in E_5$,*

$$(77) \quad C_Y([h_e^{-1}], 0) H_e = H_e C_Y([h_e^{-1}], 0) = C_Y([\text{id}_Y], 0).$$

Consequently

$$(78) \quad (\mathsf{T}_e^{K,q})^{-1} = (Q_{K,d}^q)^{-1} Z_K(C_Y([h_e^{-1}], 0)) Q_{K,d}^q.$$

Proof. For the first composite in (77), the Walker seam in (6) is

$$\mu_Y((h_e)_* \Lambda, \Lambda, ((h_e^{-1})_*)^{-1} \Lambda) = \mu_Y(\Lambda, \Lambda, \Lambda) = 0.$$

The reverse composite has the same three Lagrangians. Both composites therefore have identity mapping class and weight zero. Applying Z_K to the two cylinder identities in (77) gives mutually inverse linear maps. Conjugating both identities by the same $Q_{K,d}^q$ gives (78). $\square$

Lemma 6.6 (One fixed comparison transports both routes). *The two route products satisfy*

$$(79) \quad \mathsf{T}_{B\star}^{K,q} \mathsf{T}_{0B}^{K,q} = (Q_{K,d}^q)^{-1} Z_K(H_{B\star} H_{0B}) Q_{K,d}^q,$$

$$(80) \quad \mathsf{T}_{AC\star}^{K,q} \mathsf{T}_{AAC}^{K,q} \mathsf{T}_{0A}^{K,q} = (Q_{K,d}^q)^{-1} Z_K(H_{AC\star} H_{AAC} H_{0A}) Q_{K,d}^q.$$

Proof. Insert $Q_{K,d}^q (Q_{K,d}^q)^{-1}$ between adjacent factors and use strict functoriality. For example,

$$\begin{aligned} \mathsf{T}_{AC\star}^{K,q} \mathsf{T}_{AAC}^{K,q} \mathsf{T}_{0A}^{K,q} &= (Q_{K,d}^q)^{-1} Z_K(H_{AC\star}) (Q_{K,d}^q (Q_{K,d}^q)^{-1}) Z_K(H_{AAC}) \\ &\quad (Q_{K,d}^q (Q_{K,d}^q)^{-1}) Z_K(H_{0A}) Q_{K,d}^q \\ &= (Q_{K,d}^q)^{-1} Z_K(H_{AC\star} H_{AAC} H_{0A}) Q_{K,d}^q. \end{aligned}$$

The factor order is unchanged. The two-factor calculation is identical with one cancellation. $\square$

The pentagon has a different geometry from the disjoint cube: its three handles are reused along two routes, so its operator statement is imported from a separately coordinated source with complete markings. The role of the comparison map is therefore precise: it transports the already literal geometric route equality while preserving factor order and the Walker weight.

With one compatible comparison core fixed, the route lemma cancels the intermediate comparison maps without changing factor order. The supplied Walker route equality then gives the operator pentagon. The theorem that follows it compares different cores by correctly typed intertwiners.

Theorem 6.7 (Fixed-level operator image of the pentagon with complete markings). *For every fixed $K \geq 1$, every tuple $\mathcal{T}$ in the stated range, and every selected witness package furnished by the geometric input, choose any COMP- Q core $\mathfrak{q}$ supplied by Proposition 6.4. Then one has the literal equality*

$$(81) \quad \boxed{\mathsf{T}_{B\star}^{K,\mathfrak{q}} \mathsf{T}_{0B}^{K,\mathfrak{q}} = \mathsf{T}_{AC\star}^{K,\mathfrak{q}} \mathsf{T}_{AAC}^{K,\mathfrak{q}} \mathsf{T}_{0A}^{K,\mathfrak{q}} = (Q_{K,d}^{\mathfrak{q}})^{-1} Z_K(C_Y([h_{B\star} h_{0B}], 0)) Q_{K,d}^{\mathfrak{q}}}.$$

There is no projective scalar. Set $r_{K,d}^{\mathfrak{q}} := \dim_{E_K} \mathcal{M}_{K,d}^{\mathfrak{q}} < \infty$. In the reference-spine basis belonging to $\mathfrak{q}$, the five operators lie in $\text{Mat}_{r_{K,d}^{\mathfrak{q}}}(E_K)$ and satisfy the same equality and the inverse formulas (78).

Proof. Apply $G \mapsto (Q_{K,d}^{\mathfrak{q}})^{-1} Z_K(G) Q_{K,d}^{\mathfrak{q}}$ to all three members of (71). Lemma 6.6 identifies the first two images with the displayed products. The source equality already lives in the Walker extension at weight zero, so strict functoriality introduces no central factor. The matrix statement follows only after fixing the finite core-indexed reference-spine basis of $\mathcal{M}_{K,d}^{\mathfrak{q}}$. $\square$

Theorem 6.8 (Change of comparison data). *Let two compatible cores $\mathfrak{q}_\alpha$ and $\mathfrak{q}_\beta$ for the same fixed K , Y , and Λ give the source modules and isomorphisms*

$$\begin{aligned} \mathcal{M}_\alpha &:= \mathcal{M}_{K,d}^{\mathfrak{q}_\alpha}, & Q_\alpha &:= Q_{K,d}^{\mathfrak{q}_\alpha} : \mathcal{M}_\alpha \xrightarrow{\sim} \mathcal{V}, \\ \mathcal{M}_\beta &:= \mathcal{M}_{K,d}^{\mathfrak{q}_\beta}, & Q_\beta &:= Q_{K,d}^{\mathfrak{q}_\beta} : \mathcal{M}_\beta \xrightarrow{\sim} \mathcal{V}, \end{aligned}$$

where $\mathcal{V} = \mathcal{V}_{K,d}^+$. Define the comparison in the direction $\alpha \rightarrow \beta$ by

$$(82) \quad C_{\beta \leftarrow \alpha} := Q_\beta^{-1} Q_\alpha : \mathcal{M}_\alpha \xrightarrow{\sim} \mathcal{M}_\beta.$$

For every Walker endocobordism G of (Y, Λ) and $\gamma \in \{\alpha, \beta\}$,

$$(83) \quad \rho_\beta(G) C_{\beta \leftarrow \alpha} = C_{\beta \leftarrow \alpha} \rho_\alpha(G), \quad \rho_\gamma(G) := Q_\gamma^{-1} Z_K(G) Q_\gamma.$$

For a third core $\mathfrak{q}_\delta$, with the analogous notation, the comparisons satisfy

$$(84) \quad C_{\delta \leftarrow \beta} C_{\beta \leftarrow \alpha} = C_{\delta \leftarrow \alpha}.$$

If $\mathcal{M}_\alpha = \mathcal{M}_\beta = \mathcal{M}$ is literally fixed and $C = Q_\alpha^{-1} Q_\beta$, then

$$(85) \quad \rho_\beta(G) = C^{-1} \rho_\alpha(G) C.$$

In particular, the five edge operators and both pentagon routes are simultaneously intertwined.

Proof. Typed cancellation through the common TQFT fibre gives

$$\begin{aligned} \rho_\beta(G) C_{\beta \leftarrow \alpha} &= Q_\beta^{-1} Z_K(G) Q_\beta Q_\beta^{-1} Q_\alpha \\ &= Q_\beta^{-1} Z_K(G) Q_\alpha \\ &= Q_\beta^{-1} Q_\alpha Q_\alpha^{-1} Z_K(G) Q_\alpha = C_{\beta \leftarrow \alpha} \rho_\alpha(G). \end{aligned}$$

Likewise,

$$(Q_\delta^{-1} Q_\beta)(Q_\beta^{-1} Q_\alpha) = Q_\delta^{-1} Q_\alpha,$$

which proves (84). Equation (85) is the same identity after the two source modules have already been identified. Taking $G = H_e$ proves the last assertion. $\square$

Remark 6.9 (Independence from the disjoint-cube relations). The proof uses the root normalization, the reference-data comparison argument, and strict TQFT functoriality. It does not use the ordinary disjoint-cube edge, phase, forward/reverse trace, square, order, or telescoping results. In particular, the pentagon is not a formal consequence of commuting disjoint slides.

7. SCOPE AND FURTHER COMPARISON PROBLEMS

Both transport theorems are fibrewise at one fixed K . The cube uses pairwise-disjoint positive standard slides with distinct active labels; it does not cover overlapping sectors, an arbitrary slide groupoid, or Cerf completeness. The pentagon uses one selected fully marked three-handle witness. Proposition 6.4 supplies its comparison core, and Theorem 6.8 compares auxiliary choices by intertwiners rather than literal equality of coordinate matrices. It proves neither relations among several pentagons nor phase and trace extensions for the five pentagon edges.

7.1. Existence and choice control for pentagon comparison maps.

Existence: Lemma 6.2 and Proposition 6.4 construct a compatible comparison core for every selected geometric pentagon in the stated range. The construction uses the literal selected vertex-zero compression body directly, by choosing its cap and complementary handlebody and applying the normalized reduced pairing.

Choice control: Given two comparison maps with the same TQFT target, Theorem 6.8 defines the correctly typed comparison $Q_\beta^{-1}Q_\alpha : \mathcal{M}_\alpha \rightarrow \mathcal{M}_\beta$. All transported endocobordisms intertwine, and comparisons satisfy the groupoid cocycle law. If the source module is already literally fixed, this becomes conjugacy.

The comparison need not be the identity or centralize the operator image, and it does not identify different geometric witness data.

7.2. Further comparison problems. The disjoint-cube branch uses [Sun26b, Theorem 1.2]. The independent pentagon branch uses [Sun26a, Theorem 4.1, Proposition 12.3, Theorems 13.5 and 14.1, and Section 15]. These are external geometric source theorems; their geometry is not reproved in the present paper.

Thus Papers 1 and 2 supply the source relations, while this paper supplies their normalized transport, coefficient and anomaly control, and typed comparison intertwiners. It constructs no cross-level object, refined theory, closed invariant, or WRT–Habiro– $\widehat{Z}$ comparison. Those questions require new stabilization, completion, cap, or disk-comparison input.

REFERENCES

- [BHMV95] Christian Blanchet, Nathan Habegger, Gregor Masbaum, and Pierre Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology*, 34(4):883–927, 1995.
- [GM13] Patrick M. Gilmer and Gregor Masbaum. Maslov index, lagrangians, mapping class groups and TQFT. *Forum Mathematicum*, 25(5):1067–1106, 2013. Equation and section locators refer to arXiv:0912.4706v3.
- [MV94] Gregor Masbaum and Pierre Vogel. 3-valent graphs and the Kauffman bracket. *Pacific Journal of Mathematics*, 164(2):361–381, 1994.
- [Prz99] Józef H. Przytycki. Fundamentals of Kauffman bracket skein modules. *Kobe Journal of Mathematics*, 16:45–66, 1999.
- [Sun26a] Hao-Yu Sun. Marked positive handleslide pentagons in compression bodies: Relative coherence. Hexagon preprint, 2026. [hexagon:2610.00227v1](#).
- [Sun26b] Hao-Yu Sun. Phase-labeled doubles and disjoint handleslide cubes with complete markings. Hexagon preprint, 2026. [hexagon:2610.00251v1](#).
- [Wal91] Kevin Walker. On Witten’s 3-manifold invariants. Preliminary version no. 2 (1991); electronic version prepared 2001, minor changes 2003, 1991.

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