

Finite Projective Source Geometry for GL_2

Abel Resolvents and Central-Jet Degree Laws

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Abstract

Starting from the two trace functions $p_1 = \mathrm{Tr}(A)$ and $p_2 = \mathrm{Tr}(A^2)$ for semisimple $A \in \mathrm{GL}_2(\mathbb{C})$, we develop the associated projective source geometry. The coordinate $y = t^2/d - 1$ identifies the adjoint character, and normalized Haar measure on a specified closed subgroup $H \subset \mathrm{SO}(3)$ pushes forward to a probability measure ν_H . We describe this measure through harmonic multiplicities, moments, and its Stieltjes transform, and prove that it determines H up to conjugacy within the closed-subgroup class of $\mathrm{SO}(3)$.

A positive Abel kernel resums the projective character tower and gives a resolvent formula whose endpoint limit recovers the mass of the identity atom. We give explicit Abel transforms for the finite rotation groups and positive-dimensional benchmark sources, and derive a finite-image Frobenius-prime Abel limit from Chebotarev. For central-jet interpolation, we prove two exact degree laws: the cyclic-versus-torus minimum degree is $\max(n, M + 1)$, while the normalized tetrahedral detector against S_4 and A_5 has minimum degree 3 for $M = 1, 2$ and $M + 2$ for $M \geq 3$. We also compute the integral image of the tetrahedral averaging functional, obtaining $2\mathbb{Z}$ for every $M \geq 2$.

All results are stated with their hypotheses. The paper does not claim a general infinite-spectrum Beyond Endoscopy trace formula or transfer of central-jet vanishing to orbital kernels.

1 Introduction

The starting point is the pair of trace functions on a semisimple matrix $A \in \mathrm{GL}_2(\mathbf{C})$,

$$p_1 = \mathrm{Tr}(A), \quad p_2 = \mathrm{Tr}(A^2).$$

Together these recover the characteristic-polynomial state (t, d) , where $t = \mathrm{Tr}(A)$ and $d = \det(A)$, and hence the projective adjoint character

$$y = \frac{t^2}{d} - 1.$$

The purpose of this paper is to study the compact-source geometry and polynomial-detector complexity attached to this one-variable projective coordinate.

Relation to the companion paper. The companion paper, S. Schafer, *Finite Analytic Characters and Tannakian Source Reconstruction: A Finite Tensor–Quiver Framework for Beyond Endoscopy* [1], develops a finite-visible-spectrum analytic framework in which the two local trace channels p_1 and p_2 can be recovered under its stated separation, archimedean-isolation, and nonvanishing hypotheses. That work addresses the analytic extraction and selection stage. The present paper takes the resulting two-trace state as its algebraic starting point and studies what follows after projectivization: the character ring, compact source measures, their resolvents, and exact central-jet source filters.

The papers are complementary but their claims are logically distinct. The central-jet degree laws proved here are finite-dimensional algebraic statements and do not depend on the analytic realization in the companion paper. Conversely, defining a trace measure ν_H from a specified compact group H does not by itself show that finitely many analytic observations reconstruct that measure. Any such arithmetic reconstruction requires additional hypotheses. The local Hecke recurrences below are algebraic; the finite-image prime theorem later in the paper is a separate consequence of Chebotarev.

The architecture is

$$\begin{array}{c} (p_1, p_2) \longrightarrow (t, d) \longrightarrow y, \\ H \longmapsto \nu_H \longrightarrow R_H \longrightarrow \text{source filters.} \end{array}$$

Here $H \subset SO(3)$ is a separately specified closed subgroup, ν_H is the pushforward of normalized Haar measure by the trace character, and R_H is its Stieltjes transform. A single observed local value $y(A)$ is not the full source measure.

Prior work and scope of novelty. General minimum-degree polynomial interpolation is established; relevant work includes de Boor–Ron and Sauer, as well as Gasca–Sauer’s treatment of minimal-degree spaces for bivariate Hermite interpolation [2–4]. Prescribed-zero interpolating filters have also been studied in wavelet and refinable-function theory [5]. In a different finite-group character setting, Nozawa and Uno study least-degree polynomials whose roots are the distinct nonidentity values of specified group characters [6]. These works are important context, but they do not formulate the source-average problem considered here: one seeks a polynomial in the ideal $(y - 3)^M \mathbf{Q}[y]$ whose normalized averages against specified compact-source measures equal a prescribed target vector. We do not claim the general interpolation criterion or the classical Langlands polyhedral selectors as new.

The principal contribution is the exact degree optimization for two particular systems of source functionals: the cyclic-versus-torus law $d_{C_n}(M) = \max(n, M + 1)$ and the tetrahedral law (degree 3 for $M = 1, 2$ and $M + 2$ for $M \geq 3$). Their proofs provide matching upper and lower bounds via explicit determinants and nonvanishing estimates valid over the full parameter ranges. The roots-of-unity filtering and binomial alias expansions used in the cyclic argument are standard

tools; the result claimed here is the exact optimal degree under the specified source-average and central-jet constraints. We also compute the integral image of the tetrahedral averaging functional and develop the positive Abel transform linking harmonic generating functions to source resolvents, with an endpoint limit that recovers the identity atom. To the best of our knowledge, neither exact source-specific degree law has previously been stated in the literature we have located; this priority claim is limited to these functionals and constraints. Explicit source transforms and the finite-image Frobenius-prime Abel limit provide context and arithmetic applications.

The paper is organized as follows. Section 2 develops the projective character algebra from the two traces. Section 3 constructs the compact source measures, harmonic coordinates, Abel transform, and rigidity theorem. Section 4 contains the central-jet interpolation criterion, both exact degree laws, and the integral refinement. Section 5 presents the classical detectors and finite-image arithmetic benchmark. Section 6 states the scope and remaining analytic limitations.

2 Projective character algebra from two traces

Let $A \in GL_2(\mathbf{C})$ be semisimple with eigenvalues α, β . Define

$$t = \alpha + \beta, \quad d = \alpha\beta, \quad p_m = \alpha^m + \beta^m.$$

The characteristic polynomial is

$$X^2 - tX + d.$$

Cayley–Hamilton gives

$$A^2 = tA - dI,$$

so

$$\boxed{p_{m+2} = tp_{m+1} - dp_m.}$$

In particular

$$p_2 = t^2 - 2d.$$

Consequently,

$$\boxed{d = \frac{p_1^2 - p_2}{2}.}$$

Theorem 2.1 (Two-channel completeness). *For semisimple $A \in GL_2(\mathbf{C})$, the data (t, d) , (p_1, p_2) , and the semisimple conjugacy class $[A]_{ss}$ determine one another. Moreover, the character of every finite-dimensional algebraic representation of GL_2 is a Laurent polynomial in t and d . In particular, every symmetric-power character is a polynomial in t and d . These characters are therefore determined by p_1, p_2 , with negative powers of d allowed for determinant twists of negative exponent.*

Proof. The equivalence of (p_1, p_2) and (t, d) follows from $t = p_1$ and the displayed determinant

formula. The characteristic polynomial determines the semisimple conjugacy class. Every symmetric polynomial in α, β lies in $\mathbf{C}[t, d]$, while every algebraic irreducible representation of GL_2 has the form $\mathrm{Sym}^a(\mathrm{Std}) \otimes \det^b$ with $a \geq 0$ and $b \in \mathbf{Z}$. Its character is therefore d^b times a symmetric polynomial, hence a Laurent polynomial in t, d . $\square$

2.1 Symmetric powers

For $m \geq 1$, the Clebsch–Gordan identity

$$\mathrm{Std} \otimes \mathrm{Sym}^m(\mathrm{Std}) \simeq \mathrm{Sym}^{m+1}(\mathrm{Std}) \oplus (\mathrm{Sym}^{m-1}(\mathrm{Std}) \otimes \det)$$

gives

$$\chi_{m+1} = t\chi_m - d\chi_{m-1}, \quad \chi_0 = 1, \quad \chi_1 = t.$$

Therefore

$$\chi_{\mathrm{Sym}^m}(A) = \sum_{j=0}^{\lfloor m/2 \rfloor} (-d)^j \binom{m-j}{j} t^{m-2j}.$$

The finite formula follows by induction from the recurrence, using Pascal's identity for binomial coefficients. For the generating function, put $C(u) = \sum_{m \geq 0} \chi_{\mathrm{Sym}^m}(A) u^m$. Multiplying the recurrence by powers of u and summing gives $(1 - tu + du^2)C(u) = 1$, hence

$$C(u) = \sum_{m \geq 0} \chi_{\mathrm{Sym}^m}(A) u^m = \frac{1}{1 - tu + du^2}.$$

At a maximal-level unramified place v , let

$$\mathrm{Sat}(T_v) = x_1 + x_2, \quad \mathrm{Sat}(Z_v) = x_1 x_2.$$

Define

$$H_{0,v} = 1, \quad H_{1,v} = T_v, \quad H_{m+1,v} = T_v H_{m,v} - Z_v H_{m-1,v}, \quad m \geq 1.$$

The Satake map is an algebra homomorphism. The symmetric-power characters $s_m = \chi_{\mathrm{Sym}^m}(x_1, x_2)$ satisfy $s_0 = 1$, $s_1 = x_1 + x_2$, and

$$s_{m+1} = (x_1 + x_2)s_m - (x_1 x_2)s_{m-1},$$

by the Clebsch–Gordan identity. The Satake transforms of $H_{m,v}$ satisfy the same recurrence and initial data. Induction therefore gives

$$\mathrm{Sat}(H_{m,v}) = \chi_{\mathrm{Sym}^m}(x_1, x_2).$$

Thus each fixed-index symmetric-power character is an explicit polynomial in the trace-state coordinates t, d (or a Laurent polynomial when determinant twists are allowed).

The Hecke recursion above is an algebraic identity in the spherical Hecke algebra. Any assertion that a global limiting functional realizes these characters analytically requires a separate theorem with explicit spectral and nonvanishing hypotheses; it is not used below.

2.2 Projective collapse and the character algebra

The adjoint representation of GL_2 is

$$\mathrm{Ad}^0(\mathrm{Std}) \simeq \mathrm{Sym}^2(\mathrm{Std}) \otimes \det^{-1}.$$

Its character is

$$y = \chi_{\mathrm{Ad}^0}(A) = \frac{t^2}{d} - 1.$$

Since $d = (p_1^2 - p_2)/2$, this may also be written as

$$y = \frac{p_1^2 + p_2}{p_1^2 - p_2}.$$

Let $G = \mathrm{PGL}_2(\mathbf{C})$. The irreducible algebraic representations of GL_2 that are trivial on its scalar center are precisely

$$W_r = \mathrm{Sym}^{2r}(\mathrm{Std}) \otimes \det^{-r}, \quad r \geq 0.$$

They are the irreducible algebraic representations of G . The representation W_1 is the adjoint representation.

Theorem 2.2 (Projective character algebra). *The Grothendieck ring of finite-dimensional algebraic representations of $\mathrm{PGL}_2(\mathbf{C})$ is*

$$K_0(\mathrm{Rep}^{\mathrm{alg}} \mathrm{PGL}_2(\mathbf{C})) \simeq \mathbf{Z}[X],$$

where X corresponds to the class $[W_1]$. Under the character map this becomes

$$K_0(\mathrm{Rep}^{\mathrm{alg}} \mathrm{PGL}_2(\mathbf{C})) \simeq \mathbf{Z}[y],$$

where $y = \chi_{W_1}$. Consequently

$$K_0(\mathrm{Rep}^{\mathrm{alg}} \mathrm{PGL}_2(\mathbf{C})) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathbf{Q}[y].$$

Proof. The irreducible algebraic representations of GL_2 that act trivially on its scalar center are

precisely the W_r . For $r \geq 1$, the Clebsch–Gordan rule gives

$$W_1 \otimes W_r \simeq W_{r+1} \oplus W_r \oplus W_{r-1},$$

while $W_1 \otimes W_0 \simeq W_1$. Hence every W_r is generated recursively by W_1 . Under the character map, $[W_1]^n$ has character y^n , while the irreducible character $P_n(y) = \chi_{W_n}$ is monic of degree n . Thus the powers of $[W_1]$ are linearly independent, so no nonzero polynomial in $[W_1]$ vanishes. Therefore the representation ring is the polynomial ring $\mathbf{Z}[X]$ with $X = [W_1]$. $\square$

The character ring of finite-dimensional algebraic representations of PGL_2 over $\mathbf{C}$ is $\mathbf{C}[y]$. For GL_2 , the corresponding character ring is $\mathbf{C}[t, d, d^{-1}]$. The subring spanned by representations on which the scalar center acts trivially consists exactly of those representations that descend to PGL_2 , and it is isomorphic to $\mathbf{C}[y]$.

3 Compact source geometry, resolvents, and rigidity

3.1 Projective harmonic coordinates

Let $H \subset \mathrm{SO}(3)$ be closed. If $g \in \mathrm{SO}(3)$ is a rotation through angle $\theta \in [0, \pi]$, then

$$\mathrm{Tr}(g) = 1 + 2 \cos \theta.$$

For $r \geq 0$ write

$$W_r = \mathrm{Sym}^{2r}(\mathbf{C}^2) \otimes \det^{-r}.$$

Under the compact identification $\mathrm{PU}(2) \simeq \mathrm{SO}(3)$, W_r is the degree- r spherical-harmonic representation.

Proposition 3.1 (Harmonic character polynomials). *For a rotation g through angle θ , the character of W_r is*

$$\chi_{W_r}(g) = 1 + 2 \sum_{j=1}^r \cos(j\theta).$$

Writing

$$y = 1 + 2 \cos \theta,$$

there is a monic polynomial $P_r \in \mathbf{Z}[y]$ of degree r such that

$$\chi_{W_r} = P_r(y),$$

and

$$\boxed{P_{r+1}(y) = (y-1)P_r(y) - P_{r-1}(y), \quad P_0 = 1, \quad P_1 = y.}$$

Proof. The weights of W_r on a rotation through angle θ are $e^{-ir\theta}, \dots, e^{ir\theta}$, so its character is $\sum_{j=-r}^r e^{ij\theta}$. Multiplying this sum by $y = 1 + e^{i\theta} + e^{-i\theta}$ gives

$$yP_r(y) = P_{r+1}(y) + P_r(y) + P_{r-1}(y)$$

for $r \geq 1$, with $P_0 = 1$ and $P_1 = y$. This yields the recurrence. The recurrence also shows inductively that P_r is monic of degree r with integer coefficients. $\square$

Definition 3.2. Let μ_H denote normalized Haar probability measure on H . For $g \in H$, let $\theta(g) \in [0, \pi]$ be its rotation angle. Define the trace source measure by

$$\nu_H = (\text{Tr})_* \mu_H, \quad \text{Tr}(g) = 1 + 2 \cos \theta(g),$$

so that $\text{supp } \nu_H \subset [-1, 3]$. Equivalently, ν_H is the pushforward of the rotation-angle distribution under $\theta \mapsto 1 + 2 \cos \theta$.

Define

$$h_r(H) = \dim W_r^H, \quad m_r(H) = \int_{[-1,3]} y^r d\nu_H(y).$$

The averaging operator $\int_H W_r(g) d\mu_H(g)$ is the projection onto W_r^H ; taking its trace gives the compact-group averaging formula

$$h_r(H) = \int P_r(y) d\nu_H(y).$$

Since P_r is monic, the finite systems $(h_0, \dots, h_N)$ and $(m_0, \dots, m_N)$ determine one another by triangular linear transformations.

Define also

$$e_r(H) = h_r(H) - h_{r-2}(H), \quad h_{-1} = h_{-2} = 0.$$

Then

$$e_0 = h_0, \quad e_1 = h_1,$$

and

$$h_r = e_r + h_{r-2}.$$

Thus the full e -sequence is equivalent to the full h -sequence, but the tail e_r , $r \geq 2$, alone loses the initial datum h_1 . This loss accounts for the torus/normalizer ambiguity discussed below.

3.2 Positive commutant coordinates

Let

$$\pi : \text{SU}(2) \rightarrow \text{SO}(3)$$

be the canonical double covering and set

$$\tilde{H} = \pi^{-1}(H).$$

Define

$$a_n(H) = \dim \text{End}_{\tilde{H}}(\text{Sym}^n \mathbf{C}^2).$$

This definition is canonical and does not depend on a choice of lift.

Clebsch–Gordan gives the following decomposition as an $SU(2)$ -representation:

$$\text{End}(\text{Sym}^n \mathbf{C}^2) \simeq \text{Sym}^n \mathbf{C}^2 \otimes (\text{Sym}^n \mathbf{C}^2)^\vee \simeq \bigoplus_{r=0}^n \pi^* W_r,$$

where $\pi^* W_r$ is the pullback of the $SO(3)$ -representation W_r along $\pi : SU(2) \rightarrow SO(3)$. Taking $\tilde{H}$ -invariants gives

$$a_n(H) = \sum_{r=0}^n h_r(H).$$

The element $-I \in SU(2)$ acts by a scalar on $\text{Sym}^n \mathbf{C}^2$, hence trivially by conjugation on its endomorphism algebra; there is no parity ambiguity in a_n .

3.3 Harmonic generating functions and the source resolvent

Theorem 3.3 (Harmonic/resolvent equivalence). *Define*

$$H_H(q) = \sum_{r \geq 0} h_r(H) q^r, \quad R_H(w) = \int \frac{d\nu_H(y)}{w - y}.$$

Then, for $|q| < 1$,

$$H_H(q) = \int \frac{1 + q}{1 - (y - 1)q + q^2} d\nu_H(y).$$

If

$$w = 1 + q + q^{-1},$$

with the branch $q \rightarrow 0$ as $w \rightarrow \infty$, then

$$H_H(q) = (w - q)R_H(w).$$

Moreover, for $|w| > 3$,

$$R_H(w) = \sum_{n \geq 0} \frac{m_n(H)}{w^{n+1}}.$$

Proof. The recurrence for P_r gives

$$\sum_{r \geq 0} P_r(y) q^r = \frac{1+q}{1-(y-1)q+q^2}.$$

Integrating against ν_H gives the first identity. If $w = 1+q+q^{-1}$ then

$$1-(y-1)q+q^2 = q(w-y)$$

and

$$1+q^{-1} = w-q,$$

which gives the second identity. For $|w| > 3$ the geometric expansion

$$(w-y)^{-1} = w^{-1} \sum_{n \geq 0} (y/w)^n$$

converges uniformly on $[-1, 3]$, giving the moment expansion. $\square$

3.4 The positive Abel transform and the identity atom

For $0 < q < 1$, define

$$A_q(y) = \frac{(1-q)^2}{1-(y-1)q+q^2}, \quad w = 1+q+q^{-1} > 3,$$

and put

$$\mathcal{A}_H(q) = \int A_q(y) d\nu_H(y).$$

Since $w-3 = (1-q)^2/q$ and $w-y = [1-(y-1)q+q^2]/q$, one has

$$\boxed{A_q(y) = \frac{w-3}{w-y}}, \quad \boxed{\mathcal{A}_H(q) = (w-3)R_H(w) = \frac{(1-q)^2}{1+q}H_H(q)}.$$

The last equality follows from the harmonic/resolvent equivalence. This kernel is positive and bounded by one on $[-1, 3]$, because

$$1-(y-1)q+q^2 - (1-q)^2 = q(3-y) \geq 0.$$

For fixed $y < 3$, $A_q(y) \rightarrow 0$ as $q \rightarrow 1^-$, whereas $A_q(3) = 1$. Therefore dominated convergence yields the endpoint identity.

Theorem 3.4 (Stieltjes edge-atom formula). *For every probability measure ν supported on $[-1, 3]$, with*

$$R_\nu(w) = \int \frac{d\nu(y)}{w-y}, \quad w > 3,$$

one has

$$\lim_{w \downarrow 3} (w - 3)R_\nu(w) = \nu(\{3\}).$$

Equivalently, for a compact source $H \subset SO(3)$,

$$\lim_{q \rightarrow 1^-} \mathcal{A}_H(q) = \nu_H(\{3\}).$$

This limit is $|H|^{-1}$ when H is finite and is zero when H is infinite.

Proof. Write $\nu = a\delta_3 + \nu_0$ with $\nu_0(\{3\}) = 0$. Then

$$(w - 3)R_\nu(w) = a + (w - 3) \int \frac{d\nu_0(y)}{w - y}.$$

The integrand $(w - 3)/(w - y)$ lies in $[0, 1]$ for $y \in [-1, 3]$ and tends to zero for every $y < 3$. Dominated convergence gives the limit $a = \nu(\{3\})$. For a compact source, $y = 3$ occurs only at the identity. Haar probability has identity mass $1/|H|$ for finite H . If H is infinite, translation invariance makes all singleton masses equal; a positive singleton mass would contradict total mass one, so the identity mass is zero. $\square$

3.5 A local Hecke realization

At a maximal-level unramified place v , let $H_{m,v}$ be defined by the symmetric-power Hecke recursion in Section 2, and write T_v, Z_v for the standard Hecke generators. Put

$$\widehat{H}_{r,v} = Z_v^{-r} H_{2r,v}, \quad \text{Sat}(\widehat{H}_{r,v}) = P_r(y), \quad y = T_v^2 Z_v^{-1} - 1.$$

The character-generating identity therefore gives, as a formal power-series identity in the q -adic completion $\mathcal{H}_v[Z_v^{-1}][[q]]$ of the localized spherical Hecke algebra,

$$\sum_{r \geq 0} q^r \widehat{H}_{r,v} = \frac{1 + q}{(1 + q)^2 - qT_v^2 Z_v^{-1}}.$$

This is a local algebraic realization of the harmonic generating function. It does not identify the compact-source Cauchy transform with a global automorphic resolvent; that would require additional analytic input.

3.6 Explicit source measures and rigidity

The trace-pushforward measures for the finite subgroups used below are explicit.

For the cyclic group C_n ($n \geq 1$) generated by a rotation through $2\pi/n$,

$$\nu_{C_n} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{1+2 \cos(2\pi j/n)}.$$

For the rotational dihedral group D_n of order $2n$ ($n \geq 2$),

$$\nu_{D_n} = \frac{1}{2} \nu_{C_n} + \frac{1}{2} \delta_{-1}.$$

For the exceptional groups,

$$\nu_{A_4} = \frac{1}{12} (\delta_3 + 3\delta_{-1} + 8\delta_0),$$

$$\nu_{S_4} = \frac{1}{24} (\delta_3 + 9\delta_{-1} + 8\delta_0 + 6\delta_1),$$

$$\nu_{A_5} = \frac{1}{60} (\delta_3 + 15\delta_{-1} + 20\delta_0 + 12\delta_\phi + 12\delta_{\bar{\phi}}),$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad \bar{\phi} = \frac{1 - \sqrt{5}}{2}.$$

For these exceptional groups, the formulas follow by counting rotation angles. In A_4 the angles are 0 once, π three times, and $2\pi/3$ eight times. In S_4 they are 0 once, π nine times, $\pi/2$ six times, and $2\pi/3$ eight times. In A_5 they are 0 once, π fifteen times, $2\pi/3$ twenty times, and each of $2\pi/5$ and $4\pi/5$ twelve times. Applying $y = 1 + 2 \cos \theta$ gives the displayed atoms and masses.

For the positive-dimensional closed subgroups, the trace-pushforward measures are as follows:

$$\frac{d\nu_{SO(2)}}{dy} = \frac{1}{\pi \sqrt{(3-y)(y+1)}},$$

$$\nu_{N(SO(2))} = \frac{1}{2} \nu_{SO(2)} + \frac{1}{2} \delta_{-1},$$

and

$$\frac{d\nu_{SO(3)}}{dy} = \frac{1}{2\pi} \sqrt{\frac{3-y}{y+1}}.$$

The first and third formulas are Lebesgue densities on $(-1, 3)$; the middle formula also records the atom at -1 . For $SO(2)$, the angle folded to $[0, \pi]$ is uniform with density $1/\pi$. For normalized Haar measure on $SO(3)$, the angle density is $\frac{2}{\pi} \sin^2(\theta/2) d\theta$. In both cases, the change of variables $y = 1 + 2 \cos \theta$ gives the displayed trace density.

Proposition 3.5 (Measure-resolvent equivalence). *Let μ and η be probability measures supported on*

$[-1, 3]$, and define

$$R_\mu(w) = \int \frac{d\mu(y)}{w - y}, \quad R_\eta(w) = \int \frac{d\eta(y)}{w - y}.$$

Then $R_\mu = R_\eta$ if and only if $\mu = \eta$. For any such probability measure μ , R_μ is rational if and only if μ is finitely supported.

Proof. The expansion at infinity determines all moments, and polynomials are dense in the continuous functions on $[-1, 3]$, so equality of Cauchy transforms implies equality of measures. Suppose R_μ is rational. It is holomorphic on $\mathbb{C} \setminus [-1, 3]$, so its rational continuation has no nonremovable poles off $[-1, 3]$. On every open subinterval of $(-1, 3)$ avoiding the finite pole set, the transform extends holomorphically across the interval; Stieltjes inversion implies that the interval has μ -measure zero. Thus μ is supported on the finite set of real poles. It is therefore atomic, and $R_\mu(w) = \sum_a \mu(\{a\})/(w - a)$: each pole is simple and has residue $\mu(\{a\}) \geq 0$. Conversely, a finite atomic measure has a rational Cauchy transform. $\square$

3.7 Specialized subgroup identification in $SO(3)$

Theorem 3.6 (Principal source-rigidity theorem). *Let H, K be closed subgroups of $SO(3)$. Then*

$$\boxed{\nu_H = \nu_K \iff H, K \text{ are conjugate in } SO(3).}$$

Moreover

$$\boxed{R_H \text{ rational} \iff \nu_H \text{ finitely supported} \iff H \text{ finite}.}$$

Proof. It remains to identify the closed subgroup from its trace measure. A zero-dimensional closed subgroup of the compact Lie group $SO(3)$ is discrete and compact, hence finite. The classification of finite rotation groups (see, for example, Wolf [7, Theorem 2.6.5]) gives the finite possibilities C_n ($n \geq 1$), D_n ($n \geq 2$), A_4 , S_4 , and A_5 . For completeness, the positive-dimensional cases follow from the Lie algebra. The Lie algebra of a closed subgroup is a Lie subalgebra of $\mathfrak{so}_3$; under the identification $\mathfrak{so}_3 \simeq \mathbb{R}^3$ with bracket given by the cross product, there is no two-dimensional Lie subalgebra. Thus a positive-dimensional proper closed subgroup has one-dimensional identity component, conjugate to $SO(2)$. The full subgroup normalizes its identity component, so it lies in $N_{SO(3)}(SO(2))$, and the quotient of this normalizer by $SO(2)$ has order two. Hence the only possibilities are $SO(2)$ and $N_{SO(3)}(SO(2))$. A three-dimensional Lie algebra gives an open subgroup of the connected group $SO(3)$, hence gives $H = SO(3)$. Therefore the complete list up to conjugacy is

$$C_n \ (n \geq 1), \quad D_n \ (n \geq 2), \quad A_4, \quad S_4, \quad A_5, \quad SO(2), \quad N(SO(2)), \quad SO(3).$$

For a finite subgroup H ,

$$\nu_H(\{3\}) = \frac{1}{|H|},$$

since the identity is the unique element with trace 3.

The mass at -1 is

$$\nu_{C_n}(\{-1\}) = \begin{cases} 0, & n \text{ odd}, \\ 1/n, & n \text{ even}, \end{cases}$$

and

$$\nu_{D_n}(\{-1\}) = \begin{cases} 1/2, & n \text{ odd}, \\ 1/2 + 1/(2n), & n \text{ even}. \end{cases}$$

For the exceptional groups, the masses at -1 are $1/4$, $3/8$, and $1/4$ for A_4 , S_4 , and A_5 , respectively. The atom at 3 determines the order of a finite subgroup. For an even order $N \geq 4$, the cyclic group C_N and rotational dihedral group $D_{N/2}$ have respective masses $1/N$ and either $1/2$ or $1/2 + 1/N$ at -1 , so they are distinct; at order 2, only C_2 occurs in the classification above. The only orders at which an exceptional group can also occur are 12, 24, and 60. At those orders, the competing cyclic/dihedral masses are respectively $(1/12, 7/12)$, $(1/24, 13/24)$, and $(1/60, 31/60)$, distinct from the exceptional masses $1/4$, $3/8$, and $1/4$. At odd order only the cyclic possibility occurs. Thus the mass at -1 distinguishes all finite types once the order is known.

The positive-dimensional measures are pairwise distinct: $\nu_{N(SO(2))}$ has an atom at -1 , while the other two have none; the density ratio of $\nu_{SO(3)}$ to $\nu_{SO(2)}$ is $(3 - y)/2$, which is not constant. Thus the measure distinguishes these three conjugacy classes. Each proper positive-dimensional closed subgroup has identity component conjugate to $SO(2)$, and $SO(3)$ itself is the remaining case. Each of these three groups has trace support $[-1, 3]$, so its trace measure has infinite support. Finite groups have finite trace support. The preceding proposition now gives the rationality criterion for R_H . $\square$

3.8 Explicit Abel curves for the classified sources

The Abel transform can be evaluated directly from the atomic source measures. Put

$$b(q) = \left(\frac{1-q}{1+q} \right)^2, \quad c(q) = \frac{(1-q)^2}{1+q+q^2}.$$

Proposition 3.7 (Explicit Abel transforms). *For cyclic and rotational dihedral groups one obtains*

$$\boxed{\mathcal{A}_{C_n}(q) = \frac{1-q}{1+q} \frac{1+q^n}{1-q^n}}, \quad \boxed{\mathcal{A}_{D_n}(q) = \frac{1}{2} \mathcal{A}_{C_n}(q) + \frac{1}{2} b(q)}.$$

For the exceptional rotation groups, substituting the atomic measures gives

$$\begin{aligned} \mathcal{A}_{A_4}(q) &= \frac{1}{12} (1 + 8c(q) + 3b(q)), \\ \mathcal{A}_{S_4}(q) &= \frac{1}{24} \left(1 + 9b(q) + 8c(q) + 6 \frac{(1-q)^2}{1+q^2} \right), \end{aligned}$$

$$\mathcal{A}_{A_5}(q) = \frac{1}{60} (1 + 15b(q) + 20c(q) + 12A_q(\phi) + 12A_q(\bar{\phi})),$$

where $\phi = (1 + \sqrt{5})/2$ and $\bar{\phi} = (1 - \sqrt{5})/2$. Simplifying these rational expressions, using $\phi + \bar{\phi} = 1$ and $\phi\bar{\phi} = -1$, yields

$$\begin{aligned}\mathcal{A}_{A_4}(q) &= \frac{1 - q^2 + q^4}{(1 + q)^2(1 + q + q^2)}, \\ \mathcal{A}_{S_4}(q) &= \frac{1 - q^3 + q^6}{(1 + q)^2(1 + q^2)(1 + q + q^2)}, \\ \mathcal{A}_{A_5}(q) &= \frac{1 + q - q^3 - q^4 - q^5 + q^7 + q^8}{(1 + q)^2(1 + q + q^2)(1 + q + q^2 + q^3 + q^4)}.\end{aligned}$$

For the positive-dimensional closed subgroups, W_r has exactly one zero weight under $SO(2)$, so $h_r(SO(2)) = 1$. In its normalizer, a representative of the nontrivial Weyl element exchanges the two standard $SU(2)$ weight vectors with one minus sign; on the zero-weight vector $e_1^r e_2^r$ in Sym^{2r} it therefore acts by $(-1)^r$. Thus $h_r(N(SO(2))) = \mathbf{1}_{2|r}$. Finally, only W_0 is trivial on all of $SO(3)$, so $h_r(SO(3)) = \mathbf{1}_{r=0}$. Consequently,

$$\boxed{\mathcal{A}_{SO(2)}(q) = \frac{1 - q}{1 + q}}, \quad \boxed{\mathcal{A}_{N(SO(2))}(q) = \frac{1 - q}{(1 + q)^2}}, \quad \boxed{\mathcal{A}_{SO(3)}(q) = \frac{(1 - q)^2}{1 + q}}.$$

Proof. For a finite group, substitute the displayed atoms of ν_H into $\mathcal{A}_H(q) = \int A_q d\nu_H$. For the cyclic formula, use the absolutely convergent Fourier expansion

$$\frac{1 - q^2}{1 - 2q \cos \theta + q^2} = \sum_{m \in \mathbb{Z}} q^{|m|} e^{im\theta}.$$

Averaging at $\theta = 2\pi j/n$ retains exactly the terms with $n \mid m$, so the average is $1 + 2 \sum_{k \geq 1} q^{kn} = (1 + q^n)/(1 - q^n)$. Since $A_q(1 + 2 \cos \theta) = \frac{1 - q}{1 + q} \frac{1 - q^2}{1 - 2q \cos \theta + q^2}$, this gives the stated cyclic formula; the dihedral formula follows from $\nu_{D_n} = \frac{1}{2} \nu_{C_n} + \frac{1}{2} \delta_{-1}$ and $A_q(-1) = b(q)$. The exceptional formulas follow by bringing the finite sums over their listed trace atoms to a common denominator. In the A_5 calculation, the two nonrational denominators satisfy

$$(1 - (\phi - 1)q + q^2)(1 - (\bar{\phi} - 1)q + q^2) = 1 + q + q^2 + q^3 + q^4,$$

so the resulting expression is rational over $\mathbb{Q}(q)$. For positive-dimensional groups, use $\mathcal{A}_H(q) = \frac{(1 - q)^2}{1 + q} \sum_{r \geq 0} h_r(H) q^r$ and the stated invariant multiplicities. $\square$

Remark 3.8. Because $w = 1 + q + q^{-1}$ maps $(0, 1)$ bijectively onto $(3, \infty)$, equality of $\mathcal{A}_\mu(q)$ and $\mathcal{A}_\nu(q)$ for every $q \in (0, 1)$ implies equality of the corresponding Stieltjes transforms and hence $\mu = \nu$. Thus the Abel curve is another complete coordinate description of the source measure. Rationality as a function of q should not be confused with rationality of $R_\mu(w)$ as a function of w : the substitution

$w = 1 + q + q^{-1}$ is algebraic, and the Abel curves for positive-dimensional sources above are rational in q even though their Stieltjes transforms are not rational in w .

Remark 3.9. *The group-theoretic implication in Theorem 3.6 is specialized to $SO(3)$ and uses its closed-subgroup classification. The measure/resolvent equivalence itself is ordinary Stieltjes theory. Related rigidity and compactness questions for character measures and dimension data of compact-group subgroups are studied in [8–10]; the equivalence used here is the direct identity $h_r(H) = \int P_r d\nu_H$ together with the triangular character basis.*

3.9 Finite atomic reconstruction by moments

The rigidity theorem identifies the source once its measure is known. We now give a complementary finite-complexity result: for a measure with finitely many trace atoms, finitely many moments suffice, with the required number governed by the number of distinct projective trace values.

Suppose

$$\nu = \sum_{i=1}^s w_i \delta_{y_i}, \quad w_i > 0, \quad y_i \in [-1, 3], \quad y_i \neq y_j \ (i \neq j).$$

Let

$$m_n = \sum_{i=1}^s w_i y_i^n.$$

Define the Hankel matrices

$$H_N = (m_{i+j})_{0 \leq i, j \leq N}.$$

Proposition 3.10 (Finite atomic source inversion). *The rank satisfies $\text{rank } H_N = \min(N+1, s)$. If the number s of distinct atoms is known, the moments*

$$\boxed{m_0, m_1, \dots, m_{2s-1}}$$

determine the support points and masses uniquely.

Proof. Let V_N be the $(N+1) \times s$ matrix (y_i^j) . Then

$$H_N = V_N \text{diag}(w_1, \dots, w_s) V_N^T.$$

Because $D = \text{diag}(w_1, \dots, w_s)$ is positive definite, $H_N = V_N D V_N^T$ has rank $\text{rank } V_N$. The Vandermonde matrix V_N has rank $\min(N+1, s)$: any square submatrix using the first $\min(N+1, s)$ powers and that many distinct nodes is nonsingular. Hence

$$\text{rank } H_N = \min(N+1, s).$$

Write

$$Q(Y) = \prod_{i=1}^s (Y - y_i) = Y^s + c_{s-1}Y^{s-1} + \cdots + c_0.$$

Then

$$m_{k+s} + c_{s-1}m_{k+s-1} + \cdots + c_0m_k = 0.$$

For $0 \leq k \leq s-1$ this is a nonsingular linear system because

$$H_{s-1} = V_{s-1} \operatorname{diag}(w_i) V_{s-1}^T$$

is positive definite. Thus $m_0, \dots, m_{2s-1}$ determine Q , hence its distinct roots y_i and the support. Once the support is known, the first s moment equations form a nonsingular Vandermonde system for the weights w_i , determining them uniquely. $\square$

The reconstruction above is a classical Prony-type method for finite atomic measures; see Plonka and Tasche [11] for a survey. Here the relevant source complexity is the number s of distinct projective trace values, not the dimension of the original two-dimensional representation. There is no uniform finite moment cutoff that distinguishes every finite source: for any fixed N , every polynomial of degree at most N has the same C_n -average and maximal-torus average whenever $n > N$, because its Fourier frequencies are strictly smaller than n . Thus the first N moments of sufficiently large cyclic sources agree with those of the maximal torus.

4 Central-jet interpolation and exact degree laws

The source-rigidity theorem identifies $y = 3$ as the identity-trace point; for a finite source, it is an atom of mass $1/|H|$. The resolvent viewpoint makes this point canonical. The remaining algebraic question is source-preserving interpolation: how much vanishing can be imposed at $y = 3$ while retaining prescribed averages against a selected family of source measures? General minimum-degree interpolation and Hermite interpolation have a substantial literature [2–4]; prescribed-zero interpolating filters have been studied in refinable-function theory [5]. Nozawa–Uno consider a nearby but distinct finite-group character problem, involving least-degree polynomials with prescribed nonidentity character values as roots [6]. The present problem instead imposes a central-jet ideal constraint and prescribes values of a vector of normalized source-average functionals. We use the general interpolation criterion as a tool, and make no novelty claim for the criterion itself; the contribution claimed here is the exact degree calculation for the stated source functionals.

Definition 4.1. *For the trace-pushforward measure ν_H , write*

$$L_H(P) = \int P(y) d\nu_H(y).$$

For $M \geq 1$, let

$$J_M^{\mathbf{Q}} = (y - 3)^M \mathbf{Q}[y].$$

For the tetrahedral detector define

$$\mathcal{F}_M = \{P \in \mathbf{Q}[y] : P \in J_M^{\mathbf{Q}}, L_{A_4}(P) = 1, L_{S_4}(P) = 0, L_{A_5}(P) = 0\}.$$

Theorem 4.2 (Central-jet interpolation criterion). *Let $\mu_1, \dots, \mu_k$ be finite positive Borel measures supported on a common compact interval containing 3. Put*

$$J_M^{\mathbf{C}} = (y - 3)^M \mathbf{C}[y], \quad L = (L_1, \dots, L_k) : J_M^{\mathbf{C}} \longrightarrow \mathbf{C}^k, \quad L_i(P) = \int P(y) d\mu_i(y).$$

Then

$$\boxed{L(J_M^{\mathbf{C}}) = \mathbf{C}^k}$$

if and only if the only coefficient vector $c = (c_1, \dots, c_k) \in \mathbf{C}^k$ for which

$$\eta_c = \sum_{i=1}^k c_i \mu_i$$

is supported on $\{3\}$ is $c = 0$. Equivalently, the restricted measures $\mu_1|_{\mathbb{R} \setminus \{3\}}, \dots, \mu_k|_{\mathbb{R} \setminus \{3\}}$ are linearly independent.

Proof. A vector $c = (c_i)$ annihilates $L(J_M^{\mathbf{C}})$ precisely when

$$\int (y - 3)^M q(y) d\eta_c(y) = 0$$

for every polynomial q , where $\eta_c = \sum_i c_i \mu_i$. Thus every polynomial moment of the compactly supported complex measure $(y - 3)^M \eta_c$ vanishes. By density of polynomials in the continuous functions on the common compact support, $(y - 3)^M \eta_c = 0$, so η_c is supported on $\{3\}$. Conversely, any such coefficient vector annihilates $L(J_M^{\mathbf{C}})$. The image is all of $\mathbf{C}^k$ exactly when this annihilator is zero. $\square$

For the three source measures used here, the restrictions to $[-1, 3] \setminus \{3\}$ are linearly independent. Indeed, the atoms at ϕ and $\bar{\phi}$ force the A_5 coefficient in any linear combination supported at 3 to vanish. The atom at 1 then forces the S_4 coefficient to vanish, and the atom at 0 forces the A_4 coefficient to vanish. The criterion therefore gives surjectivity over $\mathbf{C}$ for every M .

For a subfield $K \subset \mathbf{C}$, write $J_M^K = (y - 3)^M K[y]$. If every moment entry $L_i((y - 3)^{M+j})$, for $j \geq 0$, lies in K and the complexified map is surjective, then some finite $k \times k$ moment minor is nonzero and belongs to K . Hence $L(J_M^K) = K^k$. For the present polyhedral source measures, integrating any polynomial in $\mathbf{Q}[y]$ gives a rational number: the irrational trace atoms for A_5 occur in Galois-conjugate pairs with equal masses. Thus the interpolation is solvable over $\mathbf{Q}$.

If the $k \times k$ moment minor formed from degrees $M, \dots, M + k - 1$ is nonzero, it gives a filter of degree at most $M + k - 1$. For the tetrahedral detector, Δ_M is this minor when $k = 3$; the augmented determinant below tests whether the target $(1, 0, 0)$ can be attained by a polynomial of degree at most $M + 1$.

4.1 An exact cyclic degree law

Let $T = SO(2) \subset SO(3)$ be a maximal torus, and for $n \geq 2$ let C_n be the cyclic subgroup generated by a rotation through $2\pi/n$. Put $u = 3 - y$. For an integer $M \geq 0$, define

$$\kappa_M = \binom{2M}{M}, \quad \alpha_M(n) = \frac{1}{n} \sum_{j=0}^{n-1} \left(2 - 2 \cos \frac{2\pi j}{n} \right)^M, \quad \eta_M(n) = \frac{\alpha_M(n) - \kappa_M}{2}.$$

For $u(\theta) = 2 - 2 \cos \theta = (1 - e^{i\theta})(1 - e^{-i\theta})$, the binomial expansion is

$$u(\theta)^M = \sum_{a=-M}^M (-1)^a \binom{2M}{M-a} e^{ia\theta}.$$

Averaging over $\theta = 2\pi j/n$ annihilates all terms unless $n \mid a$, while averaging over the full circle keeps only $a = 0$. Thus $L_T(u^M) = \kappa_M$ and, with out-of-range binomial coefficients interpreted as zero,

$$\alpha_M(n) = \kappa_M + 2 \sum_{k \geq 1} (-1)^{nk} \binom{2M}{M-kn}.$$

Set

$$\Delta_{n,M} = \kappa_M \eta_{M+1}(n) - \kappa_{M+1} \eta_M(n).$$

Lemma 4.3 (Cyclic determinant nonvanishing). *For every $n \geq 2$ and $M \geq n - 1$,*

$$\text{sgn}(\Delta_{n,M}) = (-1)^n.$$

In particular, $\Delta_{n,M} \neq 0$ in this range.

Proof. At $M = n - 1$ one has $\eta_M(n) = 0$ and $\eta_n(n) = (-1)^n$, so $\Delta_{n,n-1} = (-1)^n \kappa_{n-1}$. Suppose $M \geq n$ and write

$$B_a(M) = \binom{2M}{M-a}, \quad c_M = \frac{\kappa_{M+1}}{\kappa_M} = 4 - \frac{2}{M+1}.$$

If n is even, every term in $\eta_M(n) = \sum_{k \geq 1} B_{kn}(M)$ is nonnegative. For each fixed $a \geq 1$ with $a \leq M$,

$$\frac{B_a(M+1)/\kappa_{M+1}}{B_a(M)/\kappa_M} = \frac{(M+1)^2}{(M+1)^2 - a^2} > 1.$$

Any new alias entering at $M + 1$ is positive. Hence $\eta_{M+1}/\kappa_{M+1} > \eta_M/\kappa_M$, and $\Delta_{n,M} > 0$.

Suppose n is odd. Put

$$U_M = -\eta_M(n) = \sum_{k \geq 1} (-1)^{k+1} B_{kn}(M).$$

It suffices to show $U_{M+1} - c_M U_M > 0$. With $m = M + 1$, for each existing offset $a \leq M$ set

$$T_a(M) = B_a(M+1) - c_M B_a(M) = B_a(M) c_M \frac{a^2}{m^2 - a^2} > 0.$$

For every integer a with $1 \leq a < M$, a direct quotient calculation gives

$$\frac{T_{a+1}(M)}{T_a(M)} = \frac{m-a}{m+a+1} \frac{(a+1)^2}{a^2} < 1 \iff m < a(a+1).$$

The restriction $a < M$ ensures that the adjacent-offset quotient is valid. First assume $m < n(n+1)$. Since $m < a(a+1)$ for every integer $a \geq n$, each adjacent ratio needed to compare the existing sampled terms $T_n, T_{2n}, T_{3n}, \dots$ is strictly less than one. These sampled terms therefore strictly decrease in magnitude. If $n \nmid m$, the required increment is a finite alternating sum beginning with a positive term and having decreasing magnitudes, hence is positive. If $m = kn$, a new alias of magnitude one enters. For $k = 1$ the assertion follows from $U_{n-1} = 0$ and $U_n = 1$. For odd $k > 1$ the new alias is positive. For even $k \geq 2$, the preceding alternating sum ends with $+T_{(k-1)n}$ and is at least that term. At $m = kn$,

$$T_{(k-1)n}(kn-1) = \binom{2kn-2}{n-1} \left(4 - \frac{2}{kn}\right) \frac{(k-1)^2}{2k-1} > 1,$$

so the new negative unit term cannot change the sign. Thus $U_{M+1} - c_M U_M > 0$ in this first range, giving $\Delta_{n,M} < 0$.

It remains to consider $m \geq n(n+1)$. Write $A_M = \alpha_M(n)$. For odd n , the largest sampled value of $2 - 2\cos(2\pi j/n)$ is

$$a_* = 2 + 2\cos(\pi/n) = 4 - 4\sin^2(\pi/(2n)),$$

so the ratio A_{M+1}/A_M is a weighted average of the sampled values, with weights proportional to their nonnegative M th powers, and is therefore at most a_* . The elementary inequality $\sin x \geq 2x/\pi$ on $[0, \pi/2]$ gives $\sin(\pi/(2n)) \geq 1/n$, so one has

$$\frac{1}{2\sin^2(\pi/(2n))} \leq \frac{n^2}{2} < n(n+1) \leq m.$$

Consequently $c_M = 4 - 2/m > a_*$ and $A_{M+1}/A_M < c_M$. Using $\eta_M = (A_M - \kappa_M)/2$ gives

$$\Delta_{n,M} = \frac{1}{2}(\kappa_M A_{M+1} - \kappa_{M+1} A_M) < 0.$$

This completes the odd- n case. □

Theorem 4.4 (Cyclic central-jet degree law). *For every $n \geq 2$ and $M \geq 1$, the least degree of a polynomial $P \in (y-3)^M \mathbb{Q}[y]$ satisfying*

$$L_T(P) = 0, \quad (L_{C_n} - L_T)(P) = 1$$

is

$$d_{C_n}(M) = \max(n, M+1).$$

Proof. The averages L_{C_n} and L_T agree on every polynomial of degree less than n , since a degree- d polynomial in $y = 1 + 2 \cos \theta$ has Fourier frequencies at most d . Thus $\deg P \geq n$. Also, a nonzero polynomial in $(y-3)^M \mathbb{Q}[y]$ of degree at most M is a scalar multiple of $(3-y)^M$, whose torus average is nonzero; hence $\deg P \geq M+1$.

If $M < n$, the polynomial

$$P(y) = \frac{(-1)^n}{2} u^n - \frac{(-1)^n \kappa_n}{2\kappa_M} u^M$$

has degree n , lies in the jet ideal, and has torus average zero because $L_T(u^j) = \kappa_j$. Moreover, $L_{C_n}(u^n) - L_T(u^n) = 2(-1)^n$, while $L_{C_n}(u^M) - L_T(u^M) = 0$; hence $(L_{C_n} - L_T)(P) = 1$. This attains the lower bound $n = \max(n, M+1)$.

If $M \geq n$, restrict the two constraints to the span of u^M, u^{M+1} . The determinant of the resulting two-by-two matrix is a nonzero scalar multiple of $\Delta_{n,M}$, which is nonzero by the lemma. Thus a solution of degree at most $M+1$ exists. The lower bound is $M+1$, proving the claim. All matrix entries are rational, so the solution can be chosen in $\mathbb{Q}[y]$. $\square$

4.2 The exact tetrahedral degree law

Set

$$x = y - 3, \quad \rho = \frac{5 + \sqrt{5}}{2}, \quad \sigma = \frac{5 - \sqrt{5}}{2}.$$

The nonidentity support values in the x -coordinate are

$$A_4 : -4, -3, \quad S_4 : -4, -3, -2, \quad A_5 : -4, -3, -\rho, -\sigma,$$

with masses

$$A_4 : \frac{1}{4}, \frac{2}{3}, \quad S_4 : \frac{3}{8}, \frac{1}{3}, \frac{1}{4}, \quad A_5 : \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}.$$

Consequently, for each integer $n \geq 1$,

$$\begin{aligned} L_{A_4}(x^n) &= \frac{(-4)^n}{4} + \frac{2(-3)^n}{3}, \\ L_{S_4}(x^n) &= \frac{3(-4)^n}{8} + \frac{(-3)^n}{3} + \frac{(-2)^n}{4}, \\ L_{A_5}(x^n) &= \frac{(-4)^n}{4} + \frac{(-3)^n}{3} + \frac{(-\rho)^n + (-\sigma)^n}{5}. \end{aligned}$$

Substituting these source moments into the determinants below and expanding gives the stated formulas. One can make the expansion completely explicit using the Vandermonde identity: if a, b, c are selected nonidentity support points from A_4, S_4, A_5 , respectively, with their corresponding masses $w_{A_4}(a), w_{S_4}(b), w_{A_5}(c)$, then

$$\Delta_M = \sum_{a,b,c} w_{A_4}(a)w_{S_4}(b)w_{A_5}(c)(abc)^M(b-a)(c-a)(c-b),$$

where each sum ranges over that source's nonzero x -support. The identity atoms $x = 0$ contribute zero because $M \geq 1$. Grouping the remaining triples by their product abc gives the following complete list of nonzero aggregate coefficients:

abc	-24	-12σ	-12ρ	-8σ	-8ρ	-6σ	-6ρ
coefficient	$\frac{1}{24}$	$-\frac{\sqrt{5}+2}{30}$	$\frac{\sqrt{5}-2}{30}$	$\frac{1+\sqrt{5}}{80}$	$\frac{1-\sqrt{5}}{80}$	$\frac{1}{30}$	$\frac{1}{30}$

This list yields the displayed closed formula for Δ_M term by term.

For $P = x^M q(x)$ with $\deg q \leq 2$, let

$$M_M = (L_H(x^{M+j}))_{H=A_4, S_4, A_5, \substack{0 \leq j \leq 2}}.$$

Then

$$\begin{aligned} \Delta_M = \det M_M &= \frac{(-24)^M}{24} - \frac{\sqrt{5}+2}{30}(-12\sigma)^M + \frac{\sqrt{5}-2}{30}(-12\rho)^M \\ &\quad + \frac{1+\sqrt{5}}{80}(-8\sigma)^M + \frac{1-\sqrt{5}}{80}(-8\rho)^M + \frac{(-6\sigma)^M + (-6\rho)^M}{30}. \end{aligned}$$

At $M = 3$, exact evaluation gives $\Delta_3 = -616$. For $M \geq 4$ write $(-1)^M \Delta_M = A_M - B_M - C_M + D_M$, where

$$A_M = \frac{\sqrt{5}-2}{30}(12\rho)^M, \quad B_M = \frac{\sqrt{5}+2}{30}(12\sigma)^M, \quad C_M = \frac{\sqrt{5}-1}{80}(8\rho)^M,$$

where the nonnegative remainder is explicitly

$$D_M = \frac{24^M}{24} + \frac{1+\sqrt{5}}{80}(8\sigma)^M + \frac{(6\sigma)^M + (6\rho)^M}{30} > 0.$$

Then B_M/A_M is decreasing in M and at $M = 4$ equals $(3 - \sqrt{5})/2 < 1$, while C_M/A_M is decreasing

and at $M = 4$ equals $2/9 + 2\sqrt{5}/27$. Hence

$$\frac{B_M + C_M}{A_M} \leq \frac{31}{18} - \frac{23\sqrt{5}}{54} < 1, \quad M \geq 4,$$

The last strict inequality is equivalent to $39 < 23\sqrt{5}$, which follows by squaring the positive sides. Hence $(-1)^M \Delta_M > 0$ for all $M \geq 4$. Thus $\Delta_M \neq 0$ for every $M \geq 3$, and a quotient polynomial q of degree at most 2 suffices. The matrix entries are rational, since $\rho + \sigma = 5$ and $\rho\sigma = 5$ imply $(-\rho)^n + (-\sigma)^n \in \mathbb{Q}$; hence the solution obtained by Cramer's rule has coefficients in $\mathbb{Q}$.

To exclude quotient polynomials q of degree at most 1, define the augmented determinant

$$\Gamma_M = \det \begin{pmatrix} L_{A_4}(x^M) & L_{A_4}(x^{M+1}) & 1 \\ L_{S_4}(x^M) & L_{S_4}(x^{M+1}) & 0 \\ L_{A_5}(x^M) & L_{A_5}(x^{M+1}) & 0 \end{pmatrix}.$$

Expanding along the final column gives

$$\Gamma_M = L_{S_4}(x^M)L_{A_5}(x^{M+1}) - L_{S_4}(x^{M+1})L_{A_5}(x^M).$$

Equivalently, using the nonidentity supports and masses, it is $\sum_{b,c} w_{S_4}(b)w_{A_5}(c)(bc)^M(c-b)$. Grouping by bc gives the coefficient list

bc	12	6	8	2ρ	2σ	3ρ	3σ	4ρ	4σ
coefficient	$\frac{1}{24}$	$-\frac{1}{12}$	$-\frac{1}{8}$	$-\frac{1+\sqrt{5}}{40}$	$\frac{\sqrt{5}-1}{40}$	$-\frac{\sqrt{5}-1}{30}$	$\frac{1+\sqrt{5}}{30}$	$\frac{9(3-\sqrt{5})}{240}$	$\frac{9(3+\sqrt{5})}{240}$

which expands to the following formula:

$$\begin{aligned} \Gamma_M = & \frac{10 \cdot 12^M - 20 \cdot 6^M - 30 \cdot 8^M}{240} - \frac{6(1+\sqrt{5})(2\rho)^M}{240} - \frac{8(\sqrt{5}-1)(3\rho)^M}{240} \\ & + \frac{9(3-\sqrt{5})(4\rho)^M}{240} + \frac{6(\sqrt{5}-1)(2\sigma)^M}{240} + \frac{8(1+\sqrt{5})(3\sigma)^M}{240} \\ & + \frac{9(3+\sqrt{5})(4\sigma)^M}{240}. \end{aligned}$$

Direct calculation gives

$$\Gamma_3 = 35, \quad \Gamma_4 = 924, \quad \Gamma_5 = 17149.$$

For $M \geq 5$, define the ratio of the absolute values of all negative terms to the positive $(4\rho)^M$ term by

$$\mathcal{R}_M = \frac{20 \cdot 6^M + 30 \cdot 8^M + 6(1+\sqrt{5})(2\rho)^M + 8(\sqrt{5}-1)(3\rho)^M}{9(3-\sqrt{5})(4\rho)^M}.$$

At $M = 5$, this ratio is

$$\mathcal{R}_5 = \frac{3351}{1280} - \frac{5517\sqrt{5}}{6400} < \frac{7}{10},$$

and each constituent ratio decreases thereafter because

$$\frac{6}{4\rho} < 1, \quad \frac{8}{4\rho} < 1, \quad \frac{3\rho}{4\rho} = \frac{3}{4}, \quad \frac{2\rho}{4\rho} = \frac{1}{2}.$$

The bound at $M = 5$ is strict because it is equivalent to $12275 < 5517\sqrt{5}$, again verified by squaring positive sides. Since each constituent ratio decreases with M , the total ratio remains below $7/10$ for every $M \geq 5$. Thus $\Gamma_M > 0$ for every $M \geq 5$, while the displayed exact values cover $M = 3, 4$. Hence no quotient polynomial of degree at most 1 can attain the target for any $M \geq 3$.

Theorem 4.5 (Exact tetrahedral central-jet degree). *For the A_4 detector on the source family $\{A_4, S_4, A_5\}$,*

$$\min_{P \in \mathcal{F}_M} \deg P = \begin{cases} 3, & M = 1, 2, \\ M + 2, & M \geq 3. \end{cases}$$

Proof. The general central-jet criterion shows nonemptiness for every M . For $M \geq 3$, the nonvanishing of Δ_M gives a solution with $\deg q \leq 2$, hence $\deg P \leq M + 2$. The nonvanishing of Γ_M proves that no solution with $\deg q \leq 1$ exists, hence $\deg P \geq M + 2$. For $M = 1$, direct evaluation gives $L_H(x) = -3$ and $L_H(x^2) = 10$ for each $H \in \{A_4, S_4, A_5\}$. Hence every polynomial $x(a + bx)$ has the same value $-3a + 10b$ on all three sources and cannot realize the target $(1, 0, 0)$. For $M = 2$, every polynomial of degree below 3 in $J_2^{\mathbf{Q}}$ is a scalar multiple of x^2 , whose three source values are all 10; it too cannot realize $(1, 0, 0)$. The displayed filters F_1, F_2 attain degree 3. $\square$

Convenient minimal filters are

$$F_1(y) = y(y + 2)(y - 3),$$

$$F_2(y) = \frac{1}{2}(y - 3)^2(2y + 1),$$

$$F_3(y) = -\frac{(y - 3)^3(5y^2 + 25y + 12)}{88},$$

$$F_4(y) = \frac{(y - 3)^4(924y^2 + 2567y + 1132)}{28424},$$

$$F_5(y) = -\frac{(y - 3)^5(1559y^2 + 3306y + 1336)}{111216},$$

$$F_6(y) = \frac{(y - 3)^6(278135y^2 + 512579y + 193568)}{52217024}.$$

Direct substitution into the three finite source measures verifies

$$(L_{A_4}, L_{S_4}, L_{A_5})(F_M) = (1, 0, 0)$$

for each displayed $M = 1, \dots, 6$.

4.3 Integral refinement of the tetrahedral theorem

The rational central-jet theorem has an arithmetic refinement. Since

$$K_0(\mathrm{Rep}^{\mathrm{alg}} \mathrm{PGL}_2(\mathbf{C})) \simeq \mathbf{Z}[y],$$

it is natural to ask whether the same filters can be chosen integrally. The theorem below computes the image of the single A_4 -functional; it does not compute the full joint image in $\mathbb{Z}^3$.

Let

$$J_M^{\mathbf{Z}} = (y - 3)^M \mathbf{Z}[y].$$

For $P = (y - 3)^M q$, $q \in \mathbf{Z}[y]$, the A_4 functional is

$$L_{A_4}(P) = \frac{1}{4}P(-1) + \frac{2}{3}P(0).$$

Hence

$$L_{A_4}(P) = (-1)^M (4^{M-1}q(-1) + 2 \cdot 3^{M-1}q(0)).$$

Since $q(-1)$ and $q(0)$ can be prescribed independently by an integral linear polynomial, computing the image reduces to finding the ideal generated by 4^{M-1} and $2 \cdot 3^{M-1}$.

Theorem 4.6 (Integral central-jet index). *For every $M \geq 1$,*

$$L_{A_4}(J_M^{\mathbf{Z}}) = \begin{cases} \mathbf{Z}, & M = 1, \\ 2\mathbf{Z}, & M \geq 2. \end{cases}$$

Consequently

$$\mathbf{Z}/L_{A_4}(J_M^{\mathbf{Z}}) \simeq \begin{cases} 0, & M = 1, \\ \mathbf{Z}/2\mathbf{Z}, & M \geq 2. \end{cases}$$

Proof. The displayed formula shows that the image is contained in the ideal generated by 4^{M-1} and $2 \cdot 3^{M-1}$. Their gcd is 1 for $M = 1$ and 2 for $M \geq 2$. Since $q(-1)$ and $q(0)$ can be prescribed independently, both ideals are attained. $\square$

Thus, for $M \geq 2$, no integral virtual character belonging to $J_M^{\mathbf{Z}}$ realizes the normalized detector.

5 Classical detectors and finite-image arithmetic

5.1 The positive Fejér detector

Because the source measure determines the subgroup in this setting, the commutant coordinates provide a complementary positive detector. The normalized squared-character kernel is bounded by 1 and converges pointwise to 0 away from the identity. Its Haar average therefore detects the mass of the identity, separating finite sources from positive-dimensional ones without requiring the exceptional-subgroup classification.

Theorem 5.1 (Fejér source detector). *Let $H \subset SO(3)$ be closed, and let $g \in SO(3)$ be a rotation through angle θ . For $0 < \theta \leq \pi$,*

$$\sum_{r=0}^N \chi_{W_r}(g) = \left(\frac{\sin((N+1)\theta/2)}{\sin(\theta/2)} \right)^2.$$

At $\theta = 0$, the right-hand side is interpreted by continuity and equals $(N+1)^2$. Consequently

$$\frac{a_N(H)}{(N+1)^2} = \int_H \Phi_N(\theta(g)) d\mu_H(g),$$

where

$$\Phi_N(\theta) = \frac{1}{(N+1)^2} \left(\frac{\sin((N+1)\theta/2)}{\sin(\theta/2)} \right)^2 \quad (0 < \theta \leq \pi), \quad \Phi_N(0) := 1, \quad 0 \leq \Phi_N \leq 1.$$

If H is finite then

$$\lim_{N \rightarrow \infty} \frac{a_N(H)}{(N+1)^2} = \frac{1}{|H|}.$$

If H is infinite and closed, then

$$\lim_{N \rightarrow \infty} \frac{a_N(H)}{(N+1)^2} = 0.$$

For nontrivial finite H , if $\theta_{\min}$ denotes the smallest positive rotation angle in H , then

$$\left| \frac{a_N(H)}{(N+1)^2} - \frac{1}{|H|} \right| \leq \frac{|H| - 1}{|H|(N+1)^2 \sin^2(\theta_{\min}/2)}.$$

Proof. The weight expansion gives the character identity explicitly:

$$\sum_{r=0}^N \chi_{W_r}(g) = \sum_{j=-N}^N (N+1-|j|) e^{ij\theta} = \left| \sum_{j=0}^N e^{ij\theta} \right|^2,$$

which equals the displayed sine quotient by the finite geometric-series formula (with its continuous value at $\theta = 0$). Since

$$a_N(H) = \dim \text{End}_{\tilde{H}}(\text{Sym}^N \mathbf{C}^2) = \int_{\tilde{H}} |\chi_{\text{Sym}^N \mathbf{C}^2}(u)|^2 d\mu_{\tilde{H}}(u),$$

Haar averaging gives the commutant dimension. The Clebsch–Gordan decomposition

$$\text{Sym}^N \mathbf{C}^2 \otimes (\text{Sym}^N \mathbf{C}^2)^\vee \simeq \bigoplus_{r=0}^N \pi^* W_r$$

shows that the integrand descends through $\pi : \tilde{H} \rightarrow H$ as $\sum_{r=0}^N \chi_{W_r}(g)$; the pushforward of normalized Haar measure on $\tilde{H}$ is normalized Haar measure on H . This yields the displayed formula. The kernel is bounded between 0 and 1, equals 1 at the identity, and tends to 0 pointwise away from it.

For finite H ,

$$\frac{a_N(H)}{(N+1)^2} = \frac{1}{|H|} \sum_{h \in H} \Phi_N(\theta(h)).$$

The identity contributes $1/|H|$. Each other element contributes at most

$$\frac{1}{(N+1)^2 \sin^2(\theta(h)/2)} \leq \frac{1}{(N+1)^2 \sin^2(\theta_{\min}/2)},$$

which gives the error bound. If H is infinite, Haar measure has no atoms: translation invariance makes every singleton have the same mass, and a positive singleton mass would force the compact group to be finite. Dominated convergence then gives the zero limit. The trivial group is immediate from $a_N = (N+1)^2$. $\square$

This detector has a structural role: it is positive and bounded, and it does not presuppose a classification into A_4, S_4, A_5 or into cyclic and dihedral families. For finite H , its limit is the mass $1/|H|$ at the identity; for infinite closed H , the limit is zero. The signed Langlands coordinates can then be applied after the finite branch has been isolated.

5.2 Classical signed filters and the polyhedral fingerprint

The signed selectors are best understood as a coordinate realization of the same source measure. They do not add a second source theory: they extract particular low-degree finite-difference coordinates after the positive source object has been identified.

For $r \geq 2$ define

$$E_r = W_r - W_{r-2}.$$

Then

$$e_r(H) = \dim E_r^H = h_r(H) - h_{r-2}(H)$$

in the virtual representation ring. On a maximal-level unramified place, the corresponding Hecke function is obtained from the ordinary symmetric-power recursion. For clarity, write

$$\chi_m(A) = \chi_{\mathrm{Sym}^m(\mathrm{Std})}(A).$$

For a determinant-one lift $A \in \mathrm{SL}_2(\mathbf{C})$, the character specializes to

$$\boxed{\chi_{E_r}(A) = \chi_{2r}(A) - \chi_{2r-4}(A).}$$

For a general semisimple $A \in \mathrm{GL}_2(\mathbf{C})$ with $d = \det(A)$, its scalar-invariant expression is

$$\boxed{\chi_{E_r}(A) = d^{-r} \chi_{2r}(A) - d^{-(r-2)} \chi_{2r-4}(A).}$$

For determinant-one parameters,

$$E_3 = \chi_6 - \chi_2, \quad E_4 = \chi_8 - \chi_4, \quad E_6 = \chi_{12} - \chi_8.$$

These are the classical Langlands selectors; see Part I, §1.3 of Langlands [12]. The point here is to recover them as low-degree finite-difference coordinates on the one-variable source geometry, not to claim them as new results.

For clarity, the table can be computed directly from the rotation-angle class counts. Set

$$\begin{aligned} c_3(r) &= \frac{\sin((2r+1)\pi/3)}{\sin(\pi/3)}, & c_4(r) &= \frac{\sin((2r+1)\pi/4)}{\sin(\pi/4)}, \\ c_{5,1}(r) &= \frac{\sin((2r+1)\pi/5)}{\sin(\pi/5)}, & c_{5,2}(r) &= \frac{\sin(2(2r+1)\pi/5)}{\sin(2\pi/5)}. \end{aligned}$$

Averaging the character $\chi_{W_r}(\theta) = \sin((r + \frac{1}{2})\theta) / \sin(\theta/2)$ over the listed rotation classes gives

$$\begin{aligned} h_r(A_4) &= \frac{2r+1+3(-1)^r+8c_3(r)}{12}, \\ h_r(S_4) &= \frac{2r+1+9(-1)^r+6c_4(r)+8c_3(r)}{24}, \\ h_r(A_5) &= \frac{2r+1+15(-1)^r+20c_3(r)+12c_{5,1}(r)+12c_{5,2}(r)}{60}. \end{aligned}$$

For $\mathrm{SO}(3)$, $h_r(\mathrm{SO}(3)) = 0$ for $r > 0$ and $h_0(\mathrm{SO}(3)) = 1$. Subtracting h_{r-2} from h_r at $r = 3, 4, 6$ gives the table below. For the four-source family

$$\{\mathrm{SO}(3), A_4, S_4, A_5\},$$

we have

	E_3	E_4	E_6
$\mathrm{SO}(3)$	0	0	0
A_4	1	1	1
S_4	0	1	0
A_5	0	0	1

Thus

$$I_{A_4} = E_3, \quad I_{S_4} = E_4 - E_3, \quad I_{A_5} = E_6 - E_3, \quad I_{\mathrm{SO}(3)} = 1 - E_4 - E_6 + E_3.$$

These are exact indicators on the restricted four-class family $\{\mathrm{SO}(3), A_4, S_4, A_5\}$. They do not classify arbitrary closed subgroups: sufficiently large cyclic and dihedral groups can have the same three displayed selector values as $\mathrm{SO}(3)$, and the torus $\mathrm{SO}(2)$ and its normalizer also have $E_3 = E_4 = E_6 = 0$. More generally, the finite-difference map $h_r \mapsto h_r - h_{r-2}$ loses the initial datum h_1 , which explains the torus/normalizer ambiguity in the full e_r -tail.

5.3 Finite-image arithmetic benchmark

For finite-image projective Galois representations, the source coordinates have an exact arithmetic realization. For $r \geq 2$ write $E_r = W_r - W_{r-2}$ for the virtual character used below.

Let F be a number field, and let $\rho : G_F \rightarrow \mathrm{GL}_2(\mathbf{C})$ be continuous with projectivization $\bar{\rho} : G_F \rightarrow \mathrm{PGL}_2(\mathbf{C})$ of finite image. By the classical classification of finite subgroups of $\mathrm{PGL}_2(\mathbf{C})$, every such subgroup is conjugate to one of the cyclic, dihedral, tetrahedral, octahedral, or icosahedral rotation groups; see Beauville [13]. In particular, it is conjugate into the compact subgroup $\mathrm{PU}(2) \simeq \mathrm{SO}(3)$. After conjugating the image, write $H = \bar{\rho}(G_F) \subset \mathrm{SO}(3)$. For fixed $r \geq 2$, the virtual character E_r is a bounded class function on H .

Theorem 5.2 (Finite-image filtered-prime theorem). *Let $\mathcal{P}$ denote the finite places of F unramified in $\bar{\rho}$. Then*

$$\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{\substack{v \in \mathcal{P} \\ Nv \leq X}} (\log Nv) E_r(\bar{\rho}(\mathrm{Frob}_v)) = \dim W_r^H - \dim W_{r-2}^H.$$

Moreover, the sum of all prime-power terms with exponent at least two satisfies

$$\sum_{\substack{v \in \mathcal{P}, k \geq 2 \\ (Nv)^k \leq X}} (\log Nv) E_r(\bar{\rho}(\mathrm{Frob}_v)^k) = O(X^{1/2} \log X).$$

Consequently, adding these terms to the numerator does not change the limit.

Proof. The Chebotarev prime-ideal theorem in logarithmically weighted form (see, for example,

Serre [14]) gives the average of a bounded class function on the finite projective image as its uniform average over H . Hence the limit equals

$$\frac{1}{|H|} \sum_{h \in H} E_r(h),$$

which is exactly the multiplicity of the trivial representation in the virtual representation E_r . For the prime-power estimate, let $\pi_F(Y)$ count prime ideals of norm at most Y . The prime ideal theorem gives $\pi_F(Y) = O_F(Y/\log Y)$ for $Y \geq 3$. Thus the number of pairs (v, k) with $k \geq 2$ and $(Nv)^k \leq X$ is

$$\begin{aligned} \sum_{2 \leq k \leq \log_2 X} \pi_F(X^{1/k}) &= \pi_F(X^{1/2}) + \sum_{3 \leq k \leq \log_2 X} \pi_F(X^{1/k}) \\ &= O_F(\sqrt{X}/\log X) + O_F(X^{1/3} \log X) \\ &= O_F(\sqrt{X}). \end{aligned}$$

Here the terms with $k \geq 3$ are bounded using $\pi_F(Y) = O_F(Y)$, and there are $O(\log X)$ such exponents. Since E_r is bounded on the finite image and each weight satisfies $\log Nv \leq \log X$, the total contribution is $O_F(X^{1/2} \log X) = o(X)$. $\square$

Corollary 5.3. *If a finite projective source has already been shown not to be cyclic or dihedral, then it is conjugate to one of A_4 , S_4 , or A_5 . The three weighted prime averages attached to E_3 , E_4 , and E_6 identify its class exactly. The $SO(3)$ row in the preceding table is an infinite closed source and is not covered by the finite-image hypothesis of Theorem 5.2.*

Proof. By the classification of finite subgroups of $SO(3)$, a finite group that is neither cyclic nor rotational dihedral is one of A_4 , S_4 , A_5 . The displayed selector table assigns the three classes the distinct vectors $(1, 1, 1)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. The prime-average theorem identifies each coordinate with the corresponding weighted Frobenius average. The $SO(3)$ row is not a finite source, so it is outside the theorem's hypotheses. $\square$

Corollary 5.4 (Finite-image prime Abel limit). *Under the hypotheses of Theorem 5.2, for every fixed $q \in (0, 1)$,*

$$\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{\substack{v \in \mathcal{P} \\ Nv \leq X}} (\log Nv) A_q(\chi_{\text{Ad}}(\bar{\rho}(\text{Frob}_v))) = \mathcal{A}_H(q) = (w-3)R_H(w).$$

Proof. For $y \in [-1, 3]$,

$$A_q(y) = \frac{(1-q)^2}{1+q} \sum_{r \geq 0} q^r P_r(y), \quad |P_r(y)| \leq 2r+1.$$

The series converges uniformly on $[-1, 3]$. For each fixed r , the same Chebotarev argument as in Theorem 5.2, applied directly to the character $P_r = \chi_{W_r}$, gives the limiting prime average $h_r(H)$.

The tail is bounded by a constant times $\sum_{r>R} (2r+1)q^r$, uniformly in the prime cutoff because $X^{-1} \sum_{Nv \leq X} \log Nv$ is bounded. Passing first to the prime limit for a finite sum and then sending $R \rightarrow \infty$ yields

$$\frac{(1-q)^2}{1+q} \sum_{r \geq 0} q^r h_r(H) = \int A_q(y) d\nu_H(y) = \mathcal{A}_H(q).$$

□

This is a finite-image arithmetic benchmark, not an automorphic Beyond Endoscopy theorem. It gives a rigorous test case for any future analytic realization of these source filters.

6 Scope of the results

The central-jet statements are algebraic statements in the projective character coordinate: for $\lambda = t/(2\sqrt{d})$ and $y = 4\lambda^2 - 1$, divisibility by $(y-3)^M$ is exactly divisibility by $(\lambda^2 - 1)^M$ after substitution. This does not assert the same vanishing for an orbital integral or a global trace-formula kernel. In Jacquet–Zagier-type frameworks, local orbital integrals are weighted shell sums with test-dependent coefficients, and the post-Poisson formula uses integrated quantities indexed by arithmetic and Poisson variables; see Warade [15] for a related treatment. Establishing a transfer requires a separately defined transformation and analytic estimates justifying the relevant summation and limit operations; no such transfer is proved here.

The arithmetic result in this paper is the finite-projective-image prime theorem and its positive Abel corollary. We do not claim that the limiting source measure exists for every automorphic representation, that all balanced symmetric-power L -functions have the required pole interpretation, or that the global trace-formula aggregate equals an individual representation’s prime statistic. Those are separate questions requiring additional hypotheses and analytic arguments.

7 Conclusion

The projective character coordinate gives a common algebraic language for source measures, harmonic multiplicities, resolvents, and source filters. The positive Abel transform provides a bounded resolvent observable and detects the identity atom at the endpoint. The source-rigidity theorem and explicit transforms describe the closed subgroup geometry in $SO(3)$, while Chebotarev supplies a finite-image arithmetic realization. The two exact central-jet degree laws quantify the complexity of cyclic and tetrahedral source separation; the integral refinement identifies the parity obstruction for the normalized tetrahedral detector. These results give an algebraic and finite-image framework for projective source measures and source filters, without asserting a general Beyond Endoscopy transfer theorem. In relation to the companion paper [1], the present work develops the downstream projective and source-filtering layer; it neither replaces the companion paper’s conditional analytic extraction

nor assumes that extraction in the proofs of the two degree laws.

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