

THE FIFTEENTH COHOMOLOGY OF MODULI SPACES OF STABLE CURVES

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ABSTRACT. We determine the fifteenth cohomology of the moduli stack of stable curves $\overline{\mathcal{M}}_{g,n}$ in every genus g and number of markings n , with its symmetric-group action, by providing explicit multiplicity formulas for its simple constituents. We also prove the generalized Hodge conjecture in this case. The main result of this paper was obtained using the Pharos system. The author has reviewed the arguments and developed an understanding of the proof. On the basis of this review, the author believes the paper to be correct.

CONTENTS

1. Introduction	1
2. Notation and preliminaries	4
3. Motivic types and boundary Gysin maps	6
4. The genus-one and genus-two formulas	15
5. Boundary generation in genus three	27
6. The genus-three rational Gram formula	43
7. The ordinary Gysin image	51
8. The geometric base cases	62
9. Marking supports in genus five	67
10. Genus-two compact test classes	71
11. The sign and ten-row coefficients	89
12. The nine-row coefficients	98
13. The eight-row coefficients	99
14. Ordinary lifting	105
15. Proof of the main theorem	111
16. Geometric coniveau and the generalized Hodge conjecture	113
Appendix A. Use of generative AI	116
Appendix B. The problem as posed	117
References	123

1. INTRODUCTION

The Hodge conjecture and Tate conjecture are two central conjectures connecting algebraic topology of algebraic varieties, properties of algebraic cycles and categories of motives. These conjectures are still wide open in general, and only known in some special cases.

Let $X = \overline{\mathcal{M}}_{g,n}$ be the smooth proper moduli stack of stable curves of genus g with n ordered marked points, where $g, n \geq 0$ and $2g - 2 + n > 0$. For $r \geq 0$, let $N^r H^k(X)$ be the geometric coniveau filtration (classes supported on closed algebraic subsets of codimension at least r) and $N_H^r H^k(X)$ the Hodge coniveau filtration (the largest rational sub-Hodge structure of Hodge types

(p, q) with $p, q \geq r$); the precise definitions are recalled at the beginning of Section 16. The generalized Hodge conjecture asserts that these two filtrations agree.

Payne [Pay26] surveys recent progress on these conjectures on $\overline{\mathcal{M}}_{g,n}$. In particular, it is known that the generalized Hodge conjecture holds for $H^k(\overline{\mathcal{M}}_{g,n})$ for all odd $k \leq 13$. In this paper, we prove that the generalized Hodge conjecture holds for $H^{15}(\overline{\mathcal{M}}_{g,n})$, answering Payne's question (Problem 6.2 of that survey).

Theorem 1.1. *The generalized Hodge conjecture holds for $H^{15}(\overline{\mathcal{M}}_{g,n})$.*

Theorem 1.1 is proved as Corollary 16.4 in Section 16.

Put

$$\mathbf{L} = H^2(\mathbb{P}^1), \quad \mathbf{S}_{12} = H^{11}(\overline{\mathcal{M}}_{1,11}), \quad \mathbf{P} = \mathbf{L}^2 \otimes \mathbf{S}_{12},$$

and let $\mathbf{S}_{16}$ be the rank-two Hodge structure attached to the normalized level-one cusp form of weight sixteen. The Hodge types of $\mathbf{P}$ are $(13, 2)$ and $(2, 13)$, whereas those of $\mathbf{S}_{16}$ are $(15, 0)$ and $(0, 15)$. Previous work identifies these as the only possible simple constituents of $H^{15}(\overline{\mathcal{M}}_{g,n})$, with $\mathbf{S}_{16}$ occurring only in genus one [CLP24, CLPW24]; we recall the precise statements in Theorem 3.8. We use the same symbols for their compatible ℓ -adic realizations, and refer to these compatible realizations as motives.

Write $H^{15}(\overline{\mathcal{M}}_{g,n})_{\mathbf{P}}$ for the canonical $\mathbf{P}$ -isotypic sub-Hodge structure, namely the sum of all sub-Hodge structures isomorphic to $\mathbf{P}$.

Theorem 1.2 (The geometric coniveau filtration). *We have that*

$$N^1 H^{15}(\overline{\mathcal{M}}_{g,n}) = N^2 H^{15}(\overline{\mathcal{M}}_{g,n}) = H^{15}(\overline{\mathcal{M}}_{g,n})_{\mathbf{P}}, \quad N^r H^{15}(\overline{\mathcal{M}}_{g,n}) = 0 \quad (r \geq 3).$$

The subspace $H^{15}(\overline{\mathcal{M}}_{g,n})_{\mathbf{P}}$ is supported on a single closed algebraic subset of the boundary of codimension at least two in $\overline{\mathcal{M}}_{g,n}$. For $g \geq 2$, this subspace is all of $H^{15}(\overline{\mathcal{M}}_{g,n})$. If $g \geq 2$ and $H^{15}(\overline{\mathcal{M}}_{g,n}) \neq 0$, its geometric coniveau is exactly two.

Theorem 1.2 is proved as Theorem 16.3 in Section 16. The geometric input is the identification of $H^{15}(\overline{\mathcal{M}}_{g,n})_{\mathbf{P}}$ with the sum of the boundary Gysin images. The Hodge types of $\mathbf{P}$ and $\mathbf{S}_{16}$ give $N_H^1 H^{15}(X) = N_H^2 H^{15}(X) = H^{15}(X)_{\mathbf{P}}$ and $N_H^r H^{15}(X) = 0$ for $r \geq 3$; comparing with Theorem 1.2 proves Theorem 1.1.

We also determine the $\mathfrak{S}_n$ -equivariant multiplicity spaces of these Hodge constituents. The action of $\mathfrak{S}_n$ is induced by relabelling the marked points. For a partition $\lambda \vdash n$, let V_λ denote the rational Specht module. Set $K_n^{15} = V_{(n-14, 1^{14})}$ for $n \geq 15$ and $K_n^{15} = 0$ for $n < 15$, and let $\delta_{g,1}$ be one for $g = 1$ and zero otherwise.

Theorem 1.3. *There is a rational $\mathfrak{S}_n$ -module $\mathcal{E}_{g,n}$, unique up to isomorphism, and an $\mathfrak{S}_n$ -equivariant isomorphism of pure Hodge structures*

$$H^{15}(\overline{\mathcal{M}}_{g,n}) \cong \mathcal{E}_{g,n} \otimes \mathbf{P} \oplus \delta_{g,1} K_n^{15} \otimes \mathbf{S}_{16}.$$

The multiplicity modules carry Hodge type $(0, 0)$. After scalar extension of these modules to $\mathbb{Q}_\ell$, the same decomposition holds for the ℓ -adic Galois representations before semisimplification, for every prime ℓ . The exact nonvanishing range is

$$\begin{array}{c|ccccc} g & 0 & 1 & 2 & 3 & g \geq 4 \\ \hline H^{15}(\overline{\mathcal{M}}_{g,n}) \neq 0 & \emptyset & n \geq 13 & n \geq 11 & n \geq 10 & n \geq 9. \end{array}$$

Theorem 1.3 is proved as Theorem 15.2 in Section 15. The module $\mathcal{E}_{g,n}$ records the $\mathfrak{S}_n$ -equivariant multiplicities of $\mathbf{P}$. We determine these modules by computing the series

$$F_g = \sum_{\substack{n \geq 0 \\ 2g-2+n > 0}} \text{ch}_n(\mathcal{E}_{g,n}) \in \hat{\Lambda} = \prod_{n \geq 0} \Lambda_n,$$

where Λ_n is the space of rational symmetric functions homogeneous of degree n , and ch_n is the Frobenius characteristic; the conventions are given in Notations 2.3 and 2.4.

One has $F_0 = 0$. We give formulas for F_1 and F_2 in Theorems 4.7 and 4.18. In genus three, Definition 6.10 and Theorem 6.13 give a finite rational intersection algorithm for every coefficient of F_3 . For $g \geq 4$, Theorem 15.1 gives a uniform Schur-positive formula for F_g . These calculations establish the decomposition and nonvanishing range in Theorem 15.2.

Writing $e_{g,n} = \dim \mathcal{E}_{g,n}$, the decomposition also gives

$$h^{13,2} = h^{2,13} = e_{g,n}, \quad h^{15,0} = h^{0,15} = \delta_{g,1} \binom{n-1}{14},$$

with all other degree-fifteen Hodge numbers zero and the binomial coefficient taken to be zero for $n < 15$; see Corollary 15.3.

1.1. Related work to the determination of $H^{15}(\overline{\mathcal{M}}_{g,n})$. The relation between modular forms and boundary geometry has a long history. Keel’s presentation identifies the Chow ring of $\overline{\mathcal{M}}_{0,n}$ with cohomology, so odd cohomology vanishes in genus zero [Kee92]. In positive genus, Pikaart used elliptic cusp-form classes and boundary strata to construct nonzero odd cohomology in sufficiently large genus [Pik95]. Graber–Pandharipande subsequently used odd Künneth components on elliptic boundary products to detect non-tautological algebraic cycles [GP03].

Arbarello–Cornalba introduced boundary restriction as an inductive method for studying the cohomology of $\overline{\mathcal{M}}_{g,n}$ and proved vanishing in degrees one, three, and five [AC98]. Bergström–Faber–Payne established the additional base cases by point counting and extended the vanishing to degrees seven and nine [BFP22]. Canning–Larson–Payne then determined degree eleven: it is nonzero precisely for $g = 1$, $n \geq 11$, where

$$H^{11}(\overline{\mathcal{M}}_{1,n}) \cong V_{(n-10,1^{10})} \otimes \mathbf{S}_{12}$$

equivariantly [CLP23, Theorem 1.1 and Proposition 2.3]. The genus-one calculations build on Getzler’s equivariant Serre series and the correction of Consani–Faber [Getzler98, CF06], which also supply the generating-series input for our genus-one formula. Consani–Faber also realized level-one cusp-form motives as alternating summands in the cohomology of $\overline{\mathcal{M}}_{1,n}$.

In degree thirteen, Canning–Larson–Payne–Willwacher prove that every class is obtained by gluing an elliptic degree-eleven class to a second component carrying the unit class in H^0 . For $g \geq 2$, the direct sum of these Gysin maps is an isomorphism, giving the complete equivariant decomposition [CLPW, Theorem 1.7]. Earlier, Canning–Larson–Payne had attached components of distinct positive genera to an elliptic component to construct classes in higher odd degrees. Their nonvanishing theorem already implies $H^{15}(\overline{\mathcal{M}}_{g,n}) \neq 0$ for every $g \geq 4$, $n \geq 9$: take the genera of the two attached components to be 1 and $g - 2$ [CLP23, Theorem 1.5]. Thus the positive direction of our high-genus nonvanishing range is known independently of the multiplicity calculation.

The possible degree-fifteen constituents are also known. Canning–Larson–Payne prove that, for $g \geq 2$, $H^{15}(\overline{\mathcal{M}}_{g,n})$ is a direct sum of copies of $\mathbf{P}$ as a Hodge structure, and has the same form after Galois semisimplification [CLP24, Theorem 1.1]. Their framework of semi-tautological extensions supplies general finite-generation results and a generation theorem for $H_{15}(\overline{\mathcal{M}}_{g,n})$ [CLP24, Theorems 1.3 and 1.7]. This generation statement does not give a multiplicity formula or a presentation of H^{15} by the boundary Gysin maps considered here. They also isolated the primitive $\mathbf{S}_{16}$ in $H^{15}(\overline{\mathcal{M}}_{1,15})$ [CLP24, Lemma 8.3]. In genus one, Canning–Larson–Payne–Willwacher identify the $\mathbf{S}_{16}$ summand as $K_n^{15} \otimes \mathbf{S}_{16}$, with a complement whose semisimplification consists of copies of $\mathbf{P}$ [CLPW24, Theorem 1.3]. Their study of these cusp structures in the weight-graded compactly supported cohomology of the open spaces $\mathcal{M}_{g,n}$ addresses a related problem; the present calculation concerns the pure group H^{15} of the compactification. Their subsequent work places

holomorphic forms, including the degree-fifteen contribution, in a uniform framework of functors on finite marking sets, or **FA**-modules [CLPW25].

Explicit multiplicity calculations are available in finite ranges. Petersen computed the cohomology of symplectic local systems on the moduli space of principally polarized abelian surfaces in terms of Tate structures and elliptic and Siegel modular forms, up to semisimplification [Pet15]. These results provide inputs for our genus-two calculation. Bergström–Faber computed equivariant Euler characteristics in genus three through $n = 14$, conditionally for $n \geq 9$ on the predicted Galois constituents [BF23]. Canning–Larson–Payne subsequently confirmed the cases $n = 9, 10, 11$ unconditionally [CLP24, Theorem 1.9].

Payne asks for the generalized Hodge conjecture for $H^{15}(\overline{\mathcal{M}}_{g,n})$ [Pay26, Problem 6.2]. Theorem 1.2 answers it. The semisimplicity statement in [Pay26, Theorem C], applied in degree $2d - 15$ with $d = 3g - 3 + n$, also gives Galois semisimplicity in degree fifteen by Poincaré duality.

We have become aware of recent work by Bradford, Iribar López, Kretschmer, and Larson [BIKL26], whose results overlap in part with those of this paper. To clarify the chronology, Pharos completed the mathematical body of this manuscript on 23 September 2026, before their preprint appeared on arXiv on 9 October 2026. The intervening period was devoted to the author’s study and checking of the arguments.

More precisely, they study the semi-tautological extension generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$ and give explicit generators, complete relations, and bases for its degree-fifteen part in every genus $g \geq 2$. They also show that this part equals the full cohomology in genus two, and in genus three under a conjectural dimension assumption [BIKL26, Theorems 1.1–1.3]. This gives a concrete and systematic description of the relevant boundary classes and their relations.

We use a similar approach based on boundary classes, gluing maps, and restrictions to boundary strata. Our calculations also address generation of the full cohomology in higher genus and its symmetric-group multiplicities. In low genus, some technical arguments differ: we use local-system cohomology to study generation, including an unconditional argument in genus three, and intersection pairings to compute the genus-three multiplicities.

The overlap provides useful opportunities to check the calculations in both papers. In particular, our genus-two dimension formula agrees with theirs for all numbers of markings, and the higher-genus dimension checks carried out so far are also consistent with their presentation. Their explicit bases and relations make these comparisons particularly accessible, and we find their treatment a valuable complement to the calculations presented here.

Acknowledgements. The main result of this paper was obtained using the Pharos system. Pharos is an automated mathematical reasoning system that inherits selected design principles from Rethlas [Ju+26] and Danus [Liu+26], especially the fact graph and multi-agent architecture, while substantially redesigning and extending the remaining system components. The author would like to express their sincere gratitude to Bin Dong, Guoxiong Gao, Jiedong Jiang, Shurui Liu, and Bin Wu for their substantial contributions to the development and refinement of Pharos. The details of the use of generative AI are documented in Appendix A for transparency.

2. NOTATION AND PRELIMINARIES

Set-up 2.1. Throughout the paper, $g, n \geq 0$ satisfy $2g - 2 + n > 0$, and we call such (g, n) a stable pair. Let $\overline{\mathcal{M}}_{g,n}$ denote the smooth proper moduli stack of stable curves with ordered markings. The symmetric group $\mathfrak{S}_n$ acts by relabelling. Rational cohomology of a finite quotient stack is the invariant part of the rational cohomology of its cover, and averaging makes finite-group invariants exact; in particular the multiplicity functors $\mathrm{Hom}_{\mathbb{Q}[\mathfrak{S}_n]}(V_\lambda, -)$ are exact.

All tensor products of multiplicity spaces are over $\mathbb{Q}$, followed by scalar extension to $\mathbb{Q}_\ell$ in an ℓ -adic realization.

Notation 2.2. The objects $\mathbf{L}, \mathbf{S}_{12}, \mathbf{P}, \mathbf{S}_{16}$, their Hodge types, and the modules K_n^{15} and V_λ are as in Section 1.

Notation 2.3. Let Λ be the ring of symmetric functions over $\mathbb{Q}$, with the power-sum, complete homogeneous and elementary symmetric functions p_d, h_d, e_d ($d \geq 1$). Set $h_0 = e_0 = 1$ and $h_d = 0$ for $d < 0$. One has $\Lambda = \mathbb{Q}[p_1, p_2, \dots]$, graded by $\deg p_d = d$. If Λ_n denotes its degree- n component, write $\hat{\Lambda} = \prod_{n \geq 0} \Lambda_n$ for the completion by degree, and $[F]_n$ for the degree- n component of $F \in \hat{\Lambda}$. All infinite symmetric-function sums below are interpreted in this completion. In particular, set

$$E = \sum_{d \geq 0} h_d, \quad E_{\geq j} = \sum_{d \geq j} h_d \quad (j \geq 0).$$

Notation 2.4. For a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell > 0)$ of n , let V_λ be the corresponding rational Specht module and let $s_\lambda \in \Lambda_n$ be the Schur function. It is homogeneous of degree n ; in particular, $s_{(d)} = h_d$ and $s_{(1^d)} = e_d$ for $d \geq 1$. Write $p_\lambda = \prod_{j=1}^\ell p_{\lambda_j}$, with $p_\emptyset = 1$. For $\sigma \in \mathfrak{S}_n$, the partition $\text{type}(\sigma)$ records its cycle lengths. For a symmetric function f , write $f^\perp$ for the adjoint of multiplication by f under the Hall inner product, in which the Schur functions are orthonormal.

The Frobenius characteristic of a rational $\mathfrak{S}_n$ -module V is

$$\text{ch}_n(V) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \text{tr}(\sigma | V) p_{\text{type}(\sigma)}.$$

It satisfies $\text{ch}_n(V_\lambda) = s_\lambda$. Thus the coefficient $[s_\lambda] \text{ch}_n(V)$ is the multiplicity of V_λ in V .

For convenience, we set $s_{(a, 1^r)} = 0$ when $a \leq 0$. Plethysm uses $\Psi_r(p_i) = p_{ri}$ and $\Psi_r(\mathbf{L}) = \mathbf{L}^r$; ∂ denotes $\partial/\partial p_1$.

Notation 2.5. Set

$$\begin{aligned} H &= \sum_{a \geq 10} (s_{(a-10, 1^{10})} + s_{(a-9, 1^9)}), \\ G &= \sum_{a \geq 9} (s_{(a-9, 1^9)} + s_{(a-8, 1^8)}), \\ U &= E s_{(2, 1^{10})} + E_{\geq 3} H. \end{aligned}$$

We use $h_2[E] = (E^2 + \Psi_2(E))/2$ and $e_2[E] = (E^2 - \Psi_2(E))/2$.

Set-up 2.6. For a finite marking set I , we write $\mathfrak{S}_I$ for its permutation group and $\overline{\mathcal{M}}_{h,I}$ for the moduli stack with those markings. For a stable pair $(h, |I|)$, let $\pi : \mathcal{C}_{h,I} \rightarrow \overline{\mathcal{M}}_{h,I}$ be the universal curve, with sections s_i and section divisors D_i , and with ω_π the relative dualizing sheaf. We use

$$\psi_i = c_1(s_i^* \omega_\pi), \quad \kappa_1 = \pi_* c_1 \left(\omega_\pi \left(\sum_{i \in I} D_i \right) \right)^2.$$

We assign polynomial degree one to each of ψ_i and κ_1 . A homogeneous polynomial of degree d therefore represents a class in $H^{2d}(\overline{\mathcal{M}}_{h,I}; \mathbb{Q})$. Integration means evaluation on the fundamental class of the relevant proper moduli stack, with the normalization $\int_{[\text{pt}/A]} 1 = 1/|A|$ for a finite group A .

Write $S = \mathbf{S}_{12}$ and put

$$J(u, v) = \int_{\overline{\mathcal{M}}_{1, [11]}} u \smile v.$$

The form J is alternating and nondegenerate. In formulas for underlying rational vector spaces we suppress the common Tate factor of a Gysin map.

Definition 2.7 (Stable graphs and clutching classes). Recall that the strata correspond to stable graphs Γ : a vertex set V with genus function $g : V \rightarrow \mathbb{Z}_{\geq 0}$, a half-edge set H with vertex map $v : H \rightarrow V$ and involution ι ; the edges E are the 2-cycles of ι (self-edges allowed), the legs L its fixed points, in bijection with the markings; (V, E) is connected, and $2g(v) - 2 + |F(v)| > 0$ for every vertex, $F(v)$ being the flags at v . The total genus of Γ is $\sum_v g(v) + |E(\Gamma)| - |V(\Gamma)| + 1$. Graph automorphisms fix each ordinary leg label. Set

$$X_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v), F(v)}, \quad \xi_\Gamma : X_\Gamma \longrightarrow \overline{\mathcal{M}}_{g,n}, \quad [\Gamma, \alpha] = \frac{\xi_{\Gamma*} \alpha}{|\text{Aut}(\Gamma)|}.$$

Here ξ_Γ is the labelled clutching map. This normalization agrees with integration from the quotient stack for invariant classes: on an invariant class α , quotient-stack pushforward is $\xi_{\Gamma*} \alpha / |\text{Aut}(\Gamma)|$, so that half-edge exchanges and vertex exchanges are already incorporated and there is no further factor of two, and the ordinary pushforward uses the complex normal orientation, not an edge-order determinant. Let $\mathcal{G}_{g,n}^{(1)}$ denote the isomorphism classes of stable graphs of type (g, n) with one unordered edge.

Definition 2.8 (Contractions and common refinements). For a stable graph Γ and a subset $S \subseteq E(\Gamma)$, let Γ/S be the graph obtained by contracting the edges in S . Each connected component C of the spanning subgraph $(V(\Gamma), S)$ is replaced by a vertex of genus

$$\sum_{v \in V(C)} g(v) + b_1(C), \quad b_1(C) = |E(C)| - |V(C)| + 1.$$

The remaining edges and labelled legs are attached to the corresponding vertices of Γ/S .

A contraction $f : \Gamma \rightarrow \Gamma_0$ consists of a subset $S_f \subseteq E(\Gamma)$ together with a specified isomorphism

$$\phi_f : \Gamma/S_f \xrightarrow{\sim} \Gamma_0$$

of stable graphs, preserving incidence, vertex genus labels, and leg labels. An edge is *retained by* f if it does not belong to S_f .

Fix stable graphs A and B of the same type. A *common refinement* is a stable graph Γ equipped with contractions

$$A \xleftarrow{f} \Gamma \xrightarrow{g} B.$$

We call (Γ, f, g) *generic* if $S_f \cap S_g = \emptyset$. The set of edges retained by both maps is

$$H(f, g) = E(\Gamma) \setminus (S_f \cup S_g).$$

Two common refinements (Γ, f, g) and (Γ', f', g') are isomorphic if there is an isomorphism of stable graphs $\alpha : \Gamma \xrightarrow{\sim} \Gamma'$ such that

$$f = f' \circ \alpha, \quad g = g' \circ \alpha.$$

We write $\mathcal{R}(A, B)$ for the set of isomorphism classes of generic common refinements.

3. MOTIVIC TYPES AND BOUNDARY GYSIN MAPS

We retain the notation of Section 2.

Set-up 3.1. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and put $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For a smooth proper Deligne–Mumford stack X over $\mathbb{Q}$ and a prime ℓ , write

$$H_{\text{B}}^j(X) = H^j(X(\mathbb{C}); \mathbb{Q}), \quad H_{\ell}^j(X) = H_{\text{et}}^j(X_{\overline{\mathbb{Q}}}; \mathbb{Q}_{\ell}).$$

Subscripts B and ℓ on the fixed motivic pieces denote these realizations. In particular, $\mathbf{L}_{\ell} = \mathbb{Q}_{\ell}(-1)$. For commuting $\mathfrak{S}_n$ - and $G_{\mathbb{Q}}$ -actions, Galois semisimplification is taken on each Specht multiplicity space, with the Specht factor unchanged.

By [Rib77, Theorem (2.3)], for a normalized cuspidal Hecke eigenform of weight at least two and a number field K containing its eigenvalues, the attached two-dimensional K_λ -representation of $G_\mathbb{Q}$ is simple for every finite place λ of K . Applied to the two level-one eigenforms defining our cusp objects, this gives simplicity of $S_{12\ell}$ and $S_{16\ell}$ over $\mathbb{Q}_\ell$ for every prime ℓ . Taking a dual, if required by the cohomological convention, and Tate twisting both preserve simplicity. Thus P_ℓ is simple as well.

Theorem 3.2 (Keel [Kee92, Section 2, Theorem 1]). *Let $n \geq 3$. The cycle map identifies the Chow groups of $\overline{\mathcal{M}}_{0,n}$ with its integral homology. In particular, $H^{2j+1}(\overline{\mathcal{M}}_{0,n}; \mathbb{Q}) = 0$ for every $j \geq 0$, and $H^{2j}(\overline{\mathcal{M}}_{0,n}; \mathbb{Q})$ is spanned by algebraic cycle classes.*

Theorem 3.3 (Bergström–Faber–Payne [BFP22, Theorem 1.1]). *Let $g, n \geq 0$ satisfy $2g - 2 + n > 0$. Then*

$$H^j(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}) = 0 \quad (j = 1, 3, 5, 7, 9).$$

Artin comparison gives the same vanishings in every ℓ -adic realization.

Definition 3.4. A smooth algebraic stack X of finite type, stratified by quotient stacks, has the *Chow–Künneth generation property* if, for every algebraic stack Y of finite type stratified by quotient stacks, the exterior product of rational Chow groups

$$A_*(X) \otimes A_*(Y) \longrightarrow A_*(X \times Y)$$

is surjective.

Proposition 3.5 (Canning–Larson–Payne [CLP24, Proposition 4.2 and Lemma 4.3]). *For smooth finite-type algebraic stacks stratified by quotient stacks, the Chow–Künneth generation property passes to open substacks, to proper surjective images under maps representable by Deligne–Mumford stacks, and from a finite stratification to its union. An affine bundle has the property if and only if its base does. A Grassmann bundle over a base with the property has the property. The stacks BGL_r , BSL_r , and $BPGL_r$ have the property. If X is an open substack of a smooth proper complex Deligne–Mumford stack and has this property, its cycle class map surjects onto $\bigoplus_k W_k H^k(X)$. Thus $W_k H^k(X) = 0$ for odd k .*

Proposition 3.6 (Canning–Larson–Payne [CLP24, Table 1, Proposition 4.5, and Lemma 7.3]). *For $3 \leq g \leq 7$ and $0 \leq n \leq c(g)$, where*

$$c(3) = 11, \quad c(4) = 11, \quad c(5) = 7, \quad c(6) = 5, \quad c(7) = 3,$$

$\mathcal{M}_{g,n}$ has the Chow–Künneth generation property, and $W_k H^k(\mathcal{M}_{g,n}) = 0$ for every odd k . For $g = 2$ and $0 \leq n \leq 10$, $W_k H^k(\mathcal{M}_{2,n}) = 0$ for every odd $k \geq 1$. The group $H^{15}(\overline{\mathcal{M}}_{2,12})$ is generated by boundary pushforwards and is contained in the semi-tautological extension generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$.

In genus one with fifteen markings, the additional constituent is the image of the natural map from compactly supported cohomology of the open moduli stack to cohomology of its compactification: by [CLP24, Lemma 8.3], if $V \subset H^{15}(\overline{\mathcal{M}}_{1,15})$ is the image of boundary Gysin maps and W is that image, then, in both realization categories, $H^{15}(\overline{\mathcal{M}}_{1,15}) = V \oplus W$ and $W \cong S_{16}$.

Theorem 3.7 (Canning–Larson–Payne–Willwacher [CLPW24, Theorem 1.3]). *Let $n \geq 1$. In both realization categories there is an $\mathfrak{S}_n$ -equivariant decomposition*

$$H^{15}(\overline{\mathcal{M}}_{1,n}) \cong (K_n^{15} \otimes S_{16}) \oplus V_n, \quad V_n^{\text{ss}} \cong \bigoplus P.$$

The primitive $S_{16} \subset H^{15}(\overline{\mathcal{M}}_{1,15})$ is the image of $H_c^{15}(\mathcal{M}_{1,15}) \rightarrow H^{15}(\overline{\mathcal{M}}_{1,15})$. For every ordered 15-subset $A \subset [n]$, the line of the hook generated by k_A , tensored with S_{16} , maps to its forgetful

pullback from $\overline{\mathcal{M}}_{1,A}$. For each permutation σ of an ordered 15-subset A and each ordered 16-subset $(a_0, \dots, a_{15})$, their relations are precisely

$$k_{\sigma(A)} = \text{sgn}(\sigma)k_A, \quad \sum_{j=0}^{15} (-1)^j k_{(a_0, \dots, \widehat{a_j}, \dots, a_{15})} = 0.$$

Theorem 3.8 (Motivic type reduction). *For every stable (g, n) there is a rational $\mathfrak{S}_n$ -module $\mathcal{E}_{g,n}$, unique up to isomorphism, with*

$$H^{15}(\overline{\mathcal{M}}_{g,n}) \cong \mathcal{E}_{g,n} \otimes \mathbf{P} \oplus \delta_{g,1} K_n^{15} \otimes \mathbf{S}_{16}.$$

This is an $\mathfrak{S}_n$ -equivariant isomorphism of pure Hodge structures, with $\mathcal{E}_{g,n}$ and K_n^{15} given Hodge type $(0, 0)$. The same decomposition holds after Galois semisimplification of the compatible ℓ -adic realizations. For $g = 0$ the cohomology group is zero, so $\mathcal{E}_{0,n} = 0$.

Proof. *The Hodge multiplicities.* Fix a stable pair (g, n) and, for $\lambda \vdash n$, put

$$T_\lambda = \text{Hom}_{\mathbb{Q}[\mathfrak{S}_n]}(V_\lambda, H_B^{15}(\overline{\mathcal{M}}_{g,n})), \quad a_\lambda = \dim_{\mathbb{Q}} \text{Hom}_{\mathbb{Q}[\mathfrak{S}_n]}(V_\lambda, K_n^{15}).$$

The split semisimplicity of $\mathbb{Q}[\mathfrak{S}_n]$ gives an isomorphism of Hodge structures

$$H_B^{15}(\overline{\mathcal{M}}_{g,n}) \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes T_\lambda.$$

Each T_λ is a polarizable pure Hodge structure. The rank-two objects $\mathbf{P}_B$ and $\mathbf{S}_{16B}$ are simple: a proper nonzero rational sub-Hodge structure would have rank one and odd weight, whereas complex conjugation forces a rank-one pure rational Hodge structure to have type (p, p) . Their different Hodge types also show that they are not isomorphic.

For $g \geq 2$, the rational Hodge structure $H^{15}(\overline{\mathcal{M}}_{g,n})$ is a direct sum of copies of $\mathbf{P}$ [CLP24, Theorem 1.1], so every T_λ is a direct sum of copies of $\mathbf{P}_B$. For $g = 1$, Theorem 3.7 gives exactly a_λ copies of $\mathbf{S}_{16B}$ and leaves only copies of $\mathbf{P}_B$. For $g = 0$, Theorem 3.2 gives zero. Thus unique nonnegative integers m_λ satisfy

$$T_\lambda \cong \mathbf{P}_B^{\oplus m_\lambda} \oplus \mathbf{S}_{16B}^{\oplus \delta_{g,1} a_\lambda}.$$

Taking $\mathcal{E}_{g,n} = \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus m_\lambda}$ and regrouping proves the Hodge formula.

The Galois multiplicities. For a prime ℓ , put

$$T_{\lambda,\ell} = \text{Hom}_{\mathbb{Q}_\ell[\mathfrak{S}_n]}(V_\lambda \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, H_\ell^{15}(\overline{\mathcal{M}}_{g,n})).$$

Theorem 1.1 of [CLP24] for $g \geq 2$ and Theorem 3.7 for $g = 1$ describe its semisimplification as copies of $\mathbf{P}_\ell$ together with $\delta_{g,1} a_\lambda$ copies of $\mathbf{S}_{16\ell}$. Let $m_{\lambda,\ell}$ be the number of copies of $\mathbf{P}_\ell$ in this description. Comparison commutes with the rational $\mathfrak{S}_n$ -action and therefore with passage to each multiplicity space. Since both cusp objects have rank two, it gives

$$2m_{\lambda,\ell} + 2\delta_{g,1} a_\lambda = \dim_{\mathbb{Q}_\ell} T_{\lambda,\ell} = \dim_{\mathbb{Q}} T_\lambda = 2m_\lambda + 2\delta_{g,1} a_\lambda.$$

Hence $m_{\lambda,\ell} = m_\lambda$. The genus-zero vanishing transfers by the same comparison. Regrouping the Specht factors proves the semisimplified Galois formula in every genus. $\square$

Definition 3.9 (Multiplicity modules and characteristic series). For each stable (g, n) let $\mathcal{E}_{g,n}$ be the rational $\mathfrak{S}_n$ -module of Theorem 3.8 and $m_{g,n}^\lambda$ the multiplicity of V_λ in it. We call $\mathcal{E}_{g,n}$ the $\mathbf{P}$ -multiplicity module. For fixed g , define its Frobenius characteristic series by

$$F_g = \sum_{\substack{n \geq 0 \\ 2g-2+n > 0}} \text{ch}_n(\mathcal{E}_{g,n}) \in \widehat{\Lambda}.$$

Definition 3.10. For a finite set I , write $K_I = K_I^{11}$. Set $K_I = 0$ if $|I| < 11$. Otherwise K_I is generated over $\mathbb{Q}$ by ordered eleven-tuples of distinct elements, with relations

$$[a_{\sigma(1)}, \dots, a_{\sigma(11)}] = \text{sgn}(\sigma)[a_1, \dots, a_{11}], \quad \sum_{j=0}^{11} (-1)^j [a_0, \dots, \widehat{a_j}, \dots, a_{11}] = 0.$$

Choose an ordering of I and an anchor $a \in I$. The symbols $[B, a]$, with B an increasing ten-element subset of $I \setminus \{a\}$, give a basis. Relabelling acts on the tuple presentation. For an ordered tuple P , let $f_P : \overline{\mathcal{M}}_{1,I} \rightarrow \overline{\mathcal{M}}_{1,[11]}$ forget all other markings and label the retained markings in the order of P . Write $\omega_I([P], u) = f_P^* u$ for $u \in \mathbb{S}_{12}$, extended linearly.

Definition 3.11. For $T \subseteq I$ with $|T| \geq 2$ and a new flag p , put $A = I \setminus T$. The rational-tail map $r_T : K_I \rightarrow K_{A \cup \{p\}}$ retains a tuple with no entry in T , replaces its unique entry in T by p when there is one, and is zero when there are at least two. Replacement preserves the tuple order.

The anchor basis and the rational-tail rule can be checked on an exterior power. For $|I| \geq 11$, let $\mathbb{Q}^I$ have basis $(e_i)_{i \in I}$, and let $\varepsilon_I(e_i) = 1$. Contraction by ε_I induces

$$K_I \cong \bigwedge^{10} \ker(\varepsilon_I), \quad [i_1, \dots, i_{10}, a] \mapsto \bigwedge_{j=1}^{10} (e_{i_j} - e_a).$$

Indeed, the image of contraction on $\bigwedge^{12} \mathbb{Q}^I$ is $\bigwedge^{11} \ker(\varepsilon_I)$, and its kernel on $\bigwedge^{11} \mathbb{Q}^I$ is the same space. This proves the quotient description and the anchor basis. The linear map that sends e_i to itself for $i \notin T$ and to e_p for $i \in T$ preserves augmentation. Its exterior power gives r_T : a tuple with two entries in T acquires a repeated p and vanishes. Thus the rule respects all hook relations.

Proposition 3.12 (Canning–Larson–Payne [CLP23, Theorem 1.1 and Proposition 2.3]). *Let $g, n \geq 0$ satisfy $2g - 2 + n > 0$. Then $H^{11}(\overline{\mathcal{M}}_{g,n}) = 0$ unless $g = 1$ and $n \geq 11$. For $n \geq 11$, the pullbacks of the weight-twelve cusp form identify*

$$H^{11,0}(\overline{\mathcal{M}}_{1,n}) \cong K_n^{11} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Those indexed by 11-subsets containing a fixed marking form a basis.

Lemma 3.13 (Canning–Larson–Payne [CLP23, Lemma 2.2]). *Let $n \geq 11$, let $A \subset [n]$ be ordered with $|A| = 11$, and let*

$$\iota_B : D_B = \overline{\mathcal{M}}_{1,B \cup p} \times \overline{\mathcal{M}}_{0,B^c \cup p'} \longrightarrow \overline{\mathcal{M}}_{1,n}$$

be a stable separating gluing map. Let ω_A be the forgetful pullback of the cusp form. If $A \subset B$, put $\epsilon_B(A) = A$; if $|A \cap B| = 10$, replace the unique element of $A \setminus B$ by p , retaining its position in the ordering. Then

$$\iota_B^* \omega_A = \begin{cases} \text{pr}_1^* \omega_{\epsilon_B(A)}, & |A \cap B| \geq 10, \\ 0, & |A \cap B| \leq 9. \end{cases}$$

Lemma 3.14. *Let I be a finite marking set with $|I| \geq 11$, and let R be the Betti realization or the ℓ -adic realization for a prime ℓ . The maps in Definition 3.10 induce an actual $\mathfrak{S}_I$ -equivariant injection*

$$\omega_{I,R} : K_I \otimes \mathbb{S}_{12R} \longrightarrow H_R^{11}(\overline{\mathcal{M}}_{1,I}).$$

For $R = \mathbb{B}$, its complex image is $H^{11,0}(\overline{\mathcal{M}}_{1,I}) \oplus H^{0,11}(\overline{\mathcal{M}}_{1,I})$. The injections commute with forgetful pullback and with the rational-tail maps of Definition 3.11.

Proof. The source S_{12B} has only Hodge types $(11, 0)$ and $(0, 11)$. On its $(11, 0)$ line, the forgetful maps satisfy the tuple relations and send the anchor basis to the basis in Proposition 3.12. The same assertions hold on its conjugate line. The rational maps therefore descend through the tuple relations, and their complexification is an isomorphism onto the two displayed Hodge summands. This proves injectivity over $\mathbb{Q}$.

Forgetting markings acts on tuples by retaining their labels, so compatibility with forgetful pullback follows from composition of the forgetful morphisms. Lemma 3.13 gives the rational-tail formula on the $(11, 0)$ line and, by conjugation, on the $(0, 11)$ line. It therefore gives an identity of rational cohomology maps on the whole source. All the forgetful and clutching morphisms are defined over $\mathbb{Q}$. Comparison transfers these identities, the relations, and injectivity to ℓ -adic cohomology. The resulting maps commute with $G_{\mathbb{Q}}$ and with relabelling. $\square$

Proposition 3.15. *Let (g, n) be stable and let R be the Betti realization or the ℓ -adic realization for any prime ℓ . Then*

$$H_R^{11}(\overline{\mathcal{M}}_{g,n}) \cong \begin{cases} K_n^{11} \otimes S_{12R}, & g = 1, \\ 0, & g \neq 1. \end{cases}$$

These are actual equivariant isomorphisms, compatible with forgetful pullback and with the boundary formula in Lemma 3.13.

We prove this proposition after the degree-thirteen theorem.

Proposition 3.16 (Stable forgetting identities [BS22, Proposition 3.10 and the paragraph after Proposition 3.11]). *Let $2g - 2 + n > 0$, let $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ forget marking $n + 1$, and let $D_{i,n+1}$ be the image of the i -th section, the rational tail carrying $i, n + 1$. In rational Chow groups,*

$$\begin{aligned} \pi^* \psi_i &= \psi_i - D_{i,n+1}, & \pi^* \kappa_r &= \kappa_r - \psi_{n+1}^r \quad (r \geq 0), \\ \pi_* \psi_{n+1}^{r+1} &= \kappa_r \quad (r \geq 0), & \kappa_0 &= (2g - 2 + n) 1 \quad \text{on } \overline{\mathcal{M}}_{g,n}. \end{aligned}$$

Theorem 3.17 (Arbarello–Cornalba [AC98, Theorem 2.2]). *Let (g, n) be stable. Then $H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ is generated by κ_1 , the cotangent classes ψ_i , the nonseparating boundary class δ_{irr} , and the classes $\delta_{a,A}$ with $0 \leq a \leq g$, $A \subset [n]$, $2a - 2 + |A| \geq 0$, and $2(g - a) - 2 + |A^c| \geq 0$. Let $t \geq 3$ and let L be a finite marking set. Let $\mathcal{D}_{t,L}$ contain one representative of each stable separating divisor modulo $(a, A) \sim (t - a, L \setminus A)$. Then*

$$\{\kappa_1\} \cup \{\psi_i : i \in L\} \cup \{\delta_{\text{irr}}\} \cup \{\delta_{a,A} : (a, A) \in \mathcal{D}_{t,L}\}$$

is a basis of $H^2(\overline{\mathcal{M}}_{t,L}; \mathbb{Q})$.

Here $\delta_{\Gamma} = [\Gamma, 1] = \xi_{\Gamma*} 1 / |\text{Aut}(\Gamma)|$ for a one-edge graph Γ , in the normalization of Definition 2.7. Thus $\delta_{\text{irr}} = \frac{1}{2} \xi_{\text{irr}*} 1$ in positive genus, and it is zero in genus zero. An unstable symbol $\delta_{a,A}$ is zero. Complementary descriptions satisfy $\delta_{a,A} = \delta_{g-a,A^c}$ and represent one divisor. For a separating graph the normalization factor is one, except when $n = 0$ and $g = 2a$, when it is $1/2$. For a root $q \in L$, one may take $\mathcal{D}_{t,L}$ to consist of exactly the stable representatives with $q \in A$, allowing every $0 \leq a \leq t$ subject to stability. For $L = \emptyset$, take $(a, \emptyset)$ with $1 \leq a \leq \lfloor t/2 \rfloor$; the equal-genus divisor occurs once.

Each displayed generator is the class of a rational algebraic cycle defined over $\mathbb{Q}$. Consequently degree-two cohomology is a direct sum of Tate structures in both realization categories. More explicitly, take the rational vector space on these generators and its cycle-class map. The Betti map is surjective, and comparison identifies its scalar extension with the ℓ -adic cycle-class map after the common Tate twist. Its kernel therefore extends from the same rational relation space. This realizes the rational divisor multiplicity module itself, including its relabelling action, in every realization.

Proposition 3.18 (Divisor identities in genera at most two [AC98, Theorem 2.2]). *Let $(h, |L|)$ be stable with $0 \leq h \leq 2$. Write $\delta_{c,A}$ for the divisor with one side of genus c carrying A , require both sides with their attaching flags to be stable, and identify $\delta_{c,A} = \delta_{h-c, L \setminus A}$. Let δ_c sum distinct divisors of that type, counting complementary pairs once, and put $\delta_{\text{irr}} = \frac{1}{2} \xi_{\text{irr}*} 1$ when the labelled nonseparating normalization exists, and zero otherwise. Then $H^2(\overline{\mathcal{M}}_{h,L}; \mathbb{Q})$ is generated by κ_1 , all ψ_i , and the stable boundary divisors. For genus zero and distinct $x, y, z \in L$ one has*

$$\psi_z = \sum_{\substack{A \subset L, z \in A, x, y \notin A \\ |A|, |L \setminus A| \geq 2}} \delta_{0,A}, \quad \kappa_1 = \sum_{\substack{A \subset L, x, y \notin A \\ |A|, |L \setminus A| \geq 2}} (|A| - 1) \delta_{0,A}.$$

For genus one one has

$$\psi_i = \frac{\delta_{\text{irr}}}{12} + \sum_{\substack{T \subseteq L, i \in T \\ |T| \geq 2}} \delta_{0,T}, \quad \kappa_1 = \sum_{i \in L} \psi_i - \delta_0.$$

For genus two one has

$$5\kappa_1 = 5 \sum_{i \in L} \psi_i + \delta_{\text{irr}} - 5\delta_0 + 7\delta_1,$$

with node flags treated as markings.

Theorem 3.19 (Nonseparating divisor pullbacks [AC98, Lemma 3.2]). *Let $h \geq 1$, let L be finite with $2h - 2 + |L| > 0$, and let u, v be distinct labels outside L . Let*

$$\xi_{uv} : \overline{\mathcal{M}}_{h-1, L \cup \{u, v\}} \longrightarrow \overline{\mathcal{M}}_{h,L}$$

identify the markings u, v , and define

$$\mathcal{A}_{h,L}^{u|v} = \left\{ (b, S) : \begin{array}{ll} 0 \leq b \leq h-1, & S \subseteq L, \\ 2b-1+|S| \geq 0, & 2(h-1-b)-1+|L \setminus S| \geq 0 \end{array} \right\}.$$

Then, with every right-hand class taken in genus $h-1$,

$$\begin{aligned} \xi_{uv}^* \kappa_1 &= \kappa_1, \\ \xi_{uv}^* \psi_i &= \psi_i \quad (i \in L), \\ \xi_{uv}^* \delta_{\text{irr}} &= \delta_{\text{irr}} - \psi_u - \psi_v + \sum_{(b,S) \in \mathcal{A}_{h,L}^{u|v}} \delta_{b, S \cup \{u\}}. \end{aligned}$$

For every stable separating divisor $\delta_{a,A}$ on $\overline{\mathcal{M}}_{h,L}$,

$$\xi_{uv}^* \delta_{a,A} = \begin{cases} \delta_{a,\emptyset}, & h = 2a, \ A = L = \emptyset, \\ \delta_{a,A} + \delta_{a-1, A \cup \{u, v\}}, & \text{otherwise.} \end{cases}$$

Theorem 3.20 (Clutching intersections with tautological decorations [SchmittvanZelm18, Propositions 2.14 and 2.19, Lemma 2.27, Theorem 2.29 of arXiv:1808.05817v1]). *Let A, B be stable graphs of the same type and let α, β be tautological decorations on their vertex products. For each flag h , write L_h for its cotangent line bundle. Then*

$$[A, \alpha][B, \beta] = \frac{1}{|\text{Aut}(A)| |\text{Aut}(B)|} \sum_{[\Gamma, f, g] \in \mathcal{R}(A, B)} \xi_{\Gamma*} \left(f^* \alpha g^* \beta \prod_{\{h, h'\} \in H(f, g)} (-\psi_h - \psi_{h'}) \right).$$

The fibre product of the labelled clutching maps has components X_{Γ} indexed by these generic structures. Its excess bundle on such a component is the direct sum of the node smoothing lines $L_h^{\vee} \otimes L_{h'}^{\vee}$ over $\{h, h'\} \in H(f, g)$. Cotangent classes at retained flags pull back to those flags, and a logarithmic kappa class pulls back to the sum over its inverse vertices.

Lemma 3.21 (Ordered cohomological excess identity). *Let A, B be stable graphs of the same type, and let $\alpha \in H^*(X_A; \mathbb{Q})$ and $\beta \in H^*(X_B; \mathbb{Q})$ be homogeneous classes. Then the product identity of Theorem 3.20 holds for α, β , with $f^*\alpha$ preceding $g^*\beta$, including when both classes have odd degree. The sum over $\mathcal{R}(A, B)$ has no additional graph automorphism factor.*

Proof. The fibre-product components and their excess bundles are those of Theorem 3.20. On a normal-crossing chart, the Thom identity for a component with excess bundle $\mathcal{N}$ multiplies the restricted coefficient classes by $c_{\text{top}}(\mathcal{N})$. This is an identity of cohomological Gysin maps: tensoring the Thom classes with α and β does not require these coefficients to be algebraic. Each Thom class has even degree. Keeping the coefficients in the order α, β therefore gives

$$f^*\alpha g^*\beta c_{\text{top}}(\mathcal{N}) = f^*\alpha g^*\beta \prod_{\{h, h'\} \in H(f, g)} (-\psi_h - \psi_{h'}).$$

Proper pushforward and descent from the charts give the asserted identity. The factors $1/|\text{Aut}(A)|$ and $1/|\text{Aut}(B)|$ come from the normalizations of the two input classes.

An automorphism of Γ preserving both specified contractions fixes every half-edge retained by either contraction. Genericity says that these are all the half-edges. It also fixes the labelled legs and hence every vertex of a connected graph with an edge. An edgeless connected graph has just one vertex, which is fixed as well. Thus the action on specified generic structures is free. Summing structures on a fixed representative and dividing by $|\text{Aut}(\Gamma)|$ is consequently the same as summing their isomorphism classes once. $\square$

Definition 3.22. For a finite marking set J of cardinality m , define $U_{0, J} = 0$. For $h \geq 2$, define

$$U_{h, [m]} = \bigoplus_{a=10}^m \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_{m-a}}^{\mathfrak{S}_m} \left(\text{Res}_{\mathfrak{S}_a}^{\mathfrak{S}_{a+1}} K_{a+1}^{11} \boxtimes \mathbf{1} \right).$$

Define

$$\begin{aligned} U_{1, [m]} &= \mathbf{1}_{m \geq 12} \text{Ind}_{\mathfrak{S}_{12} \times \mathfrak{S}_{m-12}}^{\mathfrak{S}_m} (V_{(2, 1^{10})} \boxtimes \mathbf{1}) \\ &\quad \oplus \bigoplus_{a=10}^{m-3} \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_{m-a}}^{\mathfrak{S}_m} \left((V_{(a-10, 1^{10})} \oplus V_{(a-9, 1^9)}) \boxtimes \mathbf{1} \right). \end{aligned}$$

The first term is omitted when $m < 12$; empty sums and Specht symbols with nonpositive first part are zero. Transport these modules by bijections from $[m]$ to J ; each restriction fixes the node flag.

Theorem 3.23 (Canning–Larson–Payne–Willwacher [CLPW, Theorem 1.7 and Corollary 4.9]). *Let (g, n) be stable. Then*

$$H^{13}(\overline{\mathcal{M}}_{g, n}) \cong U_{g, [n]} \otimes \text{LS}_{12}$$

as an $\mathfrak{S}_n$ -equivariant Hodge structure and as an actual ℓ -adic Galois representation. For all stable elliptic-tail gluing maps ι_A , the map

$$\bigoplus_A H^{11}(\overline{\mathcal{M}}_{1, A \cup p}) \otimes H^0(\overline{\mathcal{M}}_{g-1, A^c \cup p'}) \otimes \mathbb{L} \xrightarrow{\sum_A \iota_{A^*}} H^{13}(\overline{\mathcal{M}}_{g, n})$$

is surjective. For $g \geq 2$ it is also injective.

Proof of Proposition 3.15. The vanishing for $g \neq 1$ and for $g = 1$, $n < 11$ follows from Proposition 3.12 and comparison. Let $g = 1$ and $n \geq 11$, and put $q = n - 1$. By Lemma 3.14, for every $m \geq 11$,

$$\dim_{\mathbb{Q}} H_{\text{B}}^{11}(\overline{\mathcal{M}}_{1, m}) \geq 2 \binom{m-1}{10}.$$

We obtain equality for $m = n$ from Theorem 3.23 at $(2, q)$. In genus two, both factors in every elliptic-tail gluing map are stable for every subset $A \subseteq [q]$. The theorem makes the direct sum of these Gysin maps an isomorphism, and each degree-zero factor has dimension one. On the other hand, the explicit formula for $U_{2,[q]}$ in Definition 3.22 gives

$$\begin{aligned} \sum_{a=10}^q \binom{q}{a} \dim_{\mathbb{Q}} H_{\mathbb{B}}^{11}(\overline{\mathcal{M}}_{1,a+1}) &= \dim_{\mathbb{Q}} H_{\mathbb{B}}^{13}(\overline{\mathcal{M}}_{2,q}) \\ &= 2 \sum_{a=10}^q \binom{q}{a} \binom{a}{10}. \end{aligned}$$

Here restriction to $\mathfrak{S}_a$ preserves the dimension $\dim K_{a+1}^{11} = \binom{a}{10}$, and induction multiplies it by $\binom{q}{a}$. Every summand in

$$\sum_{a=10}^q \binom{q}{a} \left(\dim_{\mathbb{Q}} H_{\mathbb{B}}^{11}(\overline{\mathcal{M}}_{1,a+1}) - 2 \binom{a}{10} \right) = 0$$

is nonnegative by the hook injection. In particular, the term $a = q$ vanishes. Thus $\dim_{\mathbb{Q}} H_{\mathbb{B}}^{11}(\overline{\mathcal{M}}_{1,n}) = 2 \binom{n-1}{10}$, and the injection $\omega_{[n],\mathbb{B}}$ is an isomorphism of Hodge structures.

Comparison, as in Lemma 3.14, makes $\omega_{[n],\ell}$ bijective. Its relabelling, forgetful, and boundary compatibilities are those already proved in Lemma 3.14. $\square$

Definition 3.24. For Γ of type (g, n) with one edge, with X_{Γ} as in Definition 2.7 and $A_{\Gamma} = \text{Aut}(\Gamma)$, define

$$\mathcal{D}_{g,n,R}^{13} = \bigoplus_{[\Gamma]} H_R^{13}([X_{\Gamma}/A_{\Gamma}]) \otimes \mathbb{L}_R, \quad \Phi_{g,n,R}^{15} = \sum_{[\Gamma]} \xi_{\Gamma*},$$

for $R = \mathbb{B}$ or $R = \ell$ (normalization of Definition 2.7). The Tate factor is the codimension-one Gysin twist.

Definition 3.25. For a stable (h, J) let $\Pi_{h,J}$ be the rational relabelling module underlying $H^2(\overline{\mathcal{M}}_{h,J}) \otimes \mathbb{L}^{-1}$. For a stable one-edge graph Γ with separating vertices v, w , write $J_v = F_{\Gamma}(v)$, $h_v = g(v)$, and set

$$\begin{aligned} M_{\Gamma} &= (U_{h_v,J_v} \boxtimes \mathbf{1}) \oplus (\mathbf{1} \boxtimes U_{h_w,J_w}) \\ &\oplus \mathbf{1}_{h_v=1} (K_{J_v}^{11} \boxtimes \Pi_{h_w,J_w}) \oplus \mathbf{1}_{h_w=1} (\Pi_{h_v,J_v} \boxtimes K_{J_w}^{11}). \end{aligned}$$

For a nonseparating graph put $M_{\Gamma} = U_{g-1,[n] \sqcup \{p,p'\}}$. In each case graph automorphisms act through all flag bijections and vertex exchanges. Put $B_{g,n}^{(1)} = \bigoplus_{[\Gamma]} M_{\Gamma}^{A_{\Gamma}}$.

Proposition 3.26 (Degree-thirteen boundary input). *Let (g, n) be stable and let R be the Betti realization or any ℓ -adic realization. Then*

$$\mathcal{D}_{g,n,R}^{13} \cong B_{g,n}^{(1)} \otimes \mathbb{P}_R.$$

Under this identification $\Phi_{g,n,R}^{15}$ is the sum of the ordinary quotient-stack clutching Gysin maps, with no edge-order determinant. The maps are equivariant under relabelling and, for $R = \ell$, under $G_{\mathbb{Q}}$. Comparison commutes with every Specht projection and these maps. If

$$\text{im } \Phi_{g,n,\mathbb{B}}^{15} = \mathcal{E}_{g,n} \otimes \mathbb{P}_{\mathbb{B}},$$

then for every prime ℓ there is an actual isomorphism

$$\text{im } \Phi_{g,n,\ell}^{15} \cong (\mathcal{E}_{g,n} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}) \otimes \mathbb{P}_{\ell}.$$

If the Betti map is surjective onto the full target, the same is true of the ℓ -adic map, and this is an actual decomposition of the target.

Proof. The complete source. Consider a separating one-edge graph Γ . Put $Y_v = \overline{\mathcal{M}}_{h_v, J_v}$ and $Y_w = \overline{\mathcal{M}}_{h_w, J_w}$. Theorem 3.3 and comparison give the vanishing of odd cohomology through degree nine in both realizations. In a Künneth summand of degree thirteen, exactly one factor has odd degree. It must have degree eleven or thirteen. Consequently

$$\begin{aligned} H_R^{13}(Y_v \times Y_w) &\cong (H_R^{13}(Y_v) \otimes H_R^0(Y_w)) \oplus (H_R^0(Y_v) \otimes H_R^{13}(Y_w)) \\ &\quad \oplus (H_R^{11}(Y_v) \otimes H_R^2(Y_w)) \oplus (H_R^2(Y_v) \otimes H_R^{11}(Y_w)). \end{aligned}$$

The degree-zero factors are trivial. Substitute Theorem 3.23, Proposition 3.15, and the divisor cycle-class comparison following Theorem 3.17. This identifies the displayed sum with $M_\Gamma \otimes \mathbb{L}_R \otimes \mathbb{S}_{12R}$. For a nonseparating graph, its sole factor is $\overline{\mathcal{M}}_{g-1, [n] \sqcup \{p, p'\}}$, and Theorem 3.23 gives the same identification with the prescribed M_Γ .

All these identifications respect flag bijections. A vertex exchange has no Künneth sign here: in each summand the two degrees have opposite parity, so their product is even. Passing to the invariant parts by Set-up 2.1, tensoring with the Gysin factor $\mathbb{L}_R$, and summing therefore gives

$$\mathcal{D}_{g,n,R}^{13} \cong \bigoplus_{[\Gamma]} M_\Gamma^{A_\Gamma} \otimes \mathbb{P}_R = B_{g,n}^{(1)} \otimes \mathbb{P}_R.$$

The geometric maps. The Gysin maps are the quotient-stack pushforwards normalized as in Definition 2.7.

The stacks, clutching morphisms, forgetful morphisms, and divisor cycles are defined over $\mathbb{Q}$. The hook realizations come from the forgetful maps, the degree-thirteen realizations from their elliptic-tail Gysin images in Theorem 3.23, and the degree-two realizations from the cycle-class maps. Artin comparison respects these operations, Künneth, and finite-group invariants. It therefore identifies the Betti and étale Gysin maps and commutes with relabelling and every rational Specht projector. The algebraic maps also commute with $G_\mathbb{Q}$ in their ℓ -adic realizations.

The actual image. Assume the stated Betti image equality. For $\lambda \vdash n$, put

$$C_\lambda = \mathrm{Hom}_{\mathbb{Q}[\mathfrak{S}_n]}(V_\lambda, B_{g,n}^{(1)}), \quad m_\lambda = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}[\mathfrak{S}_n]}(V_\lambda, \mathcal{E}_{g,n}).$$

Apply $\mathrm{Hom}_{\mathbb{Q}_\ell[\mathfrak{S}_n]}(V_\lambda \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, -)$ to $\Phi_{g,n,\ell}^{15}$. Its source is

$$(C_\lambda \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \otimes \mathbb{P}_\ell.$$

This multiplicity functor is exact by Set-up 2.1. Since $\mathbb{P}_\ell$ is simple, the image in this multiplicity space is a quotient of a direct sum of copies of $\mathbb{P}_\ell$, and hence is $\mathbb{P}_\ell^{\oplus r_\lambda}$ for an integer $r_\lambda \geq 0$. Comparison with the Betti multiplicity map gives

$$2r_\lambda = 2m_\lambda.$$

Thus $r_\lambda = m_\lambda$ for every partition, and regrouping the Specht factors gives the claimed actual image isomorphism.

Surjectivity onto the target. If the Betti map is surjective, comparison gives

$$\mathrm{rank}_{\mathbb{Q}_\ell} \Phi_{g,n,\ell}^{15} = \mathrm{rank}_{\mathbb{Q}} \Phi_{g,n,B}^{15} = \dim_{\mathbb{Q}} H_B^{15}(\overline{\mathcal{M}}_{g,n}) = \dim_{\mathbb{Q}_\ell} H_\ell^{15}(\overline{\mathcal{M}}_{g,n}).$$

The ℓ -adic map is therefore surjective. $\square$

Lemma 3.27 (Row-bound calculus). *Say a representation of a symmetric group has row bound r if every Specht constituent has at least r rows.*

- (1) *Restriction fixing one flag lowers a row bound by at most one. Fixing two flags and taking their ordinary exchange invariants also lowers it by at most one. Littlewood–Richardson induction preserves the row bound of either factor. Subrepresentations, quotients, and finite-group invariants preserve row bounds.*

- (2) The hooks K_J^{11} have eleven rows. Each nonzero $U_{h,J}$ of Definition 3.22 has row bound ten, and hence, by Theorem 3.23, $H^{13}(\overline{\mathcal{M}}_{h,J})$ has row bound ten for every stable (h, J) .
- (3) For every stable (g, n) , including genus one, $B_{g,n}^{(1)}$ of Definition 3.25 has row bound nine, and so does every ordinary one-edge image in degree fifteen.

Proof. (1) Restriction of a Specht module with one flag fixed removes one corner and lowers the number of rows by at most one. With two flags, ordinary invariants under their exchange satisfy

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{S}_m} \left(V_\mu, \left(\operatorname{Res}_{\mathfrak{S}_m \times \mathfrak{S}_2}^{\mathfrak{S}_{m+2}} V_\nu \right)^{\mathfrak{S}_2} \right) &\neq 0 \\ \implies \nu/\mu &\text{ is a horizontal two-strip.} \end{aligned}$$

This is the Pieri rule. A horizontal strip meets the first column in at most one box, so taking these invariants also decreases the number of rows by at most one. Finally, every constituent of $\operatorname{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{a+b}} (V_\rho \boxtimes V_\sigma)$ contains both diagrams ρ, σ . Thus induction preserves the lower row bound of either factor. Subrepresentations and quotients preserve such bounds as well, and finite invariants are direct summands.

(2) The formulas of Definition 3.22 give at least ten rows for each nonzero $U_{h,J}$. For $h \geq 2$ they fix one flag of an eleven-row hook and induce; for $h = 1$ all the displayed hooks have ten or eleven rows.

(3) In the separating terms of Definition 3.25, fixing the node therefore leaves at least nine rows in a U factor, or at least ten in a K^{11} factor. Tensoring externally with the divisor module on the other vertex and inducing in the ordinary labels preserves these bounds. In the nonseparating term, taking ordinary invariants under the two node flags leaves at least nine rows. There is no determinant character for this exchange in the ordinary Gysin source. If a graph symmetry exchanges vertices, its invariant summand is a direct summand of the representation induced from the subgroup preserving the vertices. Such symmetries cannot introduce new Specht constituents. This proves the bound for $B_{g,n}^{(1)}$, including genus one. By Proposition 3.26, the same bound holds for every ordinary one-edge image in degree fifteen. $\square$

4. THE GENUS-ONE AND GENUS-TWO FORMULAS

Definition 4.1. For $g \geq 2$, let $\mathbb{H}_g = R^1\pi_*\mathbb{Q}$ be the weight-one variation associated with the universal smooth curve over $\mathcal{M}_g$. For a partition λ of length at most g , let $\mathbb{V}_\lambda$ be its irreducible symplectic variation, normalized to weight $|\lambda|$; for $g = 2$ we also write $\mathbb{V}_{a,b} = \mathbb{V}_{(a,b)}$ for $a \geq b \geq 0$. For $n > 0$, let $\pi_i : \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n-1}$ forget the i th marking, and put

$$\mathcal{F}_{g,n}^k = \sum_{i=1}^n \pi_i^* W_k H^k(\mathcal{M}_{g,n-1}), \quad \Psi_{g,n}^k = \sum_{i=1}^n \psi_i \mathcal{F}_{g,n}^{k-2}.$$

Set both spans to zero for $n = 0$, and write $\mathcal{O}_{g,n} = W_{15} H^{15}(\mathcal{M}_{g,n})$.

Theorem 4.2 (Deligne [Del71, Section 3.2]). *Let X be a smooth proper complex algebraic variety, let $D \subset X$ be a normal-crossings divisor, and put $U = X \setminus D$. Let $D^{[p]}$ be the normalized locus of p distinct local branches, with the orientation local system ϵ_p , and set $D^{[0]} = X$. Then the ordinary weight spectral sequence is*

$$E_1^{-p,q} = H^{q-2p}(D^{[p]}, \epsilon_p) \otimes \mathbb{L}^p \implies H^{q-p}(U).$$

Its first differential is alternating Gysin pushforward. It degenerates at E_2 , and $E_2^{-p,q} = \operatorname{Gr}_q^W H^{q-p}(U)$.

Proposition 4.3 (The primitive degree-fifteen quotient). *For $g \geq 2$ and $n \geq 0$, the restriction to the open moduli space gives a right-exact sequence*

$$B_{g,n}^{(1)} \otimes \mathbb{P} \xrightarrow{\Phi_{g,n,B}^{15}} H^{15}(\overline{\mathcal{M}}_{g,n}) \longrightarrow \mathcal{O}_{g,n} \longrightarrow 0.$$

Moreover, $W_{13}H^{13}(\mathcal{M}_{g,n}) = 0$ and

$$\mathcal{O}_{g,n}/\mathcal{F}_{g,n}^{15} \cong \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq g}} W_{15}H^{15-n}(\mathcal{M}_g, \mathbb{V}_\lambda) \otimes V_{\lambda^T}.$$

For $n \geq 15$ one has $\mathcal{O}_{g,n} = \mathcal{F}_{g,n}^{15}$. Consequently, the maps $\Phi_{g,n,B}^{15}$ are surjective for all n if and only if $W_{15}H^{15-n}(\mathcal{M}_g, \mathbb{V}_\lambda) = 0$ for all $0 \leq n \leq 14$ and all $\lambda \vdash n$ of length at most g .

Proof. The boundary quotient. Apply Theorem 4.2 to local normal-crossings charts for the boundary of $\overline{\mathcal{M}}_{g,n}$. The branch complex is invariant under changes of chart. Descending its ordered branch covers and taking the alternating part gives the orientation local system; finite stabilizer invariants are exact over $\mathbb{Q}$ (Set-up 2.1). Thus the same complex computes the stack cohomology. Its terms are pure of the indicated weight, so its higher differentials vanish as in the variety theorem. At the right end of the weight-fifteen row this gives

$$H^{13}(\widetilde{\partial\mathcal{M}}_{g,n}) \otimes \mathbb{L} \longrightarrow H^{15}(\overline{\mathcal{M}}_{g,n}) \longrightarrow W_{15}H^{15}(\mathcal{M}_{g,n}) \longrightarrow 0,$$

where the normalization is taken with its quotient-stack stabilizers. Proposition 3.26 identifies the first term and its map with $B_{g,n}^{(1)} \otimes \mathbb{P}$ and $\Phi_{g,n,B}^{15}$.

The marking quotient. By Theorem 3.23, $H^{13}(\overline{\mathcal{M}}_{g,n})$ is spanned by boundary Gysin images. These restrict to zero on the open stack, whereas restriction surjects onto its lowest-weight cohomology. Hence $W_{13}H^{13}(\mathcal{M}_{g,n}) = 0$ for every n . It follows that $\mathcal{F}_{g,n}^{13} = 0$ and $\Psi_{g,n}^{15} = \sum_i \psi_i \mathcal{F}_{g,n}^{13} = 0$. By [CLP24, Lemma 3.1], for $g \geq 2$ and $n, k \geq 0$ the quotient $W_k H^k(\mathcal{M}_{g,n})/(\mathcal{F}_{g,n}^k + \Psi_{g,n}^k)$ is $\mathfrak{S}_n$ -equivariantly isomorphic to $\bigoplus_{\lambda \vdash n, \ell(\lambda) \leq g} W_k H^{k-n}(\mathcal{M}_g, \mathbb{V}_\lambda) \otimes V_{\lambda^T}$, and $W_k H^k(\mathcal{M}_{g,n}) = \mathcal{F}_{g,n}^k$ if $n > 0$ and $n \geq k$. With $k = 15$ this gives the asserted quotient and the equality $\mathcal{O}_{g,n} = \mathcal{F}_{g,n}^{15}$ for $n \geq 15$.

The finite reduction. Surjectivity of the sum of the boundary Gysin maps in every arity is equivalent to $\mathcal{O}_{g,n} = 0$ for all n . This implies the listed coefficient vanishings by the quotient formula. Conversely, assume those vanishings. At $n = 0$ the forgetful span is zero. If $\mathcal{O}_{g,n-1} = 0$, every summand defining $\mathcal{F}_{g,n}^{15}$ is zero. The quotient formula therefore gives $\mathcal{O}_{g,n} = 0$ successively for $n \leq 14$. For $n \geq 15$ the equality with the forgetful span continues the same induction. $\square$

The function μ below is the number-theoretic Möbius function.

Definition 4.4. Work over $\widehat{\Lambda} \otimes \mathbb{Q}(\mathbb{L})$. Put

$$\mathbf{a}_0 = \frac{\prod_{r \geq 1} (1 + p_r)^{\frac{1}{r} \sum_{d|r} \mu(r/d)(1+\mathbb{L}^d)} - 1}{\mathbb{L}^3 - \mathbb{L}} - \frac{h_1}{\mathbb{L}^2 - \mathbb{L}} - \frac{h_2}{\mathbb{L} + 1}.$$

Let $\mathcal{T} = h_1 +$ terms of arity at least two be the unique series with $\mathcal{T} \circ (h_1 - \partial \mathbf{a}_0) = h_1$. For a Laurent variable w , let

$$\mathcal{P}(w) = \prod_{r \geq 1} (1 + p_r)^{\frac{1}{r} \sum_{d|r} \mu(r/d)(1-w^d - \mathbb{L}^d w^{-d} + \mathbb{L}^d)}$$

and define

$$\mathcal{C}_{12} = \operatorname{Res}_{w=0} \left[\frac{\mathcal{P}(w) - 1}{(1-w)(1-\mathbb{L}/w)} \frac{w^{10}}{\mathbb{L}^{11}} \left(w - \frac{\mathbb{L}}{w} \right) dw \right].$$

Expand the products in the p_r , with $\deg p_r = r$. At each arity take the ordinary Laurent residue at $w = 0$ over $\mathbb{Q}(\mathbb{L})$; in particular,

$$\frac{1}{1-w} = \sum_{a \geq 0} w^a, \quad \frac{1}{1-\mathbb{L}/w} = -\sum_{b \geq 1} \frac{w^b}{\mathbb{L}^b}.$$

Let $\mathfrak{a}_1$ be the arity sum of the equivariant Serre characteristics of $\mathcal{M}_{1,n}$, and for $g = 0, 1$ let $\mathfrak{b}_g$ be the corresponding sum for $\overline{\mathcal{M}}_{g,n}$, in each case over stable arities. Serre characteristics use compactly supported cohomology with sign $(-1)^i$. Write $f' = \partial f$ and $\dot{f} = \partial f / \partial p_2$, with ∂ and Ψ_r as in Section 2. For $k \geq 2$ let $\mathbf{S}_k$ denote the sum of the pure Hodge structures of weight $k-1$ attached to cusp forms for $\mathrm{SL}_2(\mathbb{Z})$; it is zero when the cusp space is zero, in particular for odd k ; thus $\mathbf{S}_{12} = \mathbf{S}_{12}$ and $\mathbf{S}_{16} = \mathbf{S}_{16}$. The operator $[\mathbf{S}_{12}]$ retains all Tate powers. Let ϕ denote Euler's totient function.

Theorem 4.5 (Getzler; correction of Consani–Faber [Getzler98, Section 3], [CF06, Section 4]). *With the arity expansions of Definition 4.4, $\mathfrak{a}_0$ is the arity sum of the equivariant Serre characteristics of $\mathcal{M}_{0,n}$, $\mathcal{T} = h_1 + \mathfrak{b}'_0$, and*

$$\begin{aligned} \mathfrak{a}_1 &= \mathrm{Res}_{w=0} \left[\frac{\mathcal{P}(w) - 1}{(1-w)(1-\mathbb{L}/w)} \left(\sum_{k \geq 1} \frac{\mathbf{S}_{2k+2} + 1}{\mathbb{L}^{2k+1}} w^{2k} - 1 \right) \left(w - \frac{\mathbb{L}}{w} \right) dw \right], \\ \mathfrak{b}_1 &= \left(\mathfrak{a}_1 - \frac{1}{2} \sum_{r \geq 1} \frac{\phi(r)}{r} \log(1 - \Psi_r(\mathfrak{a}_0'')) + \frac{\dot{\mathfrak{a}}_0^2 + \dot{\mathfrak{a}}_0 + \frac{1}{4} \Psi_2(\mathfrak{a}_0'')}{1 - \Psi_2(\mathfrak{a}_0'')} \right) \circ \mathcal{T}. \end{aligned}$$

These are identities in the arity-completed Grothendieck ring of equivariant Hodge structures.

Proposition 4.6 (Genus-one coefficient extraction). *For $n \geq 1$, the genus-one Serre series satisfies*

$$[\mathbf{S}_{12}] \mathfrak{b}_1 = \mathcal{C}_{12} \circ \mathcal{T}, \quad \mathrm{ch}(\mathcal{E}_{1,n}) = [-[\mathbb{L}^2](\mathcal{C}_{12} \circ \mathcal{T})]_n.$$

Each homogeneous arity component of the first identity is a Laurent polynomial in $\mathbb{L}$, so the second formula is finite in each arity.

Proof. In the residue of Theorem 4.5, the only term containing $\mathbf{S}_{12}$ is the $k = 5$ term

$$\mathbf{S}_{12} \mathbb{L}^{-11} w^{10}.$$

The other factors contain no cusp structures. Taking this coefficient inside the residue gives $[\mathbf{S}_{12}] \mathfrak{a}_1 = \mathcal{C}_{12}$. The remaining terms in the compactification identity are genus-zero terms. Genus-zero compact cohomology is Tate by Theorem 3.2, and the explicit $\mathfrak{a}_0$ has only Tate coefficients. Differentiation, Adams operations, products, and the displayed formal expansions preserve this property. These terms have no $\mathbf{S}_{12}$ coefficient. Since $\mathcal{T} = h_1 + \mathfrak{b}'_0$ also has Tate coefficients, the compactification identity gives

$$[\mathbf{S}_{12}] \mathfrak{b}_1 = ([\mathbf{S}_{12}] \mathfrak{a}_1) \circ \mathcal{T} = \mathcal{C}_{12} \circ \mathcal{T}.$$

The coefficient of $\mathbb{L}^2 \mathbf{S}_{12}$ in the equivariant Serre characteristic of $\overline{\mathcal{M}}_{1,n}$ comes only from H^{15} , by purity. It occurs with a negative sign in the Serre characteristic. Theorem 3.8 identifies its multiplicity module with $\mathcal{E}_{1,n}$, giving the second formula.

The residue is finite with the stated convention. In fixed arity the numerator is a Laurent polynomial in w and $\mathbb{L}$, and its negative w powers have a lower bound. Both denominator expansions in Definition 4.4 have nonnegative w powers, so only finitely many terms contribute to the coefficient of $w^{-1} dw$. Their coefficients are Laurent polynomials in $\mathbb{L}$. The series $\mathcal{T}$ has no constant term and has polynomial Tate coefficients. Composition therefore uses finitely many terms at each arity and preserves Laurent-polynomial coefficients. $\square$

Theorem 4.7 (Genus one). *With the series of Definition 4.4, one has*

$$F_1 = -[\mathbb{L}^2](\mathcal{C}_{12} \circ \mathcal{T}), \quad \text{ch}_n(\mathcal{E}_{1,n}) = [F_1]_n.$$

This module vanishes for $n < 13$ and is nonzero for $n \geq 13$. Together with the primitive term in Theorem 3.8, this gives decompositions of Hodge structures and ℓ -adic Galois representations before semisimplification in every stable genus-one arity. The ordinary one-edge Gysin image is exactly $\mathcal{E}_{1,n} \otimes \mathbb{P}$.

Proof. The coefficient assertion is Proposition 4.6.

The threshold. Equivariant Poincaré duality with $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{1,n} = n$ gives

$$H^{15}(\overline{\mathcal{M}}_{1,n}) \cong H^{2n-15}(\overline{\mathcal{M}}_{1,n})^{\vee} \otimes \mathbb{L}^n.$$

For $1 \leq n \leq 7$ the degree on the right is negative. For $8 \leq n \leq 12$ it is one of 1, 3, 5, 7, 9, so Theorem 3.3 gives zero. At $n = 13$, Proposition 3.15 gives

$$H^{11}(\overline{\mathcal{M}}_{1,13}) \cong V_{(3,1^{10})} \otimes \mathbb{S}_{12}.$$

Specht self-duality and $\mathbb{S}_{12}^{\vee} \otimes \mathbb{L}^{11} \cong \mathbb{S}_{12}$ therefore give

$$H^{15}(\overline{\mathcal{M}}_{1,13}) \cong V_{(3,1^{10})} \otimes \mathbb{P}.$$

For $n \geq 13$ the forgetful map $\pi_n : \overline{\mathcal{M}}_{1,n+1} \rightarrow \overline{\mathcal{M}}_{1,n}$ has a section, so π_n^* is injective. The nonzero $\mathbb{P}$ substructure at $n = 13$ persists under these Hodge morphisms. Theorem 3.8 and the distinct Hodge types in Notation 2.2 place it in the $\mathbb{P}$ summand.

The ordinary boundary image. The lowest-weight boundary sequence, obtained by the same branch localization used in Proposition 4.3, is $\mathcal{D}_{1,n,\mathbb{B}}^{13} \xrightarrow{\Phi_{1,n,\mathbb{B}}^{15}} H^{15}(\overline{\mathcal{M}}_{1,n}) \rightarrow W_{15}H^{15}(\mathcal{M}_{1,n}) \rightarrow 0$. By [CLP24, Proposition 2.2], $W_{15}H^{15}(\mathcal{M}_{1,n}) \cong \mathbb{S}_{16} \otimes V_{(n-14,1^{14})}$ for $n \geq 15$ and $W_{15}H^{15}(\mathcal{M}_{1,n}) = 0$ for $n < 15$; this identifies the last term with $K_n^{15} \otimes \mathbb{S}_{16}$. By Proposition 3.26 the first term is entirely of type $\mathbb{P}$. The two simple Hodge structures are nonisomorphic. Hence the map to the open term vanishes on the $\mathbb{P}$ summand and is an isomorphism on the primitive summand. Exactness proves

$$\text{im } \Phi_{1,n,\mathbb{B}}^{15} = \mathcal{E}_{1,n} \otimes \mathbb{P}.$$

The actual Galois realization. The boundary-image calculation satisfies the hypothesis of Proposition 3.26. The ℓ -adic boundary image is

$$(\mathcal{E}_{1,n} \otimes \mathbb{Q}_{\ell}) \otimes \mathbb{P}_{\ell}.$$

Lemma 8.3 of [CLP24] (recalled before Theorem 3.7) and that theorem identify the primitive summand $(K_n^{15} \otimes \mathbb{Q}_{\ell}) \otimes \mathbb{S}_{16\ell}$ via algebraic forgetful maps, with the stated hook relations. The direct sum of the primitive inclusion and the boundary Gysin map is surjective in the Betti realization by the preceding decomposition. Comparison, compatible with these maps and the relabelling action, makes its ℓ -adic realization surjective. The dimensions of its two images add to the target dimension, so their intersection is zero. Consequently

$$H_{\text{et}}^{15}(\overline{\mathcal{M}}_{1,n,\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}) \cong (\mathcal{E}_{1,n} \otimes \mathbb{Q}_{\ell}) \otimes \mathbb{P}_{\ell} \oplus (K_n^{15} \otimes \mathbb{Q}_{\ell}) \otimes \mathbb{S}_{16\ell}.$$

The Hodge decomposition is the one in Theorem 3.8. □

Theorem 4.8 (Saito [Sai90, Section 4.5]). *Let X be a smooth complex algebraic variety and let $\mathbb{V}$ be a geometric polarizable variation of pure Hodge structures of weight w . Then $H^p(X, \mathbb{V})$ has weights at least $p + w$ for every $p \geq 0$.*

Definition 4.9. Let $\mathcal{A}_2$ be the moduli stack of principally polarized abelian surfaces, and put $D = \mathcal{A}_{1,1} = [\mathcal{A}_1^2/\mathfrak{S}_2]$, whose complement is $\mathcal{M}_2$. For $a \geq b \geq 0$, the irreducible symplectic variation of highest weight (a, b) on $\mathcal{A}_2$, normalized to weight $a + b$, restricts on $\mathcal{M}_2$ to the variation $\mathbb{V}_{a,b}$ of Definition 4.1; we use the same symbol on $\mathcal{A}_2$. Let $s_k = \dim S_k(\mathrm{SL}_2(\mathbb{Z}))$, let $s_{j,k}$ be the dimension of Siegel cusp forms of type $\mathrm{Sym}^j \otimes \det^k$ for $\mathrm{Sp}_4(\mathbb{Z})$, and let $\bar{S}_{j,k}$ be the weight- $(j + 2k - 3)$ part of their attached cohomological structure.

Theorem 4.10 (Petersen [Pet15, Theorem 2.1 and Remark 2.2]). *Let $a \geq b \geq 0$, assume $(a, b) \neq (0, 0)$ and $a + b$ even. Then $H_c^i(\mathcal{A}_2, \mathbb{V}_{a,b}) = 0$ for $i \notin \{2, 3, 4\}$, and*

$$H_c^4(\mathcal{A}_2, \mathbb{V}_{a,b}) = \begin{cases} s_{a+b+4} \mathbb{L}^{b+2}, & a = b \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

The odd-weight part in degree two is $\bigoplus_{r \text{ odd}} \mathrm{Gr}_r^W H_c^2(\mathcal{A}_2, \mathbb{V}_{a,b}) \cong \mathbb{S}_{b+2}$. Moreover,

$$\bigoplus_{r \text{ odd}} \mathrm{Gr}_r^W H_c^3(\mathcal{A}_2, \mathbb{V}_{a,b}) \cong \bar{\mathbb{S}}_{a-b, b+3} \oplus \mathbb{S}_{a-b+2}^{\oplus s_{a+b+4}} \otimes \mathbb{L}^{b+1} \oplus \mathbb{S}_{a+3}.$$

The last formula is an isomorphism of weight-graded rational Hodge structures and an isomorphism after semisimplification in the ℓ -adic category.

Proposition 4.11 (Petersen [Pet13, Proposition 3.4]). *Write $V_r = \mathrm{Sym}^r \mathbb{Q}^2$ for the irreducible Sp_2 -representation. On $(\mathrm{Sp}_2 \times \mathrm{Sp}_2) \rtimes \mathfrak{S}_2$, let $U_{r,s}$ restrict to $(V_r \boxtimes V_s) \oplus (V_s \boxtimes V_r)$ with exchanged summands, and let $U_r^\pm$ restrict to $V_r \boxtimes V_r$ with positive or negative interchange of factors. Let $a \geq b \geq 0$ with $a + b$ even, and let the superscript $(-1)^b$ mean $+$ for even b and $-$ for odd b . For the irreducible Sp_4 -representation $V_{a,b}$ one has*

$$\mathrm{Res}_{\mathrm{Sp}_2 \wr \mathfrak{S}_2}^{\mathrm{Sp}_4} V_{a,b} = \bigoplus_{i=0}^b \left(U_{(a-b)/2+i}^{(-1)^b} \oplus \bigoplus_{0 \leq j < (a-b)/2} U_{a-b+i-j, i+j} \right).$$

Proposition 4.12. *For every $n \geq 0$, the complete ordinary one-edge Gysin map*

$$\Phi_{2,n,B}^{15} : \mathcal{D}_{2,n,B}^{13} \longrightarrow H^{15}(\overline{\mathcal{M}}_{2,n})$$

is surjective, and $W_{15} H^{15}(\mathcal{M}_{2,n}) = 0$.

Proof. The genus-two clauses of Proposition 3.6 give $\mathcal{O}_{2,n} = 0$ for $n \leq 10$. At $n = 12$, their compact boundary-generation statement makes $\Phi_{2,12,B}^{15}$ surjective. The right-exact sequence in Proposition 4.3 therefore gives $\mathcal{O}_{2,12} = 0$.

For an odd-weight symplectic variation on $\mathcal{M}_2$, the central hyperelliptic involution acts by -1 . Rigidification by this order-two automorphism computes its rational invariants (Set-up 2.1), which are zero. All cohomology with coefficients in such a local system therefore vanishes. The quotient in Proposition 4.3, first at $n = 11$ and then at $n = 13$, gives $\mathcal{O}_{2,11} = \mathcal{O}_{2,13} = 0$.

For $a + b = 14$, localization along D gives

$$H^1(\mathcal{A}_2, \mathbb{V}_{a,b}) \longrightarrow H^1(\mathcal{M}_2, \mathbb{V}_{a,b}) \longrightarrow H^0(D, \mathbb{V}_{a,b}|_D) \otimes \mathbb{L}.$$

The first term vanishes by Theorem 4.10 and duality from $H_c^5(\mathcal{A}_2, \mathbb{V}_{a,b}) = 0$. In Proposition 4.11, a trivial representation of the wreath product can occur only when $a = b$, in the $i = 0$ equal-factor term. Interchange acts there by $(-1)^b$, so it is invariant only for even b . Here $a = b$ would force $a = b = 7$. Thus the last term is also zero, and $H^1(\mathcal{M}_2, \mathbb{V}_{a,b}) = 0$ for every $a + b = 14$. Proposition 4.3 now gives $\mathcal{O}_{2,14} = 0$.

Proposition 4.3 (its induction from $n = 14$) now gives $\mathcal{O}_{2,n} = 0$ and surjectivity in every arity. $\square$

Proposition 4.13. *For $D = \mathcal{A}_{1,1} \subset \mathcal{A}_2$, the localization Gysin map*

$$\mathrm{Gr}_{15}^W(H^1(D, \mathbb{V}_{11,1}|_D) \otimes \mathbb{L}) \longrightarrow \mathrm{Gr}_{15}^W H^3(\mathcal{A}_2, \mathbb{V}_{11,1})$$

is an isomorphism. Both its source and target are isomorphic to $\mathbb{P}$.

Proof. A sign-projected configuration space. Let $f : \mathcal{M}_{2,12} \rightarrow \mathcal{M}_{2,1}$ retain marking 12, let $\rho : \mathcal{M}_{2,1} \rightarrow \mathcal{M}_2$ be the universal proper curve, and put $\mathbb{V} = \rho^* \mathbb{H}_2$. The fiber of f over (C, p) is the ordered configuration space of eleven points on $C \setminus \{p\}$. Filter its complement in $(C \setminus \{p\})^{11}$ by collision partitions. Each proper collision partition has a block of size at least two, and a transposition in that block acts trivially on the stratum. Its sign-isotypic compactly supported cohomology is therefore zero. Localization for this finite stratification, followed by relative duality, identifies the sign parts for the configuration space and the product. This argument is functorial in (C, p) .

Since $H^\bullet(C \setminus \{p\}) = \mathbb{Q} \oplus H^1(C)[-1]$, the Koszul action gives

$$\mathrm{Hom}_{\mathfrak{S}_{11}}(\mathrm{sgn}_{11}, R^q f_* \mathbb{Q}) = \begin{cases} \mathrm{Sym}^{10} \mathbb{V}, & q = 10, \\ \mathrm{Sym}^{11} \mathbb{V}, & q = 11, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, the alternating projection permits at most one degree-zero factor and symmetrizes the odd factors.

In the sign-isotypic part of the Leray spectral sequence, the term $E_2^{4,11} = H^4(\mathcal{M}_{2,1}, \mathrm{Sym}^{11} \mathbb{V})$ has no incoming differential. Its only possible outgoing differential has target $H^6(\mathcal{M}_{2,1}, \mathrm{Sym}^{10} \mathbb{V})$. Theorem 4.8 puts the target in weights at least sixteen. To apply the variety statement, take a finite level cover of the smooth moduli stack and then finite-group invariants (Set-up 2.1). Strictness makes the differential zero on weight fifteen. Thus the weight-fifteen part of $E_2^{4,11}$ survives. It is a subquotient of $W_{15} H^{15}(\mathcal{M}_{2,12})$, which is zero by Proposition 4.12. Hence

$$\mathrm{Gr}_{15}^W H^4(\mathcal{M}_{2,1}, \mathrm{Sym}^{11} \mathbb{V}) = 0.$$

The (11, 1) coefficient. For ρ , the direct-image rows have coefficients $\mathrm{Sym}^{11} \mathbb{H}_2$, $\mathrm{Sym}^{11} \mathbb{H}_2 \otimes \mathbb{H}_2$, and $\mathbb{L} \mathrm{Sym}^{11} \mathbb{H}_2$. The hyperelliptic involution acts by minus the identity on the first and last, so their base cohomology is zero. Leray therefore identifies

$$H^4(\mathcal{M}_{2,1}, \mathrm{Sym}^{11} \mathbb{V}) \cong H^3(\mathcal{M}_2, \mathrm{Sym}^{11} \mathbb{H}_2 \otimes \mathbb{H}_2).$$

Symplectic Pieri gives the decomposition of variations

$$\mathrm{Sym}^{11} \mathbb{H}_2 \otimes \mathbb{H}_2 = \mathbb{V}_{12,0} \oplus \mathbb{V}_{11,1} \oplus \mathbb{L} \mathbb{V}_{10,0}.$$

All summands have weight twelve. It follows that $\mathrm{Gr}_{15}^W H^3(\mathcal{M}_2, \mathbb{V}_{11,1}) = 0$. Localization now makes the map in the statement surjective.

The two rank-two objects. In the restriction of $\mathbb{V}_{11,1}$ to D , only the unequal-factor term with one trivial factor contributes to H^1 . Here $H^1(\mathcal{A}_1, \mathbb{Q}) = 0$, since the coarse space is the j -line (Set-up 2.1). Proposition 4.11 gives that term as $\mathbb{L}$ times the two exchanged coefficients of degrees $(10, 0)$. The Tate factor restores the weight of the variation. By Eichler–Shimura [Pet13, Section 5], for even $r > 0$ and the standard weight-one variation $\mathbb{H}_1$ on $\mathcal{A}_1$, $H^i(\mathcal{A}_1, \mathrm{Sym}^r \mathbb{H}_1) = 0$ for $i \neq 1$, and $H^1(\mathcal{A}_1, \mathrm{Sym}^r \mathbb{H}_1)$ has weights $r + 1$ and $2r + 2$, with $\mathrm{Gr}_{r+1}^W H^1(\mathcal{A}_1, \mathrm{Sym}^r \mathbb{H}_1) = \mathbb{S}_{r+2}$ and Eisenstein weight- $(2r + 2)$ piece. With $r = 10$ this identifies the weight-fifteen part after the localization twist with $\mathbb{L}^2 \mathbb{S}_{12} = \mathbb{P}$.

Theorem 4.10 exhibits $s_{16} \mathbb{S}_{12} \otimes \mathbb{L}^2 = \mathbb{P}$ in $\mathrm{Gr}_{15}^W H_c^3(\mathcal{A}_2, \mathbb{V}_{11,1})$; here $s_{16} = 1$. Duality on the three-dimensional stack has twist $\mathbb{L}^{15}$ for this weight-twelve coefficient, so it exhibits $\mathbb{P}^\vee \otimes \mathbb{L}^{15} \cong \mathbb{P}$

in the target of our map. The target thus has dimension at least two, and surjectivity from $\mathbf{P}$ makes its dimension at most two. The map is an isomorphism. $\square$

Definition 4.14. For a stable (g, n) , let

$$G_p^{15}(g, n) = \bigoplus_{[\Gamma], |E(\Gamma)|=p} \left(H^{15-2p}(X_\Gamma) \otimes \det(\mathbb{Q}^{E(\Gamma)}) \right)^{A_\Gamma} \otimes \mathbb{L}^p.$$

The face map d_p^{face} is alternating Gysin contraction of distinct ordered branches, descended by quotient-stack pushforward. For an incidence stabilizer map $H \rightarrow K$, its transfer on a pushed forward class z is $|H|^{-1} \sum_{k \in K} k^* z$. For a rational $\mathfrak{S}_d$ -module W and $n \geq d$, write $I_n(W)$ for the induced module $\text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}^{\mathfrak{S}_n}(W \boxtimes \mathbf{1})$ of Definition 3.22, and put $I_n(W) = 0$ for $n < d$.

Proposition 4.15 (The genus-two ordinary row). *Let $n \geq 0$. Then $G_p^{15}(2, n) = 0$ for $p \geq 3$, and the row*

$$G_2^{15}(2, n) \xrightarrow{d_2^{\text{face}}} G_1^{15}(2, n) \xrightarrow{\Phi_{2,n,B}^{15}} H^{15}(\overline{\mathcal{M}}_{2,n}) \longrightarrow 0$$

has canonical homology identifications

$$\begin{aligned} R_n &= \ker d_2^{\text{face}} \cong \text{Gr}_{15}^W H^{13}(\mathcal{M}_{2,n}) \cong \left(I_n(V_{(2,1^8)}) \oplus I_n(V_{(3,1^8)}) \right) \otimes \mathbf{P}, \\ Q_n &= \ker \Phi_{2,n,B}^{15} / \text{im } d_2^{\text{face}} \cong \text{Gr}_{15}^W H^{14}(\mathcal{M}_{2,n}) \cong \bigwedge^{11} \mathbb{Q}^n \otimes \mathbf{P}. \end{aligned}$$

The first decomposition into induced modules need not be canonical. The map d_2^{face} is injective exactly for $n \leq 9$; the row is exact at $G_1^{15}(2, n)$ exactly for $n \leq 10$.

Proof. The ordinary boundary row. Apply Theorem 4.2 with the branch descent described in Proposition 4.3. Ordering the branches gives the determinant of the edge set. On quotient charts, summing over the ordered branches gives the finite transfer in Definition 4.14. Thus its weight-fifteen row is exactly the complex in the statement. For $p \geq 3$, a nonnegative degree $15 - 2p$ is one of 9, 7, 5, 3, 1. Every odd Künneth factor of such a degree vanishes by Theorem 3.3; negative degrees vanish as well. Hence $G_p^{15}(2, n) = 0$ for $p \geq 3$. Degeneration at E_2 identifies the kernel and middle homology with

$$\text{Gr}_{15}^W H^{13}(\mathcal{M}_{2,n}), \quad \text{Gr}_{15}^W H^{14}(\mathcal{M}_{2,n}),$$

respectively. The last map is surjective by Proposition 4.12.

Localization along the collision locus. The calculation uses cohomology with supports, since the collision divisors need not have normal crossings. Let $\mathcal{C}_2 \rightarrow \mathcal{M}_2$ be the universal proper curve, and put $Y_n = \mathcal{C}_2^n$, with fiber products over $\mathcal{M}_2$ and $Y_0 = \mathcal{M}_2$. The open substack $\mathcal{M}_{2,n} \subset Y_n$ is the locus of pairwise distinct points. Write $\Delta_{ij} \cong Y_{n-1}$ for a collision divisor and $Z_n = \bigcup_{i < j} \Delta_{ij}$. A collision stratum is indexed by a partition of $[n]$ into k blocks; it is $\mathcal{M}_{2,k}$ of codimension $c = n - k$.

In genus two, $W_m H^m(\mathcal{M}_{2,k}) = 0$ for every odd $m \leq 13$. For $m \leq 9$ this follows from Theorem 3.3, for $m = 11$ from Proposition 3.15, and for $m = 13$ from Theorem 3.23 and restriction. The same ordinary boundary sequence gives

$$\begin{aligned} \text{Gr}_{m+2}^W H^m(\mathcal{M}_{2,k}) &= 0, \quad m \leq 11 \text{ odd}, \\ \text{Gr}_{m+1}^W H^m(\mathcal{M}_{2,k}) &= 0, \quad m \leq 10 \text{ even}. \end{aligned}$$

Indeed, the possible E_1 terms are respectively in column -2 with compact degree $m - 2$, and column -1 with compact degree $m - 1$. Their odd degrees are at most nine.

Purity for a smooth collision stratum expresses its contribution to degree- r cohomology with supports as $H^{r-2c}(\mathcal{M}_{2,k}) \otimes \mathbb{L}^c$. Filter a union of strata by codimension and use the localization exact sequences. The preceding vanishings give

$$\mathrm{Gr}_{15}^W H_{Z_n}^{15}(Y_n) = \mathrm{Gr}_{15}^W H_{Z_n}^{13}(Y_n) = 0.$$

For a union of strata of codimension at least two they also give vanishing of cohomology with supports in degree fourteen and weight fifteen. The Mayer–Vietoris sequence for cohomology with supports, applied one divisor at a time, consequently gives

$$\mathrm{Gr}_{15}^W H_{Z_n}^{14}(Y_n) = \bigoplus_{i < j} \mathrm{Gr}_{13}^W H^{12}(\Delta_{ij}) \otimes \mathbb{L}.$$

Localization is now the exact sequence

$$(4.1) \quad \begin{aligned} 0 &\longrightarrow \mathrm{Gr}_{15}^W H^{13}(Y_n) \longrightarrow R_n \\ &\longrightarrow \bigoplus_{i < j} \mathrm{Gr}_{13}^W H^{12}(Y_{n-1}) \otimes \mathbb{L} \\ &\longrightarrow \mathrm{Gr}_{15}^W H^{14}(Y_n) \longrightarrow Q_n \longrightarrow 0. \end{aligned}$$

Cohomology on the unmarked base. The rigidification of $\mathcal{M}_2$ by the hyperelliptic involution is $[\mathcal{M}_{0,6}/\mathfrak{S}_6]$. Odd-weight coefficients have zero cohomology (proof of Proposition 4.12); for even coefficients rational cohomology is the $\mathfrak{S}_6$ -invariant part. The three forgetful fibrations

$$\mathcal{M}_{0,6} \longrightarrow \mathcal{M}_{0,5} \longrightarrow \mathcal{M}_{0,4} \longrightarrow \mathcal{M}_{0,3}$$

have punctured projective lines as fibers. Each fiber has cohomological dimension one for local systems, so the base has cohomological dimension at most three. After fixing three points at $0, 1, \infty$, $\mathcal{M}_{0,6}$ is the complement of $x_i = 0$, $x_i = 1$, and $x_i = x_j$ in affine three-space. Deletion and restriction of these hyperplanes, whose intersections are affine spaces, show that its cohomology is Hodge–Tate. The same holds for the trivial coefficient on $\mathcal{M}_2$.

For later use, we compute the relevant groups for $\mathbb{L}^t \mathbb{V}_{a,b}$, where $t \geq 0$ and $w = a + b + 2t$. For $w = 10$ or 12 one has

$$(4.2) \quad \mathrm{Gr}_{w+3}^W H^2(\mathcal{M}_2, \mathbb{L}^t \mathbb{V}_{a,b}) = \begin{cases} \mathrm{LS}_{12}, & (a, b, t) = (10, 0, 0), \\ \mathrm{P}, & (a, b, t) = (10, 0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Here and below equalities between cohomology objects mean isomorphisms. To prove this formula, first suppose $(a, b) \neq (0, 0)$. By Theorem 4.10 and duality, $H^2(\mathcal{A}_2, \mathbb{V}_{a,b})$ is zero unless $a = b$ is even; in that case it is $s_{2a+4} \mathbb{L}^{a+1}$. After the twist it has weight $w + 2$, not $w + 3$. In $H^1(D, \mathbb{V}_{a,b}|_D)$, Proposition 4.11 and Eichler–Shimura (as recalled in the proof of Proposition 4.13), together with Künneth, leave only the unequal-factor term of degrees $(a - b, 0)$, with twist $\mathbb{L}^b$. Terms with two positive symmetric degrees have zero H^0 on either factor, and the equal trivial term has $H^1(\mathcal{A}_1) = 0$. After the localization twist, the cusp contribution is

$$\mathbb{L}^{t+b+1} \mathcal{S}_{a-b+2},$$

of weight $w + 3$. The Eisenstein contribution has even weight. In the indicated range, a nonzero cusp contribution requires $a - b = 10$: the cusp spaces below weight twelve and in weight fourteen vanish. Thus only $(a, b) = (10, 0)$ and $(11, 1)$ remain. For $(10, 0)$ the weight-thirteen part of $H^3(\mathcal{A}_2, \mathbb{V}_{10,0})$ is zero: the scalar coefficient is $s_{14} = 0$, the odd elliptic coefficient is $\mathcal{S}_{13} = 0$, and $s_{10,3} = 0$ since by [Pet15, Section 2.1] the level-one Siegel cusp dimensions satisfy $\sum_{j \geq 0} s_{j,3} x^j = x^{36} / ((1 - x^6)(1 - x^8)(1 - x^{10})(1 - x^{12}))$. The same conclusion after a Tate twist shows that this boundary cusp contribution lies in the kernel of the localization Gysin map. For

(11, 1) that kernel vanishes by the isomorphism in Proposition 4.13. This proves (4.2); the trivial coefficient was treated separately above.

Two further degrees enter the calculation of R_n . For $w = 12$,

$$\mathrm{Gr}_{15}^W H^1(\mathcal{M}_2, \mathbb{L}^t \mathbb{V}_{a,b}) = 0.$$

For nontrivial coefficients, $H^1(\mathcal{A}_2, \mathbb{V}_{a,b}) = 0$ by duality from $H_c^5 = 0$; localization embeds H^1 of the complement in $H^0(D, \mathbb{L}^{t+1} \mathbb{V}_{a,b})$, which is pure of weight fourteen. The trivial case is Hodge–Tate. For $w = 10$ one has

$$(4.3) \quad \mathrm{Gr}_{15}^W H^3(\mathcal{M}_2, \mathbb{L}^t \mathbb{V}_{a,b}) = \begin{cases} \mathbb{P}, & (a, b, t) = (9, 1, 0), \\ 0, & \text{otherwise.} \end{cases}$$

In the localization sequence the $H^1(D)$ cusp contribution, including the Gysin twist, has weight thirteen. Its Eisenstein contribution has even weight. In $H^2(D)$, a cusp would require one symmetric degree at least ten. Since the total symmetric degree is at most ten, the other factor would be trivial and have zero H^1 . All remaining contributions have even weight. Thus the weight-fifteen group equals that on $\mathcal{A}_2$. Set $u = a + b = 10 - 2t$. Duality identifies the desired part with weight $u + 1$ in $H_c^3(\mathcal{A}_2, \mathbb{V}_{a,b})$, followed by duality and twist $\mathbb{L}^{u+3+t}$. The Siegel and endoscopic terms in Theorem 4.10 have weight $u + 3$. The term $\mathbb{S}_{a+3}$ has weight $a + 2 = u + 1$ exactly when $b = 1$. Then $a = 9 - 2t$, and $\mathbb{S}_{12-2t}$ is nonzero only for $t = 0$. It gives $\mathbb{S}_{12}^\vee \otimes \mathbb{L}^{13} = \mathbb{P}$, proving (4.3).

The universal-curve powers. The direct image $R^q(Y_n \rightarrow \mathcal{M}_2)_* \mathbb{Q}$ is pure of weight q . It is a direct sum of tensor products of $\mathbb{H}_2$ and $\mathbb{L}$, and its irreducible summands have the form just considered. Odd q gives zero base cohomology by the involution. The cohomological dimension bound leaves only base degree two in total degrees twelve and fourteen at weights thirteen and fifteen, respectively. A base-degree-zero term has the weight of its fiber degree and does not contribute. These degree-two terms survive Leray: an incoming d_2 has an odd fiber row, higher incoming differentials have negative base degree, and outgoing differentials exceed degree three on the base. For total degree thirteen and weight fifteen, the base-degree-one term vanishes by the preceding calculation, leaving base degree three and fiber degree ten. Its possible incoming d_2 has an odd row; an incoming d_3 originates in base degree zero and pure weight twelve, so is zero at weight fifteen.

Write $\mathbb{Q}^m$ with its permutation basis indexed by $[m]$. Formula (4.2) gives

$$\mathrm{Gr}_{13}^W H^{12}(Y_m) = \bigwedge^{10} \mathbb{Q}^m \otimes \mathbb{L} \mathbb{S}_{12}.$$

The ten degree-one factors are symmetrized as symplectic tensors. Their Koszul action makes their marking action alternating. Formula (4.3) similarly gives

$$\mathrm{Gr}_{15}^W H^{13}(Y_n) = I_n(V_{(2,1^8)}) \otimes \mathbb{P}.$$

Indeed, $\mathbb{V}_{9,1}$ first occurs in tensor degree ten, with ordinary Schur–Weyl multiplicity $V_{(9,1)}$; tensoring by the Koszul sign conjugates this partition.

Let M_n be the multiplicity space in $\mathrm{Gr}_{15}^W H^{14}(Y_n) = M_n \otimes \mathbb{P}$. There are two types of generators. For an oriented ten-subset T and $i \notin T$, let $B_{T;i}$ denote ten degree-one factors and the degree-two class in slot i . For an unordered pair $\{i, j\}$ disjoint from T , let $A_{T;ij}$ denote twelve degree-one factors, with the symplectic tensor inserted in slots i, j . These generators are independent and exhaustive. To check the latter assertion for twelve degree-one slots, the multiplicity of $\mathbb{L} \mathrm{Sym}^{10} \mathbb{H}_2$ is $\binom{12}{2}$. In the symplectic tensor rule, a path from the empty partition to (10) in twelve moves has eleven additions and one removal. There are eleven paths staying in one row. The other paths add a second-row box at a time i and remove it at a time j , with $2 \leq i < j \leq 12$, giving $\binom{11}{2}$ paths. A third row cannot be removed in the single removal. The total is $11 + 55 = 66$. The

corresponding 66 insertion maps are independent: in a symplectic basis x, y, z, w , the coefficient with z, w in a prescribed pair and x in every other slot detects just that pair. Before the Koszul sign their permutation module is induced from $\mathbf{1}_{10} \boxtimes \text{sgn}_2$; after it, the module is induced from $\text{sgn}_{10} \boxtimes \mathbf{1}_2$. This is precisely the orientation and symmetry specified for A .

The collision Gysin map. Normalize the fiber point class ν_i by $\int_C \nu_i = 1$. Put $\Omega_{ij} = [\Delta_{ij}] - \nu_i - \nu_j$. The two diagonal identities are

$$\Delta_* 1 = \nu_i + \nu_j + \Omega_{ij}, \quad \Delta_* h = h_i \nu_j + \nu_i h_j \quad (h \in H^1(C)).$$

Choose the insertion generators A using Ω_{ij} . These identities determine the middle map in (4.1), since the Leray pieces identified above are its only weight-fifteen pieces.

A source generator is an oriented ten-subset of the labels after merging i, j to a label $*$. If it excludes $*$, its image is

$$A_{T;ij} + B_{T;i} + B_{T;j}.$$

If it contains $*$, put $*$ last and write T for the preceding oriented nine-subset. Its image is

$$B_{(T,i);j} + B_{(T,j);i}.$$

The two signs agree because the point class has even degree. Each A occurs once in the first family, with coefficient one. Eliminating all these generators leaves the second family of relations among the B 's. The map

$$B_{T;i} \mapsto \left(\bigwedge_{t \in T} e_t \right) \wedge e_i$$

annihilates these relations. For a fixed eleven-element support the relations identify the eleven distinguished-slot choices with their exterior signs, leaving one alternating line. Hence

$$Q_n \cong \bigwedge^{11} \mathbb{Q}^n \otimes \mathbb{P}.$$

The kernel can be read from the same presentation. The projection of the first source family onto the A generators is an isomorphism, so a kernel vector uses only the second family. On each eleven-element support its source and target characters are

$$e_9 h_2 = s_{(3,1^8)} + s_{(2,1^9)}, \quad e_{10} h_1 = s_{(2,1^9)} + e_{11}.$$

The cokernel is the alternating line just found. Thus the image is $V_{(2,1^9)}$ and the kernel is $V_{(3,1^8)}$. Inducing over the support gives $I_n(V_{(3,1^8)}) \otimes \mathbb{P}$. The left term of (4.1) contributes $I_n(V_{(2,1^8)}) \otimes \mathbb{P}$. Pure polarizable Hodge structures are semisimple, and averaging over $\mathfrak{S}_n$ makes a splitting equivariant (Set-up 2.1). Consequently R_n is their direct sum, with no preferred splitting.

Its left homology first occurs on ten markings, and its middle homology first occurs on eleven. Both induced modules remain nonzero in every larger arity. This proves the two exactness thresholds. $\square$

Lemma 4.16. *Let B be a finite set and let $q \notin B$. Put $\lambda_1 = \delta_{\text{irr}}/12$ on $\overline{\mathcal{M}}_{1,B \cup \{q\}}$. Then λ_1 and the classes $\delta_{0,T}$ for $T \subseteq B \cup \{q\}$, $|T| \geq 2$, form a basis of $H^2(\overline{\mathcal{M}}_{1,B \cup \{q\}}; \mathbb{Q})$. As B varies, the Frobenius characteristic of this permutation basis is*

$$E + EE_{\geq 2} + EE_{\geq 1} = E(2E - 1 - h_1).$$

Proof. Proposition 3.18 expresses all generators in Theorem 3.17 in the displayed classes. We prove independence by showing that $H^1(\mathcal{M}_{1,B \cup \{q\}}; \mathbb{Q}) = 0$.

Let $\mathcal{C}_1 \rightarrow \mathcal{M}_{1,1}$ be the universal elliptic curve with origin q , and let $m = |B|$. Inside $\mathcal{C}_1^m$, the open moduli stack removes the zero sections in each slot and all pair diagonals. The base has zero rational H^1 (as at Proposition 4.13). The invariant part of $R^1(\mathcal{C}_1^m \rightarrow \mathcal{M}_{1,1})_* \mathbb{Q}$ is zero: inversion acts by -1 . Leray gives $H^1(\mathcal{C}_1^m) = 0$. Intersections of distinct removed divisors have codimension

at least two. The localization exact sequence therefore injects H^1 of the complement into the span of the divisor classes, with image equal to the kernel of their map to $H^2(C_1^m)$. These classes are independent already after restriction to one smooth fiber C^m . The zero sections give the independent point classes ν_i . Each diagonal has, in addition to $\nu_i + \nu_j$, a nonzero component in the distinct Künneth summand $H^1(C)_i \otimes H^1(C)_j$. These components first force every diagonal coefficient to vanish, and the point classes then force the remaining coefficients to vanish. Thus the open H^1 is zero.

Localization for $\mathcal{M}_{1,B \cup \{q\}} \subset \overline{\mathcal{M}}_{1,B \cup \{q\}}$ now injects the span of its irreducible boundary divisors into compact H^2 . They are δ_{irr} and the listed $\delta_{0,T}$, proving independence. The basis element λ_1 contributes E . A tail not containing q contributes $EE_{\geq 2}$; a tail containing q contributes $EE_{\geq 1}$. Their sum is the stated characteristic. $\square$

Proposition 4.17. *Let $D_+ = \frac{1}{2}\partial^2 + \partial/\partial p_2$. Let B_2 and C_2 be the arity sums of the rational multiplicity characters of $G_1^{15}(2, n)$ and $G_2^{15}(2, n)$ after removing $\mathbf{P}$. Then*

$$\begin{aligned} B_2 &= D_+U + E_{\geq 2}\partial(EH) + E\partial U + HE(2E - 1 - h_1), \\ C_2 &= (2H + G)EE_{\geq 2} + HEE_{\geq 1} + HE + (H + G)E_{\geq 2} + GE_{\geq 1}, \\ \sum_{n \geq 0} \text{ch}(\mathcal{E}_{2,n}) &= B_2 - C_2 + E(s_{(2,1^8)} + s_{(3,1^8)}) - Ee_{11}. \end{aligned}$$

Proof. One edge. Proposition 3.26 requires both Künneth families in the degree-thirteen source. A nonseparating genus-one normalization contributes D_+U : the two gluing flags are unordered and invariant, so their operator is $h_2^\perp = \frac{1}{2}\partial^2 + \partial/\partial p_2 = D_+$. A separating divisor of genera zero and two contributes $E_{\geq 2}\partial(EH)$, since the rational vertex needs at least two ordinary markings. For a divisor of genera one and one, the $H^{13} \otimes H^0$ family contributes $E\partial U$. The $H^{11} \otimes H^2$ family contributes $HE(2E - 1 - h_1)$ by Lemma 4.16. Here H is the hook module with one distinguished attaching flag, by Proposition 3.15. The odd decoration distinguishes the two elliptic vertices, so no further symmetric-square operation occurs. This gives B_2 .

Two edges. The compact degree is eleven. By Proposition 3.15, precisely one vertex has genus one and carries H^{11} ; all other factors are units. Let

$$\mathcal{K}^{11} = \sum_{m \geq 11} s_{(m-10, 1^{10})}.$$

Restriction of the exterior hook gives

$$\partial\mathcal{K}^{11} = H, \quad \partial H = H + G, \quad h_2^\perp\mathcal{K}^{11} = H, \quad e_2^\perp\mathcal{K}^{11} = G.$$

First consider a path with three vertices. If the decorated vertex is central, the remaining genera are zero and one, giving $(H + G)EE_{\geq 2}$. If it is an endpoint, a genus-one central vertex and a rational endpoint give $HEE_{\geq 2}$; a rational central vertex and a genus-one endpoint give $HE_{\geq 1}E$. Together these are

$$(2H + G)EE_{\geq 2} + HEE_{\geq 1}.$$

For a loop joined by a bridge, a loop on the rational unit vertex gives HE . That vertex has three flags and needs no ordinary marking. A loop on the decorated elliptic vertex gives $(H + G)E_{\geq 2}$. The flip of a loop fixes its edge and hence acts trivially on the edge determinant; its two half-edges take invariants. Finally, two parallel edges between the decorated elliptic vertex and a rational vertex contribute $GE_{\geq 1}$. The rational vertex needs one ordinary marking, and interchanging the two edges acts by -1 on their determinant, so the hook takes anti-invariants. A single vertex with two loops has genus zero and has no odd cohomology. This exhausts the connected two-edge graphs of total genus two, and gives C_2 .

The Euler characteristic. For a surjection at the right end,

$$[H^{15}(\overline{\mathcal{M}}_{2,n})] = [G_1^{15}(2, n)] - [G_2^{15}(2, n)] + [R_n] - [Q_n].$$

All four source and homology terms have common Hodge factor P . Proposition 4.15 gives their left and middle homology multiplicity series, respectively, as

$$E(s_{(2,1^8)} + s_{(3,1^8)}), \quad Ee_{11}.$$

Taking Frobenius characteristics proves the asserted formula. $\square$

Theorem 4.18 (Genus two). *With the series of Definition 2.5,*

$$F_2 = E(1 + E)s_{(2,1^{10})} + E^2(e_{11} + s_{(2,1^9)}) + H(E_{\geq 1} + (E + 1)E_{\geq 3} + 2EE_{\geq 2}) + EGE_{\geq 3}.$$

The homogeneous degree- n part is $\text{ch}(\mathcal{E}_{2,n})$, it is nonzero exactly for $n \geq 11$, and it gives the Hodge and ℓ -adic realizations before Galois semisimplification.

Proof. A positive expression. Put $Z = s_{(2,1^{10})}$, $V = s_{(2,1^9)}$, $X = s_{(2,1^8)}$, and $Y = s_{(3,1^8)}$ for this calculation. Definition 2.5 and Pieri's rule give $U = EZ + E_{\geq 3}H$, $H = Ee_{10}$, and $G = Ee_9$. First use $\partial(EH) = E(2H + G)$ in Proposition 4.17. Cancellation gives

$$F_2 = D_+U + E\partial U + H(E - 1)E_{\geq 2} - G(E_{\geq 2} + E_{\geq 1}) + E(X + Y - e_{11}).$$

The necessary differentiation identities are

$$\begin{aligned} \partial Z &= e_{11} + V, & D_+Z &= e_{10}, \\ D_+E &= E, & \partial E &= E, & D_+H &= H + G, \\ \partial E_{\geq 3} &= E_{\geq 2}, & D_+E_{\geq 3} &= E_{\geq 1}. \end{aligned}$$

They follow by deleting one or two boxes, with horizontal two-strips for $D_+ = h_2^\perp$. Its product rule is $D_+(fg) = (D_+f)g + (\partial f)(\partial g) + fD_+g$. Consequently

$$\begin{aligned} D_+U &= E(Z + e_{11} + V + e_{10}) + H(E_{\geq 1} + E_{\geq 2} + E_{\geq 3}) + G(E_{\geq 2} + E_{\geq 3}), \\ E\partial U &= E^2(Z + e_{11} + V) + EE_{\geq 2}H + EE_{\geq 3}(H + G). \end{aligned}$$

Substitution makes the H coefficient

$$E_{\geq 1} + (E + 1)E_{\geq 3} + 2EE_{\geq 2}$$

and the remaining G coefficient $(E + 1)E_{\geq 3} - E_{\geq 1}$. The terms not yet included in these two groups are

$$E(Z + V + e_{10} + X + Y) + E^2(Z + e_{11} + V).$$

Pieri gives $e_9h_1 = e_{10} + X$ and $e_9h_2 = V + Y$. Thus

$$GE_{\geq 1} = E(e_{10} + X + V + Y) + GE_{\geq 3}.$$

The threshold. Every factor in that expression is Schur-positive. The series H starts in degree ten, G in degree nine, and Z in degree twelve. It follows that every term starts in degree at least eleven. The degree-eleven part is $2(e_{11} + V)$, so it is nonzero. For every $n \geq 11$, the summand E^2e_{11} contains $e_{11}h_{n-11}$, a nonzero Schur-positive character. Hence no later arity vanishes.

The actual realizations. The Hodge decomposition follows from Theorem 3.8 and the multiplicity character just computed. Proposition 4.12 proves that the ordinary boundary image is the whole Betti target. Proposition 3.26 therefore identifies its actual ℓ -adic image with $(\mathcal{E}_{2,n} \otimes \mathbb{Q}_\ell) \otimes P_\ell$ and makes this image the whole target. $\square$

5. BOUNDARY GENERATION IN GENUS THREE

We prove that ordinary boundary Gysin classes generate genus-three cohomology in degree fifteen. Reduction in the number of markings leaves twelve symplectic local systems of weight thirteen. The corresponding degree-two obstruction groups inject into cohomology on the elliptic Satake stratum, where restriction to a proper collision curve detects the relevant hypercohomology with its full derived coefficients.

Proposition 5.1 (Bergström–Faber–Payne [BFP22, Proposition 2.1]). *For every stable (g, n) with $g \geq 1$,*

$$H_c^k(\mathcal{M}_{g,n}) = 0 \quad \text{if} \quad \begin{cases} k < 2g, & n = 0, 1, \\ k < 2g - 2 + n, & n \geq 2. \end{cases}$$

Definition 5.2. For a stable (g, F) and $r = 0, 1, 2$, put

$$C_{g,F}^r = \bigoplus_{[\Gamma], |E(\Gamma)|=r} \left(H^{15}(X_\Gamma) \otimes \det(\mathbb{Q}^{E(\Gamma)}) \right)^{\text{Aut}_F(\Gamma)}.$$

Let $\alpha : C_{g,F}^0 \rightarrow C_{g,F}^1$ be boundary pullback. On a labelled two-edge graph with ordered edges (e_1, e_2) , set

$$(\beta x)_\Gamma = (\xi_{e_1}^* x_{\Gamma/e_1} - \xi_{e_2}^* x_{\Gamma/e_2}) \otimes (e_1 \wedge e_2).$$

Here $\xi_e : X_\Gamma \rightarrow X_{\Gamma/e}$ glues the flags of e . Let C_{sep}^1 be the direct sum of the separating one-edge summands. For $g \geq 2$, choose new flags c, d , set $t = (cd)$, and let $\xi : \overline{\mathcal{M}}_{g-1, F \cup \{c, d\}} \rightarrow \overline{\mathcal{M}}_{g,F}$ glue them. Put

$$N_{g,F} = H^{15}(\overline{\mathcal{M}}_{g-1, F \cup \{c, d\}})^{(t)}, \quad \mathcal{Q}_{g,F} = C_{g,F}^2 / \beta(C_{\text{sep}}^1), \quad \delta(x) = [\beta(0, x)].$$

Thus $C_{g,F}^1 = C_{\text{sep}}^1 \oplus N_{g,F}$ and $\delta : N_{g,F} \rightarrow \mathcal{Q}_{g,F}$.

Proposition 5.3 (The compact restriction row). *The maps of Definition 5.2 satisfy*

$$\beta\alpha = 0, \quad \ker \alpha \cong \text{Gr}_{15}^W H_c^{15}(\mathcal{M}_{g,F}), \quad \ker \beta / \text{im } \alpha \cong \text{Gr}_{15}^W H_c^{16}(\mathcal{M}_{g,F}).$$

For $g \geq 1$, the map α is injective if $15 < 2g$ when $|F| = 0, 1$, or if $15 < 2g - 2 + |F|$ when $|F| \geq 2$.

Proof. Dualizing the spectral sequence in Theorem 4.2 by Poincaré duality gives

$$E_1^{r,q} = \bigoplus_{[\Gamma], |E(\Gamma)|=r} \left(H^q(X_\Gamma) \otimes \det(\mathbb{Q}^{E(\Gamma)}) \right)^{\text{Aut}_F(\Gamma)} \implies H_c^{q+r}(\mathcal{M}_{g,F}).$$

Indeed X_Γ has dimension $3g - 3 + |F| - r$; dualizing its ordinary Gysin term cancels its codimension twist against the dualizing twists of X_Γ and $\overline{\mathcal{M}}_{g,F}$. Thus this restriction row has no Tate twist. Labelled étale node charts give the same normal-crossings resolution on the stack: the orientation of a set of branches is precisely its edge determinant. Taking finite graph invariants is exact, so the spectral sequences on these charts descend to the displayed spectral sequence on the stack.

In weight fifteen the first two differentials are exactly α and β . On a two-edge chart the two composite restrictions agree, whereas their exterior orientations have opposite signs. This proves $\beta\alpha = 0$. Row cohomology in degrees zero and one gives the two displayed identifications. Finally Proposition 5.1, applied with $k = 15$, gives the claimed injectivity inequalities. $\square$

Lemma 5.4 (Low-degree symplectic coefficients). *Let $d \geq 1$, $g \geq d + 2$, and let λ be a nonempty partition of length at most g with $\lambda_1 > d$. Then*

$$H^d(\mathcal{M}_g, \mathbb{V}_\lambda) = 0.$$

Proof. Let Γ_g be the mapping class group of a closed oriented genus- g surface, let I_g be its Torelli group, and denote the rational fibre representation of $\mathbb{V}_\lambda$ by $\mathbf{V}_\lambda$. The extension $1 \rightarrow I_g \rightarrow \Gamma_g \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \rightarrow 1$ gives the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{a,q} = H^a(\mathrm{Sp}_{2g}(\mathbb{Z}); H^q(I_g; \mathbb{Q}) \otimes \mathbf{V}_\lambda) \Rightarrow H^{a+q}(\Gamma_g; \mathbf{V}_\lambda).$$

For $a + q = d$ one has $q \leq d \leq g - 2$, so by Gaifullin [Gai26, Theorems A and B] the homology $H_q(I_g; \mathbb{Q})$ is a finite-dimensional algebraic $\mathrm{Sp}_{2g}(\mathbb{Z})$ -representation, the restriction of a rational representation of Sp_{2g} defined over $\mathbb{Q}$, isomorphic to a sum of irreducibles $\mathbf{V}_\mu$ with $\ell(\mu) \leq g$ and $\mu_1 \leq q$, the empty partition allowed. Universal coefficients and self-duality of symplectic irreducibles give the same finite algebraic decomposition of $H^q(I_g; \mathbb{Q})$, with first parts at most q . None equals λ , including the empty partition at $q = 0$. Zariski density therefore gives

$$(H^q(I_g; \mathbb{Q}) \otimes \mathbf{V}_\lambda)^{\mathrm{Sp}_{2g}(\mathbb{Z})} = \mathrm{Hom}_{\mathrm{Sp}_{2g}}(\mathbf{V}_\lambda^\vee, H^q(I_g; \mathbb{Q})) = 0.$$

Complete reducibility and Borel’s theorem [MPPRW24, Theorem 4.6], that $H^k(\Gamma, V) = 0$ for $0 \leq k < g$ when V is a nontrivial irreducible real algebraic representation of Sp_{2g} and $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Q})$ an arithmetic subgroup, imply the vanishing of $E_2^{a,q}$: rational-to-real scalar extension is faithful, $\mathrm{Sp}_{2g}(\mathbb{Z})$ is an arithmetic subgroup, and the theorem applies since $0 \leq a \leq d \leq g - 2 < g$. Every term of total degree d is zero, so the finite filtration on the abutment gives $H^d(\Gamma_g; \mathbf{V}_\lambda) = 0$, without requiring degeneration. Contractibility of Teichmüller space identifies this group with $H^d(\mathcal{M}_g, \mathbb{V}_\lambda)$. $\square$

Lemma 5.5. *Let X be a smooth complex Deligne–Mumford stack of finite type, and let $Z \subset X$ be a smooth closed substack of pure codimension c .*

- (1) *Let $c \geq 3$, and let $\mathcal{V} \subset H^{15}(X)$ be a pure sub-Hodge structure with Hodge types contained in $\{(13, 2), (2, 13)\}$. Then restriction is injective on $\mathcal{V}$:*

$$\mathcal{V} \longrightarrow H^{15}(X \setminus Z).$$

- (2) *If $W_{15-2c}H^{15-2c}(Z) = 0$, then restriction*

$$W_{15}H^{15}(X) \longrightarrow W_{15}H^{15}(X \setminus Z)$$

is injective.

Proof. The localization sequence contains

$$H^{15-2c}(Z) \otimes \mathbb{L}^c \longrightarrow H^{15}(X) \longrightarrow H^{15}(X \setminus Z).$$

- (1) The mixed Hodge structure on $H^{15-2c}(Z)$ is effective. After tensoring with $\mathbb{L}^c$, both Hodge indices are at least c . Strictness of the Hodge and weight filtrations gives the same bound for every subquotient of the first term. Its image has zero intersection with $\mathcal{V}$, since each allowed type of $\mathcal{V}$ has an index equal to two.

- (2) The preceding term has weight-fifteen part $W_{15-2c}H^{15-2c}(Z) \otimes \mathbb{L}^c = 0$. Taking weight fifteen in localization, strictness of weights makes restriction injective in weight fifteen. $\square$

Proposition 5.6 (The open locus with twelve markings). *Let $U_{12} = \mathcal{M}_{3,12} \setminus \mathrm{Hyp}_{3,12}$, and let*

$$Z_{12} = \{(C, p_1, \dots, p_{12}) \in U_{12} : p_1 + \dots + p_{12} \sim 3K_C\}.$$

Then Z_{12} is smooth of codimension three in U_{12} , and $U_{12} \setminus Z_{12}$ has the Chow–Künneth generation property. Moreover,

$$\mathcal{O}_{3,n} = 0 \quad (0 \leq n \leq 12).$$

Proof. The exceptional locus. On the nonhyperelliptic locus the canonical model is a smooth plane quartic and $\omega_C \simeq \mathcal{O}_C(1)$. For $D = \sum p_i$, restriction of quartic equations and $H^1(C, \omega_C^4) = 0$ identify the evaluation cokernel with

$$H^1(C, \omega_C^4(-D)) \cong H^0(C, \mathcal{O}_C(D) \otimes \omega_C^{-3})^\vee.$$

The latter degree-zero line bundle has a section exactly when it is trivial: a nonzero section has an effective divisor of degree zero and is nowhere vanishing. Thus the locus where evaluation is not surjective is precisely Z_{12} . For the universal nonhyperelliptic curve π_C , Riemann–Roch gives $h^0(C, 3K_C) = 10$ and $h^1(C, 3K_C) = 0$. Consequently $\pi_{C*}\omega_{\pi_C}^3$ is a rank-ten vector bundle. The open locus of simple divisors in its projectivization, followed by the finite étale cover ordering the zeros, is Z_{12} . Scheme-theoretically this is the fibre of the relative Abel map over the section $3K_C$: in degree twelve that Abel map is a projective bundle. The separated relative Picard space makes the condition closed. The base is smooth, so Z_{12} is smooth of codimension $12 - 9 = 3$.

Chow–Künneth generation on the complement. Let $\mathcal{V}$ be the rank-three bundle on BSL_3 and set $B_{\text{ev}} = (\mathbb{P}\mathcal{V})_{BSL_3}^{12}$. On the open locus where evaluation is surjective, the map

$$\text{Sym}^4 \mathcal{V}^\vee \longrightarrow \bigoplus_{r=1}^{12} \eta_r^* \mathcal{O}_{\mathbb{P}\mathcal{V}}(4)$$

has a rank-three kernel. The open substack of its projectivization where the quartic is smooth and the marked points distinct is a μ_3 -gerbe over $U_{12} \setminus Z_{12}$. Projective bundles and open restriction preserve the Chow–Künneth generation property, by Proposition 3.5. The map from the special-linear presentation to the projective-linear one is a finite μ_3 -gerbe, so Canning–Larson [CL22, Lemma 3.6] (finite gerbes preserve the property) implies that $U_{12} \setminus Z_{12}$ has the Chow–Künneth generation property.

Lowest weight. The smooth hyperelliptic divisor has the same property by Canning–Larson [CL22, Theorem 6.2], by which $\mathcal{H}_{g,n}^{\text{irr}} \subset \overline{\mathcal{M}}_{g,n}$, parametrizing smooth or irreducible nodal hyperelliptic curves, has the Chow–Künneth generation property for $g \geq 2$ and $0 \leq n \leq 2g + 6$, and the odd lowest-weight vanishing in Proposition 3.5, whose compactification hypothesis holds because the open substacks under consideration admit smooth proper compactifications obtained by resolving their boundaries without modifying those substacks, gives $W_{13}H^{13}(\text{Hyp}_{3,12}) = 0$ and $W_{15}H^{15}(U_{12} \setminus Z_{12}) = 0$. Lemma 5.5(2), applied with $c = 1$, injects $\mathcal{O}_{3,12}$ into $W_{15}H^{15}(U_{12})$. Proposition 4.3 and Theorem 3.8 identify $\mathcal{O}_{3,12}$ as a pure quotient with only types (13, 2) and (2, 13), so Lemma 5.5(1), applied with $c = 3$, makes restriction to $U_{12} \setminus Z_{12}$ injective on the injected quotient. Its target in weight fifteen is zero, so $\mathcal{O}_{3,12} = 0$. For $n \leq 11$, Proposition 3.6 gives the assertion directly. $\square$

Definition 5.7. Set

$$\begin{aligned} \mathcal{R}_3 = \{ & (13), (12, 1), (11, 2), (11, 1, 1), (10, 3), (10, 2, 1), \\ & (9, 4), (9, 3, 1), (9, 2, 2), (8, 5), (8, 4, 1), (8, 3, 2) \}. \end{aligned}$$

For $\lambda \in \mathcal{R}_3$, put $\mathcal{Q}_\lambda = W_{15}H^2(\mathcal{M}_3, \mathbb{V}_\lambda)$.

Proposition 5.8 (Reduction to twelve local systems). *There is a canonical $\mathfrak{S}_{13}$ -equivariant isomorphism*

$$\mathcal{O}_{3,13} \cong \bigoplus_{\lambda \in \mathcal{R}_3} \mathcal{Q}_\lambda \otimes V_{\lambda^T}.$$

For a finite set F and $J \subseteq F$, let π_J forget the markings outside J . For every F with $|F| \geq 14$, the map

$$\bigoplus_{\substack{J \subseteq F \\ |J|=13}} \mathcal{O}_{3,J} \longrightarrow \mathcal{O}_{3,F}, \quad (x_J)_J \longmapsto \sum_J \pi_J^* x_J$$

is surjective. The sum of the ordinary one-edge Gysin maps is surjective in all arities if and only if $\mathcal{Q}_\lambda = 0$ for every $\lambda \in \mathcal{R}_3$.

Proof. Nine rows in the ordinary source. We use the row bounds of Lemma 3.27. By Lemma 3.27(3), the complete ordinary source $B_{3,n}^{(1)} \otimes \mathbb{P}$ of Proposition 3.26 has row bound nine in every arity.

Theorems 4.7 and 4.18 give that bound for compact H^{15} in genera one and two. In genus one the remaining S_{16} hook has fifteen rows. For genus three with at most twelve flags, Proposition 5.6 and Proposition 4.3 give surjectivity from the source with row bound nine, hence the same bound. Genus-zero H^{15} is zero by Theorem 3.8.

Eight rows at thirteen markings. Proposition 5.3 makes $H^{15}(\overline{\mathcal{M}}_{3,13}) \rightarrow C_{3,[13]}^1$ injective since $15 < 4 + 13$. Its nonseparating term is $H^{15}(\overline{\mathcal{M}}_{2,15})^{\oplus 2}$ and has row bound eight, by Lemma 3.27(1). For a separating term, Theorem 3.3 leaves exactly

$$(15, 0), (0, 15), (13, 2), (2, 13), (11, 4), (4, 11).$$

The first two have bound eight after fixing the node whenever the odd factor has genus one or two. If its genus is three, the rational factor has $b \geq 2$ external labels, so the odd factor has $14 - b \leq 12$ flags and the preceding bound applies. An odd rational factor contributes zero. In the next two sectors H^{13} has row bound ten before fixing its node, hence nine afterward. In the last two, H^{11} is elliptic and has ten rows after fixing its node, irrespective of the degree-four factor. Induction and graph invariants preserve these bounds. Thus the entire restriction target, and its compact subspace, have at least eight rows.

The primitive quotient and higher markings. By Proposition 4.3, $\mathcal{O}_{3,13}$ is a quotient of this compact space. The span of the images of pullbacks forgetting one marking is zero because $\mathcal{O}_{3,12} = 0$. The same proposition gives

$$\mathcal{O}_{3,13} \cong \bigoplus_{\lambda \vdash 13, \ell(\lambda) \leq 3} W_{15} H^2(\mathcal{M}_3, \mathbb{V}_\lambda) \otimes V_{\lambda^T}.$$

Since $\ell(\lambda^T) = \lambda_1$, the bound excludes $\lambda_1 \leq 7$. Writing the remaining parts in decreasing order gives precisely the twelve partitions in $\mathcal{R}_3$.

At fourteen markings, every partition of length at most three has first part at least five. Lemma 5.4 with $(g, d) = (3, 1)$ gives vanishing of its H^1 coefficient, so the primitive quotient is zero. At arities at least fifteen it is zero by Proposition 4.3. Iterating the one-marking forgetful maps proves the stated surjection from thirteen-element subsets. Vanishing of the twelve groups therefore implies vanishing of every $\mathcal{O}_{3,n}$, including the lower arities already treated. Conversely, surjectivity of the sum of the Gysin maps implies vanishing of $\mathcal{O}_{3,13}$; its direct sum decomposition forces each coefficient to vanish. $\square$

Definition 5.9. Let $\mathcal{A}_g$ be the stack of principally polarized abelian varieties, with coarse space A_g and Satake compactification A_g^{Sat} . Put

$$Y = A_3^{\text{Sat}}, \quad A = A_3, \quad B = A_2^{\text{Sat}}, \quad Z = A_1^{\text{Sat}},$$

and denote the inclusions by $k : A \hookrightarrow Y$, $i : B \hookrightarrow Y$, $z : Z \hookrightarrow B$, and $v : A_2 \hookrightarrow B$. Write $o = Z \setminus A_1$. Let $\mathcal{N} = \mathcal{M}_3 \setminus \text{Hyp}_3$, let $U \subset A$ be its coarse Torelli image, and write $p : \mathcal{N} \rightarrow U$ and $j : U \hookrightarrow A$.

Let $f : \mathcal{M}_3^{\text{ct}} \rightarrow \mathcal{A}_3$ be compact-type Torelli, $q : \mathcal{A}_3 \rightarrow A$ the coarse map, $a : \mathcal{M}_3^{\text{ct}} \hookrightarrow \overline{\mathcal{M}}_3$ the open immersion, and $t : \overline{\mathcal{M}}_3 \rightarrow Y$ extended Torelli. Use $\mathbb{V}_\lambda$ also for the weight-thirteen abelian

variation and its pullbacks. For $\lambda \in \mathcal{R}_3$, set

$$\begin{aligned} G_\lambda &= j_! p_*(\mathbb{V}_\lambda|_{\mathcal{N}}), & C_\lambda &= i^* Rk_* G_\lambda, \\ F_\lambda &= q_* R^1 f_* f^* \mathbb{V}_\lambda, & E_\lambda &= i^* Rk_* F_\lambda, \\ M_\lambda &= i^* Rt_* Ra_* \mathbb{V}_\lambda, & L_\lambda &= \mathcal{H}^1(C_\lambda)|_{A_1}. \end{aligned}$$

Coarse direct image means exact rational direct image, including finite stabilizer invariants. Fibre weights on a smooth stack stratum are the weights of the corresponding geometric variation before this coarse direct image.

Proposition 5.10 (The Torelli correction). *For every $\lambda \in \mathcal{R}_3$, one has*

$$H^0(A, F_\lambda) = H^1(A, F_\lambda) = 0.$$

There is a canonical distinguished triangle

$$C_\lambda \longrightarrow M_\lambda \longrightarrow E_\lambda[-1] \xrightarrow{+1}.$$

The natural maps give isomorphisms of mixed Hodge structures

$$H^2(A, G_\lambda) \cong H^2(\mathcal{M}_3^{\text{ct}}, \mathbb{V}_\lambda) \cong \mathbb{H}^2(B, C_\lambda),$$

and restriction induces

$$W_{15} H^2(\mathcal{M}_3^{\text{ct}}, \mathbb{V}_\lambda) \cong \mathcal{Q}_\lambda.$$

Proof. The fibres and their normalization. The Torelli morphism extends regularly from $\mathcal{M}_3$ to the second Voronoi compactification, and its composite with the Satake morphism is the extended Satake Torelli map [AB12, Theorem 3.4]; thus t is proper, and its base change over $\mathcal{A}_3$ is f . The Satake image of a stable curve is the product of the Jacobians of its positive-genus normalization components. Thus this base change is exactly the compact-type locus. Every indecomposable principally polarized abelian threefold is the Jacobian of a smooth projective curve of genus three [Hai02, Corollary 7]; with Torelli this identifies the coarse indecomposable locus with smooth curve moduli: the proper quasi-finite coarse map is finite and birational onto the normal indecomposable locus, hence an isomorphism. In the framed fibre a nonhyperelliptic curve has two points exchanged by the central abelian involution; a hyperelliptic curve has one.

Over $E_{\text{att}} \times J(C)$ with C smooth of genus two, the framed fibre is C : attach E_{att} at a moving point of C , fixing its own attachment by translation. The Jacobian framing retains that point, rather than its quotient by the hyperelliptic involution. Over an ordered product $E_1 E_2 E_3$, the fibre is $E_1 \cup E_2 \cup E_3$ with their origins identified. On the middle component E_r , the two attachments have a difference in $E_r \setminus \{0\}$; at zero, the common stable limit is the rational trivalent component with three elliptic tails. These exhaust the graphs: a stable tree with three genus-one vertices has at most one rational vertex, necessarily that trivalent vertex. The three proper attachment maps are a proper bijection from this union to the reduced fibre and hence a homeomorphism.

Let $\mathcal{D}_3 \subset \mathcal{A}_3$ be the decomposable locus. Its normalization is $\nu_{12} : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathcal{D}_3$. Let $\iota_{111} : \mathcal{T}_{111} = \mathcal{A}_1^3 / \mathfrak{S}_3 \hookrightarrow \mathcal{D}_3$ be the triple locus, and let $s_{111} : \mathcal{T}'_{111} = \mathcal{A}_1^3 / \mathfrak{S}_{\{2,3\}} \rightarrow \mathcal{T}_{111}$ distinguish its first factor. In framed Siegel coordinates there are exactly three normalization branches at the triple locus, one per distinguished elliptic factor. Repeated factors take the corresponding finite permutation quotient. Write H_r for the standard system of the r th elliptic factor, and H_{23} for that of $\mathcal{A}_2$. Define

$$\begin{aligned} \mathcal{K}_{12} &= \mathbb{V}_\lambda(H_1 \oplus H_{23}) \otimes H_{23}, \\ \mathcal{K}_{111} &= \mathbb{V}_\lambda(H_1 \oplus H_2 \oplus H_3) \otimes (H_1 \oplus H_2 \oplus H_3) \otimes \text{sgn}_{\mathfrak{S}_3}, \end{aligned}$$

where the second system is specified on the ordered cover, and let $\mathcal{K}'_{111}$ be the restriction of $\mathcal{K}_{12}$ to $\mathcal{T}'_{111}$.

The sign comes from the attachment coordinates. Choose on middle component r the attachment to $r + 2$ minus that to $r + 1$, cyclically. A transposition reverses that difference, acting by -1 on first cohomology. Attach the distinguished elliptic factor to the universal genus-two compact-type curve on the normalization. At its node, stable marking inserts the rational component described above. This proper family gives the three degree-one pullbacks

$$P_1(v_1, v_2, v_3) = (v_2, -v_3), \quad P_2(v_1, v_2, v_3) = (-v_1, v_3), \quad P_3(v_1, v_2, v_3) = (v_1, -v_2).$$

For example, on branch one the moving coordinate on E_2 agrees with the cyclic difference and that on E_3 has the opposite sign. Cyclic permutation gives the remaining formulas. A transposition acts by the signed permutation on the source and the usual branch exchange on the targets, so these are equivariant maps. Proper base change consequently gives the exact normalization sequence

$$(5.1) \quad 0 \longrightarrow R^1 f_* f^* \mathbb{V}_\lambda \longrightarrow \nu_{12*} \mathcal{K}_{12} \oplus \iota_{111*} \mathcal{K}_{111} \longrightarrow \iota_{111*} s_{111*} \mathcal{K}'_{111} \longrightarrow 0.$$

The last map is normalization restriction minus the three P_i . Off the triple locus the first arrow is an isomorphism onto the normalization term. At a triple point the normalization part maps identically onto the three branch spaces; its kernel is the graph of the three P_i on the actual fibre. This verifies exactness, including finite stabilizers (Set-up 2.1).

Degree zero and degree one of the correction. Write $\lambda = (a_1, a_2, a_3)$. There is no Sp_2^3 constituent of $\mathbb{V}_\lambda$ equal to one standard factor with the others trivial. Indeed, by the symplectic branching rule [Yac10, Corollary 3.3], $\mathrm{Hom}_{\mathrm{Sp}_{2m-2}}(\mathbf{V}_\mu^{(m-1)}, \mathbf{V}_\lambda^{(m)})$ is zero unless $\lambda_i \geq \mu_i \geq \lambda_{i+2}$ for $1 \leq i \leq m-1$ (double interlacing, with $\lambda_{m+1} = 0$), and it is then the representation $\bigotimes_{i=1}^m \mathrm{Sym}^{x_i - y_i} \mathbb{C}^2$ of the commuting Sp_2 , where $x_1 \geq y_1 \geq \dots \geq x_m \geq y_m$ sorts the entries of $(\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_{m-1}, 0)$. Thus the Sp_4 weights admitting Sp_2^2 invariants are (u, u) . Double interlacing requires $a_3 \leq u \leq a_2$, and the remaining rank-one multiplicity is $\mathrm{Sym}^{a_1 - a_2} \otimes \mathrm{Sym}^{a_3}$. Every constituent has highest weight at least $a_1 - a_2 - a_3 = 2a_1 - 13 \geq 3$, excluding the standard weight one. Zariski density implies the vanishing of $H^0(\mathcal{T}'_{111}, \mathcal{K}'_{111})$ and $H^0(\mathcal{T}_{111}, \mathcal{K}_{111})$. A normalization invariant would also be invariant after restriction to Sp_2^3 , so it is zero. Sequence (5.1) proves $H^0(A, F_\lambda) = 0$.

For any finite algebraic coefficient $\mathcal{E}$ on $\mathcal{A}_2$, Borel's theorem [MPPRW24, Theorem 4.6], that for an arithmetic subgroup $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Q})$ and $0 \leq k < g$ one has $H^k(\Gamma, V) = 0$ for every nontrivial irreducible real algebraic representation V of Sp_{2g} , while for trivial coefficients in this range the cohomology is the degree- k part of the polynomial algebra on generators of degrees $2, 6, 10, \dots$, gives $H^1(\mathcal{A}_2, \mathcal{E}) = 0$: decompose semisimply, use its nontrivial-coefficient assertion in degree $1 < 2$, and observe that the trivial-coefficient algebra has no degree-one generator. Rational-to-real scalar extension is faithful. On each normalization branch, Künneth therefore places degree-one cohomology only in the distinguished elliptic factor. Its restriction R_i to the triple cover is injective, since its remaining degree-zero map is $\mathcal{E}^{\mathrm{Sp}_4} \hookrightarrow \mathcal{E}^{\mathrm{Sp}_2^2}$.

The degree-one sequence of (5.1) injects $H^1(A, F_\lambda)$ into the kernel of the matching map. On the ordered cover this map has equations $P_i x - R_i b_i = 0$. Split x into its three Künneth cohomological directions $x^{[r]}$. For $i \neq r$, the $[r]$ equation says $P_i x^{[r]} = 0$. The two kernels in question are distinct direct coefficient summands, indexed by H_i , and intersect trivially. Thus every $x^{[r]} = 0$, and injectivity of the R_i then gives every $b_i = 0$. Exact permutation invariants prove $H^1(A, F_\lambda) = 0$ with the signed matching retained.

The truncation triangle. The central abelian involution acts as $(-1)^{13} = -1$ on $\mathbb{V}_\lambda$. On each connected Torelli fibre it is trivial on H^0 and H^2 : on C it is the hyperelliptic involution, and on each elliptic parameter curve it is inversion. Thus coarse invariants vanish in degree zero off U and in degree two everywhere. The fibres have dimension at most one. On U , coarse degree zero is $p_* \mathbb{V}_\lambda$; adjunction from extension by zero identifies the degree-zero sheaf with G_λ . Consequently

ordinary truncation gives

$$(5.2) \quad G_\lambda \longrightarrow R(qf)_* \mathbb{V}_\lambda \longrightarrow F_\lambda[-1] \xrightarrow{+1}.$$

Its degree-two sequence and the two established vanishings give $H^2(A, G_\lambda) \cong H^2(\mathcal{M}_3^{\text{ct}}, \mathbb{V}_\lambda)$. Since $ta = kqf$, applying $i^* Rk_*$ gives the triangle of the statement.

Comparison with the Satake boundary and with smooth curves. The coarse nonhyperelliptic locus U is affine: it is the discriminant-nonzero open in the projective GIT quotient of plane quartics. A full symplectic level cover is a smooth affine sixfold finite over U . On it the local system shifted by six is perverse; Artin vanishing [dC12, Theorem 2.0.1], [BBD82, Theorem 4.1.1 and Corollary 4.1.2] gives $H_c^p = 0$ for $p < 6$. Finite descent yields $H_c^r(U, p_* \mathbb{V}_\lambda) = 0$ for $r < 6$. Localization on proper Y has adjacent terms $H_c^2(U, -)$ and $H_c^3(U, -)$, hence identifies $H^2(A, G_\lambda)$ with $\mathbb{H}^2(B, C_\lambda)$.

Finally the complement of $\mathcal{M}_3$ in $\mathcal{M}_3^{\text{ct}}$ is a normal-crossings divisor whose normalization is $\mathcal{M}_{1,1} \times \mathcal{M}_{2,1}^{\text{ct}}$. Its coefficient has no global sections: on the dense smooth product, monodromy is Zariski dense in $\text{Sp}_2 \times \text{Sp}_4$, containing the odd central involution. Thus its degree-two support cohomology vanishes. In degree-three support cohomology the possible one-branch term is $H^1(-, \mathbb{V}_\lambda)(-1)$, of weights at least $13 + 1 + 2 = 16$ by Theorem 4.8; the two-branch term is $H^0(-, \mathbb{V}_\lambda)(-2)$, of weights at least seventeen. An r -branch intersection first contributes in support degree $2r - (r - 1) = r + 1$, so no higher intersections intervene. Localization and strictness give the asserted isomorphism on W_{15} . Every comparison above is induced by algebraic adjunction, proper direct image, or localization, and therefore respects the geometric mixed Hodge structures. $\square$

Proposition 5.11 (Localization along the Satake strata). *For $\lambda \in \mathcal{R}_3$, one has*

$$\mathcal{H}^0(C_\lambda) = 0, \quad \mathbb{H}_c^2(A_2, v^* C_\lambda) = 0.$$

Every ordinary cohomology sheaf of C_λ , M_λ , and E_λ , restricted to any of

$$\mathcal{M}_2^{\text{coarse}}, \quad (\mathcal{A}_1^2 / \mathfrak{S}_2)^{\text{coarse}}, \quad A_1,$$

is the exact coarse direct image of a local system carrying a geometric mixed Hodge variation on the indicated smooth stack. The same holds for kernels and cokernels of their cohomology-sheaf maps. All cohomology groups of the derived cusp stalk $(C_\lambda)_o$ are mixed Tate. In particular, restriction and extension from the elliptic stratum give

$$\mathbb{H}^2(B, C_\lambda) \hookrightarrow \mathbb{H}^2(Z, z^* C_\lambda),$$

and, for every odd integer b and every r ,

$$\text{Gr}_b^W \mathbb{H}_c^r(A_1, C_\lambda|_{A_1}) \xrightarrow{\sim} \text{Gr}_b^W \mathbb{H}^r(Z, z^* C_\lambda).$$

Proof. Degree-zero extension. The closure of the hyperelliptic Jacobian divisor contains B . For a smooth genus-two curve, identify two distinct conjugate non-Weierstrass points. The resulting hyperelliptic genus-three curve is a limit obtained by colliding two branch points of a double cover, and its Satake image is the genus-two Jacobian. These images are dense in B .

Choose a connected normal analytic representative V_b near $b \in B$. Removing a nowhere-dense analytic subset leaves it connected (Riemann extension on normal spaces [Sae92, Theorem 0.2, p. 336]). Apply this to the complement of the automorphism-free locus $U^\circ \subset U$, including the closure of the extra-inertia locus and $Y \setminus U$. Thus $V_b \cap U^\circ$ is connected. A section of G_λ on $V_b \cap A$ vanishes near a hyperelliptic point, since its stalk there is zero. That neighborhood meets U° , so local constancy forces the section to vanish on all of $V_b \cap U^\circ$. Finite uniformizing covers show that $p_* \mathbb{V}_\lambda$ has no section supported on the extra-automorphism locus: its lifted local-system section vanishes on a dense open. Hence the original section is zero. Taking germs proves $\mathcal{H}^0(C_\lambda) = 0$. These complexes have no negative ordinary degrees; the correction triangle also gives $\mathcal{H}^0(M_\lambda) = 0$.

The proper polar model. On a labelled étale deformation chart of $\overline{\mathcal{M}}_3$, write t_e for the independent smoothing parameters of all nodes. The compact-type open is obtained by requiring $t_e \neq 0$ for every nonbridge edge. Contracting other edges preserves a surviving nonbridge cycle, whereas a bridge remains separating. Bridge parameters are therefore ordinary disks, including zero. Blow up the noncompact-type divisor in real oriented directions: $t_e = r_e \exp(i\theta_e)$ for the excluded parameters. The resulting map $\rho : \overline{\mathcal{M}}_3^{\log} \rightarrow \overline{\mathcal{M}}_3$ is proper, and node relabellings and units glue these charts. The variation extends to a local system $\tilde{\mathbb{V}}_\lambda$ on this space, since small angular-interval charts have contractible interiors. Consequently

$$(5.3) \quad Ra_* \mathbb{V}_\lambda = R\rho_* \tilde{\mathbb{V}}_\lambda.$$

Over a graph its complex is the full Koszul complex of the commuting operators $T_e - 1$. Each nonbridge contributes its own exterior generator, even if two coedges coincide.

We verify the hypotheses of Thom's first isotopy lemma [Mat70, Proposition 11.1] (properness on a Whitney-stratified subset, submersivity on each stratum) for $t\rho$ over each of the three listed smooth stack strata, ordering elliptic factors when necessary. The retained normalization components are respectively one genus-two curve, two elliptic curves, and one elliptic curve; all others are rational. A graph stratum is a product of distinct attachment configurations on these curves and rational pointed moduli. Forgetting the attachments is smooth, and freely varying the retained curves supplies all base tangent directions. At collisions, rational insertions give further graph strata with independent node parameters, so the same surjectivity holds. Allowed contractions are exactly those preserving the indicated normalization type. They impose coordinate-vanishing conditions on node parameters and no equations on the free moduli of retained curves.

The graph strata are thus normal-crossings coordinate arrangements times smooth pointed-component deformations. Their coordinate strata are Whitney; after polar blowup they are products with radial faces and circles, and every face still maps submersively. Embed the radial orthants in real coordinate spaces to apply the isotopy lemma. Properness holds because both t and ρ are proper. In finite quotient charts average the lifting fields over stabilizers; the linear lifting and tangency conditions are preserved. The resulting local trivializations preserve all graph and face strata. On a contractible base neighborhood the local system on this product pulls back from the fibre. Proper base change in (5.3) therefore proves the assertion for every cohomology sheaf of M_λ .

For E_λ , apply the same construction to the normalization sequence (5.1). The proper models of its three terms are

$$\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_2 \longrightarrow Y, \quad \overline{\mathcal{M}}_{1,1}^3 \longrightarrow Y,$$

with the respective tensor coefficients $\mathcal{K}_{12}, \mathcal{K}_{111}, \mathcal{K}'_{111}$ defined in the proof of Proposition 5.10. Take their noncompact-type polar boundaries before permutation invariants. Retained curves distribute among the factors in finitely many ways, each with the same smooth attachment and coordinate-face description. Their proper direct images are locally constant on the stack strata. The matching long exact sequence gives the assertion for E_λ , and the correction triangle gives it for C_λ . Kernels and cokernels of local-system maps are local systems. Since all original operations are geometric mixed-Hodge operations, these systems carry their geometric variations, with the same assertion for weight subobjects and for the finite stabilizer invariants taken by coarse direct image (Set-up 2.1).

Vanishing on A_2 . The indecomposable genus-two stratum has a smooth finite affine level cover of dimension three: genus-two curves are double covers branched at six points, and $\mathcal{M}_{0,6}$ is affine. The product-elliptic stratum has a smooth finite affine cover of dimension two. On a smooth affine d -fold a local system shifted by d is perverse; Artin vanishing [dC12, Theorem 2.0.1] gives $H_c^p = 0$ for $p < d$. On either stratum R , M_λ has no cohomology sheaves below degree one and

E_λ has none below degree zero. The compact-support spectral sequences therefore give

$$\mathbb{H}_c^2(R, M_\lambda) = 0, \quad \mathbb{H}_c^0(R, E_\lambda) = 0.$$

The correction triangle gives $\mathbb{H}_c^2(R, C_\lambda) = 0$. Localization between these two strata yields $\mathbb{H}_c^2(A_2, v^*C_\lambda) = 0$.

Mixed Tate cohomology at the cusp. For a product of the proper curve spaces just used, its Satake cusp fibre has only rational normalization components. A labelled graph stratum is a product $\prod_v \mathcal{M}_{0,F(v)}$. The normalization sequence for the limiting standard coefficient on a graph Γ gives

$$\mathrm{Gr}_0^W = H^1(\Gamma; \mathbb{Q}), \quad \mathrm{Gr}_1^W = 0, \quad \mathrm{Gr}_2^W = H_1(\Gamma; \mathbb{Q})(-1).$$

These are constant on the labelled stratum. Tensor constructions, duals, and direct summands have finite filtrations by constant Tate systems. Their full link complex is

$$\Psi\mathbb{V} \longrightarrow \bigoplus_{e \text{ nonbridge}} \Psi\mathbb{V}(-1) \longrightarrow \bigoplus_{\{e,e'\} \text{ nonbridge}} \Psi\mathbb{V}(-2) \longrightarrow \cdots,$$

with alternating logarithms N_e . The substitution $T_e - 1 = N_e(1 + N_e/2! + \cdots)$ is invertible, so this computes the full torus cochains. Every node is retained independently. Each N_e has type $(-1, -1)$, and the complex has constant Tate graded coefficients. Choosing three rational flags as $0, 1, \infty$ trivializes the marked tangent lines on each labelled stratum. Changes of frames act through the nilpotent operators on the extensions and trivially on these graded pieces.

Both ordinary and compactly supported cohomology of $\mathcal{M}_{0,m}$ are mixed Tate. To verify this directly, forget one point after fixing $0, 1, \infty$: the fibre is a punctured projective line, with constant Tate H^1 generated by its labelled punctures and their sum relation. Leray, induction, and smooth duality prove the assertion. The coefficient filtration, finite graph invariants, and localization over the finite stratification by stable graphs now show that the proper cusp-fibre hypercohomology is mixed Tate. Apply this to $\overline{\mathcal{M}}_3$, $\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_2$, and $\overline{\mathcal{M}}_{1,1}^3$. Proper base change and the full matching sequence prove it for $(M_\lambda)_o$ and $(E_\lambda)_o$, and the correction triangle proves it for $(C_\lambda)_o$.

Finally, the localization triangle for $A_2 \subset B$ and the degree-two vanishing proved above give the asserted injection into Z . In the localization sequence for $A_1 \subset Z$, both adjacent cusp cohomology groups have only even weights. Taking any odd Gr_b^W gives the stated isomorphism. $\square$

Proposition 5.12 (The elliptic localization injection). *For each $\lambda \in \mathcal{R}_3$, the localization maps induce a canonical injection of pure weight-fifteen Hodge structures*

$$\mathcal{Q}_\lambda \hookrightarrow \mathrm{Gr}_{15}^W H_c^1(A_1, L_\lambda).$$

Proof. Theorem 4.8 gives weights at least $13 + 2 = 15$ in $H^2(\mathcal{M}_3, \mathbb{V}_\lambda)$, also after finite level descent. Thus $\mathcal{Q}_\lambda$ is pure of weight fifteen. The two preceding propositions inject it into $\mathrm{Gr}_{15}^W \mathbb{H}_c^2(A_1, C_\lambda|_{A_1})$. Every ordinary cohomology sheaf on this stratum is a coarse local-system direct image. Such a sheaf has $H_c^0 = 0$: on a finite smooth nonproper modular-curve cover a section of a local system with compact support is zero. In

$$E_2^{p,q} = H_c^p(A_1, \mathcal{H}^q(C_\lambda)|_{A_1}) \implies \mathbb{H}_c^{p+q}(A_1, C_\lambda|_{A_1}),$$

the $q = 0$ row is zero and the $p = 0$ column is zero. Only $(p, q) = (1, 1)$ remains in total degree two. No differential enters it; the only possible outgoing term has $q = 0$ or negative q . This canonically identifies that hypercohomology with $H_c^1(A_1, L_\lambda)$ and proves the injection as a mixed-Hodge map. $\square$

Definition 5.13. Fix a smooth elliptic curve E_0 , with Satake point $x_E \in A_1$. Let $X_E = t^{-1}(x_E)$, and let W_E be the fibre over x_E of the stabilized elliptic-Jacobian map $\overline{\mathcal{M}}_{1,4} \rightarrow A_1^{\mathrm{Sat}}$. Put

$$\mathfrak{G} = \langle (12), (34), (13)(24) \rangle \subset \mathfrak{S}_4.$$

Simultaneous clutching of the pairs $\{1, 2\}$ and $\{3, 4\}$ gives $\nu : [W_E/\mathfrak{G}] \rightarrow X_E$ and $\tilde{\nu} : W_E \rightarrow X_E$. For $\lambda \in \mathcal{R}_3$, set

$$K_{\lambda,E} = (Ra_* \mathbb{V}_\lambda)|_{X_E}, \quad \delta_{\lambda,E} : \mathbb{H}^1(X_E, K_{\lambda,E}) \longrightarrow \mathbb{H}^1(W_E, \tilde{\nu}^* K_{\lambda,E})^{\mathfrak{G}}.$$

Proposition 5.14 (Weights in the finite-clutching kernel). *The morphism ν is finite, representable, and surjective. For every $\lambda \in \mathcal{R}_3$,*

$$W_{13} \ker \delta_{\lambda,E} = \ker \delta_{\lambda,E}.$$

Proof. The finite cover. A curve in X_E has one elliptic normalization component isomorphic to E_0 , all other components rational, and graph Betti number two. Choose a spanning tree and normalize at its two complementary edges. Labelling the four preimages produces a stable four-pointed genus-one curve in W_E : replacing a node branch by a marking preserves stability. Gluing recovers the curve, so ν is surjective. Its fibres are finite, since one chooses two nodes from a finite set; the group $\mathfrak{G}$ removes pair order and endpoint order. An automorphism of such a partial normalization, allowing these relabellings, that becomes the identity on the glued curve fixes every normalized component and every branch. It is the identity, proving representability. Since W_E is proper and X_E separated, ν is proper. A proper representable quasi-finite morphism is finite.

Degree-zero link weights. At any point $x \in X_E$, the compact-type complement in a labelled deformation chart punctures precisely its nonbridge parameters. The commuting unipotent monodromies give

$$H^0((K_{\lambda,E})_x) = \bigcap_e \ker(T_e - 1) = \bigcap_e \ker N_e.$$

Choose an algebraic smooth local curve through the chart on which every node parameter vanishes to order one; its punctured part smooths all nodes. Algebraic base change on degree zero is the injective map

$$\bigcap_e \ker N_e \longrightarrow \ker N_{\text{tot}}, \quad N_{\text{tot}} = \sum_e N_e.$$

It is restriction of local sections from the product of punctured disks to the diagonal, and hence is a morphism of geometric mixed Hodge structures. Bridge operators are zero. By Schmid [Sch73, Lemma 6.4 and Theorem 6.16], the one-variable limiting mixed Hodge structure of this polarized weight-thirteen variation with unipotent monodromy has weight filtration centered at thirteen, with $N_{\text{tot}} W_b \subset W_{b-2}$ and $N_{\text{tot}}^r : \text{Gr}_{13+r}^W \xrightarrow{\sim} \text{Gr}_{13-r}^W(-r)$, and N_{tot} is of type $(-1, -1)$; hence $\ker N_{\text{tot}} \subset W_{13}$. The one-variable logarithmic complex $\mathbb{V}_{\lambda,\infty} \xrightarrow{N_{\text{tot}}} \mathbb{V}_{\lambda,\infty}(-1)$ identifies degree zero with these invariants with their induced filtration. Strictness therefore bounds every $H^0((K_{\lambda,E})_x)$ by thirteen, including its finite stabilizer invariants.

The adjunction cone. Let $\mathcal{T}_{\lambda,E}$ be the cone in

$$K_{\lambda,E} \longrightarrow R\nu_* \nu^* K_{\lambda,E} \longrightarrow \mathcal{T}_{\lambda,E} \xrightarrow{+1}.$$

Finite proper base change makes the map on each degree- q stalk the injective diagonal $H^q(K_x) \rightarrow \bigoplus_{y \in \nu^{-1}(x)} H^q(K_x)$. Thus $\mathcal{T}_{\lambda,E}$ has no negative cohomology sheaves and

$$H^0((\mathcal{T}_{\lambda,E})_x) = \text{coker} \left(H^0(K_x) \longrightarrow \bigoplus_{y \in \nu^{-1}(x)} H^0(K_x) \right)$$

has weights at most thirteen. Choose a finite constructible stratification and a point on each connected stratum. Evaluation gives an injection of mixed Hodge structures

$$\mathbb{H}^0(X_E, \mathcal{T}_{\lambda,E}) \hookrightarrow \bigoplus_s H^0((\mathcal{T}_{\lambda,E})_{x_s}),$$

because a section locally constant on a stratum is determined at one point. The long exact sequence identifies $\ker \delta_{\lambda,E}$ with an image of this bounded degree-zero group. Strictness proves the assertion. $\square$

Definition 5.15. Let $\overline{W}_E$ be the compactification of E_0^4/E_0 by stable rational tails at collisions, with the elliptic normalization identified with E_0 . Thus $W_E = [\overline{W}_E/\text{Aut}(E_0, 0)]$. For $S \subseteq \{1, 2, 3, 4\}$ with $|S| \geq 2$, let D_S be its collision divisor. Set

$$\mathcal{B} = \{13, 14, 23, 24, 123, 124, 134, 234\},$$

$$O_E = \overline{W}_E \setminus \bigcup_{S \in \mathcal{B}} D_S, \quad D_E = D_{12} \cap D_{34}.$$

Here a string denotes its set of labels. Let $\widehat{G}_E = \text{Aut}(E_0, 0) \times \mathfrak{S}$, and let $\widehat{\nu} : \overline{W}_E \rightarrow X_E$ be the clutching map. Restriction to the invariant closed locus D_E defines

$$r_{D,E} : \mathbb{H}^1(W_E, \widehat{\nu}^* K_{\lambda,E})^{\mathfrak{S}} \longrightarrow \mathbb{H}^1(D_E, \widehat{\nu}^* K_{\lambda,E}|_{D_E})^{\widehat{G}_E}.$$

Proposition 5.16 (Detection on the proper collision curve). *For every smooth elliptic curve E_0 and $\lambda \in \mathcal{R}_3$, the curve D_E is proper and isomorphic to E_0 , and $r_{D,E}$ is injective. The Torelli correction triangle gives an exact sequence*

$$0 \longrightarrow (L_\lambda)_{x_E} \longrightarrow \mathbb{H}^1(X_E, K_{\lambda,E}) \xrightarrow{\gamma_{\lambda,E}} \mathcal{H}^0(E_\lambda)_{x_E}.$$

There is a morphism induced by restriction along the proper family of middle-elliptic attachments,

$$\eta_{\lambda,E} : \mathcal{H}^0(E_\lambda)_{x_E} \longrightarrow \mathbb{H}^1(D_E, \widehat{\nu}^* K_{\lambda,E}|_{D_E})^{\widehat{G}_E},$$

such that

$$r_{D,E} \delta_{\lambda,E} = \eta_{\lambda,E} \gamma_{\lambda,E}.$$

The displayed maps are the geometric maps on the full derived coefficients and respect their mixed Hodge structures.

Proof. The collision compactification. Fix the stabilized fourth marking at zero. Then $E_0^4/E_0 = E_0^3$, and $\overline{W}_E$ is its blowup at the full-collision point followed by the four proper triple-diagonal transforms. The latter are disjoint after the first blowup: their tangent directions are the three coordinate axes and the diagonal direction. This is the wonderful compactification of the diagonal building set [Li09, Theorem 1.3], blown up in order of decreasing cardinality; pair diagonals are Cartier, so blowing them up changes nothing.

The identification with the compactification by stable marked curves follows from its charts. A cluster has a rational component with attaching point at infinity and coordinates given by rescaled differences, modulo translation and dilation. Nested clusters repeat this construction. Conversely, these relative coordinates and node parameters recover the blowup charts. The full-collision exceptional surface is $\mathbb{P}^2$ blown up at the four triple directions; the triple exceptional fibres are $\mathbb{P}^1$. Knudsen's stabilization of an extra section commutes with base change [Knu12, Theorems 5.1–5.2]; this makes these rational insertions compatible with families, also when an extra section meets a node. Translation acts freely on the stabilized fourth point and preserves the diagonal ideals. This proves the description on their quotient and includes every collision.

The eleven divisor classes $[D_S]$ are independent in $H^2(\overline{W}_E)$. Project a relation to $H^2(E_0^3)$ using the blowup decomposition. For a symplectic basis u_E, v_E of $H^1(E_0)$, the six pair classes project to

$$u_{E,r} v_{E,r} \quad (1 \leq r \leq 3), \quad (u_{E,r} - u_{E,s})(v_{E,r} - v_{E,s}) \quad (1 \leq r < s \leq 3).$$

Each off-diagonal Künneth component occurs in just its indicated difference class, so all six coefficients vanish. The five exceptional classes are independent because each connected center of

codimension at least two contributes a new $H^0(\text{center})(-1)$ summand. Degree-one localization for the eight removed divisors therefore gives

$$H^1(O_E; \mathbb{Q}) = H^1(\overline{W}_E; \mathbb{Q}) = \{(h_1, h_2, h_3, h_4) \in H^1(E_0)^4 : \sum h_r = 0\}.$$

Indeed the residue map ends in the injective map $\bigoplus_{S \in \mathcal{B}} \mathbb{Q}(-1) \rightarrow H^2(\overline{W}_E)$, and the blowups preserve H^1 .

The curve D_E is parameterized by $z \in E_0$: place the 12-tail at z and the 34-tail at zero. When $z = 0$ insert a three-pointed central rational component carrying both tails and the elliptic attachment. This is the unique additional stable point. Hence $D_E \simeq E_0$ is proper, lies in O_E , and meets only the retained additional divisor D_{1234} . Let $j = \iota_{E_0}(12)(34)$, where ι_{E_0} is inversion. It is central in $\widehat{G}_E$. On constant $H^1(O_E)$ it acts by minus the indicated permutation, and its minus eigenspace is $\{(h, h, -h, -h)\}$. Restriction to $z \mapsto (z, z, 0, 0)$ sends this vector to $2h$. Thus

$$(5.4) \quad H^1(O_E; \mathbb{Q})^{j=-1} \xrightarrow{\sim} H^1(D_E; \mathbb{Q}).$$

Stable normal lines and all corner complexes. Let $\mathcal{T}_r$ be the stable tangent line at marking r , and put

$$\mathcal{L}_1 = \mathcal{T}_1 \otimes \mathcal{T}_2, \quad \mathcal{L}_2 = \mathcal{T}_3 \otimes \mathcal{T}_4, \quad \mathcal{P}_E = \mathcal{L}_1^\times|_{O_E} \times_{O_E} \mathcal{L}_2^\times|_{O_E}.$$

These are the smoothing lines of the two nodes obtained by pairwise clutching: the equation $xy = \tau$ transforms τ by the product of the derivatives of the two branch coordinates. Orient their coedges as q_1, q_2 . A tail with label set S has coedge

$$c_S = (1_S(1) - 1_S(2))q_1 + (1_S(3) - 1_S(4))q_2.$$

This follows by closing each pair edge along its unique path in the original rational tree. Exactly the sets in $\mathcal{B}$ have nonzero coedge; the other three tails are bridges after clutching.

Plumbing an annulus gives the Picard–Lefschetz operator $N(c)v = \langle c, v \rangle c$ on the standard coefficient: rotating its parameter makes a positive Dehn twist, adding the intersection with its core. The disjoint node operators commute. On $\mathbb{V}_\lambda$ use their differentiated tensor action; horizontal symplectic projectors commute with this transport.

The stable and elliptic normal frames are related by

$$(5.5) \quad \tau_r^{\text{ell}} = u_r \tau_r^{\text{st}} \prod_S s_S^{m_r(S)}, \quad m_r(S) = |S \cap \{2r-1, 2r\}|, \quad r = 1, 2.$$

To verify it, at one tail attachment use $zy = s$ and $y = 1/w$ on the tail. Then $z = sw$, so $dz = s dw$ at a marked point. Along a tree the chain rule multiplies these derivatives over its path to E_0 , with an invertible regular unit. Multiplying the two derivatives in a pair gives precisely the displayed exponents, including exponent two when both markings lie on the tail. At fixed elliptic frame the tail logarithm is consequently

$$N(c_S) - \sum_{r=1}^2 m_r(S) N(q_r).$$

The full normal cochains have a separate exterior generator for each punctured coordinate and differential $\sum_e N_e e_e \wedge -$. The equality $T_e - 1 = N_e(1 + N_e/2! + \dots)$ identifies them with the unipotent circle resolution. Logarithmic differentiation of (5.5) makes the triangular generator change

$$e_r = f_r - \sum_S m_r(S) f_S, \quad e_S = f_S.$$

It is invertible in every exterior degree and gives exactly the corrected tail operators. Restoring units adds $-d \log u_r$ and the gauge transport $\exp(-\log(u_r) N(q_r)/(2\pi i))$ in the same flat frame. Changing a chosen logarithm changes both sides by the same integral monodromy, so these local maps descend. At a bridge, restriction of the unpunctured disk is inclusion into its zero-operator

circle complex before the change of frame. Its correction is accounted for by $d\iota_r + \iota_r d = N(q_r)$, with ι_r exterior contraction. The contractions anticommute, giving the compatible signs on iterated faces. Thus no bridge disk is punctured in the compact-type open, and equal coedges never identify their distinct smoothing coordinates.

Blow up $\overline{W}_E$ in oriented directions along the eight removed divisors. On a chart its additional angles are exactly the nonbridge-tail angles, with the two pair angles supplied by $\mathcal{L}_1, \mathcal{L}_2$. Equations (5.3) and (5.5) identify their unit-circle bundle with the full pulled-back polar link, including all nested and disjoint corners and horizontal point transport. A collar gives a homotopy equivalence with its interior relative to D_E : patch inward radial vector fields on boundary neighborhoods disjoint from D_E , average over the finite group, and lift by invariant connections on the two lines. Small positive flow moves all boundary faces inward and fixes D_E . Radial contraction in each punctured line then gives

$$(5.6) \quad \mathbb{H}^*(\overline{W}_E, \widehat{\nu}^* K_{\lambda, E}) \cong H^*(\mathcal{P}_E, \mathcal{V}_E)$$

as rational groups, with the full transported coefficient $\mathcal{V}_E$ and its filtered subsystems, compatibly with restriction to D_E .

Detection of the full coefficient. Both normal lines restrict trivially to D_E : their three-pointed rational tails are fixed, including at the coincident-attachment point. Thus $\mathcal{P}_E|_{D_E} = E_0 \times (\mathbb{C}^*)^2$. The involution j preserves the complex orientation of both normal circles. The torus-bundle Leray sequence and (5.4) therefore give

$$H^1(\mathcal{P}_E; \mathbb{Q})^{j=-1} \xrightarrow{\sim} H^1(\mathcal{P}_E|_{D_E}; \mathbb{Q})^{j=-1}.$$

The vertical H^1 has $j = 1$ and contributes no minus eigenspace. Both spaces are connected, so their constant degree-zero minus eigenspaces vanish.

The normalization filtration on the standard limiting coefficient has constant graded systems $I_E, H^1(E_0), I_E^\vee(-1)$, where $I_E = \mathbb{Q}q_1 \oplus \mathbb{Q}q_2$, of weights zero, one, and two. Labels fix the graph basis and E_0 is fixed. Point-pushing and meridian transports remain in the extensions. Pair reversal negates the graph pieces and elliptic inversion negates the elliptic piece, so j acts by -1 on all three. The degree-thirteen symplectic tensor construction has a finite filtration by constant quotients with coefficient action $j = -1$; polarization contractions have even degree and preserve this sign.

For each such constant quotient the preceding displayed map is an isomorphism on invariant degree one, and invariant degree zero vanishes on both spaces. In an extension $0 \rightarrow \mathcal{A} \rightarrow \mathcal{V} \rightarrow \mathcal{B} \rightarrow 0$, these degree-zero vanishings make the two maps from $H^1(\mathcal{A})^j$ injective. A class in $H^1(\mathcal{P}_E, \mathcal{V})^j$ restricting to zero has zero image in the $\mathcal{B}$ group by induction, so lifts to $\mathcal{A}$. Its restricted lift is zero by that injectivity, and the inductive injectivity statement then implies that the lift vanishes. This proves injectivity for the full filtered system without splitting it. Further exact $\widehat{G}_E$ invariants and (5.6) prove injectivity of the actual map $r_{D, E}$.

The proper attachment and its boundary map. The correction triangle, E_λ being nonnegative, and proper base change for t give the exact sequence of the statement. On the ordered product $\mathcal{Z}_3 = \mathcal{A}_1^3$ consider the proper universal elliptic curve $r_0 : \mathcal{E}_0 \rightarrow \mathcal{Z}_3$ attaching the other two factors at zero and at a moving point of its first factor. The rational insertion at zero is included. Denote its attachment by $c_0 : \mathcal{E}_0 \rightarrow \mathcal{M}_3^{\text{ct}}$ and the product map followed by coarse moduli by $h_0 : \mathcal{Z}_3 \rightarrow A$. Then $(qf)c_0 = h_0 r_0$. Adjunction gives the restriction morphism

$$R(qf)_* \mathbb{V}_\lambda \longrightarrow R h_{0*} R r_{0*} c_0^* \mathbb{V}_\lambda.$$

The simultaneous central involution acts as $(-1)^{13+j}$ on $\mathbb{V}_\lambda \otimes H^j(E_0)$, so exact coarse direct image through the finite product map vanishes in degrees zero and two. The target is thus a sheaf in degree one. Negative Ext vanishing applied to (5.2) factors this map uniquely through $F_\lambda[-1]$. Its degree-one map is the middle-component restriction in (5.1).

This component restriction retains the matching term. For the actual fibre $\mathcal{T} = E_0 \cup E_1 \cup E_2$ with common origin and normalization maps η_i , every rational complex $\mathcal{Q}$ has the triangle

$$\mathcal{Q} \longrightarrow \bigoplus_{i=0}^2 R\eta_{i*} \eta_i^* \mathcal{Q} \longrightarrow \mathcal{Q}|_0^{\oplus 2} \xrightarrow{+1}.$$

Its difference map is $(v_0, v_1, v_2) \mapsto (v_1 - v_0, v_2 - v_0)$. The constant-sheaf sequence is stalkwise $0 \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}^3 \rightarrow \mathbb{Q}^2 \rightarrow 0$ at the origin, and tensoring proves the derived version. Restriction to E_0 is its canonical first arrow followed by that component, with the difference term preserved. The signed cyclic descent above transports the entire triangle.

Compactify the attachment diagram by

$$\bar{\mathcal{Z}}_3 = \bar{\mathcal{M}}_{1,1}^3, \quad \bar{\mathcal{P}}_0 = \bar{\mathcal{M}}_{1,2} \times \bar{\mathcal{M}}_{1,1}^2.$$

The maps $\bar{r}_0, \bar{c}_0, \bar{h}_0$ are proper, and $t\bar{c}_0 = \bar{h}_0\bar{r}_0$. If $b_0 : \mathcal{E}_0 \hookrightarrow \bar{\mathcal{P}}_0$, then $\bar{r}_0^{-1}(\mathcal{Z}_3) = \bar{c}_0^{-1}(\mathcal{M}_3^{\text{ct}}) = \mathcal{E}_0$: a loop on any elliptic factor remains nonseparating after the separating attachments. There is the canonical algebraic comparison, defined by adjunction from the identity on the open,

$$(5.7) \quad \chi : \bar{c}_0^* Ra_* \mathbb{V}_\lambda \longrightarrow Rb_{0*} c_0^* \mathbb{V}_\lambda.$$

We prove that it is an isomorphism near the component under consideration.

At a point with distinct attachments the target node chart has coordinates $(t_1, t_2, \zeta_E, z', b_1, b_2)$, where t_i smooth the two self-nodes, b_i the separating attachments, and (ζ_E, z') deform the middle elliptic curve and its attachment difference. The source chart is (s_1, s_2, ζ_E, z') and maps to $(s_1, s_2, \zeta_E, z', 0, 0)$. At coincident attachments the target has coordinates $(t_1, t_2, \zeta_E, b', b_1, b_2)$: b' smooths the new bridge to the central rational component. The source's stable two-pointed first factor has coordinate z' smoothing this same bridge, so the map is again $(s_1, s_2, \zeta_E, z') \mapsto (s_1, s_2, \zeta_E, z', 0, 0)$. Normalizing the two ordered separating nodes recovers the source factors and proves these coordinate identifications without ramification in s_1, s_2 . These two chart types exhaust D_E .

The compact-type opens are $(\Delta^*)^2 \times \Delta^2$ and $(\Delta^*)^2 \times \Delta^4$. Contracting the last two bridge disks gives a homotopy inverse of their inclusion with the full local system. To check every corner, let $I \subset \{1, 2\}$ be the vanishing self-node coordinates and let V_{loc} be the actual fibre. Its cochains are

$$K_I^p = \bigoplus_{\substack{A' \subset I \\ |A'|=p}} V_{\text{loc}} e_{A'}, \quad d(v e_{A'}) = \sum_{i \in I \setminus A'} (T_i - 1) v e_i \wedge e_{A'}.$$

In matched frames χ is identity on every coefficient and every labelled generator. A generization to $I' \subset I$ keeps $e_{A'}$ if $A' \subset I'$ and kills it otherwise; this commutes with the differential and with χ . Both orders of two generizations agree. Moving across $z' = 0$ removes no circle, since it is a bridge disk. For arbitrary frames $t_i = u_i s_i$, polar angles change by $\text{Im} \log u_i$ with their actual lifted transport. A change of logarithm multiplies both sides by the same $T_i^{n_i}$. Permuting tails permutes the operators and exterior generators together. Hence the comparison holds on every overlap and corner for the full coefficient. Since (5.7) is an algebraic mixed-Hodge morphism and its rational realization is an isomorphism there, the same holds in the geometric mixed-Hodge category.

There is a proper map $d_E : D_E \rightarrow (\bar{h}_0 \bar{r}_0)^{-1}(x_E)$ obtained by taking the stable two-pointed first factor and two one-pointed nodal rational elliptic factors. At the coincident point the first factor includes its rational marked tail. Thus $\bar{c}_0 d_E = \hat{v}|_{D_E}$. Restriction along d_E is the canonical adjunction map on the whole proper fibre, with every other component and its matching still in its source.

The composite of adjunction for $\bar{c}_0$ with $R\bar{c}_{0*} \chi$ equals Ra_* of adjunction for c_0 , since both are adjoint to the identity on $c_0^* \mathbb{V}_\lambda$. Apply proper direct images and proper base change for $t, \bar{c}_0, \bar{h}_0, \bar{r}_0$,

then restrict along d_E . The isomorphism $\chi|_{D_E}$ identifies this map with actual restriction from X_E to D_E . Its factorization through $F_\lambda[-1]$ therefore gives $\eta_{\lambda,E}$ and exactly $r_{D,E}\delta_{\lambda,E} = \eta_{\lambda,E}\gamma_{\lambda,E}$. The nodal branch reversals and interchange of the two nodal factors are the pair symmetries in $\widehat{G}_E$, so the maps descend with their stated invariants. All Hodge comparisons in this last argument are the algebraic maps just constructed; the nonproper open comparison was proved on its corner complexes before any proper base change was used. $\square$

Proposition 5.17 (The elliptic coefficient weight bound). *For $\lambda \in \mathcal{R}_3$, one has $W_{13}L_\lambda = L_\lambda$. Moreover, there is an algebraic mixed Hodge module N_λ on A_1 with*

$$\text{rat}(N_\lambda) = L_\lambda[1], \quad W_{14}N_\lambda = N_\lambda.$$

Its weights and rational realization agree with the geometric coefficient defined above, including at the finite-quotient points.

Proof. The fibre bound. By Proposition 5.16, the image of $(L_\lambda)_{x_E}$ lies in $\ker \gamma_{\lambda,E}$ and hence restricts to zero on D_E . Injectivity of $r_{D,E}$ forces its finite-clutching pullback to vanish. It is therefore a mixed-Hodge subobject of $\ker \delta_{\lambda,E}$, whose weights are at most thirteen by Proposition 5.14. Strictness proves the fibre bound. All maps were constructed equivariantly before finite elliptic stabilizer invariants; the same calculation on each uniformizing elliptic chart gives the geometric variation bound there. The local-system assertion of Proposition 5.11 then gives $W_{13}L_\lambda = L_\lambda$ with the stipulated coarse interpretation.

An algebraic realization. After full finite level, the universal abelian scheme on the smooth six-dimensional compact-type curve space gives an algebraic first-cohomology Hodge module by proper direct image. Tensor operations, the polarization, and the symplectic idempotent construct $\mathbb{V}_\lambda^H$ inside the algebraic category. Finite descent defines it on the stack. By [PS12, Theorem 3.1] a polarizable variation of weight w on a smooth d -dimensional base is a pure smooth Hodge module of weight $w + d$ with rational perverse sheaf $\mathbb{V}[d]$, and conversely; so its weight is $13 + 6 = 19$ and its rational realization is $\mathbb{V}_\lambda[6]$. Put $\mathcal{V}_\lambda^H = \mathbb{V}_\lambda^H[-6]$, with ordinary rational realization $\mathbb{V}_\lambda$, and form

$$C_\lambda^H = i^* Rk_* j_! R p_* (\mathcal{V}_\lambda^H|_{\mathcal{N}}) \quad \text{in } D^b \text{MHM}(B).$$

These are the algebraic functors defining the coefficient; rational coarse direct image, exact by Set-up 2.1, gives $\text{rat}(C_\lambda^H) = C_\lambda$. For $\ell_1 : A_1 \hookrightarrow B$ define

$$N_\lambda = {}^p\mathcal{H}^2(\ell_1^* C_\lambda^H).$$

Every ordinary cohomology sheaf of $\ell_1^* C_\lambda$ is a coarse local-system direct image. On a curve such a sheaf shifted by $[1]$ is perverse. The ordinary-to-perverse cohomology spectral sequence has only the row ${}^p\mathcal{H}^1(\mathcal{H}^q) = \mathcal{H}^q[1]$. Its index-two term is therefore $L_\lambda[1]$, proving the required rational realization.

No punctual constituents. At a coarse elliptic orbifold point, pass to a finite uniformizing disk. Each coefficient sheaf is the invariant direct image of a local system on that disk. Its monodromy on the punctured coarse disk has finite order and its stalk is exactly that monodromy's invariant space. Shifted by one, it is the middle extension, with no punctual subquotient. The same description applies to weight subobjects and graded quotients. At other points the coefficient is a smooth variation.

A nonzero pure graded piece of N_λ of weight b thus restricts to a variation of weight $b - 1$ on the smooth one-dimensional elliptic stack, by the converse clause of [PS12, Theorem 3.1] recalled above. These are the fibre weights of the geometric complex C_λ^H . If $b > 14$, the fibre bound makes that restriction zero. The graded piece would be punctual, which the disk calculation excludes. Hence $W_{14}N_\lambda = N_\lambda$. $\square$

Theorem 5.18 (Vanishing of the coefficient obstruction). *For every $\lambda \in \mathcal{R}_3$ (Definition 5.7), one has*

$$W_{15}H^2(\mathcal{M}_3, \mathbb{V}_\lambda) = 0.$$

Proof. The injection of Proposition 5.12 places the obstruction inside $\mathrm{Gr}_{15}^W H_c^1(A_1, L_\lambda)$. By Proposition 5.17, N_λ has upper derived weight at most fourteen. For the structure map $s_1 : A_1 \rightarrow \mathrm{Spec} \mathbb{C}$, compactly supported direct image preserves such upper weight bounds, compatibly with the rational constructible functors [TS20, Definition 3.7 and Proposition 3.8]; this gives weights at most fourteen on $\mathcal{H}^0(s_{1!}N_\lambda)$. Its rational realization is $H_c^1(A_1, L_\lambda)$, because $\mathrm{rat}(N_\lambda) = L_\lambda[1]$. Its weight-fifteen graded piece is therefore zero, and the injection forces $W_{15}H^2(\mathcal{M}_3, \mathbb{V}_\lambda) = 0$. $\square$

Definition 5.19. Let $n \geq 10$. Let $\mathcal{G}_{3,n}^{(2)}$ be the isomorphism classes of stable genus-three graphs with two edges, and let $\mathcal{G}_{3,n}^{(1;1,2)}$ be those with one separating edge and vertex genera one and two. Define

$$A_n = \bigoplus_{[\Gamma] \in \mathcal{G}_{3,n}^{(2)}} \left(\bigoplus_{v: g(v)=1} K_{F(v)} \right)^{\mathrm{Aut}(\Gamma)} \oplus \bigoplus_{[\Gamma] \in \mathcal{G}_{3,n}^{(1;1,2)}} K_{F(v_1)} \otimes \mathbb{Q}[F(v_2)].$$

In the second sum, $g(v_i) = i$, and the vector indexed by $j \in F(v_2)$ corresponds to ψ_j , including the attaching flag. In the first sum, automorphisms act on the whole direct sum and may move the decorated vertex. Choose a rational basis in each graph summand by the hook basis and the finite invariant equations. Let $a_1, \dots, a_{N_n}$ be the resulting basis and let $\rho_n : \mathfrak{S}_n \rightarrow \mathrm{GL}_{N_n}(\mathbb{Q})$ be the relabelling representation in this basis. Node flags are transported by the induced graph isomorphisms.

For a decoration a on Γ , let $\alpha_a(u)$ be its class on X_Γ : it has the factor $\omega_{F(v)}(k, u)$ at its specified genus-one vertex, a factor ψ_j in the second family, and units on the other vertices, extended linearly over the invariant decoration. Define the Gysin map

$$\Phi_n : A_n \otimes \mathbb{P} \longrightarrow H^{15}(\overline{\mathcal{M}}_{3,n}), \quad \Phi_n(a, u) = \frac{\xi_{\Gamma*} \alpha_a(u)}{|\mathrm{Aut}(\Gamma)|}.$$

The notation on the right suppresses the common Tate factor as above. Both families have degree $11 + 2(|E(\Gamma)| + \deg f) = 15$, with $f = 1$ or $f = \psi_j$, respectively.

Proposition 5.20 (Genus-three boundary generation). *For every $n \geq 0$, the sum of the ordinary quotient-normalized one-edge Gysin maps is surjective:*

$$\bigoplus_{[\Gamma] \in \mathcal{G}_{3,n}^{(1)}} H^{13}(X_\Gamma)^{\mathrm{Aut}(\Gamma)} \otimes \mathbb{L} \longrightarrow H^{15}(\overline{\mathcal{M}}_{3,n}).$$

For every $n \geq 10$, the map $\Phi_n : A_n \otimes \mathbb{P} \rightarrow H^{15}(\overline{\mathcal{M}}_{3,n})$ is surjective.

Proof. Proposition 5.8 and Theorem 5.18 give surjectivity of the sum of the ordinary one-edge Gysin maps in every arity.

Degree-thirteen factors. For a separating edge its degree-thirteen Künneth sectors are exactly $(13, 0), (0, 13), (11, 2), (2, 11)$, by Theorem 3.3. The nonseparating term is $H^{13}(\overline{\mathcal{M}}_{2,n+2})$ with the exchange invariants. In every H^{13} factor, Theorem 3.23 gives generation by Gysin images of an elliptic H^{11} class tensored with the unit on its complement, including in genus one. Composing this inner gluing with the outer edge gives a stable two-edge genus-three graph, decorated at its elliptic vertex by the hook of Proposition 3.15. These graphs index the first family in A_n .

Picard factors. In the remaining sectors the H^{11} factor has genus one, so the other factor has genus two and carries a Picard class. Proposition 3.18 generates H^2 by κ_1 , the ψ_j , and the boundary divisors, and its genus-two identity expresses $5\kappa_1$ through $\sum_j \psi_j$, δ_{irr} , δ_0 , and δ_1 . Thus every Picard class is a linear combination of flag ψ classes and boundary divisors. The former give exactly the second family, including the attaching flag. The latter compose to graphs in the first family, also for the irreducible boundary divisor.

The normalized span. Labelled and quotient-normalized pushforwards differ only by their nonzero graph automorphism orders. Given an invariant class in an intermediate source, average any generating preimage over its finite group. This proves the same spanning assertion with all automorphisms, including those moving a decorated vertex. The composed classes are exactly the images under Φ_n of the generators in Definition 5.19. It is therefore surjective for every $n \geq 10$. $\square$

6. THE GENUS-THREE RATIONAL GRAM FORMULA

We compute the quotient of the finite-dimensional source A_n that gives $H^{15}(\overline{\mathcal{M}}_{3,n})$. Its intersection matrix is determined by recursions for descendant integrals and integrals containing two cusp classes. We first evaluate these entries, then identify the radical and compute the quotient character.

Definition 6.1. Write $\mathcal{P}_L = \mathbb{Q}[\kappa_1, (\psi_i)_{i \in L}]$. For a partition $L = A \sqcup B$ and distinct new node flags p, q , let

$$\Delta_{A|B} : \mathcal{P}_L \longrightarrow \mathcal{P}_{A \cup \{p\}} \otimes \mathcal{P}_{B \cup \{q\}}$$

send ψ_i to the same variable on the factor containing i and send κ_1 to $\kappa_1 \otimes 1 + 1 \otimes \kappa_1$. This is the pullback rule of Theorem 3.20 for a one-edge separating graph, and the same retained-flag rule and sum over the inverse vertices there define the polynomial pullback for a general graph contraction. A hook decoration is restricted by iterating r_T of Definition 3.11 (Lemma 3.13).

Definition 6.2. For $h \geq 0$, $m \geq 1$, and integer exponents d_i , define $T_h(d_1, \dots, d_m) = 0$ unless $2h - 2 + m > 0$, every $d_i \geq 0$, and $\sum_i d_i = 3h - 3 + m$. Negative-genus and unstable expressions are zero. Set

$$T_0(0, 0, 0) = 1, \quad T_1(1) = \frac{1}{24}, \quad (-1)!! = 1.$$

These base cases take precedence. For other dimension-compatible stable inputs with a zero exponent, use symmetry and the string recursion

$$(6.1) \quad T_h(0, d_2, \dots, d_m) = \sum_{j=2}^m T_h(d_j - 1, (d_i)_{i \neq 1, j}).$$

If all exponents are positive, use

$$(6.2) \quad \begin{aligned} & (2d_1 + 1)!! T_h(d_1, \dots, d_m) \\ &= \sum_{j=2}^m \frac{(2d_1 + 2d_j - 1)!!}{(2d_j - 1)!!} T_h(d_1 + d_j - 1, (d_i)_{i \neq 1, j}) \\ &+ \frac{1}{2} \sum_{\substack{a, b \geq 0 \\ a+b=d_1-2}} (2a+1)!! (2b+1)!! (T_{h-1}(a, b, d_2, \dots, d_m) \\ &+ \sum_{\substack{\text{stable} \\ h_1, h_2 \geq 0, h_1+h_2=h \\ I \sqcup J = \{2, \dots, m\}}} T_{h_1}(a, (d_i)_{i \in I}) T_{h_2}(b, (d_j)_{j \in J}) \Bigg). \end{aligned}$$

The inner sum runs over ordered partitions and only stable factors. Each nonzero descendant term on the right-hand side has smaller $2h - 2 + m$.

For $a \geq 0$ and a stable marked pair (h, m) , define

$$(6.3) \quad W_h(a; d_1, \dots, d_m) = \sum_{k=0}^a \frac{(-1)^{a-k}}{k!} \sum_{\substack{a_1 + \dots + a_k = a \\ a_i \geq 1}} \frac{a!}{a_1! \dots a_k!} T_h(d_1, \dots, d_m, a_1 + 1, \dots, a_k + 1).$$

There is one empty composition when $a = k = 0$ and none when $a > 0, k = 0$. Extend W_h linearly to $\mathcal{P}_{[m]}$ by evaluating the monomial $\kappa_1^a \prod_i \psi_i^{d_i}$ with this expression.

Proposition 6.3 (Descendant integral identities [LiuXu09, Theorem 1.2, Proposition 3.1, Lemma 2.2]). *For every stable pair (h, m) with $m \geq 1$ and nonnegative integers $a, d_1, \dots, d_m$, the recursions (6.1)–(6.3) evaluate the ordinary stack integrals*

$$T_h(d_1, \dots, d_m) = \int_{\mathcal{M}_{h,m}} \prod_i \psi_i^{d_i}, \quad W_h(a; d_1, \dots, d_m) = \int_{\mathcal{M}_{h,m}} \kappa_1^a \prod_i \psi_i^{d_i}.$$

Definition 6.4. For $|L| \geq 3$, define a linear functional $Z_L : \mathcal{P}_L \rightarrow \mathbb{Q}$, zero outside degree $|L| - 3$, with $Z_L(1) = 1$ at $|L| = 3$. For larger L , select a factor q of a monomial qf' . If $q = \psi_z$, choose distinct $x, y \in L \setminus \{z\}$ and set

$$\mathcal{A}_q = \{A \subset L : z \in A, x, y \notin A, |A| \geq 2, |L \setminus A| \geq 2\}, \quad c_q(A) = 1.$$

If $q = \kappa_1$, choose distinct $x, y \in L$ and set

$$\mathcal{A}_q = \{A \subset L : x, y \notin A, |A| \geq 2, |L \setminus A| \geq 2\}, \quad c_q(A) = |A| - 1.$$

For each A write $\Delta_{A|L \setminus A} f' = \sum_b f_{A,b} \otimes f_{B,b}$. The recursion is

$$Z_L(qf') = \sum_{A \in \mathcal{A}_q} c_q(A) \sum_b Z_{A \cup \{p\}}(f_{A,b}) Z_{(L \setminus A) \cup \{q'\}}(f_{B,b}).$$

Here p, q' are distinct new node flags used by Δ . Both new label sets have smaller cardinality than L .

Definition 6.5. For $|I| \geq 1$, define $\mathcal{C}_I : K_I \times K_I \times \mathcal{P}_I \rightarrow \mathbb{Q}$, bilinear in its first two arguments and linear in its last. Set it to zero for $|I| < 11$ and outside polynomial degree $|I| - 11$. For $|I| = 11$, choose an ordering of I and let $\epsilon_I : K_I \rightarrow \mathbb{Q}$ send its increasing eleven-symbol to one. For constant f put

$$\mathcal{C}_I(k, l; f) = f \epsilon_I(k) \epsilon_I(l).$$

For $|I| > 11$, select a factor q in a monomial qf' and put

$$\begin{array}{c|c|c} q & \mathcal{T}_q & c_q(T) \\ \hline \psi_i & \{T \subseteq I : i \in T, |T| \geq 2\} & 1 \\ \kappa_1 & \{T \subseteq I : |T| \geq 2\} & |T| - 1. \end{array}$$

For $A = I \setminus T$, expand $\Delta_{A|T} f' = \sum_b f_{1,b} \otimes f_{0,b}$ with new flags p, q' . Define

$$(6.4) \quad \mathcal{C}_I(k, l; qf') = \sum_{T \in \mathcal{T}_q} c_q(T) \sum_b \mathcal{C}_{A \cup \{p\}}(r_T k, r_T l; f_{1,b}) Z_{T \cup \{q'\}}(f_{0,b}).$$

The term $T = I$ is allowed: the elliptic factor has its single node marking and is stable, and its hook is zero. Every elliptic term on the right-hand side has $|A| + 1 < |I|$.

Proposition 6.6 (Vertex evaluation). *For every stable marked genus-zero space and every $f \in \mathcal{P}_L$,*

$$Z_L(f) = \int_{\mathcal{M}_{0,L}} f.$$

For every $|I| \geq 1$, $k, l \in K_I$, $u, v \in S$, and $f \in \mathcal{P}_I$,

$$\int_{\overline{\mathcal{M}}_{1,I}} \omega_I(k, u) \omega_I(l, v) f = C_I(k, l; f) J(u, v).$$

The recursion values are independent of the selected factors, auxiliary labels, and orderings. The functional C_I is symmetric in k, l and equivariant under relabelling.

Proof. We identify the genus-zero recursion with integration, and then reduce the paired-cusp integral to rational-tail divisors.

Genus-zero integration. In top degree and for $|L| > 3$, a monomial has a positive-degree factor q . Proposition 3.18 gives $q = \sum_{A \in \mathcal{A}_q} c_q(A) \delta_{0,A}$ with exactly the coefficients in Definition 6.4. The auxiliary labels distinguish the complementary component, so each divisor is counted once. Both components have labelled ordinary markings, and the graph has automorphism group one. Its labelled clutching pushforward of the unit is therefore $\delta_{0,A}$. The remaining polynomial pulls back to $\Delta_{A|L \setminus A} f'$ by Theorem 3.20. The projection formula gives

$$\int_{\overline{\mathcal{M}}_{0,L}} q f' = \sum_{A \in \mathcal{A}_q} c_q(A) \sum_b \left(\int_{\overline{\mathcal{M}}_{0,A \cup \{p\}}} f_{A,b} \right) \left(\int_{\overline{\mathcal{M}}_{0,(L \setminus A) \cup \{q'\}}} f_{B,b} \right).$$

The two factors have fewer markings than L . Together with $\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$ and the degree convention, induction proves the first assertion for every polynomial.

Paired-cusp integration. Let $x \in H^{11}(\overline{\mathcal{M}}_{1,I})$. The labelled nonseparating normalization is $\overline{\mathcal{M}}_{0,I \cup \{h, h'\}}$, whose degree-eleven cohomology is zero by Proposition 3.15. The projection formula, with $\delta_{\text{irr}} = \frac{1}{2} \xi_{\text{irr}} * 1$, gives $\delta_{\text{irr}} x = 0$. The genus-one identities in Proposition 3.18 give $q x = \sum_{T \in \mathcal{T}_q} c_q(T) \delta_{0,T} x$ with exactly the coefficients in Definition 6.5. The rational side distinguishes the two vertices, so these separating graphs have automorphism group one, including $T = I$.

For $A = I \setminus T$ let

$$j_T : \overline{\mathcal{M}}_{1,A \cup \{p\}} \times \overline{\mathcal{M}}_{0,T \cup \{q'\}} \longrightarrow \overline{\mathcal{M}}_{1,I}$$

be the labelled gluing map. Proposition 3.15 identifies its pullback of $\omega_I(k, u)$ with $\omega_{A \cup \{p\}}(r_T k, u) \boxtimes 1$; the same applies to the second hook. Select a factor in $f = q f'$ and apply the divisor identity to the first hook. Projection along each j_T , followed by the polynomial pullback $\Delta_{A|T}$, expresses the integral as

$$\sum_{T \in \mathcal{T}_q} c_q(T) \sum_b \left(\int_{\overline{\mathcal{M}}_{1,A \cup \{p\}}} \omega_{A \cup \{p\}}(r_T k, u) \omega_{A \cup \{p\}}(r_T l, v) f_{1,b} \right) \left(\int_{\overline{\mathcal{M}}_{0,T \cup \{q'\}}} f_{0,b} \right).$$

All polynomial factors have even cohomological degree, so this formula preserves the order of the odd classes without a sign change. The elliptic factor has fewer than $|I|$ markings. If $|I| < 11$, its hook is zero. At $|I| = 11$, the ordered identification with $\overline{\mathcal{M}}_{1,[11]}$ gives $f \epsilon_I(k) \epsilon_I(l) J(u, v)$. Induction and the genus-zero formula give exactly (6.4). Degree mismatches vanish on both sides.

These identifications with integration prove independence of the recursive choices. Changing the eleven-flag ordering changes both orientation signs and preserves their product. Relabelling commutes with pullback, the divisor identities, and integration. Interchanging the two degree-eleven classes changes the integral by -1 , while $J(v, u) = -J(u, v)$. Their rational coefficient is therefore symmetric in k, l . $\square$

Definition 6.7. On a graph Γ , evaluate a tensor monomial containing two restricted hook decorations as follows. Its value is zero if a hook restriction is zero or the hooks occur on different

vertices. If both occur at v , put

$$\mathrm{Ev}_\Gamma \left(k_v, l_v; \bigotimes_w f_w \right) = \mathcal{C}_{F(v)}(k_v, l_v; f_v) \prod_{w \neq v} W_{g(w)}(f_w).$$

Extend linearly, retaining the first hook before the second. Here $W_h(f)$ means the linear polynomial evaluation in Definition 6.2. In every nonzero term the other vertices have genus at most two and at least one flag.

For the basis vector a_r , let Γ_r and α_r be its graph and formal hook decoration from Definition 5.19. For $n \geq 10$ define the rational matrix C_n by

$$(6.5) \quad (C_n)_{rs} = \frac{1}{|\mathrm{Aut}(\Gamma_r)| |\mathrm{Aut}(\Gamma_s)|} \sum_{[\Gamma, f, g] \in \mathcal{R}(\Gamma_r, \Gamma_s)} \mathrm{Ev}_\Gamma(f^* \alpha_r g^* \alpha_s) \cdot \prod_{\{h, h'\} \in H(f, g)} (-\psi_h - \psi_{h'}) \left(\sum_{v \in V(\Gamma)} \kappa_{1,v} \right)^{n-9}.$$

The evaluation acts on the polynomial expansion of the displayed formal decorations. There is no further $|\mathrm{Aut}(\Gamma)|$ factor in this sum over isomorphism classes of specified generic structures. Every refinement here has at most four edges.

Theorem 6.8 (Ampleness of the logarithmic kappa class [Cor93, Theorem 2.2], [Cor95, Theorem 1, specialized to $r = d = 0$]). *Let (h, m) be stable, put $\lambda = c_1(\pi_* \omega_\pi)$, and let δ be the total boundary divisor with ordinary stack normalization. Then the logarithmic class κ_1 descends to an ample rational divisor on the coarse space $\overline{M}_{h,m}$. It is the first Chern class of the fractional line given by the Deligne pairing of $\omega_\pi(\sum_i D_i)$ with itself and satisfies*

$$\kappa_1 = 12\lambda - \delta + \sum_i \psi_i.$$

Proposition 6.9 (The Gram identity and its radical). *For every $n \geq 10$, the matrix C_n is symmetric and satisfies*

$$\rho_n(\sigma)^t C_n \rho_n(\sigma) = C_n \quad (\sigma \in \mathfrak{S}_n),$$

and for all $a, b \in A_n$ and $u, v \in S$,

$$\int_{\overline{\mathcal{M}}_{3,n}} \Phi_n(a, u) \Phi_n(b, v) \kappa_1^{n-9} = C_n(a, b) J(u, v).$$

As a map of Hodge structures,

$$\ker \Phi_n = (\ker C_n) \otimes \mathbb{P}, \quad H^{15}(\overline{\mathcal{M}}_{3,n}) \cong (A_n / \ker C_n) \otimes \mathbb{P}.$$

Proof. We identify the matrix with the pulled-back intersection form, then apply Hard Lefschetz to determine its radical.

The intersection entries. Lemma 3.21 computes the product of two boundary Gysin images in their displayed order. Under a contraction, the inverse graph of a genus-one vertex either has a unique genus-one vertex and rational trees, or consists of rational vertices with first Betti number one. In the first case the hook pullback is obtained by iterating Proposition 3.15; in the second it is zero, since each rational factor has only even cohomology by Theorem 3.2. Thus Definition 6.1 gives the actual coefficient pullbacks. The logarithmic convention and Theorem 3.20 give

$$\xi_\Gamma^* \kappa_1^{n-9} = \left(\sum_{v \in V(\Gamma)} \kappa_{1,v} \right)^{n-9}.$$

If the surviving hooks occur at different vertices, each such vertex carries an odd-degree class multiplied by even-degree decorations. Its integral is zero. If they occur at the same vertex, Proposition 6.6 evaluates that factor as $\mathcal{C}_{F(v)}(k_v, l_v; f_v)J(u, v)$. Every other vertex has genus at most two and at least one flag: total genus is three, and the refinement is connected with an edge. Its polynomial integral is $W_{g(w)}(f_w)$ by Proposition 6.3. Künneth integration gives precisely (6.5), including its two input automorphism factors, and proves the asserted Gram identity.

Every entry is a finite rational expression (Definitions 6.2–6.7).

The ambient intersection form is alternating because its arguments have degree fifteen. Since J is alternating and nondegenerate, the Gram identity gives $C_n^t = C_n$. Relabelling preserves κ_1 and integration and commutes with Φ_n . Applying the identity after relabelling both inputs gives $\rho_n(\sigma)^t C_n \rho_n(\sigma) = C_n$.

Detection of the kernel. The coarse space of $\overline{\mathcal{M}}_{3,n}$ is a projective orbifold of complex dimension $d = n + 6$. Its rational cohomology agrees with that of the stack: on a finite quotient chart the positive-degree rational cohomology of each stabilizer vanishes, so the coarse-space map has no higher rational direct images. By Theorem 6.8, a positive integral multiple of this logarithmic κ_1 is ample and hence Kähler. For $n \geq 10$ one has $15 \leq d$, and ordinary Hard Lefschetz for the projective orbifold $\overline{\mathcal{M}}_{3,n}$ with this Kähler class [Fer06, Theorem 2.2(1)] gives

$$\kappa_1^{n-9} : H^{15}(\overline{\mathcal{M}}_{3,n}; \mathbb{Q}) \xrightarrow{\sim} H^{2d-15}(\overline{\mathcal{M}}_{3,n}; \mathbb{Q}).$$

That theorem gives the isomorphism after extension to $\mathbb{R}$; its map is rational, so it is already an isomorphism over $\mathbb{Q}$. Poincaré duality makes $\int xy \kappa_1^{n-9}$ nondegenerate on H^{15} .

The map Φ_n is surjective by Proposition 5.20. A vector in the radical of its pulled-back form has image pairing to zero with the whole target and therefore has zero image. Conversely, each kernel vector lies in that radical. The pulled-back form is $C_n \otimes J$, whose radical is $(\ker C_n) \otimes S$ because J is nondegenerate. Restoring the common Tate factor gives the stated kernel and quotient as Hodge structures. $\square$

Definition 6.10. Put $\mathcal{E}_{3,n} = A_n / \ker C_n$ for $n \geq 10$, and put $\mathcal{E}_{3,n} = 0$ for $0 \leq n \leq 9$. For $n \geq 10$ with $N_n > 0$, put

$$Q_n = \sum_{\sigma \in \mathfrak{S}_n} \rho_n(\sigma)^t \rho_n(\sigma), \quad M_n = Q_n^{-1} C_n^t Q_n^{-1} C_n,$$

and define, for $\sigma \in \mathfrak{S}_n$,

$$(6.6) \quad e_{3,n}(\sigma) = \operatorname{tr} \rho_n(\sigma) - [z^0] \frac{z \operatorname{tr}(\rho_n(\sigma) \operatorname{adj}(z I_{N_n} + M_n))}{\det(z I_{N_n} + M_n)}.$$

The coefficient is the Laurent constant at $z = 0$, and adj is the classical adjugate. Set $e_{3,n} = 0$ when $N_n = 0$ or $0 \leq n \leq 9$. For a partition μ let $m_j(\mu)$ count its parts equal to j , put $z_\mu = \prod_j j^{m_j(\mu)} m_j(\mu)!$ and $p_\mu = \prod_i p_{\mu_i}$, and let $\chi_\lambda(\mu)$ be the ordinary character of V_λ on that cycle type. Define

$$F_3 = \sum_{n \geq 10} \sum_{\mu \vdash n} \frac{e_{3,n}(\mu)}{z_\mu} p_\mu, \quad m_{3,n}^\lambda = \sum_{\mu \vdash n} \frac{\chi_\lambda(\mu) e_{3,n}(\mu)}{z_\mu}.$$

Lemma 6.11 (Restriction of elliptic degree-eleven classes). *For every nonempty finite marking set L , restriction induces an $\mathfrak{S}_L$ -equivariant isomorphism*

$$H^{11}(\overline{\mathcal{M}}_{1,L}; \mathbb{Q}) \xrightarrow{\sim} W_{11} H^{11}(\mathcal{M}_{1,L}; \mathbb{Q}).$$

Proof. We apply the ordinary weight sequence to the full normalized boundary, keeping every quotient-stack component. On an étale normal-crossing chart of $\overline{\mathcal{M}}_{1,L}$, its incoming degree-eleven

term is the direct sum over all stable one-edge graphs Γ of type $(1, L)$:

$$\bigoplus_{[\Gamma]} \left(H^9(X_\Gamma; \mathbb{Q}) \otimes \mathbb{L} \right)^{\text{Aut}_L(\Gamma)} \longrightarrow H^{11}(\overline{\mathcal{M}}_{1,L}; \mathbb{Q}) \longrightarrow W_{11} H^{11}(\mathcal{M}_{1,L}; \mathbb{Q}) \longrightarrow 0.$$

For a nonseparating graph, the normalization is a genus-zero moduli stack, so its degree-nine cohomology vanishes by Theorem 3.2. For a separating graph, the vertex genera are one and zero. Künneth decomposes $H^9(X_\Gamma)$ into tensor products whose degrees add to nine. At least one factor has odd degree at most nine, and that factor vanishes by Theorem 3.3; the genus-zero factor also has no odd cohomology by the same genus-zero theorem. Thus every $H^9(X_\Gamma)$ is zero, before taking automorphism invariants.

The normalized boundary and its branch-order local system descend from the charts by finite quotient groups. Taking invariants is exact over $\mathbb{Q}$, and the quotient-stack Gysin maps commute with this descent. Consequently the displayed source is zero on the stack as well. The weight sequence therefore identifies restriction with an isomorphism onto $W_{11} H^{11}(\mathcal{M}_{1,L})$. All constructions commute with relabelling of L , which proves $\mathfrak{S}_L$ -equivariance. $\square$

Proposition 6.12 (The genus-three threshold). *For $0 \leq n \leq 8$, nonseparating pullback is injective from $H^{15}(\overline{\mathcal{M}}_{3,n})$ into $H^{15}(\overline{\mathcal{M}}_{2,n+2})$, and the latter group is zero. Also $H^{15}(\overline{\mathcal{M}}_{3,9}) = 0$. For $n \geq 10$, write $\text{Std}_{[n]} = \ker(\mathbb{Q}^{[n]} \xrightarrow{\sum} \mathbb{Q})$. Clutching two unmarked elliptic tails to an elliptic component carrying all ordinary labels, with the tail-exchange quotient, induces an injection*

$$\left(\bigwedge^{10} \text{Std}_{[n]} \oplus \bigwedge^9 \text{Std}_{[n]} \right) \otimes \mathbb{P} \hookrightarrow H^{15}(\overline{\mathcal{M}}_{3,n}).$$

Proof. We prove vanishing below arity nine by duality, prove that every boundary Gysin image vanishes at arity nine, and detect the equal-tail image in every larger arity.

Arities at most eight. The dimension of $\overline{\mathcal{M}}_{3,n}$ is $n + 6$, so the Poincaré-dual degree to fifteen is $2n - 3$. For $n = 0, 1$ this degree is negative. For $2 \leq n \leq 6$ it is one of 1, 3, 5, 7, 9, and Theorem 3.3 applies. At $n = 7$ it is eleven, which vanishes in genus three by Proposition 3.15. At $n = 8$ it is thirteen, which vanishes by Theorem 3.23: the defining sum for $U_{3,[8]}$ is empty. Thus the genus-three source of the asserted pullback is zero. Its genus-two target is zero by Theorem 4.18, since $n + 2 \leq 10$.

The separating source at arity nine. Proposition 5.20 gives its complete ordinary one-edge boundary source. By Theorem 3.3, a degree-thirteen class on a separating normalization can only have degrees $(13, 0), (11, 2), (2, 11), (0, 13)$. A degree-eleven elliptic factor would need ten ordinary labels in addition to the attaching flag, so both mixed sectors are zero. A rational component has no odd cohomology; if it carries a stable rational tail, the complementary genus-three factor has at most eight markings and has zero H^{13} . On a genus-one factor there are at most ten markings, so $U_{1,[m]} = 0$. Theorem 3.23 therefore leaves only

$$H^{13}(\overline{\mathcal{M}}_{2,[9] \cup \{p\}}) \boxtimes H^0(\overline{\mathcal{M}}_{1,\{p'\}}).$$

The same theorem describes the first factor by its unique allocation of ten labels to an elliptic degree-eleven component. Attaching the outer unmarked elliptic tail produces the two-edge star with an elliptic center carrying all nine ordinary labels and two unmarked elliptic tails. Its center has eleven flags, so its hook is the sign representation. The graph automorphism group has order two, generated by exchanging the tails and their central flags. This exchange acts by -1 on the hook and by $+1$ on the unit decorations. If γ is such a decoration and τ is the exchange, then

$$\xi_{\Gamma*} \gamma = \xi_{\Gamma*} \tau^* \gamma = -\xi_{\Gamma*} \gamma = 0.$$

Each of the two separating one-edge maps used in this factorization has graph automorphism group one; their composite is the labelled $\xi_{\Gamma*}$, twice the normalized star class. Thus its vanishing also proves that the full separating source maps to zero.

The nonseparating source at arity nine. Write $I = [9] \cup \{h, h'\}$. The remaining source is $H^{13}(\overline{\mathcal{M}}_{2,I})^{\langle(hh')\rangle}$, and its outer Gysin map is $\frac{1}{2}\xi_{\text{irr}*}$. By Theorem 3.23, the inner elliptic allocations have size ten or eleven. We may check their labelled generators before averaging under (hh') .

If a size-ten allocation omits an ordinary label, gluing h, h' makes a loop on the decorated elliptic vertex. Reversal of the loop transposes two of its eleven hook flags. If it omits h or h' , the resulting graph has two parallel edges between its elliptic vertices; their exchange transposes two of the eleven flags on the decorated vertex. In each case the graph automorphism group has order two, the other factor carries the unit, and the displayed anti-invariance calculation shows that the labelled pushforward vanishes. The outer factor $1/2$ makes it exactly the normalized two-edge class.

For the size-eleven allocation the elliptic center carries all of I and an attaching flag p , and the unmarked elliptic tail has flag p' . Let

$$\eta : \overline{\mathcal{M}}_{1,I \cup \{p\}} \longrightarrow \overline{\mathcal{M}}_{2,[9] \cup \{p\}}, \quad \iota : \overline{\mathcal{M}}_{2,[9] \cup \{p\}} \times \overline{\mathcal{M}}_{1,\{p'\}} \longrightarrow \overline{\mathcal{M}}_{3,9}$$

glue h, h' and attach the tail, respectively. For a degree-eleven class s at the center, transitivity of labelled Gysin maps writes the two-edge contribution as

$$\frac{1}{2}\xi_{\Gamma*}(s \boxtimes 1) = \iota_* \left(\left(\frac{1}{2}\eta_* s \right) \boxtimes 1 \right).$$

The factor $1/2$ on both sides is the loop-reversal normalization; the outer separating graph has automorphism group one. The class $\frac{1}{2}\eta_* s$ belongs to $H^{13}(\overline{\mathcal{M}}_{2,[9] \cup \{p\}})$, which is the image of the unique ten-label elliptic allocation already treated. Applying ι_* therefore gives the equal-tail star class, which vanishes. All one-edge sectors have now been exhausted, and surjectivity of the sum of their Gysin maps proves $H^{15}(\overline{\mathcal{M}}_{3,9}) = 0$.

The equal-tail source. Let $n \geq 10$, and choose four distinct new flags p, q, p', q' . Put $L = [n] \cup \{p, q\}$ and let τ exchange p, q and the two tail factors in

$$\mathcal{D}_n = \left[\overline{\mathcal{M}}_{1,L} \times \overline{\mathcal{M}}_{1,\{p'\}} \times \overline{\mathcal{M}}_{1,\{q'\}} / \langle \tau \rangle \right].$$

Let $\zeta_n : \mathcal{D}_n \rightarrow \overline{\mathcal{M}}_{3,n}$ glue p to p' and q to q' . Thus its degree-eleven source is $K_L^{\langle(pq)\rangle} \otimes S$, with Tate factor L^2 .

The tuple presentation identifies K_L with $\bigwedge^{10} \text{Std}_L$: the symbol $[b_1, \dots, b_{11}]$ maps to

$$(\mathbf{e}_{b_1} - \mathbf{e}_{b_{11}}) \wedge \dots \wedge (\mathbf{e}_{b_{10}} - \mathbf{e}_{b_{11}}),$$

where $\mathbf{e}_b$ is the permutation-basis vector. Exterior multilinearity gives the alternating tuple relations, and the anchor symbols map to an exterior basis. Under $\mathfrak{S}_n \times \langle(pq)\rangle$ there is a decomposition

$$\text{Std}_L = \text{Std}_{[n]} \oplus \mathbb{Q} \left(\mathbf{e}_p + \mathbf{e}_q - \frac{2}{n} \sum_{i=1}^n \mathbf{e}_i \right) \oplus \mathbb{Q}(\mathbf{e}_p - \mathbf{e}_q).$$

The last line is anti-invariant and the other two summands are fixed by (pq) . Taking the tenth exterior power and its invariants gives exactly $\bigwedge^{10} \text{Std}_{[n]} \oplus \bigwedge^9 \text{Std}_{[n]}$.

Detection by self-intersection. Restrict the central factor to $\mathcal{M}_{1,L}$, and write

$$\tilde{\mathcal{U}}_n = \mathcal{M}_{1,L} \times \overline{\mathcal{M}}_{1,\{p'\}} \times \overline{\mathcal{M}}_{1,\{q'\}}, \quad \mathcal{U}_n = [\tilde{\mathcal{U}}_n / \langle \tau \rangle].$$

Remove from $\overline{\mathcal{M}}_{3,n}$ the image under ζ_n of the locus where the central factor is singular, obtaining an open substack $\mathcal{V}_n$. This removed image is closed because the full clutching map is proper. It is disjoint from the image of $\mathcal{U}_n$: a curve in that image has a unique smooth elliptic component

carrying every ordinary label, and the two separating nodes cutting off unmarked genus-one subcurves are determined by the curve. This remains true when either one-pointed tail is nodal. Consequently ζ_n restricts to a closed regular embedding $i : \mathcal{U}_n \hookrightarrow \mathcal{V}_n$ of codimension two. In local deformation coordinates its equations are the smoothing parameters of these two nodes. The quotient accounts for their exchange, so there are no additional branches of this embedding.

For $s \in H^{11}(\overline{\mathcal{M}}_{1,L})^{(pq)}$, let $r(s)$ be its restriction to $\mathcal{M}_{1,L}$. By Lemma 6.11, this map is injective on the weight-eleven source, so $s \neq 0$ implies $r(s) \neq 0$. Pulling the self-intersection identity for i back to the ordered cover gives exactly

$$(r(s) \boxtimes 1 \boxtimes 1)(\psi_p + \psi_{p'})(\psi_q + \psi_{q'}).$$

The two normal lines have first Chern classes $-\psi_p - \psi_{p'}$ and $-\psi_q - \psi_{q'}$; their product has the displayed positive sign. Its Künneth component of degrees $(11, 2, 2)$ is

$$r(s) \boxtimes \psi_{p'} \boxtimes \psi_{q'}.$$

For $s \neq 0$ this is nonzero: $r(s) \neq 0$, and each tail cotangent class has integral $1/24$ by Proposition 6.3. The other three terms have different Künneth degrees. Pullback to the ordered cover is injective, since its trace composed with pullback is multiplication by two. For the quotient-normalized Gysin map the displayed self-intersection has coefficient one: the class descends with pullback $s \boxtimes 1 \boxtimes 1$, while its global pushforward is $\frac{1}{2}$ of the ordered pushforward. Thus every nonzero invariant s has nonzero global image. Restoring the codimension-two twist proves the asserted equivariant injection. Finally, $\dim \bigwedge^9 \text{Std}_{[n]} = \binom{n-1}{9} > 0$ and $S \neq 0$, so the injected module is nonzero for every $n \geq 10$. $\square$

Theorem 6.13 (Genus-three decomposition). *For every integer $n \geq 0$ and partition $\lambda \vdash n$, the numbers $m_{3,n}^\lambda$ are nonnegative integers and there are $\mathfrak{S}_n$ -equivariant isomorphisms*

$$H^{15}(\overline{\mathcal{M}}_{3,n}; \mathbb{Q}) \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes \mathbb{P}^{\oplus m_{3,n}^\lambda}, \quad \text{ch}_n(\mathcal{E}_{3,n}) = [F_3]_n.$$

The isomorphism is one of pure Hodge structures. For every prime ℓ the same multiplicities give an actual $\mathfrak{S}_n \times \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -equivariant isomorphism

$$H_{\text{et}}^{15}(\overline{\mathcal{M}}_{3,n,\overline{\mathbb{Q}}}; \mathbb{Q}_\ell) \cong \bigoplus_{\lambda \vdash n} (V_\lambda \otimes \mathbb{Q}_\ell) \otimes (\mathbb{Q}_\ell(-2) \otimes S_\ell)^{\oplus m_{3,n}^\lambda},$$

Moreover, $H^{15}(\overline{\mathcal{M}}_{3,n}) \neq 0$ if and only if $n \geq 10$. Every displayed multiplicity is determined by the finite graph sum and terminating recursions in Definitions 6.2–6.10.

Proof. We compute the character of the Gram quotient, identify its Hodge realization, and then apply comparison to the sum of all ordinary boundary Gysin maps.

The quotient character. Fix $n \geq 10$ with $N_n > 0$. For every nonzero real column x ,

$$x^t Q_n x = \sum_{\sigma \in \mathfrak{S}_n} \|\rho_n(\sigma)x\|^2 > 0.$$

Reindexing the sum gives $\rho_n(\sigma)^t Q_n \rho_n(\sigma) = Q_n$. Together with the corresponding identity for C_n from Proposition 6.9, this implies that $Q_n^{-1} C_n$ commutes with every $\rho_n(\sigma)$. Its adjoint for the Q_n -inner product is $Q_n^{-1} C_n^t$, so M_n also commutes with the action and is self-adjoint. Moreover,

$$x^t Q_n M_n x = (C_n x)^t Q_n^{-1} (C_n x) \geq 0, \quad \ker M_n = \ker C_n.$$

The real spectral theorem diagonalizes M_n with nonnegative eigenvalues. Therefore the rational endomorphism

$$\Pi_n = [z^0] z(zI_{N_n} + M_n)^{-1}$$

is the projection onto $\ker C_n$. On the zero eigenspace the displayed function is one, while on an eigenspace with eigenvalue $\nu > 0$ the constant coefficient of $z/(z + \nu)$ is zero. The projection is

rational because the inverse matrix has rational function entries, and it commutes with the group action.

The Laurent coefficient in (6.6) is consequently $\text{tr}(\rho_n(\sigma)\Pi_n) = \text{tr}(\sigma \mid \ker C_n)$. Subtracting it from the source character proves that $e_{3,n}$ is the character of $A_n/\ker C_n$. When $N_n = 0$ the quotient and the specified character are both zero.

Character orthogonality gives $\text{mult}(V_\lambda, A_n/\ker C_n) = m_{3,n}^\lambda$ (Definition 6.10). These are non-negative integers, and the Frobenius characteristic is $\sum_{\mu \vdash n} e_{3,n}(\mu) p_\mu / z_\mu = [F_3]_n$. Proposition 6.9 identifies the corresponding Specht decomposition tensored with $\mathbf{P}$ with the full Hodge structure $H^{15}(\overline{\mathcal{M}}_{3,n})$. For $n \leq 9$, the definitions give $e_{3,n} = 0$ and $m_{3,n}^\lambda = 0$, and Proposition 6.12 gives the same vanishing for the cohomology. This proves the Hodge assertions in every arity.

The actual Galois realization. Use the complete map $\Phi_{3,n,B}^{15} : \mathcal{D}_{3,n,B}^{13} \rightarrow H^{15}(\overline{\mathcal{M}}_{3,n})$ of Definition 3.24. It is surjective in every arity by Proposition 5.20, and its Betti image has just been identified with $\mathcal{E}_{3,n} \otimes \mathbf{P}$. The source identifications use the actual forgetful maps realizing the hooks in Proposition 3.15, the elliptic separating maps in Theorem 3.23, and the algebraic divisor classes in the mixed degree-eleven and degree-two sectors. Their composites with the outer quotient-normalized clutching maps are defined over $\mathbb{Q}$. In particular these are the geometric maps of Proposition 3.26, which commute with comparison and every Specht projection. Its surjective-image assertion applies and gives the actual equivariant isomorphism

$$H_{\text{et}}^{15}(\overline{\mathcal{M}}_{3,n,\overline{\mathbb{Q}}}; \mathbb{Q}_\ell) \cong (\mathcal{E}_{3,n} \otimes \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(-2) \otimes S_\ell.$$

The rational Specht decomposition established above gives exactly the displayed multiplicities in this isomorphism.

The nonzero injected module of Proposition 6.12 gives $H^{15} \neq 0$ for $n \geq 10$; every multiplicity is determined by the finite formulas above. $\square$

7. THE ORDINARY GYSIN IMAGE

We describe the ordinary degree-fifteen boundary image in genus at least four as the image of four explicit Gysin maps. The induced map from the direct sum of their domains is injective: a rooted degree-four quotient separates the Picard and parallel-edge contributions, and further restrictions detect the elliptic degree-thirteen and path contributions.

Definition 7.1. For every stable genus and finite marking set put

$$\mathcal{I}_{g,F} = \text{im } \Phi_{g,F,B}^{15} \subset H^{15}(\overline{\mathcal{M}}_{g,F}).$$

For $h \geq 3$ and $q \in J$, let $\Pi_{h,J;q}^\circ \subset \Pi_{h,J}$ be the divisor multiplicity subspace spanned by κ_1 , every ψ_i , δ_{irr} , and all stable separating divisors except those whose genus-zero side contains q . Complementary descriptions of a divisor are identified.

Definition 7.2. Let $g \geq 4$ and let F be finite. Using distinct auxiliary flags, set

$$\begin{aligned} \mathcal{D}_{g,F} &= \bigoplus_{A \subseteq F} U_{1,A \cup \{p\}}, \\ \mathcal{W}_{g,F} &= \bigoplus_{A \subseteq F} K_{A \cup \{p\}} \otimes \Pi_{g-1,(F \setminus A) \cup \{q\};q}^\circ, \\ \mathcal{R}_{g,F} &= \bigoplus_{A \subseteq F} K_{A \cup \{u,v\}}^{\langle uv \rangle}. \end{aligned}$$

Let $\mathcal{T}_{g,F}$ be the direct sum over stable three-vertex paths with legs F , a distinguished genus-one center carrying A , and two unordered positive-genus leaves of genera r, s with $r + s = g - 1$, of

$$K_{A \cup \{p_1, p_2\}}^{\text{Aut}_F(\Gamma)}.$$

The leaves carry units. All automorphism invariants are ordinary, without an edge determinant.

The maps $\Gamma_D, \Theta, J_{\text{path}}$ are the Gysin maps that respectively attach a genus- $(g-1)$ partner carrying the unit class to the genus-one degree-thirteen class, attach a Picard-decorated genus- $(g-1)$ partner to a hook class, and glue the two edges of the path. Let Π_{par} be the quotient-normalized gluing of the two parallel pairs on

$$[\overline{\mathcal{M}}_{1,A \cup \{u,v\}} \times \overline{\mathcal{M}}_{g-2,(F \setminus A) \cup \{u',v'\}} / \mathfrak{S}_2],$$

where $\mathfrak{S}_2$ exchanges both pairs simultaneously. Put

$$\Omega_{g,F} = (\Gamma_D, \Theta, J_{\text{path}}, 2\Pi_{\text{par}}) : (D_{g,F} \oplus W_{g,F} \oplus T_{g,F} \oplus R_{g,F}) \otimes \mathbb{P} \longrightarrow H^{15}(\overline{\mathcal{M}}_{g,F}).$$

Definition 7.3. For $h \geq 3$ and a finite set J , let $\mathcal{E}_{h,J}^{(4)}$ be the free rational vector space on Polito's Section 3 essential degree-four symbols, with realization

$$e_{h,J} : \mathcal{E}_{h,J}^{(4)} \longrightarrow H^4(\overline{\mathcal{M}}_{h,J}; \mathbb{Q}).$$

The symbols are Mumford monomials, one-edge classes decorated by a degree-two class on a normalization factor, and pure two-edge graph classes. Normalize a decoration by

$$p \mid \delta_\Gamma = \frac{\xi_{\Gamma,*}(p)}{|\text{Aut } \Gamma|}.$$

The presentation in Section 3 eliminates the ψ and κ_1 decorations on genus-zero and genus-one factors and the κ_1 decorations on genus-two factors by their divisor relations. To reduce the pure graph symbols, the presentation uses Keel relations among trees with two rational vertices; choose the retained symbols as in that section. Every other pure two-edge graph is retained. In particular, for $h \geq 3$ the Mumford symbols

$$\kappa_1^2, \quad \kappa_2, \quad \kappa_1\psi_i, \quad \psi_i^2, \quad \psi_i\psi_j \quad (i \neq j)$$

are retained, as are the distinct nonseparating mixed symbols

$$\mathbf{N}_{h,J}^{\text{irr}} = \frac{1}{2}\xi_{\text{irr},*}(\psi_u + \psi_v), \quad \mathbf{X}_{i,h,J}^{\text{irr}} = \frac{1}{2}\xi_{\text{irr},*}(\psi_i).$$

Here u, v are auxiliary node flags. The nonseparating κ_1 decoration is eliminated at $h = 3$. Let pr_M project onto the free span of the Mumford symbols, and let $[m]r$ denote the coefficient of a symbol m in $r \in \mathcal{E}_{h,J}^{(4)}$.

Theorem 7.4 (Polito [Pol03, Section 5.3 and Theorem 10]). (1) Let $h \geq 3$ be an integer and J a nonempty finite marking set.

- (a) For $h \geq 6$, the map $e_{h,J}$ is injective.
- (b) For $h = 5$, its kernel is spanned by a vector R satisfying

$$[\kappa_2]R = -180.$$

- (c) For $h = 4$, its kernel is spanned by vectors R_0 and R_i for $i \in J$, satisfying

$$\begin{array}{c|cc} & [\kappa_1^2] & [\kappa_2] \\ \hline R_0 & 45/2 & -240 \\ R_i & 5 & -30 \end{array} \quad [\kappa_1\psi_j]R_i = \begin{cases} -40, & j = i, \\ -10, & j \neq i. \end{cases}$$

- (d) For $h = 3$, its kernel is spanned by vectors R_{00}, R_{02}, R_i for $i \in J$, and R_{ij} for unordered pairs $\{i, j\} \subset J$. There exist $a \in \mathbb{Q}$ and $c \in \mathbb{Q} \setminus \{0\}$ such that

$$\begin{array}{c|cc} & [\kappa_1^2] & [\kappa_2] \\ \hline R_{00} & 1 & 0 \\ R_{02} & 0 & 1 \end{array}$$

and, for all relevant i, j ,

$$\mathrm{pr}_M(R_i) = \kappa_1 \psi_i + \alpha \psi_i^2 - \sum_{j \in J \setminus \{i\}} \psi_i \psi_j, \quad [\mathbf{N}_{3,J}^{\mathrm{irr}}] R_i = c,$$

$$\mathrm{pr}_M(R_{ij}) = \psi_i^2 + \psi_j^2 - \frac{6}{5} \psi_i \psi_j.$$

(2) On $\overline{\mathcal{M}}_{2,\{q\}}$ let $\gamma_F, \gamma_E, \gamma_{H,q}, \gamma_{H,\emptyset}, \gamma_G$ denote the ordinary pure two-edge classes with the following graphs: a rational vertex carrying q and two loops; a rational vertex carrying q joined by two parallel edges to an elliptic vertex; a looped rational vertex joined to an elliptic vertex with q on the rational side; the same graph with q on the elliptic side; and a rational center carrying q joined to two elliptic leaves. The classes are normalized by the full finite graph-automorphism groups. Then these five classes are linearly independent in $H^4(\overline{\mathcal{M}}_{2,\{q\}}; \mathbb{Q})$, and

$$\psi_q^2 = \frac{1}{120} \gamma_F + \frac{1}{5} \gamma_E + \frac{13}{120} \gamma_{H,q} - \frac{1}{120} \gamma_{H,\emptyset} + \frac{7}{5} \gamma_G.$$

Proposition 7.5. *For an integer $h \geq 3$ and disjoint finite sets P, K with stable target, let $f : \overline{\mathcal{M}}_{h,P \cup K} \rightarrow \overline{\mathcal{M}}_{h,P}$ forget K and stabilize (a composite of forgetful maps as in Proposition 3.16). In the free-symbol pullback rules of Polito's Section 4.1 and Proposition 9, the Mumford parts of $f^* \kappa_1$ and $f^* \kappa_2$ are respectively $\kappa_1 - \sum_{j \in K} \psi_j$ and $\kappa_2 - \sum_{j \in K} \psi_j^2$. Products have the corresponding product Mumford part. Pure boundary symbols pull back to pure boundary symbols, and mixed symbols to mixed and pure boundary symbols. The essential reductions of boundary symbols are linear combinations of boundary symbols. The pullback of $\mathbf{N}_{h,P}^{\mathrm{irr}}$ has coefficient one at $\mathbf{N}_{h,P \cup K}^{\mathrm{irr}}$ and otherwise only pure boundary terms; the pullback of $\mathbf{X}_{i,h,P}^{\mathrm{irr}}$ has no $\mathbf{N}_{h,P \cup K}^{\mathrm{irr}}$ coefficient.*

Definition 7.6. Let $h \geq 3$, $q \in J$, and $B = J \setminus \{q\}$. For each nonempty $A \subseteq B$, let

$$j_A : \overline{\mathcal{M}}_{0,\{q\} \cup A \cup \{x\}} \times \overline{\mathcal{M}}_{h,(B \setminus A) \cup \{y\}} \longrightarrow \overline{\mathcal{M}}_{h,J}$$

glue x, y . Let $\mathcal{N}_{\mathrm{tail}}$ be the sum of $j_{A*} H^2(\mathrm{dom} j_A)$, and let $\mathcal{N}_{\mathrm{rr}}$ be the span of ordinary two-edge tree classes with exactly two genus-zero vertices. Put

$$Q_{h,J;q}^4 = H^4(\overline{\mathcal{M}}_{h,J}) / (\mathcal{N}_{\mathrm{tail}} + \mathcal{N}_{\mathrm{rr}}).$$

For every $A \subseteq B$, including the empty set, let ζ_A be the ordinary parallel-edge class with rational vertex carrying $\{q\} \cup A$ and genus- $(h-1)$ vertex carrying $B \setminus A$.

Lemma 7.7 (Rooted divisor and parallel detection). *For every $h \geq 3$ and rooted finite set (J, q) , the rational map*

$$\Pi_{h,J;q}^{\circ} \oplus \bigoplus_{A \subseteq J \setminus \{q\}} \mathbb{Q} \longrightarrow Q_{h,J;q}^4, \quad (\eta, (c_A)) \longmapsto \left[\psi_q \eta + \sum_A c_A \zeta_A \right]$$

is injective and equivariant for permutations fixing q .

Proof. Lifts of the subspaces being divided out. In $\mathcal{E}_{h,J}^{(4)}$ let $\mathcal{N}$ be spanned by the following essential boundary symbols: mixed classes on a rational-tail divisor whose rational side contains q , decorated on its positive-genus side; pure graphs admitting a contraction to such a divisor; and retained two-edge trees with two rational vertices. Then

$$\mathcal{N}_{\mathrm{tail}} + \mathcal{N}_{\mathrm{rr}} \subseteq e_{h,J}(\mathcal{N}).$$

Indeed, the degree-two Künneth decomposition on $\mathrm{dom} j_A$ has only the two factor summands, since the rational factor has no H^1 by Theorem 3.2. Theorem 3.17 expresses the class on the positive-genus factor by divisors. Its κ_1 and cotangent decorations give the indicated mixed symbols; its boundary decorations give pure graphs retaining the rational-tail edge. Divisor

classes on the rational factor give trees with two rational vertices. Their essential reductions remain in that family. In particular, $\mathcal{N}$ has neither Mumford coordinates nor mixed coordinates on the nonseparating divisor.

Independent column coordinates. Choose a lift of each displayed divisor generator multiplied by ψ_q . The products with κ_1 and ψ_i are their Mumford symbols, and $\psi_q \delta_{\text{irr}}$ is $\mathbf{X}_{q,h,J}^{\text{irr}}$, with normalization $1/2$. For a separating divisor, specify its q -side as (d, C) , where $q \in C$. The retained cases have $d \geq 1$. If $d \geq 2$, the product is the essential external- q mixed symbol on that side. These are distinct from one another and from every mixed symbol of $\mathcal{N}$.

If $d = 1$, let x be the attaching flag on its elliptic factor. Proposition 3.18 gives

$$\psi_q = \frac{1}{12} \delta_{\text{irr}} + \sum_{\substack{T \subseteq C \cup \{x\} \\ q \in T, |T| \geq 2}} \delta_{0,T}.$$

Push this identity along the separating clutching. The term $T = \{q, x\}$ is the pure tree with rational center carrying exactly q , genus-one leaf carrying $C \setminus \{q\}$, and genus- $(h-1)$ leaf carrying $J \setminus C$. Denote its essential symbol by γ_C . Its coefficient is one: the leaf genera $1, h-1$ are distinct, and the graph has trivial automorphism group. The nonseparating term has coefficient $1/12$ in ordinary graph normalization, since its loop reversal contributes the same denominator 2 as δ_{irr} on the elliptic factor.

The symbol γ_C lies outside $\mathcal{N}$. Both its one-edge contractions put q on a positive-genus component, and it has only one rational vertex. It occurs in no other elliptic column: to have rational center carrying exactly q , the above expansion must use $T = \{q, x\}$, which recovers C from the genus-one leaf. The other terms have a rational endpoint, extra center markings, or a loop.

Finally, each ζ_A is its retained parallel-graph symbol, normalized by the simultaneous edge exchange of order two. These distinct symbols have no separating edge and only one rational vertex. They occur in neither $\mathcal{N}$ nor any preceding column. Thus the chosen lifts are linearly independent modulo $\mathcal{N}$. Write $\mathcal{V}$ for their span. Every vector in $\mathcal{V} + \mathcal{N}$ has zero coordinate at

$$\kappa_1^2, \quad \kappa_2, \quad \kappa_1 \psi_j, \quad \psi_j^2 \ (j \neq q), \quad \psi_i \psi_j \ (i, j \neq q, i \neq j), \quad \mathbf{N}_{h,J}^{\text{irr}}.$$

Its only possible Mumford coordinates are $\kappa_1 \psi_q$ and $\psi_q \psi_i$.

Exclusion of the relation space. We claim

$$\ker e_{h,J} \cap (\mathcal{V} + \mathcal{N}) = 0.$$

For $h \geq 6$ this is Theorem 7.4(1). For $h = 5$, the coordinate $[\kappa_2]R = -180$ excludes every nonzero multiple of its relation.

For $h = 4$, suppose $\alpha R_0 + \sum_{i \in J} \beta_i R_i \in \mathcal{V} + \mathcal{N}$ and put $b = \sum_i \beta_i$. The first two missing coordinates give

$$\frac{45}{2} \alpha + 5b = 0, \quad -240\alpha - 30b = 0.$$

The determinant is 525, so $\alpha = b = 0$. For $j \neq q$, the $\kappa_1 \psi_j$ coordinate is

$$-40\beta_j - 10 \sum_{i \neq j} \beta_i = -30\beta_j,$$

hence $\beta_j = 0$. The equality $b = 0$ then gives $\beta_q = 0$.

For $h = 3$, write a relation as a combination of $R_{00}, R_{02}, R_i, R_{ij}$. The coordinates κ_1^2, κ_2 first remove R_{00}, R_{02} . Next $\kappa_1 \psi_j$, for $j \neq q$, removes R_j . For distinct $i, j \neq q$, the coordinate $\psi_i \psi_j$ is now $-6/5$ times the coefficient of R_{ij} , so all these coefficients vanish. For each $j \neq q$, the coordinate ψ_j^2 then removes R_{qj} . Only a multiple of R_q remains. Its coefficient at $\mathbf{N}_{3,J}^{\text{irr}}$ is the nonzero scalar c in Theorem 7.4(1), so it too vanishes. This argument includes $J = \{q\}$.

These are free-symbol coordinate tests; by Proposition 7.5 forgetting adds no Mumford coordinate from boundary terms and preserves the node coordinate.

Passage to the cohomological quotient. Suppose a linear combination of the proposed column lifts v realizes to an element of $\mathcal{N}_{\text{tail}} + \mathcal{N}_{\text{rr}}$. Choose $w \in \mathcal{N}$ with $e_{h,J}(v) = e_{h,J}(w)$. Then $v - w \in \ker e_{h,J} \cap (\mathcal{V} + \mathcal{N})$, so $v = w$. The distinct column coordinates force every coefficient to vanish. The constructions commute with all relabellings fixing q , proving the asserted equivariance. $\square$

Corollary 7.8. *Let $h \geq 3$ and let J be a finite set containing q . The divisor generators specified in Definition 7.1 form a basis of $\Pi_{h,J;q}^{\circ}$.*

Proof. They span by definition, and they are a subset of the basis of $H^2(\overline{\mathcal{M}}_{h,J}; \mathbb{Q})$ given by the basis clause of Theorem 3.17 for $h \geq 3$ (all generators except the rational-tail divisors $\delta_{0,A}$ with $q \in A$). $\square$

Lemma 7.9. *Let $b \geq 1$, let I be a finite marking set containing q , and let $(b, |I|)$ be stable. For $b \geq 2$, the restriction of ψ_q to $\mathcal{M}_{b,I}$ is nonzero. For $b = 1$, let $\mathcal{U}_I$ be the open in $\overline{\mathcal{M}}_{1,I}$ obtained by removing all separating boundary divisors and all strata of codimension at least two. Then $\psi_q|_{\mathcal{U}_I} \neq 0$.*

Proof. Independence from the boundary classes. Choose $k \geq 3$ with $b + k \geq 6$ and $k \neq b$, and a new marking z . Glue q to a flag q' on a genus- k component carrying z :

$$\iota : \overline{\mathcal{M}}_{b,I} \times \overline{\mathcal{M}}_{k,\{q',z\}} \longrightarrow \overline{\mathcal{M}}_{b+k,(I \setminus \{q\}) \cup \{z\}}.$$

If $b \geq 2$, the class $\iota_*(\psi_q \otimes 1)$ is an essential node-cotangent mixed symbol. On the other hand, pushing any boundary divisor of the first factor gives pure two-edge symbols, whose essential reductions remain pure. Theorem 7.4(1) is injective in this target genus, with its nonempty marking set. It therefore excludes a relation expressing ψ_q as a combination of boundary divisors on $\overline{\mathcal{M}}_{b,I}$.

If $b = 1$, Proposition 3.18 instead gives

$$\iota_*(\psi_q \otimes 1) = \frac{1}{12} \iota_*(\delta_{\text{irr}} \otimes 1) + \sum_{\substack{T \subseteq I \\ q \in T, |T| \geq 2}} \iota_*(\delta_{0,T} \otimes 1).$$

The first term is $1/12$ times the essential bridge-and-loop graph with a rational looped vertex and a genus- k vertex. Its automorphism group has order two, exactly the denominator of the elliptic δ_{irr} . All other terms are trees. The same injectivity therefore excludes a relation expressing ψ_q using only separating divisors on $\overline{\mathcal{M}}_{1,I}$.

Restriction to the opens. Remove first the strata with at least two nodes. This has codimension at least two and changes no H^2 . The remaining boundary components are disjoint smooth divisors. The divisor localization sequence says that the kernel on H^2 when they are removed is their span of fundamental classes. Removing all components proves the claim for $b \geq 2$; removing just the separating components proves it for $b = 1$. $\square$

Lemma 7.10. *Let I be a nonempty finite set, and let $\overline{\mathcal{M}}_{1,I}^{\text{ct}}$ be the locus of curves with tree dual graph. The restriction maps*

$$H^{11}(\overline{\mathcal{M}}_{1,I}) \longrightarrow H^{11}(\mathcal{M}_{1,I}), \quad H^{13}(\overline{\mathcal{M}}_{1,I}) \longrightarrow H^{13}(\overline{\mathcal{M}}_{1,I}^{\text{ct}})$$

are injective.

Proof. The supports sequence. For a union Δ of boundary components in a smooth proper stack Y , let $\Delta^{[p]}$ normalize its loci of p distinct local branches, with branch-order sign local system ϵ_p . The local Mayer–Vietoris complex of supports and purity in codimension p give

$$E_1^{1-p,t} = H^{t-2p}(\Delta^{[p]}, \epsilon_p) \otimes \mathbb{L}^p \implies H_{\Delta}^{t+1-p}(Y), \quad p \geq 1.$$

Indeed, on a chart with ordered branches the term for a p -fold intersection is shifted by $p - 1$ in the supports complex and by $2p$ under purity. The alternating complex descends under branch permutations, giving the displayed local system. The normalized intersections are finite quotients of products of smooth proper vertex moduli spaces. Thus each term has weight t and Hodge indices at least p in both coordinates.

The first injectivity is Lemma 6.11.

Degree thirteen on compact type. Apply the construction with $Y = \overline{\mathcal{M}}_{1,I}$ and $\Delta_{\text{ct}} = Y \setminus \overline{\mathcal{M}}_{1,I}^{\text{ct}}$. On total supports degree thirteen the terms have $t = 12 + p$. For $p \geq 2$ their weights are at least fourteen. For $p = 1$, the possible weight-thirteen term is $H^{11}(\Delta_{\text{ct}}^{[1]}) \otimes \mathbb{L}$. The normalization here is the quotient of $\overline{\mathcal{M}}_{0,I \cup \{u,v\}}$ by node exchange, so this term vanishes by Theorem 3.2. Consequently $H_{\Delta_{\text{ct}}}^{13}(Y)$ has no weight-thirteen part, and its map to the pure weight-thirteen group $H^{13}(Y)$ is zero. Localization gives the second injectivity. $\square$

Lemma 7.11 (Positive path detection). *Let Γ be a path of total genus g with legs F , elliptic center, and positive leaves of genera $h_1, h_2 \geq 1$. Let A, B_1, B_2 be the external marking sets of its center and two leaves, with attaching flags p_i on the center and q_i on leaf i . If $h_1 = h_2$, assume $B_1 \neq B_2$, so the external labels distinguish the two leaves. Refine each genus-one leaf into a rational vertex with one loop, leave every other vertex unchanged, and call the result $\tilde{\Gamma}$. Let O_{Γ} be the open union of strata obtained by contracting edges of $\tilde{\Gamma}$, and put*

$$Z_{\Gamma} = Y_0 \times Y_1 \times Y_2, \quad Y_0 = \mathcal{M}_{1,A \cup \{p_1, p_2\}},$$

where a leaf of genus at least two contributes its smooth moduli space $Y_i = \mathcal{M}_{h_i, B_i \cup \{q_i\}}$, and a genus-one leaf contributes $Y_i = \mathcal{U}_{B_i \cup \{q_i\}}$ from Lemma 7.9, the smooth or irreducible one-nodal curves. The restricted clutching map

$$i_{\Gamma} : Z_{\Gamma} \longrightarrow O_{\Gamma}$$

is a closed regular embedding of codimension two. For every $z \in H^{11}(\overline{\mathcal{M}}_{1,A \cup \{p_1, p_2\}})$,

$$\text{pr}_{11,2,2} i_{\Gamma}^* i_{\Gamma*}(z|_{Y_0} \otimes 1 \otimes 1) = z|_{Y_0} \otimes \psi_{q_1} \otimes \psi_{q_2},$$

and this map is injective in z . The open O_{Γ} meets no different two-edge tree stratum and no two-edge tree stratum with a rational vertex.

If instead $h_1 = h_2 \geq 2$ and $B_1 = B_2 = \emptyset$, so that $\text{Aut}_F(\Gamma)$ has order two, the same conclusions hold for the induced map $[Z_{\Gamma}/\text{Aut}_F(\Gamma)] \rightarrow O_{\Gamma}$, invariant cohomology being detected on the cover Z_{Γ} by the same self-intersection formula, without sign from the edge exchange.

Proof. The labelled support. At fixed genus and markings there are finitely many stable graph types. The types not obtained by contracting $\tilde{\Gamma}$ are closed under specialization, so their union is closed. This proves that O_{Γ} is open. Keeping the two original bridges in this open permits only the specified irreducible degenerations of the genus-one leaves. Consequently the inverse image of O_{Γ} in the labelled path product is exactly Z_{Γ} . The center is distinguished by its two bridges. The leaf genera, or their different external label sets when both genera are equal, distinguish the leaves. The bridges are therefore uniquely recovered from the curve. The proper clutching map restricted to this open is a closed embedding, and its two independent node smoothing parameters show that it is regular of codimension two. Loop reversal on an irreducible leaf belongs to that leaf's moduli factor; it does not exchange the two bridges.

Every loop must be contracted to obtain a tree from $\tilde{\Gamma}$. To retain two edges one must then retain both bridges, giving exactly Γ . In fact O_Γ is disjoint from the closure of every other two-edge tree stratum: a graph in such a closure would also contract to that tree. This proves the support assertions.

The self-intersection. The normal classes are $-\psi_{p_i} - \psi_{q_i}$, so the self-intersection formula gives

$$i_\Gamma^* i_{\Gamma*}(z \otimes 1 \otimes 1) = (z \otimes 1 \otimes 1)(-\psi_{p_1} - \psi_{q_1})(-\psi_{p_2} - \psi_{q_2}),$$

with z restricted to the smooth center. Only the product of the two leaf classes has multidegree $(11, 2, 2)$. Lemma 7.10 makes restriction on the center injective, and Lemma 7.9 makes each leaf class nonzero. Tensoring with these two classes is injective.

Equal unmarked leaves. Let $h_1 = h_2 \geq 2$ and $B_1 = B_2 = \emptyset$. The pair of bridges is unique, up to the indicated leaf exchange; a second bridge with the same genus and marking partition would bound an unmarked stable rational tree with only two exterior flags. Summing $2g(v) - 2 + |F(v)|$ on such a tree gives zero, contradicting stability. The two node-smoothing parameters give regular local equations. The support assertions hold as before. The cover $Z_\Gamma \rightarrow [Z_\Gamma / \text{Aut}_F(\Gamma)]$ has degree two; rational push-pull is multiplication by two, so invariant cohomology is detected on the cover by the same self-intersection formula. $\square$

Lemma 7.12 (Minimal elliptic tail detection). *Let $h \geq 2$, let $J = A \sqcup B$ with $A \neq \emptyset$, and let $W_A \subseteq \overline{\mathcal{M}}_{h+1, J}$ be the open union of contractions of graphs obtained by gluing a compact-type genus-one curve on $A \sqcup \{p\}$ to a smooth curve of genus h on $B \sqcup \{q\}$. Then the restricted clutching*

$$j_A^{\text{ct}} : \overline{\mathcal{M}}_{1, A \sqcup \{p\}}^{\text{ct}} \times \mathcal{M}_{h, B \sqcup \{q\}} \longrightarrow W_A$$

is a regular closed embedding of codimension one, and for every $z \in H^{13}(\overline{\mathcal{M}}_{1, A \sqcup \{p\}})$,

$$\text{pr}_{13,2}(j_A^{\text{ct}})^* j_{A*}^{\text{ct}}(z|_{\text{ct}} \otimes 1) = -z|_{\text{ct}} \otimes \psi_q.$$

This map is injective in z . A genus-one tail with external marking set A' can meet W_A only if $A' \subseteq A$.

Proof. The minimal tail support. The contraction criterion used in the preceding proof makes W_A open. An allowed uncontracted graph has one genus-one vertex, one smooth genus- h vertex carrying every B label, and only rational vertices on the genus-one side. A separating genus-one side therefore carries a subset of A : it must avoid the genus- h vertex, and contractions preserve the partition of a retained edge. There cannot be two edges with the same genus-one side and marking set: the intervening unmarked rational tree would have only two external attachments, contradicting stability. This proves both the assertion about A' and the uniqueness of the attachment of type A . It also identifies the inverse image of W_A with the displayed product: a contraction retaining an edge of type $(1, A)$ retains that unique edge; every other edge is on the compact-type genus-one side, and the partner remains a single smooth vertex; conversely, every point of the product is in the defining family. The genera distinguish the two factors, so the restricted clutching map is a regular closed embedding of codimension one with normal class $-\psi_p - \psi_q$.

The self-intersection. The self-intersection formula gives

$$(j_A^{\text{ct}})^* j_{A*}^{\text{ct}}(z|_{\text{ct}} \otimes 1) = -(z|_{\text{ct}} \psi_p) \otimes 1 - z|_{\text{ct}} \otimes \psi_q.$$

Only the second term has multidegree $(13, 2)$. Its first factor depends injectively on z by Lemma 7.10; its second factor is nonzero by Lemma 7.9, since $h \geq 2$. $\square$

Lemma 7.13. *Let $g \geq 4$ and let F be finite. The map J_{path} of Definition 7.2 is injective, and*

$$\text{im } J_{\text{path}} \cap \text{im } (\Gamma_D, \Theta, 2\Pi_{\text{par}}) = 0.$$

Proof. An open isolating one path. Fix a path Γ with center markings A and leaves $(r, B_1), (s, B_2)$. Since $r + s = g - 1 \geq 3$, at most one leaf has genus one, and leaves of equal genus have genus at least two; such leaves are distinguished by their labels or are both unmarked. Lemma 7.11 therefore applies; we use its open O_Γ , product $Z_\Gamma = Y_0 \times Y_1 \times Y_2$ and embedding i_Γ . By its support assertions, every other path class with positive-genus leaves restricts to zero on O_Γ . Theorem 3.23 expresses each elliptic H^{13} class by rational-tail Gysin classes. After the outer attachment, every Γ_D class is supported on a two-edge tree with a rational vertex. Hence Γ_D also restricts to zero. The same holds for Π_{par} , since a contraction of a graph whose edges are bridges or self-loops cannot have two parallel edges.

Separation from the Picard Gysin images. A genus-one tail divisor meets O_Γ only by cutting off the genus-one leaf, when one exists. Suppose it is Y_1 , so that Y_2 has genus $g - 2 \geq 2$. Its restricted normalization is a regular divisor embedding k , through which i_Γ factors. If β is its degree-eleven leaf class and $\eta_{02} \in H^2(Y_0 \times Y_2)$ is the restricted partner class, self-intersection gives, on Z_Γ ,

$$i_\Gamma^* k_*(\beta \otimes \eta) = -(\psi_{p_1} + \psi_{q_1}) \text{pr}_1^* \beta \text{pr}_{02}^* \eta_{02}.$$

For a Künneth component of η_{02} of degrees (d_0, d_2) , where $d_0 + d_2 = 2$, the two terms have respective degrees

$$(d_0 + 2, 11, d_2), \quad (d_0, 13, d_2)$$

in the fixed order (Y_0, Y_1, Y_2) . Their projection to degrees $(11, 2, 2)$ is zero. If both leaves have genus at least two, the supports of the Picard Gysin images are disjoint from the open.

Recovery of the path coefficient. By Lemma 7.11, the $(11, 2, 2)$ component of the self-intersection of a center coefficient is injective in that coefficient; when $|\text{Aut}_F(\Gamma)| = 2$ the ordinary map is $\frac{1}{2}\xi_{\Gamma*}$ on the invariant coefficient and the lemma detects it on the cover.

Apply this detector to $J_{\text{path}}(t) + \Gamma_D(d) + \Theta(w) + 2\Pi_{\text{par}}(\rho) = 0$. The support conditions imply vanishing of the other path contributions, the elliptic degree-thirteen terms, and the parallel terms; the fixed Künneth projection vanishes on the Picard terms. This injectivity then forces the Γ coefficient of t to vanish. Repeating for all paths gives $t = 0$, proving both assertions. $\square$

Lemma 7.14. *Let $g \geq 4$ and let F be finite. The map*

$$(\Gamma_D, \Theta, 2\Pi_{\text{par}}) : (D_{g,F} \oplus W_{g,F} \oplus R_{g,F}) \otimes \mathbf{P} \longrightarrow H^{15}(\overline{\mathcal{M}}_{g,F})$$

is injective.

Proof. (We suppress the common Tate factors.)

The rooted detectors. Put $h = g - 1$. For $A \subseteq F$ with $|A| \geq 10$, let

$$i_A : Y_A = \overline{\mathcal{M}}_{1,A \cup \{p\}} \times \overline{\mathcal{M}}_{h,(F \setminus A) \cup \{q\}} \longrightarrow \overline{\mathcal{M}}_{g,F}$$

be the labelled separating clutching. It is a regular divisor embedding. Indeed, a node with this genus and marking partition is unique. Two such genus-one sides would contain the same nonempty marking set, hence be nested; their intervening region would be an unmarked rational tree with two exterior flags, excluded by stability. The normal class is $-\psi_p - \psi_q$. Define

$$d_A : H^{15}(\overline{\mathcal{M}}_{g,F}) \longrightarrow H^{11}(\overline{\mathcal{M}}_{1,A \cup \{p\}}) \otimes Q_{h,(F \setminus A) \cup \{q\};q}^4$$

by i_A^* , the Künneth projection to $(11, 4)$, and the quotient on the second factor.

Let $C \subseteq F$ index another elliptic tail. Distinct nodes of types $(1, A)$ and $(1, C)$ meet transversely. After contracting all other edges, the possibilities are the three paths corresponding to $C \subsetneq A$, $A \subsetneq C$, or $A \cap C = \emptyset$. Partial overlap without containment is impossible: separating partitions are compatible, and two genus-one sides cannot cover a curve of genus at least four. Further degenerations are already included in these normalization products.

For $\Theta_C(\alpha \otimes \eta)$, the diagonal self-intersection has $(11, 4)$ part $-\alpha \otimes \psi_q \eta$. If $C \subsetneq A$, the first factor receives the inner Gysin and has degree at least thirteen. If $A \cap C = \emptyset$, the degree-eleven decoration stays on the second factor and the first receives degree at most two. Both cases disappear under the projection. If $A \subsetneq C$, the intersection on the partner is the q -containing rational-tail map $j_{C \setminus A}$. Its surviving class is the pushforward of a degree-two class on that normalization, hence belongs to $\mathcal{N}_{\text{tail}}$.

For $\Gamma_{D,C}(z)$, with z the elliptic H^{13} class, the diagonal terms have degrees $(15, 0)$ and $(13, 2)$. The cases $C \subsetneq A$ and $A \cap C = \emptyset$ disappear by the same degree bounds. If $A \subsetneq C$, only the $(11, 2)$ part of the restriction of z to the elliptic factor and rational middle can survive. Its degree-two rational class is pushed by $j_{C \setminus A}$ and again belongs to $\mathcal{N}_{\text{tail}}$. Thus

$$d_A \Gamma_D = 0, \quad d_A \Theta_C = 0 \ (C \neq A), \quad d_A \Theta_A(\alpha \otimes \eta) = -\alpha \otimes [\psi_q \eta].$$

Parallel pullback and its factor. On the source of $\Pi_{\text{par},C}$, the two selected parallel edges remain nonseparating under every degeneration. A node of type $(1, A)$ must therefore be an additional node in a source factor. If it separates that factor's two selected flags, it becomes nonseparating after gluing. Otherwise the genus-one side must be the side away from the cycle. When this side lies in the genus- $(g-2)$ factor, the hook remains on the other side and the $(11, 4)$ projection is zero. The remaining cases have $A \subseteq C$ and the fibre product component

$$\overline{\mathcal{M}}_{1,A \cup \{p\}} \times \left[\overline{\mathcal{M}}_{0,(C \setminus A) \cup \{q,u,v\}} \times \overline{\mathcal{M}}_{g-2,(F \setminus C) \cup \{u',v'\}} / \mathfrak{S}_2 \right].$$

The new node is distinct from the two selected nodes, so the intersection is transverse, of multiplicity one, with no excess line.

Let

$$\rho_{C,A} : K_{C \cup \{u,v\}}^{\langle (uv) \rangle} \longrightarrow K_{A \cup \{p\}}$$

be the rational-tail map of Definition 3.11 that collapses $(C \setminus A) \cup \{u, v\}$ to p . By Proposition 3.15 this is the actual boundary restriction. Base change gives

$$d_A \Pi_{\text{par},C}(\beta) = \begin{cases} \rho_{C,A}(\beta) \otimes [\zeta_{C \setminus A}], & A \subseteq C, \\ 0, & A \not\subseteq C. \end{cases}$$

Here $\rho_{C,A}$ acts on the hook factor and the common S is understood. Both the parallel source and the class $\zeta_{C \setminus A}$ divide the labelled pushforward by the same simultaneous flag exchange of order two. Thus this formula has coefficient one; the coefficient two in the map of the lemma remains an explicit factor two.

The map $\rho_{C,C}$ is an isomorphism. To check the map itself, put $\mathcal{A}_I = \ker(\mathbb{Q}^I \xrightarrow{\sum} \mathbb{Q})$. The tuple presentation identifies

$$K_I = \bigwedge^{10} \mathcal{A}_I.$$

The alternating eleven-tuple relation verifies this identification, and the anchored tuples give a basis on both sides. Identifying e_u, e_v with e_p maps the invariant subspace of $\mathcal{A}_{C \cup \{u,v\}}$ isomorphically to $\mathcal{A}_{C \cup \{p\}}$. Its kernel is the anti-invariant line $\mathbb{Q}(e_u - e_v)$. Taking tenth exterior powers, the invariant part contains no copy of this line, since its square is zero. The induced map is therefore an isomorphism. On tuples it replaces a unique u or v by p and kills a tuple containing both, exactly as $\rho_{C,C}$ does. This also shows that the invariant source vanishes for $|C| < 10$.

Elimination of the Picard and parallel terms. A linear relation among the images of the three Gysin maps gives, after applying d_A ,

$$-(\text{id} \otimes (\psi_q \cdot)) w_A + 2 \sum_{C \supseteq A} \rho_{C,A}(\beta_C) \otimes [\zeta_{C \setminus A}] = 0.$$

Tensor Lemma 7.7 with $H^{11}(\overline{\mathcal{M}}_{1,A \cup \{p\}})$. Different $C \supseteq A$ give different parallel columns, so $w_A = 0$ and every $\rho_{C,A}(\beta_C) = 0$. Taking $A = C$ and using the isomorphism just proved gives $\beta_C = 0$. This covers every nonzero hook summand in the source. The relation is now $\Gamma_D(d) = 0$.

Simultaneous detection of the elliptic classes. Write this last relation as

$$\sum_{C \subseteq F} i_{C*}(z_C \otimes 1) = 0, \quad z_C \in H^{13}(\overline{\mathcal{M}}_{1,C \cup \{p\}}).$$

Suppose some z_C is nonzero, and choose A of smallest cardinality among its nonzero indices. Definition 3.22 gives $|A| \geq 11$. Let $W_A \subseteq \overline{\mathcal{M}}_{g,F}$ be the open of Lemma 7.12 for the partner genus $h = g - 1 \geq 3$ and the decomposition $F = A \sqcup (F \setminus A)$. By that lemma, a genus-one tail of type C meets W_A only if $C \subseteq A$, so minimality of A reduces the restricted relation to

$$j_{A*}^{\text{ct}}(z_A|_{\text{ct}} \otimes 1) = 0 \quad \text{in } H^{15}(W_A),$$

and the $(13, 2)$ component of its self-intersection, $-z_A|_{\text{ct}} \otimes \psi_q$, depends injectively on z_A , a contradiction. $\square$

Theorem 7.15 (Structure of the ordinary image). *For every $g \geq 4$ and finite F , $\Omega_{g,F}$ is injective and its image is $\mathcal{I}_{g,F}$. Set $t_g = \lfloor (g-2)/2 \rfloor$ and let ϵ_g be one for odd g and zero for even g . The characteristic series of its four rational multiplicity summands, over $F = [n]$, are*

$$\begin{aligned} \sum_n \text{ch}_n(\mathcal{D}_{g,[n]}) &= E \partial U, \\ \sum_n \text{ch}_n(\mathcal{W}_{g,[n]}) &= H(2E + (g-1)E^2), \\ \sum_n \text{ch}_n(\mathcal{T}_{g,[n]}) &= t_g(H + G)E^2 + \epsilon_g(Hh_2[E] + Ge_2[E]), \\ \sum_n \text{ch}_n(\mathcal{R}_{g,[n]}) &= HE. \end{aligned}$$

Proof. *Generation by the four Gysin images.* Proposition 3.26 gives the complete one-edge source. Its mixed Künneth terms are H^{11} on a genus-one vertex and H^2 on its genus- $(g-1)$ partner. Divisor generation gives the full Picard source at that partner. A divisor omitted from Π° is $\delta_{0,\{q\} \cup B_0}$ for some nonempty $B_0 \subseteq F \setminus A$. Its composite is the two-edge path with genera 1, 0, $g-1$ and the hook on its genus-one endpoint. Contracting the edge adjacent to that endpoint first gives an actual elliptic H^{13} class, followed by the outer attachment. Thus its image lies in $\text{im } \Gamma_D$. The retained Picard generators give $\text{im } \Theta$.

The separating H^{13} sectors with a decorated genus-one vertex give Γ_D directly. For every decorated vertex of genus at least two, Theorem 3.23 expresses its class as a sum of images under elliptic-tail Gysin maps. A separating outer edge then gives a two-edge tree with one hook-decorated genus-one vertex. If this vertex is an endpoint and the middle vertex has positive genus, contracting the other edge first expresses the class in the full Picard source, hence in $\text{im}(\Gamma_D, \Theta)$. If the middle vertex is rational, contracting its edge to the decorated endpoint gives Γ_D . If the decorated vertex is the center, a rational leaf can likewise be contracted first to give Γ_D ; two positive leaves give J_{path} . These exhaust the tree cases.

For a nonseparating outer edge, expand $H^{13}(\overline{\mathcal{M}}_{g-1,F \cup \{u,v\}})$ by the same theorem. If neither of u, v is on the decorated elliptic vertex, the resulting loop is on its partner, giving the Picard decoration δ_{irr} . If both are on the elliptic vertex, first glue that loop to obtain an actual genus-two H^{13} class. Expanding this class by the genus-two instance of the theorem reduces to the tree cases just listed. If exactly one of u, v is on the elliptic vertex, the two edges are parallel, giving Π_{par} .

Here are the quotient factors in these reductions. All compositions on labelled products agree by functoriality of Gysin. A loop is divided by its reversal of order two. For a path with equal unmarked leaves, the labelled two-edge map f_Γ satisfies $J_{\text{path}} = \frac{1}{2}f_{\Gamma*}$ on the invariant hook. A labelled one-edge factorization yields $f_{\Gamma*}$, and one half of that coefficient therefore represents J_{path} . Otherwise a labelled path has trivial label-fixing group. For the parallel case, the two inner elliptic-tail coordinates containing exactly one outer node are exchanged by the outer loop flip. An invariant pair has the form $x + \tau x$, with τ the flag exchange. The outer factor $1/2$ makes its composite the labelled parallel pushforward of x . That pushforward vanishes on the anti-invariant part; on $x^+ = (x + \tau x)/2$ it is

$$f_{\text{par},*}(x^+) = 2\Pi_{\text{par}}(x^+).$$

Thus the remaining parallel source is the ordinary invariant hook, with precisely the coefficient two used in $\Omega_{g,F}$.

Every map defining $\Omega_{g,F}$ factors through an ordinary one-edge source by one of these contractions. Conversely, the preceding cases exhaust that source and place its image in $\text{im } \Omega_{g,F}$. Consequently $\text{im } \Omega_{g,F} = \mathcal{I}_{g,F}$.

Simultaneous injectivity. Lemma 7.13 shows that the path coefficient vanishes in every linear relation among the four Gysin images. Lemma 7.14 then shows that the remaining three coefficients vanish. Thus $\Omega_{g,F}$ is injective. The sources are the actual degree-thirteen, hook, and divisor multiplicity spaces; their maps are induced by ordinary Gysin pushforwards along algebraic clutching morphisms. They commute with relabelling, and the two codimension-one twists, or the single twist on a degree-thirteen source, give the common factor $\mathbf{P} = \mathbf{L}^2 \otimes S$.

Character operations and the two hook restrictions. Summing a tensor product over decompositions of a label set corresponds to induction from the product of the two symmetric groups, hence to multiplication of Frobenius series. The unit decoration on an arbitrary label set has series E . Fixing one additional flag corresponds to $\partial = \partial/\partial p_1$: in the power-sum formula, differentiation removes a fixed point and replaces its centralizer factor by that for the remaining cycle type.

With $\mathcal{A}_a = \mathcal{A}_{[a]}$ as in the proof of Lemma 7.14, the tuple identification used there gives the following restrictions of the hook:

$$K_{[a] \cup \{p\}} \cong \bigwedge^{10} \mathcal{A}_a \oplus \bigwedge^9 \mathcal{A}_a,$$

$$\text{Res}_{\mathfrak{S}_a \times \mathfrak{S}_2}^{\mathfrak{S}_{a+2}} \mathcal{A}_{a+2} \cong \mathcal{A}_a \oplus \mathbf{1} \oplus \text{sgn}_2.$$

The second decomposition follows by separating the two augmentation spaces of the ordinary-label block and the flag block; the remaining augmentation-zero direction is constant on each block. Taking its tenth exterior power shows that the plus and minus flag eigenspaces are

	representation on $[a]$	series over a
+	$\bigwedge^{10} \mathcal{A}_a \oplus \bigwedge^9 \mathcal{A}_a$	H
-	$\bigwedge^9 \mathcal{A}_a \oplus \bigwedge^8 \mathcal{A}_a$	G

Here $\bigwedge^j \mathcal{A}_a = V_{(a-j, 1^j)}$ when nonzero. At $a = 0$ all the hooks involved vanish; the formulas also vanish below their stated lower arities.

The elliptic degree-thirteen summand. Definition 3.22 gives the series

$$\sum_m \text{ch}_m(U_{1,[m]}) = E s_{(2, 1^{10})} + E_{\geq 3} H = U.$$

Fixing its attaching flag and allocating arbitrary complementary ordinary labels to the unit partner therefore gives

$$\sum_n \text{ch}_n(D_{g,[n]}) = E \partial U.$$

The Picard summand. Fix $h \geq 3$ and write the partner markings as $B \cup \{q\}$. By Corollary 7.8, we count the specified divisors as a basis. The three classes $\kappa_1, \psi_q, \delta_{\text{irr}}$ give $3E$. The classes ψ_i , for $i \in B$, give h_1E . For a separating divisor, the side containing q distinguishes the two sides even when the genera agree. There are $h - 1$ choices of positive genera on both sides, each giving E^2 . The remaining retained case has genus h on the q -side and a rational opposite side with at least two ordinary labels, giving $EE_{\geq 2}$. Hence

$$\sum_{b \geq 0} \text{ch}_b(\Pi_{h,[b] \cup \{q\};q}^\circ) = 3E + h_1E + (h - 1)E^2 + E(E - 1 - h_1) = 2E + hE^2.$$

Taking $h = g - 1$ and multiplying by the one-flag hook gives

$$\sum_n \text{ch}_n(\mathbf{W}_{g,[n]}) = H(2E + (g - 1)E^2).$$

The positive paths. There are t_g unordered distinct positive-genus pairs $r < s$ with $r + s = g - 1$. The distinct genera determine an ordering of their leaves, so their contribution is $t_g(H + G)E^2$. Equal genera occur exactly when $\epsilon_g = 1$. Start with ordered leaf label sets and then take invariants under simultaneous leaf and center-flag exchange.

For a permutation σ of the leaf labels, the trace of its composition with block exchange counts decompositions $B = B_1 \sqcup B_2$ such that $\sigma(B_1) = B_2$ and $\sigma(B_2) = B_1$. This number is zero unless every cycle is even. For cycle type 2μ it is $2^{\ell(\mu)}$, since each cycle admits two alternating assignments. Writing $z_\mu = \prod_j j^{m_j} m_j!$ for the centralizer order of cycle type $\mu = (1^{m_1} 2^{m_2} \dots)$, its Frobenius series is

$$\sum_\mu \frac{2^{\ell(\mu)} p_{2\mu}}{z_{2\mu}} = \sum_\mu \frac{p_{2\mu}}{z_\mu} = \Psi_2(E).$$

The plus and minus leaf eigenspaces consequently have series $h_2[E]$ and $e_2[E]$. Pairing these with the corresponding center eigenspaces gives

$$\sum_n \text{ch}_n(\mathbf{T}_{g,[n]}) = t_g(H + G)E^2 + \epsilon_g(Hh_2[E] + Ge_2[E]).$$

The parallel summand. The simultaneous edge exchange retains the plus two-flag hook, of series H . The partner allows arbitrary labels, so

$$\sum_n \text{ch}_n(\mathbf{R}_{g,[n]}) = HE.$$

□

8. THE GEOMETRIC BASE CASES

Projectivization parametrizes one-dimensional subspaces.

We prove that the images of the ordinary one-edge Gysin maps span degree-fifteen cohomology in genus four for every number of markings, and in genus five through ten markings. The additional cases beyond the Chow–Künneth bounds are controlled by the loci where marked points fail to impose independent conditions on the equations of a cover.

Let $\mathcal{H}_{k,g,n}$ parametrize smooth connected degree- k covers of a projective line with n distinct ordered source markings, allowing arbitrary ramification, and let β_n forget the cover. Let $\mathcal{M}_{g,n}^k$ denote the locus of gonality at most k . By [CL22, Section 8.2, equation (8.2)], the forgetful morphism $\beta'_n : \mathcal{H}_{k,g,n} \setminus \beta_n^{-1}(\mathcal{M}_{g,n}^{k-1}) \rightarrow \mathcal{M}_{g,n} \setminus \mathcal{M}_{g,n}^{k-1}$ is proper. We recall the two evaluation constructions in [CL22, Sections 9.2 and 10.3].

Definition 8.1 (Trigonal and tetragonal evaluation data). Let $k \in \{3, 4\}$. For $k = 3$, let $\pi : \mathcal{P} \rightarrow \mathcal{B}$ carry the universal globally generated rank-two degree- $g + 2$ bundle $\mathcal{E}$ on the universal projective line, and set

$$\mathcal{S} = \mathbb{P}\mathcal{E}^\vee, \quad \gamma : \mathcal{S} \rightarrow \mathcal{P}, \quad \mathcal{L} = \mathcal{O}_{\mathcal{S}}(3) \otimes \gamma^* \det \mathcal{E}^\vee, \quad \mathcal{U} = \gamma_* \mathcal{L} = \text{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee.$$

For $k = 4$, let $\mathcal{B}$ parametrize globally generated bundles $\mathcal{E}, \mathcal{F}$ on a projective-line bundle $\pi : \mathcal{P} \rightarrow \mathcal{B}$, of ranks 3, 2, both of degree $g + 3$, together with an isomorphism $\det \mathcal{E} \simeq \det \mathcal{F}$, and set

$$\mathcal{S} = \mathbb{P}\mathcal{E}^\vee, \quad \gamma : \mathcal{S} \rightarrow \mathcal{P}, \quad \mathcal{L} = \mathcal{O}_{\mathcal{S}}(2) \otimes \gamma^* \mathcal{F}^\vee, \quad \mathcal{U} = \gamma_* \mathcal{L} = \text{Sym}^2 \mathcal{E} \otimes \mathcal{F}^\vee.$$

For $k = 3$, restrict to $\mathcal{B}'$ where $\mathcal{U}$ is relatively globally generated; for $k = 4$, let $\mathcal{B}' = \mathcal{B} \setminus \text{Supp } R^1 \pi_* \mathcal{U}$. Put $\mathcal{X} = \pi_* \mathcal{U}|_{\mathcal{B}'}$. For $T_n = \mathcal{S} \times_{\mathcal{B}'} \cdots \times_{\mathcal{B}'} \mathcal{S}$, let $\eta_i : T_n \rightarrow \mathcal{S}$ and $\epsilon : T_n \rightarrow \mathcal{B}'$ be the projections, and set

$$e_n : \epsilon^* \mathcal{X} \longrightarrow \bigoplus_{i=1}^n \eta_i^* \mathcal{L}, \quad T_n^{\text{ev}} = \{e_n \text{ is surjective}\}, \quad \mathcal{Y}_n = e_n^{-1}(0).$$

For $k = 4$, let $\mathcal{H}'_{4,g,n}$ be the open substack over $\mathcal{B}'$ parametrizing smooth covers with distinct markings. For an integer f , let $\mathcal{H}^f_{4,g,n} \subset \mathcal{H}'_{4,g,n}$ be the locus where $\mathcal{F} \simeq \mathcal{O}(f_1) \oplus \mathcal{O}(f_2)$, $f_1 \leq f_2$, has $f_1 \geq f$.

Proposition 8.2 (Canning–Larson [CL22, Section 9.2, Lemmas 9.1–9.2; Section 10.3, (10.9) and Lemma 10.11]). *With the preceding data, for either k , $\mathcal{X}$ is a vector bundle, and $\mathcal{Y}_n|_{T_n^{\text{ev}}} = \ker(e_n|_{T_n^{\text{ev}}})$ is a vector bundle over T_n^{ev} . For $k = 3$, $\mathcal{H}_{3,g,n}$ is the open of $\mathcal{Y}_n$ on which the zero curve is smooth and the markings are distinct, and the base $\mathcal{B}'$ has the Chow–Künneth generation property. For $k = 4$, $\mathcal{H}'_{4,g,n} \subset \mathcal{Y}_n$ is open, and the base $\mathcal{B}$ has the Chow–Künneth generation property.*

Lemma 8.3. *Let $g \geq 2$ and $n \geq 0$. Let $\mathcal{H}_3^{\text{ev}}$ be the open substack of $\mathcal{H}_{3,g,n}$ lying over T_n^{ev} in Definition 8.1 with $k = 3$, and let $\mathcal{H}_4^{\text{ev}}$ be the open substack of $\mathcal{H}'_{4,g,n}$ lying over T_n^{ev} there with $k = 4$. Then both stacks have the Chow–Künneth generation property.*

Proof. Proposition 8.2 gives the property for the respective bases $\mathcal{B}'$, for $k = 4$ by open restriction from $\mathcal{B}$. Each factor of T_n is a projective bundle over the universal projective line. The factors are taken over $\mathcal{B}'$, so their points on the projective line vary independently. Successive projective bundles and open restriction give the property for T_n^{ev} by Proposition 3.5. The kernel of e_n is a vector bundle there, and its smooth-curve, distinct-marking locus is open. Another application of that proposition proves the assertion.

One may perform these constructions on the finite gerbe of lifts of the target structure group from PGL_2 to SL_2 , where the universal line is the projectivization of a rank-two bundle. Such a gerbe has the same rational Chow groups, also after product with any test stack, and the same rational cohomology. The assertions therefore descend to the Hurwitz stacks used here. $\square$

For $g = 4, 5$, Theorem 3.8 and Proposition 4.3 show that $\mathcal{O}_{g,n}$ is pure and has only types (13, 2) and (2, 13). When applying the cycle-class assertion of Proposition 3.5, the smooth separated stacks below admit smooth proper compactifications: their coarse spaces are quasi-projective, and in characteristic zero one can compactify and resolve while preserving the given smooth open.

Theorem 8.4 (The genus-four base). *For every $n \geq 0$, the complete ordinary one-edge Gysin map $\Phi_{4,n,\mathcal{B}}^{15}$ is surjective, and $\mathcal{O}_{4,n} = 0$.*

Proof. *Removing the hyperelliptic locus.* Put $X = \mathcal{M}_{4,12}$, let $Y \subset X$ be its hyperelliptic locus, and put $X^\circ = X \setminus Y$. The hyperelliptic stack is smooth: it is a finite gerbe over the quotient

of the space of ten distinct unordered branch points on $\mathbb{P}^1$, and adding markings gives an open substack of a fiber power of the universal curve. Thus

$$\dim X = 21, \quad \dim Y = (10 - 3) + 12 = 19.$$

Y is open in the substack $\mathcal{H}_{4,12}^{\text{irr}}$ of [CL22, Theorem 6.2], which has the Chow–Künneth generation property since $12 \leq 2 \cdot 4 + 6$; with Proposition 3.5 this gives $W_{11}H^{11}(Y) = 0$. Lemma 5.5(2), applied with $c = 2$, gives an injection

$$\mathcal{O}_{4,12} \hookrightarrow W_{15}H^{15}(X^\circ).$$

Pullback along the proper trigonal morphism. Let $\mathcal{T} = \mathcal{H}_{3,4,12}$. A hyperelliptic genus-four curve cannot carry a degree-three map: together with its degree-two map it would be birational to a curve of bidegree $(2, 3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, of arithmetic genus two. The degree of the map onto its image is one because it divides both two and three. Consequently $\beta : \mathcal{T} \rightarrow X^\circ$ is proper.

This map is surjective and generically finite. Indeed, the canonical model of a nonhyperelliptic genus-four curve has degree six in $\mathbb{P}^3$. Since $h^0(2K_C) = 9 < 10$, it lies on a quadric. The quadric is irreducible, since the curve is nondegenerate. Since $h^0(3K_C) = 15$, there is a cubic containing the curve which is not a multiple of that quadric: the space of cubics has dimension twenty, and the multiples of the quadric have dimension four. The complete intersection of the quadric and cubic has degree six and hence is the curve. A ruling of the quadric cuts out a degree-three pencil; if the quadric is a cone, smoothness of the complete intersection implies that the curve avoids its vertex. Conversely, a divisor of a degree-three pencil spans a line in the canonical model by Riemann–Roch. The line lies on the quadric, since its intersection with that quadric contains a length-three divisor. The pencil therefore comes from a ruling. A general smooth quadric has two rulings, so β has degree two with the ungerbed convention.

The stack $\mathcal{T}$ is smooth of dimension twenty-one. Deformations of a finite map $\alpha : C \rightarrow \mathbb{P}^1$ are controlled by the torsion cokernel of $T_C \rightarrow \alpha^*T_{\mathbb{P}^1}$; its first cohomology vanishes and its length for a trigonal curve is $2g + 4$. Quotienting by the three-dimensional target automorphism group gives unmarked dimension $2g + 1$, and distinct markings add twelve dimensions. For the proper map β between smooth stacks of the same dimension, Poincaré duality and the projection formula give

$$\beta_*\beta^* = 2 \text{id}.$$

Thus the pullback embeds $\mathcal{O}_{4,12}$ in $H^{15}(\mathcal{T})$.

The locus of nonsurjective evaluation. For a cover $\alpha : C \rightarrow \mathbb{P}^1$, let $\mathcal{E}_\alpha$ be its rank-two bundle in Definition 8.1 with $k = 3$. Denote its ruled surface by $\Sigma = \mathbb{P}(\mathcal{E}_\alpha^\vee)$ and its projection by γ_α . Put

$$L_\alpha = \mathcal{O}_\Sigma(3) \otimes \gamma_\alpha^*\mathcal{O}_{\mathbb{P}^1}(-6), \quad D = p_1 + \cdots + p_{12}.$$

The curve is a divisor of L_α . The canonical bundle of Σ is $\mathcal{O}_\Sigma(-2) \otimes \gamma_\alpha^*\mathcal{O}(4)$, so adjunction gives

$$\omega_C \simeq \mathcal{O}_\Sigma(1)|_C \otimes \alpha^*\mathcal{O}(-2), \quad L_\alpha|_C \simeq \omega_C^{\otimes 3}.$$

Since $H^1(\Sigma, \mathcal{O}_\Sigma) = 0$, the divisor sequence shows that ambient sections of L_α surject onto $H^0(C, \omega_C^{\otimes 3})$. Also $H^1(C, \omega_C^{\otimes 3}) = 0$. The cokernel of evaluation at D is therefore

$$H^1(C, \omega_C^{\otimes 3}(-D)) \simeq H^0(C, \mathcal{O}_C(D) \otimes \omega_C^{-2})^\vee.$$

The last line bundle has degree zero. It has a section precisely when it is trivial, and then its space of sections is one-dimensional. Thus there are exactly two rank strata: evaluation has rank twelve off the locus

$$Z = \{(\alpha : C \rightarrow \mathbb{P}^1, p_1, \dots, p_{12}) : D \sim 2K_C\},$$

and rank eleven on Z .

Give Z the closed scheme structure obtained by pulling back the bicanonical section of the relative degree-twelve Picard space under the Abel map. Over $\mathcal{H}_{3,4}$ it is the open of the projective

bundle of bicanonical divisors with simple zeros, followed by the finite étale cover ordering those zeros. The bundle of bicanonical sections has rank $h^0(C, 2K_C) = 12 - 4 + 1 = 9$ and vanishing first cohomology. The degree-twelve Abel map is smooth, since $12 > 2g - 2$. It follows that Z is smooth and closed, and

$$\dim Z = 9 + 8 = 17, \quad \text{codim}_{\mathcal{Z}} Z = 4.$$

This locus is the entire complement of the evaluation open.

Lemma 8.3 and Proposition 3.5 give $W_{15}H^{15}(\mathcal{Z} \setminus Z) = 0$. By Lemma 5.5(1), applied with $c = 4$, restriction to this open is injective on the image of $\mathcal{O}_{4,12}$. That image is therefore zero. The hyperelliptic localization and the proper pullback then give $\mathcal{O}_{4,12} = 0$.

Arbitrary numbers of markings. Proposition 3.6 gives $\mathcal{O}_{4,n} = 0$ for $0 \leq n \leq 11$. For $n = 13, 14$, every partition $\lambda \vdash n$ of length at most four has $\lambda_1 \geq 4$. Applying Lemma 5.4 with $d = 15 - n \in \{2, 1\}$ gives

$$H^{15-n}(\mathcal{M}_4, \mathbb{V}_\lambda) = 0.$$

The primitive quotient in Proposition 4.3 is thus zero for these two numbers of markings. Its forgetful span is zero in each case because the quotient with one fewer marking vanishes. For $n \geq 15$, the same proposition identifies $\mathcal{O}_{4,n}$ with its forgetful span, so induction proves vanishing for every n . The right-exact one-edge sequence of that proposition now proves all the claimed surjectivities. $\square$

Proposition 8.5 (The initial genus-five range). *For $0 \leq n \leq 10$, the complete ordinary one-edge Gysin map $\Phi_{5,n,B}^{15}$ is surjective, and $\mathcal{O}_{5,n} = 0$.*

Proof. Proposition 3.6 proves the assertion for $n \leq 7$. Fix $n \in \{8, 9, 10\}$.

The hyperelliptic and trigonal strata. Put $X = \mathcal{M}_{5,n}$, let $Y \subset X$ be the hyperelliptic locus, and put $X^\circ = X \setminus Y$. The branch-divisor description gives $\dim Y = 9 + n$, whereas $\dim X = 12 + n$. Thus Y is smooth of codimension three, and Lemma 5.5(1), applied with $c = 3$, makes restriction injective on $\mathcal{O}_{5,n}$.

Let $\mathcal{D} \subset X^\circ$ be the trigonal locus and put $V = X^\circ \setminus \mathcal{D}$. We show that $\mathcal{D}$ is a smooth divisor and has the Chow–Künneth generation property. A hyperelliptic genus-five curve has no degree-three map, by the bidegree-(2, 3) argument used above. A nonhyperelliptic genus-five curve has at most one degree-three pencil: two inequivalent such pencils would give a birational model of bidegree (3, 3), whose arithmetic genus is four. If the joint map had degree three onto its image, both maps would factor through the same projective line and would be equivalent.

The map from the unmarked trigonal Hurwitz stack is unramified as well. For a cover $\alpha : C \rightarrow \mathbb{P}^1$, put $A = \alpha^*\mathcal{O}(1)$. Then $h^0(C, A) = 2$ and $h^0(C, K_C - A) = 3$. The basepoint-free pencil sequence identifies the kernel of multiplication

$$H^0(C, A) \otimes H^0(C, K_C - A) \longrightarrow H^0(C, K_C)$$

with $H^0(C, K_C - 2A)$. The dimensions of the displayed spaces are six and five, so this kernel is nonzero. Its line bundle has degree two, and nonhyperellipticity forces its space of sections to have dimension at most one. Riemann–Roch therefore gives

$$h^0(C, K_C - 2A) = 1, \quad h^0(C, 2A) = 3.$$

Deformations of α with C fixed have tangent space $H^0(C, \alpha^*T_{\mathbb{P}^1}) = H^0(C, 2A)$. The three-dimensional space of target automorphisms maps injectively to this space, so the tangent space after quotienting is zero.

Properness follows from the properness of β'_n recalled above. The map is representable, since an automorphism inducing the identity on the source induces the identity on the target. Uniqueness of the pencil makes it universally injective. A proper, universally injective, unramified morphism is

a closed immersion. It identifies the ungerbed Hurwitz stack with the reduced trigonal locus. The deformation calculation for trigonal covers above gives smoothness and dimension $2 \cdot 5 + 1 = 11$. Adding distinct markings is base change along the smooth configuration space of the universal curve. It follows that $\mathcal{D}$ is smooth of dimension $11 + n$, hence is a divisor in X° .

Since $n \leq 10 < 5 + 7$, [CL22, Section 9.2, Lemmas 9.1–9.2] give the Chow–Künneth generation property for $\mathcal{D}$. Proposition 3.5 then gives $W_{13}H^{13}(\mathcal{D}) = 0$, and Lemma 5.5(2), applied with $c = 1$, makes restriction injective in weight fifteen. We have obtained an injection

$$\mathcal{O}_{5,n} \hookrightarrow W_{15}H^{15}(V).$$

Denote its image by $\mathcal{V}_n$.

Balanced tetragonal evaluation. By [CL22, Lemma 10.9 and Section 10.5],

$$\mathcal{H}_{4,5,n}^4 = \mathcal{H}_{4,5,n} \setminus \beta_n^{-1}(\mathcal{M}_{5,n}^3),$$

and its forgetful morphism to $\mathcal{M}_{5,n} \setminus \mathcal{M}_{5,n}^3$ is proper and surjective; this gives a proper surjective morphism

$$\beta_n : \mathcal{H}_{4,5,n}^4 \longrightarrow V.$$

For any of its covers $\alpha : C \rightarrow \mathbb{P}^1$, let $\mathcal{E}_\alpha, \mathcal{F}_\alpha$ be the bundles in Definition 8.1 with $k = 4$. The rank-two bundle $\mathcal{F}_\alpha$ has degree eight and both summands have degree at least four. Hence $\mathcal{F}_\alpha \simeq \mathcal{O}(4)^{\oplus 2}$. Write

$$\Sigma = \mathbb{P}(\mathcal{E}_\alpha^\vee), \quad \gamma_\alpha : \Sigma \rightarrow \mathbb{P}^1, \quad N_\alpha = \mathcal{O}_\Sigma(1) \otimes \gamma_\alpha^* \mathcal{O}(-2).$$

The curve is the common zero locus of two sections of N_α^2 , forming a complete intersection, and the bundle defining it is $W_\alpha = (N_\alpha^2)^{\oplus 2}$. Here $\det \mathcal{E}_\alpha \simeq \mathcal{O}(8)$, so

$$\omega_\Sigma \simeq \mathcal{O}_\Sigma(-3) \otimes \gamma_\alpha^* \mathcal{O}(6).$$

Adjunction gives $\omega_C \simeq N_\alpha|_C$, and consequently $W_\alpha|_C \simeq (\omega_C^{\otimes 2})^{\oplus 2}$.

The restriction of ambient sections is surjective. With I_C the ideal sheaf of C in Σ , the twisted Koszul sequence is

$$0 \longrightarrow N_\alpha^{-2} \longrightarrow \mathcal{O}_\Sigma^{\oplus 2} \longrightarrow I_C \otimes N_\alpha^2 \longrightarrow 0.$$

Every direct image of N_α^{-2} under γ_α vanishes, since its restriction to a projective-plane fiber is $\mathcal{O}_{\mathbb{P}^2}(-2)$, which has no cohomology. Together with $H^1(\Sigma, \mathcal{O}_\Sigma) = 0$, the sequence therefore gives $H^1(\Sigma, I_C \otimes N_\alpha^2) = 0$. It follows that $H^0(\Sigma, N_\alpha^2) \rightarrow H^0(C, \omega_C^{\otimes 2})$ is surjective. Applying this to both summands, and using $H^1(C, \omega_C^{\otimes 2}) = 0$, identifies the cokernel of evaluation at $D = p_1 + \cdots + p_n$ with

$$H^1(C, W_\alpha|_C(-D)) \simeq (H^0(C, \mathcal{O}_C(D) \otimes \omega_C^{-1})^\vee)^{\oplus 2}.$$

Define the closed locus

$$Z_n = \{(C, p_1, \dots, p_n) \in V : h^0(C, \mathcal{O}_C(D) \otimes \omega_C^{-1}) > 0\}, \quad V^{\text{ev}} = V \setminus Z_n.$$

Closedness follows from upper semicontinuity. The condition is independent of the choice of tetragonal pencil. Therefore the restriction

$$\beta_n^{-1}(V^{\text{ev}}) \longrightarrow V^{\text{ev}}$$

is the base change of the proper surjective map β_n . Its source is precisely the open substack of the balanced Hurwitz stack where evaluation is surjective. The balanced Hurwitz stack is open in $\mathcal{H}'_{4,5,n}$, by the displayed equality of [CL22, Lemma 10.9]. Lemma 8.3 and open restriction thus give the Chow–Künneth generation property for $\beta_n^{-1}(V^{\text{ev}})$. The morphism is representable by Deligne–Mumford stacks, as before. Proper descent in Proposition 3.5 gives the property for V^{ev} , and hence

$$W_{15}H^{15}(V^{\text{ev}}) = 0.$$

Canonical and residual-divisor strata. Put $k = n - 8 \in \{0, 1, 2\}$. The line bundle $\mathcal{O}_C(D) \otimes \omega_C^{-1}$ has degree k . On a curve of gonality greater than three, a line bundle of degree at most two has at most one independent section. Indeed, two independent sections, after removing their common zeros, would define a map of degree at most two to $\mathbb{P}^1$. Thus evaluation has rank $2n$ off Z_n , and rank $2n - 2$ on Z_n ; no further rank drop occurs.

For $n = 8$, the degree is zero, so Z_8 is exactly the locus $D \sim K_C$. Over the unmarked open $\mathcal{M}_5 \setminus \mathcal{M}_5^3$, its unordered divisors form the projective bundle of canonical sections, of relative dimension four. Restricting to simple zeros and ordering the eight zeros gives an open substack followed by a finite étale cover. This identifies Z_8 , with its relative linear-system scheme structure, as a smooth closed substack of V .

For $n = 9, 10$, there is a unique effective divisor R of degree k such that $D \sim K_C + R$. The Abel map

$$\mathrm{Sym}^k C \longrightarrow \mathrm{Pic}^k C$$

is a closed immersion: it is proper and universally injective, and its tangent kernel at R is $H^0(C, \mathcal{O}_C(R))/H^0(C, \mathcal{O}_C) = 0$. These statements hold relatively over the unmarked open as well. Since $n > 2g - 2 = 8$, the degree- n Abel map is smooth, with fibers the complete linear systems of dimension $n - 5 = 3 + k$. Its inverse image of $K_C + \mathrm{Sym}^k C$ is the projective bundle of sections of $\omega_C(R)$ over the relative symmetric power. The direct image has constant rank $4 + k$, since

$$H^1(C, \omega_C(R)) = H^0(C, \mathcal{O}_C(-R))^\vee = 0.$$

The simple-zero open, with its zeros ordered, identifies with Z_n . In particular, Z_n is smooth and closed, of relative dimension $k + (3 + k) = 3 + 2k$ over the unmarked open. The dimensions and codimensions are

n	failure condition	$\dim Z_n$	$\mathrm{codim}_V Z_n$
8	$D \sim K_C$	$12 + 4 = 16$	4
9	$D \sim K_C + R$, $\deg R = 1$	$12 + 5 = 17$	4
10	$D \sim K_C + R$, $\deg R = 2$	$12 + 7 = 19$	3

Exclusion by Hodge type. For each of the three values of n , write $c = \mathrm{codim}_V Z_n \in \{4, 3\}$. Lemma 5.5(1), applied with this $c \geq 3$, makes restriction to V^{ev} injective on $\mathcal{V}_n$. Its target in weight fifteen is zero by proper descent from the evaluation open, so $\mathcal{V}_n = 0$. The injections obtained by removing the hyperelliptic and trigonal strata give $\mathcal{O}_{5,n} = 0$ for $n = 8, 9, 10$. The right-exact sequence of Proposition 4.3 gives the asserted one-edge surjectivity throughout this range. $\square$

9. MARKING SUPPORTS IN GENUS FIVE

We identify the open degree-fifteen quotient in genus five with the direct sum of the images of its eleven-marking pullbacks. Injectivity of the sum of these maps follows from rational-tail restriction and weighted forgetting. Compact boundary restriction and the primitive marking quotient then reduce generation to seven symplectic coefficient groups.

Theorem 9.1 (Independence of marking supports). *Let $g \geq 2$ and $d \geq 2$, and suppose $\mathcal{O}_{g,d-1} = 0$. For every finite set F with $|F| \geq d$, the sum of forgetful pullbacks is injective:*

$$\bigoplus_{\substack{S \subseteq F \\ |S|=d}} \mathcal{O}_{g,S} \longrightarrow \mathcal{O}_{g,F}, \quad (x_S)_S \longmapsto \sum_S \pi_S^* x_S.$$

Here π_S forgets the markings outside S .

Proof. For any marking set A , write

$$r_A : H^{15}(\overline{\mathcal{M}}_{g,A}) \longrightarrow \mathcal{O}_{g,A}.$$

By Proposition 4.3, this map is surjective, its kernel is $\mathcal{I}_{g,A}$, and $W_{13}H^{13}(\mathcal{M}_{g,A}) = 0$. Denote the sum in the statement by q_F , and put $n = |F|$.

Rational-tail restriction. For distinct $i, j \in F$ and a new label u , put $F' = (F \setminus \{i, j\}) \sqcup \{u\}$. Let

$$s_{ij} : \overline{\mathcal{M}}_{g,F'} \longrightarrow \overline{\mathcal{M}}_{g,F}$$

attach at u the three-pointed rational curve carrying i, j . Its image is the divisor D_{ij} , with normalization $\overline{\mathcal{M}}_{g,F'} \times \overline{\mathcal{M}}_{0,3} = \overline{\mathcal{M}}_{g,F'}$. We claim that $r_{F'} s_{ij}^*$ annihilates $\mathcal{I}_{g,F}$. It suffices to test the complete one-edge Gysin generators of that space. Over $\mathcal{M}_{g,F'}$, the image of s_{ij} has just the node defining D_{ij} , so a class supported on any other boundary divisor pulls back to zero there. For a class $s_{ij*}\eta$ from D_{ij} itself, the normal class is $-\psi_u$ because ψ vanishes on $\overline{\mathcal{M}}_{0,3}$, and hence

$$s_{ij}^* s_{ij*} \eta \Big|_{\mathcal{M}_{g,F'}} = -\psi_u \smile \eta|_{\mathcal{M}_{g,F'}} = 0.$$

Indeed, $\eta \in H^{13}(\overline{\mathcal{M}}_{g,F'})$ restricts into the zero group $W_{13}H^{13}(\mathcal{M}_{g,F'})$. The labelled rational tail has no flag-exchange factor. The claim gives a unique map

$$c_{ij} : \mathcal{O}_{g,F} \longrightarrow \mathcal{O}_{g,F'}, \quad c_{ij} r_F = r_{F'} s_{ij}^*.$$

For a d -element support $S \subset F$, one computes $c_{ij} \pi_S^* x_S$ by stabilizing the composite $\pi_S s_{ij}$. If S avoids i, j , the result is $\pi_S^* x_S$ on F' , with support avoiding u . If S contains exactly one of them, say v , the result is the pullback from $(S \setminus \{v\}) \sqcup \{u\}$ of x_S relabelled by $v \mapsto u$. If S contains both, the composite factors through the collision map with target $\mathcal{O}_{g,(S \setminus \{i,j\}) \sqcup \{u\}} = 0$.

Weighted forgetting. Let $\pi_i : \overline{\mathcal{M}}_{g,F} \rightarrow \overline{\mathcal{M}}_{g,F \setminus \{i\}}$ forget i . The degree-preserving map

$$z \longmapsto \pi_{i*}(\psi_i \smile z)$$

sends $\mathcal{I}_{g,F}$ into $\mathcal{I}_{g,F \setminus \{i\}}$. To see this, let j be a normalized one-edge clutching map. The projection formula gives

$$\pi_{i*}(\psi_i \smile j_* \eta) = (\pi_{ij})_*(j^* \psi_i \smile \eta).$$

If the node survives forgetting i , this class remains supported on the boundary. The only node that can disappear is the rational tail with ordinary labels i, k for some k . On its normalization, $j^* \psi_i = 0$, since the cotangent line comes from $\overline{\mathcal{M}}_{0,3}$. Thus there is a descended map

$$T_i : \mathcal{O}_{g,F} \longrightarrow \mathcal{O}_{g,F \setminus \{i\}}, \quad T_i r_F = r_{F \setminus \{i\}} \pi_{i*}(\psi_i \smile -).$$

Proposition 3.16 and the projection formula imply

$$(9.1) \quad T_i \pi_i^* = (2g + n - 3) \text{id}.$$

The mixed transfer identity. Fix $k \neq i$, and write

$$\begin{aligned} \mathcal{X} &= \overline{\mathcal{M}}_{g,F}, & \mathcal{Y}_i &= \overline{\mathcal{M}}_{g,F \setminus \{i\}}, \\ \mathcal{Y}_k &= \overline{\mathcal{M}}_{g,F \setminus \{k\}}, & \mathcal{Z} &= \overline{\mathcal{M}}_{g,F \setminus \{i,k\}}. \end{aligned}$$

Let $f_k : \mathcal{Y}_i \rightarrow \mathcal{Z}$ and $f_i : \mathcal{Y}_k \rightarrow \mathcal{Z}$ forget the indicated markings. On all rational cohomology, one has

$$(9.2) \quad \pi_{i*} \pi_k^* = f_k^* f_{i*}.$$

To prove this identity, we compare the corresponding algebraic cycles. Both f_i, f_k are representable proper flat universal curves. Their fiber product $\mathcal{W} = \mathcal{Y}_i \times_{\mathcal{Z}} \mathcal{Y}_k$ has the expected dimension, and its fundamental cycle represents the composition of the graph correspondences for f_{i*} and f_k^* . By [BS22, Proposition 3.10, diagram (22)], the induced map $\tau = (\pi_i, \pi_k) : \mathcal{X} \rightarrow \mathcal{W}$ is proper and

birational, and it satisfies $\tau_*[\mathcal{X}] = [\mathcal{W}]$. Flatness excludes components of excess dimension, and the degree is one over the locus of smooth curves with distinct markings. Pushing this equality to $\mathcal{Y}_i \times \mathcal{Y}_k$ gives equality of the two correspondence cycles, and hence of their cycle classes. Acting by cup product and proper pushforward gives (9.2) in every degree.

Suppose now that $\mathcal{O}_{g,n-2} = 0$. For $x_k \in \mathcal{O}_{g,F \setminus \{k\}}$, choose a compact lift z_k . The psi identity and (9.2) give

$$\pi_{i*}(\psi_i \smile \pi_k^* z_k) = f_k^* f_{i*}(\psi_i^{\mathcal{Y}_k} \smile z_k) + \pi_{i*}([D_{ik}] \smile \pi_k^* z_k),$$

where $\psi_i^{\mathcal{Y}_k}$ is the cotangent class on $\mathcal{Y}_k$. The first term restricts through $\mathcal{O}_{g,n-2}$ and vanishes. On D_{ik} , both forgetful maps contract the rational component and are isomorphisms onto their targets, with i, k interchanged. The second term is therefore the relabelled class, with coefficient one. Writing $(i k)$ for this relabelling, we obtain

$$(9.3) \quad T_i \pi_k^* x_k = (i k) x_k \quad (k \neq i).$$

Independence with $d+1$ markings. The case $n = d$ is the identity map. For $n = d+1$, index a putative relation by the omitted label:

$$\sum_{i \in F} \pi_i^* x_i = 0, \quad x_i \in \mathcal{O}_{g,F \setminus \{i\}}.$$

Applying c_{ik} and identifying its target with $\mathcal{O}_{g,F \setminus \{i\}}$ gives

$$x_i + (i k) x_k = 0.$$

All other terms vanish by the collision calculation, since their supports contain both i, k and $\mathcal{O}_{g,d-1} = 0$. Applying T_i to the relation and using (9.1) and (9.3), we find

$$0 = (2g + n - 3)x_i + \sum_{k \neq i} (i k) x_k = (2g + n - 3 - (n - 1))x_i = (2g - 2)x_i.$$

Since $g \geq 2$, every x_i is zero.

Independence in larger arities. Suppose $n \geq d+2$ and independence is known on $n-1$ markings. Let $q_F((x_S)_S) = 0$, and fix one d -element subset $S \subset F$. Choose distinct $i, j \in F \setminus S$ and apply c_{ij} . The collision calculation expresses the result as a relation among d -element supports in F' . In that relation the coefficient at S , which avoids u , is exactly x_S : a support meeting i, j either vanishes or becomes a support containing u . The induction hypothesis makes every coefficient zero, in particular x_S . This holds for each S , and proves the theorem. $\square$

Lemma 9.2. *For $0 \leq n \leq 12$, put*

$$r_n = \begin{cases} 9, & 0 \leq n \leq 10, \\ 8, & n = 11, \\ 7, & n = 12. \end{cases}$$

Then the following hold.

(i) *For every stable (g, n) with $g \geq 1$ and every partition $\mu \vdash n$ with $\ell(\mu) \leq 8$, one has*

$$\mathrm{Hom}_{\mathfrak{S}_n}(V_\mu, B_{g,n}^{(1)}) = 0.$$

(ii) *For every $0 \leq n \leq 12$ and every partition $\mu \vdash n$ with $\ell(\mu) < r_n$, one has*

$$\mathrm{Hom}_{\mathfrak{S}_n}(V_\mu, H^{15}(\overline{\mathcal{M}}_{5,n})) = 0.$$

(iii) *For $h = 2, 3, 4$, every m with (h, m) stable, and every partition $\mu \vdash m$ with $\ell(\mu) \leq 8$, one has*

$$\mathrm{Hom}_{\mathfrak{S}_m}(V_\mu, H^{15}(\overline{\mathcal{M}}_{h,m})) = 0.$$

(iv) For every stable (g, F) , the summand of C_{sep}^1 in Definition 5.2 formed by the Künneth sectors with degree pairs $(13, 2)$, $(2, 13)$, $(11, 4)$, $(4, 11)$ on the separating normalizations has row bound nine (Lemma 3.27).

Proof. The source and the lower-genus targets. Assertion (i) is Lemma 3.27(3), and every ordinary one-edge image in degree fifteen inherits the bound. Consequently Propositions 4.12 and 5.20, together with Theorem 8.4, give at least nine rows for $H^{15}(\overline{\mathcal{M}}_{h,m})$ for $h = 2, 3, 4$ and every m , which is assertion (iii). Proposition 8.5 gives the same bound for $h = 5$ and $m \leq 10$. Theorem 4.7 and the primitive term of Theorem 3.8 give

$$H^{15}(\overline{\mathcal{M}}_{1,m}) = 0 \quad (1 \leq m \leq 12), \quad H^{15}(\overline{\mathcal{M}}_{1,13}) = \mathcal{I}_{1,[13]}.$$

In particular the latter group also has at least nine rows. The genus-zero degree-fifteen group is zero by Theorem 3.8.

The mixed separating sectors. Let (g, F) be stable. By Theorem 3.3, the only possible degree pairs on a separating normalization are

$$(15, 0), (0, 15), (13, 2), (2, 13), (11, 4), (4, 11).$$

In the $(13, 2)$ and $(2, 13)$ terms, Theorem 3.23 and the bound on U give at least nine rows after fixing the node. In the $(11, 4)$ and $(4, 11)$ terms, Proposition 3.15 places the odd class on an elliptic vertex; fixing its node leaves at least ten rows. Induction and graph invariants preserve these bounds regardless of the representation on the even factor. This proves assertion (iv).

The complete compact boundary. For $n = 11, 12$, Proposition 5.3 gives an injection

$$H^{15}(\overline{\mathcal{M}}_{5,n}) \hookrightarrow C_{5,[n]}^1,$$

since $15 < 2 \cdot 5 - 2 + n$. The mixed separating terms have at least nine rows by (iv).

The nonseparating term is

$$H^{15}(\overline{\mathcal{M}}_{4,[n] \sqcup \{c,d\}})^{\langle (cd) \rangle}.$$

It has at least eight rows by the nine-row genus-four bound (iii) and Lemma 3.27(1). The exchange fixes the single edge, so its action on the edge determinant in C^1 is trivial.

Consider a separating $(15, 0)$ term; the $(0, 15)$ term is identical with the vertices interchanged. If the degree-fifteen vertex has genus $h < 5$, it has at most $n + 1 \leq 13$ flags. The lower-genus results just proved give at least nine rows before fixing its node, and hence at least eight afterward. This also covers $h = 1$, whose factor vanishes unless it has thirteen flags; the rational case is zero.

If that vertex has genus five, its rational partner carries $b \geq 2$ ordinary markings. The genus-five factor then has $m = n - b + 1$ flags. At $n = 11$ one has $m \leq 10$, so its nine-row bound leaves eight rows after fixing the node. Every summand of $C_{5,[11]}^1$ has therefore at least eight rows, which proves the bound for $n = 11$. At $n = 12$ and $b \geq 3$, the same argument applies. The remaining case is $b = 2$, $m = 11$: the bound just obtained has eight rows before fixing the node and seven afterward. Hence every summand of $C_{5,[12]}^1$ has at least seven rows. The boundary injections prove the remaining cases $n = 11, 12$ of assertion (ii). $\square$

Proposition 9.3 (The seven coefficient reduction). *For every $0 \leq r \leq 14$ and every partition $\lambda \vdash r$ of length at most five, one has*

$$W_{15} H^{15-r}(\mathcal{M}_5, \mathbb{V}_\lambda) = 0 \quad \text{unless } r = 11 \text{ and } \lambda \in \mathcal{R}_{11},$$

$$\mathcal{R}_{11} = \{(11), (10, 1), (9, 2), (9, 1, 1), (8, 3), (8, 2, 1), (8, 1, 1, 1)\}.$$

Put $\mathcal{C}_\lambda = W_{15} H^4(\mathcal{M}_5, \mathbb{V}_\lambda)$. There is a canonical equivariant isomorphism

$$\mathcal{O}_{5,11} \cong \bigoplus_{\lambda \in \mathcal{R}_{11}} \mathcal{C}_\lambda \otimes V_{\lambda^T}.$$

For every $n \geq 11$, the sum of eleven-marking forgetful pullbacks gives an isomorphism

$$\mathrm{Ind}_{\mathfrak{S}_{11} \times \mathfrak{S}_{n-11}}^{\mathfrak{S}_n} [\mathcal{O}_{5,11} \boxtimes \mathbf{1}] \xrightarrow{\sim} \mathcal{O}_{5,n}.$$

Equivalently, restriction to the smooth locus gives an exact sequence

$$0 \rightarrow \mathcal{I}_{5,[n]} \rightarrow H^{15}(\overline{\mathcal{M}}_{5,n}) \rightarrow \mathcal{O}_{5,n} \rightarrow 0$$

with $\mathcal{O}_{5,n}$ the induced module displayed above.

Proof. Throughout we use the primitive quotient of Proposition 4.3 with $g = 5$: $\mathcal{O}_{5,n}/\mathcal{F}_{5,n}^{15}$ is the direct sum over $\lambda \vdash n$ with $\ell(\lambda) \leq 5$ of $W_{15}H^{15-n}(\mathcal{M}_5, \mathbb{V}_\lambda) \otimes V_{\lambda^T}$.

The first possible support. Proposition 8.5 gives $\mathcal{O}_{5,n} = 0$ for $0 \leq n \leq 10$. For $r \leq 10$ the primitive quotient is therefore zero, and since its Specht factors are all nonzero, $W_{15}H^{15-r}(\mathcal{M}_5, \mathbb{V}_\lambda) = 0$ for every $\lambda \vdash r$ of length at most five. Thus $\mathcal{F}_{5,11}^{15} = 0$, and the primitive quotient identifies $\mathcal{O}_{5,11}$ with the full sum of size-eleven coefficients. It is a quotient of $H^{15}(\overline{\mathcal{M}}_{5,11})$, so Lemma 9.2 excludes every V_{λ^T} with $\ell(\lambda^T) = \lambda_1 \leq 7$, hence

$$W_{15}H^4(\mathcal{M}_5, \mathbb{V}_\lambda) = 0 \quad (|\lambda| = 11, \ell(\lambda) \leq 5, \lambda_1 \leq 7).$$

For first parts 11, 10, 9, 8, the remaining parts partition 0, 1, 2, 3, respectively. Their 1, 1, 2, 3 possibilities are exactly $\mathcal{R}_{11}$. This proves the first isomorphism.

No new support at twelve markings. The primitive quotient at $n = 12$ is a subquotient of $H^{15}(\overline{\mathcal{M}}_{5,12})$, so the same lemma gives

$$W_{15}H^3(\mathcal{M}_5, \mathbb{V}_\lambda) = 0 \quad (|\lambda| = 12, \ell(\lambda) \leq 5, \lambda_1 \leq 6).$$

For the complementary range $\lambda_1 \geq 7$, Lemma 5.4, applied with $d = 3$, $g = 5 = d + 2$, and $\lambda_1 > 3$, gives the stronger vanishing of the full group $H^3(\mathcal{M}_5, \mathbb{V}_\lambda)$. These two ranges cover every admissible partition of twelve. Therefore $\mathcal{O}_{5,12} = \mathcal{F}_{5,12}^{15}$.

No new support in higher arities. If $\lambda \vdash n$ with $n = 13, 14$ and $\ell(\lambda) \leq 5$, then $\lambda_1 \geq \lceil n/5 \rceil = 3 > 15 - n$, so the same lemma with $d = 15 - n$, for which $5 \geq d + 2$, gives $H^{15-n}(\mathcal{M}_5, \mathbb{V}_\lambda) = 0$ for every such λ . The primitive quotient therefore vanishes, and $\mathcal{O}_{5,n} = \mathcal{F}_{5,n}^{15}$ at $n = 13, 14$. For every $n \geq 15$ this equality is the last assertion of Proposition 4.3. This proves the first assertion.

Iterating these equalities and composing the forgetful maps gives

$$\mathcal{O}_{5,n} = \sum_{\substack{S \subseteq [n] \\ |S|=11}} \pi_S^* \mathcal{O}_{5,S} \quad (n \geq 11).$$

Indeed, at each larger arity every eleven-element subset misses a label, and forgetting that label followed by forgetting to S is precisely π_S .

The pullback isomorphism and the restriction kernel. Theorem 9.1, with $g = 5$, $d = 11$, applies because $\mathcal{O}_{5,10} = 0$. It makes the displayed sum injective, while the preceding argument makes it surjective. The stabilizer of a fixed eleven-element subset is $\mathfrak{S}_{11} \times \mathfrak{S}_{n-11}$, and its second factor acts trivially on the corresponding summand. Thus the labelled direct sum is the induced representation in the statement.

The exact sequence follows, since the kernel of compact restriction is $\mathcal{I}_{5,[n]}$ by Proposition 4.3. $\square$

10. GENUS-TWO COMPACT TEST CLASSES

We construct cohomological test maps on compact moduli spaces that distinguish the parallel and Picard Gysin images in the genus-five coefficient calculation and determine the auxiliary kernel.

Set-up 10.1. For a finite marking set F , choose distinct new flags a, b, c, d and put

$$I = F \sqcup \{c, d\}, \quad J = I \sqcup \{a, b\}, \quad t = (cd), \quad s = (ac)(bd).$$

Let $\nu_{ab} : \overline{\mathcal{M}}_{3,J} \rightarrow \overline{\mathcal{M}}_{4,I}$ glue a, b , and set

$$\mathcal{D}_F = \{y \in H^{15}(\overline{\mathcal{M}}_{3,J}) : (ab)^*y = (cd)^*y = y, \ s^*y = -y\}, \quad q_F = (1 - s^*)\nu_{ab}^* : N_{5,F} \longrightarrow \mathcal{D}_F.$$

Use the compact row and quotient $\mathcal{Q}_{5,F}$ of Definition 5.2; write δ_F for its map δ , $\xi_F : \overline{\mathcal{M}}_{4,I} \rightarrow \overline{\mathcal{M}}_{5,F}$ for its map ξ gluing c, d , and $\beta_{5,F}$ for the compact row map at $(5, F)$. Order the double-loop determinant as $\ell_{ab} \wedge \ell_{cd}$. For $\mu \vdash |F|$, put $M_\mu(V) = \text{Hom}_{\mathfrak{S}_F}(V_\mu, V)$ and

$$N_\mu = M_\mu(N_{5,F}), \quad \mathcal{Q}_\mu = M_\mu(\mathcal{Q}_{5,F}), \quad J_\mu = M_\mu(\xi_F^*)(M_\mu(\mathcal{I}_{5,F})).$$

Write $\delta_\mu = M_\mu(\delta_F)$. The equality $\beta\alpha = 0$ gives $J_\mu \subseteq \ker \delta_\mu$, and hence a map

$$\bar{\delta}_\mu : N_\mu/J_\mu \longrightarrow \mathcal{Q}_\mu.$$

Throughout, all graph classes and Gysin maps use the ordinary quotient-stack normalization of Definition 2.7; the displayed sources include their Tate twists. All sources and tests commute with relabelling of F ; hence every map below is $\mathfrak{S}_F$ -equivariant, and by Set-up 2.1 every injectivity or kernel statement passes to the exact functors M_μ and to invariants under finite groups of labelled graph automorphisms, such as $\langle t \rangle$. The group of order eight generated by the pair reversals (ab) , (cd) and by s projects onto $\mathcal{D}_F$ by the average with positive loop reversals and negative pair exchange; this projector leaves each pairing with a class of $\mathcal{D}_F$ unchanged and retains the order $\ell_{ab} \wedge \ell_{cd}$, so the tests below introduce no additional normalization factor.

Lemma 10.2. *For every finite F , projection to the double-loop coordinate induces a map $\text{pr}_{\text{dl}} : \mathcal{Q}_{5,F} \rightarrow \mathcal{D}_F$ satisfying*

$$\text{pr}_{\text{dl}} \delta_F = q_F.$$

Proof. Both contractions defining the double-loop coordinate are the two nonseparating contractions in the compact row. Hence the projection annihilates the complete separating image in the quotient, and on the remaining coordinate it is exactly $(1 - s^*)\nu_{ab}^*$. $\square$

Definition 10.3. On $\overline{\mathcal{M}}_{2,\{q\}}$ let $\gamma_F, \gamma_E, \gamma_{H,q}, \gamma_{H,\emptyset}, \gamma_G$ be the five ordinary pure two-edge classes of Theorem 7.4(2).

For a finite set B disjoint from q , put $Y_B = \overline{\mathcal{M}}_{2,B \sqcup \{q\}}$, let $\pi_B : Y_B \rightarrow \overline{\mathcal{M}}_{2,\{q\}}$ forget B , and put $\vartheta_B = \prod_{i \in B} \psi_i$. Let $\mathcal{N}_B \subseteq H^4(Y_B)$ be the sum of the following spaces: the images $j_{T*} H^2(\text{dom } j_T)$ for all nonempty $T \subseteq B$, with

$$j_T : \overline{\mathcal{M}}_{0,T \sqcup \{q,x\}} \times \overline{\mathcal{M}}_{2,(B \setminus T) \sqcup \{y\}} \longrightarrow Y_B;$$

the spans of pure two-edge trees with exactly two rational vertices; the spans of trees with a rational q -center and two elliptic leaves; and $\psi_q H^2(Y_B)$. For $T \subseteq B$, let ζ_T be the ordinary parallel class with rational vertex carrying $q \sqcup T$ and elliptic vertex carrying $B \setminus T$.

Proposition 10.4. *There exists a rational Hodge morphism $\Lambda : H^4(\overline{\mathcal{M}}_{2,\{q\}}) \rightarrow \mathbb{Q}(-2)$ with*

$$(\Lambda(\gamma_F), \Lambda(\gamma_E), \Lambda(\gamma_{H,q}), \Lambda(\gamma_{H,\emptyset}), \Lambda(\gamma_G)) = (-12, 1, 0, 12, 0).$$

For any such Λ and every finite B , define $\Lambda_B(u) = \Lambda(\pi_{B*}(u\vartheta_B))$. Then

$$\Lambda_B(\mathcal{N}_B) = 0, \quad \Lambda_B(\zeta_T) = |T|!(|B \setminus T| + 1)! \quad (T \subseteq B).$$

These maps commute with relabelling B and transport of q .

Proof. The one-marked functional. Theorem 7.4(2) gives independence of the five pure classes. The prescribed values therefore define a functional on their Tate span. A rational polarization splits off this sub-Hodge structure, extending the functional by zero on its orthogonal complement. The genus-one identities of Proposition 3.18, applied on the two genus-two divisor normalizations, give

$$\psi_q \delta_{\text{irr}} = \gamma_F/6 + 2\gamma_E + \gamma_{H,q}, \quad \psi_q \delta_1 = \gamma_{H,q}/12 + 2\gamma_G.$$

The first double-loop coefficient is $8/(2 \cdot 2 \cdot 12) = 1/6$: the two loop normalizations each contribute $1/2$, whereas the final graph has automorphism order eight. The two rational-tail choices give $2\gamma_E$. In the second identity the two unmarked elliptic leaves give $2\gamma_G$. Thus these are identities with the ordinary graph normalizations. Their images under Λ vanish, as does the expression for ψ_q^2 displayed in Theorem 7.4(2). Theorem 3.17 and Proposition 3.18 show that $\psi_q, \delta_{\text{irr}}, \delta_1$ generate the one-marked Picard group. Consequently $\Lambda(\psi_q H^2(\overline{\mathcal{M}}_{2,\{q\}})) = 0$.

Weighted forgetting. For a rational tail carrying $T \neq \emptyset$, the factor ϑ_B contains $|T|$ cotangent classes on a rational factor of dimension $|T| - 1$, hence annihilates its full H^2 pushforward. The same dimension argument annihilates every two-rational-vertex tree class. On a q -centered tree, weighted forgetting leaves a multiple of γ_G , which is annihilated by Λ . The cotangent identity under forgetting differs from $\pi_B^* \psi_q$ by the same q -containing tails, so $\Lambda_B(\psi_q H^2) = 0$.

The parallel quotient has the same simultaneous edge-exchange stabilizer before and after forgetting, so no additional factor is introduced. Finally, on a compact genus- h factor with n_0 retained markings, Proposition 3.16 gives

$$\pi_* \left(\prod_{r=1}^m \psi_{z_r} \right) = (2h - 2 + n_0)_m,$$

where $(x)_m = x(x+1) \cdots (x+m-1)$. Indeed, each section correction has its forgotten cotangent on a rigid three-pointed rational tail, where it vanishes; the remaining factors are removed successively by the projection formula. Applying this to the two factors of ζ_T gives $|T|!(|B \setminus T| + 1)!$ times γ_E , proving the claim. $\square$

Definition 10.5. For $h \geq 3$ and a finite set L , put

$$\mathcal{R}_h^{\text{par}}(L) = \bigoplus_{S \subseteq L} H^{11}(\overline{\mathcal{M}}_{1,S \sqcup \{u,v\}})^{\langle(uv)\rangle}(-2).$$

The partner factor, carrying the unit class, is $\overline{\mathcal{M}}_{h-2,(L \setminus S) \sqcup \{u',v'\}}$. Let $p_h : \mathcal{R}_h^{\text{par}}(L) \rightarrow H^{15}(\overline{\mathcal{M}}_{h,L})$ be twice the ordinary Gysin map from the quotient by simultaneous exchange $(uv)(u'v')$. For $h = 3$ the two possible odd Künneth placements on the equal-genus product are retained. Put $\mathcal{R}_F^{\text{par}} = \mathcal{R}_4^{\text{par}}(I)^t$, and let $\mathcal{R}_{0,F}^{\text{par}}$ and $\mathcal{R}_{+,F}^{\text{par}}$ be its allocation subspaces with $S \cap \{c, d\}$ empty and nonempty. Let $\text{Sep}_{h,L}$ be the complete ordinary separating one-edge Gysin image in $H^{15}(\overline{\mathcal{M}}_{h,L})$, with full degree-thirteen coefficients, and put $B_{\text{sep},F} = \text{Sep}_{4,I}^t$.

For $A \subseteq I$ with $|A| \geq 10$, put $H_A = H^{11}(\overline{\mathcal{M}}_{1,A \sqcup \{p\}})(-2)$. Let

$$\rho_A : H^{11}(\overline{\mathcal{M}}_{1,A \sqcup \{u,v\}})^{\langle(uv)\rangle}(-2) \longrightarrow H_A$$

be rational-tail restriction collapsing u, v to p . For $A \subseteq S$, let $f_{S,A} : H_S \rightarrow H_A$ be induced by the tenth exterior power of the augmentation-space map identifying every label of $S \setminus A$ with p . For $A \cap \{c, d\} \neq \emptyset$, define

$$\iota_A : \overline{\mathcal{M}}_{1,A \sqcup \{p\}} \times \overline{\mathcal{M}}_{2,(J \setminus A) \sqcup \{q\}} \rightarrow \overline{\mathcal{M}}_{3,J}, \quad \mathcal{D}_A = (1 \otimes \Lambda_{J \setminus A}) \text{pr}_{11,4}^* \iota_A^*.$$

Proposition 10.6. For every finite F , each ρ_A is an isomorphism. The formulas

$$\mathcal{L}_A(y) = \frac{\rho_A^{-1} \left(\mathcal{D}_A(y) - \sum_{i \in I \setminus A} f_{A \sqcup \{i\}, A} \mathcal{D}_{A \sqcup \{i\}}(y) \right)}{2(|J \setminus A| + 1)!}$$

define a map $\mathcal{L} : \mathcal{D}_F \rightarrow \mathcal{R}_{+,F}^{\text{par}}$ satisfying

$$\mathcal{L}q_F p_4|_{\mathcal{R}_{+,F}^{\text{par}}} = \text{id}, \quad \mathcal{L}q_F(B_{\text{sep},F}) = 0.$$

Moreover, the map

$$e_F : \mathcal{R}_F^{\text{par}} \rightarrow \mathcal{D}_{5,F}/\delta_F(B_{\text{sep},F}), \quad r \mapsto [\delta_F p_4(r)]$$

has kernel $\mathcal{R}_{0,F}^{\text{par}}$, and $\mathcal{L} \text{pr}_{\text{dl}}$ induces a left inverse for its restriction to $\mathcal{R}_{+,F}^{\text{par}}$.

Proof. The hook restriction. On augmentation spaces the map identifying u, v with p has kernel $\mathbb{Q}(e_u - e_v)$ and maps its swap-invariant complement isomorphically onto the augmentation space on $A \sqcup \{p\}$. The tenth exterior power decomposes into its invariant part and the part containing $e_u - e_v$, on which the swap acts negatively. Proposition 3.15 and Lemma 3.13 identify the invariant restriction with ρ_A , proving it is an isomorphism.

The complete separating image. Theorems 3.3 and 3.23, together with Proposition 3.15, reduce the full degree-thirteen coefficients of a genus-three separating source to three types: an elliptic H^{13} tail; an elliptic H^{11} tail with any Picard coefficient on its partner; and an elliptic H^{11} center with two unit elliptic leaves. Indeed, the only odd Künneth degrees are thirteen and eleven, and an H^{13} class on any higher-genus factor is generated by elliptic-tail Gysin classes. An outer flag on the unit side gives a Picard-decorated tail. An outer positive-genus side attached to the odd factor gives a centered path; an outer rational side there gives an elliptic H^{13} tail. This also retains rational boundaries inside every compact coefficient.

Compare the decorated tail set S with the test set A . If $S = A$, a Picard-decorated tail has $(11, 4)$ component $-z \otimes \psi_q \eta$ from the bridge normal line; an H^{13} tail has elliptic degree thirteen and does not contribute. If $S \subsetneq A$, the internal separating Gysin raises the elliptic degree to at least thirteen. If S and A are disjoint, the odd class is on the second test factor. If $A \subsetneq S$, the second test factor has a q -containing rational tail with its full degree-two coefficient. These are the nested or disjoint possibilities for the two genus-one sides in a stable tree. For a centered path, a cut isolating a unit leaf places the odd class on the other factor; a cut inside the decorated center leaves a rational q -center with the two unit elliptic leaves. Additional rational subdivisions give two-rational-vertex trees. The same statements apply after expanding an elliptic H^{13} coefficient into its rational-tail Gysin generators. Thus every surviving degree-four coefficient belongs to $\mathcal{N}_{J \setminus A}$. Proposition 10.4 proves $\mathcal{D}_A(\text{Sep}_{3,J}) = 0$.

The parallel pullback and its moments. On a labelled normal-crossings chart, a node normalized in the compact pullback of p_h is a selected parallel edge or an internal node of a vertex factor. For a selected edge its normal class is $-(\psi_x + \psi_y)$, and the other parallel edge becomes separating; this branch lies in the complete separating image with its full coefficient. An internal nonseparating node of the decorated elliptic factor gives genus-zero odd cohomology and vanishes by Theorem 3.2. Such a node in the unit factor gives p_{h-1} with both new labels on that factor. An internal separating node can be nonseparating in the glued curve only when it separates the two attaching flags. Cutting it turns the two parallel edges into a separating path. These cases exhaust the compact fiber product; distinct smoothing parameters give coefficient one and retain all deeper degenerations. Consequently, modulo $\text{Sep}_{3,J}$,

$$q_F p_4(r) = p_3((1 - s^*)H_{a,b}r),$$

where $H_{a,b}$ places a, b on the unit factor. Neither selected parallel edge is a bridge, so a genus-one tail in the test pullback must split off inside one factor. A cut in the unit factor places the odd class on the second test factor. A cut in the decorated factor must leave both attaching flags on its rational complement; it therefore contributes only for $A \subseteq S$. Its hook restriction is $f_{S,A} \rho_S$ and its other factor is $\zeta_{S \setminus A}$. This treats both odd placements when the two original vertices have

genus one. In the exchanged term the decorated vertex contains neither c nor d , so the necessary containment fails for every old-present A . Base change and Proposition 10.4 now give

$$\mathcal{D}_A(q_{FP4}(r_S)) = 2|S \setminus A|!(|J \setminus S| + 1)! f_{S,A} \rho_S(r_S).$$

The augmentation maps satisfy $f_{T,A} f_{S,T} = f_{S,A}$ for $A \subseteq T \subseteq S$. The $|S \setminus A|$ summands in the subtraction have factorial $d(d-1)!$ when $d = |S \setminus A|$; all proper supersets cancel, leaving $2(|J \setminus A| + 1)! \rho_A(r_A)$. This proves the left-inverse formula.

The factor two here is the factor in p_h : the simultaneous edge-exchange quotient is unchanged by the rational tail cut, and the bridge defining the cut is uniquely determined by its labelled side, so it contributes no further multiplicity.

Descent and the old-absent kernel. A selected separating bridge remains separating after a nonseparating normalization, so $\nu_{ab}^* \text{Sep}_{4,I} \subseteq \text{Sep}_{3,J}$. This and the preceding vanishing prove $\mathcal{L}_{qF}(B_{\text{sep},F}) = 0$. For $r_0 \in \mathcal{R}_{0,F}^{\text{par}}$, regard its coefficients as an element of $\mathcal{R}_5^{\text{par}}(F)$, placing c, d on the unit factor after pullback. Let ξ_{cd} glue these two flags. The genus-five instance of the compact pullback formula gives $\xi_{cd}^* p_5(r_0) - p_4(r_0) \in B_{\text{sep},F}$. In the compact restriction row, all other coordinates of $\alpha(p_5(r_0))$ are separating. Therefore $\beta\alpha = 0$ from Proposition 5.3 gives $\delta_F(\xi_{cd}^* p_5(r_0)) = 0$ in $\mathcal{Q}_{5,F}$, and hence $e_F(r_0) = 0$. Conversely, if $e_F(r) = 0$, Lemma 10.2 and the left-inverse formula force every old-present coordinate of r to vanish, so $r \in \mathcal{R}_{0,F}^{\text{par}}$. $\square$

Theorem 10.7 (Hodge integrals [FP03, Theorem 1 and Section 0.5]). *Let $g \geq 1$, $n \geq 1$, and $d_i \geq 0$ satisfy $\sum_i d_i = 2g + n - 3$. With $\lambda_g = c_g(\pi_* \omega_\pi)$, put*

$$b_g = \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-2}.$$

Then

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{i=1}^n \psi_i^{d_i} = \binom{2g+n-3}{d_1, \dots, d_n} b_g, \quad 1 + \sum_{g \geq 1} b_g t^{2g} = \frac{t/2}{\sin(t/2)}.$$

In particular, $b_1 = 1/24$ and $b_2 = 7/5760$.

Definition 10.8. Use the summands and maps of Definition 7.2 at $(4, I)$. Define the following sub-sums, over every allocation of I : $\mathcal{D}_F = \mathcal{D}_{4,I} \otimes \mathbf{P}$, the elliptic H^{13} tails $H^{13}(\overline{\mathcal{M}}_{1,A \cup \{p\}})(-1)$ with genus-three unit partners; $\mathcal{T}_F = \mathcal{T}_{4,I} \otimes \mathbf{P}$, the centered-path source

$$\bigoplus_{A \sqcup B \sqcup C = I} H^{11}(\overline{\mathcal{M}}_{1,A \cup \{p_1, p_2\}})(-2) \otimes H^0(\overline{\mathcal{M}}_{1,B \cup \{q_1\}}) \otimes H^0(\overline{\mathcal{M}}_{2,C \cup \{q_2\}})$$

of paths with elliptic H^{11} center and unit leaves of genera one and two; $\mathcal{E}_F \subseteq \mathcal{W}_{4,I} \otimes \mathbf{P}$, the summands whose partner coefficient is a positive separating divisor of the genus-three partner, which by composition of Gysin maps are the endpoint paths of genera $(1, h, 3-h)$, $h = 1, 2$, with H^{11} at the first endpoint; and $\mathcal{V}_F \subseteq \mathcal{W}_{4,I} \otimes \mathbf{P}$, the summands whose partner coefficient lies in $\langle \delta_{\text{irr}}, \psi_i : i \in I \setminus A \rangle$, the elliptic H^{11} tails whose genus-three partner is decorated by δ_{irr} or an external ψ_i ; the attaching cotangent is excluded from the last family. Let $\mathcal{U}_F \subseteq N_{5,F}$ be the t -invariant part of $\Omega_{4,I}(\mathcal{D}_F \oplus \mathcal{T}_F \oplus \mathcal{E}_F \oplus \mathcal{V}_F)$. Put $\mathcal{Q}_F^{\text{Pic}} = \mathcal{Q}_{5,F}/\delta_F(\mathcal{U}_F)$.

For $A \subseteq I$ meeting $\{c, d\}$, put $B = I \setminus A$, $k = |B|$, and choose fresh attaching flags p, q . Define

$$E_A = \overline{\mathcal{M}}_{1,A \cup \{p\}}, \quad Y_A = \overline{\mathcal{M}}_{2,B \sqcup \{q, a, b\}}, \quad v_A : Y_A \rightarrow \overline{\mathcal{M}}_{3,B \sqcup \{q\}},$$

$$i_A : E_A \times \overline{\mathcal{M}}_{3,B \sqcup \{q\}} \rightarrow \overline{\mathcal{M}}_{4,I}, \quad j_A : E_A \times Y_A \rightarrow \overline{\mathcal{M}}_{3,J}.$$

The map v_A glues a, b , and i_A, j_A glue p, q . On the genus-three partner put

$$Z_B = \langle \kappa_1, \psi_q, \delta_{0,T} : T \subseteq B, |T| \geq 2 \rangle.$$

Define

$$\mathcal{R}_F^{\text{Pic}} = \left(\bigoplus_{\substack{A \subseteq I, \ A \cap \{c,d\} \neq \emptyset \\ |I \setminus A| \leq 3}} H^{11}(E_A)(-1) \otimes Z_B \right)^t, \quad \Phi_F^{\text{Pic}}((z_A \otimes \eta_A)_A) = \sum_A i_{A*}(z_A \otimes \eta_A).$$

For $\theta \in H^{2k+8}(Y_A)$ define

$$\mathcal{T}_A^\theta(y) = \left(1 \otimes \left[u \mapsto \int_{Y_A} u \theta \right] \right) \text{pr}_{11,4} j_A^* y.$$

The original divisor η_A is on genus three; its tested pullback is $v_A^* \eta_A$.

Lemma 10.9. *On every pointed genus-two space let $\lambda_i = c_i(\pi_* \omega_\pi)$. The pullback of $\lambda_1 \lambda_2$ to every irreducible or positive separating normalization is zero, and it pulls back under forgetting markings. Put*

$$I_0 = \int_{\overline{\mathcal{M}}_{2,\{q\}}} \lambda_1 \lambda_2 \psi_q = \frac{1}{2880}, \quad \mathcal{J}_n(d_1, \dots, d_n) = \int_{\overline{\mathcal{M}}_{2,[n]}} \lambda_1 \lambda_2 \prod_i \psi_i^{d_i}.$$

With omitted exponents zero, the following identities hold in their indicated ranges:

$$\begin{aligned} \mathcal{J}_n(n) &= I_0 & (n \geq 1), \\ \mathcal{J}_n(n-1, 1) &= (n+1)I_0 & (n \geq 2), \\ \mathcal{J}_n(n-2, 2) &= \left(\frac{n(n+1)}{2} - 2 \right) I_0 & (n \geq 2), \\ \mathcal{J}_n(n-2, 1, 1) &= n(n+1)I_0 & (n \geq 3). \end{aligned}$$

Proof. Hodge restrictions. Write $\mathbb{E}_h$ for the genus- h Hodge bundle. The residue sequence on a nonseparating normalization is $0 \rightarrow \mathbb{E}_1 \rightarrow \nu^* \mathbb{E}_2 \rightarrow \mathcal{O} \rightarrow 0$; hence $\nu^* \lambda_1 = \lambda := c_1(\mathbb{E}_1)$ and $\nu^* \lambda_2 = 0$. On a positive separating normalization the Hodge bundle is the direct sum of the two elliptic Hodge lines. Each line pulls back from $\overline{\mathcal{M}}_{1,1}$, so its first Chern class has square zero. Thus $(\lambda_1 \lambda_2)|_{\text{irr}} = (\lambda_1 \lambda_2)|_{\text{sep}} = 0$.

The one-point integral. Writing D_0 for the rational-tail divisor sum and D_1 for the positive separating divisor sum, Proposition 3.18 and Theorem 6.8 give

$$5\kappa_1 = 5 \sum_i \psi_i + \delta_{\text{irr}} - 5D_0 + 7D_1, \quad \kappa_1 = 12\lambda_1 - \delta_{\text{irr}} - D_0 - D_1 + \sum_i \psi_i.$$

Subtracting these expressions for $5\kappa_1$ yields $60\lambda_1 = 6\delta_{\text{irr}} + 12D_1$. Hence $10\lambda_1 = \delta_{\text{irr}} + 2D_1$ and $\kappa_1 = \sum_i \psi_i + 2\lambda_1 - D_0 + D_1$. Pairing the first identity with $\lambda_2 \psi_q$ gives

$$10I_0 = 2 \left(\int_{\overline{\mathcal{M}}_{1,\{q,u\}}} \lambda \psi_q \right) \left(\int_{\overline{\mathcal{M}}_{1,\{v\}}} \lambda \right) = 2(1/24)^2.$$

The two genus-one integrals are Theorem 10.7 with respectively exponents $(1,0)$ and (0) . There is no factor $1/2$ because q distinguishes the two factors. Thus $I_0 = 1/2880$.

Forgetting recursions. Forgetting preserves the Hodge bundle: contracting unstable rational components does not change regular differentials. Under forgetting one mark, Proposition 3.16 gives $\psi_i = f^* \psi_i + D_{it}$, $D_{it}^2 = -f^* \psi_i D_{it}$, and $\psi_t D_{it} = 0$; also $f_* \psi_t = 2g - 2 + n$. Since $\lambda_1 \lambda_2$ pulls back, these are the string and dilaton recursions. Repeated string gives $\mathcal{J}_n(n) = I_0$; one dilaton gives $\mathcal{J}_n(n-1, 1) = (n+1)I_0$; the base value at $n=2$ and the string increment n give $\mathcal{J}_n(n-2, 2) = (n(n+1)/2 - 2)I_0$; two dilaton steps give $\mathcal{J}_n(n-2, 1, 1) = n(n+1)I_0$. $\square$

Definition 10.10. For the sets in Definition 10.8, with no restriction on k , put

$$b_k = \frac{4(k+7)}{3(3k^2+15k+14)}, \quad a_k = -k(k+4)b_k - 2(k+2), \quad e_k = b_k + \frac{2}{3},$$

$$\theta_{\psi,B} = \lambda_1 \lambda_2 \psi_q^k \left(a_k \psi_q + b_k \sum_{i \in B} \psi_i + \psi_a + \psi_b - \frac{4}{3} D_{ab} + e_k \sum_{i \in B} D_{abi} \right).$$

Here $D_T = \delta_{0,T}$ on Y_A ; subscripts list sets of markings. Put

$$\mathcal{W}_F = \left(\bigoplus_{A \cap \{c,d\} \neq \emptyset} H^{11}(E_A)(-2) \right)^t, \quad \Phi_{\psi,F}((z_A)_A) = \sum_A i_{A*}(z_A \otimes \psi_q),$$

and let $\mathcal{T}^\psi = (\mathcal{T}_A^{\theta_{\psi,B}})_A$.

Proposition 10.11. For every finite F ,

$$\mathcal{T}^\psi_{qF}(\mathcal{U}_F) = 0, \quad \mathcal{T}^\psi_{qF} \Phi_{\psi,F} = -\frac{1}{1080} \text{id}_{\mathcal{W}_F}.$$

Consequently $-1080 \mathcal{T}^\psi_{\text{pr}_{\text{dl}}}$ induces a left inverse for $\mathcal{W}_F \rightarrow \mathcal{Z}_F^{\text{Pic}}, z \mapsto [\delta_F \Phi_{\psi,F}(z)]$.

Proof. The local moments. For rational numbers $\alpha_0, \beta_0, \chi_0, \epsilon_0$, put

$$L_0 = \alpha_0 \psi_q + \beta_0 \sum_{i \in B} \psi_i + \psi_a + \psi_b + \chi_0 D_{ab} + \epsilon_0 \sum_{i \in B} D_{abi}, \quad [D]_0 = I_0^{-1} \int_{Y_A} \lambda_1 \lambda_2 \psi_q^{k+1} D L_0$$

for a divisor D on Y_A , and put $\Xi = \psi_a + \psi_b - \sum_{\emptyset \neq S \subseteq B} (D_{aS} + D_{bS})$. Lemma 10.9 gives

$$\begin{aligned} [\psi_q]_0 &= \alpha_0 + k(k+4)\beta_0 + 2(k+4) + \chi_0, \\ [\psi_i]_0 &= (k+4)\alpha_0 + ((k-\frac{1}{2})(k+3)(k+4) - 2)\beta_0 \\ &\quad + 2(k+3)(k+4) + (k+3)\chi_0 + \epsilon_0, \\ [D_{ab}]_0 &= \alpha_0 + k(k+3)\beta_0 - (k+3)\chi_0 + k\epsilon_0, \\ [D_{abi}]_0 &= \beta_0 + 2 + \chi_0 - \epsilon_0, \\ [\Xi]_0 &= 8(\alpha_0 + k(k+4)\beta_0 + 2(k+2)). \end{aligned}$$

Here $i \in B$ in the indexed rows. To see the boundary coefficients, on D_{ab} the two new cotangents vanish and its normal class is $-\psi_v$ on the genus-two factor. Each D_{abi} restricts there to D_{vi} , contributing one. On D_{abi} the rational factor is $\overline{M}_{0,4}$; its four contributions from $\psi_i, \psi_a + \psi_b, D_{ab}, D_{abi}$ are $\beta_0, 2, \chi_0, -\epsilon_0$. For the last row the contribution of $\psi_a + \psi_b$ is

$$2(k+4)\alpha_0 + 2k(k+3)(k+4)\beta_0 + 3(k+3)(k+4) - 4 + 2k\epsilon_0.$$

The $2k$ tails with $|S| = 1$ each contribute $\alpha_0 + (k+3)(k-1)\beta_0 + k+3 + \epsilon_0$. The $k(k-1)$ tails with $|S| = 2$ each contribute $2\beta_0 + 1$. For larger S , the degree on the rational factor is below its dimension. Subtraction gives the row for Ξ .

Specialize to $\alpha_0 = a_k, \beta_0 = b_k, \chi_0 = -4/3, \epsilon_0 = e_k$, and denote the resulting bracket by $[D]_\psi$. The D_{abi} and Ξ rows vanish by the definitions of e_k, a_k . The D_{ab} row reduces to $-4 - 3\chi_0 = 0$, and the ψ_i row reduces to $-(3k^2 + 15k + 14)b_k/2 + 2(k+7)/3 = 0$. The remaining row is

$$[\psi_q]_\psi = 4 - \frac{4}{3} = \frac{8}{3}.$$

This also covers $k = 0$, when the indexed rows are absent.

The endpoint boundary. Theorem 3.19 gives

$$v_A^* \delta_{\text{irr}} = \delta_{\text{irr}}^{Y_A} - (\psi_a + \psi_b) + \sum_{\text{separating types splitting } a \text{ from } b} \delta_{\text{type}}.$$

The irreducible and positive separating terms are annihilated by $\lambda_1\lambda_2$. A rational tail containing q and exactly one new flag has dimension at most k , so ψ_q^{k+1} restricts to zero there. The remaining terms are precisely $-\Xi$, and hence $[v_A^*\delta_{\text{irr}}]_\psi = 0$. For $C \subseteq B$, the rational factor of D_{abC} has dimension $|C|$. Only $|C| = 0, 1$ can contribute with the single divisor factor in $L_\psi = L_0|_{(a_k, b_k, -4/3, e_k)}$, and those are the two zero boundary rows above. The rational factor of D_{qabC} has dimension $|C| + 1$. If $C \neq B$, ψ_q^{k+1} restricts to zero there; if $C = B$, every term of L_ψ restricts to that rational factor, and the total degree is $k + 2 > k + 1$. Thus both families vanish.

All compact allocations. The selected ordinary edges are bridges and remain separating under degeneration. Thus ν_{ab} cuts an internal node of one source factor with transverse coefficient one. Cutting the decorated elliptic factor gives zero odd cohomology on genus zero. In a common refinement with j_A , an original edge internal to E_A raises its odd degree from at least eleven to at least thirteen; an odd factor outside E_A cannot supply its degree eleven. Consequently a surviving decorated allocation A' contains A . An exchanged branch has both c, d outside its decorated factor, so none survives. For an endpoint source with $S = A' \setminus A \neq \emptyset$, the intervening rational factor has markings S, q and its original attaching flag, and dimension $|S| - 1 \leq k - 1$. The restriction of ψ_q^k to this factor is zero. For $A' = A$, stability excludes an intervening unmarked rational tree with two attachments, and the bridges coincide. An H^{13} endpoint then has elliptic degree thirteen and does not contribute. An H^{11} endpoint contributes $-z_A \otimes \psi_q v_A^* \eta_A$. The external cotangents and irreducible divisor vanish by the local rows. The two cuts of an endpoint path give a positive divisor and either D_{abC} or D_{qabC} , whose pairings have already been shown to vanish.

A centered path has an intervening rational factor with markings S, q and two leaf attachments, of dimension $|S|$. Only $S = B$ can survive ψ_q^k . Then the original decorated factor carries all of I . Cutting the genus-two leaf leaves two elliptic leaves, on which $\lambda_1\lambda_2$ vanishes. Cutting the genus-one leaf leaves a rigid rational ab leaf and a genus-two leaf. The q and B cotangents lie on the dimension- k center, the a, b cotangents are zero, D_{ab} is minus the attaching cotangent on the center, and D_{abi} restricts to a boundary divisor on the center. Therefore every term of L_ψ lies on the center, and $\psi_q^k L_\psi$ has degree $k + 1$ there. Thus the remaining compact centered term vanishes, including $k = 0$.

The node scalar and descent. Only the matching tail decorated by the node cotangent class contributes, with value $-z_A \int_{Y_A} \theta_{\psi, B} \psi_q^2 = -(8/3)I_0 z_A = -z_A/1080$. Nonempty decorated marking sets distinguish the ordinary graph vertices; the new pair distinguishes a lowered leaf. Their transverse coefficients are one, and by Set-up 10.1 the order-eight projector onto $\mathcal{D}_F$ introduces no additional normalization factor. Lemma 10.2 gives the claimed left inverse on the full quotient. $\square$

Definition 10.12. Assume $k = |B| \leq 3$. Let $\mathcal{C}_B$ have coordinates x_T for two-element subsets of B and y_U for three-element subsets. For a vector of these coordinates define

$$\begin{aligned} X &= \sum_T x_T, & Y &= \sum_U y_U, & X_i &= \sum_{T \ni i} x_T, & Y_i &= \sum_{U \ni i} y_U, \\ d_H &= 3k^2 + 15k + 14, & m_H &= (k^2 + 7k + 14)/2, & S_H &= \frac{6(Y - (k+2)X)}{d_H}, \\ r_i &= \frac{-X - (k+3)X_i + Y_i - (k+4)S_H}{m_H}, & a_0 &= -(k+4)S_H - X, \\ L_H &= a_0\psi_q + \sum_{i \in B} r_i(\psi_i + D_{abi}) + \sum_T x_T D_T + \sum_U y_U D_U, & \theta_{H, B}(x, y) &= \lambda_1\lambda_2\psi_q^k L_H. \end{aligned}$$

Define a second test by the rational coefficients

$$\begin{aligned} d_\lambda &= 21k^2 + 105k + 50, & b_\lambda &= -28(k+3)/d_\lambda, & c_\lambda &= -19/7 - (5k/14)b_\lambda, & e_\lambda &= b_\lambda + 2 + c_\lambda, \\ f_\lambda &= 56/25, & h_\lambda &= 96/25 - (k/2)b_\lambda, & a_\lambda &= -16/5 - k(k+4)b_\lambda - 2(k+4) - c_\lambda, \end{aligned}$$

$$L_\lambda = a_\lambda \psi_q + b_\lambda \sum_{i \in B} \psi_i + \psi_a + \psi_b + c_\lambda D_{ab} + e_\lambda \sum_{i \in B} D_{abi} \\ + f_\lambda E_\emptyset + (b_\lambda + f_\lambda) \sum_{i \in B} E_i + h_\lambda (E_a + E_b), \quad \theta_{\lambda, B} = \lambda_2 \psi_q^{k+1} L_\lambda.$$

Here E_S is the genus-one/genus-one separating divisor with one side carrying S ; complementary descriptions designate the same divisor and are counted once. Put $\eta_B^0 = \kappa_1 + \sum_{T \subseteq B, |T| \geq 2} \delta_{0, T} \in Z_B$. Let $\mathcal{K}_F \subseteq \mathcal{R}_F^{\text{Pic}}$ have each divisor coefficient in $\langle \psi_q, \eta_B^0 \rangle$, and let $\mathcal{R}_{F, \psi}^{\text{Pic}}$ be the subspace with divisor coefficient in the span of ψ_q .

Proposition 10.13. *For every finite F , the families $\mathcal{T}^H = (\mathcal{T}_A^{\theta_{H, B}(x, y)})_{A, (x, y)}$ and $\mathcal{T}^\lambda = (\mathcal{T}_A^{\theta_{\lambda, B}})_A$ annihilate $q_F(\mathcal{U}_F)$ and induce maps on $\mathcal{Q}_F^{\text{Pic}}$. On the source $\mathcal{R}_F^{\text{Pic}}$,*

$$\ker(\mathcal{T}^H q_F \Phi_F^{\text{Pic}}) = \mathcal{K}_F.$$

The family $\mathcal{T}^\lambda q_F \Phi_F^{\text{Pic}}$ is diagonal in A , is zero on $\mathcal{R}_{F, \psi}^{\text{Pic}}$, and has value $z_A/75$ on a coefficient $z_A \otimes \eta_B^0$. In particular the map

$$\mathcal{R}_F^{\text{Pic}} \longrightarrow \mathcal{Q}_F^{\text{Pic}}, \quad r \longmapsto [\delta_F \Phi_F^{\text{Pic}}(r)]$$

is injective.

Proof. The first family annihilates the auxiliary classes. For a divisor D on Y_A , put

$$[D]_H = I_0^{-1} \int_{Y_A} \theta_{H, B}(x, y) \psi_q D.$$

Summing the defining numerators of r_i , using $\sum_i X_i = 2X$, $\sum_i Y_i = 3Y$, and $m_H + k(k+4) = d_H/2$, gives $\sum_i r_i = S_H$. Boundary restriction and Lemma 10.9 now give

$$[\psi_q]_H = [D_{ab}]_H = a_0 + (k+4)S_H + X = 0, \\ [D_{abi}]_H = r_i - r_i = 0, \\ [\psi_i]_H = -(k+4)S_H - X - (k+3)X_i + Y_i - m_H r_i = 0.$$

For the last row, a pair divisor not containing i contributes $(k+3)x_T$, a pair containing i contributes zero, and a triple containing i contributes y_U ; the other triple terms vanish by dimension. The cotangent terms contribute

$$(k+4)a_0 + (k+3)(k+4)S_H - \left(\frac{(k+3)(k+4)}{2} + 2 \right) r_i,$$

and the D_{abi} correction adds r_i . These are all the terms in the displayed row.

Use the full irreducible pullback $v_A^* \delta_{\text{irr}}$ of Theorem 3.19, as in the proof of Proposition 10.11. Its irreducible and positive terms are annihilated by $\lambda_1 \lambda_2$, and its q -containing rational tails by ψ_q^{k+1} . For the remaining class Ξ defined there, the cotangent and D_{abi} terms contribute $8(a_0 + (k+4)S_H)$. Each pair correction contributes $8x_T$: the contribution $2(k+3)x_T$ of $\psi_a + \psi_b$ loses $2(k-2)x_T$ from disjoint singleton tails and $2x_T$ from the two tails with subset T . Triple corrections contribute zero. Thus $[\Xi]_H = 8(a_0 + (k+4)S_H + X) = 0$. For D_{abC} only $|C| \leq 1$ can contribute; these are the two zero boundary rows. For D_{qabC} the cotangent power at q vanishes for every proper $C \subset B$. If $C = B$, all terms of L_H , including its pair and triple divisors, lie on the rational factor of dimension $k+1$, where $\psi_q^{k+1} L_H$ vanishes.

The common-refinement argument of Proposition 10.11 therefore applies to all endpoint families. For centered paths the only additional case has all original markings on the decorated center. The cut giving two elliptic leaves is annihilated by $\lambda_1 \lambda_2$. For the cut giving a rigid rational ab leaf, each added D_T, D_U has rational side contained in B , so it restricts to the rational center or to zero. It cannot include the positive-genus leaf. Thus all of L_H lies on the dimension- k center and

$\psi_q^k L_H = 0$ there. It follows that $\mathcal{T}^H_{qF}$ annihilates every summand of $\mathcal{U}_F$, and that its Picard evaluation is diagonal in A , with matching value $-I_0 z_A [v_A^* \eta_A]_H$.

The first kernel. For a pair $T = \{i, j\} \subseteq B$, restriction to its rigid rational tail gives

$$[D_T]_H = a_0 + (k+3)(S_H - r_i - r_j) - (k+3)x_T + \sum_{\substack{|T'|=2 \\ T' \cap T = \emptyset}} x_{T'} + \sum_{\substack{|U|=3 \\ U \supset T}} y_U.$$

The normal class gives $-(k+3)x_T$, disjoint pairs give two rigid tails, and containing triples give a rigid tail on the contracted core. All crossing cases are incompatible. On a triple U the rational factor is $\overline{\mathcal{M}}_{0,4}$, giving

$$[D_U]_H = \sum_{i \in U} r_i + \sum_{\substack{T \subset U \\ |T|=2}} x_T - y_U.$$

The terms are its cotangents, boundary points, and normal class. There are no tail columns for $k = 0, 1$. For $k = 2$, writing x for the only pair parameter gives $S_H = -3x/7$, $r_i = -3x/14$, $a_0 = 11x/7$, and $[D_B]_H = -24x/7$. For $k = 3$, put $y = y_B$. Substitution yields

$$[D_T]_H = \frac{308y + 94X - 2064x_T}{473}, \quad [D_B]_H = \frac{28X - 40y}{43}.$$

On the two-dimensional space $X = y = 0$ the pair part is $-48/11$ times the identity. On equal pair parameters $x_{12} = x_{13} = x_{23} = x$ and triple parameter y , the matrix is

$$\frac{1}{43} \begin{pmatrix} -162 & 28 \\ 84 & -40 \end{pmatrix}, \quad \det = \frac{96}{43}.$$

This proves invertibility of the full labelled tail matrix.

Logarithmic dualizing differentials identify $v_A^* \kappa_1$ with the logarithmic κ_1 on Y_A ; also $v_A^* \psi_q = \psi_q$ and $v_A^* \delta_{0,T} = D_T$ for $T \subseteq B$. The genus-two Picard relation from Lemma 10.9 gives

$$[v_A^* \kappa_1]_H = - \sum_{\substack{T \subseteq B \\ |T| \geq 2}} [D_T]_H.$$

Indeed, $\lambda_1^2 \lambda_2$ is pulled back from a codimension-four class on the three-dimensional unmarked genus-two space, and vanishes. Positive boundary terms vanish as above. The external and node cotangents vanish, the q -tails and abC tails have already been shown to have zero pairing, and the full irreducible-pullback equation cancels the new cotangents with the one-new-flag tails. Only the negative sum of the base rational tails remains. Consequently, for vector coefficients $z_\kappa \kappa_1 + z_\psi \psi_q + \sum_T z_T \delta_{0,T}$, all first tests vanish exactly when $z_T = z_\kappa$ for every T . Together with the zero node column this is precisely $\mathcal{K}_F$.

The second family and its positive boundary terms. Put

$$m_\lambda = k + 4, \quad X_\lambda = a_\lambda + k(k+4)b_\lambda + 2(k+4) + c_\lambda, \quad [D]_\lambda = 5760 \int_{Y_A} \lambda_2 \psi_q^{k+2} L_\lambda D.$$

Theorem 10.7 gives coefficient 7 for a genus-two factor and coefficient $5760(1/24)^2 = 10$ for two genus-one factors. In particular λ_2 restricts to $\lambda_1' \lambda_1''$ on a positive normalization; those

contributions must be included. Restriction gives the following table:

$$\begin{aligned}
[\psi_q]_\lambda &= 7X_\lambda + 10f_\lambda, \\
[\psi_i]_\lambda &= 7[m_\lambda a_\lambda + (k - \frac{1}{2})m_\lambda(m_\lambda - 1)b_\lambda + 2m_\lambda(m_\lambda - 1) \\
&\quad + (m_\lambda - 1)c_\lambda + e_\lambda] + 10[(m_\lambda - 1)f_\lambda + (b_\lambda + f_\lambda)], \\
[D_{ab}]_\lambda &= 7[a_\lambda + k(m_\lambda - 1)b_\lambda - (m_\lambda - 1)c_\lambda + ke_\lambda] + 10f_\lambda, \\
[D_{abi}]_\lambda &= 7(b_\lambda + 2 + c_\lambda - e_\lambda), \\
[E_\emptyset]_\lambda &= 10[a_\lambda + k(m_\lambda - 1)b_\lambda + 2(m_\lambda - 1) + c_\lambda \\
&\quad - (m_\lambda - 2)f_\lambda + k(b_\lambda + f_\lambda) + 2h_\lambda], \\
[E_i]_\lambda &= 10(b_\lambda - (b_\lambda + f_\lambda) + f_\lambda), \\
[E_a]_\lambda + [E_b]_\lambda &= 20(1 - h_\lambda + f_\lambda).
\end{aligned}$$

The cotangent entries are the multinomial integrals on the surviving genus-two factor. The D_{ab} and D_{abi} entries use the rigid tail and $\overline{\mathcal{M}}_{0,4}$ normalizations as in the node calculation, now with the genus-two coefficient 7. Their positive intersections and those of the cotangents give the terms with coefficient 10. In the $E_\emptyset$ row its normal class contributes $-10(m_\lambda - 1)f_\lambda$. There is also a branch with all markings on a rational component, contributing $+10f_\lambda$; together these give $-10(m_\lambda - 2)f_\lambda$. The intersections with E_i, E_a, E_b contribute $10k(b_\lambda + f_\lambda)$ and $20h_\lambda$. Each is a product of two marked genus-one Hodge integrals (counting the self-incidence of the unmarked elliptic side once).

For the full irreducible pullback of Theorem 3.19, retain the persistent normal, the one-new-flag rational tails, and the positive splitting branches. To make their signs explicit, the class Ξ from the node proof has bracket

$$[\Xi]_\lambda = 7(8a_\lambda + 8k(k + 4)b_\lambda + 16k + 36) + 60f_\lambda + 20h_\lambda.$$

Indeed, the two new cotangents contribute

$$\begin{aligned}
[\psi_a + \psi_b]_\lambda &= 7(2m_\lambda a_\lambda + 2km_\lambda(m_\lambda - 1)b_\lambda + 3m_\lambda(m_\lambda - 1) + 2ke_\lambda) \\
&\quad + 20(m_\lambda - 1)f_\lambda + 20h_\lambda.
\end{aligned}$$

Each of the $2k$ singleton rational tails contributes $7(a_\lambda + (k - 1)(k + 3)b_\lambda + k + 3 + e_\lambda) + 10f_\lambda$; each of the $k(k - 1)$ pair tails contributes $7(1 + 2b_\lambda)$; larger tails vanish by dimension. Their subtraction gives the displayed Ξ bracket. The only positive splitting sides that can receive the remaining divisor degree are $\{a\}$ and $\{b\}$. Consequently

$$\begin{aligned}
[v_A^* \delta_{\text{irr}}]_\lambda &= -[\Xi]_\lambda + [E_a]_\lambda + [E_b]_\lambda \\
&= -7(8a_\lambda + 8k(k + 4)b_\lambda + 16k + 36) - 40f_\lambda - 40h_\lambda + 20.
\end{aligned}$$

Substituting the coefficients gives

$$\begin{aligned}
X_\lambda &= -16/5, & [\psi_q]_\lambda &= [\psi_i]_\lambda = [D_{abi}]_\lambda = [E_i]_\lambda = 0, \\
[D_{ab}]_\lambda + [E_\emptyset]_\lambda &= 0, & [v_A^* \delta_{\text{irr}}]_\lambda &= 0.
\end{aligned}$$

The complete pullback of an endpoint divisor with elliptic side marked by $C \subseteq B \sqcup \{q\}$ is $D_{abC} + E_C$. If $q \in C$, its rational part has dimension at most $k + 1$ and its elliptic part has dimension at most $k + 2$ with one Hodge class; both terms are annihilated by ψ_q^{k+2} . If $q \notin C$ and $|C| \geq 2$, the side away from q needs at least two further divisor degrees, while L_λ supplies one. The cases $C = \emptyset, \{i\}$ are the cancellations in the table. This proves all endpoint vanishings.

The kappa scalar. Restriction to the remaining base rational tails gives

$$[D_T]_\lambda = \begin{cases} 7(-3(k+2)b_\lambda - 2), & |T| = 2, \\ 21b_\lambda, & |T| = 3, \\ 0, & |T| \geq 4. \end{cases}$$

For a pair, its rigid tail removes its two cotangents. The remaining genus-two terms and its intersection with $E_\emptyset$ give

$$7(a_\lambda + (k-2)(k+3)b_\lambda + 2(k+3) + c_\lambda) + 10f_\lambda = 7X_\lambda + 10f_\lambda + 7(-3(k+2)b_\lambda - 2).$$

The node row is zero, giving the first value. For a triple, only its three rational cotangents have top degree on $\overline{\mathcal{M}}_{0,4}$, giving $21b_\lambda$; larger tails vanish by dimension. Apply the full genus-two relation $\kappa_1 = \sum \psi_i + 2\lambda_1 - D_0 + D_1$. The zero rows just proved remove the old cotangents, node cotangent and endpoint pairs; the full irreducible equation removes the one-new-flag terms. Since $[\lambda_1]_\lambda = 2X_\lambda$ by Lemma 10.9, the remaining terms are

$$[v_A^* \kappa_1]_\lambda = 2([E_\emptyset]_\lambda + [E_a]_\lambda + [E_b]_\lambda) + 4X_\lambda - \sum_{\substack{T \subseteq B \\ |T| \geq 2}} [D_T]_\lambda.$$

The positive rows simplify to $[E_\emptyset]_\lambda = 10(X_\lambda - 2 - 2f_\lambda + 2h_\lambda)$ and $[E_a]_\lambda + [E_b]_\lambda = 20(1 - h_\lambda + f_\lambda)$. Their sum is $10X_\lambda = -32$. It follows that

$$[v_A^* \eta_B^0]_\lambda = 2(-32) + 4(-16/5) = -384/5.$$

Thus its local integral is $-1/75$, and the bridge minus sign gives $+1/75$ on the coefficient.

The common-refinement argument for the first family applies verbatim, ψ_q^{k+1} restricting to zero on every intervening rational factor including the centered one of dimension k ; so $\mathcal{T}^\lambda q_F$ annihilates $\mathcal{U}_F$ and the node column and is diagonal in A with matching value $-z_A \psi_q v_A^* \eta_A$.

Injectivity of the combined test maps. For a class in the common kernel, the first family forces its divisor coefficient to lie in $\langle \psi_q, \eta_B^0 \rangle$. The second, with scalar $1/75$, forces its η_B^0 coefficient to vanish. The node test, with scalar $-1/1080$ by Proposition 10.11, then forces its node coefficient to vanish. Lemma 10.2 gives descent through the complete separating denominator, and the auxiliary vanishings give descent to $\mathcal{Q}_F^{\text{Pic}}$. $\square$

Definition 10.14. Use the sets F, I, J of Set-up 10.1 and the four families of Definition 10.8. Let $\mathcal{D}_F^+, \mathcal{E}_F^+, \mathcal{V}_F^+$ be the sub-sums of $\mathcal{D}_F, \mathcal{E}_F, \mathcal{V}_F$ over the allocations whose decorated vertex's marking set A meets $\{c, d\}$, and keep the full centered-path source $\mathcal{T}_F$. Let

$$\mathcal{A}_F^{\text{aux}} = (\mathcal{D}_F^+ \oplus \mathcal{T}_F \oplus \mathcal{E}_F^+ \oplus \mathcal{V}_F^+)^{(t)}, \quad \Phi_F^{\text{aux}} : \mathcal{A}_F^{\text{aux}} \longrightarrow N_{5,F}$$

be the sum of their Gysin maps, the restriction of $\Omega_{4,I}$.

Write $\mathcal{T}_F = \mathcal{T}_{0,F} \oplus \mathcal{T}_{1,F} \oplus \mathcal{T}_{2,F}$, where $\mathcal{T}_{i,F}$ for $i = 1, 2$ has both c, d on the genus- i unit leaf, and $\mathcal{T}_{0,F}$ has all other allocations. On $\mathcal{T}_{1,F}$ define

$$\sigma_T(A, B \cup \{c, d\}, C; z) = (A, C \cup \{c, d\}, B; (p_1 p_2)^* z),$$

and let $\mathcal{T}_{1,F}^\pm$ be its eigenspaces. For $(A, B, C \cup \{c, d\}; w) \in \mathcal{T}_{2,F}$, glue the decorated center along p_1 to $\overline{\mathcal{M}}_{0, B \cup \{c, d, q_1\}}$. This gives a degree-thirteen class on $\overline{\mathcal{M}}_{1, A \cup B \cup \{c, d, p_2\}}$; placing it in the elliptic-tail source with genus-three unit partner marked by C defines $L : \mathcal{T}_{2,F} \rightarrow (\mathcal{D}_F^+)^{(t)}$.

Proposition 10.15 (The auxiliary kernel). *For every finite F , in the coordinate order $(\mathcal{D}_F^+, \mathcal{T}_{0,F}, \mathcal{T}_{1,F}, \mathcal{T}_{2,F}, \mathcal{E}_F^+, \mathcal{V}_F^+)$, one has*

$$\ker(q_F \Phi_F^{\text{aux}}) = \{(Lw, 0, u, w, 0, 0) : u \in \mathcal{T}_{1,F}^+, w \in \mathcal{T}_{2,F}\}.$$

The map $q_F \Phi_F^{\text{aux}}$ is injective on the direct complement

$$(\mathcal{D}_F^+)^{(t)} \oplus \mathcal{T}_{0,F}^{(t)} \oplus \mathcal{T}_{1,F}^- \oplus (\mathcal{E}_F^+)^{(t)} \oplus (\mathcal{V}_F^+)^{(t)}.$$

Moreover, for a source vector v with $\mathcal{V}_F^+$ coefficient $e_A \otimes \delta_{\text{irr}} + \sum_{i \in B} d_{A,i} \otimes \psi_i$ at an old-present allocation A , $B = I \setminus A$, the following hold.

- (a) (External Picard tests.) For each $C \subseteq B$, restrict $q_F \Phi_F^{\text{aux}}(v)$ to the open of the positive path with odd endpoint carrying A , middle vertex carrying $(B \setminus C) \sqcup \{a\}$, and other endpoint carrying $C \sqcup \{b\}$, and take the $(11, 2, 2)$ component of the self-intersection. The result is $-(10e_A + \sum_{i \in C} d_{A,i})|_{\mathcal{M}_{1,A \sqcup \{p\}}} \otimes x \otimes y$, with x, y the nonzero cotangents on the two partial elliptic factors; the other families of $\mathcal{A}_F^{\text{aux}}$ and the exchanged terms contribute nothing to these tests, which therefore recover the $\mathcal{V}_F^+$ coefficient.
- (b) (Positive support tests.) If the $\mathcal{V}_F^+$ coefficient of v vanishes, the same tests on the positive paths obtained by lowering the genus-two unit vertex of an endpoint path recover each $\mathcal{E}_F^+$ coefficient with scalar 1; if moreover the $\mathcal{E}_F^+$ coefficient vanishes, lowering the genus-two leaf of a centered path recovers each $\mathcal{T}_{0,F}$ coefficient with scalar 1 and each $\mathcal{T}_{1,F}^-$ coefficient with scalar 2. The elliptic H^{13} tails and all classes supported on rational strata contribute nothing to these tests.

Proof. Coefficients are recovered in the order $\mathcal{V}_F^+, \mathcal{E}_F^+, \mathcal{T}_{0,F}, \mathcal{T}_{1,F}^-, \mathcal{D}_F^+$.

Restrictions on the selected strata. A collection of stable graphs closed under contraction defines an open substack: its complement is the finite union of the proper clutching images of the excluded graphs. On charts with labelled nodes, each selected bridge has its own smoothing parameter and normal class $-\psi_h - \psi_{h'}$. A uniquely determined set of selected bridges therefore defines a regular embedding, with the curve stabilizers retained.

Let Γ be a three-elliptic-vertex path in $\overline{\mathcal{M}}_{3,J}$ with a specified odd vertex, replace its two unit vertices by rational vertices with one loop each, and let V_Γ be the open union of the strata obtained by contracting edges of the resulting graph. The normalization over this open is

$$\mathcal{M}_{1,L_0} \times \mathcal{U}_{L_1} \times \mathcal{U}_{L_2},$$

where L_0, L_1, L_2 are the respective marking and bridge-flag sets of the odd vertex and of the two unit vertices, and $\mathcal{U}_L$ is the open of Lemma 7.9, the smooth or irreducible one-nodal genus-one curves with distinct smooth markings.

If the odd vertex is the center, this is the situation of Lemma 7.11 with leaf genera $(1, 1)$; in every application below the two unit leaves carry different external label sets. That lemma makes the normalization a closed regular embedding of codimension two, gives the self-intersection with $(11, 2, 2)$ component $z|_{\mathcal{M}_{1,L_0}} \otimes \psi_{q_1} \otimes \psi_{q_2}$, injective in the odd coefficient z , and shows that V_Γ meets no other two-edge tree stratum and no two-edge tree stratum with a rational vertex.

If the odd vertex is an endpoint, the same argument applies with the roles of the vertices changed. The open meets only the specified two-edge tree: a tree contraction must contract both loops and retain both bridges. The bridges are determined by their labelled sides. Two nested bridges of the same type would enclose an unmarked rational tree with two exterior attachments. On such a tree the sum of valences minus two is zero, whereas stability makes every summand positive. Thus there is no duplicate bridge. Lemma 6.11 makes restriction of the odd coefficient to $\mathcal{M}_{1,L_0}$ injective. The cotangents on each $\mathcal{U}_L$ are equal, since Proposition 3.18 identifies them modulo rational-tail divisors with the elliptic Hodge class λ , and nonzero by Lemma 7.9. It follows that self-intersection on this path is injective on the odd coefficient in the Künneth component of degree eleven there and degree two on each unit vertex. The coefficient is $+1$: from each bridge normal factor choose a cotangent on a different unit vertex. If the odd vertex is an endpoint, choose the middle cotangent at the first bridge and the other endpoint cotangent at the second.

All other choices have different multidegrees. Odd decorations on a different vertex are excluded by parity.

In both cases, by Theorem 3.23, elliptic H^{13} is generated by rational-tail Gysin classes, so its tail sources are supported on trees with a rational vertex and do not meet these positive-path opens.

The two kernel families. Every selected source edge is a bridge, which remains separating through compact degenerations. A node normalized by ν_{ab} must therefore lie inside a source factor. Each such branch lowers that factor's genus and puts a, b on it, with transverse coefficient one. Lowering an odd elliptic factor gives zero by Theorem 3.2. The compact factors in these pullbacks already contain all deeper degenerations.

For $\mathcal{T}_{1,F}$ let R_1 denote the positive branch obtained by lowering the genus-two leaf. Its leaves carry the old pair and the new pair respectively. Pair exchange interchanges these leaves and the two center flags, so on this source

$$q_F \Phi_F^{\text{aux}} = R_1(1 - \sigma_T).$$

The other branch lowers the genus-one leaf to a rational leaf carrying all four flags, and cancels under $1 - s^*$. Exchanging ordinary bridge Thom classes has positive sign; no determinant is imposed on these bridges. Thus $\mathcal{T}_{1,F}^+$ is contained in the kernel.

For $w \in \mathcal{T}_{2,F}$ the positive branch has all four flags on the lowered leaf and cancels. The remaining rational path has $B \sqcup \{a, b\}$ on its rational leaf and $C \sqcup \{c, d\}$ on its genus-two leaf, minus its pair exchange. The composite Gysin defining Lw gives the exchanged path. Composition preserves complex orientations and the two Tate twists, so

$$q_F \Phi_F^{\text{aux}}(w) = -q_F \Phi_F^{\text{aux}}(Lw).$$

The rational factor in L is stable also for $B = \emptyset$, since it carries c, d and its node. This is an equality of compact classes with all H^{13} relations imposed. Subtracting $(Lw, 0, u_+, w, 0, 0)$ from any source vector therefore leaves its image unchanged and places it in the displayed complement.

Recovery of partner divisors. Use the path of (a). Write p, q for the first bridge flags and u, v for the second, and x, y for the common nonzero cotangents on the middle and unit endpoint partial elliptic factors. After gluing a, b , the partner divisor is restricted along the parallel gluing of those two elliptic factors. By Theorem 3.19 at the two persistent nonseparating nodes, its full irreducible-boundary restriction is

$$\delta_{\text{irr}}^{(1)} + \delta_{\text{irr}}^{(2)} - \psi_a - \psi_b - \psi_u - \psi_v = 12x + 12y - 2x - 2y = 10x + 10y.$$

Both persistent nonseparating nodes contribute their normal lines, and an additional node on either factor contributes the corresponding transverse Cartier branch. These are all branches on the chosen partial product.

Write its coefficient as in (a). The first bridge normal multiplies this by $-\psi_p - \psi_q$. Its $(11, 2, 2)$ component is the class displayed in (a). The ψ_p term raises the odd degree, and the x^2 term has multidegree $(11, 4, 0)$. Only this unexchanged partner-divisor allocation contributes. An elliptic tail can isolate either endpoint of the path, and the other endpoint places the odd class in a different Künneth summand. An exchanged odd endpoint has a new flag and no old flag, so cannot carry A . Endpoint-path branches have a, b together on the lowered unit vertex, while this test splits them. Centered paths have their odd degree at the middle vertex; rational trees miss the open. Hence $C = \emptyset$ recovers $10e_A$, and $C = \{i\}$ then recovers $d_{A,i}$. Injectivity of smooth restriction and the nonzero factors x, y force all coefficients in the $\mathcal{V}_F^+$ summand to vanish.

Recovery of the positive paths. Lower the unique genus-two unit vertex of an endpoint path. Its old-present endpoint and the vertex carrying a, b recover the original path and its marking allocation uniquely. An exchanged endpoint is old-absent, centered paths place the odd class on

the middle vertex, and classes supported on rational strata restrict to zero on the test open. The self-intersection just proved recovers every $\mathcal{E}_F^+$ coefficient with scalar 1.

After these coefficients have been shown to vanish, lower the genus-two leaf of a $\mathcal{T}_{0,F}$ source. Raising the vertex carrying a, b recovers the source uniquely. If an old flag lies on the decorated center, no exchanged branch has that center. Otherwise the old pair is split between the leaves, whereas an exchanged branch has the old pair together. The same test has scalar 1 and forces the coefficients in $\mathcal{T}_{0,F}$ to vanish. The $\mathcal{T}_{1,F}^-$ targets have the old pair on one leaf and the new pair on the other, with none on the center. These allocations exclude the preceding targets and their exchanges. The formula $R_1(1 - \sigma_T)$ gives scalar 2 and recovers the entire negative eigenspace.

Recovery of elliptic degree-thirteen tails. Only $\mathcal{D}_F^+$ remains in a relation. Choose a nonzero coefficient z_A with $|A|$ minimal, and let $W_A \subseteq \overline{\mathcal{M}}_{3,J}$ be the open of Lemma 7.12 for the partner genus $h = 2$ and the decomposition $J = A \sqcup (J \setminus A)$, with tail normalization $\mathcal{M}_{1,A \sqcup \{p\}}^{\text{ct}} \times \mathcal{M}_{2,(J \setminus A) \sqcup \{q\}}$. By that lemma, any genus-one tail meeting W_A has its external labels contained in A . Strict subsets have zero coefficients by minimality; an equal-sized set must be A . Exchanged terms contain a new flag outside A . The only remaining $(13, 2)$ component is therefore

$$-z_A | \mathcal{M}_{1,A \sqcup \{p\}}^{\text{ct}} \otimes \psi_q | \mathcal{M}_{2,(J \setminus A) \sqcup \{q\}},$$

which depends injectively on z_A by the same lemma, a contradiction.

Nonempty decorated marking sets distinguish odd endpoints, and the old and new pairs distinguish the relevant unit leaves. Thus the recovery scalars are exactly 1, 1, 2; node-flip stabilizers remain inside the partial elliptic stacks. $\square$

Lemma 10.16 (Multiplicities of the genus-four sources). *Let (F, μ) be one of $([11], (1^{11}))$, $([11], (2, 1^9))$, $([12], (2, 2, 2, 1^6))$, $([11], (3, 1^8))$. Use the sources of Definitions 10.14, 10.8, and 10.5; the κ_1 , ψ_q , and rational-tail columns give the three summands of the Picard source, and the parallel column is $M_\mu(\mathcal{R}_{+,F}^{\text{par}})$. The multiplicities of $\mathbf{P}$ in these sources, each computed after taking invariants under exchange of the old flags and applying the functor M_μ , are as follows; entries marked $-$ are not used.*

μ	$\mathcal{D}_F^+$	$\mathcal{T}_{0,F}$	$\mathcal{T}_{1,F}^+$	$\mathcal{T}_{1,F}^-$	$\mathcal{T}_{2,F}$	$\mathcal{E}_F^+$	$\mathcal{V}_F^+$	κ_1	ψ_q	tails	par
(1^{11})	25	36	—	2	—	36	15	6	6	3	6
$(2, 1^9)$	45	102	—	6	—	72	28	12	12	8	12
$(2, 2, 2, 1^6)$	5	16	1	0	1	6	2	0	0	1	0
$(3, 1^8)$	25	96	2	4	6	36	13	6	6	5	6

For $\mu = (2, 2, 2, 1^6)$ the rational-tail summand is supported precisely on $|A \cap F| = 10$, $|A \cap \{c, d\}| = 1$. For such an allocation, let e be the unique old flag outside A and write $\{x, y\} = F \setminus A$. Then its partner coefficient is the line $\mathbb{Q}(\delta_{0,\{e,x\}} - \delta_{0,\{e,y\}})$. For $\mu = (1^{11}), (2, 1^9)$ the sources $S_{0,\mu}, S_{e,\mu}, R_{0,\mu}$ of Definition 11.5 have multiplicities 114, 9, 6 and 253, 9, 6.

The multiplicity of $\mathbf{P}$ in $N_\mu = M_\mu(N_{5,F})$ is 179, 349, 32, 205, respectively. For $\rho = (2, 2, 2, 1^6)$ and $(3, 1^8)$, the cohomological maps of the sources induce an isomorphism

$$M_\rho(\mathcal{A}_F^{\text{aux}}) \oplus M_\rho(\mathcal{R}_F^{\text{Pic}}) \oplus M_\rho(\mathcal{R}_{+,F}^{\text{par}}) \xrightarrow{\sim} N_\rho,$$

whose first summand maps onto $M_\rho(\mathcal{U}_F)$.

Proof. The parent tables. By Theorems 8.4 and 7.15, the genus-four compact multiplicity series is

$$E\partial U + 3HE + 4HE^2 + GE^2.$$

The multiplicity in N_μ is the coefficient of $s_\mu h_2$ in this series: the two loop flags carry ordinary invariants. The horizontal-two-strip parents of (1^{11}) are $(3, 1^{10})$ and $(2, 1^{11})$; those of $(2, 1^9)$ are $(4, 1^9)$, $(3, 2, 1^8)$, $(3, 1^{10})$, $(2, 2, 1^9)$. For $\rho = (2, 2, 2, 1^6)$ the parents are $(4, 2, 2, 1^6)$, $(3, 2, 2, 2, 1^5)$,

$(3, 2, 2, 1^7)$, $(2, 2, 2, 2, 1^6)$, and for $\rho = (3, 1^8)$ they are $(5, 1^8)$, $(4, 2, 1^7)$, $(4, 1^9)$, $(3, 3, 1^7)$, $(3, 2, 1^8)$. The required coefficients are

	$(4, 1^9)$	$(3, 2, 1^8)$	$(3, 1^{10})$	$(2, 2, 1^9)$	$(2, 1^{11})$
$E\partial U$	12	8	17	9	9
$3HE$	12	6	18	6	6
$4HE^2$	40	32	72	36	36
GE^2	30	24	18	9	3
sum	94	70	125	60	54

The same series is $E\partial U + H(2E + 3E^2) + (H + G)E^2 + HE$, the four summands D, W, T, R of Theorem 7.15 at genus four; grouped in this way, the coefficients at the parents of the two nine-row partitions are

θ	D	W	T	R	θ	D	W	T	R
$(4, 2, 2, 1^6)$	1	0	6	0	$(5, 1^8)$	3	0	15	0
$(3, 2, 2, 2, 1^5)$	0	0	0	0	$(4, 2, 1^7)$	2	0	15	0
$(3, 2, 2, 1^7)$	4	9	12	0	$(4, 1^9)$	12	38	40	4
$(2, 2, 2, 2, 1^6)$	0	0	0	0	$(3, 3, 1^7)$	0	0	6	0
sum	5	9	18	0	$(3, 2, 1^8)$	8	28	32	2
					sum	25	66	108	6

The table can be evaluated by the following finite rule. Removing corners gives

$$E\partial U = E^2(s_{(2,1^{10})} + e_{11} + s_{(2,1^9)}) + EE_{\geq 2}H + EE_{\geq 3}(H + G).$$

For a contained core $\rho \subseteq \theta$, the coefficient from E^2s_ρ is $s_{\theta/\rho}(1, 1)$, and that from $EE_{\geq j}s_\rho$ is

$$\sum_{v=0}^{|\theta|-|\rho|-j} \langle s_{\theta/\rho}, h_v h_{|\theta|-|\rho|-v} \rangle.$$

An empty sum is zero; a single E contributes precisely when the skew diagram is horizontal. These rules give all the displayed entries by successive horizontal strips. The lists of partitions are exhaustive by $\theta_i \geq \rho_i \geq \theta_{i+1}$ and $|\theta| - |\rho| = 2$. In the first table, the second and fourth parents have four boxes in column two, while every possible core with at most ten rows acquires at most two additional boxes in that column from its two complete-function factors. In the second table, the parent $(3, 3, 1^7)$ contributes $1 + 2 + 3 = 6$ from the three nine-row cores in GE^2 ; it is included in the count. Thus the multiplicities in N_μ are $125 + 54 = 179$, $94 + 70 + 125 + 60 = 349$, $5 + 9 + 18 + 0 = 32$, and $25 + 66 + 108 + 6 = 205$.

The restriction rule. For a set with u external labels, r old flags and v node flags, let Std_u be the augmentation kernel of the permutation module on the external labels. The elliptic hook restricts as

$$\bigwedge^{10} (\text{Std}_u \oplus \mathbb{Q}^{r+v}).$$

This is the exterior-power form of the tuple presentation in Definition 3.10. When $r = 2$, the old transposition has one negative line among the auxiliary directions, and invariants omit that line. When $r = 1$, its two allocations form one orbit and the invariant coefficient is one representative. For induction over the remaining labels, the multiplicity is the pairing of the corresponding skew Specht character with the unit or divisor character on those labels.

The sign and ten-row partitions. For the old-present centered hook the auxiliary positive space has dimension three in either old-pair case. After distributing all remaining external labels over

the two unit leaves, its counts for a fixed old-pair pattern are

u	8	9	10	11
(1^{11})	0	3	6	1
$(2, 1^9)$	2	13	12	3.

There are two choices for the other old flag when just one is on the center, and one choice when both are there. With neither old flag on the center and with the pair split between the leaves, the counts for $u = 9, 10, 11$ are $(1, 4, 1)$ and $(4, 6, 2)$. Hence the $\mathcal{T}_{0,F}$ counts are $3(3 + 6 + 1) + (1 + 4 + 1) = 36$ and $3(2 + 13 + 12 + 3) + (4 + 6 + 2) = 102$.

For clarity, the entry 13 comes from three sign-nine endpoint coefficients, each with leaf counts $1 + 1 + 2$, and one further vertical-two-box skew character contributing on the split leaves. The entry 2 at $u = 8$ comes from the two leaf sizes $(1, 2)$ and $(2, 1)$. On $\mathcal{T}_{1,F}$, the leaf exchange also exchanges the node flags. Its negative-eigenspace multiplicities at $u = 9, 10, 11$ are $(0, 2, 0)$ and $(2, 3, 1)$, respectively. At $u = 9$ for the sign partition the sign of exchanging the two singleton leaf labels cancels the sign of exchanging the two exterior node directions; at $u = 10$ the two leaf locations pair off. For the ten-row partition the split two-label skew character has both signs. This gives 2 and 6 for $\mathcal{T}_{1,F}^-$.

For an old-present endpoint the auxiliary positive space has dimension two. For each genus order and each fixed old-pair pattern the $u = 9, 10, 11$ counts are $(1, 4, 1)$ and $(4, 6, 2)$. There are three such patterns and two genus orders, giving 36 and 72 for $\mathcal{E}_F^+$. For the external Picard source, its partner character is $\mathbf{1} \oplus \mathbb{Q}[B]$. The contributions with one and two old flags on the endpoint are

u	one old flag			two old flags		
	9	10	11	9	10	11
(1^{11})	1	6	2	1	4	1
$(2, 1^9)$	4	9	4	3	6	2.

For example, at $u = 9$ with one old flag the partner consists of one fixed old flag and two external labels. Its divisor character is three trivial lines and one sign line for their transposition. The sign partition uses the latter; the ten-row skew character uses both. The totals are 15 and 28.

The full elliptic H^{13} tail source has character $E\partial U$. Its two-old-flag invariant coefficients were computed in the parent tables above: they are $17 + 9 = 26$ and $12 + 8 + 17 + 9 = 46$. If neither old flag is on the elliptic factor, the threshold in Definition 3.22 forces all eleven external labels onto it. The restriction of $V_{(2,1^{10})}$ after fixing the node contains each of our two partitions once. Removing this one old-absent copy leaves 25 and 45. We have therefore obtained

	$\mathcal{D}_F^+$	$\mathcal{T}_{0,F}$	$\mathcal{T}_{1,F}^-$	$\mathcal{E}_F^+$	$\mathcal{V}_F^+$	$S_{0,\mu}$
(1^{11})	25	36	2	36	15	114
$(2, 1^9)$	45	102	6	72	28	253.

With no old flag on an endpoint, the hook is $\bigwedge^{10} \text{Std}_u \oplus \bigwedge^9 \text{Std}_u$. For either partition, only $u = 11$ and $u = 10$ contribute: once at $u = 11$, and once for each of the two unit placements of the remaining label at $u = 10$. Each of the three selected patterns in $\mathcal{S}_e$ therefore contributes three, giving $S_{e,\mu} = \mathbb{P}^9$. The same restrictions give one copy for each of the two $m = 0$ columns and each of the four $m = 1$ columns, hence $R_{0,\mu} = \mathbb{P}^6$.

It remains to count $R_{p,\mu}$. The rooted divisor generators are independent by Corollary 7.8. For either one or two old flags on the elliptic endpoint, its invariant hook is

$$\bigwedge^{10} \text{Std}_u \oplus 2 \bigwedge^9 \text{Std}_u \oplus \bigwedge^8 \text{Std}_u.$$

For a constant partner coefficient the counts at $u = 9, 10, 11$ are $(0, 2, 1)$ for the sign partition and $(1, 3, 2)$ for the ten-row partition. Each of κ_1 and ψ_q therefore contributes 6 and 12, respectively,

after the two old-pair cases. For rational-tail coefficients with one old flag on the endpoint, $u = 10$ gives the single divisor on the partner $\{d, i\}$, with multiplicities 2 and 3. At $u = 9$ the partner is $\{d, i, j\}$ and its four divisors are

$$D_{di}, D_{dj}, D_{ij}, D_{dij}.$$

Under (ij) they give three trivial lines and one sign line. The sign partition contributes one and the ten-row partition four. With both old flags on the endpoint, the partner $\{i, j\}$ has one invariant tail divisor, contributing zero and one. Thus the rational-tail counts are 3 and 8, and

$$R_{p,(1^{11})} = \mathbf{P}^{6+6+3}, \quad R_{p,(2,1^9)} = \mathbf{P}^{12+12+8}.$$

All contributing endpoints have $u \geq 9$, so their partners have at most three flags. The stated restriction on the Picard source loses no term. Finally put $R_{\text{par},\mu} = M_\mu(\mathcal{R}_{+,F}^{\text{par}})$. By Proposition 10.6, the maps ρ_A identify its coefficients with the old-present single-node elliptic hook, with a unit partner. The constant-partner count just given therefore gives

$$R_{\text{par},(1^{11})} = \mathbf{P}^6, \quad R_{\text{par},(2,1^9)} = \mathbf{P}^{12}.$$

The nine-row partitions. Let now $\rho = (2, 2, 2, 1^6)$ or $(3, 1^8)$. For the character counts put

$$\mathcal{H}_j = \sum_{r \geq 1} s_{(r, 1^{j-1})} \quad (8 \leq j \leq 11).$$

Thus $H = \mathcal{H}_{11} + \mathcal{H}_{10}$ and $G = \mathcal{H}_{10} + \mathcal{H}_9$.

Allocations without an old flag. An elliptic endpoint with no old flag has only its attaching flag fixed. On its ordinary labels its one-node hook has at least ten rows. Every Littlewood–Richardson constituent contains that hook, so neither of the nine-row partitions under consideration occurs, whatever the partner coefficient. This excludes all such endpoint-path and Picard allocations. The parallel hook has the same property, since ρ_A is an isomorphism (Proposition 10.6).

For the degree-thirteen endpoint, the only possible full flag counts without an old flag are twelve and thirteen. By Definition 3.22 and Theorem 3.23, the corresponding characters are

$$\text{ch}(U_{1,[12]}) = s_{(2,1^{10})}, \quad \text{ch}(U_{1,[13]}) = s_{(2,1^{10})}h_1 + e_{10}h_3.$$

Fixing the attaching flag applies ∂ . All resulting terms have at least ten rows except e_9h_3 in the second expression. The latter cannot contain $(2, 2, 2, 1^6)$, since the skew diagram above (1^9) is a vertical three-strip. At $|F| = 11$ only the twelve-flag case is possible. Thus these degree-thirteen allocations also vanish. Consequently the full auxiliary multiplicity is accounted for by $\mathcal{A}_F^{\text{aux}}$, and the full parallel multiplicity comes from allocations with at least one old flag on the decorated vertex.

Centered paths. Two ordered unit allocations are counted by E^2 . If a hook core is $\eta \subseteq \rho$, its contribution is $s_{\rho/\eta}(1, 1)$. The complete core calculations are

ρ	$[s_\rho]\mathcal{H}_9E^2$	$[s_\rho]\mathcal{H}_8E^2$
$(2, 2, 2, 1^6)$	$0 + 1 = 1$	$0 + 2 = 2$
$(3, 1^8)$	$3 + 2 + 1 = 6$	$6 + 4 + 2 = 12$

For the first row the skew Schur functions are e_3, e_2 over the nine-row cores, and e_3h_1, e_2h_1 over the eight-row cores. For the second row they are $h_2, h_1, 1$ and h_2h_1, h_1^2, h_1 .

If both old flags lie on the center, its augmentation representation on the ordinary center labels is the standard representation plus three trivial lines and one t -sign line. Invariants exclude the sign line. The resulting hook character is $\mathcal{H}_{11} + 3\mathcal{H}_{10} + 3\mathcal{H}_9 + \mathcal{H}_8$. The same character occurs when exactly one old flag is on the center and the other is on a specified leaf. There are two such choices of leaf; each t -orbit is counted once. With the old flags on different leaves the character is $\mathcal{H}_{11} + 2\mathcal{H}_{10} + \mathcal{H}_9$. Thus the $\mathcal{T}_{0,F}$ entries are $3(3 \cdot 1 + 2) + 1 = 16$ and $3(3 \cdot 6 + 12) + 6 = 96$. Both $\mathcal{T}_{1,F}$ and $\mathcal{T}_{2,F}$ have multiplicities one and six, respectively in the two cases.

The node exchange acts by -1 on the nine-row center hook. For $(2, 2, 2, 1^6)$ the sole $\mathcal{T}_{1,F}$ contribution has ten ordinary center labels; its complementary pair has the sign character and its split-leaf allocation changes sign under leaf exchange. Their product gives $\sigma_T = +1$. For $(3, 1^8)$, center sizes nine, ten, and eleven give respectively $(1, 2)$, $(1, 1)$, and $(0, 1)$ copies in the positive and negative eigenspaces of σ_T . For size nine, the allocations with both remaining labels on a single leaf form an exchanged pair, whereas the trivial constituent of the split allocation has leaf-exchange sign $+1$. For size ten the two possible locations of the remaining label are exchanged. For size eleven only the center sign remains. This proves the asserted $\mathcal{T}_{1,F}^\pm$ counts.

Endpoint and Picard generators. Write $a_0 = |A \cap F|$. If exactly one old flag is on the endpoint, its augmentation representation is $\text{Std}_{A \cap F} \oplus \mathbb{Q}^2$. If both old flags are there, it is $\text{Std}_{A \cap F} \oplus \mathbb{Q}^2 \oplus \mathbb{Q}_{\text{sign}}$; taking t -invariants again excludes the last line. In either case the only nine-row term is $s_{(a_0-8, 1^8)}$, once.

For $(2, 2, 2, 1^6)$ only $a_0 = 9, 10$ can contribute. At size nine the three remaining ordinary labels require their sign representation. Neither allocations to two unit vertices nor the permutation representations on markings or subsets of three markings contain that sign. At size ten the complementary pair requires sign. For endpoint paths it must split between the unit vertices. There are four choices when one old flag is on the endpoint, and two when both are there, giving six. For the external cotangents the sign line is $\psi_x - \psi_y$, once for each of these two old-flag patterns, giving two. The coefficients κ_1 and ψ_q are trivial and contribute zero. For rational tails, $B = \{e, x, y\}$ has precisely the sign line $\delta_{0, \{e, x\}} - \delta_{0, \{e, y\}}$. When $B = \{x, y\}$ its only stable subset is fixed and contributes zero. This proves the one-copy tail assertion; every contributing complement has size three and hence occurs in $\mathcal{R}_F^{\text{Pic}}$. The invariant parallel source has a trivial partner coefficient, so its multiplicity is zero.

For $(3, 1^8)$ the sizes are $a_0 = 9, 10, 11$. The complementary skew is horizontal and requires the trivial representation on the remaining ordinary labels. Endpoint paths give $4 \cdot 6 + 2 \cdot 6 = 36$ copies, using the centered-path core count above. In the non-node Picard family, one old flag on the endpoint gives 3, 3, 2 copies at these sizes: δ_{irr} , the cotangent at the other old flag, and, when present, the sum of the ordinary cotangents. Both old flags on the endpoint give 2, 2, 1 copies. The total is thirteen. Each of κ_1 and ψ_q gives one copy for every size and old-flag pattern, hence six copies. For rational tails the numbers of orbits of stable subsets of B are

old flags on the endpoint	$a_0 = 9$	$a_0 = 10$	$a_0 = 11$
one	3	1	0
both	1	0	0.

For example, when $B = \{e, x, y\}$ the three orbits are $\{x, y\}$, the pair $\{e, x\}, \{e, y\}$, and B . This gives five tail copies. All these complements have size at most three. The parallel collapse gives the same count as a single trivial Picard coefficient, namely six.

Independence for the nine-row partitions. The endpoint, non-node Picard, and remaining Picard generating spaces have total multiplicities $6 + 2 + 1 = 9$ and $36 + 13 + 17 = 66$. They surject onto the corresponding W multiplicities, which have exactly these dimensions by the tables. The generating maps are therefore isomorphisms on these multiplicity spaces. This proves the required Picard splitting without imposing any additional divisor relation. The degree-thirteen entries are five and twenty-five, and the other two ordinary summands have already been counted. Together they give $N_\rho \cong \mathbf{P}^{32}$ and $\mathbf{P}^{205}$ and the claimed source decomposition. $\square$

11. THE SIGN AND TEN-ROW COEFFICIENTS

We prove that $\mathcal{C}_{(11)}$ and $\mathcal{C}_{(10,1)}$ vanish by proving injectivity of the induced compact boundary map on the residual quotient. We begin with the compact nonseparating injectivity theorem, stated for every genus at least five, since Section 15 uses it in the genus induction.

Theorem 11.1 (Compact nonseparating injectivity). *For every $g \geq 5$ and finite F , with the maps of Definition 5.2, both maps*

$$\xi^* : H^{15}(\overline{\mathcal{M}}_{g,F}) \longrightarrow N_{g,F}, \quad \beta|_{C_{\text{sep}}^1} : C_{\text{sep}}^1 \longrightarrow C_{g,F}^2$$

are injective. They are equivariant morphisms of pure Hodge structures.

Proof. Lower-degree inputs. By [AC98, Theorem 2.10], the map $\xi^* : H^k(\overline{\mathcal{M}}_{g,F}) \rightarrow H^k(\overline{\mathcal{M}}_{g-1,F \cup \{c,d\}})$ is injective for $k \leq 2g - 2$ if $g \leq 7$ and for $k \leq g + 5$ if $g \geq 7$; this gives injectivity on H^2 and H^4 for every genus at least three. We also need injectivity of the H^{13} pullback in genus at least three. By Theorem 3.23, H^{13} is the direct sum of the images of the elliptic-tail Gysin maps with domains

$$H^{11}(\overline{\mathcal{M}}_{1,A \cup \{p\}}) \otimes H^0(\overline{\mathcal{M}}_{h-1,A^c \cup \{p'\}}) \otimes \mathbb{L},$$

with the sum over stable allocations A of the ordinary markings; for $h \geq 2$ these sources map injectively by the same theorem. Normalize a nonseparating node in such a source. If it lies on the elliptic factor, the resulting factor is a stable genus-zero moduli space, so Theorem 3.2 gives $H^{11}(\overline{\mathcal{M}}_{0,m}) = 0$ for every stable m ; hence this pullback component is zero. If it lies on the unit partner, the resulting source is the corresponding elliptic tail with the two new flags on a genus- $(h - 2)$ partner. The allocations are distinct on the labelled cover, and the unit pullback has coefficient one. Thus nonseparating pullback is injective on H^{13} for every $h \geq 3$. The same argument applies after taking graph invariants: pullback from a finite labelled quotient is injective, and the loop flags are fixed as an unordered pair.

The lower-odd vanishing of Theorem 3.3, together with this description of the degree-eleven elliptic factors, leaves on a separating normalization only the six degree pairs

$$(11.1) \quad (15, 0), (0, 15), (13, 2), (2, 13), (11, 4), (4, 11).$$

There is one odd factor in each pair; in the degree-eleven cases it has genus one. Different ordered Künneth pairs remain distinct under pullback because pullback preserves the degree on each vertex factor.

Loop-bridge detectors. Let Λ be a separating one-edge graph at (g, F) , with vertex genera u, v and ordinary-label partition $F = A \sqcup B$. Choose one vertex of positive genus, lower its genus by one, and add two new flags c, d there, joined to a loop. Keep the original separating bridge and order the two edges as $(e_{\text{loop}}, e_{\text{br}})$. The resulting two-edge graph Γ is stable, and contracting e_{loop} recovers Λ while contracting e_{br} gives the nonseparating one-edge graph at (g, F) . Consequently, for $z \in C_{\text{sep}}^1$, the Γ -coordinate of $\beta(z, 0)$ is

$$(11.2) \quad \xi_{e_{\text{loop}}}^* z_{\Lambda} \otimes (e_{\text{loop}} \wedge e_{\text{br}}),$$

because the second contraction contributes the zero nonseparating coordinate. No other separating source coordinate contributes to this labelled target. The loop and bridge are distinguished, so they cannot be exchanged by a graph automorphism. Reversal of the loop flags fixes the edge determinant and the clutching pullback. Hence (11.2) has coefficient $+1$ also after passing to $\text{Aut}_F(\Gamma)$ -invariants. If Λ has a vertex exchange symmetry, perform the test on the labelled cover and then take invariants; for a vector $w \otimes 1 + 1 \otimes w$, projection to the unchanged degree-fifteen factor returns w .

We now apply this detector to each term of (11.1). For a $(15, 0)$ term whose unit factor has positive genus, put the loop on that unit factor. The unit pullback is the identity, so (11.2) is the coefficient itself. If the unit factor is rational, stability gives it at least two ordinary labels. The degree-fifteen factor then has $n - b + 1 < n$ flags, where $b \geq 2$ is the number of labels on the rational tail (the bridge flag is included), and the same detector uses induction on that smaller marking set.

For a $(13, 2)$ term, put the loop on the H^2 factor when its genus is at least three and use the H^2 injectivity above. When that genus is at most two, the H^{13} factor has genus at least $g - 2 \geq 3$; put the loop there and use the H^{13} injectivity just established. The argument is unchanged for a rational H^2 partner. For a $(11, 4)$ term, the H^{11} factor is elliptic and the other factor has genus $g - 1 \geq 4$; putting the loop on the latter and using the full H^4 pullback detects the coefficient. Exchanging the two named factor positions gives the reverse pairs in (11.1). Thus every separating Künneth coefficient is detected, including the compact H^4 factors.

The marking induction. Fix $g \geq 5$ and induct on $n = |F|$. We first record the base range needed for the compact boundary map. For $g = 5$, the series H, G, U have least possible degrees ten, nine, and twelve, respectively (Notation 2.5). Hence all four ordinary summands of Theorem 7.15 vanish for $n \leq 8$. Proposition 8.5 then gives $H^{15}(\overline{\mathcal{M}}_{5,F}) = 0$ in that range. If $(g, n) = (6, n)$ with $n \leq 5$, or $(g, n) = (7, n)$ with $n \leq 3$, the Chow–Künneth property in Proposition 3.6 gives $W_{15}H^{15}(\mathcal{M}_{g,n}) = 0$. We check directly that the compact boundary group is zero in these arities. A separating boundary divisor with two positive-genus vertices has strictly smaller vertex genera, and each factor has at most $n + 1 \leq 6$ flags; the established genus-one, genus-two, and genus-three low-arity vanishings in Theorem 4.7, Proposition 4.12, and Proposition 6.12, together with the ordinary boundary generation in Theorem 8.4 and Proposition 8.5, show that its degree-fifteen Künneth source is zero. In genera four and five the displayed ordinary sources have the same minimum elliptic-tail arity, so their surjectivity also gives compact vanishing for these small numbers of flags. Indeed, an odd factor of degree at most nine vanishes by Theorem 3.3, while the elliptic degree-eleven and degree-thirteen sources in Theorem 3.23 require at least eleven flags. For a rational tail, stability gives at least two ordinary labels on that tail, so the remaining genus- g factor has fewer markings; induction on the marking number gives the same vanishing. Thus every one-edge source is zero. The localization sequence and $W_{15}H^{15}(\mathcal{M}_{g,n}) = 0$ then give $H^{15}(\overline{\mathcal{M}}_{g,n}) = 0$. In all these initial cases the loop–bridge detectors above prove the separating injection simultaneously, starting at the empty marking set and using the same rational-tail induction. Hence α is injective in the exceptional initial ranges.

For all remaining stable (g, F) needed below, Proposition 5.3 gives injectivity of α whenever $15 < 2g$ for $|F| \leq 1$, or $15 < 2g - 2 + |F|$ for $|F| \geq 2$; for $g = 5$ and $n \geq 9$ this is $15 < 8 + n$.

Assume now that n is beyond the initial range and that nonseparating H^{15} pullback is injective for all smaller marking sets at this genus. Let $z \in C_{\text{sep}}^1$ satisfy $\beta(z, 0) = 0$. The loop–bridge tests above, together with the induction in the rational-unit case, force every coefficient in (11.1) to vanish. Thus $z = 0$, proving injectivity of $\beta|_{C_{\text{sep}}^1}$.

Finally let $x \in H^{15}(\overline{\mathcal{M}}_{g,F})$ satisfy $\xi^*x = 0$. The nonseparating coordinate of $\alpha(x)$ is then zero, so $\alpha(x)$ lies in C_{sep}^1 . Since $\beta\alpha = 0$ by Proposition 5.3, the separating injectivity just proved gives $\alpha(x) = 0$. Since α is injective (above), $x = 0$.

All constructions are defined on labelled covers and commute with relabelling of F . Pullback along clutching maps and the Künneth projections are morphisms of pure Hodge structures. Tensoring with the determinant lines preserves this property, so both injections are equivariant morphisms of pure Hodge structures. $\square$

Proposition 11.2 (Residual interface). *Let (F, ρ) be one of*

$$([11], (1^{11})), \quad ([11], (2, 1^9)), \quad ([12], (2, 2, 2, 1^6)), \quad ([11], (3, 1^8)),$$

with $N_\rho, \mathcal{Q}_\rho, J_\rho, \delta_\rho$ of Set-up 10.1. For $\lambda \in \mathcal{R}_{11}$ set

$$\mathcal{B}_{\rho,\lambda} = \text{Hom}_{\mathfrak{S}_{11}} \left(\text{Res}_{\mathfrak{S}_{11}}^{\mathfrak{S}_{|F|}} V_\rho, V_{\lambda^T} \right).$$

(1) The kernel of $\bar{\delta}_\rho$ is canonically the last column of the following table, and the multiplicities of $\mathbf{P}$ are as in its middle columns:

ρ	J_ρ	N_ρ/J_ρ	$\ker \bar{\delta}_\rho$
(1^{11})	$\mathbf{P}^{29}$	$\mathbf{P}^{150}$	$\mathcal{C}_{(11)}$
$(2, 1^9)$	$\mathbf{P}^{37}$	$\mathbf{P}^{312}$	$\mathcal{C}_{(10,1)}$
$(2, 2, 2, 1^6)$	$\mathbf{P}^2$	$\mathbf{P}^{30}$	$(\mathcal{C}_{(9,2)} \otimes \mathcal{B}_{\rho,(9,2)}) \oplus (\mathcal{C}_{(8,3)} \otimes \mathcal{B}_{\rho,(8,3)})$
$(3, 1^8)$	$\mathbf{P}^8$	$\mathbf{P}^{197}$	$\mathcal{C}_{(9,1,1)}$

Both branching spaces in the third row have dimension one.

(2) For the two nine-row partitions put

$$\mathcal{U}_\rho = M_\rho(\mathcal{U}_F), \quad \mathcal{R}_\rho = M_\rho(\mathcal{R}_F^{\text{Pic}}) \oplus M_\rho(\mathcal{R}_{+,F}^{\text{par}}),$$

let $\Phi_\rho : \mathcal{R}_\rho \rightarrow N_\rho$ be the sum of the Picard and parallel maps, and define the residual map

$$\varepsilon_\rho : \mathcal{R}_\rho \longrightarrow \mathcal{Q}_\rho / \delta_\rho(\mathcal{U}_\rho), \quad r \longmapsto [\delta_\rho \Phi_\rho(r)].$$

Then $J_\rho \subseteq \mathcal{U}_\rho$, $J_\rho = \ker(\delta_\rho|_{\mathcal{U}_\rho})$, and $\delta_\rho(\mathcal{U}_\rho) \simeq \mathbf{P}^{29}$ and $\mathbf{P}^{174}$ for $\rho = (2, 2, 2, 1^6)$ and $(3, 1^8)$, respectively. Projection to the residual source induces a canonical isomorphism $\ker \varepsilon_\rho \cong \ker \bar{\delta}_\rho$.

Proof. The canonical kernels. For $|F| = 11, 12$ we have $16 < 2 \cdot 5 - 2 + |F|$. Proposition 5.1 and the identities in Proposition 5.3 give exactness of

$$0 \longrightarrow H^{15}(\overline{\mathcal{M}}_{5,F}) \xrightarrow{\alpha} C_{5,F}^1 \xrightarrow{\beta_{5,F}} C_{5,F}^2.$$

For $x \in \ker \delta_F$, there is a separating coordinate z with $\beta_{5,F}(z, x) = 0$. Theorem 11.1 makes z unique. Thus projection to the nonseparating coordinate identifies $\ker \beta_{5,F}$ with $\ker \delta_F$, and under row exactness this projection is ξ_F^* . Quotienting by its ordinary image gives

$$\ker \bar{\delta}_\rho \cong M_\rho(H^{15}(\overline{\mathcal{M}}_{5,F}) / \mathcal{I}_{5,F}).$$

Proposition 9.3 identifies the right side. At arity eleven this is $\mathcal{C}_{\rho^T}$, which gives the first, second, and fourth rows of the table. At arity twelve, removing either corner of $(2, 2, 2, 1^6)$ gives

$$(2, 2, 1^7) = (9, 2)^T, \quad (2, 2, 2, 1^5) = (8, 3)^T.$$

Branching is multiplicity free, giving the two indicated lines.

The sign and ten-row multiplicities. Let $\rho = (1^{11})$ or $(2, 1^9)$. By Lemma 10.16, the multiplicities of $\mathbf{P}$ in N_ρ are 179 and 349. By Theorem 7.15 at genus five, the genus-five ordinary series is

$$E\partial U + 3HE + 5HE^2 + GE^2 + Hh_2[E] + Ge_2[E].$$

Its six terms contribute, respectively,

$$\begin{array}{c|cccccc} (1^{11}) & 1 & 6 & 15 & 3 & 2 & 2 \\ (2, 1^9) & 1 & 6 & 15 & 9 & 2 & 4 \end{array}$$

by the finite Pieri rule in the proof of Lemma 10.16. In the last two columns one uses $[h_2[E]]_0 = 1$, $[h_2[E]]_1 = h_1$, $[e_2[E]]_0 = 0$, $[e_2[E]]_1 = h_1$, and $[e_2[E]]_2 = h_2 + e_2$. The ordinary image therefore has multiplicities 29 and 37. Its pullback is injective by Theorem 11.1, so these are the multiplicities in J_ρ . Subtracting gives 150 and 312. Polarizability splits these subobjects of sums of $\mathbf{P}$, giving the stated, noncanonical, pure-Hodge isomorphisms.

The nine-row ordinary pullback. Let now $\rho = (2, 2, 2, 1^6)$ or $(3, 1^8)$. Apply Theorem 7.15 at genus five with ordinary markings F . By the old-absent exclusion in the proof of Lemma 10.16, only centered paths contribute to $M_\rho(\mathcal{I}_{5,F})$. Their leaf genera are $(1, 3)$ and $(2, 2)$.

For the first pair of genera the leaves are distinguished, giving the coefficients of $(\sum_{r \geq 1} s_{(r, 1^8)})E^2$, which are the $\mathcal{T}_{2,F}$ entries of Lemma 10.16: one for $(2, 2, 2, 1^6)$ and six for $(3, 1^8)$. For the equal-genus leaves, their exchange acts by -1 on the nine-row center hook, so the allocation coefficient is its anti-invariant part. In the first case the only contributing skew is e_2 , and the split-leaf allocation has the required sign, giving one copy. In the second case the horizontal skews of sizes two, one, and zero give 1, 1, 0 copies: the first two come from the difference between putting all remaining labels on either leaf; the empty allocation has no anti-invariant part. There are therefore two copies. The injectivity in Theorem 7.15 gives ordinary multiplicities $1 + 1 = 2$ and $6 + 2 = 8$.

Pull back these ordinary paths by ξ_F . Their two selected edges remain bridges under every degeneration, so the nonseparating node being normalized lies inside a vertex factor and cannot coincide with either bridge. Independent node-smoothing parameters give transverse base change with coefficient one on the labelled products. Normalizing the decorated genus-one factor gives zero by Proposition 3.15. For leaves $(1, 3)$, normalizing the genus-three leaf gives a centered path in $\mathcal{T}_{2,F}$. Normalizing the genus-one leaf gives a rational leaf carrying c, d ; its inner Gysin is the class Lw in the degree-thirteen endpoint. Thus this pullback has coordinates

$$(Lw, 0, 0, w, 0, 0)$$

in the coordinate order of Proposition 10.15. For leaves $(2, 2)$, either normalization gives a centered path in $\mathcal{T}_{1,F}$. On the cover ordering the leaves, the two branches are exchanged; their quotient has this same support and retains the ordinary normalization. These exhaust the pullback branches because the vertex factors are compact. Hence $J_\rho \subseteq \mathcal{U}_\rho$. Theorem 11.1 preserves the two and eight ordinary copies under ξ_F^* .

The auxiliary kernel. Lemma 10.2 gives $M_\rho(q_F) = M_\rho(\text{pr}_{\text{dl}})\delta_\rho$. Proposition 10.15, together with the cohomological source isomorphism of Lemma 10.16, therefore gives

$$J_\rho \subseteq \ker(\delta_\rho|_{\mathcal{U}_\rho}) \subseteq \ker(M_\rho(q_F)|_{\mathcal{U}_\rho}).$$

The last kernel is the graph of L on $\mathcal{T}_{2,F}$ together with $\mathcal{T}_{1,F}^+$. Its multiplicities are $1 + 1 = 2$ and $6 + 2 = 8$. The outer spaces in the displayed inclusions have the same dimension, so both inclusions are equalities. Subtracting from the auxiliary multiplicities thirty-one and one hundred eighty-two of Lemma 10.16 gives the stated images. Subtracting J_ρ from the multiplicities 32 and 205 of N_ρ in the same lemma gives N_ρ/J_ρ , by the same polarizability splitting.

The residual kernels. Finally Lemma 10.16 gives

$$N_\rho/J_\rho = (\mathcal{U}_\rho/J_\rho) \oplus \Phi_\rho(\mathcal{R}_\rho),$$

and $\bar{\delta}_\rho$ maps the first summand isomorphically onto $\delta_\rho(\mathcal{U}_\rho)$. Projecting a kernel element onto its $\mathcal{R}_\rho$ coordinate is injective. Conversely, if $\varepsilon_\rho(r) = 0$, there is a unique $u \in \mathcal{U}_\rho/J_\rho$ satisfying $\bar{\delta}_\rho(u) = -\delta_\rho\Phi_\rho(r)$. Then $u + \Phi_\rho(r)$ is a kernel element. $\square$

Definition 11.3. Let $F = [11]$, $A \subseteq F$ with $|A| = 10$ or 11 , and $B = F \setminus A$, $m = |B|$. Every source in this definition carries the full coefficient $H^{11}(\mathcal{M}_{1,A \sqcup \{p\}})(-2)$. For $C \subseteq B$ let $Z_{A,C}$ be the ordinary path of genera $(1, 2, 1)$ with allocations $(A, (B \setminus C) \sqcup \{c, d\}, C)$. Let $V_A(\epsilon, \eta)$ be the elliptic tail with genus-three partner divisor $\epsilon\delta_{\text{irr}} + \eta(\psi_c + \psi_d)$. For every ordered partition $B = B_1 \sqcup B_2$, form the endpoint path correction $P_A(\epsilon, \eta)$ with coefficient η on $(1, 1, 2)$ paths with the old flags split between the unit vertices, $-\eta - 2\epsilon$ on $(1, 2, 1)$ paths with the old flags split, and $2\eta + 2\epsilon$ on $(1, 1, 2)$ paths with both old flags on the middle vertex. Both split placements occur with equal coefficients. All other coefficients are zero. Put $W_A(\epsilon, \eta) = V_A(\epsilon, \eta) - P_A(\epsilon, \eta)$ as a map into $N_{5,F}$.

On $Y = \overline{\mathcal{M}}_{2,B \sqcup \{q,a,b,c,d\}}$, write $D_T = \delta_{0,T}$ for a rational-tail divisor and set

$$\Psi = \psi_a + \psi_b - \psi_c - \psi_d, \quad \Delta = D_{ab} - D_{cd}.$$

If $B = \{i\}$ also set $\Delta_i = D_{abi} - D_{cdi}$ and $U_i = D_{ai} + D_{bi} - D_{ci} - D_{di}$. Let $j_A^0 : \overline{\mathcal{M}}_{1,A \sqcup \{p\}} \times Y \rightarrow \overline{\mathcal{M}}_{3,J}$ glue p, q , and define, for each of these test divisors v ,

$$\mathcal{B}_{A,v}(y) = \left(1 \otimes \left[u \mapsto \int_Y u \lambda_1 \lambda_2 \psi_q^{m+2} v \right] \right) \text{pr}_{11,4}(j_A^0)^* y.$$

Lemma 11.4. *For the maps in Definition 11.3, the $\mathcal{B}_{A,v} q_F$ tests annihilate the image of every elliptic H^{13} tail. On the selected endpoint sources they are diagonal in A . For $m = 0$, their matrix with rows (Ψ, Δ) and columns $(Z_{A,\emptyset}, W_A(0, 1))$ is*

$$I_0 \begin{pmatrix} 20 & -72 \\ 12 & -24 \end{pmatrix}.$$

For $B = \{i\}$, their matrix with rows $(\Psi, \Delta, \Delta_i, U_i)$ and columns $(Z_{A,\emptyset}, Z_{A,\{i\}}, W_A(1, 0), W_A(0, 1))$ is

$$I_0 \begin{pmatrix} 24 & -4 & -72 & -96 \\ 14 & -2 & -24 & -28 \\ -2 & 2 & 0 & 4 \\ 4 & -4 & 0 & -24 \end{pmatrix}.$$

Their determinants are $384I_0^2$ and $12288I_0^4$, respectively.

Proof. The branch coefficients. A nonseparating cut of an ordinary tree lowers one vertex and puts a, b together there. Putting them on different vertices would create an alternate path across one of its separating edges. The cut is transverse to the ordinary edges, with coefficient one. Lowering the decorated elliptic vertex gives zero odd cohomology in genus zero. For $Z_{A,C}$, lowering the middle vertex gives a positive path fixed by pair exchange. Lowering the last vertex gives a rational endpoint. Consequently, for every elliptic coefficient z ,

$$(11.3) \quad q_F Z_{A,C}(z) = j_{A*}^0(z \otimes (D_{abC} - D_{cdC})).$$

There is one minus sign from the ordered compact loops.

Let Σ_{xy} sum the stable genus-two separating divisors splitting x, y , with complementary descriptions counted once. The full labelled nonseparating divisor pullback is

$$\nu_{ab}^* \delta_{\text{irr}} = \delta_{\text{irr}}^Y - \psi_a - \psi_b + \Sigma_{ab}.$$

The negative terms are the normal class of the selected node; the sum consists of the branches which become nonseparating after gluing a, b . Thus the coefficients of $q_F V_A(0, 1)$ and $q_F V_A(1, 0)$ inside j_{A*}^0 are, respectively,

$$-\Psi, \quad -\Psi + \Sigma_{ab} - \Sigma_{cd}.$$

Positive-path coefficients: on a positive endpoint path let $n_{\text{old}}, n_{\text{new}}$ count the old and new pair flags on the last unit vertex, and put $j_{\text{old}} = \mathbf{1}_{n_{\text{old}} > 0}$, $j_{\text{new}} = \mathbf{1}_{n_{\text{new}} > 0}$. The coefficient of $z \otimes \psi_1 \otimes \psi_2$ for $V_A(\epsilon, \eta)$ is

$$-2\epsilon(j_{\text{old}} - j_{\text{new}}) - \eta(n_{\text{old}} - n_{\text{new}}).$$

Indeed the irreducible divisor contributes $12 - n_T - \mathbf{1}_{n_T=1} = 12 - 2\mathbf{1}_{n_T > 0}$ for a pair T ; when it is split, both persistent Cartier branches contribute a normal cotangent. The complete table is

new pair	old pair	coefficient
middle	middle	0
middle	split	$-\eta - 2\epsilon$
middle	last	$-2\eta - 2\epsilon$
last	middle	$2\eta + 2\epsilon$
last	split	η
last	last	0

(11.4)

Both-split coefficients are zero. The three patterns defining $P_A(\epsilon, \eta)$, together with their pair exchanges, have exactly this table. Hence $q_F W_A(\epsilon, \eta)$ has zero full positive endpoint projection, for each allocation of B .

Compact diagonalization. In a common refinement with the test bridge, an ordinary source edge inside the tested elliptic factor raises its least odd degree from eleven to at least thirteen. Projection to $(11, 4)$ annihilates that term. An odd component outside that factor also cannot contribute. In the remaining cases the tested set A is contained in the source's decorated set A' . If $A' \setminus A \neq \emptyset$, the refinement has a rational factor carrying q , these extra flags, and one other attachment. Its dimension is $|A' \setminus A| - 1 \leq m + 1$, including the exchanged sources. The factor ψ_q^{m+2} restricts to zero on this compact factor. Further refinements pull back this same zero cotangent power.

When $A' = A$, distinct nested bridges would enclose an unmarked rational tree with two exterior attachments, which stability excludes. The test bridge therefore coincides with the source bridge. Its normal class is $-\psi_p - \psi_q$. An H^{13} tail still has elliptic degree thirteen and has zero projection. For an H^{11} coefficient the matching contribution is

$$\text{pr}_{11,4}(j_A^0)^* j_{A*}^0(z \otimes u) = -z \otimes \psi_q u.$$

These arguments apply to the tails and every path in the correction, and prove diagonalization and the asserted H^{13} annihilation. The use of the self-intersection formula with an odd coefficient follows directly from the Thom class of the node smoothing line: its Euler class is $-\psi_p - \psi_q$, and the Thom identity is linear over the cohomology of the normalization. Thus it has the same normalization as the excess bundle in Theorem 3.20.

The compact moments. For divisors u, v on Y set

$$b_m(u, v) = \int_Y \lambda_1 \lambda_2 \psi_q^{m+3} uv, \quad R = D_{acd} + D_{bcd} - D_{abc} - D_{abd}.$$

Positive separating divisors and the irreducible divisor have zero pairing by Lemma 10.9. Rational tails containing q are annihilated by the displayed power of ψ_q . For $m = 0$ the remaining part of $\Sigma_{ab} - \Sigma_{cd}$ is R , and

$$I_0^{-1}(b_0(\Psi, \Psi), b_0(\Psi, \Delta)) = (-68, -20), \quad I_0^{-1}(b_0(R, \Psi), b_0(R, \Delta)) = (-4, -4).$$

For $m = 1$ put $R_i = D_{acdi} + D_{bcdi} - D_{abci} - D_{abdi}$. The remaining divisor is $U_i + R + R_i$. The rational factor of each R_i term has dimension two, and the test supplies just one divisor there, so its pairings vanish. The needed table is

$I_0^{-1}b_1$	Ψ	Δ	Δ_i	U_i
Ψ	-92	-24	4	-24
Δ	-24	-14	2	-4
Δ_i	4	2	-2	4
U_i	-24	-4	4	-24
R	-4	-4	0	0

By the moments of Lemma 10.9 and the stable forgetting identities of Proposition 3.16, for $m = 0$ the repeated and distinct cotangent moments in $b_0(\Psi, \Psi)$ are $13I_0$ and $30I_0$, giving $4 \cdot 13 - 4 \cdot 30 = -68$. For $m = 1$ they are $19I_0$ and $42I_0$, giving -92 . A rigid tail contributes its genus-two node cotangent with a minus sign on self-intersection; distinct tails meet only when nested or disjoint. In particular

$$\begin{aligned} b_m(D_{ab} - D_{cd}, \Psi) &= -4(m + 5)I_0, \\ b_m(D_{ab} - D_{cd}, \Delta) &= -2(m + 6)I_0, \\ b_1(\Delta_i, \Psi) &= 4I_0, \quad b_1(\Delta_i, \Delta) = 2I_0. \end{aligned}$$

The last two use the rational factor $\overline{\mathcal{M}}_{0,4}$, on which each cotangent and each boundary point has integral one. The four rigid tails in U_i each give $-6I_0$ against Ψ and on self-intersection, hence both entries $-24I_0$. Their four disjoint intersections with Δ give $-4I_0$; their four nested intersections with Δ_i give $4I_0$. The two self-intersections in Δ_i^2 contribute $-I_0$ each. Each R term has a one-dimensional rational factor: restriction of Ψ and Δ gives the entries $-4I_0$, whereas the other two tests either have incompatible boundary supports or have degree below the dimension of that factor.

The rational-middle branches of $P_A(\epsilon, \eta)$ vanish under these tests: the power of ψ_q exceeds the rational dimension, except when all flags are there, in which case the additional test divisor also lies there and exceeds it. The remaining split-old branches have coefficient $-R$ for $m = 0$ and $-(R + R_i)$ for $m = 1$. After the minus sign from the outer normal class, their test vector is the R row multiplied by $-\eta - 2\epsilon$. Thus the selected $W_A(0, 1)$ column is $(-72, -24)I_0$ for $m = 0$. For $m = 1$, the two W columns are

$$(-72, -24, 0, 0)I_0, \quad (-96, -28, 4, -24)I_0.$$

Equation (11.3) gives the negative Δ and Δ_i rows for the Z columns. The determinants are $20(-24) + 72 \cdot 12 = 384$ and 12288 , in the stated powers of I_0 ; for the latter, adding twice the third row to the fourth leaves $(0, 0, 0, -16)$ and the remaining minor is -768 . $\square$

Definition 11.5. Let $F = [11]$, and retain the ordinary sources and eigenspaces of Definition 10.14. Let $\mathcal{S}_e$ be the endpoint-path source with no old flag on the decorated endpoint, and with either the old flags on different unit vertices, for both genus orders, or both old flags on the middle vertex of a $(1, 1, 2)$ path. Include every marking allocation. For $\mu = (1^{11})$ or $(2, 1^9)$, put

$$S_{0,\mu} = M_\mu\left((\mathcal{D}_F^+ \oplus \mathcal{T}_{0,F} \oplus \mathcal{T}_{1,F}^- \oplus \mathcal{E}_F^+ \oplus \mathcal{V}_F^+)^{(t)}\right), \quad S_{e,\mu} = M_\mu(\mathcal{S}_e^{(t)}).$$

Let $R_{0,\mu}$ be the multiplicity source of the two selected $m = 0$ columns and four selected $m = 1$ columns of Lemma 11.4, over all A . Put $R_{p,\mu} = M_\mu(\mathcal{R}_F^{\text{Pic}})$. The sum of the Gysin maps on these four domains is denoted

$$\Phi_{\text{sep},\mu} : S_{0,\mu} \oplus S_{e,\mu} \oplus R_{0,\mu} \oplus R_{p,\mu} \longrightarrow N_\mu.$$

Proposition 11.6. For $\mu = (1^{11})$ and $(2, 1^9)$ the map

$$M_\mu(\delta_F)\Phi_{\text{sep},\mu} : S_{0,\mu} \oplus S_{e,\mu} \oplus R_{0,\mu} \oplus R_{p,\mu} \longrightarrow M_\mu(\mathcal{Q}_{5,F})$$

is injective, and its source maps into $M_\mu(B_{\text{sep},F})$ before applying δ_F . The multiplicities of $\mathbf{P}$ in its four source summands are

μ	$S_{0,\mu}$	$S_{e,\mu}$	$R_{0,\mu}$	$R_{p,\mu}$
(1^{11})	114	9	6	15
$(2, 1^9)$	253	9	6	32

Proof. Positive support tests. Let Γ be one of the resulting genus-three paths with three elliptic vertices, one decorated by H^{11} and two carrying units, with L_0 the full flag set on the decorated vertex and L_1, L_2 those on the unit vertices. Refine each unit vertex to a rational vertex with a self-loop, leaving the decorated vertex unchanged; call the resulting graph $\tilde{\Gamma}$. Remove from $\overline{\mathcal{M}}_{3,J}$ the proper clutching images of all graphs which are not contractions of $\tilde{\Gamma}$. There are finitely many such graphs. The remaining open V_Γ has precisely the strata whose graphs are contractions of $\tilde{\Gamma}$. The inverse image of this open in the normalization is

$$U_\Gamma = \mathcal{M}_{1,L_0} \times \mathcal{U}_{L_1} \times \mathcal{U}_{L_2},$$

where $\mathcal{U}_L$ is the open of Lemma 7.9, the smooth or irreducible one-nodal genus-one curves with distinct smooth markings. For a centered decoration this is the situation of Lemma 7.11 with

leaf genera $(1, 1)$; in every application below the two unit leaves carry different external label sets. That lemma makes $i_\Gamma : U_\Gamma \rightarrow V_\Gamma$ a closed regular embedding of codimension two, gives the self-intersection with $(11, 2, 2)$ component $z|_{\mathcal{M}_{1,L_0}} \otimes \psi_{q_1} \otimes \psi_{q_2}$, injective in the odd coefficient z , and shows that V_Γ meets no other two-edge tree stratum and no two-edge tree stratum with a rational vertex.

For an endpoint decoration the same argument applies with the roles of the vertices changed. A two-edge tree meeting V_Γ must contract the two loops and retain the two bridges, so it is Γ . Each bridge is recovered uniquely from its genus and marking partition: repeated nested bridges of that type would enclose an unstable unmarked rational tree with two exterior attachments. The bridge smoothing parameters therefore give a closed regular codimension-two embedding $i_\Gamma : U_\Gamma \rightarrow V_\Gamma$. We use the labelled cover if a finite graph quotient is present. Restriction to V_Γ , pullback by i_Γ , and projection to degree eleven on the decorated vertex and degree two on each unit give a cohomological test. For an endpoint decoration, the product of the two normal classes has exactly one term in this degree, namely

$$(11.5) \quad z|_{\mathcal{M}_{1,L_0}} \otimes \psi_1 \otimes \psi_2.$$

Here ψ_1 is at the middle side of the first bridge and ψ_2 at the last side of the second, and the coefficient is $+1$; for a centered decoration one takes the cotangent on each unit side, as in Lemma 7.11. Lemma 6.11 makes restriction of the odd coefficient to $\mathcal{M}_{1,L_0}$ injective, and Lemma 7.9 makes each cotangent nonzero on $\mathcal{U}_{L_i}$, so these maps are injective on the full elliptic coefficient. All other two-edge trees, including their further degenerations, miss this open. This defines cohomological test maps on the chosen Gysin images.

External Picard coefficients. Fix a decorated endpoint set A meeting $\{c, d\}$ and containing neither new flag, and put $B = I \setminus A$. For each $C \subseteq B$ use the preceding test on the positive path with allocations

$$(A, (B \setminus C) \sqcup \{a\}, C \sqcup \{b\}).$$

By Proposition 10.15(a), for a coefficient $z \otimes (\epsilon \delta_{\text{irr}} + \sum_{i \in B} d_i \psi_i)$ in $\mathcal{V}_F^+$ the $(11, 2, 2)$ component of this test is

$$-z|_{\mathcal{M}_{1,L_0}} \otimes \left(10\epsilon + \sum_{i \in C} d_i \right) x \otimes y,$$

with x, y the nonzero cotangents on the two partial elliptic factors; taking $C = \emptyset$ and then $C = \{i\}$ recovers every coefficient, and the other families of $\mathcal{A}_F^{\text{aux}}$ and the exchanged terms contribute nothing to these tests. The tests also annihilate the remaining sources under consideration: an old-absent path or a tail in $R_{0,\mu}$ cannot isolate the old-present endpoint. Thus the tests recover the entire $\mathcal{V}_F^+$ component in a simultaneous relation.

Positive paths and endpoint remainders. Having removed the external Picard component, take all the tests (11.5) for the selected centered and endpoint paths. Proposition 10.15(b) then recovers the $\mathcal{E}_F^+$, $\mathcal{T}_{0,F}$ and $\mathcal{T}_{1,F}^-$ coefficients in turn. For an old-split allocation with no old flag on the decoration, the original has its new pair together and its old pair split; its exchange has the reverse pattern, so raising the vertex carrying a, b recovers the original source and all its marking allocations. For the additional endpoint sources with both old flags on the middle of a $(1, 1, 2)$ path, the exchanged target would instead arise from a $(1, 2, 1)$ path with both old flags on its last vertex; that source is not selected. Centered and endpoint decorations are distinguished by their degree-eleven vertex; when the undecorated graphs agree, the flag threshold excludes a second nonzero decoration. These cases exhaust $\mathcal{T}_{0,F}$, $\mathcal{T}_{1,F}^-$, $\mathcal{E}_F^+$, and $\mathcal{S}_e$.

The positive tests annihilate all elliptic H^{13} tails by degree. They also annihilate $R_{0,\mu}$. The Z outputs have rational endpoints by (11.3), and the entire positive endpoint projection of each W is zero by the six coefficients in (11.4). A centered positive target cannot receive an old-absent

endpoint tail, since the other possible degree-eleven vertex has too few flags. This checks the added corrections on all positive targets.

Now consider a relation in the $M_\mu(q_F)$ images of $S_{0,\mu} \oplus S_{e,\mu} \oplus R_{0,\mu}$. The external Picard tests first force its $\mathcal{V}_F^+$ coefficient to vanish, and the positive tests force its selected path coefficients to vanish. The remaining relation involves only elliptic H^{13} tails and $R_{0,\mu}$. The tests in Lemma 11.4 annihilate the former and recover every coefficient of the latter, since both displayed determinants are nonzero. Finally, the map in Proposition 10.15 is injective on the selected $\mathcal{D}_F^+$ summand, so its remaining coefficient vanishes. We have proved

$$(11.6) \quad M_\mu(q_F)(\Phi_{\text{sep},\mu}|_{S_{0,\mu} \oplus S_{e,\mu} \oplus R_{0,\mu}}) \text{ is injective.}$$

The full Picard source. Each of the first three sources maps into $M_\mu(\mathcal{W}_F)$: the W columns are sums of external Picard tails and endpoint paths. Pass a relation among all four sources in $M_\mu(\mathcal{Q}_{5,F})$ to

$$M_\mu(\mathcal{Q}_{5,F}/\delta_F(\mathcal{W}_F)).$$

Proposition 10.13 forces the $R_{p,\mu}$ coefficient to vanish. Project the remaining relation to the double-loop coordinate using Lemma 10.2. Equation (11.6) makes the other three coefficients zero. This proves the asserted injection into the full compact quotient.

The Gysin image of every selected source also belongs to $B_{\text{sep},F}$. An H^{13} tail is the image of a defining one-edge map; a Picard tail has the defining degree-thirteen coefficient $H^{11} \otimes H^2$. For a two-edge path, glue one bridge first. Its coefficient then has degree thirteen, and the remaining bridge is separating. Transitivity of proper Gysin, with the quotient-stack normalization, identifies this composition with the path class. The same holds for each linear combination W . This proves the image inclusion with the full compact coefficients.

The source multiplicities. The multiplicities are those of Lemma 10.16. □

Proposition 11.7. *Let $F = [11]$ and $\mu = (1^{11})$ or $(2, 1^9)$. Then the map $\bar{\delta}_\mu : N_\mu/J_\mu \rightarrow \mathcal{Q}_\mu$ of Proposition 11.2 is injective. Consequently $\mathcal{C}_{(11)} = \mathcal{C}_{(10,1)} = 0$.*

Proof. Let $R_{\text{par},\mu} = M_\mu(\mathcal{R}_{+,F}^{\text{par}}) \cong \mathbb{P}^6, \mathbb{P}^{12}$ (Lemma 10.16) and p_4 as in Definition 10.5.

Suppose that a separating source vector w and a parallel vector r satisfy

$$\delta_\mu(\Phi_{\text{sep},\mu}(w) + M_\mu(p_4)(r)) = 0.$$

Pass to $M_\mu(\mathcal{Q}_{5,F}/\delta_F(B_{\text{sep},F}))$. The first term vanishes by the image inclusion in Proposition 11.6. The normalized left inverse of Proposition 10.6 gives $r = 0$. The separating injection then gives $w = 0$. Thus the combined source injects into the full compact quotient.

The combined multiplicities are

$$114 + 9 + 6 + 15 + 6 = 150, \quad 253 + 9 + 6 + 32 + 12 = 312.$$

Since δ_μ vanishes on J_μ , the induced map to N_μ/J_μ is injective. Proposition 11.2 shows that these are precisely the dimensions of the two reduced domains. The combined Gysin images therefore span these quotients. The reduced maps $\bar{\delta}_\mu$ are injective, and their canonical kernels $\mathcal{C}_{(11)}$ and $\mathcal{C}_{(10,1)}$ are zero. □

12. THE NINE-ROW COEFFICIENTS

We prove the vanishing of the two coefficients with first part nine and of $\mathcal{C}_{(8,3)}$, using the residual interface of Proposition 11.2. The Picard test detects $\mathcal{C}_{(9,2)}$ and $\mathcal{C}_{(8,3)}$ simultaneously; the Picard and parallel tests together detect $\mathcal{C}_{(9,1,1)}$.

Proposition 12.1. *Let $F = [12]$ and $\tau = (2, 2, 2, 1^6)$. Then $\mathcal{R}_\tau = M_\tau(\mathcal{R}_F^{\text{Pic}}) \simeq \mathbb{P}$ is the single Picard tail copy described in Lemma 10.16; and the residual map $\varepsilon_\tau : \mathcal{R}_\tau \rightarrow \mathcal{Q}_\tau/\delta_\tau(\mathcal{U}_\tau)$ of*

Proposition 11.2 is injective. Its kernel being canonically $(\mathcal{C}_{(9,2)} \otimes \mathcal{B}_{\tau,(9,2)}) \oplus (\mathcal{C}_{(8,3)} \otimes \mathcal{B}_{\tau,(8,3)})$ with one-dimensional branching spaces, $\mathcal{C}_{(9,2)} = \mathcal{C}_{(8,3)} = 0$.

Proof. By Lemma 10.16, $\mathcal{R}_\tau$ is the single tail copy, with $|I \setminus A| = 3$ and divisor in Z_B . Exactness of M_τ (Set-up 2.1) identifies the residual target with $M_\tau(\mathcal{Q}_F^{\text{Pic}})$ and ε_τ with M_τ of the map $f_F : \mathcal{R}_F^{\text{Pic}} \rightarrow \mathcal{Q}_F^{\text{Pic}}$ of Proposition 10.13. That proposition makes f_F injective; its hypotheses cover the contributing allocation, whose endpoint has an old flag and whose partner has three markings besides its attaching flag. Hence the kernel in the statement is zero. $\square$

Proposition 12.2. *Let $F = [11]$ and $\mu = (3, 1^8)$, and put $U = \mathcal{U}_\mu$, $R_W = M_\mu(\mathcal{R}_F^{\text{Pic}})$, $R_{\text{par}} = M_\mu(\mathcal{R}_{+,F}^{\text{par}})$, so that $\mathcal{R}_\mu = R_W \oplus R_{\text{par}}$; let $\Phi_W = M_\mu(\Phi_F^{\text{Pic}})$ and let Φ_{par} be the restriction of $M_\mu(p_4)$ to R_{par} . Then the residual map ε_μ of Proposition 11.2 is injective. Its kernel being canonically $\mathcal{C}_{(9,1,1)}$, one has $\mathcal{C}_{(9,1,1)} = 0$. Moreover, $\text{rank}_{\mathbb{Q}} \delta_\mu = 394$.*

Proof. By Lemma 10.16,

$$N_\mu = U \oplus \Phi_W(R_W) \oplus \Phi_{\text{par}}(R_{\text{par}}), \quad U \simeq \mathbb{P}^{182}, \quad R_W \simeq \mathbb{P}^{17}, \quad R_{\text{par}} \simeq \mathbb{P}^6,$$

all with an old flag on the decorated vertex. Proposition 11.2 gives $J_\mu = \ker(\delta_\mu|_U) \simeq \mathbb{P}^8$ and $\delta_\mu(U) \simeq \mathbb{P}^{174}$, and identifies the residual kernel with $\mathcal{C}_{(9,1,1)}$.

The Picard residual. Exactness of M_μ (Set-up 2.1) identifies the residual target with $M_\mu(\mathcal{Q}_F^{\text{Pic}})$. Proposition 10.13 therefore makes the restriction to R_W injective. Its hypotheses cover every contributing allocation: the endpoint has an old flag and the partner has at most three markings besides its attaching flag, as established in the source count.

Separation of the parallel residual. Put $B_\mu = M_\mu(B_{\text{sep},F})$. Every map defining U is a separating Gysin or a composition of separating Gysins, and every map defining Φ_W is a separating Gysin. Hence

$$U + \Phi_W(R_W) \subseteq B_\mu.$$

The complete degree-thirteen coefficient on a separating parent contains each of these factorizations. There is consequently a further quotient map

$$\mathcal{Q}_\mu / \delta_\mu(U) \longrightarrow \mathcal{Q}_\mu / \delta_\mu(B_\mu) \cong M_\mu(\mathcal{Q}_{5,F} / \delta_F(B_{\text{sep},F})).$$

The subspace by which we take this last quotient is the image under δ_F of the full ordinary separating image in genus four. It is distinct from the image of the separating term of the compact row already used to define $\mathcal{Q}_{5,F}$.

Proposition 10.6 supplies the left inverse $\mathcal{L} \text{pr}_{\text{dl}}$ after precisely this further quotient. Its kernel $\mathcal{R}_{0,F}^{\text{par}}$ has zero μ -multiplicity (Lemma 10.16). Thus the induced left inverse detects all of R_{par} .

Suppose the residual image of (w, r) is zero. Passing to the further quotient kills the Picard contribution, and applying the parallel left inverse gives $r = 0$. The Picard injection then gives $w = 0$.

Finally $\delta_\mu(U) \cong \mathbb{P}^{174}$ and the residual source is $\mathbb{P}^{17+6}$. The exact sequence of images therefore gives

$$\text{rank}_{\mathbb{Q}} \delta_\mu = 2 \cdot 174 + 2(17 + 6) = 394.$$

$\square$

13. THE EIGHT-ROW COEFFICIENTS

Proposition 12.1 already gives $\mathcal{C}_{(8,3)} = 0$. We prove the vanishing of $\mathcal{C}_{(8,2,1)}$ and $\mathcal{C}_{(8,1,1,1)}$ by support tests on the double-loop coordinate. The restriction to the separating summand is already injective, so these tests also prove injectivity of the full compact boundary map.

Lemma 13.1. *Let F be finite and let $\rho \vdash |F|$. If $M_\rho(q_F)$ is injective, then $M_\rho(\beta_{5,F})$ is injective.*

Proof. Write $C_{5,F}^1 = C_{\text{sep}}^1 \oplus N_{5,F}$. Let $\pi : C_{5,F}^2 \rightarrow \mathcal{Q}_{5,F}$ be the quotient map and put $p = \text{pr}_{\text{dl}} \pi$. By Lemma 10.2,

$$p\beta_{5,F}(z, n) = q_F(n),$$

since $\pi\beta_{5,F}(z, 0) = 0$ and $\pi\beta_{5,F}(0, n) = \delta_F(n)$. If $(z, n) \in \ker M_\rho(\beta_{5,F})$, injectivity of $M_\rho(q_F)$ gives $n = 0$. Theorem 11.1 gives injectivity on the separating summand. Hence $z = 0$ as well. $\square$

Lemma 13.2. *Let L be a finite marking set.*

(0) *Let $\mathcal{D}_L = \mathcal{D}_{4,L} \otimes \mathbb{P}$ and $\mathcal{T}_L = \mathcal{T}_{4,L} \otimes \mathbb{P}$ (Definition 10.8 with L for I). Let π_9 project to the partitions of $|L|$ with exactly nine rows. Then the ordinary clutching maps induce an isomorphism*

$$\pi_9(\mathcal{D}_L \oplus \mathcal{T}_L) \xrightarrow{\sim} \pi_9 H^{15}(\overline{\mathcal{M}}_{4,L}).$$

(i) *The constituents of $H^{15}(\overline{\mathcal{M}}_{4,L})$ have at least nine rows. Those of $H^{15}(\overline{\mathcal{M}}_{5,L})$ have at least nine rows for $|L| \leq 10$, and at least eight rows for all L .*

(ii) *For an integer $m \geq 9$ and a partition $\theta \vdash m$ with exactly nine rows, put $\gamma = (\theta_1 - 1, \dots, \theta_9 - 1)$, omitting zero parts, and $f_\theta(z) = s_\gamma(1, 1, z)$. Then*

$$M_\theta(H^{15}(\overline{\mathcal{M}}_{4,[m]})) \cong \mathbb{P}^{\oplus \left(f_\theta(1) + \sum_{j \geq 3} [z^j] f_\theta(z) \right)}.$$

In particular this group is zero if $\theta_4 \geq 2$.

(iii) *One has*

$$M_{(1^9)}(H^{15}(\overline{\mathcal{M}}_{5,9})) \cong \mathbb{P}, \quad M_{(2,1^8)}(H^{15}(\overline{\mathcal{M}}_{5,10})) \cong \mathbb{P}^4.$$

Proof. (0) Theorem 7.15 identifies the ordinary image with the four summands of Definition 7.2. Theorem 8.4 identifies that image with compact cohomology. The characters of $\mathcal{W}_{4,L}$ and $\mathcal{R}_{4,L}$ are products of H with sums of complete symmetric functions. Every term of H has at least ten rows (Notation 2.5), and induction cannot decrease the number of rows (Lemma 3.27(1)). Thus these two summands have zero π_9 -projection. The other two summands, with their realizations and twists restored, are precisely $\mathcal{D}_L$ and $\mathcal{T}_L$. The restriction of the injective map $\Omega_{4,L}$ is their displayed sum of clutching maps, proving the assertion.

(i) The genus-four bound and the genus-five bound through ten markings are Lemma 9.2(iii) and (ii). For larger marking sets, the exact sequence in Proposition 9.3 adds induced modules whose seven initial shapes have at least eight rows, and induction preserves this bound (Lemma 3.27(1)). This proves the remaining bound without using any coefficient vanishing.

(ii) The nine-row coefficients of the genus-four character are those of

$$\sum_{k \geq 1} s_{(k,1^8)}(EE_{\geq 3} + E^2).$$

Here the first term comes from $E\partial U$: only removing the bottom corner of its ten-row hook contributes. In the path term only the nine-row part of G contributes. For a fixed θ , removing the first column of $\theta/(k,1^8)$ identifies it with $\gamma/(k-1)$. In a semistandard tableau of γ on three ordered letters, the first letter occupies an initial segment of the first row. Summing over that segment gives

$$\sum_{k \geq 1} s_{\theta/(k,1^8)}(1, z) = s_\gamma(1, 1, z).$$

The two horizontal strips contributed by E^2 count $f_\theta(1)$; requiring the second strip to have at least three boxes counts $\sum_{j \geq 3} [z^j] f_\theta(z)$. If $\theta_4 \geq 2$, then γ has a column of height four, so no such three-letter tableau exists.

(iii) Through ten markings the nine-row genus-five coefficients are those of

$$\sum_{k \geq 1} s_{(k, 1^8)}(E^2 + e_2[E]).$$

The nine-row part of the D term starts in degree twelve, and the remaining terms have at least ten rows (Notation 2.5). At (1^9) the coefficient is 1. At $(2, 1^8)$ the E^2 contribution is $2 + 1 = 3$, and the $e_2[E]$ contribution is 1. This proves the two counts. $\square$

Lemma 13.3. *Let $F = [11]$ and let $\rho = (3, 2, 1^6)$ or $(4, 1^7)$. With the labelled flags and maps of Set-up 10.1, the map*

$$M_\rho(q_F) : M_\rho(N_{5,F}) \longrightarrow M_\rho(\mathcal{D}_F)$$

is injective.

Proof. Work on the ρ -isotypic subspace of $N_{5,F}$. Let x belong to this isotypic subspace of $N_{5,F}$ and suppose $q_F(x) = 0$.

The old flags on the decorated factor. By Lemma 13.2(i), compact genus-four cohomology has at least nine rows. A constituent contributing to $M_\rho(N_{5,F})$ differs from ρ by a horizontal two-strip, since the old flags c, d have ordinary invariants, and such a strip adds at most one first-column box (Lemma 3.27(1)); so the constituent has exactly nine rows. Lemma 13.2(0) therefore gives x a unique preimage

$$(z_D, z_T) \in \pi_9(\mathcal{D}_I \oplus \mathcal{T}_I), \quad (cd)^*(z_D, z_T) = (z_D, z_T).$$

In each nonzero coordinate of z_D , at least one of c, d belongs to the genus-one factor. Indeed its full H^{13} module has row bound ten, and nine after fixing its node (Lemma 3.27(2),(1)). If neither old flag were on this factor, all these rows would be on F labels, excluding ρ after induction. Similarly, the eleven-row hook on a path center has at least nine rows after its two nodes are fixed. Thus every nonzero coordinate of z_T has an old flag on its center. The forbidden allocations form submodules under $\mathfrak{S}_F \times \langle (cd) \rangle$ in the full direct sum of sources. Their ρ -projections vanish coordinate by coordinate, including for the preimage under π_9 .

The transverse pullback. As in the proof of Proposition 10.15, ν_{ab}^* of an ordinary tree class is the sum over positive-genus vertices of the branch lowering that vertex and placing a, b on it, with coefficient one.

For $\mathcal{T}_I$, pulling back on the elliptic center is zero by genus-zero odd vanishing. Pullback on the genus-one leaf gives a path with leaf genera $(0, 2)$; pullback on the genus-two leaf gives a path with leaf genera $(1, 1)$ and a, b on the lowered leaf. For $\mathcal{D}_I$, pullback on the elliptic H^{13} factor is zero for the same reason, and pullback on its partner gives an elliptic H^{13} tail with a genus-two partner carrying a, b . These are all the terms. The original leaf genera distinguish the path leaves; in each resulting positive path, the external labels a, b distinguish the two elliptic leaves. Thus all these graph groups are trivial. The output is invariant under (ab) , with no extra factor from this labelled pullback.

Eliminating the positive paths. Write $r(x) = \nu_{ab}^*x$. By Lemma 10.2, the double-loop equation is the cohomological equality

$$(13.1) \quad q_F(x) = r(x) - s^*r(x) = 0.$$

Choose a coordinate of z_T and lower its genus-two leaf. Its positive target path Γ has neither a nor b on the center, and its coefficient is the unchanged center class. Raising the leaf carrying a, b uniquely recovers the original source and its allocation. Every other positive path in $r(x)$ has different recovered data. Every positive path in $s^*r(x)$ has at least one of a, b on its center, by the old-flag condition, and so is also different.

Theorem 3.23 in genus one expresses each H^{13} coefficient as a sum of rational-tail Gysin classes. Thus every $\mathcal{D}_I$ term, after its outer attachment, is supported on a two-edge tree with a rational

vertex. Apply Lemma 7.11 to Γ , with leaf genera $(1, 1)$ and the leaves distinguished by the external labels a, b , and let O_Γ be its open. Such terms, and the rational paths in the pullback, restrict to zero on O_Γ by that lemma. Restriction of (13.1) to O_Γ and its $(11, 2, 2)$ self-intersection consequently forces the selected coefficient to vanish. Applying this argument to every labelled path gives $z_T = 0$.

Eliminating the elliptic tails. Write z_A for the remaining H^{13} coefficient with elliptic external label set $A \subseteq I$. If some z_A is nonzero, choose one with $|A|$ minimal. It contains an old flag and neither a nor b . In $r(x)$ it gives a tail with genus-two partner on $J \setminus A$. Apply Lemma 7.12 with partner genus $h = 2$ and $J = A \sqcup (J \setminus A)$, with its open $W_A \subseteq \overline{\mathcal{M}}_{3,J}$: a tail of type A' can meet W_A only for $A' \subseteq A$. The proper subsets have zero coefficient by minimality. Every tail in $s^*r(x)$ contains a or b on its elliptic side, so none meets W_A . The $(13, 2)$ self-intersection of the restriction of (13.1) is therefore

$$-z_A|_{\text{ct}} \otimes \psi_q = 0.$$

The same lemma makes this map injective, a contradiction. Hence $z_D = 0$ as well. Hence $x = 0$. $\square$

Lemma 13.4. *The following multiplicities are direct sums of the indicated numbers of copies of P :*

$ L $	θ	$M_\theta(H^{15}(\overline{\mathcal{M}}_{3,L}))$
10	$(2, 1^8)$	1
11	$(3, 1^8)$	2
11	$(2, 2, 1^7)$	2
12	$(3, 2, 1^7)$	4

Proof. *The nine-row ordinary source.* Proposition 5.20 supplies ordinary generation. The complete ordinary source has row bound nine by Lemma 3.27(3), and so has its compact image $H^{15}(\overline{\mathcal{M}}_{3,L})$ by Lemma 9.2(iii). We identify the nine-row coordinates of the source. On a separating source the degree-thirteen factor has at least nine rows after fixing its node; the mixed $H^{11} \otimes H^2$ sectors have at least ten. On the nonseparating normalization use the independent elliptic-tail coordinates of Theorem 3.23 in genus two. If neither loop flag lies on the inner elliptic factor, at least ten rows remain. If one does, fixing it and the inner node leaves at least nine. If both do, write the hook as the tenth exterior power of the standard representation and restrict its three distinguished flags: on a external flags the restricted standard representation is $\text{Std}_{[a]} \oplus \text{Std}_{[3]} \oplus \mathbb{Q}$, and external exterior degree eight is paired with $\text{sgn}_3 \oplus \text{Std}_{[3]}$. Taking invariants under a transposition eliminates the sign summand and gives a one-dimensional fixed space in $\text{Std}_{[3]}$; this is the nine-row two-loop-flag coordinate.

For the nine-row part, the hook on a center with a external flags contributes $V_{(a-8, 1^8)}$, and exchange of its two attaching flags acts by minus one. The surviving separating coordinates are elliptic H^{13} tails with genus-two partners, and the independent inner elliptic-tail coordinates in higher-genus H^{13} factors. In the latter coordinates the outer node must lie on the inner elliptic factor. They therefore recover a centered path together with its chosen inner leaf. The nonseparating coordinates divide into the one-loop-flag and two-loop-flag cases just described.

Eliminating the redundant factorizations. Use the face maps of Definition 4.14; their composites with ordinary Gysin are zero by the normal-crossings row. A parallel-edge graph has selected-edge stabilizer of order one and target loop group of order two. Its two signed faces add to twice the transfer of one labelled clutching. Projection to one of the two inner coordinates recovers twice the original hook. Thus these faces eliminate the entire one-loop-flag part.

Next take a bridge with a loop on its decorated elliptic vertex. Contracting the bridge is the identity on the two-loop-flag invariant coordinate: its source and target loop groups both have order two. The other face is minus one half of the labelled loop Gysin into a separating

coordinate, since its target group is trivial. This is the graph of a specified map out of the independent two-loop-flag coordinate, and eliminates that coordinate without imposing a relation on the remaining separating part.

A centered path with leaf genera zero and two has trivial stabilizers. Its faces identify its independent higher-genus coordinate with its rational-tail Gysin image in the elliptic H^{13} source. For two elliptic leaves distinguished by external labels, the faces similarly identify the two independent inner-tail factorizations. Each resulting coordinate recovers the center labels, the inner leaf labels, and the outer leaf labels, so no different path uses that coordinate. If both elliptic leaves are unmarked, their exchange acts by minus one on the local nine-row hook. The two oriented faces add with coefficient two and kill the sole factorization; the ordinary invariant path source is zero in exactly this case.

These eliminations leave the elliptic H^{13} tail source and the ordinary paths with elliptic center and two elliptic leaves. On the four partitions in the statement the elliptic H^{13} source is zero. Its nine-row coefficients are those of

$$\sum_{k \geq 1} s_{(k, 1^8)} EE_{\geq 3},$$

which starts in degree twelve with $s_{(1^9)}h_3$ and has zero coefficient at $(3, 2, 1^7)$. The positive-path source is injective by Lemma 7.11, applied with leaf genera $(1, 1)$: its nonzero nine-row coordinates have externally distinguished leaves, and the opens O_Γ of that lemma isolate the coefficients. Thus this quotient maps isomorphically onto the compact components in question. All further face relations, including those from double-loop graphs, map to zero in this quotient and impose no further relation.

Counting the ordinary paths. Because exchange of the node flags acts by minus one, the ordinary unordered leaves contribute $e_2[E]$. Their nine-row coefficients are therefore those of

$$\sum_{k \geq 1} s_{(k, 1^8)} e_2[E].$$

In complementary degrees zero through three, $e_2[E]$ has components $0, h_1, h_1^2, h_3 + h_1h_2$. The contributions in increasing order of the hook length k are respectively 1, $1 + 1$, $1 + 1$, and $1 + 2 + 1$ for the four partitions displayed. These give 1, 2, 2, 4. $\square$

Proposition 13.5. *Let $F = [11]$ and $\mu = (3, 2, 1^6)$. The map $M_\mu(q_F)$ is injective with source $\mathbf{P}^{26}$. The map $M_\mu(\beta_{5,F})$ is injective with source $\mathbf{P}^{59}$. Consequently*

$$M_\mu(H^{15}(\overline{\mathcal{M}}_{5,F})) = 0.$$

Proof. The nonseparating map and its source. Lemma 13.3 proves injectivity on this μ . The horizontal two-strip rule for the ordinary invariants of the two old flags (Lemma 3.27(1)) and the nine-row bound of Lemma 13.2(i) leave precisely

$$\theta_1 = (4, 2, 1^7), \quad \theta_2 = (3, 3, 1^7), \quad \theta_3 = (3, 2, 2, 1^6)$$

as partitions of thirteen contributing to $N_{5,F}$, each once. The polynomials of Lemma 13.2(ii) are

θ	$f_\theta(z)$	$f_\theta(1) + \sum_{j \geq 3} [z^j] f_\theta(z)$
θ_1	$3 + 6z + 4z^2 + 2z^3$	17
θ_2	$1 + 2z + 3z^2$	6
θ_3	$2z + z^2$	3

Thus $M_\mu(N_{5,F}) \cong \mathbf{P}^{26}$.

The entire separating source. By Lemma 9.2(iv), the mixed Künneth sectors $(13, 2)$, $(2, 13)$, $(11, 4)$, $(4, 11)$ of C_{sep}^1 have row bound nine and contribute nothing to M_μ , and a genus-zero degree-fifteen factor vanishes by Theorem 3.2. Thus only degree-fifteen factors with unit partners

contribute. A genus-one factor has at most twelve flags and vanishes by Theorems 4.7 and 3.8. For genera two, three, and four, the compact factor has at least nine rows by Lemma 9.2(iii). The same bound holds for a genus-five factor here, since its rational partner leaves at most ten flags (Lemma 13.2(i)).

Let b be the number of external labels on the unit partner. Fixing the node on the decorated factor and then inducing with h_b requires exactly the following one-corner and horizontal-strip possibilities (Lemma 3.27(1)). The last two columns give the compact multiplicities in genera three and four:

b	external shape	full decorated shape	$g = 3$	$g = 4$
0	$(3, 2, 1^6)$	$(3, 2, 1^7)$	4	8
1	$(3, 1^7)$	$(3, 1^8)$	2	6
1	$(2, 2, 1^6)$	$(2, 2, 1^7)$	2	3
2	$(2, 1^7)$	$(2, 1^8)$	1	3

Every branching and Pieri coefficient is one. Indeed an eight-row external shape must be obtained by removing the bottom singleton of a nine-row decorated shape; interlacing with μ gives precisely the four rows of the table. The genus-three entries were proved in the preceding lemma. For genus four the respective polynomials $f_\theta(z)$ are $2 + 4z + 2z^2$, $3 + 2z + z^2$, $1 + 2z$, and $2 + z$, giving the displayed entries.

All four genus-two entries are zero. In degrees at most eleven Theorem 4.18 has no nine-row constituent; in degree twelve its only possible nine-row term comes from $EGE_{\geq 3}$ and is $s_{(1^9)}h_3$. It has zero coefficient at $(3, 2, 1^7)$. Thus positive partners contribute $\mathbf{P}^9$ with decorated genus three and $\mathbf{P}^{20}$ with decorated genus four. For a rational partner stability requires $b \geq 2$, leaving only the last row of the table. Its full decorated genus-five shape is $(2, 1^8)$ on ten flags, contributing $\mathbf{P}^4$ by Lemma 13.2(iii). Consequently the complete source, including all label allocations, is

$$M_\mu(C_{5,F}^1) \cong \mathbf{P}^{26} \oplus \mathbf{P}^{20} \oplus \mathbf{P}^9 \oplus \mathbf{P}^4 \cong \mathbf{P}^{59}.$$

The full compact map. Injectivity of $M_\mu(q_F)$ and Lemma 13.1 give injectivity of $M_\mu(\beta_{5,F})$ on the source just counted. Since $15 < 19$, Proposition 5.3 makes α injective. The identity $\beta\alpha = 0$ therefore gives $M_\mu(H^{15}(\overline{\mathcal{M}}_{5,F})) = 0$. $\square$

Proposition 13.6. *Let $F = [11]$ and $\mu = (4, 1^7)$. The map $M_\mu(q_F)$ is injective with source $\mathbf{P}^{35}$, and $M_\mu(\beta_{5,F})$ is injective. Consequently*

$$M_\mu(H^{15}(\overline{\mathcal{M}}_{5,F})) = 0.$$

Proof. Lemma 13.3 gives injectivity of $M_\mu(q_F)$. A horizontal two-strip above μ (Lemma 3.27(1)), with exactly one new row by the nine-row bound of Lemma 13.2(i), leaves the two genus-four shapes

$$(5, 1^8), \quad (4, 2, 1^7),$$

each with coefficient one. Their polynomials in Lemma 13.2(ii) are $5 + 4z + 3z^2 + 2z^3 + z^4$ and $3 + 6z + 4z^2 + 2z^3$. The corresponding multiplicities are 18 and 17, so $M_\mu(N_{5,F}) \cong \mathbf{P}^{35}$.

Lemma 13.1 now gives injectivity of $M_\mu(\beta_{5,F})$ and, as in the proof of Proposition 13.5, $M_\mu(H^{15}(\overline{\mathcal{M}}_{5,F})) = 0$. $\square$

Theorem 13.7 (Vanishing of the genus-five coefficients). *For every $\lambda \in \mathcal{R}_{11}$ one has $\mathcal{C}_\lambda = 0$.*

Proof. For $\lambda = (11)$ and $\lambda = (10, 1)$ this is Proposition 11.7. Proposition 12.1 proves $\mathcal{C}_{(9,2)} = \mathcal{C}_{(8,3)} = 0$, and Proposition 12.2 proves $\mathcal{C}_{(9,1,1)} = 0$. For the two remaining coefficients, the ordinary genus-five image has at least nine rows by Theorem 7.15. At eleven markings, Proposition 9.3 therefore identifies the compact multiplicities at $(3, 2, 1^6)$ and $(4, 1^7)$ with $\mathcal{C}_{(8,2,1)}$ and $\mathcal{C}_{(8,1,1,1)}$, respectively. Propositions 13.5 and 13.6 make both multiplicities zero. $\square$

Corollary 13.8 (The genus-five base). *For every $n \geq 0$, the complete ordinary one-edge Gysin map $\Phi_{5,n,B}^{15}$ is surjective, and $\mathcal{O}_{5,n} = 0$.*

Proof. Proposition 8.5 proves the assertion for $0 \leq n \leq 10$. For $n \geq 11$, the support isomorphism in Proposition 9.3 expresses $\mathcal{O}_{5,n}$ as the induction of the seven eleven-marking coefficients. Theorem 13.7 makes all seven coefficients zero. Thus $\mathcal{O}_{5,n} = 0$. The exact sequence of the same support proposition then identifies the image of the complete ordinary one-edge map with all of $H^{15}(\overline{\mathcal{M}}_{5,n})$. $\square$

14. ORDINARY LIFTING

We recover ordinary boundary classes from their nonseparating restrictions. The two ingredients are descent for rooted divisor classes and detection of parallel classes modulo the complete separating image.

Definition 14.1. For $h \geq 2$ and $q \in J$, set

$$\mathcal{B}_{h,J;q} = \text{span}_{\mathbb{Q}}\{\delta_{0,A} : q \in A \subseteq J, |A| \geq 2\}, \quad \overline{\mathcal{P}}_{h,J;q} = H^2(\overline{\mathcal{M}}_{h,J})/\mathcal{B}_{h,J;q}.$$

Only stable divisors are included. For $h \geq 3$ the quotient map restricts to an isomorphism $\Pi_{h,J;q}^{\circ} \otimes \mathbb{L} \xrightarrow{\sim} \overline{\mathcal{P}}_{h,J;q}$, by the basis clause of Theorem 3.17 and Corollary 7.8.

Lemma 14.2 (Rooted Picard descent). *For $h \geq 4$, nonseparating pullback gives an injection*

$$\xi_{cd}^* : \overline{\mathcal{P}}_{h,J;q} \longrightarrow \overline{\mathcal{P}}_{h-1,J \cup \{c,d\};q}^{((cd))}.$$

For $h \geq 5$, choose further distinct flags a, b , let ν_{ab} glue a, b , and put $s = (ac)(bd)$. The sequence

$$0 \longrightarrow \overline{\mathcal{P}}_{h,J;q} \xrightarrow{\xi_{cd}^*} \overline{\mathcal{P}}_{h-1,J \cup \{c,d\};q}^{((cd))} \xrightarrow{(1-s^*)\nu_{ab}^*} \overline{\mathcal{P}}_{h-2,J \cup \{a,b,c,d\};q}$$

is exact at its first two nonzero terms.

Proof. All genera in the first comparison are at least three; the same holds for all three genera in the second.

One-loop injectivity. By Theorem 3.19, $\xi_{cd}^*\delta_{0,A} = \delta_{0,A}$ for every omitted rational-root tail, so the quotient map is well-defined. The coefficients of κ_1 , of ψ_i for $i \in J$, and of δ_{irr} are preserved. If a class pulls back to zero, these coefficients vanish.

Distinguish the side carrying q in each remaining source divisor. This side has positive genus. If its other side has positive genus, normalize there; the q -side remains positive. If the other side is rational, normalize on the q -side, whose genus decreases from h to $h-1$. In either case the result is a stable retained target divisor. Gluing c, d recovers the unique source divisor, so distinct source coefficients cannot cancel. The image is fixed by (cd) .

Reduction to separating divisors. Assume $h \geq 5$. Gluing the two pairs commutes, so the displayed maps compose to zero, also after passing to the rooted quotients. Let η be in the kernel of the second map, and write

$$\eta = \lambda\kappa_1 + \sum_{i \in J} \mu_i \psi_i + \varepsilon(\psi_c + \psi_d) + \gamma\delta_{\text{irr}} + \eta_{\text{sep}},$$

where η_{sep} is a linear combination of the retained separating divisor classes. The coefficient of ψ_c in $(1-s^*)\nu_{ab}^*\eta$ is $\varepsilon + \gamma$: the new cotangent coefficients in the pullback of δ_{irr} are both -1 . Thus $\varepsilon = -\gamma$. Subtract the pullback of $\lambda\kappa_1 + \sum_{i \in J} \mu_i \psi_i + \gamma\delta_{\text{irr}}$ from genus h . The remainder is a pure separating combination and still satisfies the exchange equation.

A retained divisor separating c from d has a retained pullback obtained by the choice of vertex used above. This term places a, b together and keeps c, d apart. Every exchanged pullback places

c, d together. The target term uniquely recovers its source by gluing a, b , and its coefficient is one. Hence every coefficient separating c, d vanishes.

Reconstruction of the divisor orbits. Put $m = h - 1$. Each remaining source divisor is uniquely specified by its side containing c, d and has the form

$$D_{r,A} = \delta_{r,A \cup \{c,d\}}, \quad A \subseteq J.$$

Its stable indices are $0 \leq r \leq m - 1$, together with $r = m$ exactly when $|J \setminus A| \geq 2$. The omitted indices are precisely

$$r = 0, q \in A, \quad \text{or} \quad r = m, q \notin A.$$

For $0 \leq r \leq m - 1$, normalizing the side without c, d gives the opposite-pair target with sides $(r, A \cup \{c, d\})$ and $(m - 1 - r, (J \setminus A) \cup \{a, b\})$. This target is omitted exactly when $r = 0, q \in A$ or $r = m - 1, q \notin A$. These are exactly the cases in which one member of the involution

$$(r, A) \longleftrightarrow (m - 1 - r, J \setminus A)$$

is omitted. When both members are retained, the exchange equation equates their coefficients. When just one is retained, it imposes no condition on that coefficient. The other pullback terms put all four labels on one side and are fixed individually by s ; they give no further equations.

For each orbit with retained coefficient f , take $f\delta_{r+1,A}$ in genus h , using either representative when both are retained. Its two sides have positive genus, and its pullback is

$$f(D_{r,A} + D_{m-1-r, J \setminus A}).$$

If one summand is omitted, this realizes the required single coefficient in the quotient. The choice of representative does not matter, by complementarity in genus h . Since $q \in J$, these involution orbits have no fixed point. Finally, a retained $D_{m,A}$ lifts through $\delta_{0, J \setminus A}$; its rational side excludes q . All formulas count complementary divisors once; the unmarked equal-genus exception is absent because the root is retained. $\square$

Definition 14.3. For $h \geq 3$ and finite L , let $\mathcal{R}_h^{\text{par}}(L)$ and $p_h : \mathcal{R}_h^{\text{par}}(L) \rightarrow H^{15}(\overline{\mathcal{M}}_{h,L})$ be as in Definition 10.5; thus $\mathcal{R}_h^{\text{par}}(L) = \mathcal{R}_{h,L} \otimes \mathbb{P}$ and $p_h = 2\Pi_{\text{par}}$ in the notation of Definition 7.2. For $h \geq 5$ and distinct $x, y \notin L$, let

$$H_{xy} : \mathcal{R}_h^{\text{par}}(L) \longrightarrow \mathcal{R}_{h-1}^{\text{par}}(L \sqcup \{x, y\})$$

place both new markings on the unit partner, acting identically on the displayed elliptic coefficient. For $g \geq 5$, choose $a, b, c, d \notin F$, set $I = F \sqcup \{c, d\}$, $I^{(4)} = I \sqcup \{a, b\}$, $t = (cd)$, and $s = (ac)(bd)$, write $\xi = \xi_{cd}$, let $\nu_{ab} : \overline{\mathcal{M}}_{g-2, I^{(4)}} \rightarrow \overline{\mathcal{M}}_{g-1, I}$ glue a, b , put $\mathbf{q} = (1 - s^*)\nu_{ab}^*$, and put

$$\mathcal{R}_{g,F}^{\text{rel}} = \mathcal{R}_{g-1}^{\text{par}}(I)^{\langle t \rangle}, \quad \mathcal{S}_{g,F} = \text{Sep}_{g-1, I}^{\langle t \rangle}.$$

Let $\mathcal{R}_{g,F}^{\text{rel},0}$ be the allocation subspace with neither c nor d on the decorated elliptic vertex. Define

$$e_{g,F}^{\text{rel}} : \mathcal{R}_{g,F}^{\text{rel}} \longrightarrow \mathcal{Q}_{g,F} / \delta(\mathcal{S}_{g,F}), \quad r \longmapsto [\delta(p_{g-1}(r))].$$

This is Definition 10.5 with 4 replaced by $g - 1$: for $g = 5$ these are $\mathcal{R}_F^{\text{par}}$, $B_{\text{sep}, F}$, $\mathcal{R}_{0,F}^{\text{par}}$ of that definition and the map e_F of Proposition 10.6.

Lemma 14.4. *Let $h \geq 4$ and let L be a finite marking set.*

(1) *The complete separating image is*

$$\text{Sep}_{h,L} = \Omega_{h,L} ((D_{h,L} \oplus W_{h,L} \oplus T_{h,L}) \otimes \mathbb{P}),$$

(2) *The map*

$$\mathcal{R}_h^{\text{par}}(L) \longrightarrow H^{15}(\overline{\mathcal{M}}_{h,L}) / \text{Sep}_{h,L}, \quad r \longmapsto [p_h(r)]$$

is injective.

Proof. The Künneth decomposition in Proposition 3.26 consists of $H^{13} \otimes H^0$ and $H^{11} \otimes H^2$, with both placements. For the latter, the odd factor has genus one. The full degree-two partner splits as the retained subspace $\Pi^\circ \otimes \mathbb{L}$ and the span of the rational divisors containing its attaching flag. This is a direct splitting by the basis clause of Theorem 3.17. The first summand is exactly the coefficient used in W . A divisor in the second summand composes with the elliptic factor to give a degree-thirteen elliptic Gysin class, hence a D coordinate.

An H^{13} coefficient of genus one itself gives D . For higher genus, Theorem 3.23 expresses the entire coefficient by elliptic H^{11} tails with a positive-genus unit partner. If the outer attaching flag lies on that partner, the composition is an endpoint-decorated path, hence a boundary decoration in the full Picard partner just considered. If the flag lies on the elliptic factor, two positive leaves give T , while a rational outer leaf composes into D . These exhaust the placements of the outer flag.

Conversely, the maps realizing D and W are separating Gysin maps, and a path class is obtained by composing two separating Gysin maps. The same statement holds for interchangeable leaves after rational transfer from the ordered cover. The map p_h is the $R \otimes P$ part of $\Omega_{h,L}$. Its injectivity modulo the first three summands now follows from Theorem 7.15. $\square$

Proposition 14.5 (Relative parallel detection). *For $h \geq 5$, finite L , and distinct new markings x, y , nonseparating pullback satisfies*

$$[\zeta_{xy}^* p_h(r)] = [p_{h-1}(H_{xy}r)] \quad \text{in} \quad H^{15}(\overline{\mathcal{M}}_{h-1, L \sqcup \{x, y\}}) / \text{Sep}_{h-1, L \sqcup \{x, y\}}.$$

For every $g \geq 5$ and finite F ,

$$\ker e_{g,F}^{\text{rel}} = \mathcal{R}_{g,F}^{\text{rel},0}.$$

Proof. Cohomological excess and the four branches. Let A, B be stable graphs of the same type and let $\eta \in H^*(X_B; \mathbb{Q})$ be invariant under $\text{Aut}(B)$. For a common refinement (Γ, f_A, f_B) , also write f_A and f_B for the induced maps from X_Γ to the two vertex products. The geometric fibre product and excess bundle in Theorem 3.20 give the identity

$$(14.1) \quad \xi_A^*[B, \eta] = \frac{1}{|\text{Aut}(B)|} \sum_{[\Gamma, f_A, f_B] \in \mathcal{R}(A, B)} (f_A)_* \left(f_B^* \eta \prod_{\{u_0, v_0\} \in H(f_A, f_B)} (-\psi_{u_0} - \psi_{v_0}) \right).$$

To justify this identity for arbitrary cohomology classes η , choose an étale atlas with the selected nodes ordered. On this atlas, Gysin pushforward is multiplication by their Thom classes, followed by extension from a tubular neighborhood. On each component of the fibre product, transverse branches retain their Thom classes and common branches contribute the Euler class of the excess bundle. Its summands are the smoothing lines $L_{u_0}^\vee \otimes L_{v_0}^\vee$, whose first Chern classes are $-\psi_{u_0} - \psi_{v_0}$. This construction gives base change for the Gysin maps themselves and is compatible with descent on the atlas. Thom and Euler classes have even degree, so the argument applies to odd η with its coefficient order unchanged. Rational transfer gives $1/|\text{Aut}(B)|$. The target is the labelled X_A , and the common refinements have fixed target identifications, so there is no further $1/|\text{Aut}(A)|$ or $1/|\text{Aut}(\Gamma)|$ factor.

Apply (14.1) with A the one-loop graph and B the parallel graph, first on the cover ordering its two edges. The normalized node is one of the following.

- (1) A selected edge. The remaining edge becomes separating. The excess class is $-(\psi_x + \psi_y)$, multiplying the pulled-back degree-eleven coefficient. The result is an ordinary separating Gysin with a degree-thirteen coefficient. Both selected edges and both assignments of x, y are included.
- (2) A nonseparating node inside the elliptic factor. The coefficient pulls back to H^{11} in genus zero and vanishes by Theorem 3.2.

- (3) A nonseparating node inside the genus- $(h - 2)$ unit factor. That factor becomes genus $h - 3$ and receives x, y . Its smoothing parameter is independent of the selected-edge parameters, so the labelled coefficient is one. The simultaneous edge-exchange group is the same order-two group in both genera. Since p_h and p_{h-1} are both $2\Pi_{\text{par}}$, this branch is exactly $p_{h-1}(H_{xy}r)$.
- (4) A separating node inside a vertex. It is nonseparating in the glued curve precisely when the two old attaching flags lie on opposite sides. After normalization, the selected edges form a three-vertex path and are both separating. Its class is in the complete separating image. This includes every stable genus split, every marking allocation, and the full compact restriction of the odd coefficient.

A node is either selected or internal to one of the two compact vertex factors, and an internal node is separating or nonseparating. Thus the list is exhaustive.

The same base-change argument on a separating one-edge source gives

$$(14.2) \quad \xi_{xy}^* \text{Sep}_{h,L} \subseteq \text{Sep}_{h-1,L \sqcup \{x,y\}}.$$

Indeed, its selected edge is a bridge and stays separating in every degeneration of its connected vertex factors. It cannot be the normalized nonseparating node. Base change along it is transverse, and the remaining bridge carries the full pulled-back degree-thirteen coefficient, which proves (14.2).

For $g = 5$ the second assertion is the last assertion of Proposition 10.6, by the identification in Definition 14.3.

The quotient detector. Fix $g \geq 6$, with $I^{(4)}$, s , ν_{ab} of Definition 14.3. Project $C_{g,F}^2$ onto the graph with one genus- $(g - 2)$ vertex and two loops. On its labelled cover, trivialize the determinant by $e_{ab} \wedge e_{cd}$, and call this coefficient projection pr_{dl} . Each individual pair reversal is positive, while the exchange s is negative. The definition of β in Definition 5.2 gives

$$\text{pr}_{\text{dl}} \beta(0, u) = (1 - s^*) \nu_{ab}^* u \quad (u \in N_{g,F}).$$

Both contractions of this graph are nonseparating. Consequently $\text{pr}_{\text{dl}} \beta(C_{\text{sep}}^1) = 0$, with the entire compact separating source included. By (14.2), the image of $\text{pr}_{\text{dl}} \beta(0, \mathcal{S}_{g,F})$ is contained in $\text{Sep}_{g-2, I^{(4)}}$. Hence the map

$$\begin{aligned} \mathcal{D}_{g,F} : \mathcal{Q}_{g,F} / \delta(\mathcal{S}_{g,F}) &\longrightarrow H^{15}(\overline{\mathcal{M}}_{g-2, I^{(4)}}) / \text{Sep}_{g-2, I^{(4)}}, \\ [v] &\longmapsto [\text{pr}_{\text{dl}} v] \end{aligned}$$

is well-defined, where v is a representative in $C_{g,F}^2$.

The first assertion, applied with $h = g - 1$, and relabelling equivariance give

$$\mathcal{D}_{g,F} e_{g,F}^{\text{rel}}(r) = [p_{g-2}((1 - s^*)H_{ab}r)].$$

Lemma 14.4 makes p_{g-2} injective in this quotient because $g - 2 \geq 4$. Split $\mathcal{R}_{g,F}^{\text{rel}}$ into $\mathcal{R}_{g,F}^{\text{rel},0}$ and the allocation subspace $\mathcal{R}_{g,F}^{\text{rel},+}$ whose elliptic marking set meets $\{c, d\}$. On the first subspace all four labels lie on the unit partner after H_{ab} , so $s^*H_{ab} = H_{ab}$. On the second, the image of H_{ab} has elliptic marking set meeting $\{c, d\}$ and avoiding $\{a, b\}$, while the image of s^*H_{ab} meets $\{a, b\}$ and avoids $\{c, d\}$. These are disjoint supports in the labelled direct sum. Projection onto the former support recovers $H_{ab}r$, and H_{ab} is the identity on each displayed coefficient. Thus $(1 - s^*)H_{ab}$ is injective on $\mathcal{R}_{g,F}^{\text{rel},+}$. Taking t -invariants pairs the one- c, d allocations or takes invariants within the two- c, d allocations; it never identifies the two supports just distinguished. It follows that $\ker e_{g,F}^{\text{rel}} \subseteq \mathcal{R}_{g,F}^{\text{rel},0}$.

Lifts of the remaining allocations. For $r_0 \in \mathcal{R}_{g,F}^{\text{rel},0}$, let $\tilde{r}_0 \in \mathcal{R}_g^{\text{par}}(F)$ be its identical elliptic coefficients, so $H_{cd}\tilde{r}_0 = r_0$, and put $z = p_g(\tilde{r}_0)$. The first assertion in genus g gives

$$\xi_{cd}^* z - p_{g-1}(r_0) \in \mathcal{S}_{g,F}.$$

The difference is t -invariant since both terms are. Write $\alpha(z) = (v_{\text{sep}}, \xi_{cd}^* z)$ in the full compact row. Proposition 5.3 yields $\beta\alpha(z) = 0$, whence $\delta(\xi_{cd}^* z) = [-\beta(v_{\text{sep}}, 0)] = 0$. Hence $e_{g,F}^{\text{rel}}(r_0) = 0$. $\square$

Theorem 14.6 (Ordinary lifting). *For every $g \geq 6$ and finite F , nonseparating pullback preserves the ordinary image, and the sequence*

$$0 \longrightarrow \mathcal{I}_{g,F} \xrightarrow{\xi^*} \mathcal{I}_{g-1,F \cup \{c,d\}}^{(t)} \xrightarrow{\delta} \mathcal{Q}_{g,F}$$

is exact at the first two nonzero terms.

Proof. The four image subspaces. For every $h \geq 4$ and finite L , define the subspaces of $H^{15}(\overline{\mathcal{M}}_{h,L})$

$$\begin{aligned} D_h(L) &= \Gamma_D(D_{h,L} \otimes \mathbf{P}), & W_h(L) &= \Theta(W_{h,L} \otimes \mathbf{P}), \\ T_h(L) &= J_{\text{path}}(\mathbf{T}_{h,L} \otimes \mathbf{P}), & R_h(L) &= p_h(\mathcal{R}_h^{\text{par}}(L)). \end{aligned}$$

The domains and maps here are those of Definition 7.2. In particular, for an elliptic allocation $A \subseteq L$, the degree-two partner coefficient in $W_h(L)$ lies in

$$\Pi_{h-1,(L \setminus A) \cup \{q\};q}^{\circ} \otimes \mathbf{L} \subset H^2(\overline{\mathcal{M}}_{h-1,(L \setminus A) \cup \{q\}})$$

(Definition 14.1). An omitted rational divisor containing q is excluded from the domain of Θ . Gluing its rational component to the elliptic factor first gives an H^{13} class on the resulting genus-one component; its remaining attachment is the Γ_D map.

Theorem 7.15 and Lemma 14.4 now give

$$\begin{aligned} \mathcal{I}_{h,L} &= D_h(L) \oplus W_h(L) \oplus T_h(L) \oplus R_h(L), \\ \text{Sep}_{h,L} &= D_h(L) \oplus W_h(L) \oplus T_h(L). \end{aligned}$$

All subsequent coordinate projections are projections in this cohomological direct sum, with the cohomology coefficients and their Tate factors retained.

Pullback preserves the ordinary image. Let $h \geq 5$ and let v, w be new flags. For the graphs defining D_h, W_h, T_h , every selected edge is a bridge. A selected bridge, or a separating node inside a vertex of this tree, cannot be the normalized nonseparating node. Formula (14.1) therefore gives transverse base change along the selected edges. The possible internal nonseparating normalizations give the following complete list.

On D_h , the pair v, w lies on the unit partner and the elliptic H^{13} coefficient is unchanged. The other possible branch pulls that coefficient to genus zero and vanishes. On W_h , the pair lies on the Picard partner, and its coefficient is the nonseparating H^2 pullback. Its retained projection is the map on $\overline{\mathcal{P}}$; every omitted rational-root term is a D_{h-1} correction by the preceding composition. Normalization of the elliptic H^{11} factor again vanishes. Both vanishings follow from Theorem 3.2.

On a path in T_h , normalization of a unit leaf of genus at least two lowers its genus by one and puts v, w there, leaving the elliptic coefficient unchanged. A genus-one unit leaf becomes rational; composing it into the center gives an H^{13} elliptic coefficient in D_{h-1} . Normalization of the decorated genus-one center vanishes. Every surviving branch has coefficient one on the ordered cover. If the two leaves upstairs are identical, the two branches are exchanged by the graph involution; division by its order gives one copy of the invariant coefficient. If they are distinct, each branch occurs once.

Finally, Proposition 14.5 gives the R_h pullback modulo the full separating image. Consequently the pullbacks have the triangular form

$$\begin{aligned} D_h &\longrightarrow D_{h-1}, & W_h &\longrightarrow W_{h-1} + D_{h-1}, \\ T_h &\longrightarrow T_{h-1} + D_{h-1}, & R_h &\longrightarrow R_{h-1} + \text{Sep}_{h-1}, \end{aligned}$$

with marking sets as above.

Parallel lifts and the two-loop equation. $\delta \circ \xi_{cd}^* = 0$ on $\mathcal{I}_{g,F}$ (Proposition 5.3, as at the end of the proof of Proposition 14.5).

Let $x^{(0)} \in \mathcal{I}_{g-1,I}^{(t)}$ satisfy $\delta(x^{(0)}) = 0$. Its parallel coefficient $\varrho \in \mathcal{R}_{g,F}^{\text{rel}}$ satisfies $e_{g,F}^{\text{rel}}(\varrho) = 0$ because all other coordinates lie in $\mathcal{S}_{g,F}$. Proposition 14.5 therefore places ϱ in $\mathcal{R}_{g,F}^{\text{rel},0}$. Let $\tilde{\varrho} \in \mathcal{R}_g^{\text{par}}(F)$ have those same coefficients and set $z_R = p_g(\tilde{\varrho})$. Its pullback has parallel coefficient ϱ and separating corrections. Thus

$$\begin{aligned} x^{(1)} &= x^{(0)} - \xi_{cd}^* z_R, & \delta(x^{(1)}) &= 0, \\ x^{(1)} &\in (D_{g-1}(I) \oplus W_{g-1}(I) \oplus T_{g-1}(I))^{(t)}. \end{aligned}$$

The two-loop projection annihilates $\beta(C_{\text{sep}}^1)$, as in the proof of Proposition 14.5. Its value on $\beta(0, x^{(1)})$ is $\mathbf{q}(x^{(1)})$. Hence $\mathbf{q}(x^{(1)}) = 0$. Each further subtraction below is a pullback and preserves both kernel conditions.

Picard lifts. Project $\mathbf{q}(x^{(1)})$ to $W_{g-2}(I^{(4)})$; only the W coordinate of $x^{(1)}$ contributes. If its elliptic allocation $A \subseteq I$ meets $\{c, d\}$, the ν_{ab}^* image meets $\{c, d\}$ and avoids $\{a, b\}$ on the elliptic factor, while its exchanged image has the opposite support. On this allocation the coefficient map is the identity on H^{11} tensored with rooted Picard pullback from partner genus $g-2 \geq 4$. It is injective by Lemma 14.2. The disjoint supports force all such W coefficients to vanish. Rational tensor products and the finite t -invariants preserve this injection.

For $A \subseteq F$, put $M_A = (F \setminus A) \cup \{q\}$. The elliptic coefficient is unchanged by all four flags, and the W equation is exactly the equalizer of Lemma 14.2 with higher partner genus $g-1 \geq 5$ and root q . Choose the unique retained representative of each lifted divisor coefficient and apply Θ in genus g . Their sum is a class $z_W \in W_g(F)$. The pullback $\xi_{cd}^* z_W$ has all the required W coefficients, no T coefficient, and its full rational-root D corrections. Thus $x^{(2)} = x^{(1)} - \xi_{cd}^* z_W$ lies in $D + T$ and still satisfies $\mathbf{q}(x^{(2)}) = 0$.

Path lifts. Put $k = g-2 \geq 4$. A path in $T_{g-1}(I)$ has positive leaf genera summing to k . If c, d are not together on one unit leaf, normalize a leaf of genus at least two. This gives a nonzero T_{g-2} term with a, b on that leaf and with the original placement of c, d . No exchanged term has this support, since every exchanged T term has c, d on a single unit leaf. Gluing a, b uniquely recovers the source path, so no other source coefficient cancels it. These allocations therefore vanish in the T coordinate of $x^{(2)}$.

Distinguish the leaf carrying c, d in each remaining path. Write its center markings as A , its distinguished leaf as $(r, B \cup \{c, d\})$, and its other leaf as $(k-r, C)$, where $F = A \sqcup B \sqcup C$ and $1 \leq r \leq k-1$. Write

$$z_{r,A,B,C} \in H^{11}(\overline{\mathcal{M}}_{1,A \cup \{p_1, p_2\}})(-2)$$

for its coefficient, with p_1 attaching to the distinguished leaf. That leaf is distinguished by its labels, so this source has no leaf-exchange automorphism. For $1 \leq r \leq k-2$, normalize the other leaf. Comparing the target with its exchanged target gives exactly

$$z_{r,A,B,C} = (p_1 p_2)^* z_{k-1-r,A,C,B}.$$

The other T targets put all four flags on one leaf and are individually fixed by s . At $r = k-1$, normalization of the other, genus-one leaf is a D term and gives no additional T equation.

For each two-element orbit of the displayed equation, take the genus- g path with center A and ordered leaves $(r+1, B)$ and $(k-r, C)$, with coefficient $z_{r,A,B,C}$. Both leaf genera are at least two; its two normalization branches give exactly the prescribed orbit. Changing the orbit representative exchanges the attaching flags and leaves the ordinary class unchanged. A fixed orbit occurs precisely for $B = C = \emptyset$ and $2r = k-1$. Its coefficient is invariant under $(p_1 p_2)$, exactly the ordinary automorphism condition on the two equal unmarked leaves upstairs. The quotient pullback has coefficient one by the transfer calculation above.

For each remaining $r = k-1$ coordinate, use the genus- g path with leaves (k, B) and $(1, C)$. Normalizing the first leaf gives the required T coefficient. Normalizing the genus-one leaf gives a rational leaf with markings $C \cup \{c, d\}$; composing its Gysin with the center produces a D correction, which is retained. Summing these path classes gives $z_T \in T_g(F)$. Thus $x^{(3)} = x^{(2)} - \xi_{cd}^* z_T$ has only D coordinates and $\mathfrak{q}(x^{(3)}) = 0$.

Elliptic-tail lifts and exactness. On $D_{g-1}(I)$, the map ν_{ab}^* preserves each elliptic H^{13} coefficient and places a, b on its unit partner. The allocations meeting $\{c, d\}$ and their exchanged allocations have disjoint supports in the $D_{g-2}(I^{(4)})$ direct sum. Thus $\mathfrak{q}(x^{(3)}) = 0$ forces all those coefficients to vanish. The remaining allocations are contained in F and are exactly the pullbacks of the corresponding genus- g elliptic tails. Let z_D be their Γ_D image. Adding the four pullbacks gives

$$x^{(0)} = \xi_{cd}^*(z_R + z_W + z_T + z_D).$$

Every separating correction was included in the pullback being subtracted. This proves the required kernel inclusion; the converse was established by $\beta\alpha = 0$ above.

Injectivity. Suppose $z \in \mathcal{I}_{g,F}$ has $\xi_{cd}^* z = 0$. The parallel projection of this pullback is the unchanged coefficient placed by H_{cd} , so its injectivity and the direct ordinary decomposition force the R coefficient of z to vanish. The W projection is then the rooted Picard pullback at each elliptic allocation, which is injective by Lemma 14.2; hence the W coefficient vanishes. A path in the higher genus has leaf genera summing to $g-1 \geq 5$, so at least one leaf has genus at least two. Normalizing it gives a T target that uniquely determines the original path by gluing c, d , with unchanged coefficient. Equal unmarked leaves are detected with coefficient one on their invariant subspace. Therefore the T coefficient also vanishes. Finally the D pullback is the identity on its coefficients in distinct allocations, so $z = 0$. $\square$

15. PROOF OF THE MAIN THEOREM

Theorem 15.1 (Uniform high-genus character). *Use the series of Definition 2.5. For $g \geq 4$, let $t_g = \lfloor (g-2)/2 \rfloor$ and let ϵ_g be 1 for odd g and 0 for even g . Then*

$$B_g = E \frac{\partial U}{\partial p_1} + H(3E + (g-1)E^2) + t_g(H + G)E^2 + \epsilon_g(Hh_2[E] + Ge_2[E]).$$

The degree- n part is $\text{ch}(\mathcal{E}_{g,n})$, which is nonzero exactly for $n \geq 9$, with the stated realizations in cohomology. The full ordinary Gysin map is surjective and $W_{15}H^{15}(\mathcal{M}_{g,n}) = 0$ for all $g \geq 4$ and $n \geq 0$.

Proof. Theorem 8.4 and Corollary 13.8 give surjectivity of the complete ordinary one-edge Gysin map and vanishing of the open weight piece in those two genera.

Induction in the genus. Assume $g \geq 6$ and that the complete ordinary map is surjective in genus $g-1$ for every finite marking set. Let F be finite, choose new flags c, d , and let

$$\xi : \overline{\mathcal{M}}_{g-1, F \cup \{c, d\}} \longrightarrow \overline{\mathcal{M}}_{g, F}$$

be the nonseparating clutching map. For $z \in H^{15}(\overline{\mathcal{M}}_{g, F})$ put $x = \xi^* z$. Then x is fixed by $t = (cd)$ and, by the induction hypothesis, belongs to $\mathcal{I}_{g-1, F \cup \{c, d\}}^{(t)}$.

Since $\beta\alpha(z) = 0$ (Proposition 5.3) and the remaining coordinates of $\alpha(z)$ lie in C_{sep}^1 , $\delta(x) = [\beta(0, x)] = 0$ in $\mathcal{Q}_{g,F}$. Theorem 14.6 therefore gives $y \in \mathcal{I}_{g,F}$ with $\xi^*y = x$. Theorem 11.1 implies $y = z$. Thus $\mathcal{I}_{g,F} = H^{15}(\overline{\mathcal{M}}_{g,F})$, completing the induction.

Proposition 4.3 identifies the cokernel of the complete ordinary map with $W_{15}H^{15}(\mathcal{M}_{g,F})$. Hence the established surjectivity gives the asserted vanishing for every $g \geq 4$.

Character and realizations. Adding the four characteristic series displayed in Theorem 7.15 gives the displayed formula for B_g . Since the ordinary image is the full compact cohomology group, Proposition 3.26 gives

$$H^{15}(\overline{\mathcal{M}}_{g,n}) \cong \mathcal{E}_{g,n} \otimes \mathbf{P}$$

as pure Hodge structures and, for every prime ℓ ,

$$H_{\text{et}}^{15}(\overline{\mathcal{M}}_{g,n,\overline{\mathbb{Q}}}; \mathbb{Q}_{\ell}) \cong (\mathcal{E}_{g,n} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}) \otimes \mathbf{P}_{\ell},$$

as a $\mathfrak{S}_n \times \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -representation before semisimplification. Here $\mathbf{P}_{\ell}$ denotes the ℓ -adic realization of $\mathbf{P}$ supplied by that proposition.

It remains to check the first nonzero arity. The series H starts in degree ten, G in degree nine, and U in degree twelve, so $E\partial U$ starts in degree eleven. Thus B_g has no term of degree less than nine. In degree nine, G contributes e_9 , while the constant term of $h_2[E]$ is one and that of $e_2[E]$ is zero; consequently the degree-nine part is $t_9e_9 \neq 0$. For $n \geq 9$, the product $t_9e_9h_{n-9}$ is a nonzero Schur-positive summand. Every other term is Schur-positive (the derivative and products encode restriction and induction, and $h_2[E], e_2[E]$ encode symmetric and exterior squares), so there can be no cancellation. $\square$

Theorem 15.2 (Degree-fifteen determination). *This is Theorem 1.3. For every stable (g, n) , there are $\mathfrak{S}_n$ -equivariant isomorphisms of pure Hodge structures and compatible ℓ -adic realizations before semisimplification*

$$H^{15}(\overline{\mathcal{M}}_{g,n}) \cong \bigoplus_{\lambda \vdash n} V_{\lambda} \otimes \mathbf{P}^{\oplus [s_{\lambda}]F_g} \oplus \delta_{g,1} K_n^{15} \otimes \mathbf{S}_{16},$$

where $F_0 = 0$ and F_1, F_2, F_3, F_g for $g \geq 4$ are specified in Theorems 4.7, 4.18, 6.13, and 15.1, respectively. The cohomology group is nonzero precisely in the following cases:

$$g = 0 : \emptyset; \quad g = 1 : n \geq 13; \quad g = 2 : n \geq 11; \quad g = 3 : n \geq 10; \quad g \geq 4 : n \geq 9.$$

Proof. Theorem 3.8 reduces every stable arity to the $\mathbf{P}$ -multiplicity module $\mathcal{E}_{g,n}$ and, only in genus one, the primitive summand $K_n^{15} \otimes \mathbf{S}_{16}$; in genus zero it gives $F_0 = 0$. The four theorems named in the statement give $\text{ch}_n(\mathcal{E}_{g,n}) = [F_g]_n$, so $[s_{\lambda}]F_g$ is the V_{λ} multiplicity of $\mathbf{P}$, and their nonzero thresholds $n \geq 13, 11, 10, 9$ make the five displayed cases exhaustive. The same four theorems give the realizations before semisimplification; for $g \geq 4$ this is the ℓ -adic splitting displayed in the proof of Theorem 15.1. $\square$

Corollary 15.3 (Hodge numbers). *Let (g, n) be stable and let $e_{g,n} = \dim \mathcal{E}_{g,n}$. The only possibly nonzero Hodge numbers in degree fifteen are*

$$h^{13,2} = h^{2,13} = e_{g,n}, \quad h^{15,0} = h^{0,15} = \delta_{g,1} \binom{n-1}{14},$$

where the binomial coefficient is zero for $n < 15$. In particular, $\dim H^{15}(\overline{\mathcal{M}}_{g,n}) = 2e_{g,n} + 2\delta_{g,1} \binom{n-1}{14}$.

Proof. The motivic pieces have Hodge types $(13, 2)$ and $(2, 13)$ for $\mathbf{P}$, and $(15, 0)$ and $(0, 15)$ for $\mathbf{S}_{16}$, by Notation 2.2. The $\mathbf{P}$ -multiplicity space has dimension $e_{g,n}$, while the only possible $\mathbf{S}_{16}$

summand is $K_n^{15} = V_{(n-14, 1^{14})}$. The hook-length formula gives

$$\dim V_{(n-14, 1^{14})} = \binom{n-1}{14}$$

for $n \geq 15$, and it is zero otherwise. The Hodge numbers follow. $\square$

16. GEOMETRIC CONIVEAU AND THE GENERALIZED HODGE CONJECTURE

Payne asks whether the generalized Hodge conjecture holds for $H^{15}(\overline{\mathcal{M}}_{g,n})$ [Pay26, Problem 6.2]. We prove it by determining the geometric coniveau filtration.

Throughout this section, cohomology is rational singular cohomology over $\mathbb{C}$. For a smooth proper Deligne–Mumford stack X of pure dimension with projective coarse moduli space, define

$$N^r H^k(X) = \sum_{\substack{Z \subseteq X \text{ closed algebraic} \\ \text{codim}_X Z \geq r}} \ker(H^k(X) \rightarrow H^k(X \setminus Z)), \quad r \geq 0.$$

Codimension is required to be at least r on every irreducible component of Z ; the empty support is allowed. Finite unions of supports preserve this condition. Since $H^k(X)$ is finite-dimensional, the subspace $N^r H^k(X)$ itself is supported on a single such Z . Write $N_H^r H^k(X)$ for the largest rational sub-Hodge structure whose Hodge types (p, q) satisfy $p, q \geq r$. The generalized Hodge conjecture in degree k is the equality $N^r H^k(X) = N_H^r H^k(X)$ for every $r \geq 0$.

Lemma 16.1 (Support and Hodge coniveau). *Let X and Y be smooth proper Deligne–Mumford stacks of pure dimension with projective coarse moduli spaces.*

- (1) *One has $N^r H^k(X) \subseteq N_H^r H^k(X)$.*
- (2) *Suppose that $f : Y \rightarrow X$ is proper and $\dim Y = \dim X - c$, with $c \geq 0$. Then*

$$f_*(N^r H^k(Y) \otimes \mathbb{L}^c) \subseteq N^{r+c} H^{k+2c}(X).$$

- (3) *If D is a rational algebraic divisor on Y , then its cycle class restricts to zero on $Y \setminus |D|$. In particular, for any smooth proper Deligne–Mumford stack T and any $\alpha \in H^j(T)$, the class $\alpha \boxtimes [D]$ is supported on $T \times |D|$.*

Proof. For the first assertion, localization identifies the kernel associated with a closed subset Z with the image of $H_Z^k(X) \rightarrow H^k(X)$. Choose a filtration by closed subsets $Z = Z_0 \supset Z_1 \supset \cdots \supset Z_m = \emptyset$ such that each stratum $S_i = Z_i \setminus Z_{i+1}$ is smooth of pure codimension $c_i \geq r$ in X and has smooth coarse moduli space. Such a filtration exists by noetherian induction (smooth locus of the top-dimensional components, over the smooth locus of the coarse space). Purity gives

$$H_{S_i}^k(X \setminus Z_{i+1}) \cong H^{k-2c_i}(S_i) \otimes \mathbb{L}^{c_i}.$$

The localization sequences for successive supports therefore express $H_Z^k(X)$ as successive extensions of subquotients of these groups. For each S_i , the coarse-space map induces an isomorphism on rational cohomology compatible with mixed Hodge structures. Its coarse space is a smooth locally closed algebraic subvariety of the projective coarse space of X , and ordinary cohomology of a smooth algebraic variety has nonnegative Hodge indices [Del71, Sections 3.1–3.2]. Consequently the displayed groups have Hodge indices at least c_i , hence at least r . This property is preserved by subquotients and extensions in the category of mixed Hodge structures, and therefore by the image in $H^k(X)$. We use localization, purity, and their compatibility with rational cohomology for Deligne–Mumford stacks as in [Tub25, Theorem 4.1.1 and Proposition 4.2.6]. Each kernel is a rational sub-Hodge structure, and summing proves the first assertion.

For the second assertion, suppose that α restricts to zero outside a closed subset $Z \subseteq Y$ of codimension at least r . Properness makes $f(Z)$ closed, and

$$\dim f(Z) \leq \dim Z \leq \dim Y - r = \dim X - c - r.$$

Proper pushforward commutes with restriction to the open complement of $f(Z)$; see [Tub25, Proposition 4.2.5]. The restriction of α to its inverse image is zero, so $f_*\alpha$ is supported on $f(Z)$. Linearity proves the assertion.

For the third assertion, write D as a rational linear combination of irreducible divisors. Compatibility of the cycle class map with open restriction shows that $[D]$ restricts to zero off their union. Pullback to the product and cup product with α give the stated support. $\square$

Proposition 16.2 (Codimension-two support for the ordinary image). *For every stable pair (g, n) ,*

$$\mathcal{I}_{g,[n]} \subseteq N^2 H^{15}(\overline{\mathcal{M}}_{g,n})$$

(Definition 7.1). *There is a closed algebraic subset $Z_{g,n} \subseteq \partial \overline{\mathcal{M}}_{g,n}$ of codimension at least two supporting the entire subspace $\mathcal{I}_{g,[n]}$. It may be chosen invariant under $\mathfrak{S}_n$.*

Proof. Put $X = \overline{\mathcal{M}}_{g,n}$ and $d = 3g - 3 + n$. We use the complete ordinary one-edge Gysin map of Proposition 3.26. For a one-edge graph Γ , its labelled normalization $Y = X_\Gamma$ is smooth and proper of dimension $d - 1$, and its clutching map $\xi_\Gamma : Y \rightarrow X$ is proper. We show that

$$H^{13}(Y) = N^1 H^{13}(X).$$

First, Theorem 3.23 expresses every degree-thirteen class on a single moduli factor as a sum of boundary Gysin images of degree-eleven classes. Such a class is supported on the boundary of that factor. This proves the displayed equality when Γ is nonseparating, since then $Y = \overline{\mathcal{M}}_{g-1,[n] \sqcup \{p,q\}}$.

If Γ is separating, write $Y = Y_1 \times Y_2$ for its two moduli factors. Theorem 3.3 and the Künneth formula give exactly the four summands

$$\begin{aligned} H^{13}(Y) &= H^{13}(Y_1) \otimes H^0(Y_2) \oplus H^0(Y_1) \otimes H^{13}(Y_2) \\ &\oplus H^{11}(Y_1) \otimes H^2(Y_2) \oplus H^2(Y_1) \otimes H^{11}(Y_2). \end{aligned}$$

For the first two summands, the degree-thirteen presentation just used gives a divisor support in one factor, hence a divisor support in Y . The other placement is identical.

For the last two summands, Theorem 3.17 represents every degree-two class by a rational algebraic divisor. If $\alpha \in H^{11}(Y_1)$ and D is such a divisor on Y_2 , then $\alpha \boxtimes [D]$ is supported on $Y_1 \times |D|$ by Lemma 16.1(3), of codimension at least one in Y . Interchanging the factors treats the fourth summand.

Lemma 16.1(2), with $c = 1$, now places every ordinary Gysin image in $N^2 H^{15}(X)$. The finite graph-automorphism quotients (Definition 2.7) do not change this conclusion. Summing over the finitely many one-edge graphs proves the inclusion.

Choose a finite basis of the ordinary image and the supports just constructed for its basis elements. Their union is closed, has codimension at least two, and is contained in the boundary because every outer map is a boundary clutching map. Taking the union of its finitely many $\mathfrak{S}_n$ -translates gives $Z_{g,n}$ with the stated invariance. For a zero ordinary image, take $Z_{g,n} = \emptyset$. $\square$

Theorem 16.3 (The degree-fifteen coniveau filtration). *This is Theorem 1.2. Let (g, n) be stable, put $X = \overline{\mathcal{M}}_{g,n}$, and denote by $H^{15}(X)_P$ its canonical P -isotypic rational sub-Hodge structure. Then, as $\mathfrak{S}_n$ -stable subspaces of $H^{15}(X)$,*

$$N^1 H^{15}(X) = N^2 H^{15}(X) = H^{15}(X)_P, \quad N^r H^{15}(X) = 0 \quad (r \geq 3).$$

Moreover,

$$H^{15}(X)_P \cong \mathcal{E}_{g,n} \otimes \mathbb{L}^2 \mathbb{S}_{12}.$$

In particular, for $g \geq 2$ all of $H^{15}(X)$ is supported on an algebraic subset of codimension at least two contained in the boundary. For $g \geq 2$ with $H^{15}(X) \neq 0$, its geometric coniveau is exactly two.

Proof. The ordinary image $\mathcal{I}_{g,[n]}$ is $H^{15}(X)_{\mathbf{P}}$ in genus one by Theorem 4.7, and for $g \geq 2$ it is the whole group (Proposition 4.12, Proposition 5.20, Theorem 15.1), which is entirely $\mathbf{P}$ -isotypic by Theorem 3.8; in genus zero it vanishes, so $\mathcal{I}_{g,[n]} = H^{15}(X)_{\mathbf{P}}$, which lies in $N^2 H^{15}(X)$ by Proposition 16.2.

By Theorem 3.8 and Notation 2.2, the only simple Hodge constituents are $\mathbf{P}$, with types $(13, 2), (2, 13)$, and, in genus one, $\mathbf{S}_{16}$, with types $(15, 0), (0, 15)$. Their isotypic subspaces are canonical (semisimplicity). Any sub-Hodge structure of positive Hodge coniveau has zero projection onto the $\mathbf{S}_{16}$ -isotypic subspace: its Hodge types have both indices positive, whereas that subspace has only the two types just listed. The entire $\mathbf{P}$ -isotypic subspace has Hodge coniveau at least two, and none of the Hodge types in degree fifteen has both indices at least three. Consequently

$$N_H^1 H^{15}(X) = N_H^2 H^{15}(X) = H^{15}(X)_{\mathbf{P}}, \quad N_H^r H^{15}(X) = 0 \quad (r \geq 3).$$

Combining the support inclusion with Lemma 16.1(1) gives

$$H^{15}(X)_{\mathbf{P}} \subseteq N^2 H^{15}(X) \subseteq N^1 H^{15}(X) \subseteq N_H^1 H^{15}(X) = H^{15}(X)_{\mathbf{P}}.$$

The same lemma gives $N^r H^{15}(X) = 0$ for $r \geq 3$. The multiplicity identification is Theorem 3.8, and Proposition 16.2 supplies the common boundary support. This proves all assertions. $\square$

Corollary 16.4 (Generalized Hodge conjecture in degree fifteen). *This is Theorem 1.1. For every stable pair (g, n) , the generalized Hodge conjecture holds for $H^{15}(\overline{\mathcal{M}}_{g,n})$. In particular, Problem 6.2 of [Pay26] has an affirmative answer.*

Proof. The preceding theorem proves $N^r H^{15}(X) = N_H^r H^{15}(X)$ for $r \geq 1$, and both filtrations equal $H^{15}(X)$ at $r = 0$. $\square$

APPENDIX A. USE OF GENERATIVE AI

by Guoxiong Gao¹, Shurui Liu², and Zeming Sun³

The results of this paper were obtained using Pharos. Pharos is an automated mathematical reasoning system that inherits selected design principles from Rethlas [Ju⁺26] and Danus [Liu⁺26], especially the fact graph and multi-agent architecture, while substantially redesigning and extending the remaining system components. Qizheng Yin supplied the problem: for which stable pairs (g, n) is $H^{15}(\overline{\mathcal{M}}_{g,n})$ nonzero, and what is its explicit $\mathfrak{S}_n$ -equivariant decomposition as a Hodge structure and an ℓ -adic Galois representation? Working fully autonomously and without human intervention, Pharos answered the question as posed and produced the main body of the paper. Shurui Liu and Zeming Sun participated in checking the correctness of the paper. No step of the proof relies on a numerical computation or on the system’s internal acceptance decision.

In the run reported here, agents of every role ran on GPT-6 Astra, with the main agent at effort level “ultra”, three workers at “xhigh” and six at “max”, and the verifier at “xhigh”. It took 76 hours and 24 minutes from the creation of the project to the acceptance of the target, and a further 12 hours and 7 minutes to write and check its own manuscript. Its token usage is recorded in Table 1; in all, the run consumed approximately 5.4 times the weekly usage limit of a \$200 ChatGPT Pro subscription. The final fact graph contains 667 facts verified by the system’s verifier. The global memory contains 710 conclusions, 276 identified obstacles, 137 directions, 500 proof attempts, 281 plans, 66 counterexamples, 44 recorded dead ends, and 1342 verification records, of which 379 are rejections. Table 2 records how the 667 facts divide: 214 lie in the supporting closure of the final target facts, computed from the dependencies recorded with each fact, and the remaining 453 lie outside it.

	Sessions	Input (M)	of which cached (M)	Output (M)
Main agent	3	579.3	563.4	1.8
Workers	1610	4842.1	4602.0	21.4
Verifier	1593	1323.9	1200.8	11.7
Helper agents	333	658.0	601.1	4.8
Manuscript and reports	36	115.5	106.2	0.8
Total	3575	7518.8	7073.5	40.5

TABLE 1. Token usage of the run, in millions of tokens.

	Count	Share of all facts
Main-theorem closure	214	32.1%
Outside the closure	453	67.9%
Total	667	100%

TABLE 2. Distribution of facts in the fact graph.

Table 3 lists facts underlying the main results. After computing the genus-one character and the ordinary Gysin image, Pharos established the genus-five base and compact injectivity for induction. It then completed the genus-three and Schur-positive genus-two formulas, the uniform high-genus formula, and the decomposition for all stable pairs.

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Fact	Corresponding theorem	Mathematical content	Time
A	Theorem 4.7	Equivariant character in genus one.	T+3:03
B	Theorem 7.15	Character of the ordinary Gysin image, $g \geq 4$.	T+23:45
C	Corollary 13.8	Ordinary Gysin generation in genus five.	T+74:56
D	Theorem 11.1	Nonseparating pullback injectivity, $g \geq 6$.	T+75:53
E	Theorem 6.13	Finite Gram formula in genus three.	T+75:57
F	Theorem 4.18	Schur-positive character in genus two.	T+76:00
G	Theorem 15.1	Uniform character for every $g \geq 4$.	T+76:12
H	Theorem 15.2	Decomposition and nonvanishing for all stable (g, n) .	T+76:24

TABLE 3. Selected facts and their acceptance times, in hours and minutes from project creation.

Table 4 records the research and writing timeline. Parallel exploration led from image calculations to the genus-five reduction and the final assembly. During writing, proof gaps identified in reviews led the main agent to expand V1 into V2; a further audit led to V3, separating shared detectors, coefficient proofs, and the inputs to descent.

Stage	Description	Time
<i>Research</i>		
Image	The ordinary Gysin image and its character are determined (B).	T+23:45
Reduction	Genus-five generation is reduced to seven coefficient vanishings (inputs to C, G).	T+42:56
Coefficients	All seven vanishings needed for the genus-five base are established (inputs to C, G).	T+71:05
Assembly	The genus formulas give decomposition and nonvanishing for every stable pair (H).	T+76:24
<i>Writing</i>		
V1	Seven sections organize the low-genus formulas, genus-three Gram formula, and high-genus induction.	T+83:22
V2	Ten sections separate genus-three generation from its formula, and the ordinary image and base cases from high-genus descent.	T+84:38
V3	Sixteen sections separate shared detectors and genus-five coefficient proofs, then compact restriction, ordinary lifting, and final assembly.	T+85:09
Review	The assembled manuscript passes the system’s final review.	T+88:32

TABLE 4. Research and writing milestones, timed from project creation. Letters refer to facts in Table 3.

APPENDIX B. THE PROBLEM AS POSED

The following is the full supplied problem, including its original background and bibliographic conventions.

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# The Odd Cohomology of  $\overline{\mathcal{M}}_{g,n}$ : Known Results and the Problem for
↪ Degree 15

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## 1. Setup

### 1.1 The moduli space  $\overline{\mathcal{M}}_{g,n}$ 
```

For integers $g \geq 0$, $n \geq 0$ with $2g - 2 + n > 0$, the **Deligne--Mumford compactification** $\overline{\mathcal{M}}_{g,n}$ is the moduli stack of **stable curves**: connected, at-worst-nodal algebraic curves of arithmetic genus g equipped with n distinct ordered marked points, satisfying the stability condition that the automorphism group is finite. It is a smooth proper Deligne--Mumford stack over $\operatorname{Spec} \mathbb{Z}$ of dimension $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$.

The **boundary** $\partial \mathcal{M}_{g,n} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a normal crossing divisor. Its irreducible components are images of **gluing maps**:

- **Non-separating**: $\xi \colon \overline{\mathcal{M}}_{g-1, n+2} \rightarrow \overline{\mathcal{M}}_{g,n}$, identifying two new marked points to form a node.
- **Separating**: $\vartheta \colon \overline{\mathcal{M}}_{g_1, S \cup p} \times \overline{\mathcal{M}}_{g_2, S^c \cup p} \rightarrow \overline{\mathcal{M}}_{g,n}$, for a partition $\{1, \dots, n\} = S \sqcup S^c$ and $g_1 + g_2 = g$, gluing at the points p and p' .

1.2 Motivic structures on cohomology

The rational singular cohomology $H^k(\overline{\mathcal{M}}_{g,n}) := H^k(\overline{\mathcal{M}}_{g,n} \times_{\mathbb{C}} \mathbb{A}^1, \mathbb{Q})$ is a finite-dimensional $\mathbb{Q}$ -vector space. Because $\overline{\mathcal{M}}_{g,n}$ is smooth and proper over $\mathbb{Q}$, each $H^k(\overline{\mathcal{M}}_{g,n})$ carries two compatible structures:

1. A **pure Hodge structure** of weight k : a decomposition $H^k \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$ with $\overline{H^{p,q}} = H^{q,p}$.
2. A continuous **ℓ -adic Galois representation**: an action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^k_{\text{ét}}(\overline{\mathcal{M}}_{g,n}, \overline{\mathbb{Q}})$, $\mathbb{Q}_{\ell}$ -adic, pure of weight k .

Both are encoded by saying $H^k(\overline{\mathcal{M}}_{g,n})$ is a **pure motive** of weight k . The symmetric group S_n acts on $\overline{\mathcal{M}}_{g,n}$ by permuting marked points, making $H^k(\overline{\mathcal{M}}_{g,n})$ an S_n -equivariant motive.

1.3 The Tate motive

The **Tate motive** $\mathcal{L} := H^2(\mathbb{P}^1)$ is a 1-dimensional pure motive of weight 2 and Hodge type $(1,1)$. Each Frobenius Frob_p acts on $\mathcal{L}_{\mathbb{Q}_{\ell}}$ by multiplication by p . We say $H^k(\overline{\mathcal{M}}_{g,n})$ is **pure Hodge--Tate** if it is isomorphic to a direct sum of copies of $\mathcal{L}^{\otimes k/2}$ (possible only when k is even).

1.4 Cusp form motives

For each even integer $w \geq 2$, let $S_w(\operatorname{SL}_2(\mathbb{Z}))$ denote the space of cuspidal modular forms of weight w for $\operatorname{SL}_2(\mathbb{Z})$. The ring of modular forms for $\operatorname{SL}_2(\mathbb{Z})$ is $\mathbb{C}[E_4, E_6]$, and the space of cusp forms is the principal ideal $S_w = \Delta \cdot M_{w-12}$, where Δ is the Ramanujan discriminant of weight 12 and M_k is the space of modular forms of weight k . In particular:

$\dim S_w(\operatorname{SL}_2(\mathbb{Z})) = \begin{cases} \lfloor w/12 \rfloor & \text{if } w \equiv 2 \pmod{12}, \\ \lfloor w/12 \rfloor - 1 & \text{if } w \not\equiv 2 \pmod{12} \text{ and } w \geq 12, \\ 0 & \text{if } w < 12. \end{cases}$

(More precisely, $\dim S_w = 0$ for $w < 12$ and $w = 14$; $\dim S_w = 1$ for $w = 12, 16, 18, 20, 22, 26$.)

To each normalized Hecke eigenform $f \in S_w(\operatorname{SL}_2(\mathbb{Z}))$, Deligne [Del71] attaches a 2-dimensional pure motive $\mathcal{S}_w(f)$ of weight $w - 1$ and Hodge type $\{(w-1, 0), (0, w-1)\}$. When $\dim S_w = 1$, we write $\mathcal{S}_w$ for the unique such motive.

Weight w	$\dim S_w$	Motivic weight $w - 1$	Hodge type
12	1	11	$(11, 0) + (0, 11)$
14	0	---	---
16	1	15	$(15, 0) + (0, 15)$
18	1	17	$(17, 0) + (0, 17)$
20	1	19	$(19, 0) + (0, 19)$

The motive $\mathcal{S}_{12}$ has Betti and ℓ -adic realizations equal to $H^{\{1\}}(\overline{\mathcal{M}}_{1,1})$ [CLP23].

1.5 Gysin pushforward

If $\iota: D \hookrightarrow X$ is a codimension- c closed embedding of smooth Deligne--Mumford stacks, the **Gysin pushforward** is:
 $\iota_*: H^k(D) \rightarrow H^{k+2c}(X)$,
 a morphism of pure Hodge structures that shifts weight by $2c$ (i.e., tensors with $\mathcal{L}^c$).

1.6 Specht modules and hook representations

For a partition λ of n , the **Specht module** V_λ is the corresponding irreducible $\mathbb{Q}$ -representation of $\mathfrak{S}_n$. The **hook representation**
 $K_n := V_{n-1, 1}$
 has dimension $\binom{n-1}{m}$ and is generated by elements k_P indexed by ordered subsets $P \subset \{1, \dots, n\}$ of size m , with relations:
 $k_{\sigma(P)} = \operatorname{sign}(\sigma) k_P, \quad \sum_{j=0}^m (-1)^j k_{\{a_0, \dots, \hat{a}_j, \dots, a_m\}} = 0$.

For subgroups $\mathfrak{S}_m \times \mathfrak{S}_{n-m} \subset \mathfrak{S}_n$ and representations U of $\mathfrak{S}_m$, W of $\mathfrak{S}_{n-m}$, the **induced representation** is:
 $\operatorname{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}}^{\mathfrak{S}_n}(U \boxtimes W) := \mathbb{Q}[\mathfrak{S}_n] \otimes_{\mathbb{Q}[\mathfrak{S}_m] \otimes \mathbb{Q}[\mathfrak{S}_{n-m}]} (U \otimes W)$.

2. Known Results: Odd Cohomology of $\overline{\mathcal{M}}_{g,n}$

2.1 Vanishing for $k = 1, 3, 5$

Theorem (Arbarello--Cornalba [AC98]). For all g, n with $2g - 2 + n > 0$:
 $H^k(\overline{\mathcal{M}}_{g,n}) = 0$ for $k = 1, 3, 5$.

Arbarello and Cornalba introduced an inductive method based on excision of the boundary
 $\hookrightarrow$ and pullback to its normalization. They applied it to establish the vanishing for k
 $\hookrightarrow$ in $\{1, 3, 5\}$.

2.2 Vanishing for $k = 7, 9$

Theorem (Bergström--Faber--Payne [BFP22]). For all g, n with $2g - 2 + n > 0$:
 $H^k(\overline{\mathcal{M}}_{g,n}) = 0$ quad $\text{for } k = 7, 9$.

The same inductive method of Arbarello--Cornalba was used, after establishing the
 $\hookrightarrow$ additional base cases needed for the induction via point counting over finite fields.

2.3 Even degree cohomology for $k \leq 12$

Theorem 1.2 (Canning--Larson--Payne [CLP23]). For any even $k \leq 12$,
 $H^k(\overline{\mathcal{M}}_{g,n})$ is pure Hodge--Tate for all g, n .

2.4 Degree $k = 11$

Theorem 1.1 (Canning--Larson--Payne [CLP23]). The cohomology group
 $H^{11}(\overline{\mathcal{M}}_{g,n})$ is nonzero if and only if $g = 1$ and $n \geq 11$.
 $\hookrightarrow$ 11\$.

For $g = 1$ and $n \geq 11$, there is an $\mathfrak{S}_n$ -equivariant isomorphism of
 $\hookrightarrow$ Hodge structures (or ℓ -adic Galois representations):

$$H^{11}(\overline{\mathcal{M}}_{1,n}) \cong K_n^{11} \otimes_{\mathbb{Q}} \mathbb{Q} \otimes \mathfrak{S}_{12},$$

where $K_n^{11} = V_{n-10, 1^{10}}$ is the hook Specht module of dimension

$$\hookrightarrow \binom{n-1}{10}, \text{ and } \mathfrak{S}_{12} \text{ is the motive associated to the Ramanujan } \Delta\text{-function.}$$

The proof uses: (i) Getzler's generating functions [Get98] to compute

$\hookrightarrow H^{11}(\overline{\mathcal{M}}_{1,n})$ and identify the $\mathfrak{S}_n$ -action; (ii)
 $\hookrightarrow$ the Arbarello--Cornalba induction to reduce to the genus 1 case; (iii) the weight
 $\hookrightarrow$ spectral sequence to prove vanishing for $g = 2$.

2.5 Degree $k = 13$

Theorem 1.7 (Canning--Larson--Payne--Willwacher [CLPW]). The cohomology group
 $\hookrightarrow H^{13}(\overline{\mathcal{M}}_{g,n})$ is spanned by Gysin pushforward maps
 $\hookrightarrow \iota_{A^*}$ from boundary divisors of the form $\overline{\mathcal{M}}_{1, A \cup p}$
 $\hookrightarrow \times \overline{\mathcal{M}}_{g-1, A^c \cup p}$, for $A \subset \{1, \dots, n\}$.
 $\hookrightarrow$ If $g \geq 2$, then $\bigoplus_A \iota_{A^*}$ is also injective, and there is an
 $\hookrightarrow$ $\mathfrak{S}_n$ -equivariant isomorphism of Hodge structures:
 $H^{13}(\overline{\mathcal{M}}_{g,n}) \cong \bigoplus_{m=10}^n \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_m} (\mathfrak{S}_m \times \mathfrak{S}_{n-m})$
 $\hookrightarrow \left(\operatorname{Res}^{\mathfrak{S}_{m+1}}_{\mathfrak{S}_m} K^{11}_{m+1} \right) \otimes_{\mathbb{Q}} \mathfrak{S}_L \otimes \mathfrak{S}_{12}.$

In particular, $H^{13}(\overline{\mathcal{M}}_{g,n})$ is a direct sum of copies of

$$\hookrightarrow \mathfrak{S}_L \otimes \mathfrak{S}_{12} := \mathfrak{S}_L \otimes \mathfrak{S}_{12}.$$

For $g = 0$: $H^{13}(\overline{\mathcal{M}}_{0,n}) = 0$. For $g = 1$: the kernel of
 $\hookrightarrow \text{\texttt{\$bigoplus_A \iota_{A*}}}$ and the resulting presentation of
 $H^{13}(\overline{\mathcal{M}}_{1,n})$ are described in [CLPW, Section 4.4 and
 $\hookrightarrow$ Corollary 4.9]. For $n < 10$ and any g : $H^{13}(\overline{\mathcal{M}}_{g,n}) = 0$.
 $\hookrightarrow$ For $g \geq 2$: $\dim H^{13}(\overline{\mathcal{M}}_{g,n}) = 2 \sum_{m=10}^n \binom{n}{m} \binom{n}{m-10}$.
 $\hookrightarrow \text{\texttt{\$binom\{n\}\{m\}\binom\{m\}\{10\}}}$.

2.6 Nonvanishing of higher odd cohomology

Theorem 1.5 (Canning--Larson--Payne [CLP23]). Let $g_1, \dots, g_k$ be distinct
 $\hookrightarrow$ positive integers, and $g = 1 + g_1 + \dots + g_k$. Then:
 $H^{11+2k}(\overline{\mathcal{M}}_{g,n}) \neq 0 \quad \text{for } n \geq 11 - k$.

In particular:

- $H^{13}(\overline{\mathcal{M}}_{g,n}) \neq 0$ for $g \geq 2$ and $n \geq 10$.
- $H^{15}(\overline{\mathcal{M}}_{g,n}) \neq 0$ for $g = 1 + g_1 + g_2$ (with $g_1 \neq$
 $\hookrightarrow g_2$, $g_1, g_2 \geq 1$) and $n \geq 9$.
- $H^{17}(\overline{\mathcal{M}}_{g,n}) \neq 0$ for $g = 1 + g_1 + g_2 + g_3$ (with g_i
 $\hookrightarrow$ distinct, $g_i \geq 1$) and $n \geq 8$.

The proof uses the push-pull formula (self-intersection formula) along codimension- k
 $\hookrightarrow$ gluing maps $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{g_1,1}$
 $\hookrightarrow \dots \times \overline{\mathcal{M}}_{g_k,1} \rightarrow \overline{\mathcal{M}}_{g,11-k}$ to show the Gysin pushforward of $\alpha \otimes 1 \otimes \dots \otimes 1$
 $\hookrightarrow$ is nonzero, where $\alpha \in H^{11,0}(\overline{\mathcal{M}}_{1,11})$.

2.7 Tate type for open moduli spaces

Theorem 1.5 (Canning--Larson--Payne--Willwacher [CLPW]). The rational cohomology of
 $\hookrightarrow \mathcal{M}_{g,n}$ is of Tate type if and only if $g = 0$ or $3g + 2n < 25$.

2.8 Weight 13 Euler characteristics

Corollary 1.8 ([CLPW]). The weight 13 Euler characteristics of $\mathcal{M}_{12}$ and
 $\hookrightarrow \mathcal{M}_{8,1}$ are:
 $\chi_{13}(\mathcal{M}_{12}) = -6$, $\chi_{13}(\mathcal{M}_{8,1}) = -2$.

These nonzero values are used to show non-polynomiality of the point counts $N_{g,n}(q)$
 $\hookrightarrow$ in cases where the weight 11 Euler characteristic vanishes.

3. Summary

Degree k (odd)	$H^k(\overline{\mathcal{M}}_{g,n})$	Status	Reference
1	$= 0$ for all g, n	Proved	[AC98]
3	$= 0$ for all g, n	Proved	[AC98]
5	$= 0$ for all g, n	Proved	[AC98]
7	$= 0$ for all g, n	Proved	[BFP22]
9	$= 0$ for all g, n	Proved	[BFP22]
11	$\neq 0$ iff $g = 1, n \geq 11$; all copies of $\mathcal{S}_{12}$	Fully computed	[CLP23]
13	All copies of $\mathcal{L} \otimes \mathcal{S}_{12}$; explicit $\mathcal{S}_n$ -structure	Fully computed	[CLPW]
15	Nonzero for some (g,n) with $g \geq 3$	Partial (nonvanishing only)	[CLP23, Thm 1.5]
17	Nonzero for some (g,n) with $g \geq 4$	Open	[CLP23, Thm 1.5]

4. The Problem

****Problem.**** Determine the cohomology group $H^{15}(\overline{\mathcal{M}}_{g,n})$ for
 $\hookrightarrow$ all $g \geq 0$ and $n \geq 0$ (with $2g - 2 + n > 0$), as an
 $\hookrightarrow \mathbb{Q}$ -equivariant pure motive of weight 15.

More precisely:

(A) For which (g, n) is $H^{15}(\overline{\mathcal{M}}_{g,n})$ nonzero?

(B) Give an explicit decomposition of $H^{15}(\overline{\mathcal{M}}_{g,n})$ into
 $\hookrightarrow$ simple motivic pieces, together with their $\mathbb{Q}$ -equivariant multiplicity
 $\hookrightarrow$ spaces.

Note that H^{15} is the first odd-degree cohomology group of
 $\hookrightarrow \overline{\mathcal{M}}_{g,n}$ for which neither a full computation nor a complete
 $\hookrightarrow$ vanishing result is known, and for which multiple independent types of motives (of
 $\hookrightarrow$ different Hodge types) are expected to contribute simultaneously.

The computation of H^{15} depends on the computation of
 $\hookrightarrow H^{13}(\overline{\mathcal{M}}_{g,n})$, because Gysin pushforwards from codimension-1
 $\hookrightarrow$ boundary divisors map H^{13} into H^{15} .

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