

The Willow Transform and the Blind Zone Theorem: A Sharp Information-Theoretic Obstruction in the Turán Hierarchy for the Riemann xi-Function

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Abstract

Let $\Xi(t) = \xi(\frac{1}{2} + it)$ be the Riemann xi-function, let $\Xi(t) = \sum_{k \geq 0} (-1)^k c_k t^{2k}$ be its even Taylor expansion at the origin, and let $D(n, r) = \det[c_{n+i-j}]_{i,j=0}^{r-1}$ be the Toeplitz minors whose positivity for all $n \geq 0, r \geq 1$ is equivalent to the Riemann Hypothesis (RH). We prove a sharp and quantitative obstruction in the resulting hierarchy of criteria.

Parametrising the zeros of $G(z) = \sum c_k z^k$ by $\beta_j = -1/z_j$, we establish by the dual Jacobi-Trudi identity the exact identity $D(n, r) = s_{(r^n)}(\beta)$, the Schur function of the rectangular partition (r^n) in the variables β . A zero of zeta off the critical line contributes to β a conjugate pair $\rho e^{\pm i\theta}$ whose angle θ is exactly the vertical angle of the violating pair of zeros, while zeros on the critical line contribute positive real variables. For two variables we prove the closed form

$$s_{(a,b)}(\rho e^{i\theta}, \rho e^{-i\theta}) = \rho^{a+b} \frac{\sin((a+1-b)\theta)}{\sin \theta} = \rho^{a+b} U_{a-b}(\cos \theta),$$

U_k being the Chebyshev polynomial of the second kind, so that the sign is governed by the classical criterion $U_k(\cos \theta) > 0$ iff $(k+1)\theta < \pi$.

Since a Schur function in two variables can be expanded through the coproduct $s_\lambda(X \sqcup Y) = \sum_\mu s_\mu(X) s_{\lambda/\mu}(Y)$, and since $s_\mu(X) > 0$ for positive real X , positivity reduces to the two-variable factors, whose index $k = a - b$ is bounded by the number of columns r , not by the shift n . This yields:

Blind Zone Theorem. If a violating pair subtends a vertical angle $\theta < \pi/(r+1)$, then every Toeplitz minor $D(n, r')$ with $r' \leq r$ is strictly positive, notwithstanding the presence of non-real zeros.

The critical strip together with $\gamma_1 = 14.134725141 \dots$ forces $\theta \leq \theta_{\max} = 2 \arctan(1/(2\gamma_1)) = 0.07071826 \dots$ rad, whence $\pi/\theta_{\max} = 44.424 \dots$ and therefore every order $r \leq 43$ is completely non-discriminating.

We emphasise that the bound is on the order r and not on the shift n ; we exhibit the mechanism and show that the contrary n -based bound is false. We also record the refutation of a “universal covering law” previously conjectured by the author, together with a floating-point artefact that nearly produced a false refutation of the main theorem.

No claim of a proof of RH is made.

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1 Introduction

The Turán hierarchy for the Riemann xi-function asks for the positivity of the Toeplitz minors $D(n, r)$ built from the Taylor coefficients c_k of $\Xi(t) = \xi(\frac{1}{2} + it)$. Positivity of all such minors is equivalent to RH. This paper does not advance that programme; it measures, sharply and quantitatively, how much information the hierarchy can carry at all.

Our main result is a *blind zone*: a violating pair of zeros whose vertical angle is smaller than $\pi/(r+1)$ is invisible to every minor of order at most r . Combining this with the critical strip and the first ordinate γ_1 , we obtain that all orders $r \leq 43$ carry no discriminating power whatsoever.

2 Notation

Write

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right) = \sum_{k \geq 0} (-1)^k c_k t^{2k}, \quad G(z) = \sum_{k \geq 0} c_k z^k.$$

Let $\beta_j = -1/z_j$ where z_j runs over the zeros of G , and define

$$D(n, r) = \det [c_{n+i-j}]_{i,j=0}^{r-1}.$$

3 The Willow correspondence

Theorem 1 (Dual Jacobi–Trudi). *For all $n \geq 0$ and $r \geq 1$,*

$$D(n, r) = s_{(r^n)}(\beta),$$

the Schur function of the rectangular partition (r^n) evaluated in the variables $\beta = (\beta_j)$.

4 Two-variable closed form

Lemma 2. *For integers $a \geq b \geq 0$,*

$$s_{(a,b)}(\rho e^{i\theta}, \rho e^{-i\theta}) = \rho^{a+b} \frac{\sin((a+1-b)\theta)}{\sin \theta} = \rho^{a+b} U_{a-b}(\cos \theta).$$

Lemma 3. *$U_k(\cos \theta) > 0$ if and only if $(k+1)\theta < \pi$.*

5 Coproduct reduction and the index bound

The coproduct expansion

$$s_\lambda(X \sqcup Y) = \sum_{\mu} s_\mu(X) s_{\lambda/\mu}(Y),$$

together with $s_\mu(X) > 0$ for positive real X , reduces the positivity question to the two-variable factors. The surviving index $k = a - b$ is bounded by the number of columns r , and *not* by the shift n .

Remark 4 (Status). *The treatment of skew shapes (Lemma 7.2 in the full manuscript) and the index bookkeeping (Lemma 7.3) are the two steps the author invites scrutiny on.*

6 The Blind Zone Theorem

Theorem 5 (Blind Zone Theorem). *If a violating pair subtends a vertical angle $\theta < \pi/(r+1)$, then every Toeplitz minor $D(n, r')$ with $r' \leq r$ is strictly positive, notwithstanding the presence of non-real zeros.*

7 The number 43

For a zero $\rho = \beta + i\gamma$ off the critical line one has $t = \gamma - i(\beta - \frac{1}{2})$, hence

$$\theta = 2 \arctan \frac{|\beta - \frac{1}{2}|}{\gamma} \leq 2 \arctan \frac{1}{2\gamma_1} = \theta_{\max},$$

using $|\beta - \frac{1}{2}| < \frac{1}{2}$ and $\gamma \geq \gamma_1 = 14.134725141\dots$ Numerically

$$\theta_{\max} = 0.07071826\dots \text{ rad} = 4.051858\dots^\circ, \quad \frac{\pi}{\theta_{\max}} = 44.424\dots$$

so $r = 43$ satisfies $(43+1)\theta_{\max} < \pi$ while $r = 44$ does not. Hence every order $r \leq 43$ is completely non-discriminating.

8 Refuted conjecture and a numerical artefact

The author previously conjectured a “universal covering law” for the hierarchy; this was refuted and is recorded as such. A floating-point artefact in the numerical exploration nearly produced a false refutation of the main theorem; it is recorded for reproducibility.

9 Disclosure

This work was carried out with the assistance of generative AI tools. Every step stated as proved was independently checked by the author, and the steps that are not proved are flagged explicitly above.

References

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