

THE COLOURED SAND THEOREM

*The p -part of the sandpile group of the Hamming graph,
for every prime p dividing q and every dimension d*

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Abstract

Let $H(d, q) = K_q^d$ be the Hamming graph and $K(H(d, q))$ its sandpile group (critical group, Jacobian). For primes $p \nmid q$ its p -part was determined by Jacobson, Niedermaier and Reiner in 2003, who wrote that the other primes appear “to be difficult, due to bad behavior of its p -primary structure”. **We determine the Sylow p -subgroup of $K(H(d, q))$ for every prime p dividing q and every d , by an explicit rule.** After the splitting of Jacobson, Niedermaier and Reiner, the group is a sum of cokernels of operators $q(c + H) + U$ on the Boolean lattice (Theorem A). These split over the indecomposable tilting modules $T(\lambda)$ of SL_2 in characteristic p that occur in a tensor power of the natural module. The contribution of $T(\lambda)$ consists of explicit small cyclic factors and of $2^{|I|}$ large ones, where I is the set of positions of the nonzero base- p digits of $\lambda + 1$ below the leading one. The large ones are obtained by pooling explicit integers, built from binomial valuations and base- p carries, into a non-increasing sequence (Theorems I and J). The proof writes down explicit $\mathbb{Z}_p$ -lattices for all tilting modules of SL_2 , one move per base- p digit; it reduces each contribution to the Smith form of an explicit $2^{|I|} \times 2^{|I|}$ matrix, and computes that form as in our proof for the hypercube [DST], the case $p = 2$. We also give closed formulas that need no pooling: for $p > d$ with ballot numbers (B), for the exponent with every prime (C), and for the number of cyclic factors (D). This proves Conjecture I of version 3. **Every result of the paper, Theorems A–J, is checked in Lean 4 with Mathlib, for every prime, and the Lean project was rebuilt from its sources on GitHub’s machines, on Mathlib’s published build (§12).** This work has not been refereed by a human expert.

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Summary of results

Throughout, p is a prime, $q \geq 2$, $e = v_p(q)$, and $\text{Syl}_p A$ is the p -primary part of a finite abelian group A .

Result	What it says	Where	Lean
Theorem A	$\text{coker } L(H(d, q))$ is a sum of pieces $X_q(m, c) = \text{coker}(q(c + H) + U)$ (Jacobson–Niedermaier–Reiner)	§1, §2	checked
Theorem B	$\text{Syl}_p K(H(d, q))$ for $p > d$, $p \mid q$, with ballot numbers	§3	checked
Theorem C	the exponent of $\text{Syl}_p K(H(d, q))$, every $p, q \geq 3$	§4	checked
Theorem D	the number of cyclic factors, $p \mid q$	§5	checked
Theorem E	every piece of size $m < p$	§3	checked
Theorems F, G	every piece with $m \leq p^2 + p - 1$ (two Weyl factors)	§6	checked
Theorem H	$p = 2$, every d , from [DST]	§7	checked
Theorem I	on each tilting block, exactly $2^{ I }$ cyclic summands of order $> p^k$; the others form an explicit group independent of q and of the shift (Conjecture I of version 3)	§8, §9	checked
Theorem J	the whole $\text{Syl}_p K(H(d, q))$, every $p \mid q$, every d , in closed form	§8, §9	checked

Status. This is a preprint. For $d = 2$ the whole group $K(H(2, q))$ was known before (Ducey, Gerhard and Watson [DGW16]); our results are new, to the best of our knowledge, for $d \geq 3$ (see the section on provenance at the end). The proofs of Theorems A–H were re-derived by an independent reader with no access to our audits; that reading found one priority omission, three citation and presentation gaps and one wrong example in a remark, all corrected here (§13). The proofs of Theorems I and J were written by one reader, re-derived by a second and read cold by a third, who wrote the riskiest step, the floor of the Smith form for odd p , out in full; it found no error and no gap. The Lean certificate and version 5 were read cold by a fourth reader, who found three claims of version 5 wider than the Lean proof of that day: the cases Theorem E for $p \nmid q$ and Theorems F, G and I for $p = 2$ are now checked, and §12 now says on which lattices Theorem I and the blocks of Theorem J are proved. The changes of version 6 were read cold by a fifth reader, who found no error in the mathematics or in the Lean statements; its corrections of wording are applied (§13). All these readers are instances of the AI system Claude; no human expert has read the proofs. Every result of the paper, Theorems A–J, is checked in Lean 4 with Mathlib for every prime (§12).

1. Introduction

For a connected graph Γ with Laplacian $L(\Gamma)$, the sandpile group $K(\Gamma)$ is the torsion subgroup of $\text{coker } L(\Gamma)$, and $\text{coker } L(\Gamma) \cong \mathbb{Z} \oplus K(\Gamma)$; its order is the number of spanning trees [Big99]. The Hamming graph $H(d, q)$ has vertex set $(\mathbb{Z}/q)^d$, two vertices being adjacent when they differ in exactly one coordinate; it is the Cartesian power K_q^d of the complete graph. Its Laplacian eigenvalues are qs , $0 \leq s \leq d$, with multiplicity $\binom{d}{s}(q-1)^s$.

Jacobson, Niedermaier and Reiner [JNR03] determined the Sylow p -subgroup of $K(K_{n_1} \times \cdots \times K_{n_k})$ for every prime p that divides none of the n_i [JNR03, Theorem 2], and bounded the number of invariant factors [JNR03, Theorem 3]. About the other primes they wrote: “Describing the exact structure of the critical group $K(K_{n_1} \times \cdots \times K_{n_k})$ appears to be difficult, due to bad behavior of its p -primary structure for primes p that divide some of $n_1, \dots, n_k$.” Ducey and Jalil [DJ14] recovered the results of [JNR03] by another method. For $d = 2$ the whole group $K(H(2, q)) = K(K_q \times K_q)$ was determined by Ducey, Gerhard and Watson [DGW16, Theorem 1.1]; it also follows from [JNR03, Theorem 13 and Example 11]. Our results agree with it at $d = 2$, and are new, to the best of our knowledge, for $d \geq 3$. For $q = 2$ the graph is the hypercube; its odd part is due to Bai [Bai03] and its 2-part was determined in [DST].

In this paper $q \geq 2$ is arbitrary, p is a prime dividing q , and $e := v_p(q) \geq 1$, except where we say otherwise.

1.1 The pieces

Let B_m be the free abelian group on the subsets of $[m] = \{1, \dots, m\}$, graded by size: $B_m = \bigoplus_{s=0}^m B_m(s)$. Let H act on $B_m(s)$ as the scalar s , and let U and D be the up and down operators,

$$U(S) = \sum_{x \notin S} S \cup \{x\}, \quad D(S) = \sum_{x \in S} S \setminus \{x\}.$$

For integers $c \geq 0$ put

$$X_q(m, c) := \text{coker}(q(c + H) + U \text{ on } B_m).$$

It is finite when $c \geq 1$; for $c = 0$ it is $\mathbb{Z}$ plus a finite group.

Theorem A (Jacobson–Niedermaier–Reiner [JNR03, Theorem 13]). For every $q \geq 2$ and $d \geq 1$, as abelian groups,

$$\text{coker } L(H(d, q)) \cong \bigoplus_{j=0}^d X_q(d - j, j)^{\oplus \binom{d}{j} (q-2)^j}.$$

This is the case $n_1 = \dots = n_d = q$ of [JNR03, Theorem 13]: there the pieces are upper triangular matrices $M(N; n_S)$ of size $2^{|S|}$ with identity blocks above the diagonal, and after reversing the order of the subsets of S they are the matrices $q(c + H) + U$. We include the short proof below, in the form we use. For $q = 2$ only $j = 0$ survives and the theorem says $\text{coker } L(Q_d) \cong X_2(d, 0)$, the starting point of [DST].

Theorem B (the primes $p > d$). Let $d \geq 1$, let $p > d$ be a prime dividing q , and $e = v_p(q)$. Then

$$\text{Syl}_p K(H(d, q)) \cong \bigoplus_{v=1}^d (\mathbb{Z}/p^{e v})^{n_v}, \quad n_v = \sum_{\substack{j+2k=d+1-v \\ (j,k) \neq (0,0)}} \binom{d}{j} (q-2)^j b(k, v),$$

the sum over integers $j, k \geq 0$ (the condition $(j, k) \neq (0, 0)$ is automatic, since $j + 2k = d + 1 - v \geq 1$), where $b(k, v) = \frac{v}{v+2k} \binom{v+2k}{k} = [x^k] C(x)^v$ are the ballot numbers and $C(x) = \sum_n \frac{1}{n+1} \binom{2n}{n} x^n$ is the Catalan series.

Equivalently, with $c(z) = C(z^2)$ and $u = q - 2$, $\sum_v n_v y^v = [z^{d+1}] (1 + uz)^d \frac{yz c(z)}{1 - yz c(z)} - y^{d+1}$. For instance, with $u = q - 2$:

$$\begin{aligned} d = 2 : & \quad (\mathbb{Z}/p^e)^{u^2+1} \oplus (\mathbb{Z}/p^{2e})^{2u}, \\ d = 3 : & \quad (\mathbb{Z}/p^e)^{u^3+3u} \oplus (\mathbb{Z}/p^{2e})^{3u^2+2} \oplus (\mathbb{Z}/p^{3e})^{3u}, \\ d = 4 : & \quad (\mathbb{Z}/p^e)^{u^4+6u^2+2} \oplus (\mathbb{Z}/p^{2e})^{4u^3+8u} \oplus (\mathbb{Z}/p^{3e})^{6u^2+3} \oplus (\mathbb{Z}/p^{4e})^{4u}. \end{aligned}$$

Theorem B is the case $m < p$ of a statement about single pieces (Theorem E in Section 3), which also gives, for every prime p , the contribution of all pieces with $d - j < p$.

Theorem C (the exponent). Let $q \geq 3$, let p be any prime and $e = v_p(q)$. The exponent of $\text{Syl}_p K(H(d, q))$ is p^E , where

$$E = \max_{1 \leq s \leq d} (e s + v_p(s)) \quad \text{if } e \geq 1, \quad E = \lfloor \log_p d \rfloor \quad \text{if } e = 0.$$

For $p > d$ this gives $E = ed$, as in Theorem B. For $p \leq d$ the maximum need not be at $s = d$: for $q = 3$ and $d = 10$ it is at $s = 9$ and $E = 11$. The case $e = 0$ ($p \nmid q$) can also be read off from [JNR03, Theorem 2]; we include it because the proof below covers it at no cost.

Theorem D (the number of cyclic factors). Let $p \mid q$. The number of cyclic factors of $\text{Syl}_p K(H(d, q))$ is

$$N = \sum_{j=0}^d \binom{d}{j} (q-2)^j (2^{d-j} - \rho_p(d-j)) - 1, \quad \rho_p(m) = \sum_{s=0}^{m-1} r_p(m, s),$$

where $r_p(m, s) = \sum_i \left(\binom{m}{i} - \binom{m}{i-1} \right)$, the sum over $0 \leq i \leq t$ with $p \nmid k - i$, and $(t, k) = (s, s + 1)$ if $2s \leq m - 1$, $(t, k) = (m - 1 - s, m - s)$ otherwise.

Here $\rho_p(m)$ is the rank over $\mathbb{F}_p$ of U on B_m . For $q = p = 3$ the theorem gives $N = 1, 4, 12, 34, 96, 274, \dots$, and $N = 3^{d-1} + 2^{d-1} - 1$ for every d (Corollary 6). For $q = p$ the value of N equals the bound $\frac{1}{2}(q^d + (q - 2)^d) - 1$ of [JNR03, Theorem 3] for $d \leq 2$, is one less for $d = 3$, and is strictly smaller for $4 \leq d \leq 8$ and $p \leq 11$ (where we compared).

Corollary 1. $\text{Syl}_p K(H(d, q))$ depends on q only through $e = v_p(q)$ and the multiplicities $\binom{d}{j}(q - 2)^j$.

1.2 The rest of the paper

Theorems E–H (Sections 3, 6 and 7) compute the pieces $X_q(m, c)$ for $m < p$, for $m \leq p^2 + p - 1$ and for $p = 2$. Section 8 states Theorems I and J, which compute every piece for every odd p , and Section 9 proves them. Section 10 lists the computations, Section 12 the Lean proof, and Section 13 the changes from versions 3 and 4.

2. The splitting (after Jacobson, Niedermaier and Reiner)

Proof of Theorem A. Let $K_q = qI - J$ be the Laplacian of the complete graph K_q , acting on $\mathbb{Z}^q$ with basis $e_0, \dots, e_{q-1}$, and put $f_i = e_i - e_0$ ($1 \leq i \leq q - 1$) and $\sigma = f_1 + \dots + f_{q-1}$. Then $e_0, \sigma, f_1, \dots, f_{q-2}$ is a basis of $\mathbb{Z}^q$ (unimodular change from $e_0, f_1, \dots, f_{q-1}$), and

$$K_q e_0 = -\sigma, \quad K_q \sigma = q\sigma, \quad K_q f_i = qf_i.$$

So $\mathbb{Z}^q = A \oplus W$ with $A = \mathbb{Z}e_0 \oplus \mathbb{Z}\sigma$ and $W = \bigoplus_{i \leq q-2} \mathbb{Z}f_i$, both stable under K_q , and K_q is the scalar q on W . The Laplacian of $H(d, q)$ is $L = \sum_{i=1}^d 1 \otimes \dots \otimes K_q \otimes \dots \otimes 1$ on $(\mathbb{Z}^q)^{\otimes d} = \bigoplus_{T \subseteq [d]} A^{\otimes(|d| \setminus |T|)} \otimes W^{\otimes |T|}$, and each summand is L -stable. On the summand of T with $|T| = j$, L acts as $(L_A + qj) \otimes 1$, where $L_A = \sum_i K_q^{(i)}$ on $A^{\otimes(d-j)}$ and $W^{\otimes j} \cong \mathbb{Z}^{(q-2)^j}$. There are $\binom{d}{j}$ sets T of size j .

Index the basis of $A^{\otimes m}$ by the subsets $S \subseteq [m]$, the factor in position i being σ if $i \in S$ and e_0 otherwise. Then $L_A(S) = q|S|S - \sum_{x \notin S} S \cup \{x\}$, that is, $L_A = qH - U$, and $L_A + qj = q(j + H) - U$. Conjugating by the diagonal matrix $(-1)^{|S|}$ turns $-U$ into U . Hence the summand contributes $\text{coker}(q(j + H) + U \text{ on } B_{d-j}) = X_q(d - j, j)$, $\binom{d}{j}(q - 2)^j$ times. ■

Proof of Corollary 1. By Theorem A it suffices to see that $\text{Syl}_p X_q(m, c)$ depends only on the p -adic valuations of the diagonal entries $a_s = q(c + s)$. This is Lemma 2. ■

Lemma 2 (only the valuations matter). Let $a_0, \dots, a_m$ be elements of $\mathbb{Z}_{(p)}$ and $M = \text{diag}(a_{|S|}) + U$ on $B_m \otimes \mathbb{Z}_{(p)}$. If $a_s = p^{w_s} u_s$ with units u_s (or $a_s = 0$, with $u_s = 1$), then M is equivalent over $\mathbb{Z}_{(p)}$ to

$$\text{diag}(p^{w_{|S|}}) + U \quad (\text{with } p^w := 0 \text{ when } a_s = 0).$$

Proof. Look for units λ_s, μ_s such that $P = \text{diag}(\lambda_{|S|})$ and $Q = \text{diag}(\mu_{|S|})$ give $PMQ = \text{diag}(p^{w_{|S|}}) + U$. The diagonal of PMQ on rank s is $\lambda_s a_s \mu_s$ and its block from rank s to rank $s + 1$ is $\lambda_{s+1} \mu_s U$, so we need $\lambda_s \mu_s u_s = 1$ and $\lambda_{s+1} \mu_s = 1$. Put $\mu_0 = 1, \lambda_0 = u_0^{-1}$, and recursively $\lambda_{s+1} = \mu_s^{-1}, \mu_{s+1} = (\lambda_{s+1} u_{s+1})^{-1}$. ■

3. Strings, and the primes $p > m$

On B_m one has $[D, U] = m - 2H$: on a set S of size s , $DU(S) = (m - s)S + \Sigma$ and $UD(S) = sS + \Sigma$ with the same sum $\Sigma = \sum_{y \in S, x \notin S} (S \setminus \{y\}) \cup \{x\}$. For $0 \leq k \leq m/2$ let $\Lambda_k = \ker D \cap B_m(k)$. By induction on i , for $v \in \Lambda_k$,

$$D U^i v = i(m - 2k - i + 1) U^{i-1} v. \quad (1)$$

Over $\mathbb{Q}$ this is the classical $\mathfrak{sl}_2$ -structure of the Boolean lattice [Pro82]:

$$B_m \otimes \mathbb{Q} = \bigoplus_{k \leq m/2} \bigoplus_{i=0}^{m-2k} U^i \Lambda_k^{\mathbb{Q}},$$

$\dim \Lambda_k^{\mathbb{Q}} = \binom{m}{k} - \binom{m}{k-1}$, and $U^{m-2k+1} \Lambda_k = 0$. We call $\Lambda_k, U\Lambda_k, \dots, U^{m-2k} \Lambda_k$ a *string*; its ranks are $k, \dots, m - k$ and its length is $m - 2k + 1$.

Lemma 3 (strings over $\mathbb{Z}_{(p)}$). If $p > m$, then $B_m \otimes \mathbb{Z}_{(p)} = \bigoplus_{k \leq m/2} \bigoplus_{i=0}^{m-2k} U^i \Lambda_k$, each Λ_k is free of rank $\binom{m}{k} - \binom{m}{k-1}$, and $U^i : \Lambda_k \rightarrow U^i \Lambda_k$ is an isomorphism for $i \leq m - 2k$.

Proof. All lattices below are over $R = \mathbb{Z}_{(p)}$. First, for $r \leq m/2$ we show $B_m(r) = UB_m(r-1) \oplus \Lambda_r$, by induction on r (for $r = 0$, $\Lambda_0 = B_m(0)$). By induction $B_m(r-1) = \bigoplus_{k < r} U^{r-1-k} \Lambda_k$, and by (1) the operator DU acts on $U^{r-1-k} \Lambda_k$ as the scalar $(r-k)(m-k-r+1)$; both factors lie in $[1, m]$, so they are units. Hence DU is invertible on $B_m(r-1)$, $\pi = U(DU)^{-1}D$ is an idempotent on $B_m(r)$ with image $UB_m(r-1)$, and its kernel is $\ker D \cap B_m(r) = \Lambda_r$ because $U(DU)^{-1}$ is injective. This gives the decomposition, and Λ_r is a direct summand, so free, of rank $\binom{m}{r} - \binom{m}{r-1}$.

Now fix any rank r and consider $\Phi : \bigoplus_{k \leq \min(r, m-r)} \Lambda_k \rightarrow B_m(r)$, $(v_k) \mapsto \sum_k U^{r-k} v_k$. For $k' \leq r$ let $\psi_{k'} : B_m(r) \rightarrow \Lambda_{k'}$ be $D^{r-k'}$ followed by the projection of $B_m(k') = UB_m(k'-1) \oplus \Lambda_{k'}$ onto $\Lambda_{k'}$. By (1), $D^{r-k'} U^{r-k} v_k$ is 0 if $k > k'$ (since $Dv_k = 0$), lies in $U^{k'-k} \Lambda_k \subseteq UB_m(k'-1)$ if $k < k'$, and equals $\prod_{i=1}^{r-k} i(m-2k-i+1) v_k$ if $k = k'$, a unit multiple of v_k (all factors lie in $[1, m]$ because $k \leq m-r$). So $(\psi_{k'}) \circ \Phi$ is triangular with unit diagonal, Φ is split injective, and its image is a direct summand of $B_m(r)$ of full rank (compare ranks over $\mathbb{Q}$). Hence Φ is an isomorphism. Finally $U^{m-2k+1} \Lambda_k = 0$ holds over $\mathbb{Q}$, hence over R . ■

Each string is stable under $q(c+H) + U$: in the basis $v, Uv, \dots, U^{\ell-1}v$ ($\ell = m - 2k + 1$) the operator is lower bidiagonal with diagonal entries $q(c+k+i)$, $0 \leq i < \ell$, and all subdiagonal entries 1. The cokernel of such a matrix over a principal ideal domain is cyclic of order its determinant, because the minor obtained by deleting the first row and last column equals 1; if one diagonal entry is 0 (only possible for $c = k = i = 0$) the cokernel is $\mathbb{Z}_{(p)}$. This proves:

Theorem E (pieces with $m < p$). Let p be any prime, $e = v_p(q)$, and $m < p$. Then

$$\text{Syl}_p X_q(m, c) \cong \bigoplus_{\substack{0 \leq k \leq m/2 \\ (c, k) \neq (0, 0)}} (\mathbb{Z}/p^{A_k})^{\binom{m}{k} - \binom{m}{k-1}}, \quad A_k = \sum_{r=k}^{m-k} (e + v_p(c+r)),$$

and for $c = 0$ the string $k = 0$ contributes only the free summand $\mathbb{Z}$.

Proof of Theorem B. If $p > d$, every piece $X_q(d-j, j)$ has $m = d-j < p$ and $c+r \leq d < p$, so $v_p(c+r) = 0$ for $c+r \geq 1$ and $A_k = e(m-2k+1)$. Put $v = m-2k+1 = d+1-j-2k$; then $\binom{m}{k} - \binom{m}{k-1} = \binom{v+2k-1}{k} - \binom{v+2k-1}{k-1} = \frac{v}{v+2k} \binom{v+2k}{k} = b(k, v)$. Summing over j with Theorem A gives the formula. The generating function follows from $\sum_k b(k, v) x^k = C(x)^v$. ■

4. The exponent

Lemma 4. Let $c \geq 1$ and $M = q(c + H) + U$ on B_m . Then M is invertible over $\mathbb{Q}$, and for $S \subseteq T$ with $|S| = s$, $|T| = s + k$,

$$(M^{-1})_{T,S} = (-1)^k k! \prod_{i=0}^k \frac{1}{q(c + s + i)},$$

while $(M^{-1})_{T,S} = 0$ if $S \not\subseteq T$.

Proof. With $\Delta = q(c + H)$, $M^{-1} = (1 + \Delta^{-1}U)^{-1}\Delta^{-1} = \sum_{k \geq 0} (-\Delta^{-1}U)^k \Delta^{-1}$, a finite sum. A path from S to T in $(\Delta^{-1}U)^k \Delta^{-1}$ is a maximal chain $S = S_0 \subset \dots \subset S_k = T$, there are $k!$ of them, and each collects Δ^{-1} once at every rank from s to $s + k$. ■

For a nonsingular square matrix over a discrete valuation ring with uniformizer p , the exponent of the cokernel is p^ϵ with $-\epsilon$ the least valuation of an entry of M^{-1} (write M in Smith form). Hence, for $c \geq 1$, the exponent of $\text{Syl}_p X_q(m, c)$ is p to the power

$$\max_{\substack{s, k \geq 0 \\ s+k \leq m}} \left(\sum_{i=0}^k (e + v_p(c + s + i)) - v_p(k!) \right) = \max_{c \leq a \leq b \leq c+m} \left(eL + v_p(L) + v_p\binom{b}{L} \right), \quad L = b - a + 1,$$

where $a = c + s$, $b = c + s + k$, and we used $\frac{b!}{(a-1)!(b-a)!} = L\binom{b}{L}$.

Lemma 5 (carries). For $d \geq 1$ and $e \geq 1$,

$$\max_{1 \leq L \leq b \leq d} \left(eL + v_p(L) + v_p\binom{b}{L} \right) = \max_{1 \leq s \leq d} (es + v_p(s)),$$

and for $e = 0$ the left side equals $\lfloor \log_p d \rfloor$.

Proof. Taking $L = b = s$ gives $\geq$. Let $\lambda = \lfloor \log_p b \rfloor$. By Kummer's theorem $v_p\binom{b}{L}$ is the number of carries in the addition $L + (b - L)$ in base p . No carry occurs at a position below $v_p(L)$, where the digits of L vanish, and a carry out of position i forces $b \geq p^{i+1}$, so $i \leq \lambda - 1$. Hence $v_p(L) + v_p\binom{b}{L} \leq \lambda$. For $e = 0$ this gives the bound $\lfloor \log_p d \rfloor$, attained at $L = b = p^{\lfloor \log_p d \rfloor}$.

Let $e \geq 1$; we induct on b , the case $b = 1$ being clear. Let $s^* = p^\lambda \lfloor b/p^\lambda \rfloor$, so $1 \leq s^* \leq b < s^* + p^\lambda$ and $v_p(s^*) \geq \lambda$. If $L \leq s^*$, then $eL + v_p(L) + v_p\binom{b}{L} \leq es^* + \lambda \leq es^* + v_p(s^*)$. If $L > s^*$, write $b = s^* + r$ and $L = s^* + r'$ with $1 \leq r' \leq r < p^\lambda$. Then $v_p(L) = v_p(r')$, and since $(b - L) + r' = r < p^\lambda$, the additions $L + (b - L)$ and $r' + (r - r')$ have the same carries, so $v_p\binom{b}{L} = v_p\binom{r}{r'}$. By induction there is $1 \leq t \leq r$ with $er' + v_p(r') + v_p\binom{r}{r'} \leq et + v_p(t)$. Since $t < p^\lambda \leq p^{v_p(s^*)}$, $v_p(s^* + t) = v_p(t)$, and so $eL + v_p(L) + v_p\binom{b}{L} \leq e(s^* + t) + v_p(s^* + t)$ with $s^* + t \leq b$. ■

Proof of Theorem C. By Theorem A the exponent is the largest exponent of the pieces $\text{Syl}_p X_q(d - j, j)$ that occur. For $j \geq 1$ the piece occurs ($q \geq 3$) and the computation above gives the maximum over $j \leq a \leq b \leq d$; for $j = 1$ this is the full range $1 \leq a \leq b \leq d$. For $j = 0$, the row of the empty set in $M_0 = qH + U$ is zero and M_0 maps into $B' = \bigoplus_{s \geq 1} B_d(s)$, so

$$\text{coker } M_0 = \mathbb{Z} \oplus B' / (M''B' + \mathbb{Z}u),$$

where M'' is the restriction of M_0 to B' (nonsingular, lower triangular by rank with diagonal qs) and $u = M_0(\emptyset)$. Its torsion is a quotient of $\text{coker } M''$, whose exponent is given by Lemma 4 (with $c = 0$ and $s \geq 1$) as the same maximum over $1 \leq a \leq b \leq d$. So the exponent is p to $\max_{1 \leq a \leq b \leq d} (eL + v_p(L) + v_p\binom{b}{L})$, and Lemma 5 finishes. ■

For $q = 2$ the same argument gives only an upper bound, and it is not attained: for Q_2 the group is $\mathbb{Z}/4$ while the bound is 2^3 .

5. The number of cyclic factors

Proof of Theorem D. The number of cyclic factors of $\text{Syl}_p K$ is $\dim_{\mathbb{F}_p}(\text{coker } L \otimes \mathbb{F}_p) - 1$, the 1 for the free summand. Modulo p , $q(j+H) + U \equiv U$, so by Theorem A $\dim_{\mathbb{F}_p}(\text{coker } L \otimes \mathbb{F}_p) = \sum_j \binom{d}{j} (q-2)^j (2^{d-j} - \text{rank}_{\mathbb{F}_p} U|_{B_{d-j}})$. The rank of U is the sum over s of the ranks of the inclusion matrices $W_{s,s+1}$ of s -subsets versus $(s+1)$ -subsets of $[m]$. By Wilson's diagonal form [Wil90, Theorem 2], for $t \leq \min(k, m-k)$ the matrix $W_{t,k}$ is equivalent over $\mathbb{Z}$ to a diagonal matrix with entries $\binom{k-i}{t-i}$ of multiplicity $\binom{m}{i} - \binom{m}{i-1}$, $0 \leq i \leq t$ (the summary in [CSX15, (2)] states only the range $t \leq k \leq m/2$, which misses the middle rank $2s = m-1$ of odd m ; Wilson's range covers it); so its rank over $\mathbb{F}_p$ is $\sum_{i: p \nmid \binom{k-i}{t-i}} (\binom{m}{i} - \binom{m}{i-1})$. For $W_{s,s+1}$ with $2s \leq m-1$ take $(t, k) = (s, s+1)$, so $\binom{k-i}{t-i} = s+1-i$; otherwise complementation identifies $W_{s,s+1}$ with the transpose of $W_{m-s-1, m-s}$ and we take $(t, k) = (m-1-s, m-s)$. This is $r_p(m, s)$. $\blacksquare$

Corollary 6 (the number of factors when $3 \mid q$). If $3 \mid q$, the number of cyclic factors of $\text{Syl}_3 K(H(d, q))$ is

$$N = (q-1)^d + \frac{q^d - (q-1)^d}{3} - \frac{(q-1)^d - (q-3)^d}{6} - 1.$$

In particular $N = 3^{d-1} + 2^{d-1} - 1$ for $q = 3$.

Proof. By Theorem D it suffices to know $f(m) = \dim_{\mathbb{F}_3} \text{coker}(U|_{B_m \otimes \mathbb{F}_3})$. Identify $B_m \otimes \mathbb{F}_3$ with $V^{\otimes m}$, $V = \mathbb{F}_3^2$, so that $U = \sum_i e_i$ with e the nilpotent 2×2 Jordan block acting on the i -th factor. The e_i commute, so $U^3 = \sum_i e_i^3 = 0$, and $V^{\otimes m}$ is a module over the Hopf algebra $\mathbb{F}_3[e]/(e^3)$ with e primitive. Then $f(m)$ is the number of Jordan blocks of U . Write J_a for the indecomposable module of dimension $a \leq 3$. Then:

- $J_1 \otimes J_2 = J_2$;
- $J_2 \otimes J_2 = J_3 \oplus J_1$, since $e^2(v_0 \otimes v_0) = 2v_1 \otimes v_1 \neq 0$;
- $J_3 \otimes J_2 = J_3^2$, because J_3 is free, a projective tensored with any module is projective, and projectives over this local algebra are free.

So $V^{\otimes m} = a_m J_1 \oplus b_m J_2 \oplus c_m J_3$ with $(a_{m+1}, b_{m+1}, c_{m+1}) = (b_m, a_m, b_m + 2c_m)$, starting from $(1, 0, 0)$. Hence $a_m + b_m = 1$, $b_m = [m \text{ odd}]$, and counting dimensions, $c_m = (2^m - 1 - [m \text{ odd}])/3$. This gives $f(m) = 1 + (2^m - 1 - [m \text{ odd}])/3$. Summing $\sum_m \binom{d}{m} (q-2)^{d-m} f(m)$ gives the formula. $\blacksquare$

6. The range $p \leq d$: tilting lattices

By Theorem E the only pieces not yet described are $X_q(m, c)$ with $m \geq p$. For $p \leq m$ the strings of length $> p$ break: by (1) the coefficient $i(m-2k-i+1)$ vanishes modulo p . The way they break is governed by the representation theory of SL_2 in characteristic p .

6.1 The dictionary

Let $V = \mathbb{Z}_p v_0 \oplus \mathbb{Z}_p v_1$, where $\mathbb{Z}_p$ is the ring of p -adic integers, and let f act on V by $f v_0 = v_1$, $f v_1 = 0$; give v_0 weight 1 and v_1 weight -1 . Identify $B_m \otimes \mathbb{Z}_p$ with $V^{\otimes m}$ by $S \mapsto v_{1_S}$. Then U is the action of f on the tensor power, a subset of size s has weight $m-2s$, and so $H = (m-h)/2$ with h the weight. The divided powers $U^{(r)}$ (add r new elements, in all ways) are integral, and so is the analogous operator $e^{(r)}$ (remove r elements). Hence $B_m \otimes \mathbb{Z}_p$ is a module over the distribution algebra $\text{Dist}(SL_2)_{\mathbb{Z}_p}$, and

$$q(c+H) + U = q\left(c + \frac{m-h}{2}\right) + f$$

is built from that action alone.

We use three standard facts.

(T1) $V^{\otimes m}$ is a tilting module, and it decomposes over $\mathbb{Z}_p$ as $\bigoplus_{\lambda} T_{\mathbb{Z}_p}(\lambda)^{n_{\lambda}}$, where $T_{\mathbb{Z}_p}(\lambda)$ is the unique lift of the indecomposable tilting module $T(\lambda)$ over $\mathbb{F}_p$. The reason is that $\text{Ext}^1(\Delta, \nabla) = 0$, so decompositions over $\mathbb{F}_p$ lift; this is classical [And97, §1.2], [Jan03, II.E.6, II.E.19–E.22]. For $p = 2$ the argument is written out, and checked in Lean, in [DST, Lemma 3.3, Corollary 3.4].

(T2) Let $\lambda \leq p^2 + p - 1$. If $\lambda \leq p - 1$, or if $\lambda + 1 \equiv 0 \pmod{p}$ and $\lambda \leq p^2 - 1$, then $T(\lambda) = \Delta(\lambda)$ is simple. Otherwise write

$$\lambda = p - 1 + r + pa, \quad 1 \leq r \leq p, \quad 0 \leq a \leq p - 1,$$

where $r = p$ (and then $a = p - 1$) occurs only for $\lambda = p^2 + p - 1$. Then $T(\lambda)$ has exactly two Weyl factors, $\Delta(\lambda)$ and $\Delta(\lambda^-)$ with $\lambda^- := \lambda - 2r$, and it is a non-split extension $0 \rightarrow \Delta(\lambda) \rightarrow T(\lambda) \rightarrow \Delta(\lambda^-) \rightarrow 0$; so is its lift over $\mathbb{Z}_p$. Indeed, by Donkin's tensor product theorem [Don93, Prop. 2.1] (for SL_2 in every characteristic see [STWZ23, Prop. 3.4(a)] and [And97, Remark 1.1(ii)]), $T(p - 1 + r' + pb) \cong T(p - 1 + r') \otimes T(b)^{[1]}$ for $0 \leq r' \leq p - 1$. For $r \leq p - 1$ take $r' = r$, $b = a \leq p - 1$, so that $T(a) = \Delta(a)$. For $\lambda = p^2 + p - 1$ take $r' = 0$, $b = p$: then $T(p - 1)$ is the Steinberg module and $T(p)$ has the Weyl factors $\Delta(p)$, $\Delta(p - 2)$. In both cases the product of characters, expanded by Brauer's formula with $\chi(-n - 2) = -\chi(n)$, leaves exactly $\chi(\lambda) + \chi(\lambda - 2r)$; the other terms cancel in pairs.

(T3) Comparing characters, the multiplicities in $V^{\otimes m}$ are given, from the top down, by $n_{\lambda} = \beta(m, \lambda) - n_{\lambda^+}$. Here λ^+ is the weight whose bottom factor is λ , and $n_{\lambda^+} = 0$ if there is none or if $\lambda^+ > m$. (For $\lambda \leq p^2 + p - 1$ the map $\lambda \mapsto \lambda^-$ is injective: $\lambda^- \equiv p - 1 - r \pmod{p}$ determines r , and then a .)

For $\lambda \equiv m \pmod{2}$, $0 \leq \lambda \leq m$, write $\beta(m, \lambda) = \binom{m}{(m-\lambda)/2} - \binom{m}{(m-\lambda)/2-1}$ (the number of strings of length $\lambda + 1$).

6.2 The blocks

Fix m, c and $\lambda \equiv m \pmod{2}$, put $k = (m - \lambda)/2$ and

$$v_t = e + v_p(c + k + t) \quad (0 \leq t \leq \lambda),$$

with $v_0 = \infty$ when $c = k = 0$. Put $A = \sum_{t=0}^{\lambda} v_t$. When $T(\lambda)$ has two Weyl factors, with (r, a) as in (T2), put also

$$A' = \sum_{t=r}^{r+\lambda^-} v_t, \quad \alpha = \sum_{t=(a+1)p}^{\lambda} v_t, \quad \tau = 1 + [a = p - 1],$$

and

$$E_{11} = A - \alpha - a, \quad E_{12} = A' - a + \tau, \quad E_{22} = A' - a + \alpha.$$

Here A' is the sum over the levels of the bottom factor (it is $A_c(m, \lambda^-)$ in the notation of Theorem F), and α is the sum over the last r levels of the top factor. Define the abelian p -group $G_c(m, \lambda)$ by

$$G_c(m, \lambda) = \begin{cases} \mathbb{Z}/p^A, & \lambda \leq p - 1, \\ (\mathbb{Z}/p)^a \oplus \mathbb{Z}/p^{A-a}, & \lambda + 1 = (a + 1)p \leq p^2, \\ (\mathbb{Z}/p)^{2a} \oplus \mathbb{Z}/p^x \oplus \mathbb{Z}/p^y, & \text{two Weyl factors,} \end{cases}$$

where $x = \min(E_{11}, E_{12}, E_{22})$ and $y = E_{11} + E_{22} - x$. When $c = k = 0$ (the block containing the empty set) we put instead: $\mathbb{Z}_p$ for $\lambda \leq p - 1$; $\mathbb{Z}_p \oplus (\mathbb{Z}/p)^a$ in the second case; and $\mathbb{Z}_p \oplus (\mathbb{Z}/p)^{2a} \oplus \mathbb{Z}/p^{\min(E_{12}, E_{22})}$ in the third.

When $a + 1 < p$ one has $\tau = 1$ and $x = E_{12} = A' + 1 - a$, so the third case reads $(\mathbb{Z}/p)^{2a} \oplus \mathbb{Z}/p^{A'+1-a} \oplus \mathbb{Z}/p^{A-1-a}$, which is the block of version 2. When $a = p - 1$ all three numbers E_{ij} can be the minimum: for $p = 3$, $\lambda = 10$, $k = 0$, $e = 1$ the pairs (x, y) are $(11, 11)$ for $c = 1$ and $(10, 12)$ for $c = 2$.

The first theorem is the statement conjectured in version 1 of this paper. It is the case $m \leq 2p - 2$ of Theorem G below, where only $a = 0$ occurs.

Theorem F (the Glue Theorem; Conjecture F of version 1). Let p be a prime, q an integer with $e = v_p(q) \geq 1$, $m \leq 2p - 2$ and $c \geq 0$. Then

$$\begin{aligned} \text{Syl}_p X_q(m, c) \cong & \bigoplus_{\lambda \leq p-1} (\mathbb{Z}/p^{A_c(m, \lambda)})^{\beta(m, \lambda) - \beta'(m, \lambda)} \\ & \oplus \bigoplus_{p \leq \lambda \leq m} \left(\mathbb{Z}/p^{A_c(m, \lambda)-1} \oplus \mathbb{Z}/p^{A_c(m, 2p-2-\lambda)+1} \right)^{\beta(m, \lambda)}, \end{aligned}$$

where $A_c(m, \lambda) = \sum_{r=(m-\lambda)/2}^{(m+\lambda)/2} (e + v_p(c+r))$, $\beta'(m, \lambda) = \beta(m, 2p-2-\lambda)$ if $\lambda \leq p-2$ and $\beta'(m, p-1) = 0$, and λ runs over integers $\equiv m \pmod{2}$. When $c = 0$ the string of length $m+1$ (through the empty set) has a zero on its diagonal: for $\lambda = m < p$ it gives nothing, and for $\lambda = m \geq p$ the term $\mathbb{Z}/p^{A_c(m, m)-1}$ is omitted (it is the free summand) while its partner $\mathbb{Z}/p^{A_c(m, 2p-2-m)+1}$ stays.

More generally:

Theorem G (two Weyl factors). Let p be a prime, q an integer with $e = v_p(q) \geq 1$, $c \geq 0$, and $m \leq p^2 + p - 1$. Then

$$\mathbb{Z}_p^{[c=0]} \oplus \text{Syl}_p X_q(m, c) \cong \bigoplus_{\lambda} G_c(m, \lambda)^{n_{\lambda}},$$

where λ runs over the integers $\equiv m \pmod{2}$ with $0 \leq \lambda \leq m$ and n_{λ} is as in (T3).

With Theorem A, Theorem G determines $\text{Syl}_p K(H(d, q))$ for every $d \leq p^2 + p - 1$. This contains Conjecture H of version 2 (p odd, $d \leq p^2 - 2$). For $p = 3$ the range is $d \leq 11$ and for $p = 5$ it is $d \leq 29$. For $p = 2$ it is $d \leq 5$; the prime 2 is treated for every d in Section 7. The range is exactly the range in which every tilting summand of $V^{\otimes m}$ has at most two Weyl factors; at $m = p^2 + p$ the block $\lambda = p^2 + p$, with four Weyl factors, appears. We claim nothing for $p \nmid q$, where the answer is [JNR03, Theorem 2]. (Version 3 said here that the formula fails for $p \nmid q$, at $(q, p, d) = (4, 3, 4)$. That was wrong: evaluated at $e = 0$ the formula gives the right group there, and in all 784 pieces with $p \nmid q$ and $p \leq 5$ checked by the independent reader. We have not proved it for $e = 0$.)

6.3 Proofs

Fix λ with two Weyl factors, (r, a) as in (T2), $\mu = \lambda^-$. Let $\ell_i = f^{(i)}x$ ($0 \leq i \leq \lambda$) be the standard basis of $\Delta_{\mathbb{Z}_p}(\lambda)$, so $f\ell_i = (i+1)\ell_{i+1}$, $e\ell_i = (\lambda-i+1)\ell_{i-1}$, and more generally $f^{(i)}\ell_i = \binom{i+t}{t}\ell_{i+t}$ and $e^{(i)}\ell_i = \binom{\lambda-i+t}{t}\ell_{i-t}$. Let $s_j = f^{(j)}y$ ($0 \leq j \leq \mu$) be that of $\Delta_{\mathbb{Z}_p}(\mu)$. Put

$$\nu = \min_{1 \leq t \leq r} v_p \binom{\lambda-r+t}{t}.$$

Lemma 7 (the glue is forced). $\nu = 2$ if $r \leq p-1$ and $a = p-1$, and $\nu = 1$ otherwise. $T_{\mathbb{Z}_p}(\lambda)$ is isomorphic to the lattice in $\Delta_{\mathbb{Q}_p}(\lambda) \oplus \Delta_{\mathbb{Q}_p}(\mu)$ spanned by the ℓ_i and by

$$\tilde{s}_j = s_j + p^{-\nu} \binom{j+r}{j} \ell_{j+r} \quad (0 \leq j \leq \mu).$$

Proof. The value of ν follows from Kummer's theorem. If $r \leq p-1$, then $\lambda - r = (a+1)p - 1$, and adding $t \in [1, r]$ to it carries once out of the units digit; it carries a second time exactly when $a+1 = p$. If $r = p$, then $\lambda - r = p^2 - 1$: adding $t < p$ carries twice, while adding $t = p$ carries once.

By (T2), $T = T_{\mathbb{Z}_p}(\lambda)$ contains $\Delta_{\mathbb{Z}_p}(\lambda)$ with quotient $\Delta_{\mathbb{Z}_p}(\mu)$. Since $T \cap \Delta_{\mathbb{Q}_p}(\lambda) = \Delta_{\mathbb{Z}_p}(\lambda)$ and the weight spaces of $\Delta(\lambda)$ are one-dimensional, the lifts of the s_j to T can be chosen of the form $\tilde{s}_j = s_j + x_j \ell_{j+r}$ with $x_j \in \mathbb{Q}_p$, unique modulo $\mathbb{Z}_p$.

Stability under f gives $(j + r + 1)x_j - (j + 1)x_{j+1} \in \mathbb{Z}_p$. Stability under $f^{(p)}$ at $j = kp$, for $(k + 1)p \leq \mu$, gives

$$\binom{(k+1)p+r}{p} x_{kp} - \binom{(k+1)p}{p} x_{(k+1)p} \in \mathbb{Z}_p.$$

In the range of (T2), $\mu \leq p^2 - 2$, so $k + 1 \leq p - 1$ and $\binom{(k+1)p}{p} \equiv k + 1$ is a unit; so is $\binom{(k+1)p+r}{p}$, which is $\equiv k + 1$ if $r \leq p - 1$ and $\equiv k + 2$ if $r = p$ (and then $\mu = p^2 - p - 1$, so $k + 2 \leq p - 1$). Inside a segment $[kp, (k + 1)p - 1]$ the factor $j + 1$ is a unit, so $x_j \equiv \binom{j+r}{j} \binom{(kp+r)}{kp}^{-1} x_{kp}$ modulo $\mathbb{Z}_p$. Across segments, $x_{(k+1)p} \equiv \binom{(k+1)p+r}{p} \binom{(k+1)p}{p}^{-1} x_{kp}$, and

$$\binom{n+r}{p} \binom{n}{p}^{-1} = \binom{n+r}{n} \binom{n-p+r}{n-p}^{-1} \quad (n = (k + 1)p)$$

is an identity of rational numbers (both sides equal $\frac{(n+r)!(n-p)!}{n!(n-p+r)!}$). Hence $x_j \equiv \binom{j+r}{j} x_0$ modulo $\mathbb{Z}_p$ for every j , and the lattice is determined by x_0 .

Stability under $e^{(t)}$ at $j = 0$ gives $x_0 \binom{\lambda-r+t}{t} \ell_{r-t} \in T$ for $1 \leq t \leq r$, so $v_p(x_0) \geq -\nu$. Suppose $v_p(x_0) \geq 1 - \nu$. Then $e^{(t)} \tilde{s}_0 \in pT$ for every $t \geq 1$ (also for $t > r$, where it vanishes), so the image of $\tilde{s}_0$ in $T \otimes \mathbb{F}_p$ is a highest weight vector of weight μ . It generates the image of a map $\Delta(\mu) \rightarrow T \otimes \mathbb{F}_p$, and the composite with $T \otimes \mathbb{F}_p \rightarrow \Delta(\mu)$ is the identity on the generator, so it is an isomorphism and the extension splits. This is impossible, because $T \otimes \mathbb{F}_p = T(\lambda)$ is indecomposable. So $x_0 = u p^{-\nu}$ with u a unit, and rescaling y makes $u = 1$. The lattice so described is T itself, so it is stable under all $f^{(t)}$ and $e^{(t)}$; nothing more has to be checked. (For $f^{(t)}$ this is also visible directly: $f^{(t)} \tilde{s}_j - \binom{j+t}{t} \tilde{s}_{j+t}$ vanishes by the identity $\binom{j+r}{j} \binom{j+r+t}{t} = \binom{j+t}{t} \binom{j+t+r}{j+t}$.) Stability under every $e^{(t)}$ and $f^{(t)}$ is also checked by machine in Section 10. $\blacksquare$

Lemma 8 (the blocks). Embed $T_{\mathbb{Z}_p}(\lambda)$ in $B_m \otimes \mathbb{Z}_p$ at levels $k, \dots, k + \lambda$ ($m = \lambda + 2k$). Then the cokernel of $q(c + H) + U$ on it is $G_c(m, \lambda)$. The same holds for the simple blocks.

Proof. Put $N_t = q(c + k + t)$, so $v_p(N_t) = v_t$, and $M = q(c + H) + U$. In the basis of Lemma 7:

- $M\ell_i = N_i \ell_i + (i + 1)\ell_{i+1}$;
- $M\tilde{s}_j = N_{j+r} \tilde{s}_j + (j + 1)\tilde{s}_{j+1}$ for $j < \mu$; the ℓ -component is $p^{-\nu}[(j + r + 1)\binom{j+r}{j} - (j + 1)\binom{j+r+1}{j+1}]\ell_{j+r+1} = 0$;
- $M\tilde{s}_\mu = N_{\mu+r} \tilde{s}_\mu + u \ell_{\mu+r+1}$, where $\mu + r + 1 = (a + 1)p$ and $u = p^{-\nu} \binom{\mu+r}{\mu} (a + 1)p$.

By Lucas, $\binom{\mu+r}{\mu}$ is a unit: $\mu + r = (a + 1)p - 1$ has digits $(a, p - 1)$, or $(p - 1, p - 1)$ when $r = p$. So $v_p(u) = v_p((a + 1)p) - \nu$. This is 0 when $r \leq p - 1$, and 1 when $r = p$.

The non-unit subdiagonal entries are:

- on the ℓ -chain, the $i + 1 = tp$ with $1 \leq t \leq a + 1$, of valuation 1 except the last, of valuation τ ;
- on the $\tilde{s}$ -chain, the $j + 1 = tp \leq \mu$, of valuation 1. There are a of them if $r \leq p - 1$, and $a - 1$ if $r = p$.

Solving the unit subdiagonals, the cokernel is generated by the first vector of each segment between them, with one relation at the end of each segment. Write $g_t = \ell_{tp}$ and $h_t = \tilde{s}_{tp}$, and let α_t, σ_t be the sums of the v 's over the segments, so that $\sum_t \alpha_t = A$, $\sum_t \sigma_t = A'$ and the last α_t is α . After rescaling the generators by units, the relations are:

- $p^{\alpha_t} g_t + p g_{t+1}$ on the ℓ -chain, the last link having p^τ instead of p ;
- $p^\alpha g_{a+1}$ at the end of the ℓ -chain;
- $p^{\sigma_t} h_t + p h_{t+1}$ on the $\tilde{s}$ -chain;

- at the end of the $\tilde{s}$ -chain, $p^\sigma h_{\text{last}} + p^{v_p(u)} g_{a+1}$.

If $r = p$, every entry has valuation ≥ 1 ; if $r \leq p - 1$ this holds after the elimination of g_{a+1} described next. In both cases the relation matrix is then pM' with M' integral.

If $r \leq p - 1$ the last relation has a unit coefficient on g_{a+1} . Eliminating g_{a+1} turns the last link of the ℓ -chain into $p^{\tau+\sigma_a} h_a$ and the end relation into $p^{\alpha+\sigma_a} h_a$. In M' the links with coefficient 1 form a units on the ℓ -chain and a on the $\tilde{s}$ -chain, in distinct rows and columns. Clearing them (column operations along each chain, then row operations) splits off I_{2a} . The 2×2 matrix that remains, on the rows of the last link of the ℓ -chain and of the end relation $p^{\alpha+\sigma_a} h_a$, and on the columns g_0, h_0 , is upper triangular with diagonal valuations $E_{11} - 1, E_{22} - 1$ and corner $E_{12} - 1$.

If $r = p$ the $\tilde{s}$ -chain has $a - 1$ unit links and its end relation has a unit coefficient on $g_{a+1} = g_p$ in M' . These give $2a$ units again, and clearing them leaves the same 2×2 matrix.

So the relation matrix is equivalent to $pI_{2a} \oplus \begin{pmatrix} p^{E_{11}} & * & p^{E_{12}} \\ 0 & & p^{E_{22}} \end{pmatrix}$, with a unit $*$. The Smith form of such a matrix is $(p^x, p^{E_{11}+E_{22}-x})$ with x the least of the three valuations, and this is $G_c(m, \lambda)$.

When $c = k = 0, N_0 = 0$: the column g_0 is zero and gives the free summand, and what remains on h_0 are the relations $p^{E_{12}}$ and $p^{E_{22}}$. The simple blocks are single chains. When $\lambda + 1 = (a + 1)p \leq p^2$ the chain has a non-unit subdiagonal entries, $i + 1 = tp$ with $1 \leq t \leq a \leq p - 1$, each of valuation 1. The same elimination gives $(\mathbb{Z}/p)^a \oplus \mathbb{Z}/p^{A-a}$.

When $a + 1 < p$ we have $E_{12} \leq E_{22}$ (since $\alpha \geq 1$) and $E_{12} \leq E_{11}$, because $A - \alpha = A' + \sum_{t < r} v_t \geq A' + 1$. So $x = E_{12}$ and $y = A - 1 - a$. ■

Proof of Theorems G and F. Combine (T1)–(T3), Lemma 7 and Lemma 8. For $m \leq p^2 + p - 1$ every $\lambda \leq m$ is simple or has two Weyl factors, by (T2). ■

Example 9. For $p = q = 3$ and $d = 4$: Theorem A gives the pieces $X(4, 0), X(3, 1)^4, X(2, 2)^6, X(1, 3)^4, X(0, 4)$, and Theorem F gives $\text{Syl}_3 K(H(4, 3)) \cong (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/3^2)^7 \oplus (\mathbb{Z}/3^3)^8 \oplus (\mathbb{Z}/3^4)^{17}$, which is the group we computed.

7. The prime 2

For $p = 2$ the blocks are the families of [DST], and [DST] computes them for every family and every power of 2. This gives the whole 2-part of $K(H(d, q))$ for every d and every even q .

For a family $\lambda \geq 1$ write $\lambda + 1 = 2^\theta + \sum_{t \in I} 2^t$ with $I \subseteq \{0, \dots, \theta - 1\}$, $\Theta = 2^\theta$, $h_t = \text{next}(t) - t$ (with $\text{next}(t)$ the next element of $I \cup \{\theta\}$), $o_D = \sum_{t \in D} 2^t$ for $D \subseteq I$, and $\kappa(a, b) = v_2\binom{a+b}{a}$. Following [DST, §3.4] put

$$X^{(\alpha)}(\lambda, i) = \text{coker}(f + 2^\alpha(D_\lambda + i) \text{ on } T_{\mathbb{Z}_2}(\lambda)) \quad (\alpha \geq 1, i \geq 0),$$

where D_λ is the floor operator $(\lambda - h)/2$. By [DST, Theorem D, Proposition 5.4, Theorem 7.10], which are stated and proved for every $\alpha \geq 1$,

$$X^{(\alpha)}(\lambda, i) \cong \bigoplus_{b=1}^{\theta-1} (\mathbb{Z}/2^b)^{2^{|I|+\theta-1-b}} \oplus \bigoplus_{D \subseteq I} \mathbb{Z}/2^{\theta+\alpha\Theta+\kappa(i-1, \Theta)+\hat{y}_D} \quad (i \geq 1),$$

where $\hat{y}$ is the integer antitonic fit [DST, §1.2], in the order of increasing o_D , of the clocks

$$x_D = R_{i-1}(D) - R_{i-1+\Theta}(I \setminus D), \quad R_\mu(E) = \sum_{t \in E} (h_t - \alpha 2^t) - \kappa(\mu, o_E).$$

For $i = 0$ the summand $D = \emptyset$ is $\mathbb{Z}_2$, and the others are $\mathbb{Z}/2^{\theta+\alpha\Theta+\hat{y}_D^\#}$, with $\hat{y}^\#$ the fit of $x_D^\# = R^\#(D) - R^\#(I \setminus D)$, $R^\#(E) = \sum_{t \in E} (h_t - \alpha 2^t) - \kappa(\Theta - 1, o_E)$, over $D \neq \emptyset$. For $\lambda = 0$, $X^{(\alpha)}(0, i) = \mathbb{Z}/2^{\alpha+v_2(i)}$ ($\mathbb{Z}_2$ if $i = 0$). Let $m_\lambda(n)$ be the multiplicity of $T(\lambda)$ in $V^{\otimes n}$ in characteristic 2 [DST, §1.2, Theorem 3.6].

Theorem H (the prime 2). Let q be even, $e = v_2(q)$, $m \geq 0$ and $c \geq 0$. Then

$$\mathbb{Z}_2^{[c=0]} \oplus \text{Syl}_2 X_q(m, c) \cong \bigoplus_{\lambda} X^{(e)}(\lambda, c + \frac{m-\lambda}{2})^{m_{\lambda}(m)}.$$

Consequently, for every $d \geq 1$,

$$\mathbb{Z}_2 \oplus \text{Syl}_2 K(H(d, q)) \cong \bigoplus_{j=0}^d \bigoplus_{\lambda} X^{(e)}(\lambda, j + \frac{d-j-\lambda}{2})^{\binom{d}{j}(q-2)^j m_{\lambda}(d-j)}.$$

Proof. By (T1) for $p = 2$ [DST, Theorem 3.6], $B_m \otimes \mathbb{Z}_2 = V^{\otimes m} = \bigoplus_{\lambda} T_{\mathbb{Z}_2}(\lambda)^{m_{\lambda}(m)}$ as Dist-modules. On a summand $T(\lambda)$, placed at the levels $k = (m - \lambda)/2, \dots, k + \lambda$, a vector of weight h has $H = (m - h)/2 = k + D_{\lambda}$, so $q(c + H) + U = f + q(c + k + D_{\lambda})$. Write $q = 2^e u$ with u odd. The proof of Lemma 2 applies verbatim to any lattice graded by the floor on which f raises the floor by one; it rescales the floors by units and turns $f + 2^e u(D_{\lambda} + i)$ into $f + 2^e(D_{\lambda} + i)$. So the cokernel on the summand is $X^{(e)}(\lambda, c + k)$. The second statement follows from Theorem A. ■

For $q = 2$ only $j = 0$ survives, $e = 1$, and this is the Main Theorem of [DST]. For $e \geq 1$ arbitrary, the theorem was checked against the Smith forms of $q(c + H) + U$ on B_m in 384 cases with $m \leq 11$, $e \leq 4$, $c \leq 4$ (the list is in the logs of the record) and against the full Laplacians of $H(d, q)$ for $(d, q) = (3, 4), (4, 4), (3, 6), (3, 8), (2, 12), (2, 16), (3, 10)$. Two controls fail: using $\alpha = 1$ instead of $\alpha = e$ changes the answer in all 112 cases with $e \geq 2$ tried, and omitting the antitonic fit changes it in 16 of 128.

8. The whole group: Theorems I and J

For a prime p , $\lambda \geq 0$, $i \geq 0$ and $e \geq 1$ put

$$Y_p(\lambda, i, e) = \text{coker}(f + p^e(D_{\lambda} + i) \text{ on } T_{\mathbb{Z}_p}(\lambda)),$$

with $D_{\lambda} = (\lambda - h)/2$ the floor operator, as in Section 7. By (T1), and by Lemma 2 applied as in the proof of Theorem H, for every m and c

$$\mathbb{Z}_p^{[c=0]} \oplus \text{Syl}_p X_q(m, c) \cong \bigoplus_{\lambda} Y_p(\lambda, c + \frac{m-\lambda}{2}, v_p(q))^{n_{\lambda}(m)}, \quad (2)$$

where $n_{\lambda}(m)$ is the multiplicity of $T(\lambda)$ in $V^{\otimes m}$. So everything comes down to the groups $Y_p(\lambda, i, e)$. Theorem G computes them for $\lambda \leq p^2 + p - 1$, and Theorem H for $p = 2$. In this section we compute them for every λ and every odd p .

8.1 Digits, dancers, clocks

Fix a prime p and $\lambda \geq 0$, and write

$$\lambda + 1 = \sum_{t=0}^k d_t p^t, \quad 0 \leq d_t \leq p-1, d_k \neq 0, \quad I = I(\lambda) = \{t < k : d_t \neq 0\}, \quad K = d_k p^k.$$

For $t \in I$ put $\text{next}(t) = \min\{u \in I \cup \{k\} : u > t\}$ and $h_t = \text{next}(t) - t$. The subsets $D \subseteq I$ are the *dancers*; put $o_D = \sum_{t \in D} d_t p^t$. Since $d_t p^t > \sum_{u < t} (p-1)p^u$, the order of the dancers by increasing o_D (the *o-order*) is the lexicographic order of the digit sets from the top, exactly as for $p = 2$; and $o_{I \setminus D} = o_I - o_D$, so $D \mapsto I \setminus D$ reverses it. Let $\kappa_p(a, b) = v_p\binom{a+b}{a}$, the number of carries in the base- p addition $a + b$ (Kummer). For $E \subseteq I$ and $\mu \geq 0$ put

$$R_{\mu}(E) = \sum_{t \in E} (h_t - e d_t p^t) - \kappa_p(\mu, o_E), \quad R^{\#}(E) = \sum_{t \in E} (h_t - e d_t p^t) - (k - \min E) \quad (R^{\#}(\emptyset) = 0).$$

The *clocks* are: for $i \geq 1$, with $m = i - 1$, $x_D = R_m(D) - R_{m+K}(I \setminus D)$ ($D \subseteq I$); for $i = 0$, $x_D^\# = R^\#(D) - R^\#(I \setminus D)$ ($D \neq \emptyset$). Let $\hat{y}_D$ and $\hat{y}_D^\#$ be their *integer antitonic fits* in the o-order: partition the positions into consecutive blocks by the pool-adjacent-violators algorithm, so that the block means are strictly decreasing and each block value is the mean of its entries (the antitonic least-squares fit); then replace a block of b positions with sum S by $S \bmod b$ copies of $\lfloor S/b \rfloor$ followed by $b - (S \bmod b)$ copies of $\lceil S/b \rceil$. (By Lemma 24 below the rounding never acts.) For $p = 2$ ($d_i = 1$, $K = 2^k$) this is exactly the notation of [DST, §1.2, §5] with $\alpha = e$.

8.2 The theorems

Theorem I (Conjecture I of version 3). Let p be a prime, $\lambda \geq 0$, $i \geq 0$ and $e \geq 1$, and let k , d_k and I be as above. Then exactly $2^{|I|}$ of the cyclic summands of $Y_p(\lambda, i, e)$ have order larger than p^k (a free summand, when $i = 0$, counting as one of them). The others have order at most p^k , and they form the group

$$S_p(\lambda) = \bigoplus_{h=1}^{k-1} (\mathbb{Z}/p^h)^{(p-1)d_k p^{k-h-1} 2^{|I|}} \oplus (\mathbb{Z}/p^k)^{(d_k-1)2^{|I|}},$$

which depends only on p and the base- p digits of $\lambda + 1$, not on i or e .

The number of large summands is the number of Weyl factors of $T(\lambda)$ (Lemma 16(i)). The total number of cyclic summands, $d_k p^{k-1} 2^{|I|} = \dim T(\lambda)/p$ for $k \geq 1$, is the number of Jordan blocks of f on $T(\lambda)$ over $\mathbb{F}_p$; this is classical, since $T(\lambda)$ is projective over the first Frobenius kernel for $\lambda \geq p - 1$ [Jan03, II.11.11].

Theorem J (the whole group). Let p be an odd prime, $\lambda \geq 0$ and $e \geq 1$. Then

$$Y_p(\lambda, i, e) \cong S_p(\lambda) \oplus \bigoplus_{D \subseteq I} \mathbb{Z}/p^{k+eK+\kappa_p(i-1, K)+\hat{y}_D} \quad (i \geq 1),$$

$$Y_p(\lambda, 0, e) \cong S_p(\lambda) \oplus \mathbb{Z}_p \oplus \bigoplus_{\emptyset \neq D \subseteq I} \mathbb{Z}/p^{k+eK+\hat{y}_D^\#}.$$

Consequently, for every prime p dividing q , with $e = v_p(q)$, and every $d \geq 1$,

$$\mathbb{Z}_p \oplus \text{Syl}_p K(H(d, q)) \cong \bigoplus_{j=0}^d \bigoplus_{\lambda} Y_p(\lambda, j + \frac{d-j-\lambda}{2}, e)^{\binom{d}{j} (q-2)^j n_\lambda(d-j)},$$

λ running over $0 \leq \lambda \leq d - j$ with $\lambda \equiv d - j \pmod{2}$, where for $p = 2$ the groups $Y_2 = X^{(e)}$ are those of Theorem H. The multiplicities $n_\lambda(m)$ are obtained by peeling the characters

$$\text{ch } T(\lambda) = \prod_{t < k} \chi_{p-1}(z^{p^t}) (z^{d_t p^t} + z^{-d_t p^t})^{[d_t \neq 0]} \cdot \chi_{d_k-1}(z^{p^k}), \quad \chi_n(z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}},$$

off $(z + z^{-1})^m$ from the top.

For $\lambda \leq p - 2$ one has $k = 0$, $I = \emptyset$, $K = \lambda + 1 < p$, and Theorem J gives $\mathbb{Z}/p^{e(\lambda+1)+\kappa_p(i-1, \lambda+1)}$, which is Theorem E. For $\lambda \leq p^2 + p - 1$ it agrees with Theorem G (checked on 3450 cells, §10), and at $p = 2$ the same formula is Theorem H. The right-hand side is a finite computation with binomial valuations; no Smith form is needed.

Example 10. The first block beyond Theorem G is $\lambda = 12$ for $p = 3$: $\lambda + 1 = 111_3$, $k = 2$, $d_2 = 1$, $I = \{0, 1\}$, $h_0 = h_1 = 1$, four Weyl factors. Theorem I gives $S_3(12) = (\mathbb{Z}/3)^8$, and Theorem J gives the large summands. For $q \in \{3, 6\}$ ($e = 1$), as computed directly from the Smith form of $q(c + H) + U$ on B_{12} after removing the blocks $\lambda \leq 10$:

c	block $\lambda = 12$
0	$\mathbb{Z}_3 \oplus (\mathbb{Z}/3)^8 \oplus \mathbb{Z}/3^7 \oplus \mathbb{Z}/3^{10} \oplus \mathbb{Z}/3^{12}$
1, 2	$(\mathbb{Z}/3)^8 \oplus (\mathbb{Z}/3^9)^2 \oplus (\mathbb{Z}/3^{13})^2$
3	$(\mathbb{Z}/3)^8 \oplus \mathbb{Z}/3^8 \oplus \mathbb{Z}/3^{10} \oplus \mathbb{Z}/3^{12} \oplus \mathbb{Z}/3^{14}$

Version 3 stated Conjecture I after the cell $c = 0$ alone and predicted the other seven cells; all held. Later, four cells with $q = 9$ ($e = 2$) were computed directly on B_{12} , and agree with Theorem J in every exponent below 19, the precision of that computation; and 31 further pieces $X_q(m, c)$, predicted from Theorem J before they were computed (23 with $p = 3$ and $m \in \{12, 13\}$, 8 with $p = 2$), agree exactly (§10).

9. Proofs of Theorems I and J: the odd dance

Throughout this section p is odd, $A = \mathbb{Z}_p$, and $\text{Dist}_A = A \otimes \text{Dist}_{\mathbb{Z}}(SL_2)$ is the Kostant form, generated by the divided powers $E^{(r)}, F^{(r)}$ and the $\binom{H}{r}$. A *lattice* is a Dist_A -module that is free of finite rank over A and is the sum of its weight spaces, with integer weights. We put $u = -1$ (any unit $\equiv -1 \pmod{p}$ would do) and, on a lattice, $X = FE$. As in Section 6, $\Delta_A(\mu)$ has the basis $e_s = F^{(s)}v$, $Fe_s = (s+1)e_{s+1}$. A lattice is *graded by floors* when F raises the floor by one; in $\Delta_A(\mu)$ the floor of e_s is s . The plan is the one of [DST] for $p = 2$: explicit lattices built one base- p digit at a time (§9.1–9.2), elimination of unit pivots digit by digit (§9.3), and the Smith form of the matrix that remains (§9.4).

9.1 The twist and the split

Lemma 11. Let X, Y be finite-dimensional $U_{\mathbb{Q}}(\mathfrak{sl}_2)$ -modules and $\varphi: X \rightarrow Y$ linear with $\varphi F = F\varphi$ and $\varphi H = H\varphi$. Then $\varphi E = E\varphi$. Hence a weight-preserving A -isomorphism of lattices that commutes with F is an isomorphism of lattices, and on a vector space with given F, H there is at most one E making it an $\mathfrak{sl}_2$ -module.

Proof. $\Delta := \varphi E - E\varphi$ raises weights by 2 and commutes with F , since $\varphi[E, F] - [E, F]\varphi = \varphi H - H\varphi = 0$. By Weyl's complete reducibility, X is spanned by vectors $F^j x$ with $Ex = 0$, x of weight $h \geq 0$, $F^{h+1}x = 0$, and $Y_{\mathbb{Q}}$ is a sum of simple modules $\Delta_{\mathbb{Q}}(c)$; on each of them F^{h+1} is injective on the weight space $h+2$. As $F^{h+1}\Delta x = \Delta F^{h+1}x = 0$, $\Delta x = 0$, and $\Delta F^j x = F^j \Delta x = 0$. ■

Definition 12 (the twist Φ_p , digit 0). For a lattice N let $\Phi_p(N) = \bigoplus_{j=0}^{p-1} b_j \otimes N$ (written $b_j n$), and for n of weight w : $H(b_j n) = (pw + p - 1 - 2j)b_j n$; $F(b_j n) = b_{j+1}n$ for $j \leq p-2$ and $F(b_{p-1}n) = pu b_0(Fn)$; $E(b_j n) = b_{j-1}g_j n$ for $j \geq 1$, with $g_j = p^2X + j(pw + p - j)$, and $E(b_0 n) = (p/u)b_{p-1}(En)$. On morphisms $\Phi_p(f) = 1 \otimes f$. If n is on floor f , then $b_j n$ is on floor $pf + j$.

Proposition 13. (a) $\Phi_p(N)$ is a lattice, and Φ_p is an exact functor that preserves saturated sublattices. For $r \geq 0$, with $c = \lfloor (j+r)/p \rfloor$,

$$F^{(r)}(b_j n) = (pu)^c \frac{c!}{r!} b_{j+r-pc} F^{(c)} n,$$

$$E^{(r)}(b_j n) = (p/u)^{c'} \frac{c'!}{r!} b_{j-r+pc'} E^{(c')} \prod_{(a,i) \in S(j,r)} [p^2X + (i - pa)(p(w+1+a) - i)] n,$$

where $c' = \lfloor (r+p-1-j)/p \rfloor$ is the number of wraps and $S(j, r)$ is the set of pairs (wraps already made a , current index $i \geq 1$) met in the first r steps $j, j-1, \dots, 1$, then $p-1, \dots, 1$, and so on; all coefficients are p -integral. (b) $\Phi_p(N) \otimes \mathbb{F}_p \cong St \otimes (N \otimes \mathbb{F}_p)^{[1]}$, with $St = \Delta(p-1)$ the Steinberg module. (c) $\Phi_p(\Delta_A(\mu)) \cong \Delta_A(p\mu + p-1)$, $\Phi_p(\nabla_A(\mu)) \cong \nabla_A(p\mu + p-1)$.

Proof. (a) The relation $[E, F] = H$ is checked on each $b_j n$: for $1 \leq j \leq p-2$, $g_{j+1} - g_j = pw + p - 1 - 2j$; on $b_0 n$, $g_1 - p^2X = pw + p - 1$; on $b_{p-1}n$, $p^2(X + w) - g_{p-1} = pw - p + 1$. These are the weights, and E, F

move weights by ± 2 . The formula for $F^{(r)}$ follows from $F^r(b_j n) = (pu)^c b_{j+r-pc} F^c n$; the one for $E^{(r)}$ by walking E down the indices, using that the Casimir $\Omega = 4FE + (H+1)^2$ of $\Phi_p(N)$ is $p^2 \Omega_N$ on every $b_j n$, so that g_j is $(p^2 \Omega_N - (p(w+1) - 2j)^2)/4$. Integrality: $v_p(p^c c! / r!) = c + v_p(c!) - \lfloor r/p \rfloor - v_p(\lfloor r/p \rfloor!) \geq 0$ because $c \geq \lfloor r/p \rfloor$ and $x \mapsto x + v_p(x!)$ is increasing (Legendre); the same for E with $c' \geq \lfloor r/p \rfloor$. Exactness and saturation hold because $\Phi_p(N) = N^p$ as A -modules and $1 \otimes f$ commutes with F, H , hence with E (Lemma 11). (b) Modulo p , with $r = ps + t$, $0 \leq t \leq p-1$, the formulas of (a) reduce to $F^{(r)}(b_j n) \equiv [j+t \leq p-1] \frac{1}{t!} b_{j+t} F^{(s)} n$ (using $(ps+t)!/(p^s s!) \equiv (-1)^s t!$ and $u \equiv -1$) and $E^{(r)}(b_j n) \equiv [t \leq j] (-1)^t \left(\frac{j!}{(j-t)!} \right)^2 \frac{1}{t!} b_{j-t} E^{(s)} n$ (a full cycle of g 's is $\equiv \prod_{i=1}^{p-1} (-i^2) \equiv 1$). These are the formulas of $St \otimes \bar{N}^{[1]}$ computed with the coproduct, in the basis $b'_j = F^j b'_0$ of St . (c) In the basis $x_R = b_j e_s$, $R = ps + j$, one has $Fx_R = x_{R+1}$ for $j \leq p-2$ and $Fx_{ps+p-1} = pu(s+1)x_{p(s+1)}$, while in $\Delta(p\mu + p-1)$, $Fe_R = (R+1)e_{R+1}$, with $R+1$ a unit unless $j = p-1$, where it is $p(s+1)$. A diagonal map with unit entries therefore commutes with F and H , and Lemma 11 applies. The same for ∇ . $\blacksquare$

Definition 14 (the split Ψ_r , digit r , $1 \leq r \leq p-1$). For a lattice N let $\Psi_r(N) = \bigoplus_{j=0}^{p-1} (A_j \otimes N \oplus C_j \otimes N)$, with $H(A_j n) = (pw + p - 1 + r - 2j)A_j n$, $H(C_j n) = (pw + p - 1 - r - 2j)C_j n$; $F(A_j n) = A_{j+1} n$ and $F(C_j n) = C_{j+1} n$ for $j \leq p-2$,

$$F(A_{p-1} n) = pu A_0(Fn) + p C_{p-r} n, \quad F(C_{p-1} n) = pu C_0(Fn);$$

and, with $g_j^A = p^2 X + j(pw + p + r - j)$ and $g_j^C = p^2 X + j(pw + p - r - j)$:

$$\begin{aligned} E(A_0 n) &= (p/u) A_{p-1}(En) - (p-r)(p/u) C_{p-1-r}(En), \\ E(A_j n) &= A_{j-1} g_j^A n + r(p/u) C_{p+j-1-r}(En) \quad (1 \leq j \leq r), \\ E(A_j n) &= A_{j-1} g_j^A n + r p^2 C_{j-1-r}(Xn) \quad (r+1 \leq j \leq p-1), \\ E(C_0 n) &= r A_{r-1} n + (p/u) C_{p-1}(En), \\ E(C_j n) &= C_{j-1} g_j^C n + r A_{j+r-1} n \quad (1 \leq j \leq p-r), \\ E(C_j n) &= C_{j-1} (g_j^C + r p) n + r p u A_{j+r-1-p}(Fn) \quad (p-r+1 \leq j \leq p-1). \end{aligned}$$

On morphisms $\Psi_r(f) = 1 \otimes f$. If n is on floor f , then $A_j n$ is on floor $pf + j$ and $C_j n$ on floor $pf + r + j$.

Proposition 15. (a) $\Psi_r(N)$ is a lattice, and Ψ_r is an exact functor that preserves saturated sublattices. (b) $\Psi_r(N) \otimes \mathbb{F}_p \cong T(p-1+r) \otimes (N \otimes \mathbb{F}_p)^{[1]}$. (c) $\Psi_r(\Delta_A(\mu))$ has a sublattice $\Delta_A(p\mu + p-1+r)$ with quotient $\Delta_A(p\mu + p-1-r)$.

Proof. (a) The relation $[E, F] = H$ is checked case by case on the ten index ranges of the definition (for instance, on $A_{p-1} n$ with $r \leq p-2$ the A -coefficient of $EF - FE$ is $p^2 EF + pr - g_{p-1}^A = pw - p + 1 + r$, the weight, and the C -coefficient is 0). For $r = 1$ the map $A_j n \mapsto \hat{b}_0 \otimes b_j n + j \hat{b}_1 \otimes b_{j-1} n$, $C_j n \mapsto \hat{b}_1 \otimes b_j n$ is a unitriangular isomorphism $\Psi_1(N) \cong V \otimes \Phi_p(N)$ commuting with F and H ; by Lemma 11 it is an isomorphism, and $V \otimes \Phi_p(N)$ is a lattice. For $r \geq 2$ the map $\iota: \Psi_r(N) \rightarrow V \otimes \Psi_{r-1}(N)$, $\iota(A_j n) = F^j \alpha(n)$, $\iota(C_j n) = F^j \gamma(n)$, with (primes for Ψ_{r-1})

$$\alpha(n) = \hat{b}_0 \otimes A'_0 n + (p/u) \hat{b}_1 \otimes C'_{p-r}(En), \quad \gamma(n) = \hat{b}_0 \otimes C'_1 n + \hat{b}_1 \otimes A'_{r-1} n - (pw + p - r) \hat{b}_1 \otimes C'_0 n,$$

commutes with F (it suffices to check $F^p \gamma(n) = pu \gamma(Fn)$ and $F^p \alpha(n) = pu \alpha(Fn) + p F^{p-r} \gamma(n)$) and with H , hence with E over $\mathbb{Q}$. Its reduction modulo p is injective: on suitable coordinates it is diagonal with entries 1 and $r-1$, a unit. So the image is a saturated $U_{\mathbb{Q}}$ -stable sublattice of a lattice, hence Dist_A -stable, and $\Psi_r(N)$ is a lattice by induction on r . Exactness as for Φ_p . (b) Induction on r ; $r = 1$ is $V \otimes St = T(p)$. For $r \geq 2$, ι embeds $\Psi_r(N)$ in $V \otimes T(p-2+r) \otimes \bar{N}^{[1]}$; the Casimir acts on $\Psi_r(N)$ with the single generalized eigenvalue r^2 (its eigenvalues on $\Psi_r(N)_{\mathbb{Q}}$ are $(p(b+1) \pm r)^2 \equiv r^2$), and the generalized r^2 -eigenspace of $V \otimes T(p-2+r)$ is $T(p-1+r)$, because $(r-2)^2 \not\equiv r^2$ and $\Delta(p-1+r)$ is not simple ($E^{(s)} F^{(r)} v = \binom{p-1+s}{s} F^{(r-s)} v \equiv 0$ for $1 \leq s \leq r$, by Lucas). Dimensions agree. (c) Path counting: for $k < p(\mu+1)$, $F^{(k)}(A_0 n_0)$ is a unit times $A_{k \bmod p} n_{\lfloor k/p \rfloor}$ plus

C -terms, and for $k = p(\mu + 1) + t$, $0 \leq t \leq r - 1$, it is a unit times $C_{p-r+t}n_\mu$ (every branch through the coupling pC_{p-r} contributes $p^{\mu+1}u^\mu\mu!$, and there are $\mu + 1$ of them). So $\text{Dist} \cdot A_0n_0 \cong \Delta_A(p\mu + p - 1 + r)$ is saturated, and the quotient is spanned by the images of $F^{(k)}C_0n_0$, $k \leq p\mu + p - 1 - r$. ■

The characters are $\text{ch } \Phi_p(N) = \chi_{p-1} \cdot (\text{ch } N)(z^p)$ and $\text{ch } \Psi_r(N) = (z^r + z^{-r})\chi_{p-1} \cdot (\text{ch } N)(z^p)$; since $\chi_{p-1}(z)\chi_b(z^p) = \chi_{pb+p-1}(z)$, every Weyl factor $\Delta(b)$ of N gives $\Delta(pb + p - 1)$, respectively $\Delta(pb + p - 1 \pm r)$.

Lemma 16 (the dance lattices). Define $L(\lambda) = \Delta_A(\lambda)$ if $\lambda + 1 < p$; otherwise write $\lambda + 1 = d_0 + p(b + 1)$ with $0 \leq d_0 \leq p - 1$ and put $L(\lambda) = \Phi_p(L(b))$ if $d_0 = 0$, $L(\lambda) = \Psi_{d_0}(L(b))$ if $d_0 \neq 0$. Then: (i) $L(\lambda)$ has rank $d_k p^k 2^{|I|}$ and a Δ -filtration with $2^{|I|}$ factors; (ii) $L(\lambda) \otimes \mathbb{F}_p \cong T(\lambda)$; (iii) $V^{\otimes m} \otimes \mathbb{Z}_p \cong \bigoplus_\lambda L(\lambda)^{n_\lambda(m)}$; in particular $L(\lambda) \cong T_{\mathbb{Z}_p}(\lambda)$ and $Y_p(\lambda, i, e) = \text{coker}(F + p^e(D + i) \text{ on } L(\lambda))$.

Proof. (i) follows from Propositions 13(c) and 15(c) and exactness. (ii) By Propositions 13(b) and 15(b), $\overline{L(\lambda)} \cong T(p-1+d_0) \otimes \overline{L(b)}^{[1]}$ (with $T(p-1) = St$), which is $T(\lambda)$ by Donkin's tensor product theorem $T(p-1+r+pb) \cong T(p-1+r) \otimes T(b)^{[1]}$, $0 \leq r \leq p-1$ [Don93, Prop. 2.1]; for SL_2 in every characteristic p see [STWZ23, Prop. 3.4(a)] and [And97, Remark 1.1(ii)]. (iii) The lattice $\bigoplus_\lambda L(\lambda)^{n_\lambda(m)}$ has a Δ -filtration, $V^{\otimes m} \otimes \mathbb{Z}_p$ has a ∇ -filtration, and their reductions are isomorphic by (ii) and the decomposition of the tilting module $V^{\otimes m} \otimes \mathbb{F}_p$ [Jan03, E.6]. Since $\text{Ext}^1(\Delta_A, \nabla_A) = 0$, an isomorphism of the reductions lifts to a homomorphism over $\mathbb{Z}_p$, whose determinant is a unit. The existence and uniqueness of $T_{\mathbb{Z}_p}(\lambda)$ and the lifting of decompositions are classical [And97, §1.2], [Jan03, II.E.19–E.22]; what is new here is the explicit form of the lattices. The last statement is (2). ■

Remark 17. In the Lean proof (§12), the decomposition $V^{\otimes m} \otimes \mathbb{Z}_p \cong \bigoplus_\lambda L(\lambda)^{n_\lambda(m)}$ of (iii), with $n_\lambda(m)$ given by the recursion of the tensor rules, is proved without Donkin's theorem and without the lifting lemma: for $1 \leq r \leq p - 1$ there are explicit short exact sequences $0 \rightarrow \Psi_{r+1}(N) \rightarrow V \otimes \Psi_r(N) \rightarrow Q_{r-1}(N) \rightarrow 0$ (with $\Psi_p(N) := \Phi_p(V \otimes N)$, $Q_{r-1} = \Psi_{r-1}$ for $r \geq 2$, $Q_0 = \Phi_p(N)^2$), and they split over $\mathbb{Z}_p$ by a Fitting idempotent of the Casimir, whose two eigenvalue classes $(r + 1)^2$ and $(r - 1)^2$ differ modulo the odd prime p . Induction on m then gives it. The identification $L(\lambda) \cong T_{\mathbb{Z}_p}(\lambda)$ is not formalized.

9.2 Payment systems and the moves

Let N be a lattice graded by floors. A *payment system* of size R on $N^{\oplus R}$ is the operator $G(x_{t_{s'}}) = Fx_{t_{s'}} + \sum_s P(\varphi)_{ss'} x_{t_s}$ for x on floor φ , with $P(\varphi) \in M_R(A)$ lower triangular; it is *even* if every $P(\varphi) \in pM_R(A)$. Two systems are equivalent ($\sim$) when they have the same Smith form. The operator $F + p^e(D + i)$ on $L(\lambda)$ is an even system of size 1 with $P_0(\varphi) = p^e(\varphi + i)$, since $e \geq 1$.

Proposition 18 (twist move). Let $N = \Phi_p(N')$ and let G be even of size R . Then $G \sim I_{(p-1)R \text{ rank } N'} \oplus pG'$, with G' even of size R on N' and $P'(f) = (-1)^{p-1}u^{-1}p^{-1}P(pf)P(pf + 1) \cdots P(pf + p - 1)$.

Proof. For $j \leq p - 2$ the relation of $b_j n_{t_{s'}}$ has coefficient 1 on $b_{j+1} n_{t_{s'}}$ and its other terms on b_j ; eliminating these unit pivots gives, for the rows of generators $g_j(n) = (b_j n_{t_1}, \dots, b_j n_{t_R})$, $g_{j+1}(n) = -g_j(n)P(pf + j)$. The remaining relation, of $b_{p-1} n$, is $pu g_0(Fn) + g_{p-1}(n)P(pf + p - 1) = 0$, that is $pu [g_0(Fn) + g_0(n)P'(f)] = 0$. ■

Proposition 19 (split move). Let $N = \Psi_r(N')$ and let G be even of size R . Then $G \sim I_{2(p-1)R \text{ rank } N'} \oplus pG''$, with G'' even of size $2R$ on N' (slots (s, A) , then (s, C)) and

$$P''(f) = \begin{pmatrix} Q_A(f) & 0 \\ K(f) & Q_C(f) \end{pmatrix}, \quad \begin{aligned} Q_A(f) &= (-1)^{p-1}u^{-1}p^{-1}P(pf) \cdots P(pf + p - 1), \\ Q_C(f) &= (-1)^{p-1}u^{-1}p^{-1}P(pf + r) \cdots P(pf + r + p - 1), \\ K(f) &= (-1)^{p-r}u^{-1}P(pf + r) \cdots P(pf + p - 1). \end{aligned}$$

Proof. As for the twist: eliminate the $2(p-1)$ unit pivots per n and slot; the relation of $A_{p-1}n$, from $F(A_{p-1}n) = puA_0(Fn) + pC_{p-1}n$, becomes $pu[a_0(Fn) + a_0(n)Q_A(f) + c_0(n)K(f)] = 0$, and that of $C_{p-1}n$ becomes $pu[c_0(Fn) + c_0(n)Q_C(f)] = 0$. ■

Proposition 20 (last level). For G even of size R on $\Delta_A(d-1)^{\oplus R}$, $1 \leq d \leq p-1$: $G \sim I_{(d-1)R} \oplus M$ with $M = (-1)^{d-1}((d-1)!)^{-1}P(0)P(1) \cdots P(d-1) \in p^d M_R(A)$.

Proof. The pivots $s+1 \leq d-1 \leq p-2$ are units: $g_{s+1} = -g_s P(s)/(s+1)$, and the last relation is $g_{d-1}P(d-1) = 0$. ■

Theorem D_p (the odd dance). Let $\lambda \geq 0$, $e \geq 1$, $i \geq 0$. Put $P_0(\varphi) = p^e(\varphi + i)$ and, for $t = 0, \dots, k-1$, let P_{t+1} be the twist (Proposition 18) of P_t if $d_t = 0$ and the split with $r = d_t$ (Proposition 19) if $d_t \neq 0$. Let $M_p(\lambda, i, e) = P_k(0)P_k(1) \cdots P_k(d_k-1)$, a lower triangular matrix of size $2^{|I|}$ with entries in $p^{d_k} \mathbb{Z}_p$. Then

$$Y_p(\lambda, i, e) \cong S_p(\lambda) \oplus \text{coker}(p^k M_p(\lambda, i, e)).$$

Proof. By Lemma 16, $Y_p(\lambda, i, e)$ is the cokernel of an even system of size 1 on $L(\lambda)$. Apply the move of digit t for $t = 0, \dots, k-1$: before move t the lattice is $L(\lambda_t)$, $\lambda_t + 1 = \sum_{t' \geq t} d_{t'} p^{t'-t}$, carrying $2^{|I \cap [0, t]|}$ slots, and the move eliminates $(p-1)d_k p^{k-t-1} 2^{|I|}$ unit pivots and multiplies the rest by p . After k moves the system lives on $\Delta_A(d_k-1)$ with $2^{|I|}$ slots, and Proposition 20 applies. So the operator is equivalent to $I_{n_0} \oplus p(I_{n_1} \oplus p(\cdots \oplus p(I_{(d_k-1)2^{|I|}} \oplus M) \cdots))$ with $n_t = (p-1)d_k p^{k-t-1} 2^{|I|}$. ■

Proof of Theorem I. For odd p , by Theorem D_p, $Y_p(\lambda, i, e) \cong S_p(\lambda) \oplus \text{coker}(p^k M)$ with M of size $2^{|I|}$ and all entries in $p \mathbb{Z}_p$. Every Smith exponent of M is then ≥ 1 or ∞ , so $\text{coker}(p^k M)$ is the sum of exactly $2^{|I|}$ cyclic modules, each $\mathbb{Z}_p$ or of order $\geq p^{k+1}$, while every summand of $S_p(\lambda)$ has order $\leq p^k$. By the uniqueness of elementary divisors the statement follows. For $p = 2$, [DST, Theorem D] gives $Y_2(\lambda, i, e) = X^{(e)}(\lambda, i) \cong \bigoplus_{a=1}^{k-1} (\mathbb{Z}/2^a)^{2^{|I|+k-1-a}} \oplus \text{coker}(2^k M)$ with M the final matrix of an even payment system [DST, Proposition 4.3], of size $2^{|I|}$ with even entries; the same argument applies, and $S_2(\lambda)$ is the displayed group with $d_k = 1$. ■

9.3 The entries of M_p

Let $\mathcal{C} = \mathbb{Q}[x_t : t \in I]/(x_t^2)$. Define $c^{(0)} = 1$ and $c^{(h+1)} = a(c^{(h)})^p + [h \in I] b_h x_h (c^{(h)})^{p-d_h}$, with $a = (-1)^{p-1}/(up)$ and $b_h = (-1)^{p-d_h}/u$. For $\Delta \subseteq I$ write $[\cdot]_\Delta$ for the coefficient of $x^\Delta = \prod_{t \in \Delta} x_t$.

Lemma 21 (windows). After h moves the slots are the $D \subseteq I \cap [0, h]$, and $P_h(f)_{DD'} = [c^{(h)}]_{D \setminus D'} \prod_{d \in W} p^e(i+d)$ for $D' \subseteq D$ (and 0 otherwise), with $W = [p^h f + o_D, p^h f + o_{D'} + p^h - 1]$. Consequently

$$M_p(\lambda, i, e)_{DD'} = \frac{(-1)^{d_k-1}}{(d_k-1)!} [(c^{(k)})^{d_k}]_{D \setminus D'} \prod_{d=o_D}^{o_{D'}+K-1} p^e(i+d).$$

With $c^{(h)} = a^{(p^h-1)/(p-1)} q_h(y)$ and $y_t = -b_t(-p)^{1+d_t(p^t-1)/(p-1)} x_t$, the polynomials q_h satisfy $q_0 = 1$, $q_{t+1} = q_t^p - [t \in I] y_t q_t^{p-d_t}$.

Proof. As [DST, Theorem 4.4]: an entry of a product $P_h(\varphi_1) \cdots P_h(\varphi_s)$ over consecutive floors is a sum over chains $D = T_0 \supseteq \cdots \supseteq T_s = D'$; the windows of consecutive factors are adjacent, so every chain gives the same product of payments, and the coefficients add up to $[(c^{(h)})^s]_{D \setminus D'}$. The twist uses $s = p$ from floor pf ; the split uses $s = p$ from pf (Q_A), $s = p$ from $pf + r$ (Q_C , slot $D \cup \{h\}$, $o_{D \cup \{h\}} = o_D + d_h p^h$) and $s = p - r$ from $pf + r$ (K); the last level uses $s = d_k$ from floor 0. The normalization is a substitution. ■

Lemma 22 (deficit law). $[q_k^{d_k}]_\emptyset = 1$, and $v_p([q_k^{d_k}]_\Delta) = k - |\Delta| - \min \Delta$ for $\emptyset \neq \Delta \subseteq I$.

Proof. Induction on $|\Delta|$ and on $t > \max \Delta$, as in [DST, Lemma 4.5]. With $m = \max \Delta$, $[q_{m+1}]_\Delta = -[q_m^{p-d_m}]_{\Delta \setminus \{m\}}$. For a power N of q , $[q^N]_\Gamma$ is the sum over the set partitions of Γ into $j \leq N$ blocks of $N(N-1) \cdots (N-j+1) \prod_B [q]_B$. For $N = p - d_m \in [1, p-1]$ the single-block term has a unit factor; for $N = p$ (every later move) every falling factorial $(p)_j$, $1 \leq j \leq p$, has valuation 1. In both cases a partition into $j \geq 2$ blocks has valuation larger than the single-block one, by induction. So the valuation is $m - (|\Delta| - 1) - \min \Delta$ at $t = m + 1$ and grows by exactly 1 at each later move; the final power d_k is a unit. ■

Proposition 23 (cost form). Let $M = M_p(\lambda, i, e)$ and $b \subseteq a \subseteq I$, and let $\gamma(\Delta) = \sum_{u \in I \setminus \Delta, u > \min \Delta} h_u$ ($\gamma(\emptyset) = 0$). If $i \geq 1$, then $v_p(M_{ab}) = eK + \kappa_p(m, K) + R_m(a) - R_{m+K}(b) + \gamma(a \setminus b)$ with $m = i - 1$. If $i = 0$, the row $\emptyset$ is zero and, for $a \neq \emptyset$, $v_p(M_{ab}) = eK + R^\sharp(a) - R^\sharp(b) + \gamma(a \setminus b)$. All entries with $b \subseteq a$ ($a \neq \emptyset$ when $i = 0$) are nonzero.

Proof. By Lemmas 21 and 22, $v_p(M_{ab}) = -d_k \frac{p^k-1}{p-1} + \sum_{t \in \Delta} (1 + d_t \frac{p^t-1}{p-1}) + \delta(\Delta) + e(K - o_\Delta) + \sum_{d=o_a}^{o_b+K-1} v_p(i+d)$, $\Delta = a \setminus b$, $\delta(\Delta) = k - |\Delta| - \min \Delta$. Legendre's formula $v_p(n!) = (n - s_p(n))/(p-1)$, the identity $s_p(x + o_E) = s_p(x) + \sum_{t \in E} d_t - (p-1)\kappa_p(x, o_E)$, $s_p(K) = d_k$, and $|\Delta| + \delta(\Delta) = \sum_{t \in a} h_t - \sum_{t \in b} h_t + \gamma(\Delta)$ [DST, Lemma 5.1] collapse this to the stated form. For $i = 0$ use also $\kappa_p(K-1, o_E) = k - \min E$. ■

9.4 The Smith form of M_p

Lemma 24. The real antitonic fits of the clocks x and $x^\sharp$ of §8 take integer values.

Theorem (the Smith form of M_p). For $i \geq 1$ the Smith exponents of $M_p(\lambda, i, e)$ are $eK + \kappa_p(i-1, K) + \hat{y}_D$, $D \subseteq I$; for $i = 0$ they are ∞ (once) and $eK + \hat{y}_D$, $\emptyset \neq D \subseteq I$.

Proof of the Smith form and of Lemma 24. The proof of the case $p = 2$, [DST, §6–§7], goes through for odd p with the changes listed below; every step not listed is used verbatim. A complete line-by-line version for odd p , written by the independent reader, is part of the record of this paper (file THEOREM_F_ODD_P.md).

(O1) *The order.* Since $d_t p^t > \sum_{u < t, u \in I} d_u p^u$, the o-order is the binary order of the digit sets, as for $p = 2$; in particular all dancers without $c = \max I$ precede all dancers with c , and the two halves have $2^{|I|-1}$ elements each.

Ceiling [DST, §6]. The tropical bound (Lemma 6.1), the unique golden matching (Theorem 6.2), which is a statement about the binary numbers of (O1) and the quantity γ only, the natural minor (Corollary 6.3), which needs the cost form (Proposition 23) and the non-vanishing of the entries, convexity (Lemma 6.4) and Theorem 6.5 hold verbatim.

Floor [DST, §7]. Lemmas 7.1–7.3 and Theorem 7.4 are statements about sequences and hold verbatim. A *house* is $H = (I, k, \ell, e, (d_t))$ with $0 \leq \ell < p^k$, with rulers $R_H(E) = \sum_{t \in E} (h_t - e d_t p^t) - \kappa_p(\ell, o_E)$ and $J_H(E) = [\ell + o_E \geq p^k]$, an up-set.

- **Lemma 7.5_p.** With $\ell = m \bmod p^k$ and $r_k(\mu)$ the number of consecutive digits $p-1$ of μ from digit k on, $R_m = R_H - r_k(m)J_H$ and $R_{m+K} = R_H - r_k(m+K)J_H$.
- **Lemma 7.6_p (gates).** In the window $W_t = [t, \text{next}(t))$ the addend has the single digit $[t \in E] d_t$ at t ; put $\theta_t = p - \ell_t \in [1, p]$, $\kappa_t = 1 + \min(r_{t+1}(\ell), h_t - 1)$ and $\text{sat}_t = [\kappa_t = h_t]$. The window makes κ_t carries if and only if $[t \in E] d_t + c_t \geq \theta_t$ (c_t the incoming carry), so it has one of four types: OR ($\theta_t = 1$), DIG ($2 \leq \theta_t \leq d_t$, carries iff $t \in E$), AND ($\theta_t = d_t + 1$) and NEV ($\theta_t \geq d_t + 2$, never). A carry leaves the window if and only if it carries and is saturated. For $p = 2$ only OR and AND occur.
- **Lemma 7.7_p.** For $c = \max I$, $H' = (I \setminus \{c\}, c, \ell \bmod p^c, e)$, $h = k - c$ and $\tau = h - e d_c p^c - [\text{OR} \vee \text{DIG}] \kappa$: $R_H(D_0) = R_{H'}(D_0) - [\text{OR}] \kappa J'(D_0)$, $R_H(D_0 \cup \{c\}) = \tau + R_{H'}(D_0) - [\text{AND}] \kappa J'(D_0)$, $J_H(D_0) = [\text{OR}] \text{sat } J'(D_0)$, $J_H(D_0 \cup \{c\}) = \text{sat}([\text{OR} \vee \text{DIG}] + [\text{AND}] J'(D_0))$.

- *Theorem 7.8_p* (the induction of [DST, Theorem 7.8], steps 1–8) with the bound (Λ) replaced by (Λ_p) : $|\hat{y}_H| \leq e_{O_I}$. Step 2 uses $d_1 \leq ed_c p^c - h + \kappa \leq ed_c p^c$ and $o_I + d_c p^c = o_I$; in steps 3(a), 3(b) and 5 the quantity $\alpha 2^c - \alpha(2^c - 1) = \alpha$ becomes $e(d_c p^c - o_I) \geq e$, and $h - \alpha$ becomes $h - e$; in 3(c), $X_2(b') = \hat{y}'(b') - ed_c p^c \leq -e$. The two new window types fall into existing cases: DIG with $\text{sat} = 1$ has $J_H = (0, 1)$ and constant shift $d_1 = ed_c p^c$, which is case 3(b); DIG with $\text{sat} = 0$, and NEV, have $J_H \equiv 0$, which is case 3(d). Steps 4, 6, 7 and 8 hold with α replaced by e , with the inherited potential $v'(D_0) = V'(D_0) - [\text{OR}]\kappa J'(D_0)$, $v'(D_0 \cup \{c\}) = \tau + V'(D_0) - [\text{AND}]\kappa J'(D_0)$.
- Lemma 7.9 (integrality of the real fits, which is Lemma 24) and Theorem 7.10 follow verbatim, with the penalty cells $\text{Cell}(H; r_k(m), r_k(m + K))$; for $i = 0$ the house is $\ell = p^k - 1$, all windows are of type OR, and $J = [D \neq \emptyset]$.

Every claim of this list was also checked numerically, in 56 246 houses by the constructor and in 47 310 by the independent reader (§10). ■

Proof of Theorem J. Combine Theorem D_p with the Smith form of M_p : $\text{coker}(p^k M_p)$ has the Smith exponents of M_p raised by k . The statement on $K(H(d, q))$ follows from Theorem A and (2), with Lemma 16(iii) for the multiplicities; for $p = 2$ it is Theorem H. ■

10. Verification

All computations were exact. The Smith forms over $\mathbb{Z}_{(p)}$ were computed modulo p^M with $p^M < 2^{31}$, by unit pivots and Schur complements, level by level in the p -adic valuation; factors of valuation at least M were then determined separately, by the matrix $p^E M^{-1}$ of Lemma 4, whose elementary divisors are p^{E-a} for the divisors p^a of M . The scripts, the sealed prediction files and the logs are at <https://github.com/tretoef-estrella/coloured-sand-theorem> and in the Zenodo record doi:10.5281/zenodo.23327608 (all versions).

1. Theorem A: the Sylow p -subgroups for $p \in \{2, 3, 5, 7\}$ computed directly from the Laplacian and from the pieces agree in all 90 comparisons (27 cells, $q \in \{3, \dots, 10, 12\}$, $q^d \leq 729$, including $p \mid q$ and $p \leq d$); adding one extra copy of a piece changes the answer in 49 of them. Independently, the cells $(d, q) = (3, 7)$ and $(2, 11)$ were recomputed with Sage from the full Laplacian.
2. Theorem B: found first as a conjecture from the cells with $d \leq 4$; before the proof, ten predictions for $(d, p) = (2, 11), (2, 13), (2, 17), (2, 19), (2, 23), (3, 17), (3, 19), (3, 23), (3, 29), (4, 13)$ were written to a file (MD5 d5c36be3736079cbc2349f469751c627) and then computed: ten agreements.
3. Theorem C: 143 cells, for $q \in \{3, \dots, 10, 12\}$ and $p \in \{2, 3, 5, 7\}$, both with $p \mid q$ (including $p = 2$) and with $p \nmid q$, all in agreement; the guess $ed + v_p(d)$ fails in one of them and the guess ed in 26.
4. Theorem D: the rank formula for $W_{s,s+1}$ checked directly in 264 cases ($p \leq 7, m \leq 11$); the value of N agrees with the computed groups in 23 cells with $p \leq d$.
5. Theorem F, before its proof (version 1): found from 88 pieces with $p = 3, 5$. Then 30 pieces with $p = 7, 7 \leq m \leq 11, 1 \leq c \leq 6$ were predicted before being computed (MD5 of the file 1f5cc12e88f12e375022658da3eea6ce): 30 agreements. Then 14 full groups $\text{Syl}_p K(H(d, q))$ with composite $q \in \{6, 9, 10, 12, 15, 20, 25, 27\}$ and $p \leq d \leq 2p - 2$ (MD5 341a2c489aef1c30349bae079ddc81): 14 agreements. It also agrees with the 10 full groups computed for $q = p \in \{3, 5, 7\}$ and $p \leq d \leq 2p - 2$.
6. Lemmas 7 and 8, by machine:

- the lattices of Lemma 7 are stable under every $e^{(t)}$ and $f^{(t)}$ for $p \in \{3, 5, 7\}$ and every $\lambda \in [p, 3p - 2]$ (27 lattices; 2200 checks for $\lambda \leq 2p - 2$);
 - the cokernel of $q(c + H) + U$ on the model equals the formula of Lemma 8 in 292 random cases with $\lambda \leq 2p - 2$ ($p \leq 7$), in 180 cases with $\lambda \leq p^2 - 2$ ($p = 3, 5; e \in \{1, 2\}$), and in 60 cases with $c = 0$;
 - removing the glue (the split lattice) changes the answer in 293 of 293 cases, and the formula of Theorem F applied beyond $2p - 2$ fails in 12 of 18 pieces.
7. Theorem G against direct computation (version 2): the predicted pieces equal the Smith forms of $q(c+H)+U$ computed on B_m in 154 cases ($p = 3, m \leq 7; p = 5, m \leq 10; e \in \{1, 2\}; 0 \leq c \leq 6$) and in 5 cases with $p = 3, m = 8 = 3p - 1$. The full groups for $p = q = 3$ and $d = 5, \dots, 8$ computed in version 1 from the full Laplacian (Section 11) are reproduced exactly.
8. Corollary 6: $\dim_{\mathbb{F}_p} \text{coker } U$ on B_m computed by ranks agrees with the count of Jordan blocks in 37 cases ($p \in \{3, 5, 7\}, m \leq 12$); the characteristic-0 count $\binom{m}{\lfloor m/2 \rfloor}$ differs in 7 of them.
9. Theorem G up to $m \leq p^2 + p - 1$ (version 3):
- the formula of Lemma 8 against the Smith form on the model lattice of Lemma 7, for $p \in \{5, 7\}$ and every $\lambda \leq p^2 - 1, c = k = 0$ included: 216 of 216; the split lattice differs in 52 of 52; the lattices of Lemma 7 are stable under every $e^{(t)}$ and $f^{(t)}$ for every $\lambda \leq p^2 - 1$ with two Weyl factors, $p \in \{5, 7\}$ (52 lattices);
 - a first sealed prediction (MD5 9ef6b4fec30a93192b0de995519d75f3), written after the formula was derived and before the computation: 36 pieces ($p = 3, m \in \{9, 10\}, q \in \{12, 15, 21\}, 5 \leq c \leq 8; p = 2, m = 4, q \in \{6, 8, 10\}, 5 \leq c \leq 8$), the four full groups $\text{Syl}_3 K(H(10, 3)), \text{Syl}_3 K(H(9, 6)), \text{Syl}_2 K(H(4, 6)), \text{Syl}_2 K(H(4, 10))$, and $\text{Syl}_2 K(H(4, 6))$ again from its full Laplacian: 41 agreements;
 - a second one (MD5 a092c60102cfca163c8449b60680b1e1): 18 pieces ($p = 3, m = 11, q \in \{12, 15\}, 4 \leq c \leq 6; p = 2, m = 5, q \in \{6, 8, 10\}, 4 \leq c \leq 7$), the full groups $\text{Syl}_3 K(H(11, 3)), \text{Syl}_2 K(H(5, 4)), \text{Syl}_2 K(H(5, 6))$, and $\text{Syl}_2 K(H(5, 4))$ from its full Laplacian: 22 agreements;
 - the row $d = 9$ of Section 11, which version 2 did not explain, is reproduced exactly.
10. Theorem H: 384 pieces against Smith forms on B_m ($m \leq 11, e \leq 4, c \leq 4$) and seven full Laplacians, with the two controls described after the theorem.
11. Theorem I, before its proof: on the formulas of Theorems G and H, 9045 groups for $p = 2$ ($\lambda \leq 200, e \leq 5, i \leq 8$) and 4704 for $p \in \{3, 5, 7\}$, with no exception; the number of summands equals $\dim T(\lambda)/p$ in 286 of 286 cases; the count $2^{|\Gamma(\lambda)|}$ of Weyl factors from the characters for $p \leq 5, \lambda \leq 120$ (363 cases). For the block $\lambda = 12, p = 3$: a third sealed prediction (MD5 a739d9c18622ab0c5dfc811f4cbd49f7), written after one cell, for seven cells; seven agreements.
12. Lemma 16 and the moves (version 4): the lattices $L(\lambda)$ satisfy the $\mathfrak{sl}_2$ relations, have the character of $T(\lambda)$, and are stable under every $E^{(r)}$ and $F^{(r)}$, for every λ with $\dim T(\lambda) \leq 600$ and $p \in \{3, 5, 7\}$ (569 lattices, checked twice, by the constructor and by the independent reader with its own engine); the closed formula for $E^{(r)}$ of Proposition 13 on 273 812 pairs; $L(\lambda) \otimes_{\mathbb{F}_p} \cong T(\lambda)$ identified without Donkin's theorem in 62 cases. Controls (the twist without its factor p , the split without its coupling, the coupling p^2) break integrality or change the answer.
13. Theorem D_p : the right-hand side computed by the moves equals a direct p -adic Smith form of $F + p^e(D + i)$ on $L(\lambda)$ in 1856 cells ($p = 3, \lambda \leq 80; p = 5, \lambda \leq 150; e \in \{1, 2\}, i \leq 3$); among them 1520 cells sealed before the direct computation (MD5 0a9c3442e54c54c8cc880d9b2bc070b0), 1513 of them blind: all hold.

14. Theorem J: 10 710 cells ($p \in \{3, 5, 7, 11, 13\}$) sealed before the computation (MD5 2f931de9eb3933c51a4ff3efb03274ab), against the exact Smith form of M_p : all hold. The formula contains Theorem G (3450 cells, $p \leq 11$), Theorem H (5628 cells) and Theorem B (128 groups), and reproduces the rows $d = 5, \dots, 11$ of Section 11.
15. Independent of the lattices $L(\lambda)$: on each block, the length of $Y_p(\lambda, i, e)$ equals $v_p \det(F + p^e(D + i))$, computed from the character, and the number of its cyclic summands equals $\dim T(\lambda)/p$ (20 310 cells, $p \leq 11$); for whole Hamming graphs, the length of $\text{Syl}_p K(H(d, q))$ equals the p -adic valuation of Kirchhoff's count and the number of cyclic factors equals the corank of the Laplacian over $\mathbb{F}_p$ minus one (180 groups: $p = 3, d \leq 20$; $p = 5, d \leq 14$; $p = 7, d \leq 10$; $p = 11, d \leq 8$; and 12 more, up to $d = 60$, by the independent reader). Direct Smith forms on B_{12} at $p = 3, q = 9, c \leq 3$ (4 cells, every exponent below 19), and 31 pieces with $m \in \{12, 13\}$ ($p = 3$) or $p = 2$, sealed by the independent reader before computing them: all agree.
16. The floor of the Smith form (Theorem 7.8_p): every claim of the list in §9.4 was checked numerically in 56 246 houses ($p = 3, k \leq 4$; $p = 5, k \leq 3$; $p = 7$ sampled), all four window types occurring, and again in 47 310 houses by the independent reader.

11. Data for $p = q = 3$

For $q = p = 3$ the computed groups, written as $(n_1, n_2, \dots)$ with n_v the number of factors $\mathbb{Z}/3^v$, are:

d	$(n_1, n_2, \dots)$
5	(12, 10, 16, 53, 5)
6	(38, 37, 36, 117, 24, 16, 6)
7	(86, 105, 127, 285, 85, 63, 41)
8	(250, 211, 286, 1078, 174, 133, 174, 8)
9	(942, 412, 624, 3339, 449, 292, 495, 133, 84, 37, 9)
10	(3314, 1388, 1768, 7310, 1690, 1364, 1388, 1392, 316, 190, 74)
11	(11035, 4345, 5127, 11339, 10099, 4994, 4432, 7162, 704, 419, 416)

Theorem G gives every row exactly. In version 2 the row $d = 9$ was outside the range: its piece $X_3(9, 0)$ contains the block $\lambda = 9$, which has $a = p - 1$, and the version-2 formula, with glue $1/p$, gave (942, 414, 623, 3339, 448, 293, 495, 134, 84, 36, 9). With the glue $1/p^2$ of Lemma 7 and the minimum in the definition of $G_c(m, \lambda)$ the row comes out right. The rows $d = 10$ and $d = 11$ were computed piece by piece and assembled by Theorem A; both were predicted before they were computed (the first two sealed predictions of item 9 of Section 10 contain them).

12. Formal verification in Lean

The proofs are checked in the proof assistant Lean 4 with the Mathlib library. The text was cut into short self-contained pieces; Aristotle, an AI system made by Harmonic [Har], wrote the Lean proofs; on the author's machine each returned project was compared with the previous one, searched for forbidden constructs (sorry, new axioms, native_decide), compiled module by module, and its main statements were restated by hand in a check file whose axioms were printed. Every certified piece uses only the standard axioms propext, Classical.choice and Quot.sound.

Version 3 is checked. Theorems A–H, Lemmas 2–5, 7 and 8, Corollaries 1 and 6, and the Example of Section 6 are proved in Lean, in the generality of version 3 (Theorem E for $p \mid q$, Theorems F and G for odd p ; the other cases are checked by the pieces of the next paragraph), with nothing cited: the tilting lattices of Section 6, their

tensor rules and the decomposition of $V^{\otimes m} \otimes \mathbb{Z}_p$ for $m \leq p^2 + p - 1$ are proved from scratch, as is Wilson’s rank formula in the range used by Theorem D; Theorem H uses the Lean proof of [DST]. At that stage the pieces of version 3 had 110 files, 20 765 lines and 1 032 lines beginning with `theorem` or `lemma`, with no `sorry`. **Theorems I and J are checked.** Their proof was cut into 19 pieces, all certified: the twist Φ_p (Definition 12, Proposition 13) and its reduction modulo p ; the split Ψ_r ; the Casimir and the short exact sequences of the Remark after Lemma 16, and their splitting over $\mathbb{Z}_{(p)}$; the dance lattices $L(\lambda)$ and $V^{\otimes m} \otimes \mathbb{Z}_p \cong \bigoplus L(\lambda)^{n_\lambda(m)}$ for every m , without Donkin’s theorem; the twist move (Proposition 18), the split move (Proposition 19) and the last level; Theorem D_p and Theorem I; the deficit law, the windows, the cost form, the ceiling, the houses, Theorem F at p , the integrality and the floor of every cell; the Smith form of the final matrix; and Theorem J. The statement checked for Theorem J is about the sandpile group itself: the torsion of the cokernel of Mathlib’s Laplacian matrix of the Hamming graph $H(d, q)$, for every odd prime p , every $q \geq 2$ with $p \mid q$ and every $d \geq 0$. The multiplicities $n_\lambda(m)$ enter through the recursion given by the tensor rules, which is proved to describe $V^{\otimes m} \otimes \mathbb{Z}_p$; the description by characters at the end of Theorem J is a way to compute them and is not formalized.

Every prime. Three further pieces check the cases that the pieces above left out: Theorem E for $p \nmid q$ (the pieces above assume $p \mid q$); Theorem I for $p = 2$, stated on the groups $X^{(e)}(\lambda, i)$ of [DST], which on paper are the blocks $Y_2(\lambda, i, e)$ of Theorem H; and Theorems F and G for $p = 2$, with the bounds $m \leq 2$ and $m \leq 5$. With them, every result of the paper is checked for every prime. In Lean, Theorem I and the block formulas of Theorems J and H are proved for explicit dance lattices: $L(\lambda)$ for odd p , and the lattices of [DST] for $p = 2$; their identification with $T_{\mathbb{Z}_p}(\lambda)$ (Lemma 16(ii); [DST, Remark (1) after Theorem 3.6]) is not formalized and is not needed for the statements on $K(H(d, q))$. Theorem E is stated in Lean for the $\mathbb{Z}_p$ -module $\text{coker}(q(c + H) + U)$ on $B_m(\mathbb{Z}_p)$, free summand included; its form for $\text{Syl}_p X_q(m, c)$ follows in a few lines and is checked too (checks/CheckTheoremEPaperForm.lean, written by the fifth reader).

All 36 pieces live in one Lean project, which also holds the Lean proof of [DST]. Two groups of pieces defined the same names; the names were changed on one side only: in the three newer pieces of the first clash and in the files that use them (14 files in five pieces), and, for the second, in the piece that no other module imports (five files in one piece), and both renames were checked to be pure (undoing them gives back the original files byte for byte). The final check is six files; each loads all 36 pieces together and repeats the hand restatements of a part of them. They compile with no error, and each of their 451 `#print axioms` commands answers with the standard axioms only. The pieces of this paper have 254 files, 46 787 lines and 2 370 lines beginning with `theorem` or `lemma`, with no `sorry`. The whole Lean project was rebuilt from its sources, on Mathlib’s published build, with the final check, in GitHub Actions run 38724562833 of the repository <https://github.com/tretoef-estrella/coloured-sand-theorem>, at commit 51c25f0, on 15 October 2026. The Lean project is in the folder `lean/` of that repository; the Lean certificate, with the full statements, the dependency cone of the main theorems and the data of every piece, is deposited with this version in its Zenodo record.

13. Changes

From version 4 to version 6 (version 5 was not published)

1. The Lean proof is complete, for every prime (§12): Theorems I and J, and three further pieces for Theorem E with $p \nmid q$, Theorem I with $p = 2$, and Theorems F and G with $p = 2$. The Lean project was rebuilt from its sources on GitHub’s machines, on Mathlib’s published build. The summary table, the abstract and the status paragraph say so. No statement and no proof of the paper changes.
2. The Lean section says that Theorem I and the block formulas of Theorem J are checked for explicit dance lattices ($L(\lambda)$ for odd p , and those of [DST] for $p = 2$), and that their identification with $T_{\mathbb{Z}_p}(\lambda)$ is not

formalized; it describes the two renames correctly, and gives the size of the version-3 pieces by the same counting rule as the rest of the section.

3. Bibliography: [DST] with its concept DOI and the DOI of its version 11, labelled, and the DOI of its Lean certificate.
4. The status paragraph and the section on provenance describe the cold readings of the Lean work. A fifth reader read the changes of this version (the Lean certificate v2 and this paper) cold; its corrections are applied: the description of the two renames, the form in which Theorem E is stated in Lean (and a check file for the paper's form), the lattices at $p = 2$, the remark after Lemma 16, the words “rebuilt from source”, the generality of the version-3 pieces, and the subsection on digits now fixes any prime.
5. The layout of [DST].

From version 3 to version 4

1. New Theorems I (Conjecture I of version 3, every prime) and J (the whole group, every d), Sections 8 and 9, with their checks (§10, items 12–16) and the state of the Lean proof (§12). New title page and summary.
2. Priority: the case $d = 2$ was known (Ducey–Gerhard–Watson [DGW16, Theorem 1.1]; it also follows from [JNR03, Theorem 13 and Example 11]). Version 3 claimed Theorems B, C, D, G and H as new without excluding $d = 2$. The existence of the $\mathbb{Z}_p$ -forms of tilting modules and the lifting of decompositions are classical [And97, §1.2], [Jan03, II.E]; what is ours is their explicit form.
3. Theorem D: Wilson’s diagonal form is used in the range $t \leq \min(k, m - k)$, which is Wilson’s own [Wil90, Theorem 2]. Version 3 attributed it to the summary in [CSX15, (2)], which states only $t \leq k \leq m/2$ and misses the middle rank $2s = m - 1$ of odd m .
4. Remark after Theorem G: version 3 said that for $p \nmid q$ the formula fails at $(q, p, d) = (4, 3, 4)$. This was wrong: evaluated at $e = 0$, the formula of Theorem G gives the right group there.
5. Proof of Lemma 8: the sentence “every entry has valuation ≥ 1 ” was placed before the elimination that makes it true.
6. Theorem B: the condition $(j, k) \neq (0, 0)$ is automatic, since $j + 2k = d + 1 - v \geq 1$.
7. The count of predictions: version 3 said 107 (44 in version 1 and 63 in version 3); two groups of version 3 were counted twice (once from the pieces and once from the full Laplacian), so they are 105 distinct predicted groups.
8. Version 3 said that Conjecture I for $p = 2$ follows from Theorem H; that needed every big clock to exceed p^k , which was only checked by machine. It is now part of the proof of Theorem I.
9. Precise references for (T1) and (T2): [Jan03, II.E.6, II.E.19–E.22], [Don93, Prop. 2.1], [STWZ23, Prop. 3.4(a)], [And97, Remark 1.1(ii), §1.2].

Provenance, priority and use of AI

The work was done for the author by Claude (Anthropic), an AI system, in several sessions. Version 1 (night of 12–13 October 2026): choice of the problem after a search of the literature, computations, Theorems B–E and Conjecture F, with Theorem A recognized as [JNR03, Theorem 13] before publication. Versions 2 and 3 (the same night): Theorems F, G and H, Corollary 6 and Conjecture I. Version 3 was read cold by a separate instance with no access to our notes; its corrections are listed in Section 13. Version 4 (13–14 October 2026): Theorems I

and J were found and proved by a separate instance working as constructor, re-derived and re-computed by the instance that audits the project, and read cold by a third instance, with no access to the audit, which wrote the floor of the Smith form out in full, re-ran its own computations and searched the literature again. Version 6 (15 October 2026): the Lean proof was completed for every prime and the Lean project rebuilt from its sources on GitHub’s machines; version 5 and the Lean certificate were read cold by a fourth instance, which had the paper, the certificate, the Lean project and the author’s logs but not the working sessions; the changes of version 6 were read cold by a fifth instance with the same kind of material, which found no error in the mathematics or in the Lean statements. Their corrections are listed in Section 13. Lean proofs are written by Aristotle [Har] and compiled and audited on the author’s machine (Section 12). No human expert has read the proofs. The author takes responsibility for the paper.

To the best of our knowledge, after a search of the works citing [JNR03], [DGW16], [DJ14], [CSX15], [Bai03] and the main works on tilting modules of SL_2 , of arXiv, zbMATH, OpenAlex, Zenodo, HAL, CORE, MathOverflow and the Hexagon preprint server: Theorem A is due to Jacobson, Niedermaier and Reiner; the case $d = 2$ of every result is due to Ducey, Gerhard and Watson [DGW16]; the case $e = 0$ of Theorem C follows from [JNR03, Theorem 2]; the tilting theory used (Donkin’s tensor product theorem, the existence and uniqueness of $\mathbb{Z}_p$ -forms of tilting modules, the count of Weyl factors, projectivity over the first Frobenius kernel) and the decomposition of tensor powers in the proof of Corollary 6 are classical [Don93, Jan03, And97]. Theorems B–J for $d \geq 3$, Corollary 6, and the explicit lattices of Section 9 were not found in the literature.

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