hexagon:2610.00260v1

STABLE ALGEBRAIC FAMILIES AND EXCEPTIONAL FIBRATIONS

Abstract. We extend McMullen's theorem on stable algebraic families of rational maps to polarized endomorphisms of normal projective varieties over an irreducible quasi-projective curve. We prove that J-stability on a Zariski open subset is equivalent to isotriviality or to the existence, after a finite base change, of a positive-dimensional exceptional factor whose generic fiber has simultaneous local product coordinates with one isotrivial polarized fiber system. The algebraic definition of exceptional maps in arbitrary dimension was recently introduced by Ji, Song, and Xie. The proof draws on Ji and Xie's use of homoclinic orbits and the multipliers of adjoint periodic points in their new proof of McMullen's theorem. We combine this approach with induction on the phase dimension and algebraization of local return structures.

1. Introduction and definitions

McMullen's rigidity theorem asserts that a stable irreducible algebraic family of rational maps of degree at least two is either isotrivial or consists of flexible Lattès maps; see [McM87, Theorem 2.2] and the modern formulation [JX23, Theorem 1.2]. Ji and Xie give a new proof using homoclinic orbits and the multipliers of adjoint periodic points [JX23, Sections 2.2–2.3 and 6]. Their viewpoint motivates the local return arguments used here to study generic J-stability in higher phase dimension, while allowing the normal projective fibers to vary.

Throughout, varieties are integral and the ground fields have characteristic zero. The parameter space S is a smooth connected complex algebraic curve, and $K = \mathbf{C}(S)$. A finite parameter extension means a finite extension K'/K, realized by the normalization of S in K' and restriction to a nonempty open subset. The resulting parameter curve is again smooth and connected. For a complex variety X, write $X_{K'} = X \times_{\mathbf{C}} \operatorname{Spec} K'$. For a family $p : \mathcal{X} \to S$, write $X_\eta = \mathcal{X} \times_S \operatorname{Spec} K$ and $X_{\eta,K'} = X_\eta \times_K \operatorname{Spec} K'$.

Definition 1.1 (Polarized families)**.** A *q-polarized endomorphism* of a normal projective variety V is a finite surjective morphism $u : V \to V$ together with an ample line bundle L such that $u^*L \simeq L^{\otimes q}$, where $q > 1$ is an integer. An *algebraic family* consists of a flat projective morphism $p : \mathcal{X} \to S$ with geometrically integral normal fibers, an S-endomorphism f of $\mathcal{X}$, and a relatively ample line bundle L satisfying $f^*L \simeq L^{\otimes q}$. A family on a fixed X has $\mathcal{X} = S \times X$.

Definition 1.2 (J-stability)**.** Let $(p : \mathcal{X} \to S, f, L)$ be an algebraic family as above, with fibers of dimension N. For $s \in S(\mathbf{C})$, let T_s be the Green current of f_s and set

$$\mu_s = \frac{T_s^N}{(L_s^N)}, \qquad J_s = \operatorname{supp} \mu_s.$$

Here $T_s = \lim_{m\to\infty} q^{-m}(f_s^m)^*\omega_s$ for a smooth positive representative ω_s of $c_1(L_s)$. The continuous local potentials of T_s define its Monge–Ampère measure; on a normal fiber this can equivalently be computed on a resolution and pushed down. For $n \geq 1$, write

$$R_n(s) = \left\{ x \in X_{s,\mathrm{reg}} \cap J_s : \begin{array}{l} f_s^n(x) = x, \\ |\lambda| > 1 \text{ for every eigenvalue } \lambda \text{ of } D_x f_s^n \end{array} \right\}.$$

Thus $R_n(s)$ consists of the smooth repelling Julia points fixed by f_s^n; their exact periods may divide n.

The family is *J-stable* if, for every $s_0 \in S(\mathbf{C})$, there is an analytic disk $B \subset S^{\mathrm{an}}$ centered at s_0 such that, for every $n \geq 1$ and every $x \in R_n(s_0)$, there is a holomorphic section

$$\gamma_{n,x} : B \longrightarrow \mathcal{X}_B, \qquad p \circ \gamma_{n,x} = \mathrm{id}_B, \qquad \gamma_{n,x}(s_0) = x,$$

and, for every $s \in B$, evaluation induces a bijection

$$R_n(s_0) \longrightarrow R_n(s), \qquad x \longmapsto \gamma_{n,x}(s).$$

For each n, these sections are disjoint, remain smooth and repelling, and exhaust the smooth repelling Julia points fixed by f_s^n in every fiber. The same disk B works for all n. This is also called *strong J-stability.*

The family is *generically J-stable* if its restriction to $S \setminus E$ is J-stable in this sense for some finite subset $E \subset S$. Equivalently, it is J-stable on a nonempty Zariski open subset of S.

Definition 1.3 (Isotriviality and rational factors)**.** A family is *isotrivial* if, after a finite extension of K, its generic dynamical system is isomorphic to a dynamical system defined over $\mathbf{C}$. A *rational fibration* $\pi : V \dashrightarrow Z$ is a dominant rational map whose geometric generic fiber is integral; equivalently, $k(Z)$ is relatively algebraically closed in $k(V)$.

It is a *dynamical factor* of (V, u) if Z is normal projective and has an endomorphism g satisfying

$$\pi u = g\pi.$$

Equalities of rational maps are understood on their common domain of definition. The *join* of finitely many rational factors is the normalization of their joint image. Its connected version is obtained by taking the relative algebraic closure of its function field in $k(V)$.

Definition 1.4 (Exceptional maps of abelian variety type)**.** An *abelian variety* is a connected projective algebraic group. An *isogeny* is a surjective homomorphism of abelian varieties with finite kernel. It is *Q-polarized* if it pulls an ample line bundle back to its Qth tensor power.

A q-polarized system (Z, g) is *exceptional of abelian variety type* if there exist an abelian variety A, a finite surjective morphism $\eta : A \to Z$, an integer $e \geq 1$, and a q^e-polarized isogeny $\alpha : A \to A$ such that

$$\eta\alpha = g^e\eta. \tag{1.1}$$

We call (A, α, η, e) an *abelian presentation*. These are the abelian variety type among the exceptional systems of [JSX26, Definitions 1.2–1.4]. The presentation data are defined after a finite extension of the ground field.

Definition 1.5 (Simultaneous local product coordinates)**.** A *phase open* is a Zariski open subset of a dynamical variety over the parameter function field. Let $K = \mathbf{C}(S)$, let $\pi : X \dashrightarrow Z$ be a dominant rational map of normal projective K-varieties with geometrically integral generic fiber, and suppose that

$$\pi \circ f = g \circ \pi.$$

Choose an equivariant normal projective graph model $\rho : \Gamma \to Z$ with induced map F, and let L be an ample line bundle on Γ satisfying $F^*L \simeq L^{\otimes q}$. Write $\eta = \operatorname{Spec} K(Z)$ and $\sigma = g^* : K(Z) \hookrightarrow K(Z)$. The *generic fiber path of length n* is the polarized diagram

$$\Gamma_\eta \longrightarrow \sigma^*\Gamma_\eta \longrightarrow \cdots \longrightarrow (\sigma^n)^*\Gamma_\eta,$$

whose arrows are induced by F and whose polarizations are the corresponding pullbacks of $L|_{\Gamma_\eta}$.

The generic fiber arrows have *simultaneous local product coordinates with one fixed polarized fiber system over* $\mathbf{C}$ if there are a normal projective complex variety W, an ample line bundle L_W, and a finite endomorphism $h : W \to W$ satisfying $h^*L_W \simeq L_W^{\otimes q}$ such that every generic

fiber path is geometrically polarized isomorphic to

$$W \xrightarrow{h} W \xrightarrow{h} \cdots \xrightarrow{h} W.$$

A polarized path isomorphism consists of one polarized isomorphism at each vertex, commuting with every arrow. Such an isomorphism from the fixed chain to the actual path is called a *frame*. These frames admit the following simultaneous realization. For some integer n_0, the scheme of frames of the length-n_0 generic path represents frames of every longer forward path: restriction to its first n_0 arrows is an isomorphism.

In geometric coordinates, choose a dense open $V \subset Z$ over which ρ has normal geometrically integral fibers. For each $n \geq 1$, after restricting to a dense open

$$V_n \subseteq \bigcap_{i=0}^{n} g^{-i}(V),$$

there are an *étale* surjection $T_n \to V_n$ and polarized isomorphisms

$$c_i : T_n \times W \xrightarrow{\sim} \big((g^i)^*\Gamma\big)|_{T_n}, \qquad 0 \leq i \leq n,$$

for which

$$F_i \circ c_i = c_{i+1} \circ (\mathrm{id}_{T_n} \times h).$$

Here F_i is the induced ith fiber arrow, and all products with W include its base change from $\mathbf{C}$. The frames can be chosen by restriction of the same finite-length frame torsor. After spreading over a parameter open, its analytic local sections give holomorphic product coordinates. Intersecting these charts with the locus where $\Gamma \dashrightarrow X$ is an isomorphism gives the corresponding coordinates on dense phase opens of X. For a zero-dimensional fiber we take $W = \operatorname{Spec} \mathbf{C}$ and $h = \mathrm{id}_W$.

Theorem 1.6 (Stable families and exceptional fibrations)**.** *Let $f : \mathcal{X} \to \mathcal{X}$ be an algebraic family of q-polarized endomorphisms of relative dimension $N \geq 1$ over a smooth connected complex algebraic curve S. Put $K = \mathbf{C}(S)$, and write (X_η, f_η) for its generic dynamical system. Then f is generically J-stable if and only if one of the following holds:*

1. *The family f is isotrivial.*
2. *After a finite extension K'/K, there are a normal projective K'-variety Z, a q-polarized endomorphism $g : Z \to Z$ exceptional of abelian variety type, and a rational fibration*

$$\pi : X_{\eta,K'} \dashrightarrow Z, \qquad \pi f_{\eta,K'} = g\pi, \qquad 1 \leq r := \dim Z \leq N,$$

whose generic fiber arrows have simultaneous local product coordinates with one fixed polarized complex fiber system (W, L_W, h) of dimension $N - r$, in the sense of Definition 1.5.

The rational fibration has source $X_{\eta,K'}$. On a dense open U where π is regular, equivariance gives the map to the dynamical pullback

$$U \cap f^{-1}(U) \longrightarrow Z \times_{g,Z,\pi} U, \qquad x \longmapsto (\pi(x), f(x)).$$

Its fiber over z is the arrow from the fiber over z to the fiber over $g(z)$.

Corollary 1.7 (Families on projective space)**.** *Let S be a smooth connected complex algebraic curve and let $f : S \times \mathbf{P}^N \to S \times \mathbf{P}^N$ be an algebraic family of endomorphisms of degree $d \geq 2$. Put $K = \mathbf{C}(S)$. Then f is J-stable if and only if, after a finite extension K'/K, one of the following holds:*

(1) *There are $f_0 \in \mathrm{End}_d(\mathbf{P}^N)(\mathbf{C})$ and $M \in \mathrm{PGL}_{N+1}(K')$ such that $f_{K'} = M^{-1} f_0 M$.*
(2) *There are a normal projective geometrically unirational K'-variety Z, a d-polarized endomorphism $g : Z \to Z$ exceptional of abelian variety type, and a rational fibration*

$$\pi : \mathbf{P}^N_{K'} \dashrightarrow Z, \qquad \pi f_{K'} = g\pi, \qquad 1 \leq \dim Z \leq N,$$

whose generic fiber arrows have the simultaneous local product coordinates of Definition 1.5, with one fixed polarized fiber system over $\mathbf{C}$.

Here the polarization on the source is $\mathcal{O}_{\mathbf{P}^N}(1)$ and $q = d$. If $\dim Z = 1$, then $Z_{\overline{K}} \simeq \mathbf{P}^1_{\overline{K}}$ and g is a Lattès map. If $\dim Z = N$, then $f_{\overline{K}}$ is itself a Lattès map.

Here *geometrically unirational* means dominated by a projective space after passage to an algebraic closure. A *Lattès map* on $\mathbf{P}^r$ is an endomorphism admitting a finite surjective semiconjugacy from an affine isogeny of an abelian variety of dimension r.

Constant repelling spectra give local return structures, whose algebraization produces polarized rational factors. Uniformly bounded covers of periodic fibers close the induction on the phase dimension. A maximal varying abelian factor has constant finite fiber-path classes, and a stationary path representative gives the fixed system (W, L_W, h). For the converse, compatible frames extend over the abelian phase base. Periodic sections of the resulting bundle give an equilibrium web, and the relative web criterion proves J-stability on a nonempty parameter open.

2. Equilibrium webs and removable parameters

The main result of this section is Theorem 2.3: for a polarized family on a fixed normal projective variety, the existence of an arbitrary equilibrium web, the holomorphic motion of all repelling Julia cycles, and

the vanishing of the Lyapunov bifurcation current dd^cL are equivalent. Here L is the sum of the Lyapunov exponents of the equilibrium measure. We distinguish the new statements from the existing projective-space theory. Equilibrium webs were introduced by Berteloot–Bianchi–Dupont [BBD18]. Their Theorem 1.1 proves this equivalence for arbitrary families on $\mathbb{P}^2$ over simply connected parameter spaces, and for simply connected open subsets of the full parameter space in every dimension. Berteloot–Buff [BB26, Proposition 1.2 and Theorem 4.1] prove in every dimension that μ-stability, defined by the existence of an equilibrium lamination, implies the holomorphic motion of all repelling Julia cycles. The passage from acritical ergodic webs to laminations is established in [BBD18, Theorem 4.1]; see also [BB26, Theorem 2.2]. In particular, the equivalence between $dd^cL = 0$, an equilibrium lamination, and the motion of all repelling Julia cycles on $\mathbb{P}^N$ in arbitrary dimension is already contained in [BB26, Theorem 4.1]; the Lyapunov criterion itself is not a new assertion of this paper.

The new implication in Theorem 2.3 starts from an arbitrary equilibrium web, without assuming acriticity, ergodicity, or a pre-existing equilibrium lamination. This strengthens the cited results even for arbitrary one-parameter holomorphic families on $\mathbb{P}^N$ with $N \geq 3$: the implication from such a web to the motion of every repelling Julia cycle was not previously available in that generality. Consequently, an arbitrary equilibrium web can also be added to the equivalent Lyapunov and lamination criteria in this setting. The key input is the jet rigidity of Lemma 2.6, which gives uniqueness of a section through almost every point at each fixed parameter for any equilibrium web. This is a pointwise-in-parameter uniqueness statement for each given web, stronger than merely constructing one acritical web. Lemma 2.7 then supplies the inverse histories used in the argument of [BB26, Proposition 1.2]. This yields the criterion on normal projective varieties, including repelling cycles at singular points. The normal-space Lyapunov identity and its connection with this criterion are proved in Subsection 2.4, rather than inferred from results stated only for $\mathbb{P}^N$. Further extensions here are the removability result for a fixed normal phase variety (Lemma 2.20) and the relative criterion for varying normal fibers on a nonempty parameter open (Proposition 2.23).

Work over a coordinate disk B in a smooth complex parameter curve, with fixed normal projective phase variety X of dimension k. We first treat this fixed phase space; the final subsection gives the relative criterion for varying normal phase families. Put $J_\lambda = \operatorname{supp} \mu_\lambda$. For this section, a periodic point is called repelling when the eigenvalues on its Zariski tangent space all have modulus greater than one. Write $R_n(\lambda)$

for these points in J_λ fixed by f_λ^n, and $R_n^{\mathrm{reg}}(\lambda) = R_n(\lambda) \cap X_{\mathrm{reg}}$. The notation $\mathcal{O}(B, X)$ means the space of holomorphic sections, with the topology of locally uniform convergence. Set

$$(\mathcal{F}\gamma)(\lambda) = f_\lambda(\gamma(\lambda)), \qquad e_\lambda(\gamma) = \gamma(\lambda).$$

Definition 2.1 (Equilibrium web)**.** An equilibrium web is a compactly supported Borel probability measure $\mathfrak{m}$ on $\mathcal{O}(B, X)$ such that

$$\mathcal{F}_*\mathfrak{m} = \mathfrak{m}, \qquad (e_\lambda)_*\mathfrak{m} = \mu_\lambda \quad (\lambda \in B).$$

Definition 2.2 (Lyapunov stability)**.** Let $\chi_1(\lambda), \dots, \chi_k(\lambda)$ be the Lyapunov exponents of (f_λ, μ_λ) on the smooth locus, counted with multiplicity, and put

$$L(\lambda) = \sum_{j=1}^{k} \chi_j(\lambda) = \int_X \log |\operatorname{Jac} f_\lambda| \, d\mu_\lambda.$$

The Jacobian norm uses the Hermitian metric induced by a fixed projective embedding; Proposition 2.12 justifies the integral and metric independence. The family is *Lyapunov stable* on B if $dd^c L = 0$ on B, in the sense of currents. Since the parameter is a curve, this says that L is harmonic. In phase dimension greater than one the definition uses the sum of the exponents, not one individual exponent. The positive current $dd^c L$ is called the *bifurcation current*.

Theorem 2.3 (Equilibrium webs, J-stability, and Lyapunov stability)**.** *The following are equivalent:*

1. *an equilibrium web exists over B;*
2. *every $R_n(\lambda)$ is represented by disjoint holomorphic sections over B, exhausting $R_n(\lambda)$ at every parameter;*
3. *the same assertion holds for $R_n^{\mathrm{reg}}(\lambda)$;*
4. *$dd^c L = 0$ on B.*

Thus J-stability and Lyapunov stability give equivalent definitions of stability on B.

2.1. **Polarized dynamics and periodic sections.** We record the dynamical properties and the compactness of periodic sections used in the web criterion.

Proposition 2.4. *Let $h : X \to X$ satisfy $h^*L \simeq L^{\otimes q}$. Its equilibrium measure μ has the following properties.*

1. *It is mixing and satisfies $h_*\mu = \mu$ and $h^*\mu = q^k\mu$. It gives no mass to proper local analytic subsets of X.*

(2) *If R_n is the set of repelling points of h^n in $X_{\mathrm{reg}} \cap \operatorname{supp}\mu$, then*

$$q^{-kn} \sum_{x \in R_n} \delta_x \longrightarrow \mu, \qquad \#R_n = q^{kn} + o(q^{kn}). \tag{2.1}$$

(3) *Fix a projective embedding of X and its induced Hermitian norm on TX_{reg}. There is a completely invariant Borel set $E \subset \operatorname{supp}\mu \cap X_{\mathrm{reg}}$ of full μ-measure such that h is unramified along every orbit in E, every point of E has q^k distinct preimages in E, and*

$$\sup_{n \geq 0} \|D(h^n)_x v\| = +\infty \quad (x \in E,\ 0 \neq v \in T_x X). \tag{2.2}$$

Proof. Let $\pi : Y \to X$ be a projective resolution which is an isomorphism over X_{reg}, and put $g := \pi^{-1} \circ h \circ \pi$. The dynamical degrees of g are

$$d_j(g) = q^j, \qquad 0 \leq j \leq k.$$

Indeed, on X the degree of order j of h^n, computed with L, is $q^{nj}(L^k)$, and dynamical degrees are birationally invariant [Dan20, Theorem 1]. Thus g has dominant topological degree q^k.

Let ν be the equilibrium measure of g. The construction and properties of this measure are given in [Gue05, Theorem 2.1] and [DNT15, Section 1]. It is balanced and mixing, and gives no mass to pluripolar sets. Its pushforward $\pi_*\nu$ is the equilibrium measure μ of h. This description is independent of the resolution: outside a countable union of proper analytic sets, the normalized preimages of a point correspond under π, and their limits are given by [DNT15, Theorem 1.2]. The balance and mixing identities descend to X. The inverse image of a proper local analytic subset is a proper local analytic subset of Y, so its μ-mass is zero.

By [DNT15, Theorem 1.1], the repelling periodic points of g, including those in $\operatorname{supp}\nu$, are equidistributed. The exceptional locus of π is closed and has zero ν-mass. Its contribution to these normalized counting measures therefore tends to zero. A repelling periodic point in X_{reg} has its entire orbit in X_{reg}: the return germ has local degree one, hence every finite germ along the orbit has local degree one and is an isomorphism, by normality. Such cycles correspond exactly to the cycles of g disjoint from the exceptional locus. Away from that locus, supports also correspond under π. This proves (2.1).

The Lyapunov exponents of ν are at least $\frac{1}{2}\log q$ by [DNT15, Theorem 5.4]. Choose a compact set $Q \subset Y \setminus \pi^{-1}(\operatorname{Sing} X)$ of positive ν-measure. For ν-almost every y, all Lyapunov exponents are positive, the orbit is defined and avoids the exceptional and critical loci,

and there are arbitrarily large n with $g^n(y) \in Q$. On Q, the differential $D\pi$ has a uniformly positive conorm. Hence, for $x = \pi(y)$ and $0 \neq v \in T_xX$,

$$D(h^n)_x v = D\pi_{g^n(y)} D(g^n)_y (D\pi_y)^{-1} v$$

is unbounded along these return times. This proves (2.2) on a set of full measure.

Both direct and inverse images of null sets are null. The inverse-image assertion follows from invariance; for the direct image one uses

$$\mu(h(A)) \leq q^k \mu(A).$$

This inequality follows from balance by counting preimages. Delete the grand orbit of the exceptional null set just obtained, together with the singular and ramification loci and their images. Intersect the remainder with the support of μ. Balance and the local inverse theorem give complete invariance of this full-measure set. Its points have q^k distinct preimages, and all of them belong to the same set. This gives E. □

Lemma 2.5. *Let B be a coordinate disk. The set of all holomorphic periodic sections $\gamma : B \to X$ is relatively compact in $\mathcal{O}(B, X)$.*

Proof. After shrinking the parameter disk, replace the relative polarization by one sufficiently high power. Its direct image is a holomorphic vector bundle, and a local trivialization gives a relative projectively normal embedding $B \times X \hookrightarrow B \times \mathbb{P}^s$; see [Laz04, Chapter 1]. Denote the affine cone of its fiber at λ by $C_\lambda \subset \mathbb{C}^{s+1}$. The fiber embeddings may vary with λ.

The relative polarization identity, together with a local trivialization on B, lifts the family to homogeneous maps $F_\lambda : \mathbb{C}^{s+1} \to \mathbb{C}^{s+1}$ of degree q. Relative projective normality and a holomorphic splitting of the restriction map on sections give lifts depending holomorphically on λ. They satisfy

$$F_\lambda(C_\lambda) \subset C_\lambda, \qquad F_\lambda^{-1}(0) \cap C_\lambda = \{0\}, \qquad [F_\lambda(Z)] = f_\lambda([Z]).$$

For every $U \Subset B$, there is $C_U \geq 1$ with

$$C_U^{-1} \|Z\|^q \leq \|F_\lambda(Z)\| \leq C_U \|Z\|^q \quad (\lambda \in U,\ Z \in C_\lambda). \tag{2.3}$$

Let γ have period n. Since B is contractible and Stein, the pullback of the tautological line bundle is trivial. Thus γ has a nowhere zero holomorphic lift $Z_\gamma : B \to \mathbb{C}^{s+1}$, $\quad Z_\gamma(\lambda) \in C_\lambda$, and

$$F_\lambda^n(Z_\gamma(\lambda)) = a_\gamma(\lambda) Z_\gamma(\lambda)$$

for some $a_\gamma \in \mathcal{O}(B)^*$. Choose a holomorphic logarithm of a_γ and set

$$H_\gamma := a_\gamma^{-1/(q^n-1)} Z_\gamma.$$

Then $F_\lambda^n(H_\gamma) = H_\gamma$. Iterating (2.3) gives

$$C_U^{-1/(q-1)} \le \|H_\gamma(\lambda)\| \le C_U^{1/(q-1)}. \tag{2.4}$$

These bounds are independent of n and γ. Montel's theorem and a diagonal argument give a locally uniformly convergent subsequence of the lifts of any sequence of periodic sections. Its limit belongs fiberwise to $C_\lambda \setminus \{0\}$ by (2.4). Projectivizing proves local normality. Applying the construction locally on B and taking a diagonal subsequence proves the assertion on the original disk. □

2.2. **Rigidity and inverse histories.** The main results of this subsection are Lemma 2.6, which gives uniqueness of sections through expanding points, and Lemma 2.9, which describes exponentially converging inverse histories near a fixed point. For acritical ergodic equilibrium webs, almost-everywhere unbranching is part of the lamination theory of [BBD18, Theorem 4.1]. The jet argument below obtains the pointwise uniqueness needed here directly from expansion in any compact forward-invariant family of sections. We continue to use the metric induced by a projective embedding of X.

Lemma 2.6. *Let B be connected and let $\mathcal{K} \subset \mathcal{O}(B, X)$ be compact with $\mathcal{F}(\mathcal{K}) \subset \mathcal{K}$. Fix $\lambda_0 \in B$ and $x \in X$. Suppose that, on the Zariski tangent space,*

$$\sup_n \|D(f_{\lambda_0}^n)_x v\| = +\infty \quad \textit{for every } 0 \ne v \in T_x X.$$

Then $e_{\lambda_0}^{-1}(x) \cap \mathcal{K}$ contains at most one element.

Proof. Suppose that distinct $\gamma, \eta \in \mathcal{K}$ agree at λ_0. Choose a local parameter t centered at λ_0. Their first different Taylor coefficient has some order $r \ge 1$. In a local embedding at x,

$$\gamma(t) - \eta(t) = t^r v + O(t^{r+1}), \qquad 0 \ne v \in T_x X.$$

The coefficient belongs to $T_x X$ because both maps take values in X and their lower-order coefficients agree.

Put $x_n = f_0^n(x)$. Choose projective unitary coordinates ψ_n centered at x_n. The order-r coefficient of

$$\psi_n(f_t^n(\gamma(t))) - \psi_n(f_t^n(\eta(t)))$$

is

$$D\psi_n(x_n) D(f_0^n)_x v. \tag{2.5}$$

This follows by composing the two Taylor expansions: at the first differing order only the differential in the phase variable contributes.

The sections $\mathcal{F}^n\gamma$ and $\mathcal{F}^n\eta$ belong to $\mathcal{K}$. Equicontinuity therefore gives a parameter disk, independent of n, on which the above coordinate maps are uniformly bounded. Cauchy's estimates uniformly bound (2.5). The unitary coordinates compare their norms with the fixed projective metric by constants independent of n. This contradicts the assumption. □

Lemma 2.7. *Let $\mathfrak{m}$ be an equilibrium web over a connected B, and put $\mathcal{K} := \operatorname{supp}\mathfrak{m}$. Fix $\lambda_0 \in B$ and let E be as in Proposition 2.4 for f_{λ_0}. Set*

$$\mathcal{K}_E := \{\gamma \in \mathcal{K} : \gamma(\lambda_0) \in E\}.$$

Then $\mathfrak{m}(\mathcal{K}_E) = 1$, evaluation induces a bijection $\mathcal{K}_E \to E$, and $\mathcal{F} : \mathcal{K}_E \to \mathcal{K}_E$ is q^k-to-one. More precisely, each preimage of $\gamma(\lambda_0)$ determines a unique preimage of γ in $\mathcal{K}_E$.

Proof. Compactness and the defining identities of a web give

$$\mathcal{F}(\mathcal{K}) = \mathcal{K}, \qquad e_\lambda(\mathcal{K}) = J_\lambda \quad (\lambda \in B). \tag{2.6}$$

Thus evaluation onto E is surjective, and Lemma 2.6 gives injectivity. The full-measure assertion follows from $(e_{\lambda_0})_*\mathfrak{m} = \mu_{\lambda_0}$.

For $x \in f_{\lambda_0}^{-1}(\gamma(\lambda_0))$, choose the unique $\eta \in \mathcal{K}_E$ with $\eta(\lambda_0) = x$. Then $\mathcal{F}\eta$ and γ agree at λ_0, so Lemma 2.6 gives $\mathcal{F}\eta = \gamma$. The q^k preimages of $\gamma(\lambda_0)$ are distinct and belong to E. Conversely, two preimage sections with the same value at λ_0 coincide. This proves the claim. □

Lemma 2.8. *Let B be a connected Riemann surface, let $\mathcal{K} \subset \mathcal{O}(B, X)$ be compact, and let $p \in \mathcal{K}$. For each $n \geq 0$, let $\mathscr{H}_n \subset \mathcal{K}$. Suppose that on some nonempty open subset $V \subset B$ there are $C > 0$ and $0 < a < 1$ such that*

$$\sup_{\gamma \in \mathscr{H}_n} \sup_{\lambda \in V} \operatorname{dist}(\gamma(\lambda), p(\lambda)) \leq C a^n.$$

For every compact $Q \subset B$, there exist $C_Q > 0$ and $0 < a_Q < 1$ such that

$$\sup_{\gamma \in \mathscr{H}_n} \sup_{\lambda \in Q} \operatorname{dist}(\gamma(\lambda), p(\lambda)) \leq C_Q a_Q^n.$$

Proof. First, convergence to p is locally uniform, uniformly over $\mathscr{H}_n$. Otherwise, a sequence of violating sections with $n \to \infty$ would have a limit in $\mathcal{K}$ which equals p on V and differs from it elsewhere. This contradicts the identity theorem.

Join a relatively compact disk in V to a point of Q by a finite chain of parameter coordinate disks. Choose these disks small enough that the image of p over each slightly larger disk lies in one projective coordinate chart. The uniform convergence just proved puts every section in $\mathscr{H}_n$ in the same chart for all sufficiently large n.

We use the following consequence of the three-circle theorem. If $0 < r < s < R$ and a scalar holomorphic function u on a disk of radius R satisfies $\sup |u| \le M$ and $\sup_{|z|\le r} |u| \le \epsilon$, then

$$\sup_{|z|\le s} |u| \le M^{1-\theta}\epsilon^{\theta}, \qquad \theta = \frac{\log(R/s)}{\log(R/r)} > 0.$$

Apply this estimate to each coordinate of $\gamma - p$. A bound Ca^n on a smaller disk gives a bound $C'(a^\theta)^n$ on a larger disk. Successive overlapping disks propagate an exponential bound along the chain. A finite number of chains covers Q, and finitely many initial values of n are absorbed into C_Q. □

Lemma 2.9. *Let $H : (\mathbb{C}^e, 0) \to (\mathbb{C}^e, 0)$ be a holomorphic germ. Fix $r > 1$ which is not the modulus of an eigenvalue of $A := DH_0$, and let E^u be the sum of the generalized eigenspaces with eigenvalues of modulus greater than r. There is a local complex submanifold W through 0, tangent to E^u, with the following property. If*

$$H(z_{n+1}) = z_n, \qquad \|z_n\| = O(b^n), \qquad 0 < b < r^{-1},$$

then $z_n \in W$ for all sufficiently large n.

Proof. Write $\mathbb{C}^e = E^u \oplus E^s$ and $H(z) = Az + R(z)$, where $R(0) = DR(0) = 0$. Choose an adapted norm with

$$\|A_u^{-1}\| < r^{-1}, \qquad \|A_s\| < r.$$

Use the Banach space of sequences $z = (z_n)_{n\ge 0}$, with $z_n = (u_n, s_n)$ and norm

$$\|z\|_r := \sup_{n\ge 0} r^n \|z_n\|.$$

For a sufficiently small prescribed u_0, consider the equations

$$u_n = A_u^{-n} u_0 - \sum_{j=1}^{n} A_u^{-(n-j+1)} R_u(z_j), \tag{2.7}$$

$$s_n = \sum_{j\ge 1} A_s^{j-1} R_s(z_{n+j}). \tag{2.8}$$

Both convolution operators are bounded in the weighted norm. On a sufficiently small ball, the Lipschitz constant of R makes the right-hand side a contraction. Its fixed point depends holomorphically on u_0. The values at $n = 0$ form a holomorphic graph $s_0 = \varphi(u_0)$ with $\varphi(0) = D\varphi(0) = 0$; denote this graph by W.

Any sufficiently small inverse history with finite weighted norm satisfies (2.7) by iteration. It satisfies (2.8) because $\|A_s\| < r$ makes the remainder at infinity vanish. Uniqueness of the fixed point places its

initial value in W. For the history in the statement, every sufficiently late tail has arbitrarily small weighted norm, since $rb < 1$. Apply the preceding assertion to each such tail. □

2.3. **Proof of Theorem 2.3.** We first prove that an equilibrium web makes every repelling J-cycle move holomorphically. The argument also proves that a section through a regular point remains in X_{reg}. We then construct a web from moving regular periodic points.

Proposition 2.10. *Suppose that B is connected and that f admits an equilibrium web over B. For every $\lambda_0 \in B$ and every $p_0 \in R_n(\lambda_0)$, there is a unique $p \in \operatorname{supp} \mathfrak{m}$ such that*

$$p(\lambda_0) = p_0, \qquad p(\lambda) \in R_n(\lambda) \quad \textit{for every } \lambda \in B.$$

Proof. Put $\mathcal{K} := \operatorname{supp} \mathfrak{m}$. Replace the family by its nth iterate, and continue to denote it by f, writing q for its polarization factor. Its equilibrium measures and the web are unchanged. Thus p_0 is fixed and repelling.

By (2.6), some $p \in \mathcal{K}$ satisfies $p(\lambda_0) = p_0$. Lemma 2.6, applied to this repelling fixed point, gives uniqueness. Applying it to $\mathcal{F}p$ and p yields

$$f_\lambda(p(\lambda)) = p(\lambda) \quad (\lambda \in B). \tag{2.9}$$

We will use the local inverse theorem also at singular points. In a minimal embedding $(X, p_0) \subset (\mathbb{C}^e, 0)$, the germ of f_{λ_0} extends to an ambient germ with invertible derivative. The ambient inverse theorem applies, and the resulting biholomorphism sends the irreducible germ of X onto itself. Its inverse restricts to an inverse germ on X. Since all inverse eigenvalues have modulus less than one, this inverse is contracting in an adapted ambient norm. The same construction applies to the nearby family and its fixed section p.

Construction of inverse histories. Choose a parameter neighborhood V of λ_0 and a neighborhood of $p|_V$ in a local embedding on which the inverse branch ϕ_λ fixing $p(\lambda)$ is defined and uniformly contracting. An adapted ambient norm gives a contraction constant $0 < a < 1$. Let E and $\mathcal{K}_E$ be as in Lemma 2.7, for the iterated family now under consideration.

Equicontinuity of $\mathcal{K}$ allows us to choose a small relative neighborhood Ω of p_0 and shrink V so that every $\gamma \in \mathcal{K}$ with $\gamma(\lambda_0) \in \Omega$ lies in this inverse-branch neighborhood over V. We choose Ω invariant under ϕ_{λ_0}. Set

$$\mathcal{A} := \{\gamma \in \mathcal{K}_E : \gamma(\lambda_0) \in \Omega\}.$$

Then

$$\mathfrak{m}(\mathcal{A}) = \mu_{\lambda_0}(\Omega) > 0. \tag{2.10}$$

Lemma 2.7 supplies, for every $\gamma \in \mathcal{A}$, a unique sequence $\gamma_{-j} \in \mathcal{K}_E$, with $\gamma_0 = \gamma$, such that

$$\mathcal{F}\gamma_{-j-1} = \gamma_{-j}, \qquad \gamma_{-j}(\lambda_0) = \phi^j_{\lambda_0}(\gamma(\lambda_0)).$$

All these sections belong to $\mathcal{A}$. The local inverse theorem and the identity theorem give

$$\gamma_{-j}(\lambda) = \phi^j_\lambda(\gamma(\lambda)) \quad (\lambda \in V).$$

In particular, for a constant independent of $\gamma \in \mathcal{A}$,

$$\sup_{\lambda \in V} \operatorname{dist}(\gamma_{-j}(\lambda), p(\lambda)) \le Ca^j. \tag{2.11}$$

Repulsion on the Zariski tangent space. Fix an arbitrary parameter $\lambda_1 \in B$ and put $p_1 := p(\lambda_1)$. Lemma 2.8, applied to all the sections γ_{-j} above, gives constants $C_1 > 0$ and $0 < b < 1$ such that

$$\operatorname{dist}(\gamma_{-j}(\lambda_1), p_1) \le C_1 b^j \quad (\gamma \in \mathcal{A},\ j \ge 0). \tag{2.12}$$

Choose a minimal local embedding $(X, p_1) \hookrightarrow (\mathbb{C}^e, 0)$, where $e = \dim_{\mathbb{C}} T_{p_1}X$. In these coordinates $T_{p_1}X = \mathbb{C}^e$. Extend the germ of f_{λ_1} to a holomorphic germ $H : (\mathbb{C}^e, 0) \to (\mathbb{C}^e, 0)$. The derivative DH_0 is the action of Df_{λ_1} on the Zariski tangent space. Local holomorphic embedding coordinates are Lipschitz with respect to a fixed projective chart after shrinking, so (2.12) remains an exponential bound in these coordinates.

Choose $1 < r < b^{-1}$ avoiding the moduli of the eigenvalues of DH_0. Let W be the submanifold supplied by Lemma 2.9. Suppose that DH_0 has an eigenvalue of modulus at most 1. Then $T_0W \subsetneq \mathbb{C}^e$. The germ of X is irreducible, and it cannot be contained in W, since this would give

$$T_{p_1}X \subseteq T_0W \subsetneq \mathbb{C}^e.$$

Thus $W \cap X$ is a proper local analytic subset of X.

For every $\gamma \in \mathcal{A}$, the values $\gamma_{-j}(\lambda_1)$ form an exponentially converging inverse history for H. Lemma 2.9 therefore gives

$$\gamma(\lambda_1) \in \bigcup_{j \ge 0} f^j_{\lambda_1}(W \cap X).$$

Proposition 2.4 gives $\mu_{\lambda_1}(W \cap X) = 0$. The inequality $\mu_{\lambda_1}(f_{\lambda_1}(A)) \le q^k \mu_{\lambda_1}(A)$ then gives zero mass to the union above. On the other hand,

its inverse image under evaluation contains $\mathcal{A}$, contradicting (2.10). We have proved

(2.13) every eigenvalue of $Df_{\lambda_1}|_{T_{p_1}X}$ has modulus greater than 1.

The parameter λ_1 was arbitrary.

Persistence of smoothness. We show that $p^{-1}(\operatorname{Sing} X)$ is open. Suppose that $p_1 \in \operatorname{Sing} X$. In the minimal embedding just used, extend the nearby family to holomorphic maps H_λ of an ambient neighborhood. By (2.13), the total map

$$(\lambda, z) \longmapsto (\lambda, H_\lambda(z))$$

is a local biholomorphism at $(\lambda_1, 0)$. It maps the irreducible germ $B \times X$ into itself, hence onto the same germ. Its inverse therefore preserves $B \times X$ and its singular locus.

The inverse germs fixing $p(\lambda)$ are uniformly contracting after shrinking the parameter and phase neighborhoods and choosing an adapted norm. For every nearby λ, start the inverse iteration at the point $p_1 \in \operatorname{Sing} X$, chosen independently of λ. All iterates belong to $\operatorname{Sing} X$, and they converge to $p(\lambda)$. Hence $p(\lambda) \in \operatorname{Sing} X$. This proves openness. The set $p^{-1}(\operatorname{Sing} X)$ is also closed. Connectedness of B shows that it is either empty or all of B. In particular, $p_0 \in X_{\mathrm{reg}}$ implies $p(B) \subset X_{\mathrm{reg}}$.

Together with (2.9), (2.13), and $p \in \mathcal{K}$, this proves the proposition. □

Proof of Theorem 2.3. Assume first that an equilibrium web exists. Fix $\lambda_0 \in B$. For each n, apply Proposition 2.10 to all points of $R_n(\lambda_0)$, obtaining a finite set $\mathcal{R}_n$ of sections. Distinct sections cannot meet: at a common value, which is a repelling periodic point, Lemma 2.6 would make them equal. To see that their values exhaust $R_n(\lambda)$ at another parameter, apply Proposition 2.10 with that parameter as basepoint. The resulting section is repelling at λ_0 and therefore is one of the sections already constructed. This proves (1) ⇒ (2).

The persistence-of-smoothness argument in Proposition 2.10 uses only that the section is periodic and repelling on the Zariski tangent space at every parameter. Hence every section in (2) is either everywhere regular or everywhere singular. Keeping the regular sections proves (2) ⇒ (3).

Finally, suppose that the regular repelling J-cycles move holomorphically. Denote their sections by $\mathcal{R}_n^{\mathrm{reg}}$, write $N_n := \#\mathcal{R}_n^{\mathrm{reg}}$, and, for

all sufficiently large n, set

$$\mathfrak{m}_n := \frac{1}{N_n} \sum_{\gamma \in \mathcal{R}_n^{\mathrm{reg}}} \delta_\gamma.$$

Lemma 2.5 places the supports of all these probability measures in one compact subset of $\mathcal{O}(B, X)$. Take a weakly convergent subsequence with limit $\mathfrak{m}$. The map $\mathcal{F}$ permutes $\mathcal{R}_n^{\mathrm{reg}}$, as follows from the uniqueness of the local continuation of a regular repelling periodic point. Thus $\mathcal{F}_* \mathfrak{m}_n = \mathfrak{m}_n$ and $\mathcal{F}_* \mathfrak{m} = \mathfrak{m}$.

For every $\lambda \in B$, Proposition 2.4 gives

$$(e_\lambda)_* \mathfrak{m}_n = \frac{1}{N_n} \sum_{x \in R_n^{\mathrm{reg}}(\lambda)} \delta_x \longrightarrow \mu_\lambda.$$

Evaluation is continuous, so $(e_\lambda)_* \mathfrak{m} = \mu_\lambda$ for every λ. Hence $\mathfrak{m}$ is an equilibrium web, proving (3) $\Rightarrow$ (1). The equivalence with (4) is proved in Proposition 2.17 below, using only the equivalence of (1)–(3) just established. □

Proposition 2.11 (Periodic sections give a web)**.** *Let $\mathcal{M}_j$ be probability measures on holomorphic periodic sections of a polarized family on a fixed normal projective X. Suppose they are invariant under the same iterate $\mathcal{F}^e$ and that $(e_\lambda)_* \mathcal{M}_j \to \mu_\lambda$ for every λ. Then the family is J-stable.*

Proof. Lemma 2.5 puts their supports in one compact set, locally on the parameter disk. A weak limit $\mathcal{M}$ is $\mathcal{F}^e$-invariant and has evaluations μ_λ. The average $e^{-1} \sum_{i=0}^{e-1} \mathcal{F}_*^i \mathcal{M}$ is an equilibrium web. Apply Theorem 2.3. □

2.4. **The Lyapunov criterion.** We prove the equivalence with assertion (4) of Theorem 2.3. On projective space the Lyapunov criterion and the critical-mass argument are classical; see [BBD18, Theorem 1.6 and Proposition 3.12] and [BB26, Theorem 4.1]. We give the additional arguments needed for a normal phase variety. In this subsection we denote the polarizing line bundle by $\mathscr{L}$, reserving L for the Lyapunov function, and put $v = (\mathscr{L}^k)$ and $d = q^k$.

Proposition 2.12 (The bifurcation current on a normal variety)**.** *Let $F(\lambda, x) = (\lambda, f_\lambda(x))$ and $p : B \times X \to B$. Let $\mathcal{T}$ be the relative Green current, normalized by $F^* \mathcal{T} = q\mathcal{T}$ and $\mathcal{T}|_{X_\lambda} = T_\lambda$, and let R_F be the effective ramification Weil divisor of the finite map F. The Lyapunov function is continuous and subharmonic, and*

$$dd^c L = \frac{1}{v} p_*(\mathcal{T}^k \wedge [R_F]). \tag{2.14}$$

Here dd^c is normalized by $dd^c \log|z| = \delta_0$ on $\mathbb{C}$. The formula does not require X to be $\mathbb{Q}$-Gorenstein.

Proof. We first specify the products and integrability used below. In a projective embedding the polarization series gives $\mathcal{T} = \omega + dd^c G$, with locally uniformly Hölder continuous potentials. Indeed, on a compact parameter disk the maps f_λ and the initial metric potential have uniform Lipschitz bounds. Splitting the series with coefficients q^{-j} at the scale where the Lipschitz growth reaches the inverse distance gives a positive Hölder exponent, uniformly on that disk. This argument applies on the embedded normal space. Pulling back to a fixed resolution $Y \to X$ gives the same local Hölder bounds on $B \times Y$.

The exponential integrability theorem for Monge–Ampère currents with Hölder potentials [DNS10, Theorem 1.1 and Corollary 1.2] therefore applies both to these currents and to their fiber measures. On compact parameter sets it gives uniform integrability of the logarithms of meromorphic functions whose zeros and poles have no fiber component. To check the uniformity, use finitely many charts of the fixed resolution: the local logarithms of the holomorphic numerators and denominators form compact families of plurisubharmonic functions in L^1, and the Hölder constants of the Green potentials are uniform. Truncating these logarithms and then using the exponential tail estimate justifies slicing, passage to limits, and integration by parts. For a rational differential form, its norm in the induced projective metric has at most logarithmic growth on the resolution: its coefficients and the minors of the differential of the resolution are algebraic, so this follows from Łojasiewicz inequalities. The same integrability therefore applies to that norm.

For a Weil divisor A, the product $\mathcal{T}^k \wedge [A]$ means the Bedford–Taylor product on a resolution of each component of A, pushed down to $B \times X$. This definition agrees with the usual one on the regular locus. A divisor exceptional over the singular locus of $B \times X$ has image of dimension at most $k-1$. The restriction of $(\pi^*\mathcal{T})^k$ to that divisor is zero: on the open strata this follows from the dimension of the image, and the remaining analytic subsets have zero mass for the locally bounded-potential product. Thus exceptional divisors contribute nothing, and all ensuing divisor identities can be checked in codimension one. The same observation proves the projection formula for these products with Weil divisors.

Choose a nonzero rational k-form η on X, viewed as a relative meromorphic form on $B \times X$, and set $D = \operatorname{div} \eta$. Write $F^*\eta = J\eta$, where J is a meromorphic function on the total space. At the generic point

of every prime divisor the source and target are smooth, so the usual Jacobian identity gives

$$\operatorname{div} J = R_F + F^*D - D. \tag{2.15}$$

The pullback here is the valuation-theoretic pullback of Weil divisors under a finite map; no Cartier assumption is used. None of these divisors has a fiber component. If $b = \log \|\eta\|$ for the metric induced by a projective embedding, then, away from a set of measure zero,

$$\log |J_\lambda| = \log |\operatorname{Jac} f_\lambda| + b \circ f_\lambda - b.$$

The integrability just established and invariance of μ_λ show that

$$L(\lambda) = \int_X \log |J_\lambda| \, d\mu_\lambda. \tag{2.16}$$

They also justify Oseledets' formula for the sum of the exponents and its independence of the projective metric or of a smooth metric on a resolution: the logarithms of the changes of metric are integrable and hence give tempered changes of coordinates. Weak continuity of the fiber Green measures, together with the uniform logarithmic tail estimate, proves continuity in (2.16).

Apply Poincaré–Lelong on the resolution and push down. Equation (2.15) gives

$$dd^c L = \frac{1}{v} p_* \big(\mathcal{T}^k \wedge ([R_F] + F^*[D] - [D]) \big).$$

Since F has degree $d = q^k$, the identity $F^*\mathcal{T}^k = d\mathcal{T}^k$ implies $F_*\mathcal{T}^k = \mathcal{T}^k$. The projection formula and $p \circ F = p$ cancel the last two terms. This proves (2.14); its right side is positive. □

Lemma 2.13 (Periodic approximation of the Lyapunov sum)**.** *Let $f : X \to X$ be a q-polarized endomorphism of a normal projective k-fold, put $d = q^k$, and let $L(f)$ be the sum of its equilibrium Lyapunov exponents. If R_n^{reg} denotes the smooth repelling Julia points fixed by f^n, then*

$$L(f) = \lim_{n\to\infty} \frac{1}{nd^n} \sum_{p \in R_n^{\mathrm{reg}}} \log |\det D_p f^n|. \tag{2.17}$$

Proof. We give the determinant version of the argument of [BDM08, Theorem 1.5 and Section 4.2], including the inverse-branch input on a normal phase variety. Choose a resolution $\pi : Y \to X$ and put $g = \pi^{-1} f \pi$. Its equilibrium measure ν integrates quasi-plurisubharmonic functions [Gue05, Theorem 2.1], and its Lyapunov exponents are positive [DNT15, Theorem 5.4]. Work on the full-measure set of inverse histories avoiding the critical, indeterminacy, and exceptional loci. The

changes of tangent metric induced by π have logarithmically integrable norms and inverse norms, so the exponents computed there agree with those of f.

Here are the required local estimates. Fix a finite coordinate atlas on Y. Algebraic Łojasiewicz inequalities and the quantitative inverse function theorem give a proper algebraic set A, constants $a, C > 0$, and the following bounds at every unramified point $y \notin A$: an inverse germ at $g(y)$ is defined on a ball of radius at least $C^{-1}\operatorname{dist}(y, A)^a$, and its first two derivatives, in the chosen charts, are bounded by $C\operatorname{dist}(y, A)^{-a}$ after reducing that radius by a fixed factor. Enlarge A to include the exceptional loci used above. The function $\log\operatorname{dist}(\cdot, A)$ is ν-integrable: locally an ideal norm for A has plurisubharmonic logarithm, and Łojasiewicz inequalities compare this norm with a power of the distance. Consequently all these chart bounds are tempered along almost every inverse history.

For clarity, this produces contracting branches before any normalization theorem is applied. Oseledets' theorem gives tempered linear coordinates in which every one-step inverse derivative has norm at most $e^{-\chi+\epsilon}$, where $\chi > 0$ is the smallest exponent and $0 < 3\epsilon < \chi$. The preceding second-derivative bounds allow us to choose positive tempered radii on which the full inverse maps have derivative norm at most $e^{-\chi+2\epsilon}$. Replace these radii by an ϵ-slow minorant: if r_0 is a tempered radius and τ is the backward shift, take $r(\hat{x}) = \inf_{j\in\mathbb{Z}} e^{\epsilon|j|} r_0(\tau^j\hat{x})$. It is positive almost everywhere and $r(\tau\hat{x}) \geq e^{-\epsilon} r(\hat{x})$. Thus each inverse map sends its radius ball into the next one, and all inverse iterates are defined. Returning to the fixed atlas gives exponentially contracting inverse branches on a ball of positive history-dependent radius. This is also the inverse-bundle construction in [BDM08, Section 4.1].

Given $\epsilon > 0$, restriction to a measurable history block H of measure greater than $1 - \epsilon/2$, whose zeroth coordinates lie in a compact subset above X^{reg}, makes the radius at least a fixed $r > 0$ and, for every sufficiently large n, makes the Lipschitz constant of its nth inverse branches $h_{\hat{x},n}$ on the full radius-r ball at most e^{-cn} for some $c > 0$. Birkhoff's theorem and Egorov's theorem allow the same block to satisfy, uniformly,

$$-\frac{1}{n}\log|\det Dh_{\hat{x},n}(x_0)| = L(f) + o(1).$$

The coordinate and metric determinants differ by uniformly bounded endpoint terms in the fixed atlas. For a branch returning to one small regular coordinate neighborhood, $u_n = -\log|\det Dh_{\hat{x},n}|$ is positive pluriharmonic on the radius-r ball for all sufficiently large n. Harnack's inequality shows that its values within distance δr of x_0 differ from

$u_n(x_0)$ by at most $C_k\delta u_n(x_0)$. Choosing δ sufficiently small therefore gives

$$\left|\frac{1}{n}\log|\det D_p f^n| - L(f)\right| < 2\epsilon \tag{2.18}$$

at the fixed point of every such returning branch. All constructions may be made over the regular locus of X.

We recall the counting step to specify how many points satisfy (2.18). Choose finitely many pairwise disjoint convex coordinate boxes $B_i^+ \Subset X^{\mathrm{reg}}$, with inner boxes $B_i \Subset B_i^+$, diameter less than δr, and

$$\mu\Big(\bigcup_i B_i\Big) > 1-\epsilon/2, \qquad \mu(B_i) > (1-\epsilon)\mu(B_i^+).$$

Write $\widehat{B}_i$ for their inverse images in the natural extension. Mixing gives, for all sufficiently large n,

$$\widehat{\mu}\big(\widehat{f}^{-n}(H\cap\widehat{B}_i)\cap\widehat{B}_i\big) > (1-\epsilon)\widehat{\mu}(H\cap\widehat{B}_i)\mu(B_i).$$

Each associated inverse branch is defined on B_i^+; its image meets B_i and, by contraction, lies relatively compactly inside B_i^+. Different branch images are disjoint and each has mass $d^{-n}\mu(B_i^+)$ by balance. Each contains one repelling fixed point in the Julia set, since it is obtained by iterating an inverse branch from a Julia point. The displayed inequality therefore supplies at least

$$d^n(1-\epsilon)^2\widehat{\mu}(H\cap\widehat{B}_i)$$

such points in B_i^+. Summing gives at least $d^n(1-\epsilon)^3$ points satisfying (2.18), exactly as in [BDM08, Lemma 4.5].

Finally, in a fixed projective embedding $\|Df\| \le C$ on X^{reg}: local ambient holomorphic extensions and a finite cover of X give a uniform bound. Hence every smooth repelling fixed point of f^n satisfies

$$0 \le \frac{1}{n}\log|\det D_p f^n| \le k\log\max\{C,1\}.$$

Together with $\#R_n^{\mathrm{reg}} = d^n + o(d^n)$, the preceding count proves (2.17) by letting ϵ tend to zero. □

Corollary 2.14 (Stability implies harmonicity)**.** *On a parameter disk where the smooth repelling Julia cycles move holomorphically and exhaust the corresponding cycles in each fiber, their Lyapunov sum $L(s)$ is harmonic. This holds both for a fixed normal phase variety and for the relative families considered in this section.*

Proof. Put $d = q^k$. For the finite collection $\mathcal{R}_n^{\mathrm{reg}}$ of moving sections define

$$L_n(s) = \frac{1}{nd^n} \sum_{\gamma \in \mathcal{R}_n^{\mathrm{reg}}} \log \left| \det D_{\gamma(s)} f_s^n \right|.$$

The return derivative is an endomorphism of the relative tangent bundle along γ; its determinant is a nowhere zero holomorphic function. Thus L_n is nonnegative and harmonic. Relative projective embeddings and local ambient extensions give a uniform bound for the vertical derivatives on compact parameter disks. Since $\#\mathcal{R}_n^{\mathrm{reg}} = d^n + o(d^n)$, the functions L_n are locally uniformly bounded. Lemma 2.13 gives $L_n(s) \to L(s)$ at every parameter. Compactness of locally bounded harmonic functions then gives locally uniform convergence to L. In particular $dd^c L = 0$. □

Lemma 2.15 (Critical mass estimate)**.** *Let $\mathcal{T}$ be the relative Green current of a polarized family over a disk, let R_f be its ramification divisor, and suppose that*

$$p_*(\mathcal{T}^k \wedge [R_f]) = 0.$$

For each relatively compact parameter disk B_0, one has

$$\left\| (f^n)_*[R_f] \right\|_{p^{-1}(B_0)} = O\big(q^{(k-1)n}\big).$$

In particular the ramification series in Lemma 2.16 has locally finite mass.

Proof. Choose a smooth relatively positive representative ω of $c_1(\mathscr{L})$ and a positive form β on the parameter disk. Enlarging the coefficient of $p^*\beta$, we may use $\omega + p^*\beta$ to compute masses on a fixed relatively compact parameter neighborhood. Put

$$\omega_n = q^{-n}(f^n)^*\omega.$$

The Green-potential series gives

$$\omega_n - \mathcal{T} = dd^c u_n, \qquad \|u_n\|_\infty = O(q^{-n})$$

on that neighborhood. Choose a smooth cutoff χ on the parameter disk, identically one near $\overline{B_0}$. Integration by parts on the ramification cycle gives

$$\begin{aligned} &\int p^*\chi\, [R_f] \wedge (\omega_n^k - \mathcal{T}^k) \\ &\quad = \int u_n\, p^*(dd^c\chi) \wedge [R_f] \wedge \sum_{j=0}^{k-1} \omega_n^{k-1-j} \wedge \mathcal{T}^j = O(q^{-n}). \end{aligned}$$

The last bound is uniform: after the base form is inserted, the remaining intersections are fiberwise intersections of fixed cohomology classes with the ramification cycle. These identities can be computed on a resolution of that cycle; all potentials being continuous, Bedford–Taylor products and integration by parts apply. The hypothesis makes the integral containing $\mathcal{T}^k$ zero. Multiplication by q^{kn} therefore bounds the contribution of $(f^n)^*\omega^k$ by $O(q^{(k-1)n})$.

The remaining term in the mass is

$$k \int p^*(\chi\beta) \wedge [R_f] \wedge (f^n)^*\omega^{k-1},$$

which has the same bound by the fiberwise polarization identity. There are no terms containing two base forms. This proves the estimate. Finally $[Y] \leq f_*[R_f]$, so the jth term of the ramification series has mass $O(q^{-j})$ on every such neighborhood. □

Lemma 2.16 (Summable ramification and equilibrium webs)**.** *Let $p : \mathcal{X} \to B$ be a flat projective family with normal geometrically integral fibers of dimension k over a disk. Let $f : \mathcal{X} \to \mathcal{X}$ be a polarized endomorphism over B, of fiberwise topological degree $d = q^k$. Write $\Sigma = \mathcal{X} \setminus \mathrm{Sm}(p)$, let R_f be the ramification Weil divisor of f, and put $Y = (f(\operatorname{Supp} R_f))_{\mathrm{red}}$. Suppose that*

$$\mathscr{R} = \sum_{j \geq 0} d^{-j} (f^j)_*[Y]$$

has locally finite mass on $\mathcal{X}$. Then every parameter admits a neighborhood carrying an equilibrium web.

Proof. The inverse-branch counting argument is that of [BBr23, Lemma 3.1]. We give its application to transverse disks, which also accounts for the singular fibers.

Fix $s_0 \in B$, and use a coordinate t with $t(s_0) = 0$. Choose $a \in \mathcal{X}_{s_0,\mathrm{reg}}$ outside the exceptional set for backward equidistribution and outside

$$\bigcup_{j \geq 0} f^j(Y \cup \Sigma).$$

Such a point exists: the divisors $f^j(Y)$ have no vertical components, and $f^j(\Sigma)$ has fiberwise codimension at least two. The backward-equidistribution exceptional set is a countable union of proper analytic sets; use Proposition 2.4 and [DNT15, Theorem 1.2] on a resolution. Take relative smooth coordinates (t, z) centered at $a = (0, 0)$. We first check that $\nu(\mathscr{R}, a) = 0$. Every finite partial sum has Lelong number zero at a, by its choice. For a fixed ball Q centered at a, the Lelong number of the positive tail is at most a constant depending on Q times

its mass on Q. The latter tends to zero by local mass summability. Additivity of Lelong numbers for the sum of the finite partial sum and its tail proves the assertion.

Fix $0 < \epsilon < 1/4$. The radial Crofton formula, or the slicing argument in [BBr23, Lemma 3.1], now gives a radius and a set of complex line directions of arbitrarily large measure on which

$$(*) \qquad \sum_{j\geq 0} d^{-j} \big\| (f^j)_*[Y] \wedge [\Delta] \big\| \leq \epsilon.$$

For completeness, apply the radial slicing formula to every divisorial summand, and then sum by monotone convergence. The average of the intersection masses on disks of radius r is, up to the fixed normalization, the Lelong quotient of $\mathscr{R}$ at radius r, because no summand contains a. It tends to zero as $r \to 0$. Markov's inequality proves (*). We also shrink the radius so that the ball avoids Y. Restrict to directions transverse to $t = 0$ and bounded away from the vertical directions. After a common restriction of the parameter disk, these lines are graphs

$$a_v(t) = (t, tv), \qquad |t| < \rho,$$

where v belongs to a bounded measurable set $V \subset \mathbb{C}^k$ of positive Lebesgue measure, and (*) holds on every graph. Discarding a null set of directions ensures that these graphs avoid every $f^j(\Sigma)$. Indeed, radial projection of each analytic set $f^j(\Sigma)$ to the space of line directions has measure zero, since its codimension in the total space is at least two and it does not contain a.

For each $v \in V$, construct the inverse branches along a_v in the backward tree, retaining only those for which every intermediate branch avoids Y. Let $\Gamma_n(v)$ be the retained sections γ satisfying $f^n \circ \gamma = a_v$. Off $Y \cup f(\Sigma) \cup \Sigma$, the map f is an unramified cover of degree d. This follows from purity of the branch locus on the smooth target, applied to the finite map with normal source. Thus every retained branch at level n has d lifts at level $n+1$ before branches meeting Y are removed. The number removed is at most the intersection number with $(f^{n+1})_*[Y]$. More explicitly, with $I_n(v) = \|(f^n)_*[Y] \wedge [a_v]\|$ and $c_n(v) = d^{-n}\#\Gamma_n(v)$, the recurrence is

$$\#\Gamma_n(v) \geq d\#\Gamma_{n-1}(v) - I_n(v), \qquad c_n(v) \geq c_{n-1}(v) - d^{-n} I_n(v).$$

Since $c_0(v) = 1$, (*) gives, as in [BBr23, Lemma 3.1],

$$(**) \qquad \#\Gamma_n(v) \geq (1-\epsilon)d^n, \qquad f\Gamma_{n+1}(v) \subseteq \Gamma_n(v),$$

and each section has at most d predecessors. All these sections remain in the relative smooth locus, because their images under f^n lie on graphs avoiding every $f^j(\Sigma)$.

The union of these sections is relatively compact on every smaller parameter disk. To see this, use a relative projectively normal embedding and the homogeneous lifts F_t constructed in Lemma 2.5 (and, for varying fibers, in the proof of Lemma 2.22). The target sections a_v have homogeneous lifts A_v with uniformly bounded, nowhere zero values on compact parameter sets. A lift of a section in $\Gamma_n(v)$ may be multiplied by a holomorphic scalar so that

$$F_t^n(H_\gamma(t)) = A_v(t).$$

The required q^n-th root exists on the parameter disk. Iterating the homogeneous norm bounds in Lemma 2.5 bounds $\|H_\gamma\|$ both above and away from zero, independently of n, v, and γ. Montel's theorem proves the assertion.

Normalize Lebesgue measure on V to a probability ℓ, and set

$$m_n = \int_V d^{-n} \sum_{\gamma\in\Gamma_n(v)} \delta_\gamma \, d\ell(v), \qquad M_N = \frac{1}{N}\sum_{n=1}^{N} m_n.$$

The inverse branches and the condition that a branch avoids Y are Borel in the slope; the integrals can equivalently be defined on the finite analytic inverse-cover spaces and then pushed to the space of sections. By (**), the masses $b_n = \|m_n\|$ decrease and belong to $[1-\epsilon, 1]$, while

$$m_{n-1} - \mathcal{F}_* m_n \geq 0, \qquad \|m_{n-1} - \mathcal{F}_* m_n\| = b_{n-1} - b_n.$$

Consequently $\|M_N - \mathcal{F}_* M_N\| \leq 3/N$. Compactness gives a weak limit M with $\mathcal{F}_* M = M$ and mass $b \geq 1-\epsilon$.

For each t, evaluation is dominated by the full inverse-image measures:

$$(e_t)_* m_n \leq \int_V d^{-n} (f_t^n)^* \delta_{a_v(t)} \, d\ell(v).$$

At $t = 0$ the measures on the right converge to μ_0 by the choice of a. At any fixed $t \neq 0$, the map $v \mapsto tv$ is invertible and complex linear. Its image measure is absolutely continuous in the relative smooth coordinate chart, so backward equidistribution and dominated convergence again give the limit μ_t. Hence $(e_t)_* M \leq \mu_t$ for every t. The left side is f_t-invariant and has mass b; ergodicity of μ_t therefore gives $(e_t)_* M = b\mu_t$. The probability $b^{-1}M$ is an equilibrium web. □

Proposition 2.17 (Equivalence of the Lyapunov criterion)**.** *For a polarized holomorphic family on a fixed normal projective variety over a disk, $dd^c L = 0$ if and only if the three equivalent conditions* (1)–(3) *of Theorem 2.3 hold.*

Proof. Condition (3) implies $dd^cL = 0$ by Corollary 2.14. Conversely, suppose $dd^cL = 0$. Formula (2.14) and Lemma 2.15 give locally finite mass of the ramification series. By Lemma 2.16, every parameter has a neighborhood carrying an equilibrium web. The already proved implication (1) $\Rightarrow$ (3) gives holomorphic motion of all smooth repelling Julia cycles on each such neighborhood. For each period these local continuations are unique, disjoint, and exhaustive. They therefore define a finite covering of B. Since B is simply connected, this covering is trivial, so condition (3) holds over all of B. The equivalence of (1)–(3) completes the proof. □

Remark 2.18 (The bifurcation locus)**.** It follows that the complement of the stability locus is exactly $\operatorname{supp}(dd^cL)$. The equation $dd^cL = 0$ is a local equation of currents on an open parameter set; it is not a pointwise assertion about one chosen parameter. On a parameter curve, harmonicity need not mean that L is constant.

2.5. **Removable parameters.**

Lemma 2.19 (Persistence of a repelling Julia point)**.** *Let f_s be a holomorphic polarized family on a fixed normal projective variety X. A smooth repelling periodic point in J_{s_0} has a holomorphic continuation which remains smooth, repelling, and in J_s near s_0.*

Proof. The implicit function theorem gives the continuation and preserves smoothness and repulsion. Write $d = q^{\dim X}$. A sufficiently high iterate of the return has inverse branches $\psi_s : B \to B$, defined on one ball in the smooth locus, whose images contract uniformly to the continued point. The polarization series converges locally uniformly and gives jointly continuous Green potentials. Bedford–Taylor continuity on a fixed resolution [BT76, Proposition 2.3] gives weak continuity of the equilibrium measures. Together with $\mu_{s_0}(B) > 0$, this gives $\mu_s(B) > 0$ after restriction. The balance identity implies

$$\mu_s\big(\psi_s^j(B)\big) = d^{-\ell j}\mu_s(B) > 0,$$

where ℓ is the chosen return time. These images eventually lie in any prescribed neighborhood of the continued point. Hence that point belongs to $\operatorname{supp}\mu_s$. □

Lemma 2.20 (Removable parameters for stability)**.** *Let f_s be a holomorphic family of q-polarized endomorphisms of a fixed normal projective variety X, parametrized by a disk D. If the family is J-stable on $D \setminus \{0\}$, it is J-stable on D.*

Proof. Put $d = q^{\dim X}$. Let R_n be the smooth repelling Julia points of f_0 fixed by f_0^n. Periodic-point equidistribution, applied on a resolution and then restricted to the smooth locus, gives

$$\#R_n = d^n + o(d^n), \qquad d^{-n} \sum_{x \in R_n} \delta_x \longrightarrow \mu_0;$$

see [DNT15, Theorem 1.1]. By Lemma 2.19, each $x \in R_n$ defines a smooth repelling Julia section near 0. Stability on the punctured disk continues this section along every path there. Its monodromy around 0 is trivial, since the section is already defined near 0. We obtain a holomorphic section $\gamma_x : D \to X$, smooth repelling and in J_s for every s. Different points of R_n give disjoint sections. Moreover, f permutes these sections, because it does so at 0 and continuation is unique.

Lemma 2.5 makes these periodic sections a relatively compact family on every smaller parameter disk.

For large n set

$$\mathcal{M}_n = \frac{1}{\#R_n} \sum_{x \in R_n} \delta_{\gamma_x}.$$

These are invariant probability measures on the space of holomorphic sections. Their supports lie in one compact set after restriction to a smaller disk. For each s, their evaluations are uniform measures on $d^n + o(d^n)$ distinct smooth repelling Julia points of f_s. The upper bound and equidistribution in [DNT15, Theorem 1.1] show that these evaluations converge to μ_s. A weak limit of $\mathcal{M}_n$ is therefore an equilibrium web. The equilibrium-web criterion (Theorem 2.3) proves stability near 0. □

Corollary 2.21 (Extension across finitely many parameters)**.** *Let S be a smooth connected complex algebraic curve and let f be an algebraic family of polarized endomorphisms of a fixed normal projective variety X. If f is J-stable on a nonempty Zariski open subset S°, then it is J-stable on S.*

Proof. The complement $S \setminus S^\circ$ is finite. For each omitted parameter s, choose an analytic disk D centered at s such that $D \setminus \{s\} \subset S^\circ$. The restricted family is stable on this punctured disk, so Lemma 2.20 makes it stable on D. Together with stability on S°, this proves the assertion. □

2.6. **Equilibrium webs for varying normal fibers.** Let S be a smooth connected complex algebraic curve and let $p : \mathcal{X} \to S$ be flat and projective, with geometrically integral normal fibers of dimension k. Let $f : \mathcal{X} \to \mathcal{X}$ be an S-endomorphism satisfying $f^*L \simeq L^{\otimes q}$ for

a relatively ample line bundle L and $q > 1$. On a parameter disk $B \subset S^{\mathrm{an}}$, write $\mathcal{O}_B(\mathcal{X})$ for the space of holomorphic sections, with its compact-open topology, and put $\mathcal{F}\gamma = f \circ \gamma$ and $e_s(\gamma) = \gamma(s)$. An equilibrium web is a compactly supported probability measure $\mathfrak{m}$ on this space with

$$\mathcal{F}_*\mathfrak{m} = \mathfrak{m}, \qquad (e_s)_*\mathfrak{m} = \mu_s \quad (s \in B).$$

The construction by periodic sections follows [BBD18, Lemma 2.1 and Proposition 2.2(2)]. We establish the relative criterion needed on a nonempty parameter open.

Lemma 2.22 (Compactness of relative periodic sections)**.** *The holomorphic periodic sections over a parameter disk B form a relatively compact subset of $\mathcal{O}_B(\mathcal{X})$.*

Proof. Locally on B, a sufficiently high power of L gives a relative projectively normal embedding $\mathcal{X} \hookrightarrow B \times \mathbb{P}^M$. Relative Serre vanishing and cohomology and base change [Har77, Chapter III, Sections 5 and 12] provide holomorphic splittings of the restriction maps for homogeneous polynomials. The polarization therefore lifts f to homogeneous degree-q maps F_s preserving the varying affine cones C_s and having no zero on $C_s \setminus \{0\}$.

Compactness of their unit links over compact parameter sets gives the uniform estimate (2.3). The normalization of periodic lifts in Lemma 2.5 then gives (2.4), independently of the period. Montel's theorem produces limits in $C_s \setminus \{0\}$, since the relative cone is closed. Projectivization and a diagonal argument over parameter disks prove the assertion. □

Proposition 2.23 (The web criterion on a parameter open)**.** *There is a nonempty Zariski open $S^\circ \subset S$ such that, on every parameter disk $B \subset (S^\circ)^{\mathrm{an}}$, the following conditions are equivalent: the existence of an equilibrium web, J-stability, and $dd^cL = 0$, where $L(s) = \sum_{j=1}^k \chi_j(s)$ is the equilibrium Lyapunov sum.*

Proof. Let

$$\Sigma = (\mathcal{X} \setminus \mathrm{Sm}(p))_{\mathrm{red}}.$$

Its intersection with X_s is set-theoretically $\mathrm{Sing}(X_s)$. By generic flatness [Stacks, Tag 052A], choose $S^\circ \subset S$ so that $\Sigma|_{S^\circ} \to S^\circ$ is flat. Its analytification is then open. Choose also a resolution $\pi : \mathcal{Y} \to \mathcal{X}$ which is an isomorphism over the relative smooth locus, and a nonzero rational relative k-form on $\mathcal{X}$. After deleting finitely many further parameters, the map $\mathcal{Y} \to S^\circ$ is smooth, each π_s is a resolution of X_s, and the chosen form and its pullback under f restrict to nonzero rational

forms on every fiber. These restrictions follow from generic smoothness and the finiteness of the vertical components of their divisors.

We explain the changes to the fixed-variety argument. In a relative projective embedding, the first differing coefficient of two sections with equal lower jets is in the vertical Zariski tangent space. Subtracting their defining equations cancels the parameter terms; after composition with f^n, the coefficient becomes $D(f_s^n)v$. Thus projective Cauchy estimates prove Lemma 2.6 for relative sections. The coordinate differences are also holomorphic functions of the parameter, so the three-circle argument of Lemma 2.8 is unchanged. These two facts, the support identities, and the fiberwise dynamics in Proposition 2.4 give the relative lift lemma 2.7.

Suppose a web exists. Its support and jet uniqueness give a unique periodic section γ through each smooth repelling Julia point at s_0. The inverse-history construction in Proposition 2.10 starts in relative smooth coordinates near that point. The lift lemma provides a positive-measure set of section histories which approach γ exponentially near s_0; propagation gives an exponential bound at every parameter $s \in B$.

In a minimal embedding of $(X_s, \gamma(s))$, apply Lemma 2.9 to an ambient extension of the return. If a tangent eigenvalue had modulus at most one, the strong unstable manifold would have tangent space smaller than $T_{\gamma(s)}X_s$. The irreducible fiber germ cannot be contained in this manifold. Its intersection with that manifold is consequently a proper local analytic subset. The inverse histories would place a positive-measure set in the union of its forward images, contradicting Proposition 2.4. Hence the section is repelling on its full Zariski tangent space at every parameter. This is the inverse-history argument underlying [BB26, Proposition 1.2], with minimal embeddings allowing singular fibers.

We must also show that γ remains smooth. Suppose $\gamma(s_1) \in \Sigma$. Eliminate the independent linear fiber equations in a relative embedding at this point. This gives a relative embedding into $B \times \mathbb{C}^e$, minimal on the fiber at s_1. Translate γ to zero and extend the return to ambient maps H_s fixing zero. The vertical derivative at $(s_1, 0)$ is expanding, so $(s, z) \mapsto (s, H_s(z))$ is a local biholomorphism. The total space is normal, because the base is smooth and the fibers are geometrically normal. Its irreducible local germ is mapped into itself and hence onto itself. The ambient inverse therefore preserves the family and Σ.

In an adapted norm these inverse branches are uniformly contracting on a small neighborhood of the section. Openness of $\Sigma \to B$ ensures that each nearby fiber contains a singular point in that neighborhood. Its inverse iterates stay singular and converge to $\gamma(s)$. Thus $\gamma^{-1}(\Sigma)$

is open; it is also closed and does not contain s_0, so it is empty. Jet uniqueness makes the resulting smooth repelling Julia sections disjoint. Starting at any parameter proves that they exhaust all such points in every fiber. This proves J-stability.

Conversely, take the uniform measures on the moving smooth repelling Julia points fixed by f^n. They are invariant, their evaluations converge to μ_s by Proposition 2.4, and Lemma 2.22 gives compact support for their union. Any weak limit is an equilibrium web.

It remains to include the Lyapunov criterion. On each disk in S°, the relative polarization series and the smooth family of resolutions just chosen give uniform Hölder potentials and uniform logarithmic integrability exactly as in Proposition 2.12. A rational relative top form replaces the fixed form there. The identity $\operatorname{div} J = R_f + f^*D - D$ holds in codimension one; exceptional divisors again map to sets of dimension at most $k-1$. Thus, with $v = (L_s^k)$, which is independent of s,

$$dd^c L = \frac{1}{v} p_*(\mathcal{T}^k \wedge [R_f])$$

holds on the disk, with continuous subharmonic L. If it vanishes, Lemmas 2.15 and 2.16 give a web locally at each parameter. The relative web argument just proved then gives local J-stability, and uniqueness of continuation and simple connectedness of the disk give J-stability on the whole disk. Conversely, Corollary 2.14 applies fiberwise and proves that J-stability implies $dd^c L = 0$. □

Proposition 2.24 (Generic periodic-section criterion)**.** *Suppose that, over a nonempty Zariski open $S_1 \subset S$, each sufficiently small parameter disk carries probability measures $\mathcal{M}_j$ on holomorphic periodic sections such that*

$$\mathcal{F}_*^e \mathcal{M}_j = \mathcal{M}_j, \qquad (e_s)_* \mathcal{M}_j \longrightarrow \mu_s \quad \text{for every } s,$$

where $e \geq 1$ is independent of j. Then f is J-stable on a nonempty Zariski open subset of S.

Proof. Intersect S_1 with the open in Proposition 2.23. Lemma 2.22 gives a weak limit $\mathcal{M}$ with evaluations μ_s and $\mathcal{F}^e$-invariance. As in [BBD18, Lemma 2.1], continuity of evaluation and of $\mathcal{F}$ passes these identities to the limit. The measure

$$\frac{1}{e} \sum_{i=0}^{e-1} \mathcal{F}_*^i \mathcal{M}$$

is an equilibrium web, so Proposition 2.23 applies. □

3. Repelling spectra in arbitrary dimension

Proposition 3.1. *Let B be a connected smooth algebraic curve over $\mathbb{C}$, and let $\pi : \mathcal{X} \to B$ be flat and projective with normal fibers of dimension r, and let $f : \mathcal{X} \to \mathcal{X}$ be a morphism over B. Suppose that a relatively ample line bundle L satisfies $f^*L \simeq L^{\otimes q}$ for an integer $q > 1$. For each $b \in B$, let J_b be the support of the equilibrium measure. Assume that, locally on B, the smooth repelling Julia periodic points move holomorphically and that, for every period, these graphs exhaust the smooth repelling Julia periodic points in every fiber. Then every local branch x_b of such points satisfying $f_b^n(x_b) = x_b$ has constant characteristic polynomial*

$$\det\bigl(T\,\mathrm{id} - D_{x_b} f_b^n\bigr).$$

The eigenvalues are counted with their algebraic multiplicities.

Proof. Fix $n \geq 1$ and let $Z_n = \mathrm{Fix}(f^n)$ be the relative fixed-point scheme. It is closed in $\mathcal{X}$, hence proper over B. Its fibers are zero-dimensional. Indeed, a positive-dimensional component of $(Z_n)_b$ would contain an integral projective curve C on which f_b^n is the identity. Restriction of the polarization would give

$$\deg(L_b|_C) = q^n \deg(L_b|_C),$$

contrary to ampleness. Thus $Z_n \to B$ is proper and quasi-finite, and therefore finite.

Let $R_n \subset Z_n^{\mathrm{an}}$ consist of the points $x \in X_b^{\mathrm{reg}}$ which lie in J_b and for which every eigenvalue of $D_x f_b^n$ has modulus greater than one. We first show that R_n is both open and closed in Z_n^{an}.

Fix $b_0 \in B$. The motion hypothesis supplies a neighborhood V of b_0 and finitely many holomorphic sections $s_1, \ldots, s_N$ whose graphs have union $R_n \cap \pi^{-1}(V)$. There are only finitely many sections because $(Z_n)_{b_0}$ is finite. Each section has a closed graph in $\mathcal{X}_V^{\mathrm{an}}$, since the latter is separated over V. Consequently $R_n \cap \pi^{-1}(V)$ is closed in $Z_n^{\mathrm{an}}|_V$. Such neighborhoods cover B, so R_n is closed.

For openness, take $x_0 \in R_n$ above b_0. Flatness and smoothness of the fiber at x_0 imply that π is smooth near x_0. In relative local coordinates the fixed-point equations are $f_b^n(z) - z = 0$, and their derivative in z at (b_0, x_0) is $D_{x_0} f_{b_0}^n - \mathrm{id}$. This matrix is invertible. The holomorphic implicit function theorem identifies a neighborhood of x_0 in Z_n^{an} with a single section over a neighborhood of b_0. It is the section provided by the motion hypothesis, by uniqueness, and is contained in R_n. Hence R_n is open.

Let C be the irreducible component of $(Z_n)_{\mathrm{red}}$ containing the given local branch. The preceding local description shows that C dominates B. Let $\nu : S \to C$ be its normalization. The curve S is connected, and $S \to B$ is finite and surjective. The inverse image $\nu^{-1}(R_n)$ is nonempty, open and closed in S^{an}. Since S^{an} is connected, it equals S^{an}. Thus every point on this entire algebraic periodic cover is a smooth repelling Julia point. Moreover, the implicit-function description applies at every point of C: the normalization is an isomorphism there and $S \to B$ is finite étale.

Let $\iota : S \to \mathcal{X}$ be the tautological map. Along $\iota(S)$ the relative tangent sheaf is locally free, so

$$E = \iota^* T_{\mathcal{X}/B}$$

is a rank-r vector bundle on S. The identity $f^n \iota = \iota$ makes the relative derivative an algebraic bundle endomorphism $A : E \to E$. Therefore

$$P(T) = \det(T\,\mathrm{id} - A)$$

is a monic polynomial with regular coefficients on S.

Take a finite splitting field of P over $\mathbb{C}(S)$ and let $\tau : S' \to S$ be the normalization in that field. This is a finite surjective morphism of smooth connected algebraic curves. The roots $\lambda_1, \dots, \lambda_r$ of P, repeated according to multiplicity, are regular on S': they are integral over the coordinate ring of every affine open of S, and S' is its normalization in the splitting field. At every $u \in S'$ they specialize to eigenvalues of $A_{\tau(u)}$. The preceding paragraph therefore gives

$$|\lambda_j(u)| > 1 \qquad (u \in S',\ 1 \le j \le r).$$

In particular $1/\lambda_j$ is a bounded holomorphic function on S'.

Let $\overline{S'}$ be the smooth projective completion of S'. Its complement is finite. The removable-singularity theorem extends $1/\lambda_j$ across every point of this complement. A holomorphic function on the compact connected Riemann surface $\overline{S'}$ is constant. Each λ_j is consequently constant. Hence $P(T) = \prod_{j=1}^{r}(T - \lambda_j)$ has constant coefficients on S', and therefore on S. Restricting to the original local branch proves the proposition. □

4. Horizontal fields and projective rigidity

Lemma 4.1 (Rational horizontal fields on polarized normal families)**.** *Let $\pi : \mathcal{X} \to S$ be a flat projective family of normal geometrically integral varieties over a smooth complex curve, and let $F : \mathcal{X} \to \mathcal{X}$ be a finite S-morphism. Suppose an ample line bundle H on the generic*

fiber satisfies $F^*H \simeq H^q$ with $q > 1$. Let v be a nonzero rational vector field on S and let W be a rational vector field on $\mathcal{X}$ satisfying

$$d\pi(W) = v, \qquad dF(W) = W \circ F.$$

After shrinking S, the field W is holomorphic on all of $\mathcal{X}$. Its local flows trivialize the projective dynamical family. In particular the family is algebraically isotrivial after a finite base change.

Proof. If the fibers have dimension zero, the assertion is immediate after restriction. Assume henceforth that their dimension is positive. The total space is normal. At the generic point of a horizontal prime divisor it is smooth, since the generic fiber is normal and the base has characteristic zero. Define the polar Weil divisor P of W there by the minimum valuation of its coefficients in a regular local tangent frame. Changes of regular frame are invertible over the corresponding discrete valuation ring, so this definition is intrinsic.

Let D be a horizontal prime and let $D' = F(D)$. At their generic points choose regular tangent frames on the total space. The matrix of dF has regular coefficients. Therefore the pole order of $dF(W)$ at D is no greater than the pole order P_D of W. The pole order of $W \circ F$, regarded as a section of $F^*T_{\mathcal{X}}$, is exactly $e_F(D)P_{D'}$. The transport identity thus gives

$$F^*P \leq P$$

on horizontal divisors. Restrict to the generic fiber X of dimension n. For an effective Weil divisor and an ample Cartier polarization, the finite projection formula and $F^*H \simeq H^q$ give

$$q(P|_X \cdot H^{n-1}) = (F^*(P|_X) \cdot H^{n-1}) \leq P|_X \cdot H^{n-1}.$$

Ample positivity and $q > 1$ force $P|_X = 0$. Hence W has no horizontal divisorial poles. Removing finitely many points of S also removes its vertical divisorial poles and the poles and zeros of v.

For any local regular function a, the rational function $W(a)$ is now regular at every codimension-one point. Normality makes it regular on the whole open set. Thus W extends as a regular derivation, and hence as a global holomorphic vector field, across the remaining codimension-at-least-two subset. Horizontality and the transport identity persist by equality on a dense open.

Local holomorphic derivations integrate also on a singular complex space: in a local embedding, lift the coefficients to the ambient space; preservation of the defining ideal makes the ambient flow preserve the subspace. Uniqueness glues these flows. Compactness of a projective fiber gives a common time interval, so the horizontal flow identifies

nearby whole fibers. The transport identity makes these identifications conjugacies of their endomorphisms.

Fix one fiber in the resulting analytic disk. The space of dynamical isomorphisms from this fixed fiber to fibers over S is a countable union of finite-type algebraic spaces: use the graph loci in relative Hilbert schemes, over their countably many Hilbert polynomials, and impose the closed conjugacy identity. Each image in the algebraic curve S is constructible. The uncountably many points of the disk belong to their union, so one finite-type piece has dense image in S. A closed point of its generic fiber is defined over a finite extension of $\mathbf{C}(S)$ and supplies the required algebraic trivialization after finite base change and restriction. □

Corollary 4.2 (An isotrivial iterate)**.** *If one fixed positive iterate of a polarized algebraic family is isotrivial, then the family itself is isotrivial.*

Proof. After a finite base change, trivialize that iterate as a fixed polarized map $h : X \to X$. Write the resulting original maps as f_s, so $f_s^e = h$. Each f_s commutes with h. For every $x \in \mathrm{Fix}(h^n)$, its image $f_s(x)$ belongs to the finite set $\mathrm{Fix}(h^n)$ and is consequently constant on the connected parameter curve. These fixed-point sets are finite because a positive-dimensional fixed subvariety would have its ample class both fixed and multiplied by a number greater than one. Periodic points of a polarized map are Zariski dense; therefore the morphisms f_s agree on a dense set and are all equal. □

5. A spectral class for induction

5.1. **A reference-parameter spectral class.** Fix a countable algebraically closed subfield $k_0 \subset \mathbf{C}$ over which the initial family and all its prescribed algebraic data are defined. Work with pointed parameter curves (S, s_0) defined over k_0, where $s_0 \in S(\mathbf{C})$ is generic over k_0. Every nonempty k_0-defined Zariski open contains s_0. Every lift of s_0 under a finite cover of parameter curves defined over k_0 is again generic over k_0. All algebraic operations below are made over this fixed field.

Definition 5.1 (Constant spectra on good reference cycles)**.** A one-parameter algebraic family $(\mathcal{V}, F)$ of polarized endomorphisms of normal projective varieties has the *good reference-cycle spectral property* if there are a pointed reference parameter s_0 and a dense Zariski open $U \subset V_{s_0}$ obtained from a k_0-defined open of the total family, such that the following holds. For every smooth repelling Julia periodic point whose orbit meets U, the characteristic polynomial of its return is constant as an algebraic function on the normalization of the periodic

parameter component through that point. Equivalently, it is constant as a germ at the marked reference point of that component.

Julia membership is specified at s_0; each marked cycle is continued on its own parameter germ. This defines the *pointed reference spectral class*.

Proposition 5.2 (Closure of the reference spectral class)**.** *The property in Definition 5.1 is preserved under the following operations, after finite parameter extension and restriction over k_0 whenever required:*

(1) *finite compatible polarized dynamical covers;*
(2) *connected rational polarized factors and equivariant birational polarized models;*
(3) *return families on good marked smooth repelling Julia periodic fibers of a connected rational polarized factor.*

In the last assertion the good base points are Zariski dense in the reference phase base. Every strongly J-stable algebraic polarized family has this spectral property after choosing a countable field of definition and a parameter generic over it.

Proof. All the normality, flatness, smoothness and spreading restrictions in the proof are nonempty opens defined over k_0; thus none removes the reference parameter. Characteristic-polynomial coefficients on a smooth marked periodic branch are algebraic functions on its normalized parameter component. A constant analytic germ is therefore a constant function on that component. It is enough throughout to prove constancy on a possibly point-dependent neighborhood of the reference parameter.

Finite covers. Let $p : \mathcal{W} \to \mathcal{V}$ be a finite compatible polarized cover. Choose the good reference open of W_{s_0} to lie above U and in the locus where both varieties are smooth and p_{s_0} is *étale*. This open is nonempty, since characteristic zero gives generic separability. A repelling Julia periodic point in it maps to a smooth repelling Julia periodic point of V_{s_0}. Julia membership follows from finite equivariance of the equilibrium measures; repulsion and the equality of return spectra follow from the tangent isomorphism given by dp.

The two local periodic branches exist by the implicit function theorem. They are compatible with p by uniqueness. If the image has smaller exact period, take the indicated power of its constant return polynomial. On a sufficiently small disk, dp remains invertible, and the return matrices upstairs and downstairs are conjugate. Their characteristic polynomials are therefore constant. A periodic orbit merely meeting the good open has the same return spectrum at every phase, so this proves the cover assertion.

Birational models and factors. For an equivariant birational polarized model choose an open on which the birational map is an isomorphism. Equilibrium measures correspond on that open, and the same conjugacy argument applies. In particular a rational factor may first be replaced by its normalized equivariant graph. The tensor product of the source and base polarizations pulls back to an ample polarization on that graph normalization.

Let now $\pi : \mathcal{V} \to \mathcal{B}$ be a connected factor morphism. Shrink a dense open $B_0 \subset B_{s_0}$ so that its fibers are normal, geometrically integral and flat, and so that each fiber meets the original good open in the smooth locus of π. Such an open exists by generic normality, generic smoothness and dominance of the image of the good open. Choose a smooth repelling Julia periodic point $b \in B_0$, of period n. The polarized return on $Y = \pi^{-1}(b)$ has a smooth repelling Julia periodic point x in the chosen good open: its equilibrium measure gives no mass to a proper algebraic subset, and its repelling Julia periodic points equidistribute. Write ℓ for the period of x under the fiber return.

Lemma 15.1 places x in $J_{F_{s_0}}$. Smoothness of π at x and the tangent exact sequence show that x is an ambient smooth repeller for $F^{n\ell}$. Its ambient return polynomial $P(T)$ is constant by the given reference spectral property. The local motion $x(s)$ exists by the implicit function theorem, and its projection is the local period-n motion $b(s)$, by uniqueness of the repelling base branch. Smoothness of π persists at $x(s)$. Hence

$$\det(TI - D_{b(s)} g_s^{n\ell}) \quad \text{divides } P(T).$$

The set of monic divisors of P is finite. Holomorphic dependence makes this factor constant. Taking ℓth roots of its finite eigenvalue set shows that $\det(TI - D_{b(s)} g_s^{n})$ is constant as well. This proves the factor property with good open B_0. Smooth repelling Julia periodic points meeting B_0 are Zariski dense, again by repelling equidistribution and the non-pluripolarity of the equilibrium measure.

Periodic-fiber returns. Fix such a good base point b, normalize its algebraic periodic parameter component, and use its marked section $b(s)$. After restriction, the corresponding phase fibers form a family of normal projective varieties, with a polarized return H_s. Use as good reference open in Y the intersection of the ambient good open with the smooth locus of π.

Let x be any repelling Julia periodic point of H_{s_0} whose orbit meets this open, of fiber period ℓ. The same mixed measure inclusion and tangent exact sequence make it an ambient repelling Julia point at the reference parameter. Its local ambient motion exists and lies over $b(s)$.

Both the ambient polynomial and the base polynomial are constant by the preceding arguments. The exact tangent sequence gives

$$\det(TI - D_{x(s)}F_s^{n\ell}) = \det(TI - D_{x(s)}H_s^{\ell})\ \det(TI - D_{b(s)}g_s^{n\ell}).$$

The fiber characteristic polynomial is their quotient and is constant. This proves the required property for the return family. No Julia-persistence assertion away from the reference parameter is used in this calculation.

Finally, the constant-spectrum theorem for strongly stable algebraic families, Proposition 3.1, gives the property with the full available smooth reference locus. □

5.2. **A spectral input stable under the induction operations.**

Proposition 5.3 (Reference-cycle spectra suffice for the local return construction)**.** *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ be an algebraic curve family of polarized endomorphisms of normal projective varieties. Fix a parameter s_0 and a dense Zariski open $U \subset X_{s_0}$. Suppose that, for every $m \geq 1$, the characteristic polynomial of Df_s^m is constant on each algebraic component of the marked m-periodic locus whose fibre at s_0 contains a smooth repelling Julia point with an orbit meeting U. Then all the hypotheses used in the weighted-character and local return-geometry arguments hold at a suitable reference repeller. In particular Theorem 9.9 applies with this spectral hypothesis in place of J-stability.*

Proof. Choose a smooth repelling Julia periodic point $p \in U$ satisfying the return-centre conclusion of Proposition 6.2. Replace the family by its return iterate $F_s = f_s^{\ell}$ for this construction. Since $DF_{s_0}(p) - I$ is invertible, the holomorphic implicit function theorem gives a marked fixed point p_s on one parameter disc Δ about s_0. The marking belongs to the algebraic periodic component through p. The hypothesis therefore makes its characteristic polynomial constant. Holomorphic primary spectral projectors and the parameter Poincaré–Dulac theorem provide coordinates σ_s on a common phase neighbourhood, with

$$F_s\sigma_s = \sigma_s\mathcal{N}_s, \qquad \mathcal{N}_s = SU_s = U_sS,$$

where S is fixed and expanding. The nilpotent linear and nonlinear resonant coefficients of U_s may vary. On relatively compact parameter discs the iterates $\mathcal{N}_s^{-n}$ contract bounded phase sets uniformly to 0, by Lemma 6.3. Shrink the phase neighbourhood so that its reference image under σ_{s_0} is contained in U.

Construct the entire inverse-return collection first in the single reference fibre, as in Proposition 6.2. Each branch is unramified along its defining finite inverse orbit. The implicit function theorem therefore

continues that branch as a holomorphic parameter germ. The continuation can be taken on a fixed relatively compact phase ball: compactness gives finitely many inverse-function charts, and uniqueness makes their continuations agree on overlaps. Compositions and normal-form inverse prefixes continue by the same identities. No common parameter disc for the infinite collection is needed.

Fix any finite list of branches $h_1, \dots, h_r$. Their germs have a common parameter disc, and there are fixed polydiscs $D_r \Subset D_R$ such that each $h_{i,s}$ is defined on a neighbourhood of $\overline{D_r}$ and maps it into D_R. After shrinking that parameter disc, choose n_0 so that $\mathcal{N}_s^{-n}(D_R)$ is relatively compact in D_r for all $n \geq n_0$, uniformly in s. The same estimate for first derivatives, or the Cauchy estimate on slightly larger polydiscs, makes each suitably ordered inverse-prefix composition a strict contraction. Hence every inverse word with independently chosen $n_1, \dots, n_r \geq n_0$ has a unique holomorphic fixed point on this same parameter disc. Increasing n_0 once works for the whole collection of these integer tuples.

At $s = s_0$, iterate this inverse word starting at the marked Julia point. Every iterate is a backward image of a Julia point along an unramified inverse branch, and so lies in the Julia set. The limit fixed point belongs to the Julia set because that set is closed. It is smooth and repelling for the forward return, and lies in $\sigma_{s_0}(D_r) \subset U$. Its holomorphic marking belongs to an algebraic periodic component covered by the hypothesis. Consequently its forward return characteristic polynomial, and hence its inverse fixed-point spectrum, is constant on the common parameter disc. This proves exactly the simultaneous spectral assertion needed for the nuclear traces with independently large integers n_i.

The remaining infinite collection is used only through its germs. Its compressed matrices have entries in the field of meromorphic parameter germs. The Grassmannian invariance equations are generated by finitely many polynomial equations, and Burnside reconstruction uses finitely many words. Thus the construction of the module and its simultaneous gauge in Lemma 6.5 takes place on one parameter germ. Every other branch intertwines that gauge as an identity of germs. Finally all relative uniqueness assertions use the reference centres coefficient by coefficient, as established in Lemma 5.4 below. These are all the stability inputs in the local return argument. □

Lemma 5.4 (Relative uniqueness with individually continued centres)**.** *Let $\{a_i\}$ be a holomorphic uniqueness set in a phase domain D. For each i let $a_i(s)$ be a holomorphic parameter germ with $a_i(s_0) = a_i$; these germs need not have a common domain. Let $G(s, z)$ be holomorphic on*

a product neighbourhood of $\{s_0\} \times D$. If $G(s, a_i(s)) = 0$ as a parameter germ for every i, then $G = 0$ as a germ along $\{s_0\} \times D$. The same assertion holds for a uniqueness set of centres in every sufficiently small phase neighbourhood.

Proof. Write $G(s, z) = \sum_{j\geq 0}(s - s_0)^j G_j(z)$. Evaluation at s_0 and uniqueness give $G_0 = 0$. If $G_0, \dots, G_{j-1}$ vanish identically, the coefficient of $(s - s_0)^j$ in $G(s, a_i(s))$ is $G_j(a_i)$. Hence $G_j = 0$ by uniqueness. Induction annihilates every coefficient. The convergent parameter expansion proves the assertion. The local version is the same argument on the chosen phase neighbourhood. □

Lemma 5.5 (The one-dimensional spectral base case)**.** *Let S be a smooth connected algebraic curve and let $f : S\times\mathbb{P}^1 \to \mathbb{P}^1$ be an algebraic family of rational maps of degree $d > 1$. Suppose that, at a parameter $s_0 \in S$, there is a dense Zariski open $U \subset \mathbb{P}^1$ such that the multiplier of every repelling periodic cycle of f_{s_0} meeting U is constant on its marked algebraic periodic component. Then the family is isotrivial or a family of flexible Lattès maps.*

Proof. Write $Z = \mathbb{P}^1 \setminus U$. The Fatou–Shishikura inequality gives only finitely many nonrepelling cycles of f_{s_0}; see [Shi87, Corollary 1]. Choose an integer M larger than $|Z|$, the exact periods of all those cycles, and every integer mr, where m is the exact period of one of those cycles and its multiplier is a root of unity of order r.

For $n \geq 1$ let Fix_n be the relative fixed point divisor of f^n, with its fixed point multiplicities, and form the relative dynatomic divisor

$$D_n = \sum_{k|n} \mu(n/k)\,\mathrm{Fix}_k,$$

where μ denotes the Möbius function. This is an effective relative divisor finite and flat over S. We recall the local calculation, including the parabolic case. If a has exact period m and $h = f^m$ is its return germ, put $\lambda = h'(a)$. If λ is not a root of unity, every germ $h^j(z) - z$ has a simple zero at a. If λ has order r, then $h^j(z) - z$ has order one for $r \nmid j$, whereas for $j = rk$ its order is a fixed integer $e > 1$, independent of $k \geq 1$. Indeed, in a coordinate centered at a,

$$h^r(z) = z + cz^e + O(z^{e+1}), \quad c \neq 0, \qquad h^{rk}(z) = z + kcz^e + O(z^{e+1}).$$

The return germ cannot be the identity because the rational map has degree greater than one. Möbius inversion therefore shows that a occurs in D_n only for $n = m$, or for $n = mr$ when $r > 1$. In particular, a parabolic point of multiplier one occurs only at its actual period, not at arbitrary multiples of that period. The same calculation over the

algebraic closure of the generic function field proves effectiveness of the relative divisor. Its components are horizontal, because the fixed point divisors have no vertical components; as an effective Cartier divisor on $S \times \mathbb{P}^1$ with finite fibers, it is finite and flat over S. Restriction of the Cartier-divisor identity to a fiber gives precisely the local calculation just made.

It follows that, for every $n > M$, the fiber $(D_n)_{s_0}$ consists entirely of simple points of actual period n which are repelling. Their cycles meet U: a cycle contained in Z has period at most $|Z|$. Let C be any irreducible component of the support of D_n. Since $C \to S$ is finite and surjective, it meets the reference fiber at one of these points. The multiplier

$$\Lambda_n(s, z) = (f_s^n)'(z)$$

is an intrinsic regular function on the relative fixed point scheme: the derivative acts on the relative tangent line at the fixed point. Near $C \cap (\{s_0\} \times \mathbb{P}^1)$, the fixed point scheme is smooth and *étale* over S, and C is the marked actual-period-n component. The hypothesis makes Λ_n constant on its normalization, with constant value of modulus greater than one. The same value holds at every point of C.

Every actual-period-n point of every f_s belongs to D_n. Consequently every periodic point of exact period $n > M$ is repelling, for every $s \in S$. All attracting cycles in the whole family have period at most M. This is precisely the stability hypothesis in [McM87, Section 2]; its Theorem 2.2 says that the family is trivial or affine. The classification of affine families immediately following that theorem identifies every nontrivial algebraic family as a flexible Lattès family. In the trivial case, the generic Möbius-conjugacy scheme is nonempty and of finite type over $\mathbb{C}(S)$. A closed point supplies a conjugacy after a finite function-field extension, which is isotriviality. □

Corollary 5.6 (Normal curves in the spectral induction class)**.** *A family of polarized endomorphisms of normal projective curves with the good reference-cycle spectral property of Definition 5.1 is isotrivial or exceptional. In genus zero the nonisotrivial alternative is a family of flexible Lattès maps.*

Proof. Riemann–Hurwitz gives genus at most one. In genus zero, a finite extension of the parameter function field identifies the generic curve with $\mathbb{P}^1$. Spread the identification over a nonempty open of the resulting parameter curve. The fixed-field convention preceding Definition 5.1 retains the marked reference parameter under this operation. The spectral property is unchanged by conjugacy and finite parameter extension. Lemma 5.5 applies.

In genus one, choose an origin after a finite extension. The endomorphism is $x \mapsto \alpha(x)+a$, where α is an isogeny of degree greater than one. The isogeny $\alpha - \mathrm{id}$ is nonzero, hence surjective. After a further finite extension choose a fixed point and take it as origin. The endomorphism becomes the polarized isogeny α, so the family is exceptional. □

6. Choosing the local returns

The construction follows the homoclinic-orbit viewpoint of Ji–Xie [JX23, Section 2.2]: long returns produce adjoint periodic points, and their multipliers constrain the return maps. Their exceptional-map criterion and new proof of McMullen's theorem appear in [JX23, Theorem 2.11 and Section 6]. Here we use repelling spectra to control local returns in arbitrary phase dimension.

Lemma 6.1 (A finite section over a smooth ball)**.** *Let $q : T \to B$ be a finite holomorphic map, where T is normal and pure dimensional and B is a complex ball of the same dimension. Every holomorphic section $s : B \to T$ identifies B biholomorphically with an open and closed component of T.*

Proof. The image of a section is closed: if $s(b_j) \to t$, then $b_j = q(s(b_j)) \to q(t)$ and $t = s(q(t))$. The section is proper as a map into T, and its image is therefore a closed analytic subset of dimension $\dim T$. Normality makes T locally irreducible. Thus this image is a union of connected components of T, and is open. The restriction of q and s are mutually inverse holomorphic maps. □

Proposition 6.2 (Selecting a repeller with unique return centres)**.** *Let $f : X \to X$ be a finite endomorphism of a normal projective variety. Assume its meromorphic lift to a smooth projective resolution has dominant topological degree. Let μ be its equilibrium measure. In every open set of positive μ-measure one can choose a smooth repelling periodic point p, of period r, with the following property. Put $F = f^r$. There is a coordinate ball B about p and a composition-closed collection of biholomorphic inverse branches*

$$h : B \longrightarrow B, \qquad F^{n(h)}h = \mathrm{id}_B,$$

such that the centres $h(p)$ form a uniqueness set for holomorphic functions on B. Every neighbourhood of p contains the images of a subcollection whose centres are again a uniqueness set. The same local uniqueness holds near any return centre, using compositions with that return. Moreover, the unramified backward orbit of p under F is dense in $\operatorname{supp}\mu$.

Proof. Let $\pi : Y \to X$ be a resolution which is an isomorphism over X_{reg}, and write $g = \pi^{-1} f \pi$. Let d_t be its topological degree. The equilibrium measure on Y does not charge pluripolar sets, by Guedj [Gue05], *Annals of Mathematics* **161** (2005), Theorem 2.1. Its push-forward therefore gives every proper local analytic subset of X_{reg} measure zero. We use Dinh–Nguyên–Truong, *Indiana University Mathematics Journal* **64** (2015), Theorem 1.1 and Proposition 3.1. The latter supplies one positive closed $(1,1)$-current R on Y such that, if $\nu(R, y) < \varepsilon < 1$, a ball about y admits at least $(1-\varepsilon)d_t^n$ inverse branches of g^n, for every n, whose diameters tend uniformly to zero.

Fix, for example, $\varepsilon = 1/4$. The Siu level set $\{\nu(R, \cdot) \geq \varepsilon\}$ is a single proper analytic set. The equilibrium measure on Y gives it and the exceptional locus of π measure zero. Repelling periodic equidistribution therefore gives a repeller $\tilde{p}$ outside these sets, in the prescribed positive-measure open set. Put $p = \pi(\tilde{p})$ and let r be its period. The local conjugacy by π makes $F = f^r$ biholomorphic and expanding at p.

The good inverse branches on Y project to genuine inverse branches on X. Indeed, on a ball in the isomorphism locus, their projections satisfy $f^n s = \mathrm{id}$. This identity holds on the dense set where the meromorphic conjugacy is represented by ordinary maps, and hence everywhere by holomorphic continuation. Lemma 6.1 makes each projected section a biholomorphism onto a smooth open subset. Distinct branches remain distinct, since π is generically one-to-one. Their diameters still tend uniformly to zero, by uniform continuity of π on compact Y.

Restrict to inverse branches of orders rn, so that they are inverse branches of F^n. Write $\delta = d_t^r = \deg F$. Shrink the initial ball B if necessary. We can arrange that the local inverse branch v of F fixing p maps B relatively compactly into itself and that $v^\ell(B)$ shrinks to p. All the preceding good branches remain defined after restriction to this ball.

Choose a smooth probability measure σ compactly supported in B. Let $b_{n,j} : B \to X$ be the good inverse branches and define the two subprobability measures

$$\nu_n = \delta^{-n} \sum_j \delta_{b_{n,j}(p)}, \qquad \zeta_n = \delta^{-n} \sum_j (b_{n,j})_* \sigma.$$

They have the same mass, at least $1 - \varepsilon$, and their difference tends weakly to zero because the branch diameters tend uniformly to zero. Moreover

$$0 \leq \zeta_n \leq \delta^{-n} (F^n)^* \sigma.$$

The measures on the right converge to μ. For completeness, this smooth-measure version also follows by integrating the pointwise

equidistribution theorem, Dinh–Nguyên–Truong, Theorem 1.2: all its exceptional and indeterminacy sets form a countable union of proper analytic sets and therefore have zero σ-measure; bounded test functions permit dominated convergence. Passing to a subsequence gives

$$\nu_n \longrightarrow \nu, \qquad 0 < \nu \leq \mu. \tag{6.1}$$

In particular ν gives every proper analytic set measure zero. No pointwise equidistribution assertion at p has been used.

Let $\mathcal{B}_p$ denote the set of points which map to p under some iterate of F unramified at that point, including p itself. Let $C = \overline{\mathcal{B}_p}$. Every measure ν_n is supported in C, so (6.1) gives $\mu(C) > 0$. The set C is closed and forward invariant: $F(\mathcal{B}_p) \subset \mathcal{B}_p$ and F is continuous. Invariance of μ makes C invariant modulo a null set; ergodicity of F gives $\mu(C) = 1$. Hence

$$\operatorname{supp} \mu \subset C.$$

In particular, $\mathcal{B}_p \cap B$ is a uniqueness set on B. Indeed, a holomorphic function vanishing there vanishes on $B \cap \operatorname{supp} \mu$, which has positive μ-measure, whereas a proper local analytic zero set has zero μ-measure.

For each $z \in \mathcal{B}_p \cap B$, choose the unramified inverse germ b_z with $b_z(p) = z$ and $F^{n_z} b_z = \mathrm{id}$. Its initial domain may depend on z. Since $v^\ell(B)$ shrinks to p, for sufficiently large ℓ_z the composition

$$h_z = b_z v^{\ell_z}$$

is defined on the same ball B, with image relatively compact in B and centre still equal to z. It is an inverse branch of $F^{n_z+\ell_z}$, because $F^{n_z+\ell_z} b_z v^{\ell_z} = F^{\ell_z} v^{\ell_z} = \mathrm{id}$. Thus a common domain has been obtained without discarding any of the centres in $\mathcal{B}_p \cap B$.

Adjoin the branch v and all finite compositions. They remain inverse branches of iterates of F on B. For every fixed ℓ, the centres of $v^\ell h_z$ are a uniqueness set, because v^ℓ is biholomorphic on B. Their whole images lie in $v^\ell(B)$, which shrinks to p. Composing once more on the left with a fixed return gives the corresponding uniqueness in every sufficiently small neighbourhood of its centre. This proves all the assertions. □

6.1. **Weighted characters with varying resonant normal forms.** Fix a parameter disk germ $(B, 0)$ and a holomorphically marked smooth repelling Julia point. Constant spectra and holomorphic Poincaré–Dulac normalization give

$$f_s^\ell \sigma_s = \sigma_s \mathcal{N}_s, \qquad \mathcal{N}_s = SU_s = U_s S,$$
$$S = \operatorname{diag}(\lambda_1, \ldots, \lambda_N), \quad |\lambda_i| > 1.$$

Here S is fixed and U_s is a holomorphic family of unipotent resonant polynomial automorphisms. The constant-spectrum primary projectors give holomorphic generalized eigenspace bundles; choose holomorphic frames for them and keep the varying nilpotent linear parts in U_s. No constancy of their Jordan type or of the nonlinear resonant coefficients is imposed. Convergent normalization in the Poincaré domain is locally uniform in the parameter; the homological denominators outside the finitely many resonant spaces are fixed and bounded away from zero with the usual degree estimates; see [RR88, Appendix] and [BDM08].

For a multiplier $\chi = \lambda^\alpha$, let V_χ be the span of the monomials of multiplier χ, and let P_χ be Taylor projection. Every V_χ is finite dimensional. Put

$$a = \min_i \log|\lambda_i| > 0, \qquad b = \max_i \log|\lambda_i|, \qquad W_R = \bigoplus_{0<\log|\chi|\le R} V_\chi.$$

For an actual inverse return h_s, set $T_{h,s} = P_R C_{h_s}|_{W_R}$, where $C_h\varphi = \varphi\circ h$.

Lemma 6.3 (Polynomial bounds for resonant iterates)**.** *There are holomorphic nilpotents $L_{\chi,s}$ such that*

$$C_{\mathcal{N}_s^{-n}}|_{V_\chi} = \chi^{-n}\exp(nL_{\chi,s}).$$

On any relatively compact parameter disc, there are constants C, D such that the Taylor coefficients contributed by a monomial of ordinary degree m are bounded by

$$C^m(1+n)^{Dm}|\chi|^{-n}.$$

The constants are independent of m, n, and s.

Proof. Order the variables by $\log|\lambda_i|$. A nonlinear resonant monomial in coordinate i uses only variables of strictly smaller absolute weight. The diagonal blocks of the linear part of U_s are unipotent. Induction over these weight blocks shows that every coordinate of U_s^n is polynomial in n: on a linear block the binomial expansion of a unipotent matrix is finite, and at the next weight block the additional terms are finite sums of polynomial sequences in the iterate index. Their degrees admit a bound depending only on the fixed list of resonant monomials. The same assertion holds for U_s^{-n}.

Substitution by U_s is unipotent on each V_χ. Its logarithm is therefore a finite sum, and $L_{\chi,s} = \log(C_{U_s^{-1}}|_{V_\chi})$ is holomorphic and nilpotent. The first identity follows from commutation with S. The preceding coordinate bounds, multiplied for the m factors of a degree-m monomial, give the stated estimate, after increasing C, D. All coefficient

bounds are uniform on a relatively compact parameter disc. Notice that $m \le a^{-1} \log|\chi|$. □

Proposition 6.4 (Weighted character constancy)**.** *Every trace of a word in*

$$T_{h,s}, \qquad P_\chi|_{W_R}, \qquad L_{\chi,s}P_\chi$$

is independent of s. This holds for every finite cutoff R.

Proof. Fix the finite collection of returns occurring in a word. Choose polydiscs $D_r \Subset D_R$ so that these returns send D_r compactly into D_R. For all sufficiently large n, $\mathcal{N}_s^{-n}(D_R) \Subset D_r$, locally uniformly in s. The composition operators between the corresponding Hardy spaces are bounded, and the inverse-normal-form operators have absolutely summable Taylor expansions. Consequently the holomorphic fixed-point trace formula [BJ08, Theorem 4.2, formula (11)] applies to

$$\operatorname{Tr}\big(C_{h_1,s}C_{\mathcal{N}_s^{-n_1}} \cdots C_{h_r,s}C_{\mathcal{N}_s^{-n_r}}\big). \tag{6.2}$$

Its value is $1/\det(I - Dg_s(p_s))$, where g_s is the corresponding contracting inverse word. Starting its iterates at the marked Julia point shows that p_s is a Julia point. The forward periodic spectra are constant by Proposition 3.1. Thus (6.2) is constant for all independently sufficiently large integers $n_1, \dots, n_r$.

Set $A_{\chi\psi}(h_s) = P_\chi C_{h_s}|_{V_\psi}$. Expanding (6.2) gives exponential frequencies $\chi_1^{-n_1} \cdots \chi_r^{-n_r}$ and coefficients

$$\operatorname{tr}\big(A_{\chi_r\chi_1}(h_{1,s})e^{n_1 L_{\chi_1,s}} \cdots A_{\chi_{r-1}\chi_r}(h_{r,s})e^{n_r L_{\chi_r,s}}\big),$$

which are polynomials in the independent integers n_i. There are only finitely many multiplier weights below a given modulus. Distinct exponential frequencies of the same modulus, together with their polynomial coefficients, are linearly independent; finite-difference annihilators isolate each frequency. They can be extracted in order of increasing $|\chi|$.

For completeness, the infinite tail does not interfere with this extraction. Lemma 6.3 and the Cauchy estimates bound the terms of weight χ by

$$\exp\big((-n + C_1 \log(1+n) + C_2)\log|\chi|\big).$$

The number of monomials with $\log|\chi| \le t$ grows at most polynomially in t. If χ_* is the next weight modulus after the finitely many retained ones, the remaining sum is bounded by $|\chi_*|^{-n}$ times a fixed power of n, up to a harmless constant after increasing the initial n. It is exponentially smaller than every retained frequency. This estimate is locally uniform in s and is preserved by the other fixed bounded substitutions. Extraction in each independent integer n_i therefore proves

constancy of every coefficient of the displayed polynomial. These coefficients are exactly the cyclic block traces with arbitrary inserted powers of the nilpotents. Inserting the weight projectors and expanding a compressed word proves the assertion. In the full word (6.2) the constant block contributes the fixed trace 1; a positive-weight projector or a nilpotent supported on a positive block annihilates that block. Passing to W_R therefore gives the asserted traces. □

Lemma 6.5 (An irreducible holomorphic summand)**.** *After a finite change of parameter germ there is a nonzero holomorphic subbundle $E_s \subset W_R$, invariant under all the operators in Proposition 6.4, on which they act irreducibly. Their restrictions are simultaneously constant in a holomorphic frame. In particular there are $A_s : E_0 \xrightarrow{\sim} E_s$, $A_0 = I$, intertwining every compressed return, every weight projector, and every $L_{\chi,s}$.*

Proof. Over an algebraic closure of the meromorphic parameter-germ field, choose a common invariant subspace of least positive dimension. The invariance equations on its Grassmannian are polynomial and are generated by finitely many of them. Thus a finite extension of the parameter-germ field suffices. Properness of the Grassmannian extends the resulting meromorphic map holomorphically across the reference point. Invariance extends by closedness. This argument also accommodates return germs with different parameter domains: each invariance identity is an identity of meromorphic germs.

For any word, the eigenvalues of its restriction to E_s are among the constant eigenvalues of its action on W_R. The latter are constant because traces of all powers are constant. Its trace on E_s is therefore a holomorphic function taking values in a finite set, hence is constant. Generic irreducibility and Burnside's theorem give m^2 words whose restrictions form a basis of $\mathrm{End}(E_s)$, where $m = \dim E_s$. Their trace-pairing Gram determinant is nonzero generically and constant, hence nonzero at the reference point as well. The multiplication table of this word basis is determined by the same constant trace pairing. A fixed system of matrix units consequently yields a holomorphic trivialization in which every generator is constant. The finite word basis works for the entire collection of return germs. □

7. Geometric reconstruction from finite return data

On a smooth open of a normal variety, a *distribution* is a subbundle of the tangent bundle. Its rational extension is the saturated subsheaf of the reflexive tangent sheaf determined at the generic point. A *foliation* is a distribution closed under the Lie bracket. Its *algebraic part*

is the maximal subfoliation whose general leaves are algebraic. A foliation is *purely transcendental* if its algebraic part has rank zero. A *finite web* is a finite unordered collection of generically distinct distributions; invariance means that the differential permutes its members wherever it is invertible. A *Killing field* of a local geometric structure is a local vector field whose flow preserves that structure. The structure is of *finite type* if a fixed finite jet determines each local infinitesimal symmetry.

Lemma 7.1 (An invariant subspace gives an invariant foliation)**.** *Let $D \subset \mathbb{C}^N$ be a connected neighborhood of 0 and let $W \subset \mathcal{O}(D)$ be finite dimensional, with every element vanishing at 0. Suppose a linear Taylor projection P_W onto W retains all first derivatives:*

$$d(P_W\varphi)(0) = d\varphi(0).$$

Let $\mathcal{H}$ be a collection of local biholomorphisms from neighborhoods of 0 into D, closed under every composition that is defined as a germ at 0. Assume its centers are a holomorphic uniqueness set locally at 0: in every sufficiently small neighborhood $D' \ni 0$, a holomorphic function vanishing at all $h(0) \in D'$ vanishes identically near 0.

Write $T_h = P_W C_h|_W$. If a nonzero subspace $E \subset W$ is invariant under every T_h, set

$$E^{(1)} = \{de(0) : e \in E\} \subset (\mathbb{C}^N)^*, \qquad d = \dim E^{(1)}.$$

Then $1 \le d \le N$, the differentials of E have constant rank d near 0, and their common kernel defines a regular integrable distribution there. Its extension as a singular foliation on D is invariant under every return germ $h \in \mathcal{H}$. In particular, if $d < N$, it is a nonzero proper invariant foliation.

Proof. Let $a_h = h(0)$. Invariance of E and the first-jet property of P_W give

$$dE(a_h)\, Dh(0) \subset E^{(1)}. \tag{7.1}$$

The derivative $Dh(0)$ is invertible. Hence every $(d+1)$-minor of a matrix of differentials of a basis of E vanishes at all centers sufficiently close to 0. Uniqueness makes these minors vanish identically near 0, and then on D by connectedness. The rank at 0 is d, so it is exactly d in a neighborhood of 0. If $d = 0$, the same argument says $de = 0$ on D for all $e \in E$. As $e(0) = 0$, this would give $E = 0$, a contradiction.

Choose $e_1, \dots, e_d \in E$ with independent differentials at 0. Their joint map is a submersion near 0, and every differential from E lies in their span there. The common kernel is therefore the tangent distribution to its local fibers and is integrable. On D take the saturated foliation

given generically by the same kernel; equivalently it is the kernel of the finite matrix of differentials of a basis of E.

Fix $h \in \mathcal{H}$. For every $k \in \mathcal{H}$ whose center a_k is sufficiently close to 0, the composition $h \circ k$ is defined and belongs to $\mathcal{H}$. Equation (7.1) for that composition gives

$$d(E \circ h)(a_k)\, Dk(0) \subset E^{(1)}.$$

On the other hand, since the rank of dE is d near 0, (7.1) for k is an equality:

$$dE(a_k)\, Dk(0) = E^{(1)}.$$

Thus $d(E \circ h)(a_k) \subset dE(a_k)$. The $(d+1)$-minors of the combined matrix of dE and $d(E \circ h)$ vanish at all these centers. Local uniqueness makes this inclusion an identity of germs at 0. On a dense open subset both sides have rank d, because h is locally biholomorphic. Their annihilator distributions agree there, and hence their saturated foliations agree. This is the required invariance. □

Proposition 7.2 (Compressed invariance yields exact jet geometry)**.** *Let $D \subset \mathbb{C}^N$ be connected, let $A = \mathbb{C}1 \oplus E$ be a finite-dimensional space of holomorphic functions on D, and suppose that the compressed pullbacks of a collection $\mathcal{H}$ of local biholomorphisms preserve E. Assume that the compression retains the complete k-jet at 0 modulo constants. Assume the following composition and uniqueness conditions: the centres $a_h = h(0)$ determine holomorphic functions on D; for each fixed h, the maps $h \circ u$ are among the tested maps for a locally unique set of sufficiently small centres a_u in the domain of h.*

For $0 \le d \le k$ put

$$L^d_x = j^d_x A \subset J^d_x(\mathcal{O}_D), \qquad r_d = \dim L^d_0.$$

Then $\dim L^d_x \le r_d$ on D. On the common open locus D° where all these ranks are maximal, the L^d are holomorphic subbundles and every actual return preserves them:

$$h^*_x : L^d_{h(x)} \xrightarrow{\sim} L^d_x \quad (x, h(x) \in D^\circ). \tag{7.2}$$

Here h^*_x is the ordinary pullback isomorphism of full jet spaces. If A consists of polynomials, these are algebraic subbundles on a Zariski open set and the displayed conditions are algebraic differential equations of order at most k for h.

Proof. Choose a basis of A. Its derivatives of orders at most d form a holomorphic jet matrix. Compressed invariance gives

$$j^d_0 C_h A \subset j^d_0 A = L^d_0.$$

Pullback by the local biholomorphism h is an isomorphism $J^d_{a_h} \to J^d_0$. Therefore $\dim L^d_{a_h} \le r_d$. Every minor of size $r_d + 1$ of the jet matrix vanishes at the centres. Centre uniqueness makes it vanish identically, proving the rank bound. The bound is attained at 0, so the maximal-rank locus contains a neighbourhood of 0. Intersect these finitely many loci for $d \le k$.

Fix h. For all sufficiently small tested u we have $a_u \in D^\circ$ and hence

$$j^d_0 C_u A = L^d_0, \qquad j^d_0 C_{h\circ u} A \subset L^d_0.$$

The composition convention is $C_{h\circ u} = C_u C_h$. The invertible jet pullback of u then gives

$$j^d_{a_u} C_h A \subset L^d_{a_u}.$$

Adjoin the columns of the jet matrix of $C_h A$ to those of A. All minors of size $r_d + 1$ vanish at these small centres. Local centre uniqueness forces them to vanish throughout the domain of h. At any $x, h(x) \in D^\circ$ the inclusion they express is an equality, since both spaces have dimension r_d and h^*_x is invertible. This proves (7.2).

For polynomial A, its jet matrices have polynomial entries. The maximal-rank loci and the corresponding Grassmannian maps are algebraic. In coordinates, the last inclusion is expressed by the same finite collection of minors; the chain rule makes their entries polynomial in $x, h(x)$ and the derivatives of h of orders at most d. Thus the equations are algebraic. □

Corollary 7.3 (Rank stagnation gives actual finite-dimensional closure)**.** *Under Proposition 7.2, suppose that*

$$r_d = r_{d-1} \qquad (1 \le d \le k).$$

Then $\dim A = r_{d-1}$ *and every tested return satisfies* $C_h A = A$ *as a space of holomorphic functions on its domain. In particular, if* A *is polynomial, compressed invariance has upgraded to actual invariance of that polynomial space.*

Proof. Work first in a small simply connected open set in D°. Projection of jets induces an isomorphism $L^d \to L^{d-1}$. It follows by holomorphic linear algebra that every derivative of order d of an element $f \in A$ is a holomorphic linear combination of the derivatives of f of orders at most $d-1$. Choose such coefficient functions and write the resulting equations as

$$\partial^\alpha f = \sum_{|\beta| \le d-1} c_{\alpha\beta}(x) \partial^\beta f, \qquad |\alpha| = d. \tag{7.3}$$

Every holomorphic function whose d-jets lie in L^d satisfies these same equations.

Solutions of (7.3) are determined by their $(d-1)$-jet at one point. Indeed, collect all derivatives of orders at most $d-1$ into a vector Y_f. Differentiation and (7.3) give first-order linear systems $\partial_i Y_f = B_i(x) Y_f$. Along any sufficiently short path this is a linear ordinary differential equation, so zero initial data imply $Y_f = 0$. Analytic continuation proves uniqueness on the connected domain; no integrability assertion for arbitrary initial data is required.

Consequently $j_x^{d-1} : A \to L_x^{d-1}$ is injective, and therefore is an isomorphism. In particular $\dim A = r_{d-1}$. For $f \in A$, exact jet-bundle invariance says that $f \circ h$ satisfies the same system wherever source and target lie in D°. At a point there choose $g \in A$ with the same $(d-1)$-jet as $f \circ h$. Uniqueness gives $f \circ h = g$ near that point and then throughout the connected domain of h. Thus $C_h A \subset A$. Pullback is injective because h is a local biholomorphism, so equality follows from finite dimension. □

Corollary 7.4 (The first missing jet has an invariant symbol)**.** *Suppose that $r_{d-1} = \dim J^{d-1}$ but $r_d < \dim J^d$, with $d \le k$. Then on D°*

$$Q_x := \ker(L_x^d \longrightarrow J_x^{d-1}) \subset \operatorname{Sym}^d T_x^* D$$

is a proper subbundle satisfying

$$(Dh(x))^* Q_{h(x)} = Q_x.$$

If $Q = 0$, actual closure $C_h A = A$ follows from Corollary 7.3. If A is polynomial, the bundle Q is algebraic.

Proof. Projection onto lower jets is surjective by hypothesis and has constant rank. Thus its kernel is a subbundle of the indicated symmetric power. Pullback of a jet whose lower terms vanish acts on its leading term only by the derivative of h. The assertion follows from exact jet-bundle invariance. The case $Q = 0$ is exactly $r_d = r_{d-1}$. □

Lemma 7.5 (Integrating a closed tuple of differential operators)**.** *Let $N \ge 2$. Suppose polynomial-coefficient differential operators $A_1, \dots, A_N$ on $\mathbb{C}^N$, of order at most r, satisfy*

$$\partial_j \circ A_i = \partial_i \circ A_j \quad (1 \le i, j \le N).$$

Then there is a unique polynomial-coefficient differential operator B, of order at most $r-1$, such that $A_i = \partial_i \circ B$. If $r = 0$, this means that all A_i vanish.

Proof. Suppose first that $r \geq 1$, and let $a_i(x, \xi)$ denote the degree-r principal symbol of A_i, allowing $a_i = 0$. The degree-$(r+1)$ part of the displayed identity gives

$$\xi_j a_i = \xi_i a_j.$$

For each i, choose $j \neq i$. Primality of ξ_i shows that ξ_i divides a_i. The quotients a_i/ξ_i coincide, so $a_i = \xi_i b$ for a polynomial $b(x, \xi)$ homogeneous of degree $r-1$ in ξ. Choose a differential operator B_{r-1} with symbol b. Each $A_i - \partial_i \circ B_{r-1}$ has order at most $r-1$, and these new operators satisfy the same closedness identity. Iterating reduces to order zero. At order zero write $A_i = a_i(x)$. The coefficients of ∂_j in $\partial_j \circ A_i - \partial_i \circ A_j$ give $a_i = 0$ for $j \neq i$. This proves existence. If $\partial_i \circ B = 0$ for every i, the highest symbol of any nonzero B would be annihilated by multiplication by each ξ_i, an impossibility. This proves uniqueness. □

Proposition 7.6 (Geometric recovery from jet separation)**.** *Let* $S = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ *be expanding, with* $N \geq 2$*, and put*

$$a = \min_i \log|\lambda_i|, \qquad b = \max_i \log|\lambda_i|,$$
$$d_0 = \max\{2, \lfloor b/a \rfloor\}, \qquad M = d_0 + 1.$$

Let $E_s \subset W_R$ *be a holomorphic family of* S*-invariant polynomial subspaces, preserved by the compressed returns* $T_{h,s} = P_R C_{h_s}$*, with the usual relative uniqueness of the moving centres. Assume*

$$j_0^M : \mathbb{C}1 \oplus E_s \longrightarrow \mathcal{O}_{\mathbb{C}^N,0}/\mathfrak{m}^{M+1} \quad \textit{is surjective.}$$

Suppose there are holomorphic isomorphisms $A_s : E_0 \to E_s$*, with* $A_0 = I$*, intertwining all compressed returns and the action of* S*. Then there are weighted polynomial automorphisms* Q_s *commuting with* S*, and nonzero holomorphic scalars* c_s*, such that*

$$A_s = c_s^{-1} C_{Q_s}|_{E_0}.$$

All first derivatives of $Q_s h_s Q_s^{-1}$ *at* 0 *are constant in the parameter.*

Proof. Every monomial whose multiplier equals one of the λ_i has absolute weight at most b, hence degree at most $\lfloor b/a \rfloor < M$. Since the jet map is S-equivariant and surjective, E_s contains each complete multiplier eigenspace of this kind: no higher-degree monomial with that multiplier can be invisible to the displayed jet map. In particular E_s contains all coordinate functions.

Set $Q_s^{(1)} = (A_s x_1, \ldots, A_s x_N)$. Each component has the same positive multiplier weight as its coordinate. Its linear part is invertible near the reference parameter. Ordering the weights makes the nonlinear terms

triangular, so successive substitution gives a weighted polynomial inverse. The normalized spaces $E'_s = C^{-1}_{Q_s^{(1)}} E_s$ still separate M-jets, since $Q_s^{(1)}$ is a local coordinate change fixing 0. They still contain the complete coordinate-multiplier eigenspaces. The normalized isomorphisms $B_s : E_0 \to E'_s$ fix x_i. Put

$$D_s = \dot{B}_s B_s^{-1} : E'_s \longrightarrow W_R.$$

For the normalized returns, differentiation of the intertwining identity gives

$$\dot{T}_h e = D_s(T_h e) - T_h(D_s e), \qquad e \in E'_s.$$

Let Π_{c} project onto the complete coordinate-multiplier spaces and define $\ell_D(f) = d(D_s \Pi_{\mathrm{c}} f)(0)$. Extend D_s to constants by $D_s 1 = 0$. The functional ℓ_D has order $m \le \lfloor b/a \rfloor$ and annihilates constants and linear functions. Put $m = 0$ if it vanishes and $r = \max\{2, m\}$. First derivatives of the commutator give

$$d(D_s e)(a_h) Dh(0) = \ell_D(e \circ h) - d\big(de(h)\dot{h}\big)(0). \tag{7.4}$$

Thus $d(D_s e)(a_h)$ depends on the r-jet of e at a_h.

Choose finitely many elements of $\mathbb{C}1 \oplus E'_s$ whose r-jets at 0 form a basis. Their jet matrix remains invertible on a neighbourhood of 0. Interpolation therefore gives unique differential operators A_i of order at most r, with holomorphic coefficients near 0, agreeing with $\partial_i D_s$ on these tests. Equation (7.4) and uniqueness of the small centres give

$$\partial_i D_s e = A_i e \qquad (e \in E'_s).$$

The commutator $\partial_j \circ A_i - \partial_i \circ A_j$ has order at most $r + 1 \le M$ and annihilates $\mathbb{C}1 \oplus E'_s$. The M-jet surjectivity, which is open near 0, forces that differential operator to vanish identically. The proof of Lemma 7.5 works unchanged for holomorphic coefficients. It gives a unique holomorphic differential operator G of order at most $r - 1$ with $A_i = \partial_i \circ G$.

Testing on the same jet-separating space shows $A_i C_S = \lambda_i C_S A_i$ as genuine differential-operator identities. Uniqueness gives $G C_S = C_S G$. Its zeroth coefficient is therefore constant, and every positive-order coefficient vanishes at 0; indeed these coefficients are weighted polynomials. Since $(D_s - G)e$ is constant and has positive multiplier weight for each weight eigenvector $e \in E'_s$, it vanishes. Hence $D_s = G$ on E'_s.

Now $\ell_D(e) = d(Ge)(0)$ for every $e \in E'_s$. The difference between these two functionals has order at most r, and $\mathbb{C}1 \oplus E'_s$ separates all r-jets. Consequently

$$\ell_D(f) = d(Gf)(0)$$

for every holomorphic germ f. The right side has order at most $r-1$ because the positive-order coefficients of G vanish at 0. If $m \geq 2$ then $r = m$, a contradiction. Thus $m = 0$, G has order at most one. Put $\mathcal{E} = \sum_i x_i \partial_i$, the Euler vector field. The identities $D_s x_i = 0$ give

$$D_s = \gamma_s (1 - \mathcal{E})|_{E'_s}.$$

Integration yields $B_s e = c_s^{-1} e(c_s x)$, where $\dot{c}_s / c_s = -\gamma_s$ and $c_0 = 1$. Composing with $Q_s^{(1)}$ proves the proposition. All interpolations can be made holomorphically in the parameter, and relative centre uniqueness is used coefficient by coefficient by Lemma 5.4. □

Proposition 7.7 (Recovery with a varying resonant normal form)**.** *Assume Proposition 7.6. In addition, let $\mathcal{N}_s$ be holomorphic expanding Poincaré–Dulac polynomial automorphisms with the same semisimple part S, and assume*

$$C_{\mathcal{N}_s} A_s = A_s C_{\mathcal{N}_0} \quad \text{on } E_0.$$

The return collection is assumed closed under compositions which are defined near 0, and under left composition by sufficiently large negative powers of $\mathcal{N}_s$. Then the weighted polynomial automorphisms supplied by that proposition satisfy

$$Q_s \mathcal{N}_s Q_s^{-1} = \mathcal{N}_0, \qquad Q_s h_s Q_s^{-1} = h_0$$

as parameter germs for every marked actual return h.

Proof. The preceding geometric-recovery proof uses only the fixed semisimple operator S, and supplies $A_s = c_s^{-1} C_{Q_s}|_{E_0}$. Since $\mathcal{N}_s$ commutes with S, its pullback preserves each multiplier space and there is no truncation in its action on W_R. Substitution in the additional intertwining identity gives

$$e \circ Q_s \circ \mathcal{N}_s = e \circ \mathcal{N}_0 \circ Q_s \qquad (e \in E_0).$$

Every coordinate function belongs to E_0. Testing these functions proves the first assertion. Henceforth use the coordinates Q_s; thus $\mathcal{N}_s = \mathcal{N}_0$, $E_s = E_0 =: E$, and every compressed return on E is constant.

Choose a basis $e_1, \ldots, e_t$ of E and define

$$\Phi(z) = \big(de_1(z), \ldots, de_t(z)\big).$$

The surjection onto 2-jets at 0 implies that $d\Phi_0$ is injective: if $v \neq 0$, a quadratic polynomial with nonzero contraction of its Hessian against v has the same 2-jet as an element of $\mathbb{C}1 \oplus E$. Thus Φ is an embedding on a neighbourhood U of 0, after shrinking U.

Let k be a return with reference centre $a_k(0) \in U$. Constancy of the linear Taylor coefficient of $T_{k,s}e$ gives

$$de(a_k(s))Dk_s(0) = de(a_k(0))Dk_0(0) \qquad (e \in E).$$

Testing coordinate functions first gives $Dk_s(0) = Dk_0(0)$. This matrix is invertible, so $\Phi(a_k(s)) = \Phi(a_k(0))$. Injectivity on U proves $a_k(s) = a_k(0)$ as a parameter germ.

For an arbitrary return h and a sufficiently small-centred return k, the same first-derivative constancy for $h \circ k$ gives

$$Dh_s(a_k(0))Dk_0(0) = D(h_0 \circ k_0)(0).$$

Consequently $Dh_s(a_k(0)) = Dh_0(a_k(0))$. Uniqueness of the small reference centres, applied coefficient by coefficient in the parameter, gives $Dh_s = Dh_0$. Every small-centred return therefore has both constant derivative and constant value at 0, and is itself constant. Finally choose m with $\mathcal{N}_0^{-m}(a_h(0)) \in U$. The marked return $\mathcal{N}_0^{-m} \circ h_s$ is small-centred and hence constant. Since $\mathcal{N}_0$ is constant, so is h_s. □

8. Continuation and algebraization in arbitrary dimension

Throughout this section X is a normal complex projective variety of dimension N, and $f : X \to X$ is q-polarized. Put $d = q^N$. We use the inverse-branch and backward-equidistribution theorems of [DNT15, Theorem 1.2 and Proposition 3.1]. On a resolution the meromorphic lift has dynamical degrees $1, q, \ldots, q^N$; in particular its topological degree dominates the others. The set Σ_0 in the construction of [DNT15, Section 3] may be enlarged by the exceptional divisors of the resolution. The resulting branches avoid them at every intermediate step and descend to the smooth locus of X. Thus backward equidistribution holds outside a countable union of proper algebraic sets. Also, for every $\varepsilon > 0$ there is a proper algebraic set A, containing $\operatorname{Sing} X$, such that every point of $X \setminus A$ has a fixed ball admitting at least $(1-\varepsilon)d^m$ shrinking unramified inverse branches of f^m for all sufficiently large m. The set A is obtained from a positive Lelong-number locus of the inverse-branch current; it is algebraic by Siu's theorem and properness of the resolution.

Proposition 8.1 (Majority continuation)**.** *Let $O \subset X$ be a forward-invariant analytic open set of full equilibrium measure. Suppose a geometric datum is defined near $J_f \cap O \cap X^{\mathrm{reg}}$ and is invariant there as a germ. The datum may be a holomorphic section with holomorphic affine forward transport, a meromorphic distribution of any fixed rank,*

or a meromorphic affine connection. It extends uniquely to $X^{\mathrm{reg}} \setminus E$ for a proper totally invariant algebraic set E. The extension is holomorphic in the first case and meromorphic in the other two cases.

Proof. Write $U_0 \subset O \cap X^{\mathrm{reg}}$ for the original domain. The ramification current

$$R = \sum_{j \geq 1} d^{-j} (f^j)_* [R_f]$$

has finite mass. Indeed its jth summand has mass $O(q^{(N-1)j}/q^{Nj}) = O(q^{-j})$ against an ample polarization. The ramification formula gives

$$d^{-m} (f^m)_* [R_{f^m}] \leq R.$$

Choose a generic $a \in J_f \cap O$ with backward equidistribution and a generic target x. A sufficiently ample smooth complete- intersection curve C through a, x may be chosen to avoid the forward images of $\operatorname{Sing} X$ and every codimension-two bad ramification stratum of every iterate, and to meet all remaining branch divisors transversally. These are countably many dense open conditions in the parameter space of complete intersections. Consequently $(f^m)^{-1} C \to C$ is a branched covering between smooth curves. For $N = 1$, take $C = X$.

The restriction $R|_C$ is a finite atomic measure, with neither endpoint an atom. Choose an arc from a to x and simply connected strips $\Omega_0 \Subset \Omega \subset C$ around it such that $(R|_C)(\Omega) < \varepsilon$. One obtains such a strip by avoiding finitely many atoms carrying all but ε of the mass and then shrinking around the arc. The total ramification above Ω is at most εd^m. A connected covering of a disk of degree $e \geq 2$ has ramification at least $e - 1$, and $e \leq 2(e-1)$. Hence at least $(1 - 2\varepsilon) d^m$ sheets are unramified sheets of degree one.

The total area of $(f^m)^{-1} C$ is $O(q^{(N-1)m})$. For example, discarding the sheets with area greater than $q^{-m/2}$ removes only $O(q^{(N-1/2)m}) = o(d^m)$ sheets. The small-area diameter estimate on the smaller strip, along a finite chain of disks, makes the diameters of the retained sheets tend to zero; see [DNT15, Lemma 3.4]. Backward equidistribution at a, together with a compact core of U_0 of arbitrarily large equilibrium measure, shows that an arbitrarily large proportion of these branches maps all of Ω_0 into U_0. At each fixed depth they extend to a common ambient tube around the arc.

Transport the datum along each branch. At a the transported germs coincide: their source points and intermediate points belong to $J_f \cap O$, where the invariance identity holds. The identity theorem identifies their transports throughout the tube. If two constructions end at x and each retains more than half of the d^m germs at one sufficiently

large depth, they share an inverse germ. They therefore give the same germ at x, independently of the arc and the starting point.

To obtain a fixed domain, take an inverse-branch ball $B \subset X \setminus A$. Backward equidistribution at a generic point and shrinking of the branches show that an arbitrarily large proportion of its branches maps into U_0. At a generic point $b_0 \in B$, more than half of these branches belong to the equality class furnished by the arc construction. The identity theorem identifies their transports on all of B. It is the unique class containing more than half of the germs. Comparing with the arc construction at any other generic point gives the same class. The resulting data glue on a Zariski open $W \subset X \setminus A$ and satisfy the invariance identity wherever source and target belong to W.

Put $U = \bigcup_{m\geq 0} f^m(W)$ and $E = X \setminus U$. Finite maps to a normal variety are open, so U is Zariski open. We have $f^{-1}E \subset E$. The descending closed sets $f^{-m}E$ stabilize, and applying a suitable power of the surjective map f gives $f^{-1}E = E$.

For $x \in U \cap X^{\mathrm{reg}}$, use a finite proper local piece of an iterate from a relatively compact subset of W onto a neighborhood of x. Normalized trace extends holomorphic affine transports across the discriminant. For a connection, trace the meromorphic Christoffel coefficients; their affine transformation law commutes with normalized trace. For a rank-r distribution, use its Plücker line in $\bigwedge^r T_X$. Trace a local meromorphic generator with scalar weights chosen by interpolation on one unramified fiber so that the trace is nonzero. All terms are generically collinear, so the resulting line is the transported Plücker line. Decomposability holds by continuation. These extensions agree on W and therefore glue on $U \cap X^{\mathrm{reg}}$. □

Lemma 8.2 (Depth at a normal boundary)**.** *Let E be a nonempty reduced totally invariant divisor of X. There are an iterate F, an ideal norm η of E, and a constant c such that $\rho = -\log \eta \geq 1$ is proper on $X \setminus E$ and*

$$\rho \circ F \geq 2\rho - c.$$

On a log resolution of the ideal of E, the function ρ is smooth off the boundary and $dd^c\rho$ extends as a smooth real form. At a general point of E, it equals $-\log|z_1| + O(1)$.

Proof. Pass to an iterate F_0 fixing every prime component D of E, and let Q be its polarization weight. Write $F_0^*D = m_D D$. The finite projection formula gives

$$Q^{N-1}(F_0^*D \cdot H^{N-1}) = Q^N(D \cdot H^{N-1}),$$

so $m_D = Q$. For $I = \mathcal{I}_E$, divisorial valuations on the normal variety give

$$(F_0^m)^* I \subset I^{(Q^m)} = \mathcal{O}_X(-Q^m E).$$

The symbolic and ordinary topologies of this fixed ideal are cofinal, as follows from the next ideal-sheaf calculation. On an affine normal chart, write $I = \bigcap_i P_i$ with P_i the prime component ideals. The rings are excellent, so the completions of their normal localizations are normal local domains. For a prime P_i and a strictly larger prime Q', the quotient of $\widehat{R_{Q'}}$ by $P_i\widehat{R_{Q'}}$ has positive dimension. Thus Schenzel's criterion [Sch85, Theorem 1] gives, for each a, an integer b such that $P_i^{(b)} \subset P_i^a$. Write $I^2 = \bigcap_j Q_j$ as a primary decomposition and choose a with $(\sqrt{Q_j})^a \subset Q_j$ for every j. Each $\sqrt{Q_j}$ contains some P_i, and therefore, for large b,

$$I^{(b)} = \bigcap_i P_i^{(b)} \subset \bigcap_i P_i^a \subset I^2.$$

A finite affine cover makes this choice uniform. Hence an iterate F satisfies $F^* I \subset I^2$.

Combine squared norms of finite local generating tuples of I using a smooth partition of unity. Their square root is an ideal norm η, and the containment gives $\eta \circ F \leq C\eta^2$. On a log resolution the pulled-back generators have a common divisorial factor and no common zero after that factor is removed. The norm is therefore the norm of the resolved defining section times a positive smooth factor. Rescaling η gives all the assertions. □

Proposition 8.3 (Rationality of projective cocycles)**.** *Let $E \subset X$ be a proper totally invariant algebraic set. If a single-valued meromorphic map $\Phi : X \setminus E \dashrightarrow \mathbb{P}^M$ satisfies*

$$\Phi \circ f = [B]\Phi, \qquad B \in \mathrm{GL}_{M+1}(\mathbb{C}(X)),$$

then Φ is rational.

Proof. Meromorphic Hartogs extension on the normal variety removes the codimension-at-least-two part of E. Its reduced divisorial part is again totally invariant. If that part is empty, projective GAGA applies. Otherwise use Lemma 8.2 and replace F by a further iterate so that its polarization weight Q is greater than two. Let ω be a smooth curvature form of an ample polarization, and put

$$\omega_j = Q^{-j}(F^j)^*\omega.$$

On a resolution $\pi : Y \to X$ of the boundary ideal these are smooth semipositive forms, and

$$\omega_j = \omega_0 + dd^c u_j, \qquad \sup_j \|u_j\|_\infty < \infty.$$

Indeed the potentials are partial sums $\sum_{i<j} Q^{-i}\psi \circ F^i$ pulled back to Y, where $Q^{-1}F^*\omega = \omega + dd^c\psi$. We suppress π^* in the integrals; no lift of F to Y is used.

For a positive closed $(1,1)$-current S on $X \setminus E$, define

$$a_{j,S}(t) = \int_{\{\rho<t\}} S \wedge \omega_j^{N-1}, \qquad T_{j,S}(s) = \int_0^s a_{j,S}(t)\,dt.$$

Put $\vartheta = dd^c\rho$, a smooth form on Y. At regular levels, integration by parts gives

$$T_{j,dd^c u}(s) = \int u\,d\beta_{j,s} - \int_{\{\rho<s\}} u\vartheta \wedge \omega_j^{N-1}, \quad \beta_{j,s} = \delta_{\{\rho=s\}} d\rho \wedge d^c\rho \wedge \omega_j^{N-1}.$$

The positive measures $\beta_{j,s}$ have uniformly bounded mass: Stokes expresses that mass as the integral of $\vartheta \wedge \omega_j^{N-1}$ over the sublevel, and ϑ is bounded by a fixed Kähler form on Y.

For a nonzero algebraic section σ of an ample multiple, choose $g = C - \log\|\sigma\| \geq 0$. For a Kähler form ω_Y on Y one has

$$\sup_j \int_Y g\,\omega_Y \wedge \omega_j^{N-1} < \infty.$$

To see this, use the telescoping identity

$$\omega_j^{N-1} - \omega_0^{N-1} = dd^c u_j \wedge \sum_{a=0}^{N-2} \omega_j^a \wedge \omega_0^{N-2-a}$$

and integrate by parts. Since $dd^c g = m\omega_0 - [\operatorname{div}\sigma]$, boundedness of u_j bounds all the resulting positive mixed intersection masses. For $N = 1$ the assertion is immediate. The preceding Stokes identity, applied to g, also gives $\int g\,d\beta_{j,s} \leq C(s+1)$. Consequently

$$|T_{j,dd^c u}(s)| \leq C(s+1) \quad \text{if } |u| \leq g,$$

uniformly in j. These identities may be obtained by truncation and then used at almost every level.

Take $S = \Phi^*\omega_{\mathrm{FS}}$, defined by the meromorphic graph. For local reduced homogeneous lifts P of Φ, the function $u = \log(\|BP\|/\|P\|)$ is independent of the lift. Bounds for B and B^{-1} give $|u| \leq C - \log\|\sigma\|$ for a suitable algebraic section. The curvature identity, allowing the common factors of BP, gives

$$F^*S \leq S + dd^c u + [P_B],$$

where P_B is a fixed effective algebraic divisor. Hence $T_{j,F^*S}(s) \leq T_{j,S}(s) + C(s+1)$ uniformly in j.

Now $F^*\omega_j = Q\omega_{j+1}$, whereas $\deg F = Q^N$. Together with $F^{-1}\{\rho < 2t - c\} \subset \{\rho < t\}$, finite change of variables therefore gives

$$T_{j,S}(R) \leq \frac{2}{Q} T_{j+1,S}\big((R+c)/2\big) + C(R+1).$$

The factor is $2/Q$ in every dimension: the $N-1$ source forms contribute Q^{N-1} and finite degree contributes Q^N. On each compact sublevel, the Chern–Levine–Nirenberg estimate and the uniformly bounded potentials give a bound independent of j. Iterating the recurrence proves $T_{0,S}(R) = O(R)$. Monotonicity of $a_{0,S}$ implies

$$\int_{X \setminus E} S \wedge \omega^{N-1} < \infty.$$

Average linear projections $\mathbb{P}^M \dashrightarrow \mathbb{P}^1$ over the unitary group. Their averaged curvature is ω_{FS}, so almost every scalar projection of Φ has finite spherical mass. Its graph has finite volume: on a graph in $X \times \mathbb{P}^1$ the volume form has only the terms

$$\omega^N + N\omega^{N-1} \wedge \omega_{\mathrm{FS}},$$

because $\omega_{\mathrm{FS}}^2 = 0$ on $\mathbb{P}^1$. Bishop's finite-volume extension theorem [Bis64] extends the graph across $E \times \mathbb{P}^1$. A generic unitary coordinate system makes all finitely many coordinate ratios extend simultaneously. They recover a meromorphic map on X, which is rational by projective GAGA. □

Corollary 8.4. *The distributions of every rank and the affine connections obtained in Proposition 8.1 are rational. Integrability, flatness, and zero torsion persist whenever they hold initially.*

Proof. Use a rational tangent frame. A rank-r distribution is represented by its Plücker line, with rational cocycle $\bigwedge^r Df$. The Christoffel coefficients of an affine connection satisfy an invertible rational affine cocycle; homogenization gives a rational projective cocycle. Apply Proposition 8.3. All the stated differential and Plücker identities extend by the identity theorem. □

Proposition 8.5 (Horizontal fields)**.** *Let a projective algebraic curve family of normal N-fold systems admit a holomorphic horizontal field near its Julia data. Suppose its fiberwise continuation is single-valued off proper totally invariant algebraic sets and satisfies the forward differential transport identity. Then the family is locally holomorphically*

trivial and is algebraically isotrivial after finite parameter base change and restriction.

Proof. Embed the family in a fixed relative projective space. Extend the ambient components of the field across excluded sets of codimension at least two on each normal fiber by Hartogs. Tangency and the transport identity persist. If a reduced divisorial boundary E remains, it is totally invariant. Use the depth function in Lemma 8.2.

On a compact fiber the differential of F and the deformation term of the affine transport have bounded ambient norms. Thus

$$M(t) := \sup_{\rho \le t}(\|W\| + 1) \le CM\big((t+c)/2\big).$$

Indeed every target of depth at most t has a preimage of depth at most $(t+c)/2$. Iteration gives $M(t) = O((1+t)^a)$. Near a general point of E, the growth is polynomial in $\log(1/|z_1|)$. Every negative Laurent coefficient vanishes, since its bound is $Cr^k(1+\log(1/r))^a \to 0$. The field therefore extends holomorphically over the general boundary point. Normal Hartogs extends its ambient components over the remaining codimension-at-least-two set. The resulting field remains tangent and horizontal.

After shrinking the parameter curve, cohomology and base change make the pushforward of the restricted ambient tangent bundle locally free. Finitely many marked points in the initial common domain separate the space of its sections on a reference fiber. A nonvanishing evaluation minor stays nonzero nearby. Solving this finite linear system shows that the extended fields depend holomorphically on the parameter. Their flows exist for a uniform time on a compact fiber, preserve the defining ideal, and commute with the dynamics. They give local holomorphic trivializations. The graphs lie in one finite-type relative Hilbert scheme, and the conjugacy identity is algebraic there. A dominating component has a closed point over a finite extension of the parameter function field. This gives algebraic isotriviality after restriction. □

Proposition 8.6 (Descent from a normal-form source)**.** *Let $F = f^\ell$ and let $\Pi : \mathbb{C}^N \to X$ be a Poincaré map at a smooth repelling Julia point p, with*

$$F\Pi = \Pi\mathcal{N},$$

where $\mathcal{N}$ is an expanding polynomial normal-form automorphism. Suppose the good unramified backward centers of p are holomorphic uniqueness sets in the small coordinate neighborhoods meeting J_f, also after deletion of a proper analytic subset.

A regular $\mathcal{N}$-invariant source distribution, preserved by all actual inverse returns, descends to a rational F-invariant distribution on X. Its rank is arbitrary. If $\mathcal{N}$ is linear and all the actual returns are affine, the standard source connection descends to a rational F-invariant flat torsion-free connection. A holomorphic family of polynomial gauges commuting with a fixed $\mathcal{N}$ and identifying all actual returns gives isotriviality of the family.

Proof. If σ is the original injective chart, then near any source point

$$\Pi = F^m \sigma \mathcal{N}^{-m}$$

for sufficiently large m. Hence Π is locally finite and open, including over singular target points. Its image O is F-forward invariant and contains a neighborhood of p. Ergodicity of the equilibrium measure for F gives $\mu_f(O) = 1$.

Let g be a local deck germ from the original chart to a regular lift of p. Choose m so that $\mathcal{N}^{-m} g$ takes values in the original chart, and then choose ℓ so that $h = \mathcal{N}^{-m} g \mathcal{N}^{-\ell}$ is defined on the fixed coordinate ball and maps it into itself. The identity $F\Pi = \Pi\mathcal{N}$ makes h an actual inverse return. Thus $g = \mathcal{N}^m h \mathcal{N}^\ell$. Invariance under the normal form and the actual returns shows that g preserves the given datum. Let two regular inverse branches of Π be defined over a small smooth ball meeting J_f. At a good backward center b choose k such that $F^k(b) = p$ and DF_b^k is invertible. Applying $\mathcal{N}^k$ to both lifts reduces their comparison to a deck germ over p. Pullback by DF_b^k shows that they give the same datum at b. The uniqueness hypothesis, applied to local Plücker equations or to connection coefficients, identifies the two germs on the ball.

The same conclusion holds for finite local pieces over a branch locus. Indeed, a nonzero holomorphic function vanishing on the lifted center set would have a nonzero finite norm vanishing on the center set downstairs. This contradicts uniqueness. Weighted trace for the Plücker line, or normalized trace for the connection, extends the common datum meromorphically across the discriminant. Proposition 8.1 and Corollary 8.4, applied to F, finish the descent. Integrability, flatness, and zero torsion are identities and thus persist.

For a conjugating gauge Θ_t, write $R_t = \Pi_t \Theta_t$. The actual-return identities identify the branches of R_t over the good centers. The implicit function theorem for the unramified forward return, followed by the same uniqueness argument, gives holomorphic conjugacy germs near the Julia data. Their parameter derivatives are horizontal fields

with holomorphic affine forward transport. Proposition 8.1 followed by Proposition 8.5 gives isotriviality for $F_t = f_t^\ell$.

After this trivialization, $F_t = F$ is constant. If x is periodic for F, then $f_t(x)$ lies in the finite set $\mathrm{Fix}(F^m)$ for a suitable m, and hence is constant in t. These fixed-point sets are finite because a positive-dimensional fixed subvariety contradicts polarization. Periodic points are Zariski dense by repeller equidistribution. Therefore f_t itself is constant. □

9. General symbols and local return geometry

9.1. Expansion algebraizes invariant formal geometry.

Lemma 9.1 (An invariant formal distribution is polynomial)**.** *Let F be an expanding holomorphic germ at $0 \in \mathbb{C}^N$ in convergent Poincaré–Dulac normal form. Write*

$$F = S \circ U = U \circ S, \qquad S = \mathrm{diag}(\lambda_1, \dots, \lambda_N), \qquad |\lambda_i| > 1,$$

where the linear part of U is unipotent and all its monomials are resonant with S. Let $\widehat{\mathcal{D}}$ be a regular formal distribution at 0, invariant under F. Then $\widehat{\mathcal{D}}$ is the completion of an algebraic regular distribution on $\mathbb{A}^N$. If it is formally integrable, that algebraic distribution is integrable.

Proof. For every k, the k-jet of a regular distribution is a point of an algebraic jet space on which the algebraic group of $(k+1)$-jets of invertible germs acts. Its stabilizer is a closed algebraic subgroup. Since it contains the jet of F, it contains its semisimple Jordan component. In Poincaré–Dulac coordinates this component is the jet of S: the commuting jet of U is unipotent, as follows either from its action on truncated power series or from the degree filtration. Thus every finite jet of $\widehat{\mathcal{D}}$ is S-invariant, and so is the formal distribution itself.

Put $V = \widehat{\mathcal{D}}_0$ and choose an S-invariant complement W. A regular formal distribution with this value is the graph of a formal matrix

$$B(x) : V \longrightarrow W, \qquad B(0) = 0.$$

The invariance identity is

$$B(Sx) = S_W B(x) S_V^{-1}.$$

Use eigenbases on V and W. If a coefficient of x^α in the matrix entry from the λ_i-eigenspace to the λ_j-eigenspace is nonzero, then

$$\lambda^\alpha = \lambda_j / \lambda_i.$$

Let $a = \min_i \log|\lambda_i| > 0$ and $b = \max_i \log|\lambda_i|$. Every such coefficient satisfies $|\alpha|a \leq b - a$. Only finitely many multi-indices occur. Consequently B is polynomial and its graph defines an algebraic regular distribution on all of $\mathbb{A}^N$.

Formal integrability says that the brackets of the polynomial basis fields of this graph lie in their span to every order at 0. The corresponding polynomial Frobenius identities therefore vanish identically. Hence the distribution is integrable globally. □

Lemma 9.2 (Finite invariant formal Lie algebras converge)**.** *Let F and S be as in Lemma 9.1. Let $\mathfrak{g}$ be a finite-dimensional complex space of formal vector fields, invariant under F_* and F_*^{-1}. Then every element of $\mathfrak{g}$ is polynomial. In particular, if $\mathfrak{g}$ is a transitive formal Lie algebra, it is a locally transitive Lie algebra of holomorphic vector fields.*

Proof. For every k, the space of k-jets $j^k\mathfrak{g}$ is invariant under the action of the jet of F. The stabilizer of this subspace is algebraic, so it is invariant under the semisimple component, namely the jet of S. A finite-dimensional subspace of formal series is closed for the adic topology: choose k_0 for which its k_0-jet map is injective, and use this map to identify every compatible inverse system of its jets with a single element of the subspace. Thus $S_*\mathfrak{g} = \mathfrak{g}$.

The induced action on $\mathfrak{g}$ is diagonalizable, since it embeds by a sufficiently long jet map into the diagonal action on polynomial vector fields. Let Z be an eigenvector, with nonzero eigenvalue μ. A monomial $x^\alpha \partial/\partial x_j$ can occur in Z only when

$$\lambda_j/\lambda^\alpha = \mu.$$

Taking absolute values gives $|\alpha|a \leq b - \log|\mu|$. Thus only finitely many monomials occur in every eigenvector, proving the assertion. □

Lemma 9.3 (Equivariant polynomial Frobenius coordinates)**.** *Let $S = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $|\lambda_i| > 1$, preserve a regular integrable holomorphic distribution $\mathcal{D}$ near 0. There is a polynomial automorphism Q commuting with S, fixing 0, and tangent to the identity after an S-equivariant linear choice of coordinates, such that $Q_*\mathcal{D}$ is a constant coordinate distribution.*

Proof. Choose an S-invariant splitting $\mathbb{C}^N = V \oplus W$, with $V = \mathcal{D}_0$. The graph calculation in Lemma 9.1 shows that $\mathcal{D}$ is polynomial and is preserved by the algebraic diagonal group $T = \overline{\langle S \rangle}^{\mathrm{Zar}}$. Let $K \subset T$ be its maximal compact subgroup, including the finitely many components. It acts diagonally and unitarily, and its Zariski closure is T.

Holomorphic Frobenius supplies first integrals $p_j = x_j + O(|x|^2)$, one for every coordinate x_j of W. Take their common domain to be a sufficiently small K-invariant ball, and let $\chi_j : K \to \mathbb{C}^*$ be the character on x_j. Put

$$P_j(x) = \int_K \chi_j(k)^{-1} p_j(kx)\, dk.$$

Every integrand is constant along $\mathcal{D}$, so the same is true of P_j, and $dP_j(0) = dx_j$. Character orthogonality shows that a Taylor monomial x^α in P_j satisfies $\lambda^\alpha = \lambda_j$. There are finitely many such monomials because $|\lambda_i| > 1$. Thus P_j is a polynomial and $P_j \circ S = \lambda_j P_j$.

Set $Q(x) = (x_V, (P_j(x))_{j\in W})$. Its linear part is the identity. Every nonlinear monomial in its jth coordinate uses variables of strictly smaller absolute multiplier weight. Solving successively in these weights gives a polynomial inverse. Hence Q is a polynomial automorphism commuting with S. Its last coordinates are independent first integrals of $\mathcal{D}$, so it sends $\mathcal{D}$ to the constant distribution V. □

9.2. **Scalar jet equations exclude primitive infinite symmetry.**

Lemma 9.4 (The highest scalar jet detects a leading vector field)**.** *Let M be a complex manifold, let $p \in M$, and let $L \subset J^d(\mathcal{O}_M)$ be a holomorphic subbundle, where $d \geq 2$. Suppose that its projection onto $J^{d-1}(\mathcal{O}_M)$ is surjective. Write*

$$Q_p = L_p \cap \operatorname{Sym}^d T_p^* M.$$

Let V be an infinitesimal local symmetry of L whose coefficients vanish to order at least d at p, and denote by $B_d \in T_pM \otimes \operatorname{Sym}^d T_p^* M$ its degree-d term. Then

$$\ell \circ B_d \in Q_p \qquad (\ell \in T_p^* M).$$

The same assertion holds for formal symmetries and formal jet bundles.

Proof. Choose local coordinates centered at p. The local flow ϕ_t of V fixes p and has $(d-1)$-jet equal to the identity. For a jet $j_p^d u \in L_p$, invariance gives $j_p^d(u \circ \phi_t) \in L_p$. Differentiation at $t = 0$ gives $j_p^d(Vu) \in L_p$. Its terms of orders less than d vanish and its degree-d term is $du(p) \circ B_d$. Every covector occurs as $du(p)$, by the surjectivity onto the lower jets. This proves the assertion. The computation involves finitely many coefficients and is unchanged in the formal category. □

Proposition 9.5 (Exclusion of the contact pseudogroup)**.** *Let M have dimension $2m + 1$, where $m \geq 1$, and let $L \subset J^d(\mathcal{O}_M)$ be as in Lemma 9.4. If L is preserved by all local contact transformations for*

a contact structure on M, then $L = J^d(\mathcal{O}_M)$. The conclusion holds formally as well.

Proof. Use contact Darboux coordinates $(q_1, \dots, q_m, p_1, \dots, p_m, z)$ centered at the point in question, with contact form

$$\alpha = dz - \sum_{i=1}^{m} p_i \, dq_i.$$

For a polynomial H, its contact Hamiltonian vector field, with $\alpha(X_H) = H$, is

$$X_H = -\sum_i H_{p_i} \frac{\partial}{\partial q_i} + \sum_i (H_{q_i} + p_i H_z) \frac{\partial}{\partial p_i} + \left(H - \sum_i p_i H_{p_i} \right) \frac{\partial}{\partial z}.$$

Indeed, $\mathcal{L}_{X_H} \alpha = H_z \alpha$. Let P be an arbitrary homogeneous polynomial of degree d in all $2m+1$ variables. Choose a homogeneous polynomial H of degree $d+1$ with $H_{p_1} = -P$. Then X_H vanishes to order at least d, and the q_1-component of its degree-d term is P. Lemma 9.4, applied with $\ell = dq_1$, gives $P \in Q_p$. Thus $Q_p = \operatorname{Sym}^d T_p^* M$. Surjectivity onto the lower jets now gives $L_p = J_p^d(\mathcal{O}_M)$ at every point. The formal argument uses the same polynomial Hamiltonians. □

Proposition 9.6 (A primitive scalar-jet symmetry is of finite type)**.** *Let Γ be a regular transitive complex analytic Lie pseudogroup on a connected manifold M, generated locally by the flows of its infinitesimal transformations. Suppose that Γ preserves a proper holomorphic subbundle*

$$L \subsetneq J^d(\mathcal{O}_M), \qquad d \geq 2,$$

whose projection onto $J^{d-1}(\mathcal{O}_M)$ is surjective. If Γ is primitive, then it is of finite type. Consequently an infinite-type Γ as above preserves a regular foliation of rank strictly between 0 and $\dim M$.

Proof. The Cartan classification, in its complex analytic form proved by Guillemin–Quillen–Sternberg [GQS66], lists six primitive infinite-type pseudogroups: all local diffeomorphisms; those preserving a volume form; those preserving a volume form up to a constant; those preserving a symplectic form; those preserving a symplectic form up to a constant; and the contact pseudogroup.

Put $Q = L \cap \operatorname{Sym}^d T^* M$. The first five cases have linear isotropy containing $\mathrm{SL}(T_p M)$ or a standard symplectic group. Each positive symmetric power of the standard dual representation is irreducible. Hence their invariant subspace Q_p is either zero or all of $\operatorname{Sym}^d T_p^* M$. The second possibility contradicts the properness of L.

The first possibility is also excluded. In dimension at least two, a volume-preserving infinitesimal symmetry such as $x_2^d\partial/\partial x_1$ has a nonzero degree-d term. In the symplectic cases, a nonconstant homogeneous Hamiltonian of degree $d+1$ gives such a symmetry. For the full pseudogroup in dimension one use $x^d\partial/\partial x$. The leading-vector lemma gives a nonzero element of Q_p in each case. The one-dimensional volume-preserving and constant-volume-scaling pseudogroups are already of finite type.

The contact case is excluded by Proposition 9.5. Thus Γ cannot be primitive and of infinite type. The last assertion is the negation of primitivity for a transitive pseudogroup: an imprimitive pseudogroup preserves a nontrivial local foliation. □

9.3. **The regular symmetry pseudogroup of a rational jet equation.**

Proposition 9.7 (Return centres force regular transitive symmetry)**.** *Let $D \subset \mathbb{C}^N$ be a connected neighbourhood of 0, and let $L \subset J^d(\mathcal{O}_D)$ be a holomorphic subbundle which is the restriction of a rational subbundle on $\mathbb{A}^N$. Let $\mathcal{H}$ be local biholomorphisms preserving L, and suppose that their centres are a holomorphic uniqueness set in every sufficiently small neighbourhood of 0. Then the full local symmetry pseudogroup of L is regular and its infinitesimal transformations are locally transitive near 0. Its finite-order isotropy may be chosen to be an algebraic group.*

Proof. On a Zariski open set where L is an algebraic subbundle, its invariance equations define an algebraic subgroupoid of the groupoid of invertible d-jets. Their radical differential closure is an algebraic $\mathcal{D}$-groupoid; see [Cas06, Theorem 1.12]. Its analytic solutions are exactly the local symmetries of L: differentiating invariance adds only necessary equations, and the initial invariance equations remain among the defining equations.

We use the generic-base form of Malgrange's regularity theorem, stated in [Mal10, Chapter I, §§5.1–5.4]. On one nonempty Zariski open base D_*, all its finite-order truncations G_k are smooth strict algebraic groupoids, their truncation maps are smooth and surjective, and, from some order onward, the successive systems are their full prolongations. Prolonging to an involutive order gives regular analytic Lie equations. Their linearizations form the corresponding regular equations for vector fields [Mal10, Chapter I, §5.6]. After a further generic base restriction and prolongation, the linearized system is itself involutive and its projections to the retained lower jets are smooth and surjective. We use this involutive order to realize infinitesimal jets below.

Choose a return h with centre $a = h(0) \in D_*$. This is possible by the uniqueness assumption. Conjugate the small return germs by h. Their centres near a remain a holomorphic uniqueness set, and we may work in a ball contained in D_*. The order-zero groupoid $G_0 \subset D_* \times D_*$ is a closed algebraic equivalence relation. Its fibre at a contains this uniqueness set, hence equals D_*. The equivalence-relation identities give $G_0 = D_* \times D_*$. Smooth surjectivity of the truncation maps then says that the infinitesimal target map of the source fibre of G_ℓ has rank N at the identity over a.

Cartan–Kähler applied to the linearized involutive equations realizes these infinitesimal jets as local infinitesimal symmetries; compare [Cas06, Theorem 1.9] and the solution argument in [Mal10, Chapter II, proof of Theorem 2.2]. Thus the local Killing algebra evaluates surjectively onto T_aD. Pulling the regular local pseudogroup back by h proves regularity and infinitesimal transitivity near 0. Composition by a fixed solution preserves the completed equations and their regularity.

Finally, the isotropy $G_\ell(a, a)$ is an algebraic subgroup of the algebraic group of invertible ℓ-jets fixing a. Conjugation by the fixed jet of h identifies it with an algebraic finite-order isotropy at 0. □

Proposition 9.8 (The local alternative for an arbitrary first missing jet)**.** *Assume the hypotheses of Proposition 9.7, and suppose*

$$L \subsetneq J^d(\mathcal{O}_D), \qquad d \geq 2, \qquad L \longrightarrow J^{d-1}(\mathcal{O}_D) \text{ is surjective.}$$

Suppose the symmetries include an expanding germ F fixing 0. Then one of the following holds in Poincaré–Dulac coordinates.

(1) *There is a finite-dimensional transitive Lie algebra of polynomial vector fields normalized by every local symmetry of L.*
(2) *There is a finite nonempty collection of proper positive-rank integrable algebraic distributions, regular near 0, which is permuted by every local symmetry of L. An iterate of F preserves each member of this collection.*

Proof. Let Γ^0 be the regular local pseudogroup generated by the flows of infinitesimal symmetries. It is transitive by Proposition 9.7. If it is of finite type, its infinitesimal symmetry algebra $\mathfrak{g}$ is finite dimensional. Every local symmetry of L normalizes this algebra. Lemma 9.2 makes its vector fields polynomial, giving the first case.

Suppose Γ^0 is of infinite type. Proposition 9.6 gives a proper positive-rank regular foliation $\mathcal{F}$ preserved by Γ^0. A transitive invariant foliation is determined by its tangent space $V = T_0\mathcal{F}$: its tangent spaces elsewhere are transported by the infinitesimal symmetries. Let H be

the algebraic finite-order isotropy at 0 furnished by the previous proposition. Its identity component preserves V. Indeed its Lie algebra consists of isotropy jets of actual infinitesimal symmetries, by involutivity, and these preserve $\mathcal{F}$.

The group H/H^0 is finite. Therefore the tangent spaces obtained from V by actual isotropy germs form a finite collection. Their transported foliations form a finite collection $\mathcal{F}_1, \ldots, \mathcal{F}_s$ preserved by every local symmetry. To check finiteness for an arbitrary symmetry, first compose it with a local transformation in Γ^0 to bring its centre back to 0; the remaining transformation is an isotropy germ, so acts through the stated finite collection. For a centre outside this regular neighbourhood, first compose with a high inverse iterate of F. The new centre lies in the neighbourhood. Since F fixes 0 and acts through the same finite isotropy orbit, cancellation proves the assertion for the original return. Each foliation is integrable because it is the image of $\mathcal{F}$ by an actual local biholomorphism.

An iterate F^r fixes every member. Apply Lemma 9.1 to these foliations and to F^r. Each becomes an algebraic integrable distribution in the chosen normal-form coordinates. The finite permutation identities hold as identities of germs, and therefore hold wherever their algebraic distributions and the local symmetries are defined. This proves the second case. □

9.4. **The local return alternative in arbitrary dimension.**

Theorem 9.9 (Local return geometry without a symbol irreducibility assumption)**.** *Let f be a J-stable algebraic curve family of polarized endomorphisms of normal projective N-folds, with $N \geq 2$. Choose a good smooth Julia repeller as in Proposition 6.2. After restricting the parameter germ and making a finite parameter change, one of the following occurs.*

(1) *The marked inverse-return collection is holomorphically conjugate to its reference collection by a family of weighted polynomial automorphisms. These automorphisms conjugate the varying expanding normal form to its reference normal form.*
(2) *At the reference parameter, every marked inverse return normalizes one finite-dimensional transitive Lie algebra of polynomial vector fields on $\mathbb{C}^N$.*
(3) *At the reference parameter, every marked inverse return permutes a finite nonempty collection of proper positive-rank integrable algebraic distributions, regular on $\mathbb{C}^N$. An iterate of the normal form preserves each member. Each member admits*

polynomial Frobenius coordinates commuting with the semisimple part of that iterate.

Proof. Use the varying normal forms and the notation of Proposition 6.4. Let $M = 1 + \max\{2, \lfloor b/a \rfloor\}$ and fix $R \geq Mb$. Lemma 6.5 gives a nonzero irreducible module $E_s \subset W_R$ and a holomorphic simultaneous gauge. The space W_R consists only of polynomials vanishing at 0: it is the canonical zero-constant representative of the truncated substitution quotient by the constants. Thus E_s contains a nonconstant polynomial. We use $\mathcal{A}_s = \mathbb{C}1 \oplus E_s$ for scalar jet geometry and its truncated substitution action.

At the reference parameter let r be the generic rank of dE_0. Lemma 7.1 shows that this rank is attained at 0, that $r \geq 1$, and that the common kernel of dE_0 is preserved by all actual returns. If $r < N$, this is a proper positive-rank regular integrable distribution near 0. Its invariance under the normal form, Lemma 9.1, and Lemma 9.3 give the third alternative with one member.

Suppose $r = N$. If $\mathcal{A}_0$ separates M-jets, this remains true after shrinking the parameter germ. Proposition 7.6 applies. The nilpotent intertwiners also identify the actions of $C_{\mathcal{N}_s}$, so Proposition 7.7 gives the first alternative.

Otherwise choose the least $d \leq M$ for which $j_0^d \mathcal{A}_0$ is proper. Since $r = N$, one has $d \geq 2$. Proposition 7.2 and Corollary 7.4 give a rational holomorphic jet subbundle

$$L_x = j_x^d \mathcal{A}_0 \subsetneq J_x^d(\mathcal{O}), \qquad L_x \longrightarrow J_x^{d-1}(\mathcal{O}) \text{ surjective near } 0,$$

preserved by every actual return. The uniqueness of the small return centres is precisely the hypothesis of Proposition 9.7. Proposition 9.8 then gives either the second alternative or the finite web in the third. In the latter case apply Lemma 9.3 to each member after a common iterate. This proves the theorem. □

Corollary 9.10 (Global data supplied by the local alternative)**.** *In the first alternative of Theorem 9.9, the original family is isotrivial by Proposition 8.6. In its second alternative there is an equivariant rational flat Killing bundle with surjective evaluation onto TX, as constructed in Proposition 9.12. In its third alternative the source-descent and projective-cocycle arguments give a finite algebraic web of integrable rational distributions on X; a finite component cover selects one member.*

Proof. For the first assertion, use the polynomial conjugacy to make the normal form and all return identifications constant, and apply the horizontal part of the source-descent proposition. The second assertion

is exactly Proposition 9.12. For the third, let $\ell_1, \dots, \ell_t$ be the Plücker lines of the members, of common rank r. Their unordered product is a line in $\mathrm{Sym}^t(\bigwedge^r T\mathbb{C}^N)$. The natural action is the invertible rational linear cocycle $\mathrm{Sym}^t(\bigwedge^r Df)$, and unique factorization of a product of linear forms recovers the unordered set. The majority-continuation comparison and projective-cocycle rationality therefore apply to this datum. The resulting algebraic finite subset of the Grassmannian over $\mathbb{C}(X)$ splits over a finite field extension. Normalization in that extension gives the component cover, on which a selected distribution is rational and integrable. Equivariance of this cover requires taking the dynamically compatible component; it is the finite-cover issue handled separately in the relative construction. □

9.5. **The rational Killing connection.**

Lemma 9.11 (Continuation of natural jet subspaces)**.** *The majority continuation and rationality statements of Proposition 8.1 and Corollary 8.4 apply to a fixed-rank subbundle of $J^k(TX^{\mathrm{reg}})$, transported by the natural jet action of f on its unramified locus. The source-descent statement of Proposition 8.6 applies to these subbundles as well.*

Proof. On a rational coordinate chart the natural pushforward of a k-jet of a vector field is an invertible linear map whose coefficients are rational functions of the derivatives of f up to order $k+1$ and of $(Df)^{-1}$. Thus the Grassmannian-valued map defining a rank-r subspace has the same projective Plücker transport law as a distribution, with a rational invertible cocycle.

The branch comparison in the majority-continuation proof uses only this invertible transport and equality of germs. Across a finite discriminant, trace a local meromorphic Plücker generator with interpolation weights. The summands are generically collinear, so their nonzero trace defines the transported Plücker line. The Plücker equations persist, yielding a meromorphic subspace. Proposition 8.3 makes it rational. The comparison of two regular lifts in the source-descent proof is unchanged for this natural jet action. This proves all three assertions. □

Proposition 9.12 (Finite local symmetry produces a flat algebraic bundle)**.** *Let $\Pi : \mathbb{C}^N \to X$ and F be as in Proposition 8.6. Suppose a finite-dimensional Lie algebra $\mathfrak{g}$ of holomorphic vector fields near 0 acts locally transitively there, is normalized by the normal form $\mathcal{N}$, and is normalized by every actual inverse return. Then, on a nonempty Zariski open subset $X^\circ \subset X^{\mathrm{reg}}$, there are an algebraic vector bundle $\mathcal{K}$, a flat algebraic connection ∇, and a surjective algebraic morphism*

$$\varepsilon : \mathcal{K} \longrightarrow TX^\circ.$$

They have natural rational F-equivariance. The horizontal local sections of $\mathcal{K}$ evaluate to the local Killing fields. There is also a parallel algebraic Lie bracket on $\mathcal{K}$.

The pullback of this flat bundle through Π extends over $\mathbb{C}^N$ as a trivial flat bundle. In Poincaré–Dulac coordinates, its horizontal sections are the jets of polynomial vector fields.

Proof. Apply Lemma 9.2 to $\mathfrak{g}$. It consists of polynomial vector fields and is normalized by the semisimple part S of $\mathcal{N}$. Choose k so that $j_0^k : \mathfrak{g} \to J_0^k(T\mathbb{C}^N)$ is injective. Put

$$R_j(x) = \{j_x^j Z : Z \in \mathfrak{g}\} \subset J_x^j(T\mathbb{C}^N) \qquad (j = k, k+1).$$

The jet matrices of a polynomial basis of $\mathfrak{g}$ have full rank near 0. Because S normalizes $\mathfrak{g}$, their ranks at x and $S^{-n}x$ agree. Since $S^{-n}x \to 0$, the rank is $\dim \mathfrak{g}$ everywhere on $\mathbb{C}^N$. The same argument for evaluation shows that $\mathfrak{g}$ is transitive at every source point. Consequently R_k and R_{k+1} are algebraic subbundles, and truncation induces an isomorphism

$$R_{k+1} \xrightarrow{\sim} R_k.$$

All actual returns preserve both bundles, by the normalization hypothesis. Lemma 9.11 descends them to rational subbundles of the corresponding jet bundles on X. Shrink X° so that both are algebraic subbundles, the truncation is an isomorphism, and evaluation of R_k has rank N. Set $\mathcal{K} = R_k$ on this open set.

Let $\sigma : R_k \to R_{k+1}$ be the inverse of truncation. The Spencer operator defines

$$\nabla s = D(\sigma s).$$

This takes values in $\Omega^1 \otimes R_k$. Indeed, in a source chart choose a basis $Z_1, \dots, Z_m$ of $\mathfrak{g}$. For $s = \sum_i a_i j^k Z_i$ one has

$$D\left(\sum_i a_i j^{k+1} Z_i\right) = \sum_i da_i \otimes j^k Z_i.$$

It follows that ∇ is a connection and is flat. Its coefficients are algebraic on X°, because R_k, R_{k+1}, truncation, and the Spencer operator are algebraic. This formula also shows that the pullback connection extends over all of $\mathbb{C}^N$ with the global parallel basis $j^k Z_1, \dots, j^k Z_m$.

The zeroth-jet projection gives ε. The k-jet of $[Z, W]$ is a universal bilinear expression in the $(k+1)$-jets of Z and W. Composing this expression with $\sigma \otimes \sigma$ defines an algebraic bracket on $\mathcal{K}$. In the displayed parallel basis its coefficients are the constant structure constants of $\mathfrak{g}$, so it is parallel. Finally, all these constructions are natural under local

biholomorphisms. They therefore respect the equivariance of R_k and R_{k+1} under F. □

10. Regularity of invariant rational connections

Proposition 10.1 (Regularity of invariant rational connections)**.** *Let X be a normal complex projective variety, and let $f : X \to X$ be a q-polarized endomorphism, where $q > 1$. Let $(\mathcal{E}, \nabla)$ be a finite-rank integrable connection over the function field of X. If*

$$f^*(\mathcal{E}, \nabla) \simeq (\mathcal{E}, \nabla),$$

then $(\mathcal{E}, \nabla)$ is regular singular. In particular, an f-invariant rational flat connection on T_X is regular singular.

Proof. Write $n = \dim X$. At the generic point of each prime divisor D, normality gives a discrete valuation ring. Complete its fraction field, extend the residue coefficient field algebraically, and consider the formal connection in a transverse parameter. If its slopes are s_i with rank multiplicities r_i, set

$$\mathrm{irr}_D(\nabla) = \sum_i r_i s_i.$$

For an integrable connection this is intrinsic to the divisorial valuation. It is nonnegative, vanishes precisely when the connection is regular singular at that valuation, and is zero outside the finitely many divisorial poles of a rational connection matrix. Consequently

$$I(\nabla) = \sum_D \mathrm{irr}_D(\nabla) D$$

is an effective Weil divisor with finite support.

Let C be a prime divisor mapping to D, and let e_C be the ramification index. After extension of the residue coefficient field and a change of uniformizer, the extension of complete discrete valuation fields has the form $t = u^{e_C}$. In a Levelt–Turrittin decomposition, substitution of u^{e_C} for t multiplies every exponential slope by e_C and leaves its rank multiplicity unchanged. Thus

$$\mathrm{irr}_C(f^*\nabla) = e_C \,\mathrm{irr}_D(\nabla), \qquad I(f^*\nabla) = f^* I(\nabla).$$

The given isomorphism therefore implies $I = f^* I$.

Choose an ample Cartier divisor H such that $f^*H \sim qH$. The projection formula for a finite morphism and Weil cycles gives

$$\begin{aligned} q^{n-1}(f^*I \cdot H^{n-1}) &= (f^*I \cdot (f^*H)^{n-1}) \\ &= \deg(f)(I \cdot H^{n-1}) = q^n (I \cdot H^{n-1}). \end{aligned}$$

Since $f^*I = I$ and $q > 1$, we obtain $(I \cdot H^{n-1}) = 0$. Effectivity and ampleness yield $I = 0$.

Take a smooth dense open set on which the connection is nonsingular. Deligne's regularity criterion applies to the fixed normal compactification X: regular singularity at the generic points of its boundary prime divisors is equivalent to regular singularity on every algebraic curve; see [Del70, II, Proposition 4.4(iii)] and [And07, Theorems 1.1.1 and 1.3.1]. Hence the connection is regular singular, including at all divisorial valuations on any resolution. For a rational tangent connection the differential Df supplies the required isomorphism over the function field. □

11. Exceptional factor calculus

Definition 11.1 (Exceptional presentations used in the induction)**.** We use the exceptional systems of [JSX26, Definitions 1.2–1.4]. An *AT-variety* is a torsor $P \to A$ under an algebraic torus T, with A an abelian variety. A torsor is a variety with a group action which becomes the translation action on the group after étale localization on the base. A *Q-lift* $\Psi : P \to P$ covers a Q-polarized isogeny of A and satisfies $\Psi(tp) = [Q]_T(t)\Psi(p)$, where $[Q]_T$ is multiplication by Q on the torus. A q-polarized system (Z, g) is *exceptional* if, over an algebraic closure of its ground field, some iterate has a finite surjective presentation

$$\eta : P \longrightarrow Z^\circ, \qquad \eta\Psi = g^e\eta,$$

where $Z^\circ \subset Z$ is dense open and Ψ is a q^e-lift.

An abelian variety over $\overline{K}$ has *geometric* $\mathbf{C}$*-trace zero* if

$$\mathrm{Hom}_{\overline{K}}(A_0 \times_{\mathbf{C}} \overline{K}, A) = 0 \quad \text{for every abelian variety } A_0/\mathbf{C}.$$

A factor has a *trace-zero abelian presentation* if its presenting abelian variety has this property. A *maximal* connected factor of this type contains the function fields of all other such connected factors.

Proposition 11.2 (A polarized map and an invariant linear system)**.** *Let $F : V \to V$ be a d-polarized endomorphism of a normal projective variety, where $d > 1$. Suppose a rational map $\gamma : V \dashrightarrow Z$ satisfies $\gamma F = u\gamma$, where Z is projective and u is an automorphism preserving an ample line bundle on Z. Then γ is constant.*

Proof. The assertion is immediate if $\dim V = 0$. Otherwise replace Z by the normalization of the image of γ; the automorphism u lifts to this normalization. By [MZ18, Theorem 1.3],

$$(\deg F)^{\dim Z} = (\deg u)^{\dim V} = 1.$$

Since $\deg F > 1$, the image has dimension zero. $\square$

Proposition 11.3 (Recovering a factor for the original map)**.** *Let $f : X \to X$ be a q-polarized endomorphism of a normal projective variety. Suppose a positive iterate has a dominant rational factor*

$$\pi f^r = h\pi,$$

where $h : Y \to Y$ is exceptional and q^r-polarized. Then f itself has an exceptional rational factor whose function field contains $\mathbf{C}(Y)$. In particular a positive-dimensional exceptional factor for an iterate gives one for the original map.

Proof. On Y^r consider the finite cyclic map

$$\Phi(y_0, \dots, y_{r-1}) = (y_1, \dots, y_{r-1}, h(y_0)).$$

Choose an ample line bundle L with $h^*L \simeq L^{q^r}$ and set

$$\mathcal{L} = \bigotimes_{i=0}^{r-1} p_i^* L^{q^{r-1-i}}.$$

Then $\Phi^*\mathcal{L} \simeq \mathcal{L}^q$. The normalization Z of the closure of the image of

$$\Pi = (\pi, \pi f, \dots, \pi f^{r-1})$$

is preserved by Φ and carries the induced finite q-polarized map ϕ, with $\Pi f = \phi\Pi$. Moreover $\Phi^r = h \times \cdots \times h$. Take a common AT presentation of this product and a periodic dominant component over the image of Z. Proposition 11.4 makes its normalization an AT variety with scalar return lift. It gives an exceptional presentation for an iterate of ϕ^r, hence for ϕ. The first projection $Z \to Y$ is dominant, proving the containment of factor fields. $\square$

Proposition 11.4 (Periodic subvarieties of an AT lift)**.** *Let $P \to A$ be a complex AT variety with torus T, and let $\psi : P \to P$ be a d-lift. If a closed irreducible subvariety $Q \subset P$ satisfies $\psi(Q) = Q$, then its normalization Q^ν is an AT variety. The induced endomorphism of Q^ν is a d-lift after choosing the origin of its abelian base. The same conclusion applies to a periodic subvariety and its corresponding return.*

Proof. Write $T \simeq \mathbf{G}_m^r$ and $P = L_1^\times \times_A \cdots \times_A L_r^\times$. If $\alpha : A \to A$ is the isogeny covered by ψ, its equivariance with $[d]_T$ gives $\alpha^* L_i \simeq L_i^{\otimes d}$. Consequently ψ extends by the pure d-th powers on the homogeneous fiber coordinates to a finite endomorphism of

$$\overline{P} = \mathbf{P}(\mathcal{O}_A \oplus L_1) \times_A \cdots \times_A \mathbf{P}(\mathcal{O}_A \oplus L_r).$$

Each relative $\mathcal{O}(1)$ pulls back to its d-th power. If H_A polarizes α, the tensor product of these relative bundles with $\pi^* H_A^{\otimes N}$ is ample for

$N \gg 0$ and is d-polarized. Let $\overline{Q}$ be the normalization of the closure of Q in $\overline{P}$. It carries a finite d-polarized lift F, and the projection $\overline{Q} \to A$ is a morphism.

Let $C \subset A$ be the image of $\overline{Q}$. It is invariant under α. Every positive-dimensional invariant abelian subquotient carries a d-polarized induced map, by restriction and [MZ18, Theorem 1.3], so this map cannot have finite order. Thus (A, α) is strictly mixed in the sense of [PR04, Definition 2.1]. By [PR04, Theorem 3.4], C is a translate of an abelian subvariety. In particular it is smooth, and $\alpha_C : C \to C$ is finite étale and d-polarized.

Take the Stein factorization

$$\overline{Q} \longrightarrow C' \xrightarrow{\rho} C.$$

Its base C' is normal and projective, and ρ is finite. The map F induces a finite endomorphism α' of C' with $\rho\alpha' = \alpha_C\rho$. Pullback of the restricted polarization on C directly polarizes α' with weight d. Apply Ji–Song–Xie [JSX26, Lemma 3.11] with the whole projective variety C' as its invariant open, and with the finite map ρ to the smooth variety C. It follows that ρ is étale. Thus C' is an abelian variety after choosing an origin. The polarized endomorphism α' can be made a polarized isogeny by choosing a fixed point as origin.

Put $k = \dim Q - \dim C$. The vertical tangent bundle of P/A is canonically $\mathcal{O}_P \otimes \operatorname{Lie} T$. On the smooth locus where $Q \to C$ has maximal rank, its vertical tangent spaces define a rational logarithmic Gauss map

$$\Gamma : Q \dashrightarrow \operatorname{Gr}(k, \operatorname{Lie} T).$$

The vertical differential of ψ is $d\,\mathrm{id}$. Therefore $\Gamma\psi = \Gamma$, and the first proposition, applied to $\overline{Q}$, makes Γ constant. Call its value W.

A subvariety of a torus with constant logarithmic tangent space W is a coset of a subtorus. Indeed, the invariant vector fields in W are tangent to the subvariety, so W lies in the Lie algebra of its algebraic translation stabilizer. Freeness of the torus action bounds the stabilizer dimension by the dimension of the subvariety; the dimensions are equal, and its connected orbit is the whole irreducible subvariety. Applied to the geometric generic fiber over C', this gives a fixed subtorus $S \subset T$ of dimension k, with $\operatorname{Lie} S = W$. The generic fiber is geometrically integral because C' is the Stein base of the normal projective variety $\overline{Q}$. It is thus a single S-coset.

Pull $P|_C$ back to C' and quotient its torus by S. This gives a (T/S)-torsor $R \to C'$ with a lift θ over α' acting by $[d]_{T/S}$. The generic S-coset defines a rational section σ of R, and invariance of Q gives $\theta\sigma = \sigma\alpha'$. Choose a basis of characters of T/S. In the corresponding description

$R = M_1^\times \times_{C'} \cdots \times_{C'} M_t^\times$, the rational section gives nonzero rational sections s_j of the M_j. Put $D_j = \mathrm{div}(s_j)$. The equivariance gives

$$(\alpha')^* D_j = dD_j.$$

Since α' is finite étale, pullback preserves the maximum of the absolute values of the nonzero coefficients of a Weil divisor. Multiplication by $d > 1$ strictly increases that maximum. Hence $D_j = 0$ for every j. The sections s_j therefore extend as nowhere-vanishing regular sections on the smooth variety C'. Thus σ is a global section.

The inverse image of $\sigma(C')$ in $P \times_C C'$ is a smooth S-torsor $Q_S \to C'$. The normalization Q^ν maps finitely to $P \times_C C'$: its map to P is finite, and $P \times_C C' \to P|_C$ is finite. Its image is closed and has generic fiber equal to the prescribed S-coset, so that image is Q_S. The resulting finite birational map $Q^\nu \to Q_S$ is an isomorphism, since Q_S is normal. The induced endomorphism acts by $[d]_S$ on the torus and covers the d-polarized isogeny α'. It is the required d-lift. □

Lemma 11.5 (The exceptional locus of an equivariant birational morphism)**.** *Let $p : V \to W$ be a proper birational morphism between normal projective varieties, and let $F : V \to V$ and $f : W \to W$ be finite surjective endomorphisms satisfying $pF = fp$. Let $E \subset V$ be the non-quasi-finite locus of p, and put $Z = p(E)$. Then*

$$E = p^{-1}(Z), \qquad F^{-1}(E) = E, \qquad f^{-1}(Z) = Z.$$

Suppose the endomorphisms are polarized. If either V or W has a completely invariant AT open on which its endomorphism is a scalar lift, p is an isomorphism on that open onto an open of the other variety. This other open is also completely invariant.

Proof. A proper birational morphism to a normal variety has connected fibers. Every positive-dimensional fiber therefore has positive local dimension at each of its points: a zero-dimensional irreducible component would be an isolated component. This proves $E = p^{-1}(Z)$; properness also makes Z closed.

If $x \in E$, a positive-dimensional fiber subvariety through x has a positive-dimensional image under the finite map F, contained in a fiber of p. Thus $F(x) \in E$. Conversely, suppose $F(x) \in E$. Choose a positive-dimensional fiber subvariety through $F(x)$. The going-down theorem for an integral extension over a normal domain supplies an irreducible component of its inverse image through x which dominates that subvariety. This component has positive dimension. Its image under p is contained in the finite set $f^{-1}(p(F(x)))$, hence is the point

$p(x)$. Thus $x \in E$. The equality for Z follows using $E = p^{-1}(Z)$ and the surjectivity of p.

Suppose first that V contains the stated AT open P. The closure of $E \cap P$, if nonempty, is a proper completely invariant closed subset on whose generic points F is étale. Closure commutes with inverse image here because finite dominant maps to normal varieties are open, again by going-down. Ji–Song–Xie [JSX26, Lemma 3.10] rules out this closed subset. Hence $E \cap P = \varnothing$. Connected fibers and Zariski's main theorem now show that p identifies P with an open $U \subset W$, with $p^{-1}(U) = P$. The identity $pF = fp$ and complete invariance of P give $f^{-1}(U) = U$. If instead the AT open P lies in W, apply the same argument to the closure of $Z \cap P$. Then p is an isomorphism over P, and its inverse image is completely invariant. □

Proposition 11.6 (Birational invariance of exceptionality)**.** *Two birationally conjugate polarized endomorphisms of normal projective varieties are simultaneously exceptional. More generally, if a generically finite dominant rational map semiconjugates two polarized endomorphisms of the same weight, either is exceptional if and only if the other is exceptional.*

Proof. First consider a birational conjugacy between (X, f) and (Y, g), and suppose g is exceptional. The normalization G of its graph has a finite lifted endomorphism, polarized by the sum of the pullbacks of polarizations of X and Y. Take an iterate admitting an AT presentation $P \to U \subset Y$, and normalize Y in its function field. This gives a finite normal projective cover $\overline{P} \to Y$ with a polarized lift and completely invariant AT open P.

Normalize the dominant component of $G \times_Y \overline{P}$, calling it G'. It is finite over G and birational over $\overline{P}$; the lifted dynamics is finite and polarized. By Lemma 11.5, the latter birational morphism is an isomorphism over P. Thus G' contains a completely invariant AT open with scalar dynamics.

The morphism $G' \to X$ is proper and generically finite. Its Stein factorization is

$$G' \longrightarrow X' \longrightarrow X,$$

where the first map is birational and the second finite, with X' normal. The induced map on X' is finite and polarized by the pullback of a polarization of X. Apply the same lemma to $G' \to X'$, this time using its source AT open. The resulting AT open on X' is completely invariant, so X' is exceptional. Ji–Song–Xie [JSX26, Theorem 3.9] applied to $X' \to X$ makes X exceptional. Interchanging X and Y gives the converse.

For a generically finite rational semiconjugacy, take its normalized graph G and the Stein factorization of $G \to Y$. The projection $G \to X$ and the first Stein map are birational, while the second Stein map is finite. The sum polarization polarizes the graph, and the finite Stein target is polarized by pullback from Y. The birational statement just proved and the finite statement of [JSX26, Theorem 3.9] give the equivalence. □

Proposition 11.7 (Galois AT presentations)**.** *Let $f : X \to X$ be a polarized endomorphism of a normal complex projective variety. Suppose an iterate f^ℓ has an exceptional presentation*

$$h : P \longrightarrow U \subset X, \qquad h\psi = f^\ell h,$$

where h is finite surjective, P is an AT variety, and ψ is a q-lift. Then f^ℓ has such a presentation which is also Galois. The new AT lift is again a q-lift. Its deck group G is compatible with the lift: there is an automorphism $\delta : G \to G$ such that $\psi\gamma = \delta(\gamma)\psi$.

Proof. Replace f by f^ℓ. We first note that $f^{-1}(U) = U$. Indeed, $f(U) \subseteq U$, and finiteness of $h\psi = (f|_U)h$ descends along the finite surjective map h, so $f|_U : U \to U$ is finite. The inclusion $U \hookrightarrow f^{-1}(U)$ is therefore a finite open immersion. The latter open is irreducible, so the inclusion is an equality.

Put $K = \mathbf{C}(X)$ and $L = \mathbf{C}(P)$. Normalize X in L, obtaining a finite map $p : W \to X$ with $p^{-1}(U) = P$. The field embedding $\sigma = \psi^* : L \hookrightarrow L$ extends the embedding f^* of K. By the universal property of normalization it gives a finite endomorphism Ψ of W with $p\Psi = fp$. Pullback of a polarization of f polarizes Ψ.

The decreasing fields $K\sigma^j(L)$ stabilize, since L/K is finite. Write their stable value as $L_0 = K\sigma^s(L)$. Then

$$\sigma(L_0) \subseteq L_0, \qquad K\sigma(L_0) = L_0.$$

Let $p_0 : Y \to X$ be the normalization in L_0, and let F_0 be its polarized lift. The embedding $\sigma^s : L \hookrightarrow L_0$ induces a finite morphism $j : Y \to W$ satisfying

$$pj = f^s p_0, \qquad jF_0 = \Psi j.$$

Regularity follows from integrality and normality; finiteness follows because pj is finite. Complete invariance of U gives

$$Y^\circ := p_0^{-1}(U) = j^{-1}(P).$$

Apply Ji–Song–Xie [JSX26, Lemma 3.11] to the normal projective variety Y, its polarized map F_0, its completely invariant open Y°, the finite surjection $j : Y^\circ \to P$, and the finite étale endomorphism ψ of

the smooth variety P. The lemma makes j and $F_0|_{Y^\circ}$ étale. Their Proposition 3.7 gives Y° an AT structure for which j is an AT-isogeny. Lemma 3.2 and finiteness make the self-map $F_0|_{Y^\circ}$ an AT-isogeny, and Lemma 3.8 makes it a q-lift after changing the origin of its abelian base.

Let M/K be the normal closure of L_0/K in an algebraic closure. We verify that σ extends to an embedding $M \hookrightarrow M$. Choose a primitive element α of L_0/K, with minimal polynomial $A(T)$. The equality $K\sigma(L_0) = L_0$ gives $K(\sigma\alpha) = L_0$. Thus $\sigma(A)$ is still irreducible over K and its splitting field is the same normal closure M. Extend σ to an algebraic closure; images of all roots of A are the roots of $\sigma(A)$. Consequently

$$\sigma(M) \subseteq M, \qquad K\sigma(M) = M.$$

Normalize X in M, obtaining $p_Z : Z \to X$ and its polarized lift F_Z. The map $Z \to Y$ is finite and equivariant. On $Z^\circ = p_Z^{-1}(U)$, a second application of Lemma 3.11, followed by Proposition 3.7 and Lemma 3.8, shows that Z° is AT and that $F_Z|_{Z^\circ}$ is a q-lift. The map $Z^\circ \to U$ is finite Galois, as required.

For the final assertion, write $G = \mathrm{Gal}(M/K)$. Since $K\sigma(M) = M$ and $[\sigma(M) : \sigma(K)] = [M : K]$, the fields K and $\sigma(M)$ are linearly disjoint over $\sigma(K)$. Hence conjugation by σ on $\sigma(M)$ extends uniquely to an automorphism of the group G over the compositum M. This is the required compatibility $\psi\gamma = \delta(\gamma)\psi$. □

Proposition 11.8 (Removing the translation kernel)**.** *Let $P \to A$ be a torsor under a split torus T of rank r, where A is an abelian variety of dimension a. Let a finite group G act faithfully on P. Put $n = a + r$ and*

$$M_n = |\,\mathrm{GL}_{2n}(\mathbf{F}_3)|.$$

There is a normal subgroup $N \triangleleft G$ such that P/N is an AT variety and

$$[G : N] \le M_n.$$

Consequently the quotient P/G admits a finite Galois AT cover of degree at most M_n.

Suppose moreover that $\psi : P \to P$ is a q-lift and that, for each $\gamma \in G$, there is $\delta(\gamma) \in G$ with $\psi\gamma = \delta(\gamma)\psi$. Then ψ descends to a q-lift on P/N, and the preceding finite cover is equivariant.

Proof. Every morphism from a torus to an abelian variety is constant. Thus every automorphism of P descends, by faithfully flat descent, to an automorphism of A. On every torus fiber it is a translation followed by a monomial automorphism. The exponent matrix is locally

constant, hence constant on the connected base. We obtain a homomorphism

$$\lambda : G \longrightarrow \mathrm{GL}(H_1(A, \mathbf{Z})) \times \mathrm{GL}(X^*(T)) \subseteq \mathrm{GL}_{2a+r}(\mathbf{Z}).$$

Let $N = \ker \lambda$. Every element of N induces a translation on A and is T-equivariant on P. Let $H \subset A$ be the finite translation subgroup obtained in this way, and let C be the kernel of $N \to H$. An equivariant automorphism of a T-torsor over the identity of A is a morphism $A \to T$. Such a morphism is constant, so C is a finite subgroup of T. In particular it is central in N.

The quotient $P/C \to A$ is a torsor under the torus T/C. The group $N/C = H$ acts on it equivariantly and covers the free translation action of H on A. Descent along the finite étale cover $A \to A/H$ shows that

$$(P/C)/(N/C) \longrightarrow A/H$$

is a T/C-torsor. Its total space is P/N. This also explains why a noncommutative central extension inside N causes no obstruction: its commutators disappear in the first quotient by C.

The finite group G/N embeds in $\mathrm{GL}_{2a+r}(\mathbf{Z})$. Reduction modulo three is injective on finite subgroups of this group, since the principal congruence subgroup of level three is torsion-free. Padding by identity blocks gives $|G/N| \leq M_n$.

Write α for the isogeny of A covered by ψ. If $\gamma \in N$ covers translation by h, then $\delta(\gamma)$ covers translation by $\alpha(h)$: this follows from $\alpha\gamma_A = \delta(\gamma)_A\alpha$ and surjectivity of α. If M_γ is the torus exponent matrix, the same equivariance identity gives

$$qM_\gamma = M_{\delta(\gamma)}q.$$

Thus $\delta(N) \subset N$, and ψ descends to P/N. The descended map acts by $[q]$ on T/C and covers the induced isogeny of A/H. This isogeny is q-polarized, since polarization descends through an isogeny. Hence the descended map is a q-lift. □

Proposition 11.9 (Bounded branch divisor of a dynamical cover)**.** *Let X be a normal projective variety of dimension m, let H be ample, and let $f : X \to X$ satisfy $f^*H \simeq H^{\otimes q}$, where $q > 1$. Let $p : W \to X$ be a finite Galois morphism from a normal variety, and suppose that a finite endomorphism $F : W \to W$ satisfies $pF = fp$. For a prime divisor $D \subset X$, denote by e_D its inertia index and set*

$$\Delta_p = \sum_D (1 - e_D^{-1})D.$$

Then

$$f^*\Delta_p - \Delta_p \le R_f, \qquad 0 \le \Delta_p \cdot H^{m-1} \le -K_X \cdot H^{m-1}.$$

In particular the reduced branch divisor satisfies

$$\deg_H \mathrm{Br}(p)_{\mathrm{red}} \le -2K_X \cdot H^{m-1}.$$

These bounds are unchanged if f is replaced by any positive iterate. If $\deg p \le M$, all coefficients of Δ_p belong to the fixed finite set $\{1-1/e : 1 \le e \le M\}$.

Proof. All pullbacks of Weil divisors in this proof are finite-map pullbacks, defined by ramification indices of discrete valuations. They are identities of Weil divisors. Since p is Galois, those indices are constant above each prime divisor of X, and the codimension-one ramification formula gives $R_p = p^*\Delta_p$. Since the characteristic is zero, R_F is effective. The ramification identity for the equal compositions $pF = fp$ gives

$$R_F + F^*R_p = R_p + p^*R_f, \qquad R_F = p^*(R_f + \Delta_p - f^*\Delta_p).$$

Pullback by a finite surjective morphism detects effectiveness of Weil divisors. This proves the first inequality. Intersections of a Weil divisor with H^{m-1} are cycle intersections. The identities $f_*f^*D = (\deg f)D$ and $f^*H = qH$, followed by the projection formula, give $\deg_H(f^*D) = q \deg_H D$ for every Weil divisor D. The codimension-one canonical formula $K_X \sim f^*K_X + R_f$ consequently gives

$$(q-1)\Delta_p \cdot H^{m-1} \le R_f \cdot H^{m-1} = (1-q)K_X \cdot H^{m-1}.$$

Every nonzero coefficient of Δ_p is at least $1/2$, proving the reduced-branch bound. The same computation with q^s proves the assertion about iterates. □

12. Compatible covers along periodic fiber chains

Definition 12.1 (Compatible covers and periodic returns)**.** A *finite dynamical cover* of (X, f) is a finite surjective map $p : Y \to X$ from a connected normal variety, together with $F : Y \to Y$ satisfying $pF = f^e p$ for some $e \ge 1$. It is *Galois* when $k(Y)/k(X)$ is Galois. It is *Cartesian* when the normalized full fiber product $X \times_{f^e,X,p} Y$ is connected and the natural map from Y identifies Y with that normalization. The same terminology is used for the commuting squares in a finite chain.

For an equivariant fibration, a *marked periodic return* is the induced map f^ℓ on the fibers over a local periodic section $b(s)$ satisfying $g^\ell b(s) = b(s)$. It is *good* when these normalized fibers form a flat normal geometrically integral family on a parameter open, the fiber arrows

are finite and polarized, and the required rational diagrams are defined generically. In spectral arguments its marked reference point on the base is a smooth repelling Julia point.

The *birational Albanese* of a normal projective variety is the Albanese variety of a smooth projective resolution, with its induced rational map from the original variety. This is independent of the resolution.

Lemma 12.2 (Stable fields and normalized pullbacks)**.** *Let $K \subset L$ be a finite separable extension of degree D, and let $\sigma : L \hookrightarrow L$ be a field embedding preserving K. Suppose*

$$K\sigma(L) = L.$$

Then the natural homomorphism

$$K \otimes_{\sigma(K)} \sigma(L) \longrightarrow L$$

is an isomorphism. Consequently, for a finite cover of normal varieties with a compatible dynamical lift, this field condition means that the normalization of the full dynamical pullback is connected and is the original cover.

Suppose, moreover, that a return arrow is a composition of finite dominant arrows around a cycle. If its normalized pullback of a degree-D cover is connected, then all intermediate normalized pullbacks are connected and have degree D.

Proof. The tensor product has dimension D over K, since $[\sigma(L) : \sigma(K)] = D$. Its image in L is $K\sigma(L) = L$, which has the same dimension D. The homomorphism is therefore an isomorphism. In characteristic zero the generic algebra of any intermediate pullback is a finite product of separable field extensions. If such a product had a nontrivial idempotent, that idempotent would remain nontrivial under each subsequent faithfully flat field base change. The full pullback could not be a field. Thus each intermediate algebra is a field, and base change preserves its total rank D. □

Lemma 12.3 (A fixed path bounds the location of branching)**.** *Let $a_i : X_i \to X_{i+1}$, $0 \le i < N$, be finite maps between normal projective m-folds, with ample Cartier polarizations of equal volume and $a_i^* H_{i+1} \simeq H_i^q$. Let $p_i : W_i \to X_i$ be connected finite covers such that every square is a connected normalized pullback square. Suppose that their reduced branch divisors satisfy*

$$B_i \cdot H_i^{m-1} \le C.$$

If $q^N > C$, then every prime component of B_N is a divisorial critical value of

$$A = a_{N-1} \circ \cdots \circ a_0.$$

Proof. Suppose that a prime component D of B_N is not a critical value of A. The map A has ramification index one at every prime above D. At those codimension-one discrete valuation rings it is unramified, since the characteristic is zero. Unramified base change preserves the nontrivial ramification index of p_N over D. Since the successive pullbacks are connected, their composition is the full normalized pullback; therefore every prime in $A^{-1}(D)$ occurs in B_0. All coefficients of A^*D are one, so

$$C \geq B_0 \cdot H_0^{m-1} \geq A^*D \cdot H_0^{m-1} = q^N D \cdot H_N^{m-1} \geq q^N,$$

a contradiction. The final inequality uses that the degree of a nonzero effective integral divisor against an ample Cartier class is a positive integer. □

Lemma 12.4 (Finite parameter space for covers with prescribed branch support)**.** *Let $\pi : \mathcal{X} \to B$ be a projective family over a smooth integral complex variety, with normal geometrically integral generic fiber, and let $V \subset \mathcal{X}$ be a closed subset whose generic fiber is a divisor. Fix $D \geq 1$. After replacing B by a dense open, the geometric isomorphism classes of connected normal covers*

$$W \longrightarrow X_b$$

of degree at most D, unramified away from $V_b \cup (X_b)_{\mathrm{sing}}$, are the geometric points of a finite étale algebraic space $H \to B$. The locus of Galois covers is open and closed. The covers themselves form a locally constant Deligne–Mumford stack over B with finite geometric fiber groupoids and finite étale coarse space H.

Proof. Resolve the generic pair and spread the resolution. After shrinking B, this gives a smooth projective family $\widetilde{\mathcal{X}} \to B$ and a relative simple normal crossings divisor E, containing the inverse image of V and all exceptional divisors, such that

$$\widetilde{\mathcal{X}} \setminus E \simeq \mathcal{X}_{\mathrm{reg}/B} \setminus V.$$

Set $e = \mathrm{lcm}(1, \ldots, D)$. Form the eth root stack along every component of E. This is a proper smooth Deligne–Mumford stack $\mathscr{X} \to B$. The root-stack form of Abhyankar's lemma [LO10, Appendix A] identifies degree-at-most-D covers of the displayed open with finite étale covers of $\mathscr{X}$ of the same degree: all inertia orders are orders of permutations of at most D letters and therefore divide e.

For each $d \leq D$, the stack of degree-d finite étale covers is the stack $\mathrm{Hom}_B(\mathscr{X}, \mathrm{B}\mathfrak{S}_d)$. It is algebraic by the Hom-stack theorem [Ols06]. Finite étale covers have unique infinitesimal liftings, and proper smooth

invariance of the geometric fundamental group [LO10, Theorem A.11] makes this a locally constant stack. Its geometric fiber is a finite groupoid: the fundamental group of a proper smooth complex Deligne–Mumford stack is topologically finitely generated, so it has only finitely many actions on a set of d letters. Its coarse space is consequently finite étale over B. Connectedness and the Galois property are locally constant. Finally, normalization in the corresponding function field extends a cover uniquely from the displayed open to X_b. Purity on $(X_b)_{\mathrm{reg}}$ gives the inverse construction.

□

Lemma 12.5 (A finite class correspondence has a periodic branch)**.** *Let B be a normal integral projective variety and let $g : B \to B$ be finite and dominant. Let $\pi : \mathcal{X} \to B^\circ$ be a projective family with normal geometrically integral fibers over a dense open $B^\circ \subset B$. Set $U = B^\circ \cap g^{-1}(B^\circ)$, and suppose there is a finite morphism over U*

$$a : \mathcal{X}|_U \longrightarrow (g|_U)^*\mathcal{X}$$

whose fiber arrows $a_b : \mathcal{X}_b \to \mathcal{X}_{g(b)}$ are dominant. Let $H \to B^\circ$ be a finite-type quasi-finite algebraic space parametrizing geometric isomorphism classes of connected finite covers of these fibers by normal varieties. Suppose a constructible relation

$$\mathcal{R} \subset H \times_{B,g} H$$

has the following pullback property. For every geometric pair $(h, h') \in \mathcal{R}$ over $b \in U$, the normalization of the full fiber product

$$\mathcal{X}_b \times_{\mathcal{X}_{g(b)}} W_{h'}$$

is connected and represents h, where $W_{h'} \to \mathcal{X}_{g(b)}$ represents h' and the fiber product uses a_b. In particular, a target class $h' \in H_{g(b)}$ determines at most one source class $h \in H_b$ with $(h, h') \in \mathcal{R}$. Suppose a set of selected vertices has Zariski-dense image in B, and selected edges in $\mathcal{R}$ have both endpoints among those vertices. Assume that the selected sources are dense in every dominant component of the closure of the selected vertices. This holds, in particular, for all the vertices and edges of a collection of finite cyclic paths with dense image in B.

Then there are a finite dominant map $p : B' \to B$ from a normal integral projective variety, an integer $s > 0$, and a finite dominant map $g' : B' \to B'$ such that

$$pg' = g^s p,$$

and a geometric cover class over the geometric generic fiber after base change to B' which is invariant under the corresponding s-step normalized pullback. The base square is itself a connected normalized pullback square; equivalently, for $K = \mathbf{C}(B)$ and $K' = \mathbf{C}(B')$ one has $K(g')^*K' = K'$. If g is q-polarized, g' is q^s-polarized.

Proof. Let Z be the reduced closure in H of the selected vertices and discard components whose image is not dense in B. Write its remaining irreducible components as $Z_1, \dots, Z_t$, and put

$$d_i = [\mathbf{C}(Z_i) : \mathbf{C}(B)].$$

Let R be the closure of the selected edges. For every i, some irreducible component R_α dominates Z_i by the source projection. Its generic point belongs to $\mathcal{R}$, because $\mathcal{R}$ is constructible and the selected edges are dense in R_α. Its target dominates some Z_j.

Forgetting the source class maps R_α generically injectively to $B \times_{g,B} Z_j$, by the uniqueness assumption. Thus its function field is that of one component of this base change, and

$$d_i\,[\mathbf{C}(R_\alpha) : \mathbf{C}(Z_i)] = [\mathbf{C}(R_\alpha) : \mathbf{C}(B)] \le d_j.$$

Choose a component of maximal d_i. Every dominating outgoing edge from it has $d_j = d_i$ and source degree one. Repeating the choice within the finite set of maximal-degree components eventually gives a directed cycle. Each edge of that cycle therefore defines a dominant rational map $Z_i \dashrightarrow Z_j$ over g; their composition gives a dominant rational self-map of one Z_i over g^s. Along each edge, the pullback condition on $\mathcal{R}$ identifies the source cover with the full connected normalized pullback of the target cover under the corresponding fiber arrow. Composing these identifications around the cycle gives invariance of the selected geometric cover class under the s-step normalized pullback. Composition is compatible with normalized pullback by associativity of the corresponding tensor products of function fields. Equality in the degree estimate also says that the selected component of g^*Z_j accounts for its entire generic degree. Thus each base pullback is connected, and so is their composition around the cycle. This is exactly the stated stable-field equality.

Let B' be the normalization of B in $\mathbf{C}(Z_i)$. The function-field embedding defining this rational self-map preserves the integral closure of every affine coordinate ring under the corresponding pullback by g^s. It consequently extends to a morphism $g' : B' \to B'$. Since $pg' = g^s p$ is finite, g' is proper and quasi-finite, hence finite. Finally, if $g^*L \simeq L^q$, then

$$(g')^*p^*L \simeq (p^*L)^{q^s},$$

and p^*L is ample. □

Lemma 12.6 (Conjugate composita descend a selected class)**.** *Let k be algebraically closed of characteristic zero, let $K \subset L$ be a regular extension of finitely generated fields over k, and let $\sigma : L \hookrightarrow L$ preserve K. Fix an algebraic closure Ω of L and an extension of σ to an automorphism of Ω, preserving the algebraic closure $\overline{K}$ of K in Ω. Suppose a finite extension K'/K in $\overline{K}$ satisfies*

$$\sigma(K') \subset K', \qquad K\sigma(K') = K'.$$

Let $E/\overline{K}L$ be finite Galois and satisfy $\sigma(E) \subset E$. Suppose its cover class is invariant under the group $\mathrm{Gal}(\overline{K}/K')$. Then the compositum of all conjugates of E under $\Gamma = \mathrm{Gal}(\Omega/L)$ is a finite Galois extension $E^{\mathrm{c}}/\overline{K}L$ satisfying

$$\gamma(E^{\mathrm{c}}) = E^{\mathrm{c}} \quad (\gamma \in \Gamma), \qquad \sigma(E^{\mathrm{c}}) \subset E^{\mathrm{c}}.$$

If a smooth projective model of E over $\overline{K}$ has positive irregularity, the same holds for E^{c}.

Proof. Let K'' be the normal closure of K'/K inside $\overline{K}$. Then $\sigma(K'') \subset K''$ and $K\sigma(K'') = K''$. Indeed, if a is a primitive element of K'/K with minimal polynomial P, then $K(\sigma a) = K'$ makes $\sigma(P)$ irreducible over K of the same degree. Its splitting field is both K'' and $K\sigma(K'')$; all its roots belong to K''.

Put $\Gamma' = \mathrm{Gal}(\Omega/LK'')$. This is an open normal subgroup of Γ. With $\theta(\gamma) = \sigma^{-1}\gamma\sigma$, the stable-field equality shows that θ induces an automorphism of the finite quotient Γ/Γ'. The class-invariance hypothesis and the Galois property of $E/\overline{K}L$ imply that Γ' normalizes the subfield E: an isomorphism between two normal extensions inside Ω forces equality of the fields. Thus there are finitely many Γ-conjugates of E, and their compositum is finite Galois over $\overline{K}L$ and is normalized by Γ.

For any $\gamma \in \Gamma$, choose $\gamma' \in \Gamma$ such that $\theta(\gamma')\Gamma' = \gamma\Gamma'$. Since Γ' normalizes E, this gives $\gamma E = \theta(\gamma')E$, and hence

$$\sigma(\gamma E) = \gamma'\sigma(E) \subset \gamma' E.$$

Taking the compositum proves $\sigma(E^{\mathrm{c}}) \subset E^{\mathrm{c}}$. Finally, the finite dominant rational map from a smooth projective model of E^{c} to one of E makes pullback on holomorphic one-forms injective, proving the last assertion. □

12.1. **Descent of an irregular geometric covering class.** We work over an algebraically closed field k of characteristic zero. For a normal projective variety V, write $\mathrm{Alb}_{\mathrm{bir}}(V)$ for the Albanese variety of a

smooth projective model. Its torsor and its first rational ℓ-adic cohomology are birational invariants. For a polarized abelian variety (A, λ), the notation $\mathrm{Aut}_{\mathrm{gp}}(A, \lambda)$ means the group of polarization-preserving group automorphisms; every member fixes the origin. Affine automorphisms of an Albanese torsor will always be distinguished from this group.

Lemma 12.7 (Constancy of self-isogenous abelian moduli)**.** *Let $g\colon B \to B$ be a q-polarized endomorphism of a normal projective variety, with $q > 1$, and put $K = k(B)$ and $\sigma = g^*$. Let*

$$\nu\colon B \dashrightarrow \mathcal{A}_{a,\delta}$$

be a coarse moduli map for polarized abelian varieties of fixed dimension and polarization type. Suppose that the geometric abelian varieties represented by ν and $\nu \circ g$ are isogenous over $\overline{K}$. Then ν is constant.

Proof. The assertion is immediate when $\dim B = 0$, so assume $\dim B > 0$. The minimal compactification of $\mathcal{A}_{a,\delta}$ carries an ample rational Hodge line bundle. Take an ample integral power $\mathcal{L}$. If ν is nonconstant, a general section of a high power of $\mathcal{L}$ pulls back to a nonzero effective Weil divisor D on B. Here the pullback is first taken outside the indeterminacy locus, which has codimension at least two, and then closed in B.

Choose an ample Cartier divisor H with $g^*H \sim qH$, and put $b = \dim B$. A general complete-intersection curve in a high multiple of H avoids the indeterminacy loci of both moduli maps. After a finite cover of this curve, the two abelian families exist, their generic isogeny exists, and both have semistable reduction. Their extended Hodge bundles have the same degree. Indeed, an isogeny and a quasi-inverse extend to the Néron models; the composites on invariant differentials are multiplication by a nonzero integer, which is a unit in characteristic zero. Thus these differential bundles are isomorphic. Dividing by the degree of the finite cover gives

$$(g^*D \cdot H^{b-1}) = (D \cdot H^{b-1}).$$

The finite projection formula and $g^*H \sim qH$ give, on the other hand,

$$(g^*D \cdot H^{b-1}) = q(D \cdot H^{b-1}).$$

Hence $D \cdot H^{b-1} = 0$, contrary to ampleness and $D \neq 0$. □

Theorem 12.8 (An irregular Galois covering class gives an independent factor)**.** *Let X and B be normal projective varieties, let $f\colon X \to X$ and $g\colon B \to B$ be polarized endomorphisms, and let $\pi\colon X \dashrightarrow B$ be an*

equivariant dominant rational map with geometrically integral generic fiber Y. Put

$$K = k(B), \qquad L = k(X) = K(Y), \qquad \sigma = f^*\colon L \hookrightarrow L.$$

Fix an algebraic closure Ω of L, and let $\overline{K}$ be the algebraic closure of K in Ω. Suppose there is a finite Galois extension

$$E/\overline{K}L$$

with the following properties.

(1) *The associated connected geometric cover $W \to Y_{\overline{K}}$ has an isomorphism class invariant under $\mathrm{Gal}(\overline{K}/K)$.*
(2) *There is an automorphism $\widetilde{\sigma}$ of Ω extending σ such that*

$$\widetilde{\sigma}(E) \subset E.$$

(3) $\dim \mathrm{Alb}_{\mathrm{bir}}(W) > 0$.

Then there are an abelian variety A over k, a finite group H acting affinely on A, a polarized endomorphism h of A/H, and an equivariant dominant rational map

$$\psi\colon X \dashrightarrow A/H$$

whose restriction to the geometric generic fiber of π is nonconstant. In particular, ψ is an exceptional factor independent of the given phase base.

Proof. Write $f^*H_X \sim qH_X$, with H_X ample and $q > 1$. If B has positive dimension, polarization descent and the degree formula of Meng–Zhang, Theorem 1.3, give the same polarization weight q on B. Replacing X by the normalization of the graph of π makes π a morphism. This replacement admits the induced finite endomorphism and a polarization obtained from polarizations on X and B; the function fields in the statement do not change. Thus the generic fiber and its normalizations in finite extensions may be taken projective.

The arithmetic action. Set

$$\Gamma = \mathrm{Gal}(\Omega/L), \qquad \Gamma_{\mathrm{geo}} = \mathrm{Gal}(\Omega/\overline{K}L), \qquad N = \mathrm{Gal}(\Omega/E).$$

Geometric integrality gives the exact sequence

$$1 \longrightarrow \Gamma_{\mathrm{geo}} \longrightarrow \Gamma \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow 1.$$

The subgroup N is open and normal in Γ_{geo}. It need not be open in Γ.

For $\gamma \in \Gamma$, the fields $\gamma(E)$ and E represent isomorphic covers over $\overline{K}L$, by the first hypothesis. Since $E/\overline{K}L$ is Galois, two such embedded

copies are equal. Hence $\gamma(E) = E$ and N is normal in Γ. Consequently Γ/N acts semilinearly on W and, functorially, on

$$J = \mathrm{Alb}_{\mathrm{bir}}(W).$$

Fix a prime ℓ and put

$$V = T_\ell(J) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

These semilinear automorphisms induce a genuine representation

$$r\colon \Gamma \longrightarrow \mathrm{GL}(V), \qquad N \subset \ker r.$$

On $\Gamma_{\mathrm{geo}}/N = \mathrm{Gal}(E/\overline{K}L)$, this is exactly the linear action of the deck group on J. We retain the kernel of this deck action in the descent.

The representation is continuous. Indeed, the cover and its Albanese descend to a finite extension of K. At every finite torsion level, all the data and the finitely many torsion points are defined over a finite extension of L. The subgroup fixing that extension acts trivially at that level, so the corresponding kernel is open.

Constancy of the Albanese and finiteness of the image. The polarization of X induces an ample class on Y, whose pullback to W induces a polarization on J. One construction uses the intersection pairing with the $(\dim W - 1)$-st power of this class on a resolution. The pulled-back class is nef and big; the resulting Hermitian form on nonzero holomorphic one-forms is positive, since it is positive on the dense open set where the resolution is an isomorphism. This construction is birationally invariant and invariant under isomorphisms of the cover. Thus the polarized coarse moduli point of J is fixed by $\mathrm{Gal}(\overline{K}/K)$ and defines a moduli map over K.

Because K/g^*K is finite, $\widetilde{\sigma}(\overline{K}) = \overline{K}$. The second hypothesis gives a dominant semilinear rational arrow

$$W \dashrightarrow \widetilde{\sigma}^* W.$$

It induces an isogeny

$$u\colon J \longrightarrow \widetilde{\sigma}^* J.$$

Indeed, its image contains the target Albanese image, which generates the target Albanese variety, and the source and target dimensions are equal. Lemma 12.7 therefore makes the polarized moduli of J constant in the phase base.

Here is the finite-image argument, including the necessary field of definition. Choose a finite extension K_1/K and a projective model W_1/K_1 of the cover, with function field E_0 finite over L, such that

$$E = \overline{K} E_0.$$

After enlarging K_1 and E_0 by finite extensions, the polarized Albanese of W_1 is isomorphic to $A_0 \times_k K_1$ for a fixed polarized abelian variety A_0/k. The group

$$\mathrm{Gal}(\Omega/E_0)$$

is an open subgroup of Γ. Its elements need not fix E pointwise. They fix the model over E_0 and act on its geometric Albanese through the constant model A_0; all torsion points of A_0 lie in the algebraically closed field k. They therefore act trivially on V. Thus r has finite image.

Equivalently, the residual polarized twisting takes values in $\mathrm{Aut}_{\mathrm{gp}}(A_0, \lambda_0)$. This group is finite, since its integral homology representation preserves a positive definite Hermitian form and hence lies in the intersection of a discrete integral matrix group with a compact group. The action of this automorphism group on rational Tate homology is faithful.

The invariant kernel. Define

$$\theta(\gamma) = \widetilde{\sigma}^{-1}\gamma\widetilde{\sigma}.$$

For $a \in L$ and $\gamma \in \Gamma$, the element $\widetilde{\sigma}(a)$ lies in L and is fixed by γ. Hence $\theta(\gamma)(a) = a$, so θ maps Γ into itself.

The isogeny u, with the standard semilinear identification of Tate modules, gives an invertible linear operator P on V. Functoriality and $\gamma\widetilde{\sigma} = \widetilde{\sigma}\theta(\gamma)$ give

$$r(\theta(\gamma))P = P\, r(\gamma).$$

For example, with the convention that the semilinear automorphism attached to γ has comorphism γ^{-1}, the corresponding identity of rational maps is $F\gamma = \theta(\gamma)F$, which gives precisely this formula. Since P is invertible,

$$\theta^{-1}(\ker r) = \ker r.$$

Thus the forward inclusion $\widetilde{\sigma}(E) \subset E$ gives the invariant kernel needed for descent.

Put $R = \ker r$ and $L' = \Omega^R$. Then L'/L is finite Galois. For $a \in L'$ and $\gamma \in R$,

$$\gamma\widetilde{\sigma}(a) = \widetilde{\sigma}\theta(\gamma)(a) = \widetilde{\sigma}(a),$$

so $\widetilde{\sigma}(L') \subset L'$. Let $p\colon Z \to X$ be the normalization in L'. Normality extends the field embedding to a regular finite endomorphism $F_Z\colon Z \to Z$, and

$$pF_Z = fp, \qquad F_Z^* p^* H_X \sim q\, p^* H_X.$$

The induced map on the finite group Γ/R is injective, hence an automorphism. Accordingly the deck action satisfies $F_Z\eta = \delta(\eta)F_Z$ for an automorphism δ of the deck group.

The geometric fiber and its constant Albanese. Let K' be the relative algebraic closure of K in L', and normalize B in K'. The geometric generic fiber of Z over this new phase base has function field $\overline{K}L'$. It is birational to

$$W/G_0, \qquad G_0 = (R \cap \Gamma_{\mathrm{geo}})/N.$$

Every element of G_0 acts trivially on V. Faithfulness of the Tate representation of homomorphisms of abelian varieties implies that its linear action on J is the identity. Its action on the Albanese torsor is consequently a translation. The translation images form a finite subgroup $T \subset J$.

The universal property of the Albanese shows that the Albanese torsor of W/G_0 is the quotient of the Albanese torsor of W by T, and its abelian variety is J/T. Indeed, maps from W/G_0 to abelian torsors correspond to the maps from W whose induced homomorphisms kill T. In particular this Albanese has positive dimension, and its rational Tate module identifies with V.

Let J'/K' be the Albanese of the generic fiber of Z over the new base. The exact sequence for L'/K' identifies its arithmetic Galois action with the action induced by R. This action on V is trivial. Moreover J' is isotrivial: geometrically it is the quotient of the constant J by a finite subgroup, whose torsion points are constant. A polarized trivialization of J' over a finite extension has a descent cocycle in $\mathrm{Aut}_{\mathrm{gp}}(A', \lambda')$ for its constant polarized model (A', λ'). The cocycle acts trivially on the rational Tate module, so faithfulness makes it trivial. Thus

$$J' \simeq A' \times_k K'$$

already over K', for an abelian variety A'/k of positive dimension.

The Albanese torsor of the generic fiber has finite order in $H^1(K', A')$. A closed point splits it over a finite extension, and restriction–corestriction kills its class by the degree of that extension. Pushing out by a suitable multiplication map therefore gives a rational map

$$Z \dashrightarrow A'$$

which is nonconstant on the geometric generic phase fiber. This map induces multiplication by that same integer on the Albanese $J' = A'_{K'}$ of the generic fiber. This map need not intertwine the dynamics.

Global Albanese and descent to X. Apply the birational Albanese theorem of Meng–Zhang (Corollary 1.4 of *Building blocks of polarized endomorphisms of normal projective varieties*) to (Z, F_Z). The map

$$a\colon Z \dashrightarrow A := \mathrm{Alb}_{\mathrm{bir}}(Z)$$

is dominant, and the induced affine endomorphism of A is polarized. The preceding map $Z \dashrightarrow A'$ factors through a, so a is nonconstant on the geometric generic phase fiber. More precisely, if $j\colon J' \to A_{K'}$ is the homomorphism induced by the generic-fiber inclusion, the factorization supplies a homomorphism $b\colon A \to A'$ with $b_{K'} \circ j = [n]$ for some $n > 0$. Thus j has finite kernel.

The finite deck group of $Z \to X$ acts affinely on A, and $F_Z\eta = \delta(\eta)F_Z$ descends to the corresponding identity there. Consequently a induces an equivariant dominant rational map

$$\psi\colon X \dashrightarrow A/H,$$

where H is the image of the deck group on A. The induced endomorphism of A/H is polarized by Theorem 1.3 of the same paper. The polarized affine endomorphism on A has a fixed point, since its linear part minus the identity is an isogeny; translation by that point conjugates it to a polarized isogeny. Thus the factor is exceptional. The finite quotient $A \to A/H$ cannot contract a positive-dimensional image of a generic phase fiber. Hence ψ is nonconstant on that fiber, as required. □

Theorem 12.9 (Gluing irregular covers of periodic fibers)**.** *Let $F : X \to X$ and $g : B \to B$ be q-polarized endomorphisms of normal complex projective varieties, with $q > 1$, and let $\pi : X \to B$ be a dominant equivariant morphism with geometrically integral generic fiber. Fix a dense open $B_0 \subset B$ over which the fibers are normal and geometrically integral of dimension $m > 0$. Suppose a Zariski-dense set $\mathcal{P} \subset B_0$ of periodic points has the following property. For each $b \in \mathcal{P}$, some return $F^{n_b}|_{X_b}$ has a connected finite Galois cover*

$$p_b : W_b \longrightarrow X_b$$

of degree at most one fixed integer D, of positive birational irregularity, whose lifted return is a connected normalized pullback square. Equivalently, its function fields satisfy

$$\mathbf{C}(X_b)\,(F_{W_b}^{n_b})^*\mathbf{C}(W_b) = \mathbf{C}(W_b).$$

Then an iterate of F has an exceptional rational factor

$$\rho : X \dashrightarrow Z$$

which is nonconstant on the geometric generic π-fiber. The factor can be enlarged to one for F itself, retaining this nonconstancy.

Proof. Let H polarize F. Shrink B_0 so that the ample volumes of $H_b = H|_{X_b}$ and the numbers $K_{X_b} \cdot H_b^{m-1}$ are constant. For the latter, one may use the genus of a general complete-intersection curve avoiding

the singular locus. Restricting a polarized return to a fiber, and then lifting it to a finite cover, preserves its polarization. Proposition 11.9 therefore gives a uniform constant $C \geq 0$ such that every selected cover above a point of B_0 has reduced branch degree at most C.

Complete chains, including phases outside the good open. For a selected b, consider the normalizations of the successive closed images $F^i(X_b)$, with $0 \leq i < n_b$. These are projective m-folds, and the induced arrows are finite and surjective. They close up because $F^{n_b}(X_b) = X_b$. Take successive normalized pullbacks of p_b backwards along this chain. The stable-field condition and Lemma 12.2 imply that all these pullbacks are connected, Galois, and of the same degree. They form a cyclic chain of normalized Cartesian squares. Each has a polarized return, and each has positive birational irregularity, since a partial lifted cycle maps it finitely onto W_b.

At a phase $c \in B_0$, the image in question is a closed m-dimensional subvariety of the irreducible m-fold X_c, and is therefore the whole fiber. Thus every state of this cyclic chain whose phase lies in B_0 is an ordinary cover of X_c with the same degree and branch bounds. At a phase outside B_0 no bound is needed; the intermediate normalization still supplies the required finite arrow.

One branch divisor and a finite space of classes. Choose N with $q^N > C$. Form the relative critical-value divisor V_N of the N-step arrow. After shrinking a dense open $B_1 \subset B_0$, its formation commutes with the fibers, and every predecessor under g^N of a point of B_1 belongs to B_0. For any state over $b \in B_1$, use the predecessor of b in its own cyclic chain. The first and last fibers of this length-N path are good fibers of equal ample volume. Its full normalized pullback is connected, and the source branch degree is at most C. Lemma 12.3 puts the terminal branch inside $(V_N)_b$. The proof of that lemma uses only the source bound and the full length-N arrow, so none of the intervening phases is required to belong to B_0.

By Lemma 12.4, after another shrink there is a finite étale coarse parameter space $\mathcal{H} \to B_1$ for the Galois covers of degree at most D with this branch support. Retain every state of every selected cyclic chain whose phase lies in B_1, and let Z_i be the dominant components of their reduced closure in $\mathcal{H}$. All these components have positive birational irregularity generically. To check this, take a finite atlas of the cover stack, resolve the generic cover, and spread its resolution to a smooth projective family. On a dense open its $h^1(\mathcal{O})$ is constant. The selected positive-irregularity classes are dense in each retained component.

Take the closure of the one-step edges for which both phase points are in B_1. Every dominant Z_i has a dominating outgoing edge: its

selected vertices are dense, and discarding those above $g^{-1}(B \setminus B_1)$ removes only a proper closed subset. The successor of every remaining selected vertex is retained, since the complete cyclic chain was retained before restriction. On $U_1 = B_1 \cap g^{-1}(B_1)$, the restrictions

$$a_b = F|_{X_b} : X_b \longrightarrow X_{g(b)}$$

form finite dominant fiber arrows. Let $\mathcal{R} \subset \mathcal{H} \times_{B,g} \mathcal{H}$ consist of those pairs for which the source cover is the full connected normalized pullback of the target cover along a_b. This relation is constructible, and a target class determines its source class uniquely. Consequently the maximum-degree argument of Lemma 12.5 applies. Notice that this argument needs dense outgoing edges, as just verified; it does not need a single entire selected cycle to remain in the final open.

We obtain a finite phase extension K'/K, where $K = \mathbf{C}(B)$, and an iterate with $K\sigma(K') = K'$, together with a dynamically invariant geometric Galois cover class of positive irregularity. Here σ denotes the induced embedding for that iterate. Lemma 12.6 replaces it by the compositum of its finitely many arithmetic conjugates. This cover class is invariant over the original field K, is preserved by σ, and still has positive birational irregularity.

From the class to an absolute factor. Apply Theorem 12.8 to this geometric cover of the generic π-fiber. Its first-cohomology representation produces an actual finite equivariant Galois cover of X; the global birational Albanese of that cover descends to an exceptional quotient-of-abelian factor of an iterate of F. The theorem also gives the required nonconstancy on the geometric generic fiber. Finally Proposition 11.3 enlarges this to an exceptional factor for F itself. Its factor field contains the preceding one, so it remains nonconstant there. □

13. Periodic observations and the constant quotient

13.1. **Rational first integrals of the Betti foliation.** Let S be a smooth connected complex algebraic curve and let $p\colon \mathcal{A} \to S$ be a polarized abelian scheme. All statements permit replacing S by a nonempty open subset and then by a finite cover. A constant quotient means a surjective homomorphism of abelian schemes

$$c\colon \mathcal{A} \longrightarrow C \times S,$$

where C is an abelian variety over $\mathbf{C}$. The maximal such quotient, in the isogeny category, is dual to the maximal constant abelian subvariety of $\mathcal{A}^\vee$.

Lemma 13.1 (Betti first integrals)**.** *Let $F \in \mathbf{C}(\mathcal{A})$ be a rational function. Suppose that, on a simply connected analytic disc $\Delta \subset S$, F is*

constant along every Betti leaf on which it is defined. Then there are a constant quotient $c\colon \mathcal{A} \to C \times S$ *and a rational function* $F_0 \in \mathbf{C}(C)$ *such that*

$$F = F_0 \circ \mathrm{pr}_C \circ c.$$

In particular, if the geometric generic fiber of $\mathcal{A}$ *has zero* $\mathbf{C}$*-trace, then* F *is constant.*

Proof. We first record the analytic calculation. Choose normalized period coordinates on Δ, so that locally on the universal cover of the fibers

$$z = x + \tau(s)y, \qquad x, y \in \mathbf{R}^a,$$

where τ is holomorphic and $\operatorname{Im}\tau > 0$. For a polarization of arbitrary type the first period block is a fixed diagonal matrix; absorbing this matrix into x gives the displayed formula. Write $\Phi(s, z)$ for the meromorphic lift of F. Constancy along Betti leaves gives, away from the polar locus,

$$\partial_s \Phi(s, z) + d_z\Phi(s, z)\, \tau'(s)y = 0. \tag{13.1}$$

At fixed s one has

$$y = (\tau(s) - \overline{\tau(s)})^{-1}(z - \bar{z}).$$

Differentiate (13.1) with respect to $\bar{z}$. Since Φ is meromorphic in z, this gives

$$d_z\Phi(s, z)\, \tau'(s)(\tau(s) - \overline{\tau(s)})^{-1} = 0.$$

Consequently

$$d_z\Phi(s, z)\, \tau'(s) = 0, \qquad \partial_s\Phi(s, z) = 0. \tag{13.2}$$

Let K_η be the identity component of the translation stabilizer of the rational function F_η on the generic abelian variety. This is an abelian subvariety: one can take the stabilizer of the closure of the graph of F_η in $\mathcal{A}_\eta \times \mathbf{P}^1$ under translation on the first factor. After shrinking S, it extends to an abelian subscheme $\mathcal{K} \subset \mathcal{A}$. Form

$$q\colon \mathcal{A} \longrightarrow \mathcal{Q} := \mathcal{A}/\mathcal{K}.$$

The function F descends to a rational function $\bar{F}$ on $\mathcal{Q}$, and the translation stabilizer of $\bar{F}_s$ is finite for general s. Indeed, a positive-dimensional connected stabilizer in the quotient would pull back to a connected translation stabilizer of F_η strictly larger than K_η.

A homomorphism of abelian schemes respects Betti leaves. The quotient function $\bar{F}$ is therefore constant along Betti leaves of $\mathcal{Q}$. Apply (13.2) to normalized period coordinates on $\mathcal{Q}$. Every column of its period derivative is a constant vector field annihilating $\bar{F}_s$ identically. Such a vector field lies in the Lie algebra of the translation stabilizer of

$\bar{F}_s$: its local translation flow preserves the meromorphic function. This Lie algebra is zero. Hence the period derivative of $\mathcal{Q}$ vanishes.

It follows that the polarized moduli map of $\mathcal{Q}$ is constant on Δ, hence on S. After a finite cover, $\mathcal{Q} = C \times S$. One way to see the finiteness is to choose a polarized isomorphism with a fixed fiber; the residual monodromy lies in $\mathrm{Aut}_{\mathrm{gp}}(C, \lambda)$, the finite group of polarization-preserving group automorphisms fixing the origin. The second equality in (13.2), or directly constancy on constant sections, now shows that $\bar{F}$ belongs to $\mathbf{C}(C)$.

Finally, a positive-dimensional isotrivial quotient yields a positive-dimensional isotrivial abelian subvariety after an isogeny, by Poincaré reducibility. Thus geometric trace zero forces C to have dimension zero. □

Lemma 13.2 (Dense periodic Betti labels)**.** *Let $\alpha\colon \mathcal{A} \to \mathcal{A}$ be a family of q-polarized isogenies, with $q > 1$. On a simply connected disc the local sections fixed by some iterate of α are Betti leaves, and their Betti labels are dense in the real torus of labels.*

Proof. The action on integral first homology is a constant integer matrix M. All its complex eigenvalues have modulus $\sqrt{q} > 1$: the linear action of a polarized isogeny scales the positive definite Hermitian form of the polarization by q. Consequently

$$\|(M^n - I)^{-1}\| \longrightarrow 0.$$

This implication only uses that every eigenvalue has modulus greater than one; it also follows from the Jordan normal form. The labels of the fixed points of α^n are

$$(M^n - I)^{-1}\mathbf{Z}^{2a}/\mathbf{Z}^{2a}.$$

For any $v \in \mathbf{R}^{2a}$ choose an integer vector k_n within a fixed bounded distance of $(M^n - I)v$. Then $(M^n - I)^{-1}k_n \to v$. These labels are independent of s, which proves both assertions. □

Corollary 13.3 (Periodic observations descend to the constant quotient)**.** *Under the hypotheses of Lemma 13.2, let $F\colon \mathcal{A} \dashrightarrow M$ be a rational map to a fixed complex algebraic variety. Suppose that for every local periodic section u on which F is generically defined, $F \circ u$ is constant. Then, after a finite cover of S, F factors through a constant abelian quotient of $\mathcal{A}$. If the geometric generic fiber has zero $\mathbf{C}$-trace, F is constant.*

Proof. First take $M = \mathbf{P}^1$. Away from the indeterminacy and polar loci, the derivative of F in a Betti-leaf direction depends continuously

on the Betti label. It vanishes for the dense set of periodic labels, hence for every label. Lemma 13.1 applies.

In general, choose rational coordinates on the closure of the image of F and apply that lemma to their pullbacks. There are finitely many resulting constant quotients. Their product is again constant after a finite cover; the image of $\mathcal{A}$ in this product is a constant abelian subvariety, because the corresponding integral homology sublattice is locally constant. This image is the required constant quotient. □

13.2. **Finite chains with constant return.**

Lemma 13.4 (Simultaneous triviality of a finite chain)**.** *Let S be a smooth connected complex algebraic curve. Let $\mathcal{X}_i \to S$, $0 \le i < n$, be families of normal projective varieties, and let*

$$A_i : \mathcal{X}_i \longrightarrow \mathcal{X}_{i+1}, \qquad \mathcal{X}_n = \mathcal{X}_0,$$

be finite surjective S-morphisms. If the return map $R = A_{n-1} \cdots A_0$ is isotrivial, then, after a finite base change and restriction to a nonempty open subset, the whole diagram is constant. In particular every arrow has constant isomorphism class under independent source and target isomorphisms.

Proof. Make the finite base change trivializing the return, and write $(\mathcal{X}_0, R) = (X_0, R_0) \times S$. Put $K = \mathbf{C}(S)$, $E = \mathbf{C}(X_0)$ and $D = R_0^* E \subset E$. The extension E/D is finite and separable. Let $\widetilde{E}/D$ be its finite Galois closure, with group G, and let G_0 fix E.

Extension of constants from $\mathbf{C}$ to K preserves this Galois closure and its group. Indeed, function fields of integral complex varieties are geometrically integral over the algebraically closed field $\mathbf{C}$. Thus $\widetilde{E}K/DK$ is Galois with group G, and EK is its G_0-fixed field. Every intermediate field of EK/DK is therefore $(\widetilde{E}^J)K$ for a subgroup $G_0 \subset J \subset G$. In particular there are only finitely many, and each is obtained by extending constants from $\mathbf{C}$.

Define the prefixes and suffixes

$$\pi_i = A_{i-1} \cdots A_0 : X_0 \times S \longrightarrow \mathcal{X}_i, \qquad \rho_i = A_{n-1} \cdots A_i : \mathcal{X}_i \longrightarrow X_0 \times S.$$

The field $L_i = \pi_i^* K(\mathcal{X}_i)$ lies between DK and EK. Write $L_i = L_{i,0}K$ with $D \subset L_{i,0} \subset E$, and let Y_i be the normalization of X_0 in the extension $L_{i,0}/D$, where D is identified with $\mathbf{C}(X_0)$ using R_0^*. The normal variety $\mathcal{X}_{i,K}$ is finite over $X_{0,K}$ via ρ_i, so uniqueness of normalization gives

$$\mathcal{X}_{i,K} \simeq (Y_i)_K.$$

The inclusions $L_{i+1} \subset L_i$ are extensions of the inclusions $L_{i+1,0} \subset L_{i,0}$. Hence all arrows are the base changes of the finite morphisms determined by these inclusions.

These finitely many generic-fiber isomorphisms and their inverses spread over a nonempty open subset of S and satisfy the required commuting identities there. Equivalently, after making the total spaces normal, the finite maps ρ_i identify them directly with the normalizations of $X_0 \times S$ in $L_{i,0}K$, namely $Y_i \times S$. □

Lemma 13.5 (Polarized refinement)**.** *In Lemma 13.4, suppose ample relative line bundles H_i satisfy*

$$A_i^* H_{i+1} \simeq H_i^{\otimes q}, \qquad q > 1.$$

Then the simultaneous trivialization can be chosen so that the polarized arrows are constant, after shrinking S.

Proof. First use Lemma 13.4 to make the varieties and arrows constant. The Néron–Severi class of $H_{0,s}$ is constant. Since $R_0^* H_{0,s} = H_{0,s}^{q^n}$, its variation in $\mathrm{Pic}^0(X_0)$ belongs to

$$\ker\bigl(R_0^* - [q^n]\bigr).$$

This kernel is finite: the pullback of a Q-polarized endomorphism on Pic^0 has analytic eigenvalues of absolute value $\sqrt{Q}$, so $Q = q^n > 1$ is not an eigenvalue. Thus $H_{0,s}$ is constant in the Picard scheme. A line bundle from S can be removed after shrinking S.

For each i, the now constant prefix satisfies

$$\pi_i^* H_{i,s} = H_0^{q^i}.$$

The kernel of $\pi_i^* : \mathrm{Pic}(Y_i) \to \mathrm{Pic}(X_0)$ is finite. The norm shows that it is killed by $\deg \pi_i$; the torsion subgroup of any fixed order in the Picard group of a normal projective complex variety is finite. Therefore $H_{i,s}$ is constant as well. This proves constancy of the arrows with their source and target polarizations. □

13.3. **A rational moduli map for polarized fiber arrows.**

Proposition 13.6 (A separating arrow-moduli map)**.** *Let B be an irreducible complex algebraic variety, let $g\colon B \dashrightarrow B$ be dominant, and let $\pi\colon \mathcal{X} \to B$ be a projective family with geometrically integral normal generic fiber. Let H be a relatively ample line bundle. Suppose that, over a nonempty open subset of B, there is a finite surjective fiber arrow*

$$a_b\colon X_b \longrightarrow X_{g(b)}$$

varying algebraically with b, and satisfying

$$a_b^* H_{g(b)} \simeq H_b^{\otimes q}, \qquad q > 1.$$

Then there exist an integer $N > 0$, a complex algebraic variety M, and a rational map

$$\mu\colon B \dashrightarrow M$$

with the following property on a nonempty open subset B°:

$$\mu(b) = \mu(c)$$

if and only if there are polarized isomorphisms

$$u\colon (X_b, H_b^N) \xrightarrow{\sim} (X_c, H_c^N), \qquad v\colon (X_{g(b)}, H_{g(b)}^N) \xrightarrow{\sim} (X_{g(c)}, H_{g(c)}^N)$$

such that

$$v \circ a_b = a_c \circ u.$$

The two isomorphisms are independent source and target coordinate changes.

Proof. Restrict to the flat normal-fiber locus and take a sufficiently high power $L = H^N$ so that $E = \pi_* L$ is locally free of rank r and gives the relative complete embedding. Write $p(t) = \chi(X_b, L_b^t)$. A graph of a_b has Hilbert polynomial $p((q+1)t)$ for $\mathcal{O}(1,1)$, since $L_b \otimes a_b^* L_{g(b)} \simeq L_b^{q+1}$. Independent frames of E and g^*E give a map to

$$\mathcal{H} = \mathrm{Hilb}_p(\mathbf{P}^{r-1})^2 \times \mathrm{Hilb}_{p((q+1)t)}(\mathbf{P}^{r-1} \times \mathbf{P}^{r-1}).$$

Let $G = \mathrm{PGL}_r \times \mathrm{PGL}_r$ and take the closure of the G-saturation of the image. Rosenlicht's theorem [Ros56, Theorem 2] gives a geometric quotient of a G-invariant dense open of this closure. The resulting rational map $\mu : B \dashrightarrow M$ is independent of the chosen frames. Its geometric fibers are exactly the G-orbits of graph triples. Such an orbit records precisely the polarized source and target isomorphisms commuting with the arrow, because the embeddings use complete linear systems. This proves the assertion. □

13.4. **Trace-zero abelian factors and their joins.** We work over $k = \overline{\mathbf{C}(S)}$, retaining the indicated field of constants $\mathbf{C}$. For an abelian variety A/k, write A_{var} for its quotient by the image of its $\mathbf{C}$-trace. Up to isogeny this is the complementary factor to the largest constant abelian subvariety. It has zero $\mathbf{C}$-trace. These constructions are preserved by isogenies and by all homomorphisms of abelian varieties.

Lemma 13.7 (Variation survives the irregular-cover construction)**.** *In the construction of Theorem 12.8, let J/k be the abelian variety to which the Albanese variety of the geometric generic phase-fiber cover*

becomes isogenous. Suppose $J_{\mathrm{var}} \neq 0$. *The independent exceptional factor can then be chosen in the form*

$$A_{\mathrm{var}}/\Gamma,$$

where A_{var} *is a trace-zero abelian variety and* Γ *is a finite group of affine automorphisms. The resulting map is nonconstant on the geometric generic phase fiber.*

Proof. Keep the actual normal projective Galois cover Z constructed in that theorem. Its construction gives a rational map $Z \dashrightarrow J'$ whose restriction to the geometric generic phase fiber induces an isogeny on Albanese varieties; here J' is isogenous to J. Indeed, the geometric subgroup killed in the descent acts on the Albanese torsor by translations. Quotienting by these translations is an isogeny, and killing the remaining torsor replaces its Albanese map by multiplication by a positive integer. Neither operation removes a positive-dimensional abelian quotient.

Put $A = \mathrm{Alb}(Z)$. By the universal property of the Albanese, the preceding map factors through a homomorphism $A \to J'$ after a translation. If J_Z is the Albanese variety of the geometric generic phase fiber of Z, the composite

$$J_Z \longrightarrow A \longrightarrow J'$$

is an isogeny. Consequently the first arrow has finite kernel. If the image of J_Z in A_{var} were zero, that image in A would be contained in the constant trace of A. An abelian subvariety of a constant abelian variety is constant after finite extension: its integral homology lattice is one of the countably many sublattices, and an algebraic family of such subvarieties has constant lattice on a connected parameter open. Thus J_Z, and hence J, would be isotrivial, contrary to $J_{\mathrm{var}} \neq 0$. The map to A_{var} is therefore nonconstant on the phase fiber.

Every deck transformation of Z acts affinely on A and preserves the image of its constant trace. It consequently acts affinely on A_{var}. Taking the quotient by the finite image Γ descends the map to the original variety. The induced endomorphism of A_{var} is polarized: the polarized Albanese endomorphism descends to this invariant abelian quotient. A finite quotient of this system is exceptional. Since the quotient map is finite, nonconstancy on the phase fiber is preserved. □

Remark 13.8 (Detecting variation on selected periodic covers)**.** The same refinement applies to Theorem 12.9 when its selected periodic cover families have nonisotrivial Albanese varieties. Here is the specialization point in that assertion. Spread the finitely many dominant

cover-class components, choose finite covers carrying actual covering objects, and shrink so that their relative Albanese schemes and their isogenies are defined. The phase-isogeny and Hodge-height argument in Lemma 12.7 makes the abelian moduli of each dynamically selected component independent of phase. Thus this moduli map depends only on the parameter S; restriction to a good periodic section gives precisely the moduli of its selected covering family. If every selected dominant component had constant abelian moduli over S, all such specializations would be isotrivial. A Zariski-dense set of selected periodic families with varying Albanese therefore supplies a component with $J_{\mathrm{var}} \neq 0$. The finite conjugate compositum and finite translation quotients used subsequently preserve its abelian quotient up to isogeny. The preceding lemma applies.

Lemma 13.9 (Joins of trace-zero abelian-quotient factors)**.** *The join of two polarized rational factors which admit finite presentations by trace-zero abelian varieties again admits such a presentation, after an iterate. Such factors have a maximal join. Its function field can be taken relatively algebraically closed in the function field of the source.*

Proof. Lift the normalized joint image to the product of the two finite abelian presentations. After an iterate a dominant component is periodic. A periodic subvariety of a polarized abelian endomorphism is an abelian coset, by the abelian case of Proposition 11.4. Its abelian variety has trace zero: a homomorphism from a constant abelian variety to it, followed by either projection, is zero. The finite map from this coset gives the required presentation of the join.

Repeated joins increase dimension until the maximal dimension is reached; at that dimension they only produce finite extensions. Take the relative algebraic closure of the resulting factor field inside the source field, a finite extension. This extension is again a trace-zero abelian-quotient factor. To see this, normalize its pullback to the abelian presentation and take a periodic component. Its map to the abelian variety is finite and polarized. Since the abelian endomorphism is *étale*, Ji–Song–Xie's Lemma 3.11 makes this cover finite *étale*; it is an abelian variety isogenous to the original one, and therefore still has trace zero. Maximality now implies that every other factor of the indicated kind is contained in this one: otherwise its join would have larger dimension or enlarge a relatively algebraically closed subfield by a finite extension. □

13.5. **Finite gauge ambiguity and its horizontal derivation.**

Proposition 13.10 (Horizontal descent through finite ambiguity)**.** *Let $K = \mathbf{C}(B)$ and let $M = K\mathbf{C}(S)$, where S is a smooth complex parameter curve. Let $\sigma : K \hookrightarrow K$ be a finite field embedding, extended to M by fixing $\mathbf{C}(S)$. Suppose that a family of geometrically integral generic phase fibers has a nonempty finite-type gauge torsor I under G_M, where G/K is an algebraic group. Suppose first-arrow transport sends I algebraically to σ^*I and sends $T = G^0$ isogenously to σ^*T, with T a torus. Assume these gauges intertwine the first varying arrow with its fixed reference arrow.*

On the rational quotients by the descended T-actions, there is a rational horizontal derivation W which annihilates K, projects to a nonzero parameter derivation, and commutes with the dynamics. In particular, whenever these quotient fields have normal projective polarized models, the quotient family is isotrivial over B.

Proof. The torus T is normal in G. Changing a gauge by an element of G therefore leaves the invariant function field unchanged. The induced quotient gauges have ambiguity in the finite group scheme $H = G/T$ over K. They are parameterized by the finite H_M-torsor I/T, possibly with an ineffective subgroup in its action on the quotient field. The ineffective subgroup does not affect what follows.

Choose a nonzero derivation ∂ of $\mathbf{C}(S)/\mathbf{C}$ and extend it to M by annihilating K. It extends uniquely to an algebraic closure $\overline{M}$. A quotient gauge C is defined over a finite extension of M. On the reference quotient field extended to $\overline{M}$, define ∂ to annihilate the reference field over K, and transport it through C to obtain W_C. The geometric integrality of the generic phase fiber makes this extension of scalars a field; equivalently, its function field is a regular extension of K.

Every geometric point of H has coordinates algebraic over K. The derivation ∂ annihilates all such coordinates. Thus if $C' = C\gamma$ for $\gamma \in H(\overline{K})$, then

$$(C')_*\partial = C_*\partial.$$

All Galois conjugates of C give the same derivation. Consequently W_C descends to a rational derivation W on the quotient family. It annihilates K and restricts to ∂ on the parameter.

Extend σ to an automorphism $\overline{\sigma}$ of $\overline{M}$. Such an extension exists because $M/\sigma(M)$ is finite. The equality $\partial\sigma = \sigma\partial$ on M and uniqueness of algebraic extension give

$$\partial\overline{\sigma} = \overline{\sigma}\partial \quad \text{on } \overline{M}.$$

The transported target gauge and $\overline{\sigma}(C)$ are both points of $\sigma^*(I/T)$, hence differ by an element whose coordinates are algebraic over K and

are killed by ∂. The fixed reference arrow commutes with ∂. Differentiating the gauge identity therefore gives the exact rational identity

$$DFW = W \circ F.$$

For completeness, the normal polarized model removes the poles of W. If D is its polar Weil divisor, the regular differential on the left cannot increase pole order, while the right multiplies pole orders by the ramification indices. Hence $F^*D \leq D$. Intersecting on the generic parameter fiber with a polarization forces its horizontal part to vanish. Delete the finitely many remaining parameter fibers. Reflexivity of the tangent sheaf extends W over codimension two. Its local flow along the compact fibers trivializes the family and commutes with F. Since W annihilates K, these flows preserve the original map to B. The algebraic conjugacy locus in a relative graph Hilbert scheme then dominates S; a closed point of its generic fiber provides an algebraic trivialization after a finite parameter extension. □

13.6. **The dynamical splitting field of a torus.**

Proposition 13.11 (Minimal splitting fields carry the dynamics)**.** *Let M be a characteristic-zero field, let $\sigma : M \hookrightarrow M$ be finite, and let T/M be a torus with an isogeny $T \to \sigma^*T$. Its minimal Galois splitting field L/M admits an extension $\widetilde{\sigma} : L \hookrightarrow L$ of σ, and*

$$L = M\widetilde{\sigma}(L).$$

If M is the function field of a normal projective polarized system whose pullback is σ, its normalization in L carries a finite polarized lift of the same weight.

Suppose moreover that E/M is regular and that E is the function field of a normal projective variety V equipped with a finite q-polarized endomorphism F, with $F^*|_M = \sigma$. Then the normalization $p : \widetilde{V} \to V$ in EL carries a finite q-polarized endomorphism $\widetilde{F}$ satisfying

$$p\widetilde{F} = Fp.$$

Its pullback on EL extends both F^* on E and $\widetilde{\sigma}$ on L.

Proof. Extend σ to an automorphism $\overline{\sigma}$ of an algebraic closure $\overline{M}$. The character lattice of T has finite Galois image. Its rational representation determines the kernel of its integral representation, so isogenous tori have the same minimal Galois splitting field. The splitting field of σ^*T is $M\overline{\sigma}(L)$: its character representation is obtained by restricting the conjugated representation, whose kernel fixes exactly that compositum. Therefore

$$L = M\overline{\sigma}(L),$$

and $\widetilde{\sigma} = \overline{\sigma}|_L$ has image in L.

Let $h : Y \to Y$ realize σ, and let Y' be the normalization of Y in L. On an affine $U = \operatorname{Spec} A \subset Y$, write $h^{-1}U = \operatorname{Spec} R$. The image under $\widetilde{\sigma}$ of an element integral over A is integral over $\sigma(A) \subset R$. Hence the field embedding gives regular maps between the two integral closures, and these glue to $h' : Y' \to Y'$. It is proper and has finite fibers because $\rho h' = h\rho$; hence it is finite. The pullback of an ample polarization of h polarizes h' with the same weight.

Regularity of E/M makes E and L linearly disjoint over M, so $E \otimes_M L = EL$ is a field. Since F^* and $\widetilde{\sigma}$ agree on M, the rule

$$a \otimes b \longmapsto F^*(a)\widetilde{\sigma}(b)$$

defines a nonzero field homomorphism $EL \hookrightarrow EL$. Let $p : \widetilde{V} \to V$ be the normalization in this finite extension of E. The integral-closure argument above, now applied to the given morphism F on V, extends this homomorphism to $\widetilde{F} : \widetilde{V} \to \widetilde{V}$ with $p\widetilde{F} = Fp$. It is proper, and each of its fibers is contained in a fiber of the finite map Fp. Thus $\widetilde{F}$ is finite; it is surjective because its pullback on function fields is injective. If H is an ample line bundle with $F^*H \simeq H^{\otimes q}$, then p^*H is ample and

$$\widetilde{F}^*p^*H = p^*F^*H \simeq (p^*H)^{\otimes q}.$$

This gives the asserted q-polarized lift on the specified normalization. □

13.7. **Descent of isotriviality through a finite cover.**

Proposition 13.12 (A constant finite dynamical cover fixes the quotient)**.** *Let $u : Z \to Z$ be a fixed polarized endomorphism of a normal projective complex variety. Let $F : \mathcal{X} \to \mathcal{X}$ be an algebraic family of endomorphisms of normal projective varieties over a smooth curve S. Suppose there is a finite surjective S-morphism*

$$p : Z \times S \longrightarrow \mathcal{X}, \qquad p(u \times \mathrm{id}_S) = Fp.$$

Then, after finite parameter base change and restriction, the family of diagrams (p_s, F_s) is constant up to isomorphism of the targets. In particular $(\mathcal{X}, F)$ is isotrivial.

Proof. Put $n = \dim Z$ and $v = u \times u$ on $Z \times Z$. The product endomorphism v is polarized. For each parameter, let R_s be the closure of the generic reduced equivalence relation of p_s, namely the union of the n-dimensional components of

$$(Z \times_{X_s} Z)_{\mathrm{red}} \subset Z \times Z.$$

Each such component dominates Z under either projection. Equivariance gives $v(R_s) \subset R_s$. Since v is finite, the image of any irreducible component has dimension n and is an irreducible component of R_s.

After finite parameter base change, label the finitely many geometric generic components, spread them out and shrink S so that each gives a flat family of reduced irreducible subvarieties. Their finite combinatorial transition under v is constant after shrinking again. Every component therefore maps after a fixed number of steps to a component which is periodic under a fixed iterate of v.

Lemma 14.17 makes every periodic component family constant. A preperiodic component is an irreducible component of the inverse image of one of these fixed subvarieties under a fixed finite iterate. There are only finitely many choices, so its irreducible parameter family is constant as well. Hence R_s is constant.

Fix s_0. Equality of these generic equivalence relations says that p_s and p_{s_0} determine the same subfield of $\mathbf{C}(Z)$: in a finite separable extension, the functions constant on its generic equivalence classes are precisely the target field. Equivalently, the reduced image W_s of (p_{s_0}, p_s) in $X_{s_0} \times X_s$ is finite and birational over either target. Finiteness follows because a fiber of either projection is the image of a finite fiber of the corresponding p_s. Normality makes both projections isomorphisms. Thus there is a unique isomorphism c_s satisfying

$$p_s = c_s p_{s_0}.$$

The same construction with relative graphs spreads these isomorphisms algebraically over S. The equality $p_s u = F_s p_s$ and the surjectivity of p_{s_0} then give $F_s c_s = c_s F_{s_0}$. □

13.8. **Toric extensions of a fixed system.**

Lemma 13.13 (Weighted vertical divisors)**.** *Let $\pi : X \to B$ be a surjective morphism of normal projective varieties with geometrically integral generic fiber of dimension r. Let F and g be finite endomorphisms with $\pi F = g\pi$. Suppose L is big and nef, $F^*L \simeq L^q$, and g is q-polarized. Set $e = (L^r \cdot X_{\eta_B}) > 0$. For a vertical Weil divisor V define*

$$\mathcal{W}_L(V) = \frac{1}{e} \sum_{D \subset B} \left(\sum_{E \to D} \operatorname{ord}_E(V)(L^r \cdot E_{\eta_D}) \right) D.$$

Only divisors E dominating the prime divisor D occur in the inner sum. Then, as actual Weil divisors,

$$\mathcal{W}_L(\pi^* D) = D, \qquad \mathcal{W}_L(F^* V) = g^* \mathcal{W}_L(V).$$

All coefficients have denominator dividing e.

Proof. Localize at a prime divisor of the normal base. Its local ring is a discrete valuation ring. The integral total space is flat over this ring, since its local rings are torsion-free over it. Flatness and constancy of the Hilbert polynomial identify the fiber cycle degree with e. Thus, if $\pi^*D = \sum m_E E$ at this generic point,

$$\sum m_E(L^r \cdot E_{\eta_D}) = e.$$

This proves the first identity, including all fiber multiplicities.

For the second, fix D' with $g(D') = D$. Form the base change $Z = X \times_{B,g} B$ over the corresponding discrete valuation rings. It is integral: it is flat and its generic fiber is geometrically integral. Write $p : Z \to X$ for the projection and $u = (F, \pi) : X \to Z$. The map u is finite of degree q^r and $u^*p^*L = L^q$. The projection formula therefore gives

$$u_*\big(F^*V \cdot L^r\big) = q^{-r}u_*u^*\big(p^*V \cdot (p^*L)^r\big) = p^*\big(V \cdot L^r\big).$$

Proper pushforward to the base and flat base change yield

$$\pi_*\big(F^*V \cdot L^r\big) = g^*\pi_*\big(V \cdot L^r\big)$$

at D'. These are the coefficients in the asserted formula. For a Weil divisor, perform this calculation on its Cartier locus. General intersections computing the fiber degree avoid the non-Cartier locus: over the generic base divisor it has dimension at most $r-1$, whereas there are r intersections. Expressing L as a difference of sufficiently ample line bundles gives the same calculation for big and nef L. Thus no factoriality assumption is being used. The degree numerators are integers, which proves the denominator assertion. □

Lemma 13.14 (A fixed-base divisor equation)**.** *Let (B, g) be a fixed normal projective q-polarized system over $\mathbf{C}$. Suppose D_s is an algebraic curve family of rational Weil divisors with a common coefficient denominator, and $c_s \in \mathbf{C}(B)^*$ varies rationally with s, satisfying*

$$g^*D_s - qD_s = \operatorname{div}(c_s).$$

After a finite parameter extension and restriction there exist $b_s \in \mathbf{C}(B)^*$ and $a_s \in \mathbf{C}^*$, rational in s, such that, for one fixed reference s_0,

$$\frac{c_s}{c_{s_0}} = a_s \frac{b_s \circ g}{b_s^q}.$$

Proof. First reduce D_s modulo integral Weil divisors, writing its coefficients in a fixed finite subgroup of $\mathbf{Q}/\mathbf{Z}$. Its fractional part α_s satisfies $g^*\alpha_s = q\alpha_s$. For the primary parts whose primes divide q, choose m with $q^m\alpha_s = 0$. Their supports lie among the branch-value divisors

of g^m: an unramified generic inverse image of a nonzero coefficient would survive pullback. For the prime-to-q part, a nonzero coefficient at D forces a nonzero coefficient at $g(D)$; its finite support is therefore forward invariant. These divisors are preperiodic.

A fixed polarized endomorphism has only countably many preperiodic subvarieties. One way to see the countability for periodic subvarieties is to use density of periodic points on each such subvariety and the finite-type incidence spaces of subvarieties of bounded degree: a given subvariety is uniquely determined among subvarieties of its dimension and bounded degree by finitely many of its periodic points. The periodic points of the ambient map form a countable set. Taking all iterated inverse images gives the preperiodic assertion. An algebraic curve in a Chow variety with countable image is constant. Thus the fractional part of D_s is constant after labeling its components and shrinking the parameter curve. Choose a fixed representative D_{frac} and write $D_s = D_{\mathrm{frac}} + A_s$ with A_s integral.

Choose a projective resolution $\nu : \widetilde{B} \to B$. The strict transforms of $A_s - A_{s_0}$ form an algebraic divisor family and define a connected family

$$M_s \in \mathrm{Pic}^0(\widetilde{B}).$$

The divisor equation implies that $(\widetilde{g}^* - [q])M_s$ lies in the subgroup generated by the ν-exceptional divisors. Here $\widetilde{g}$ is the rational lift, whose action on Pic^0 is defined by a graph resolution. This action agrees with finite pullback of Weil classes after pushforward to B.

An exceptional divisor representing an algebraically trivial line bundle is zero: it is numerically trivial, and applying the negativity lemma to it and its negative proves the assertion. Hence $(\widetilde{g}^* - [q])M_s = 0$ in $\mathrm{Pic}^0(\widetilde{B})$. The endomorphism induced by g on the birational Albanese is q-polarized; consequently the eigenvalues of its dual on Pic^0 have absolute value $\sqrt{q}$. Thus $\widetilde{g}^* - [q]$ is an isogeny. Its finite kernel contains the connected family M_s, which contains the identity, so M_s is trivial.

The relative Picard interpretation now makes the line bundle of the strict-transform difference a pullback from the parameter curve. After shrinking, this bundle is trivial. Its rational section gives a rational family b_s with $A_s - A_{s_0} = \mathrm{div}(b_s)$ after pushforward to B. Subtract the two original divisor equations. The ratio

$$(c_s/c_{s_0})\big/\big((b_s \circ g)/b_s^q\big)$$

has zero divisor. A rational function of zero divisor on a normal projective variety is constant, giving a_s. $\square$

Lemma 13.15 (Power transport for a polarized torus action)**.** *Suppose a split torus T acts effectively and rationally on a normal projective polarized system (X, F, L), and*

$$F(tx) = \tau(t)F(x)$$

for an isogeny $\tau : T \to T$. The action is regular, and for some $m > 0$ one has $\tau^m = [q^m]$.

Proof. Let $v_1, \ldots, v_a$ be the rational fundamental vector fields of a basis of $\mathrm{Lie}(T)$ and let P be their common pole divisor, taking the largest pole order at every prime. Equivariance gives $DF(v_i) = \sum_j c_{ji} v_j \circ F$, with an invertible constant matrix (c_{ji}). At a codimension-one point the differential has regular coefficients. Invertibility of the constant matrix therefore gives $F^*P \leq P$. Intersection with $L^{\dim X - 1}$ yields $q \deg_L P \leq \deg_L P$, whence $P = 0$. Normality extends the derivations across codimension two. Their complete flows on the projective variety extend the given rational action; agreement on the initial rational-action domain makes this an algebraic torus action.

A positive power of L is T-linearizable. Let $\mathcal{P}$ be its weight polytope. This polytope has full affine span, since the action is effective. The finite map F maps the fixed locus onto itself: a finite inverse-image set of a fixed point is stable under the connected torus and consists of fixed points. Comparing the linearizations of F^*L and L^q gives

$$\tau^*\mathcal{P} = q\mathcal{P} + u$$

for a character u. The affine automorphism $x \mapsto q^{-1}(\tau^* x - u)$ permutes the finitely many vertices of the full-dimensional polytope, so a positive power is the identity. Its linear part then gives $(\tau^*)^m = q^m I$, which is the assertion on τ. □

Proposition 13.16 (Toric rigidity over a fixed quotient)**.** *Let $\mathcal{X} \to S$ be an algebraic curve family of normal projective varieties with a relatively q-polarized endomorphism F. Suppose it admits an equivariant dominant rational map to one fixed normal projective polarized system (B, g), with geometrically integral generic fiber. Assume that a fixed split torus T acts rationally over $B \times S$, has a dense orbit in the generic fiber, and satisfies*

$$F(tx) = \tau(t)F(x)$$

for one fixed isogeny τ of T. Then the family is isotrivial after finite parameter extension and restriction.

Proof. Let H be the full kernel of the action of T on the geometric generic fiber, including its finite components. A subgroup of a split

torus is defined by character equations, so H is the base change of a fixed subgroup of T. Equivariance and dominance of F give $\tau(H) \subseteq H$. Hence τ induces a surjective endomorphism, and therefore an isogeny, of the split torus T/H. Replace T and τ by this quotient and its induced isogeny. The action is now effective. Since T is commutative, the stabilizer of a point of its dense geometric orbit fixes that entire orbit and is trivial. The generic orbit is consequently a torsor under this effective split torus.

Normalize the equivariant graph of the rational map to B. Its lifted endomorphism is finite, and the sum of the pullbacks of polarizations on $\mathcal{X}$ and B is a polarization. We may therefore use a morphism π and a relatively ample polarization L on this graph. The rational torus action lifts to it. Lemma 13.15, applied over the geometric generic parameter, makes its transport $[q]$ after a common iterate; rename the new weight q.

A torsor under a split torus over a field is trivial. Choose character coordinates $z_1, \dots, z_r$ on its generic orbit. They identify the full function field with $\mathbf{C}(B)(S)(z_1, \dots, z_r)$, and the map is

$$(b, z_1, \dots, z_r) \longmapsto (g(b), c_1(b,s)z_1^q, \dots, c_r(b,s)z_r^q).$$

For one character coordinate, separate its divisor into the horizontal and vertical parts over B: $\operatorname{div}(z) = H_s + V_s$. The generic normal projective toric fiber gives $F^*H_s = qH_s$. No vertical component can lie over a horizontal divisor under a finite equivariant map. Thus this identity holds as an identity of horizontal divisors on the whole model, and

$$F^*V_s - qV_s = \pi^* \operatorname{div}(c_s).$$

Apply Lemma 13.13. The divisors $D_s = \mathcal{W}_{L_s}(V_s)$ satisfy

$$g^*D_s - qD_s = \operatorname{div}(c_s).$$

They form an algebraic family with one common denominator. Indeed the divisor of the chosen rational coordinate has finitely many components on the total family; their images, multiplicities, and fiber intersection degrees specialize after shrinking. The denominator is the fixed integer $(L_s^r \cdot X_{s,\eta_B})$.

Lemma 13.14, applied to each coordinate, gives

$$\frac{c_i(b,s)}{c_i(b,s_0)} = a_i(s)\frac{b_i(g(b),s)}{b_i(b,s)^q}.$$

After finite parameter extension choose $d_i(s)^{1-q} = a_i(s)$. The rational coordinate change $z_i \mapsto d_i(s)b_i(b,s)z_i$ conjugates the fixed reference skew system to the parameter system. Hence the original projective

family is dynamically birationally isotrivial. Lemma 4.1 extends the transported parameter derivation over the normal projective models and makes the family isotrivial. Finally, Corollary 4.2 removes the iterate. □

14. Flat bundles, foliations, and rational factors

14.1. Degrees of connected rational factors.

Lemma 14.1 (Degree of a rational factor)**.** *Let $f : X \to X$ be a q-polarized endomorphism of a normal projective variety of dimension n. Suppose that $\pi : X \dashrightarrow Y$ is a dominant rational map with geometrically integral generic fiber, that Y has dimension r, and that a dominant rational map $g : Y \dashrightarrow Y$ satisfies $\pi f = g\pi$. Then*

$$\deg(g) = q^r.$$

For general y, the finite map between the closures of the fibers through y and $g(y)$ has degree q^{n-r}.

Proof. The Chow reduction of π is a normal projective model of $\mathbf{C}(Y)$ on which g is a surjective endomorphism, by [MZ22, Proposition 7.1 and Theorem 7.2]. Applying the degree identity in [MZ18, Theorem 1.3] gives $\deg g = q^r$. For general fibers the finite restriction of f has degree $\deg f / \deg g = q^{n-r}$. □

14.2. Projective models of algebraic foliation quotients.

Proposition 14.2 (Polarized regularization of a connected factor)**.** *Let $f : X \to X$ be a q-polarized endomorphism of a normal projective variety, and let $\pi : X \dashrightarrow Y$ be a dominant rational map with geometrically integral generic fiber. Suppose that a dominant rational map $g : Y \dashrightarrow Y$ satisfies $\pi f = g\pi$. Then there is a normal projective model Y^{c} of $\mathbb{C}(Y)$ on which g is a q-polarized endomorphism. The normalized graph $X^{\mathrm{c}} \to X \times Y^{\mathrm{c}}$ carries a finite q-polarized lift of f and an equivariant surjective morphism $X^{\mathrm{c}} \to Y^{\mathrm{c}}$.*

Proof. Take the Chow reduction of π. Its existence and the regularity of the descended map are [MZ22, Proposition 7.1 and Theorem 7.2]. The resulting normal projective base is birational to Y. The descended endomorphism is q-polarized by [MZ18, Theorem 1.3], whose degree identity fixes the weight. The finite product map $f \times g$ preserves the graph and lifts to its normalization. The tensor product of the pullbacks of the two ample polarizations is ample on this normalized graph and is pulled back to its qth power. □

Proposition 14.3 (Two algebraic reductions of an invariant foliation)**.** *Let $f : X \to X$ be a q-polarized endomorphism of a normal projective complex variety of dimension n. Let $\mathcal{F}$ be an invariant rational foliation of rank strictly between zero and n. Each of the following conditions produces an equivariant connected rational quotient to a positive-dimensional normal projective variety of dimension less than n, equipped with a q-polarized endomorphism:*

(1) *the Zariski closure of a very general leaf is a proper subvariety of X;*
(2) *the maximal algebraic part $\mathcal{F}_{\mathrm{alg}}$ has positive rank.*

Consequently, the case in which neither construction gives a proper factor consists of purely transcendental foliations with Zariski dense very general leaves.

Proof. Bonnet's leaf-closure theorem [Bon06, Theorem 1.4] gives the first quotient. The quotient by the maximal algebraic part is [LBPRT22, Section 2.6 and Proposition 2.4]. Both constructions are birational, so they apply on a smooth projective model and define subfields of $\mathbf{C}(X)$. Their fibers are irreducible of dimension strictly between zero and n; taking relative algebraic closure makes the quotient connected. Invariance of $\mathcal{F}$ makes its leaf closures and algebraic part invariant, so f^* preserves the quotient field. Apply Proposition 14.2 to obtain the normal projective polarized quotient. □

14.3. **Invariant flat bundles and algebraic flows.**

Lemma 14.4 (Entire lines and a codimension-two exceptional set)**.** *Let Z be a normal quasi-projective complex variety and let $\theta : \mathbb{C}^n \to Z$ be a holomorphic map which is locally finite and has full dimension. Then Z_{reg} is h-special.*

Proof. The analytic set $A = \theta^{-1}(Z_{\mathrm{sing}})$ has complex codimension at least two. Almost every ordered pair of points of $\mathbb{C}^n$ spans an affine complex line avoiding A. Indeed, the incidence space of a point of A, a line through that point, and an ordered pair on the line has complex dimension at most $(n-2)+(n-1)+2 = 2n-1$. Covering this incidence space by countably many relatively compact analytic charts shows that its image in $\mathbb{C}^n \times \mathbb{C}^n$ has Lebesgue measure zero. Every remaining line gives an entire curve in Z_{reg}. On an open set where θ is biholomorphic, the resulting pairs have full measure in a nonempty open subset of $Z_{\mathrm{reg}} \times Z_{\mathrm{reg}}$. Thus the entire-curve equivalence relation is Zariski dense. This is h-specialness in the sense of [CDY25, Definition 2.2]. □

Proposition 14.5 (The algebraic closure of invariant flows)**.** *Let $f : X \to X$ be a polarized endomorphism of a normal projective complex variety. Suppose a nonzero finite-dimensional Lie algebra of global vector fields is preserved by invertible dynamical transport. The connected Zariski closure H of its flows in $\mathrm{Aut}^0(X)$ is a positive-dimensional semiabelian group. There is an isogeny $\psi : H \to H$ satisfying $f(hx) = \psi(h)f(x)$.*

Proof. Global holomorphic vector fields on the compact complex space X are complete. Dynamical transport gives the stated equivariance on the analytic subgroup generated by their flows. In the closed subgroup of $H \times H$ defined by $fh = kf$, projection onto the first factor is surjective: its image is an algebraic subgroup containing the Zariski-dense flow subgroup. The second coordinate is unique since f is surjective. In characteristic zero this projection is an isomorphism. The second projection is likewise surjective because transport is invertible. Its kernel is finite, being contained in the automorphism group of the finite map f. This defines ψ.

We first rule out a ψ-invariant subgroup $V \simeq \mathbb{G}_a$ on which ψ is an automorphism. A general V-orbit is $\mathbb{A}^1$, and f maps it isomorphically onto another general orbit. The closures of general orbits have one fixed positive degree e against an ample polarization L. This follows, for example, by taking their generic cycle in the Chow variety and restricting to its flat parameter locus. For x and $f(x)$ in that locus, the induced map of orbit closures has degree one. The projection formula gives $qe = e$, a contradiction.

Let L_H be the connected affine part of H. Its unipotent radical is preserved by ψ, and the restriction of ψ to it is an automorphism in characteristic zero. If that radical is nontrivial, its center has a nonzero vector subgroup invariant under ψ; an eigenline gives the forbidden V.

Thus L_H is reductive. If its derived subgroup is nontrivial, the induced self-isogeny of that semisimple group is an automorphism. After a power it is inner, say conjugation by s. Choose a Borel containing s, and a maximal torus containing its semisimple Jordan component. In a simple factor, a highest-root subgroup is centralized by the unipotent radical of the Borel and normalized by that torus. It is therefore normalized by s, and is a forbidden $\mathbb{G}_a$ subgroup for this iterate. Consequently L_H is a torus. A connected algebraic group whose affine part is a torus is semiabelian, which proves the result. □

Lemma 14.6 (First-cohomology weights on a normal invariant open)**.** *Let $f : X \to X$ be q-polarized on a normal projective variety, let E be a reduced divisor satisfying $f^{-1}E = E$, and put $Y = (X \setminus E)_{\mathrm{reg}}$.*

The rational self-map of Y induced by f defines an endomorphism of $H^1(Y, \mathbb{Q})$. All its eigenvalues have absolute value $\sqrt{q}$ or q.

Proof. The map is a morphism from $Y \setminus f^{-1}(X_{\mathrm{sing}})$ to Y. Its omitted subset has codimension at least two, so inclusion induces an isomorphism on H^1, compatibly with mixed Hodge structures. This defines the asserted endomorphism.

Take a log resolution $\overline{Y} \to X$ that is an isomorphism over Y. The weight-one part of $H^1(Y)$ is $H^1(\overline{Y})$; its action is induced by the birational Albanese endomorphism, whose polarization weight is q by [MZ18, Corollary 1.4 and Theorem 1.3]. Its eigenvalues have absolute value $\sqrt{q}$. For the weight-two part use the residue description. Projection of a cohomologically trivial boundary divisor onto the strict transforms of the components of E is injective: an element of the kernel is an exceptional numerically trivial divisor, and the negativity lemma applied to it and its negative makes it zero. After a fixed iterate, every component of E is fixed and its pullback multiplicity is q^k, by the finite projection formula. Residue functoriality therefore makes this iterate multiplication by q^k on the weight-two part. This proves the assertion. □

Proposition 14.7 (Unipotent monodromy of an invariant flat bundle)**.** *Let $f : X \to X$ be a polarized endomorphism of a normal projective complex variety. Let $(\mathcal{E}, \nabla)$ be a rational integrable connection with an isomorphism $f^*(\mathcal{E}, \nabla) \simeq (\mathcal{E}, \nabla)$. Suppose there is a Poincaré map $\theta : \mathbb{C}^n \to X$, locally finite everywhere, such that $f^r\theta = \theta P$, where P is an automorphism of $\mathbb{C}^n$ whose inverse iterates converge locally uniformly to 0. Suppose the connection is holomorphic near $\theta(0)$. Then, after a finite dynamical cover of X and an iterate, the monodromy group of the pulled-back connection has connected unipotent Zariski closure. In particular, the pulled-back rational flat bundle has a nonzero rational horizontal section.*

Proof. Replace f by the indicated return. A horizontal basis near $\theta(0)$ transforms under the pullback isomorphism by a constant invertible matrix. Transporting it to the increasing domains $P^j(B)$ gives a meromorphic horizontal basis for $\theta^*(\mathcal{E}, \nabla)$ on all of $\mathbb{C}^n$. The bases agree on overlaps by their construction. Thus the pullback local system has trivial monodromy.

Choose a smooth dense open $U \subset X$ on which the connection is holomorphic, with monodromy $\rho : \pi_1(U) \to \mathrm{GL}(V)$ and image Γ. The group Γ is finitely generated. The dynamical isomorphism induces conjugation by a matrix C on its image, giving an injective endomorphism

$\alpha(\gamma) = C\gamma C^{-1}$ of Γ with finite-index image. For the finite-index assertion, remove the critical values and the images of the singular locus, and use the resulting finite étale covering; removing the additional connection divisors induces surjections on fundamental groups.

For a prime divisor D in $X \setminus U$, let T_D denote the image of a small transverse meridian, defined up to conjugacy. Let E_∞ be the reduced union of those D for which T_D has infinite order. This divisor is totally invariant. Indeed, at a prime D' mapping to D with ramification index e, the connection isomorphism identifies the local monodromy at D' with a conjugate of T_D^e; finite and infinite order are unchanged under taking positive powers.

The image of θ avoids E_∞. If it met one component, local finiteness and openness would give a general smooth point of that component in the image. A transverse local branch above it would show that $T_D^e = 1$ for some positive integer e, because the pulled-back local system is trivial. This is a contradiction.

By Selberg's lemma choose a torsion-free subgroup of Γ of finite index d. The intersection of all subgroups of index at most d is a finite-index normal subgroup Γ_0 contained in it and preserved by α. Indeed, $[\Gamma : \alpha^{-1}H] \leq [\Gamma : H] \leq d$ for each such H, so $\Gamma_0 \subset \alpha^{-1}H$ for every H and hence $\alpha(\Gamma_0) \subset \Gamma_0$. The corresponding finite cover of U algebraizes and extends by normalization to a finite dynamical cover $p : X_1 \to X$. The lifted endomorphism is polarized by the pullback of an ample polarization.

The finite cover may moreover be chosen so that an iterate of its lift commutes with every deck transformation. Replace Γ_0 by its inverse saturation

$$\Gamma_1 = \bigcup_{j \geq 0} \alpha^{-j}(\Gamma_0).$$

The subgroups in this union form an increasing chain of finite index, so the chain stabilizes. Hence $\Gamma_1 = \alpha^{-r}(\Gamma_0)$ for some r, and $\alpha^{-1}(\Gamma_1) = \Gamma_1$. It is normal and torsion free: a torsion element would inject under α^r into the torsion-free group Γ_0. The map induced by α on the finite group Γ/Γ_1 is therefore an automorphism. After an iterate this automorphism is the identity. Use this saturated subgroup, denoted again by Γ_0, for the cover.

The map θ lifts holomorphically to $\theta_1 : \mathbb{C}^n \to X_1$. On $\theta^{-1}(U)$ its lift exists because the pulled-back monodromy is trivial. Its graph extends across the omitted analytic set: normalize the finite fiber product $\mathbb{C}^n \times_X X_1$ and take the component selected by this lift. That component is finite and birational over the normal space $\mathbb{C}^n$, hence isomorphic to

it. The lift remains locally finite, since its composition with p is locally finite.

Put $Y = (X_1 \backslash p^{-1}E_\infty)_{\mathrm{reg}}$. The restricted monodromy factors through $\pi_1(Y)$. Indeed, at each of the remaining boundary prime divisors its inertia is both finite and contained in the torsion-free group Γ_0, and is therefore trivial. The kernel of the surjection from the fundamental group of the original smooth open to $\pi_1(Y)$ is normally generated by these meridians. Its image is still Γ_0.

The map θ_1 takes values in $X_1 \setminus p^{-1}E_\infty$. Lemma 14.4 makes Y h-special. Theorem A of [CDY25] therefore makes Γ_0 virtually nilpotent. The finite-index intersection construction, repeated with a nilpotent subgroup and then with the identity component of its Zariski closure, gives a compatible cover whose monodromy closure is a connected nilpotent linear algebraic group $G = U_G \times T$. At these later steps perform all finite-index intersections inside the original monodromy group, and use inverse saturation as well. Nilpotence is preserved, since the saturated group injects into the original nilpotent subgroup under a power of α. Connectedness of the Zariski closure is preserved because this injected image has finite index, and conjugation by the transport matrix preserves the identity component of the original monodromy closure. Thus all covers can be taken Galois with commuting deck action after an iterate.

Conjugation by C normalizes G and permutes the finitely many T-weights of its faithful matrix representation. A fixed power fixes every such character. The character restricted to the monodromy image factors through the abelianization of $\pi_1(Y)$ and is fixed by the corresponding dynamical action. Lemma 14.6 shows that this action minus the identity is invertible on rational first homology. Thus every such fixed character has finite image. Its extension to the connected group G is trivial by Zariski density. Faithfulness now gives $T = 1$.

A unipotent group acting on the nonzero space V has a nonzero fixed vector. It gives a single-valued horizontal section. The connection is regular singular by Proposition 10.1. Its horizontal sections have moderate growth on a log resolution; the single-valued section is consequently meromorphic there and rational on the projective model. □

Corollary 14.8 (A finite-type Killing connection supplies global fields)**.** *In the preceding proposition suppose that the horizontal sections are the local Killing fields of a rational finite-type geometric structure, through a rational evaluation map $\mathcal{E} \dashrightarrow T_X$, and that evaluation is injective on germs of horizontal sections. Suppose the local Killing algebra is nonzero. After a finite compatible cover there is*

a nonzero finite-dimensional space of rational vector fields V_0 preserved by dynamical transport. Every member of V_0 is regular on the whole normal projective variety.

Proof. Take the full fixed space of the connected unipotent monodromy closure. It is nonzero and is preserved by the transport matrix, which normalizes that closure. Its horizontal sections are rational by the proposition; rational evaluation gives rational vector fields. Injectivity on horizontal germs makes the resulting space nonzero. For a basis W_i one has $Df(W_i) = \sum_j c_{ij} W_j \circ f$ with a constant invertible matrix (c_{ij}). The maximum of their divisorial pole orders therefore satisfies $f^*P \leq P$. Polarization kills P, and normality extends each resulting derivation across codimension two. Thus the fields are regular everywhere. □

14.4. **Semi-abelian rational factors of polarized maps.**

Lemma 14.9 (Scalar torus action)**.** *Let $f : X \to X$ be a q-polarized endomorphism of a normal projective variety. Suppose there is a dominant rational map $\pi : X \dashrightarrow G$ to a semi-abelian variety and an affine isogeny $a : G \to G$ such that $\pi f = a\pi$. Write*

$$1 \longrightarrow T \longrightarrow G \xrightarrow{p} A \longrightarrow 0$$

for its torus and abelian parts. Then an iterate a^k is equivariant with multiplication by $Q = q^k$ on T, and its induced affine endomorphism a_A^k on A is Q-polarized. The extension of A by T is torsion. Moreover, G has a normal projective compactification on which a^k extends to a Q-polarized endomorphism. A finite group of affine automorphisms of G through which a^k descends can be included in this compactification.

Proof. The map a_A is a surjective endomorphism of the projective variety A. By [MZ18, Theorem 1.3], applied to the rational map $p\pi$, it is q-polarized. Take the normalization X_1 of the closure of the graph of $p\pi$ in $X \times A$. The product map $f \times a_A$ preserves this graph and induces a finite endomorphism $f_1 : X_1 \to X_1$. The sum of the pullbacks of polarizations on X and A is ample on X_1 and has weight q. We may therefore replace X by X_1 and assume that $p\pi : X \to A$ is a morphism.

Let M be the integral matrix of the homomorphism induced by a on the torus. Trivialize the T-torsor $G \to A$ over the function field of A, and choose a basis of its character lattice. Pullback by π gives rational functions $z_1, \ldots, z_t$ on X, with

$$z_i \circ f = c_i \prod_j z_j^{M_{ij}}, \qquad c_i \in \mathbb{C}(A)^*.$$

For a horizontal prime divisor D on X, put $v_D = (\operatorname{ord}_D z_i)_i$. Let $\mathcal{D}$ be the finite set of horizontal prime divisors for which $v_D \neq 0$. The displayed equations, and the invertibility of M over $\mathbb{Q}$, imply

$$f^{-1}\mathcal{D} = \mathcal{D}.$$

Indeed, horizontal divisors remain horizontal under f, vertical divisors remain vertical under inverse image, and the factors c_i have zero order along every horizontal divisor.

After an iterate, every member of $\mathcal{D}$ is totally invariant. If $f^{k*}D = e_D D$, intersection with the $(n-1)$st power of an ample polarization gives $e_D = q^k$. The valuation equations therefore read

$$M^k v_D = q^k v_D \qquad (D \in \mathcal{D}).$$

These vectors span $\mathbb{Q}^t$. Otherwise a nonzero integral character χ would have zero divisor after pullback to the projective generic fiber of $X \to A$. It would be algebraic over $\mathbb{C}(A)$, whereas dominance of π makes every nonzero torus character transcendental over that field. Thus $M^k = q^k I$.

Replace a by this iterate and write $Q = q^k$. Translating the origin in A makes a_A an isogeny: its linear part minus the identity is an isogeny, since the eigenvalues of a Q-polarized abelian isogeny have modulus $\sqrt{Q}$. The extension G is given by classes $L_\chi \in \operatorname{Pic}^0(A)$, one for each torus character. Compatibility with a gives

$$a_A^* L_\chi = L_\chi^{\otimes Q}.$$

The endomorphism $a_A^* - [Q]$ of $\operatorname{Pic}^0(A)$ is an isogeny, again because its analytic eigenvalues have modulus $\sqrt{Q}$ before subtracting Q. Hence every L_χ is torsion. A common multiplication isogeny of A splits the extension.

Choose a complete projective toric variety V_Σ for T. The associated bundle

$$\overline{G} = G \times^T V_\Sigma \longrightarrow A$$

is normal and projective. The scalar map on the torus preserves every fan, so a extends to a finite endomorphism of $\overline{G}$. Pull back by a multiplication isogeny $[m] : A \to A$ which kills all the extension classes. The resulting cover is $A \times V_\Sigma$. The affine map a_A lifts through $[m]$ after choosing a preimage of its translation part; the lift is Q-polarized by pulling back a polarization on A. The induced map on the split torus bundle has the form $(b, t) \mapsto (a'_A(b), c(b)t^Q)$. The morphism $c : A \to T$ is constant because A is projective. Consequently this map extends to a Q-polarized product map on $A \times V_\Sigma$. Its finite map to $\overline{G}$ is equivariant, so a on $\overline{G}$ is Q-polarized by [MZ18, Theorem 1.3].

For the final assertion, the finite group induces a finite group on the cocharacter lattice. Choose a projective complete fan invariant under this group, by taking a common refinement of the translates of a projective complete fan. Translations in G, as well as the affine group automorphisms preserving this fan, extend to $\overline{G}$. Thus the finite quotient is projective, a descends to it, and the descended map is polarized by [MZ18, Theorem 1.3]. Its open G-quotient has the finite semi-abelian presentation required for exceptionality. □

14.5. **A unitary flat socle on a normal variety.**

Lemma 14.10 (An invariant flat subsheaf)**.** *Let $f : X \to X$ be a q-polarized endomorphism of a normal projective complex variety, and let $\mathcal{E} \neq 0$ be reflexive. If $f^{[*]}\mathcal{E} \simeq \mathcal{E}$, there is a nonzero saturated reflexive subsheaf $\mathcal{S} \subset \mathcal{E}$ with the following properties:*

1. *$\mathcal{S}|_{X_{\mathrm{reg}}}$ is a Hermitian flat vector bundle;*
2. *the given dynamical isomorphism restricts to $f^{[*]}\mathcal{S} \simeq \mathcal{S}$ and preserves its flat connection;*
3. *the flat connection is rational and has regular singularities.*

Proof. Choose an ample Cartier divisor H with $f^*H \sim qH$. By [MZ24, Theorem 4.2], both $\mathcal{E}$ and $\mathcal{E}^*$ are weakly positive. Taking determinants and intersecting with H^{n-1} gives $\deg_H \mathcal{E} = 0$. Every torsion-free quotient of a weakly positive sheaf has nonnegative degree. Thus $\mathcal{E}$ is H-semistable of slope zero.

Let $\mathcal{S}$ be its socle: the unique maximal saturated polystable subsheaf of slope zero. The socle is nonzero and reflexive; write

$$\mathcal{S} = \bigoplus_{i=1}^{s} \mathcal{S}_i$$

with each $\mathcal{S}_i$ stable of slope zero. The existence and uniqueness of this saturated socle hold on normal projective varieties; see [Wei23, Lemma 1.3.15]. The generically surjective map $\mathcal{E}^* \to \mathcal{S}_i^*$ makes $\mathcal{S}_i^*$ weakly positive. Since its degree is zero, its first Chern class vanishes as a functional on closed $(n-1, n-1)$-forms. Precisely, apply [CDM26, Proposition 4.7] to its weakly positive rank-one determinant. The zero degree implies that the first Chern class of the resolved determinant is represented by an exceptional divisor; its pushforward is therefore the zero functional. The stable flatness theorem [CDM26, Theorem 1.2], whose ambient hypothesis is normality, now shows that every $\mathcal{S}_i^*|_{X_{\mathrm{reg}}}$ is Hermitian flat. Consequently $\mathcal{S}|_{X_{\mathrm{reg}}}$ is Hermitian flat.

Finite separable pullback in characteristic zero preserves polystability for reflexive sheaves on normal projective varieties, with the pulled-back polarization; see [Wei24, Lemma 2.1(iv)]. Since $f^*H \sim qH$, the sheaf $f^{[*]}\mathcal{S}$ is polystable of H-slope zero. Its image under the given dynamical isomorphism is therefore contained in $\mathcal{S}$. Both have the same rank and degree. The inclusion is an isomorphism in codimension one, hence everywhere by reflexivity. Thus

$$f^{[*]}\mathcal{S} \simeq \mathcal{S}.$$

We next check compatibility with the flat connection. Put $U = X_{\mathrm{reg}} \cap f^{-1}(X_{\mathrm{reg}})$. Its complement in X_{reg} has codimension at least two. The pullback of the unitary local system of $\mathcal{S}$ on U extends uniquely to a unitary local system on X_{reg}: removing a complex analytic subset of codimension at least two from a complex manifold does not change its fundamental group. Its holomorphic bundle is the restriction of $f^{[*]}\mathcal{S}$, by uniqueness of reflexive extension.

If A is a global holomorphic morphism between two Hermitian flat bundles on X_{reg}, its squared norm is plurisubharmonic. It extends across $X \setminus X_{\mathrm{reg}}$ by normal Hartogs extension for plurisubharmonic functions [CDM26, Theorem 2.1]. Since X is compact, that extension is constant. In local unitary flat frames the Bochner identity then gives $\nabla A = 0$. Apply this to the dynamical isomorphism to obtain preservation of the flat connection.

Finally this connection is algebraic. Indeed, subtract a rational connection on $\mathcal{S}$; the difference is a holomorphic section of $\mathcal{H}om(\mathcal{S}, \mathcal{S} \otimes \Omega^1)$ away from the fixed polar divisor of the rational connection. After twisting by that divisor it extends across codimension two as a section of a coherent reflexive sheaf on X. Projective GAGA makes it algebraic. Regular singularity follows from the normal compactification criterion for integrable connections: the connection has no pole at any codimension-one point of X; see [And07, Theorem 1.1.1]. □

Theorem 14.11 (An invariant foliation gives a factor)**.** *Let $f : X \to X$ be a q-polarized endomorphism of a normal projective complex variety. Suppose that f preserves a nonzero proper rational foliation $\mathcal{F}$, and that f admits the locally finite Poincaré map used in Proposition 14.7. Then either f has a connected rational factor of dimension strictly between zero and $\dim X$, admitting a normal projective q-polarized model, or f is exceptional.*

Proof. The algebraic reductions in Proposition 14.3 already give a factor if the very general leaf has proper Zariski closure or if $\mathcal{F}$ has a

nonzero algebraic part. We may therefore assume that $\mathcal{F}$ is purely transcendental and that its very general leaf is Zariski dense.

We first show that

$$K_{\mathcal{F}} \cdot H^{n-1} \geq 0.$$

Let $\rho : Z \to X$ be a projective resolution, let $\widetilde{\mathcal{F}}$ be the induced foliation, and put $\alpha = (\rho^* H)^{n-1}$. If its maximal Harder–Narasimhan slope were positive, the maximal destabilizing subsheaf $\mathcal{A} \subset \widetilde{\mathcal{F}}$ would be a foliation. Indeed,

$$2\mu_{\alpha,\min}(\mathcal{A}) > \mu_{\alpha,\max}(\widetilde{\mathcal{F}}/\mathcal{A}),$$

and the slope inequality annihilates the bracket morphism $\wedge^2\mathcal{A} \to \widetilde{\mathcal{F}}/\mathcal{A}$. The positive-slope algebraicity theorem [CP19, Theorem 1.1] makes $\mathcal{A}$ algebraically integrable with rationally connected leaves, contradicting pure transcendence. Thus every Harder–Narasimhan slope is nonpositive, and the displayed inequality follows from

$$K_{\widetilde{\mathcal{F}}} \cdot \alpha = K_{\mathcal{F}} \cdot H^{n-1}.$$

On the other hand, the foliated ramification identity is

$$K_{\mathcal{F}} = f^* K_{\mathcal{F}} + R_{\mathcal{F}}, \qquad R_{\mathcal{F}} \geq 0.$$

It gives $(1-q)K_{\mathcal{F}} \cdot H^{n-1} = R_{\mathcal{F}} \cdot H^{n-1}$, so $R_{\mathcal{F}} = 0$. The differential therefore induces a reflexive isomorphism

$$f^{[*]}\mathcal{F} \simeq \mathcal{F}.$$

Apply Lemma 14.10 to obtain a nonzero invariant Hermitian flat subsheaf $\mathcal{S} \subset \mathcal{F} \subset T_X$.

Proposition 14.7 makes its monodromy virtually unipotent. Since that monodromy is unitary, it is finite. Let

$$\rho : \pi_1(X_{\mathrm{reg}}) \longrightarrow U(s)$$

be its representation and take the finite Galois cover of X_{reg} defined by $\ker \rho$. Its normalization over X is a finite Galois quasi-étale cover $p : X' \to X$. This cover is dynamically compatible. Indeed, after choosing a base point and a path, the flat dynamical isomorphism gives

$$\rho(f_*\gamma) = C\rho(\gamma)C^{-1}.$$

Here f_* is defined using $X_{\mathrm{reg}} \cap f^{-1}(X_{\mathrm{reg}})$, whose inclusion in X_{reg} induces an isomorphism of fundamental groups. Thus $f_*^{-1}(\ker \rho) = \ker \rho$, so f lifts to an endomorphism $f' : X' \to X'$. Normality extends the lift from the common smooth locus, and p^*H is a q-polarization of f'.

The pulled-back flat bundle is trivial, with a nonzero finite-dimensional space V of rational horizontal sections. Their evaluations are regular vector fields on X': on the étale smooth locus this follows

from $\mathcal{S} \subset T_X$, and normality extends the derivations across codimension two. Dynamical transport acts on V by an invertible constant matrix. The deck group Γ also preserves V, since it preserves the pulled-back connection. The Lie algebra generated by V is finite-dimensional as a subspace of $H^0(X', T_{X'})$, and is preserved both by dynamical transport and by Γ.

By Proposition 14.5, the connected algebraic group generated by these flows is a positive-dimensional semiabelian group G, normalized by f' through an isogeny and by Γ through automorphisms. If G has an open orbit, that orbit is isomorphic to G, because the action is effective and G is commutative. The lifted map is an affine isogeny there. Lemma 14.9 and birational invariance make f' exceptional; exceptionality descends under the finite equivariant map p by [JSX26, Theorem 3.9].

If the general G-orbit is proper, put $K = \mathbb{C}(X')^G$. The deck group acts on K and

$$L = K^\Gamma \subset \mathbb{C}(X')^\Gamma = \mathbb{C}(X).$$

This field has transcendence degree strictly between zero and n. It is relatively algebraically closed in $\mathbb{C}(X)$: an element of $\mathbb{C}(X)$ algebraic over L is algebraic over K, hence belongs to K by connectedness of the Rosenlicht quotient; being deck-invariant, it belongs to L. Conjugation by C induces an automorphism β of the finite deck group, with $f'\delta = \beta(\delta)f'$. Together with the isogeny normalization of G, this gives $f^*L \subset L$. Thus L defines a connected rational factor of f, and Proposition 14.2 provides its normal projective q-polarized model. □

14.6. **Finite webs and descent of the transcendental branch.**

Lemma 14.12 (The splitting cover of an invariant web)**.** *Let $f : X \to X$ be a polarized endomorphism of a normal projective variety. Suppose that, on a dense smooth open, a finite unordered set of distinct integrable distributions is algebraic and is preserved by Df. There is a finite Galois cover $p : Y \to X$ on which the members become rational foliations. The map f lifts to a polarized endomorphism $F : Y \to Y$. Its action on the Galois group Γ is an automorphism β, in the sense that*

$$F\gamma = \beta(\gamma)F.$$

After an iterate, each member foliation is preserved.

Proof. On an open where f is étale and the members are distinct, Df acts bijectively on the finite set of members. Take the finite étale cover of orderings and a connected component of its splitting cover. Its function field is the splitting field of the algebraic coordinates of the

members, and its normalization over X is finite Galois. The derivative transports an ordering bijectively to an ordering over $f(x)$; applying its inverse recovers the original ordering. If the selected connected component is transported to another component, compose transport with a constant permutation of the labels taking that component back to the chosen one. Thus the cover is identified with its pullback under f on the common generic open. This identification lifts f and acts bijectively on the finite deck group. Normalization extends the lift across the boundary. The pullback of a polarization of X polarizes F with the same weight. The lift permutes a finite set of foliations, so an iterate fixes each one. □

Proposition 14.13 (The transcendental member of a finite web)**.** *In the preceding setting assume that one member foliation on Y is purely transcendental with Zariski dense very general leaves. Assume the locally finite Poincaré hypothesis used in Theorem 14.11 on the relevant finite dynamical cover. Then the original map f has a proper connected rational polarized factor, or f is exceptional.*

Proof. Take the finite orbit of the chosen member under both the deck group and dynamical transport. Every member in this orbit is purely transcendental with dense very general leaves; these properties are preserved by finite pullback and by deck transformations. Write the members as $\mathcal{F}_i$, with F inducing a permutation σ of their indices. The slope argument of Theorem 14.11 gives $\deg_H K_{\mathcal{F}_i} \geq 0$, whereas single-step ramification gives

$$K_{\mathcal{F}_i} = F^* K_{\mathcal{F}_{\sigma(i)}} + R_i, \qquad R_i \geq 0.$$

Iterating this identity around each finite cycle shows that all these degrees and all R_i vanish. Thus the differential identifies $\mathcal{F}_i$ with $F^{[*]}\mathcal{F}_{\sigma(i)}$.

Let $\mathcal{S}_i$ be the nonzero unitary flat socle of $\mathcal{F}_i$. Pullback polystability puts $F^{[*]}\mathcal{S}_{\sigma(i)}$ inside $\mathcal{S}_i$. The resulting rank inequalities around a cycle are all equalities, so these inclusions are isomorphisms. Normal Hartogs and the Bochner argument show that they preserve the flat connections. The deck group also permutes these connections. Applying the monodromy argument to an iterate fixing a member shows that every member socle has finite holonomy.

Take the compositum of their finite holonomy covers over Y. It is finite and Galois over X: the deck group of Y/X permutes the finite set of conjugate covers, and the holonomy covers themselves are Galois. Call its normalization Z.

This common cover is dynamical. For each socle the holonomy kernel is exactly inverse-invariant under the induced homomorphism of

fundamental groups. The endomorphism F permutes these kernels through the deck automorphism β. Their finite intersection is therefore exactly inverse-invariant. Thus F lifts to a polarized endomorphism $\widetilde{F} : Z \to Z$, and its action on the full deck group of Z/X is an automorphism.

The map $Z \to Y$ is quasi-étale. The horizontal sections of all conjugate socles therefore give regular vector fields on Z, by extension across codimension two. Their span is nonzero, finite-dimensional, and preserved by both dynamical transport and the full deck group. The Lie algebra it generates has the same invariance properties. Its connected algebraic flow closure is semiabelian by Proposition 14.5. If it has an open orbit, $\widetilde{F}$ and hence f are exceptional. Otherwise its Rosenlicht field, followed by deck invariants, is a proper relatively algebraically closed subfield of $\mathbb{C}(X)$ preserved by f^*. The invariant-field argument at the end of Theorem 14.11 gives the required connected rational factor, and Chow regularization gives its polarized projective model. □

Lemma 14.14 (The algebraic branch of a finite web)**.** *Let $p : Y \to X$ be the finite Galois dynamical splitting cover of Lemma 14.12. Suppose that a member foliation supplies a proper connected rational factor of Y. After an iterate let*

$$\pi_i : Y \dashrightarrow B_i, \qquad 1 \le i \le m,$$

be its distinct translates under the deck group and dynamical transport, with polarized endomorphisms g_i on normal projective models of the B_i. Put

$$K = \mathbb{C}\big(\pi_1^*\mathbb{C}(B_1), \dots, \pi_m^*\mathbb{C}(B_m)\big) \subset \mathbb{C}(Y).$$

Then precisely one of the following constructions applies:

(1) *If $\operatorname{trdeg} K < \dim X$, its relative algebraic closure K^{a} in $\mathbb{C}(Y)$, followed by deck invariants, gives a proper connected rational polarized factor of X.*
(2) *If $\operatorname{trdeg} K = \dim X$, the product map $(\pi_1, \dots, \pi_m)$ is generically finite onto its image $T \subset \prod_i B_i$. The normalization of T is projective and carries a polarized endomorphism induced by $\prod_i g_i$. Thus, for that common iterate, X has a finite rational correspondence with this invariant subvariety of a product of lower-dimensional polarized systems.*

Proof. The finite set of fields is permuted by the deck group and by single-step dynamical transport; the individual factor maps are endomorphisms after the stated iterate. Hence K and its relative algebraic closure K^{a} are invariant under both. The field $(K^{\mathrm{a}})^{\Gamma}$ is relatively algebraically closed in $\mathbb{C}(X)$ by the same field argument as before. In

the first case its transcendence degree is positive and strictly less than $\dim X$, proving the assertion.

In the second case the image has dimension $\dim X$ and the product map has finite generic fiber. Take its reduced projective closure and normalize. The product of the polarized maps is finite and preserves this closure, and so lifts to its normalization. The restriction of the sum of the product polarizations is ample and is multiplied by the common polarizing weight. The two generically finite rational arrows from Y to X and to T give the asserted correspondence. Each arrow can be made finite onto a dense target open after normalizing the target in the source function field and shrinking. □

14.7. **Spreading finite algebraic presentations.**

Lemma 14.15 (Countable incidence spreading)**.** *Let S be an irreducible algebraic curve over $\mathbb{C}$, and let $f : \mathcal{X} \to \mathcal{X}$ be an algebraic family of polarized endomorphisms of normal projective varieties over S. Consider any of the following finite presentations of a fiber:*

(1) *a finite dynamical cover, optionally with a specified finite Galois group and deck-equivariance;*
(2) *a connected rational factor with a normal projective polarized model;*
(3) *a finite equivariant presentation by an AT-system, including its finite affine group action and chosen projective model;*
(4) *a finite rational correspondence with a projective invariant subvariety of a product of finitely many lower-dimensional polarized systems.*

If such a presentation exists for uncountably many $s \in S(\mathbb{C})$, then, after a finite dominant base change $S' \to S$ and replacement of S' by a nonempty open, one such presentation exists algebraically over S'. No uniform bound on the degrees of the fiberwise covers or on their Hilbert polynomials is required. The same conclusion holds if a presentation exists at a point geometrically generic over a countable field of definition of the original family.

Proof. Choose projective embeddings and graph embeddings for all the varieties and maps in a presentation. There are countably many choices of dimensions, Hilbert polynomials, degrees, finite groups, integers specifying the polarization, and numbers of factors. For any fixed choice, the graphs belong to finitely many Hilbert schemes of finite type. Incidence imposes the commuting identities, group laws, and equivariance identities by algebraic equations. After the usual flattening stratifications, the required normality, dominance, finiteness,

and geometric integrality of generic fibers are constructible conditions. Thus their loci are finite unions of locally closed parameter spaces. For rational maps, one records their graphs and dense domains of definition; for finite maps on dense opens, one also records the excluded closed sets. All these are finite algebraic data.

For an AT-presentation, record the polarized abelian variety with its group law, the torus rank, the torsor and extension data, and the affine endomorphisms and finite group action. These data can likewise be recorded by graphs on a finite affine covering. In the scalar torus presentation of Lemma 14.9, one may instead record a finite cover by an abelian variety times a fixed toric compactification; the toric fans form a countable set. Either encoding gives countably many finite-type parameter spaces.

Consequently all presentations under consideration are parameterized by a countable union

$$\coprod_{j\geq 1} P_j \longrightarrow S$$

of finite-type incidence spaces. If no P_j dominates S, each constructible image is a finite set, since S is a curve. Their union is countable. This contradicts the hypothesis. Hence some irreducible component P dominates S.

The generic fiber P_η is a nonempty finite-type scheme over $\mathbb{C}(S)$. A closed point of it has residue field a finite extension $K'/\mathbb{C}(S)$. Normalize S in K'. The chosen point gives a rational section of the pulled-back incidence space, which becomes a morphism after shrinking the resulting curve S'. Pulling back its universal data gives the desired algebraic presentation. Shrinking once more makes all the constructible conditions valid on the whole remaining base.

For the last assertion, enlarge the chosen countable field to its algebraic closure k_0 and define the family and all incidence spaces over k_0. If s_0 is geometrically generic over k_0, a constructible k_0-image containing s_0 is dense. Choose a dominant component over k_0 and a closed point of its generic fiber over $k_0(S)$, rather than over $\mathbb{C}(S)$. Its residue field is finite over $k_0(S)$. Normalizing the k_0-model of S in that field and then extending scalars to $\mathbb{C}$ gives the required finite base change defined over k_0. Every lift of s_0 is again geometrically generic over k_0, so none is removed by the nonempty k_0-opens used to spread the data. □

Corollary 14.16 (Spreading with a fixed reference parameter)**.** *Fix a countable algebraically closed field $k_0 \subset \mathbb{C}$ over which the parameter*

curve, the polarized family, and all previously constructed algebraic diagrams are defined. Fix $s_0 \in S(\mathbb{C})$ geometrically generic over k_0. If one of the finite presentations in Lemma 14.15 exists for the fiber at s_0, it can be spread after a finite base change defined over k_0, while retaining a lift of the same reference parameter. Every further finite dominant base change and nonempty open restriction defined over k_0 again retains every chosen lift of this reference parameter. The presentation spread need not coincide with the initially selected presentation of that fiber; its source dynamical system at the reference parameter remains the prescribed one.

Proof. All the incidence spaces are defined over k_0. The point representing the chosen fiberwise presentation lies in a component whose constructible image contains s_0 and hence dominates S. The closed generic-fiber point and normalization in the preceding proof are chosen over k_0. If s_0' is any point over s_0 in a finite base change $S' \to S$, then

$$\operatorname{trdeg}_{k_0} k_0(s_0') = \operatorname{trdeg}_{k_0} k_0(s_0) = 1.$$

Thus s_0' lies in every nonempty k_0-open of its irreducible parameter curve. Repeating this argument proves the assertion for all subsequent steps, without changing the reference fiber. □

14.8. **Algebraic members of a finite invariant web.**

Lemma 14.17 (Invariant subvarieties of a fixed polarized system are constant)**.** *Let (W, h) be a polarized projective system over $\mathbb{C}$, and let $K/\mathbb{C}$ be a field extension. Every geometrically integral closed subvariety $Z \subset W_K$ with $h_K(Z) = Z$ is defined over $\mathbb{C}$. Consequently an algebraic family of reduced invariant subvarieties of one fixed polarized system is constant after restriction and, if its components must be marked, finite parameter extension.*

Proof. The restriction $h_K|_Z$ is polarized by the restriction of the ambient ample polarization. Its periodic points over $\overline{K}$ are Zariski dense by Fakhruddin's periodic-point theorem [Fak03, Theorem 5.1].

For every $m \geq 1$, the fixed-point scheme of h^m is zero-dimensional. Indeed a positive-dimensional fixed locus would contain a curve on which an ample polarization is both preserved and multiplied by $q^m > 1$, a contradiction. That fixed-point scheme is defined over the algebraically closed field $\mathbb{C}$. Thus all its geometric points over $\overline{K}$ are constant $\mathbb{C}$-points. In particular the dense set of periodic points of $h_K|_Z$ consists of $\mathbb{C}$-points of W.

The Zariski closure, after any field extension, of a set of $\mathbb{C}$-points is the base change of its closure over $\mathbb{C}$. For clarity, express the finitely

many coefficients of any polynomial vanishing on the set in a basis linearly independent over $\mathbb{C}$; evaluation at a constant point then shows that every coefficient polynomial over $\mathbb{C}$ vanishes. Applying this observation to the dense set above shows that Z is the base change of a subvariety over $\mathbb{C}$. The family assertion follows by applying the result to its geometric generic components and then spreading the equality over a parameter open. □

15. Stable periodic fibers and varying covers

15.1. **The relative equilibrium measure.** The relative equilibrium-measure identity supplies the Julia-set inclusion used for the spectral closure under periodic-fiber returns. For fibered endomorphisms of projective space, see [DT21].

Lemma 15.1 (Continuous fiber measures)**.** *Let $\pi : X \to B$ be a surjective morphism of normal projective varieties, with $\dim B = r$ and $\dim X = m + r$. Let $F : X \to X$ and $g : B \to B$ be q-polarized endomorphisms, with $\pi F = g\pi$. Choose polarizations L and M, and denote their canonical currents by T and S. Let $B^0 \subset B_{\mathrm{reg}}$ be an open set over which π is flat with normal geometrically integral fibers. Put $d = (L^m \cdot X_b)$, for $b \in B^0$, and set*

$$\nu_b = d^{-1}(T|_{X_b})^m.$$

Then $b \mapsto \nu_b$, viewed as a family of measures on X, is weakly continuous on B^0, and

$$\mu_F = \int_{B^0} \nu_b \, d\mu_g(b), \qquad \mu_F = \frac{T^m \wedge \pi^* S^r}{L^m \cdot \pi^* M^r}. \tag{15.1}$$

For every $b \in B^0 \cap \operatorname{supp} \mu_g$,

$$\operatorname{supp} \nu_b \subset \operatorname{supp} \mu_F. \tag{15.2}$$

If $g^n(b) = b$, then ν_b is the equilibrium measure of $F^n|_{X_b}$.

Proof. Canonical currents of polarized endomorphisms have continuous local potentials. The normalized top wedge of any polarization of F is its equilibrium measure. This assertion also holds on normal varieties: on a projective resolution, normalized pullbacks of a smooth probability measure converge to the equilibrium measure, by [DNT15, Theorem 1.2]; the exceptional and indeterminacy sets have measure zero. The uniform convergence of the canonical potentials identifies this limit with the normalized top wedge. In particular this measure is independent of the chosen polarization.

For every nonnegative rational t, the ample rational class $L + t\pi^*M$ is q-polarized and its canonical current is $T + t\pi^*S$. Consequently

$$(T + t\pi^*S)^{m+r} = (L + t\pi^*M)^{m+r}\mu_F.$$

Comparing the coefficients of t^r proves the second identity in (15.1). Notice that $L^m \cdot \pi^*M^r = d(M^r)$.

Flatness gives the continuity of the fiber cycles $b \mapsto [X_b]$. Wedge product with a fixed current having continuous local potentials preserves this continuity. Indeed, locally write $T = dd^c u$. If $R_j \to R$ are positive closed currents, then $uR_j \to uR$ because u is continuous and the masses are locally bounded. Hence

$$dd^c(uR_j) \longrightarrow dd^c(uR).$$

Iteration applies to $T^m \wedge [X_b]$; the local mass bounds follow from positivity and the Chern–Levine–Nirenberg inequality. This also proves the assertion on a normal analytic space by embedding it locally and using a continuous extension of u to define the products with currents supported on that space. Thus the probability measures ν_b depend weakly continuously on b.

On the submersion locus of π, Fubini's formula for currents with continuous potentials gives

$$T^m \wedge \pi^*S^r = \int_{B^0} (T|_{X_b})^m \, d(S^r)(b).$$

One may verify this in product coordinates by smooth approximation of the local potentials. The relative singular locus has zero mass for every fiber measure, and also for the measure on the left, which is a multiple of μ_F. The inverse image of $B \setminus B^0$ likewise has zero equilibrium measure. The displayed formula therefore gives the first identity in (15.1) on all of X.

Let $x \in \operatorname{supp} \nu_b$, with $b \in B^0 \cap \operatorname{supp} \mu_g$. In any neighborhood of x, choose a nonnegative continuous function χ with $\int \chi \, d\nu_b > 0$. Continuity makes this integral positive on a neighborhood of b, which has positive μ_g-measure. The disintegration formula gives $\int \chi \, d\mu_F > 0$. This proves (15.2) at every such b. Finally, on a periodic fiber the restricted current satisfies

$$(F^n|_{X_b})^*(T|_{X_b}) = q^n T|_{X_b}.$$

Uniqueness of the continuous canonical metric identifies its normalized top wedge with the equilibrium measure of the return. $\square$

15.2. **Finite dynamical covers of a fixed system.**

Lemma 15.2 (A finite dynamical cover of a fixed system)**.** *Let (Y, g) be a fixed polarized normal projective complex system. Let $p : \mathcal{Z} \to Y \times S$ be a finite surjective map from a family of normal projective varieties over a smooth complex curve, with a finite endomorphism F_Z satisfying*

$$pF_Z = (g \times \mathrm{id}_S)p.$$

Then the family $(\mathcal{Z}, F_Z)$ is isotrivial. After finite parameter extension and restriction, the whole diagram is constant up to isomorphism.

Proof. If Y is a point, a finite parameter extension trivializes the finite source and the assertion is immediate. Assume $\dim Y > 0$. Work first over $k = \mathbf{C}(S)$ and put $K = k(Y)$ and $L = k(Z)$. Let σ be the embedding induced by F_Z. The decreasing sequence of finite intermediate extensions

$$L_j = K\sigma^j(L)$$

stabilizes; equality at one step implies equality at all later steps by applying σ and adjoining K. Let L_0 denote its stable value. Then

$$\sigma(L_0) \subset L_0, \qquad K\sigma(L_0) = L_0.$$

Its normalization $p_0 : Z_0 \to Y_k$ carries a finite polarized lift. The embedding $\sigma^j : L \hookrightarrow L_0$ at a stable index gives a finite equivariant map $Z_0 \to Z$, covering g^j on Y. It is enough to prove that the Cartesian cover Z_0 is constant, and then apply Proposition 13.12.

Let D be the reduced branch divisor of p_0 over $\overline{k}$, let H polarize g, and choose N with $q^N > (D \cdot H^{\dim Y - 1})$. The equality $K\sigma(L_0) = L_0$ makes every dynamical pullback connected by Lemma 12.2. Applying Lemma 12.3 to the length-N constant path shows that D is supported on the fixed divisor $\mathrm{CritVal}(g^N)$.

Spread this cover over a parameter open. By Lemma 12.4, applied to the constant pair $(Y, \mathrm{CritVal}(g^N))$, it belongs to a fixed finite set of geometric cover classes. After finite parameter extension and restriction the cover is one fixed cover $p_0 : Z_0 \to Y$: the class is constant, and its finite isomorphism torsor is trivialized by a further finite extension. For each $z \in Z_0$, the image $F_{Z_0,s}(z)$ belongs to the fixed finite fiber over $g(p_0(z))$, so it is independent of s. The lifted system is therefore fixed. The finite equivariant map $Z_0 \to Z$ now gives the claimed isotriviality of Z.

Once Z has been trivialized, the maps $p_s : Z \to Y$ form a family of equivariant finite maps. The graphs are periodic subvarieties of the fixed polarized product system on $Z \times Y$. Lemma 14.17 makes them constant, proving the assertion about the whole diagram. $\square$

Corollary 15.3 (A finite rational factor cannot hide variation)**.** *A family of polarized normal projective systems admitting a dominant generically finite rational semiconjugacy to a fixed polarized projective system is isotrivial.*

Proof. Take the equivariant normalized graph and its Stein factorization over the fixed target. The finite Stein target is constant by Lemma 15.2. Its birational conjugacy with the graph and then with the original family transports a rational parameter-horizontal derivation. Lemma 4.1 extends that derivation and makes the original projective family isotrivial. □

15.3. **The group of all forward-compatible gauges.**

Proposition 15.4 (Finite-dimensional forward gauge torsor)**.** *Let (B, g) be a fixed normal projective polarized system. Suppose an algebraic family over $B \times S$ gives normal projective generic phase fibers $X_{s,b}$, ample line bundles $L_{s,b}$, and finite arrows*

$$a_{s,b} : X_{s,b} \longrightarrow X_{s,g(b)}, \qquad a_{s,b}^* L_{s,g(b)} \simeq L_{s,b}^{\otimes q}.$$

Assume that these polarized fibers have bounded projective embeddings and that a Zariski-dense set of good periodic base points has isotrivial return families. Fix a reference parameter s_0 which is good for every selected cycle. Over $\mathbf{C}(B)(S)$ there is a nonempty finite-type torsor I of source gauges which descend through every forward arrow composition and preserve the chosen line bundles at every target. Its group G is defined over $\mathbf{C}(B)$ by the reference system. First-arrow transport defines algebraic maps

$$I \longrightarrow g^* I, \qquad G \longrightarrow g^* G.$$

The latter has finite kernel and is an isogeny on identity components. The connected group G^0 is a torus.

Proof. For a length-n arrow composition $A_{s,b}^{(n)}$, let

$$R_{s,b,n} = \left(X_{s,b} \times_{X_{s,g^n(b)}} X_{s,b}\right)_{\mathrm{red}}.$$

A source isomorphism $C : X_{s_0,b} \to X_{s,b}$ descends to a unique target isomorphism C_n if and only if it identifies $R_{s_0,b,n}$ and $R_{s,b,n}$. Indeed equality of the generic reduced equivalence relations identifies the two target function fields inside the source field. The image of the pair of finite maps in the product of the targets is finite and birational over either target; normality makes both projections isomorphisms.

For each N, require this condition for $1 \leq n \leq N$ and also require C_n to preserve the specified target line bundle. These are finite-type algebraic incidence conditions on polarized isomorphisms and their graphs. Denote their source-gauge locus by I_N. Its corresponding reference automorphism group G_N is a closed subgroup of the finite-type linear group $\mathrm{Aut}_{L_{s_0,b}}(X_{s_0,b})$. To see closedness, preservation of each reduced relation is a closed Hilbert-scheme condition. Descent to its finite normal quotient gives an algebraic group homomorphism to the target automorphism group, and preservation of the target Picard class is closed. The descent homomorphism is algebraic by the graph construction; equivalently its graph is a finite-type algebraic subgroup with trivial first-projection kernel, which in characteristic zero identifies it with its image. Whenever I_N is nonempty, two of its points differ by exactly one element of G_N; it is therefore a torsor under G_N.

An isotrivial periodic return trivializes its full finite cycle, including its polarized intermediate arrows, by Lemmas 13.4 and 13.5. Thus every selected good cycle supplies a point of each finite incidence problem. If an intermediate phase is not in the normal-fiber open, use the normalization of the corresponding closed image of the starting fiber. These normalizations form a complete finite cycle. For any fixed N, discard the proper closed set where one of the first N phases is bad; at the retained phases the normalizations are the whole prescribed fibers. The image of the incidence locus in $B \times S$ is constructible and contains a parameter open above a Zariski-dense set of such base points. Its closure contains their full cylinders. Hence its image is dense and contains an open, proving that I_N is nonempty over the algebraic closure of the generic field.

The descending chain of closed subgroups G_N stabilizes in this finite-type group. After stabilization, the nested nonempty I_N are torsors under one group and hence are equal. Their common value I satisfies every forward condition and is nonempty over a finite extension of the generic field. Cancellation in the length-$(n+1)$ identities shows that the first target gauge belongs to g^*I. It gives the stated transport maps. The kernel of group transport lies in the finite deck group of the first arrow. The source and target identity components have the same dimension, because g is dominant. The image of a group homomorphism is closed, so transport is an isogeny on the identity components.

Choose one fixed very ample power of the polarizations, with projective embedding dimension $M-1$. If G^0 contained an additive subgroup, choose a nontrivial orbit and its closure C. A unipotent one-parameter subgroup in a projective representation of dimension M has polynomial

orbit coordinates of degree at most $M-1$. Consequently

$$0 < \deg C \leq M-1.$$

Every iterated transport carries this additive group isomorphically to another additive group, since its kernel is finite and characteristic zero has no nontrivial finite additive subgroup. The corresponding finite arrow maps C birationally to an orbit closure C_n with the same degree bound. Polarization gives $\deg C_n = q^n \deg C$, a contradiction for large n. Thus G^0 has no additive one-parameter subgroup. It is linear because it preserves an ample line bundle. A connected linear group with no additive subgroup is a torus. □

15.4. **Fixed-base periodic-return rigidity.**

Theorem 15.5 (Isotrivial periodic returns over a fixed base)**.** *Let $F : \mathcal{X} \to \mathcal{X}$ be an algebraic one-parameter family of polarized endomorphisms of normal projective varieties. Suppose it has an equivariant dominant rational map to one fixed normal projective polarized system (B, g), with geometrically integral generic phase fiber. If the return families on a Zariski-dense set of good periodic base points are isotrivial, then F is isotrivial.*

Proof. Normalize the equivariant graph of the rational factor. The sum of the source and base polarizations gives a finite polarized lift. Its fibers over a dense phase open are normal and geometrically integral, and one fixed sufficiently high power of the polarization gives bounded embeddings. Since the periodic points of a fixed polarized system are countable, choose a reference parameter outside the exceptional sets of all selected return trivializations. Proposition 15.4 applies.

Write $T = G^0$. Transporting the reference action through a gauge gives a torus action on the varying generic phase fiber. It is defined independently of the gauge as a form of T: changing the gauge conjugates the action by an element of G, which preserves its identity component. Its invariant field is finitely generated and relatively algebraically closed in the source field. The first-arrow isogeny preserves this field.

Take the Chow reduction of the resulting connected rational quotient. Meng–Zhang's Chow-reduction theorem [MZ22, Proposition 7.1 and Theorem 7.2] makes its induced rational dynamics a morphism; polarization descent [MZ18, Theorem 1.3] gives a polarized normal projective model. Spread this model over the parameter curve. On this quotient the gauge ambiguity is the finite group G/T. Proposition 13.10 gives a rational parameter-horizontal derivation, annihilating

$\mathbf{C}(B)$, which commutes with the quotient dynamics. Hence this quotient is isotrivial, with its trivialization over B. Denote its fixed model by (Y, h).

The generic source fiber over Y is a torus orbit closure. Indeed a faithful torus action on a projective variety has trivial generic stabilizer, as is seen from the characters in a faithful projective embedding; its rational quotient therefore has a dense torus orbit in the generic fiber. Pull the torus to $\mathbf{C}(Y)(S)$ and take its minimal Galois splitting field. Proposition 13.11 gives a compatible finite polarized cover of $Y \times S$ with Cartesian normalized pullback. After finite parameter extension, take an iterate preserving one geometric component. Lemma 15.2 makes this cover, including its lifted dynamics, constant after a finite parameter extension. Write it as (Y', h').

Normalize the original source in the compositum with this splitting field. Geometric integrality of the generic quotient fiber makes its function field a regular extension of $\mathbf{C}(Y)(S)$, so the compositum is a field. The original source already carries a finite polarized endomorphism. The last assertion of Proposition 13.11, applied to this source model, therefore gives an actual finite polarized cover $(\mathcal{X}', F')$ of the source family. The split torus acts rationally on $\mathcal{X}'$ over $Y' \times S$ and has a dense generic fiber orbit. Its first-arrow transport is one integral exponent matrix, hence a fixed isogeny of the split torus. Proposition 13.16 now gives isotriviality of $(\mathcal{X}', F')$.

This last application handles the line-bundle and compactification data as well. After the scalar-power iterate, the weighted vertical divisors satisfy $h'^* D_s - qD_s = \operatorname{div}(c_s)$. Their fractional parts are fixed, and the integral differences belong to the finite kernel of $h'^* - [q]$ on the birational Pic^0. The resulting rational character coboundaries conjugate the toric systems; the horizontal-field lemma extends these conjugacies to the normal projective models.

The constant finite cover descends isotriviality to the source by Proposition 13.12. The rational-horizontal-field lemma transfers it through the initial birational graph, and Corollary 4.2 removes any iterate used to preserve a geometric component or make the torus transport scalar. □

15.5. **Variation in an exceptional family is abelian.**

Lemma 15.6 (The birational Albanese of an AT compactification)**.** *Let $P \to A$ be a torsor under a split torus over an abelian variety in characteristic zero, and let Z be a normal projective compactification of P. Then*

$$\mathrm{Alb}_{\mathrm{bir}}(Z) \simeq A$$

after choosing origins. The assertion is compatible with finite AT-isogenies and with algebraic families after restriction of the parameter base.

Proof. A rational map from Z to an abelian variety is regular on a smooth projective model. Its restriction to a geometric torus fiber of $P \to A$ is constant: a morphism from $\mathbf{G}_m$ to an abelian variety extends to $\mathbf{P}^1$ and is constant, and the coordinate one-parameter subgroups generate the torus. Hence it factors rationally through A. A rational map from the smooth variety A to an abelian variety extends to a morphism, which, after translation, is a homomorphism. Conversely the projection $P \to A$ is a rational map from Z and generates A. These are precisely the universal properties of the birational Albanese. The construction is functorial. In a family, resolve the generic compactification and spread the resolution and the maps over a nonempty parameter open; the same universal identities then hold fiberwise. □

Theorem 15.7 (Rigidity of an exceptional family with constant abelian part)**.** *Let $F : \mathcal{X} \to \mathcal{X}$ be an algebraic one-parameter family of q-polarized endomorphisms of normal projective varieties. Suppose that its geometric generic system has an exceptional presentation. If the abelian base of one algebraic family of full AT presentations is isotrivial, then F is isotrivial.*

Proof. All constructions are made after a finite extension of the parameter field and then spread over a nonempty parameter open. Replace F by the fixed iterate occurring in the presentation, and write Q for its polarization weight. We have

$$P \longrightarrow \mathcal{X}^\circ, \qquad \Psi : P \longrightarrow P, \qquad P \longrightarrow \mathcal{A},$$

where the first map is finite and surjective, P is a torus torsor over an abelian scheme $\mathcal{A}$, and Ψ is a Q-lift. Normalize $\mathcal{X}$ in the function field of P:

$$p : \mathcal{Z} \longrightarrow \mathcal{X}.$$

The map p is finite and surjective, its source is normal, and $P = p^{-1}(\mathcal{X}^\circ)$. The lift Ψ extends to a finite endomorphism $F_{\mathcal{Z}}$; the pullback of the polarization of F polarizes $F_{\mathcal{Z}}$ with weight Q.

Trivialize $\mathcal{A}$ as $A \times S$. The endomorphism induced by Ψ has the form $\alpha_s + a_s$, where α_s is a group endomorphism of the fixed abelian variety. The rigidity lemma for homomorphisms of abelian schemes makes $\alpha_s = \alpha$ constant. The endomorphism α is polarized with weight Q, so $1 - \alpha$ is an isogeny. A finite parameter extension supplies c_s with $(1 - \alpha)c_s = a_s$; translating the base by c_s makes its dynamical system the fixed pair (A, α).

Split the torus after a finite parameter extension. Its action on P gives a rational action on $\mathcal{Z}$ with a dense orbit in the geometric generic fiber over A. The Q-lift identity is

$$F_{\mathcal{Z}}(tz) = [Q](t)F_{\mathcal{Z}}(z).$$

Proposition 13.16 applies to the rational quotient $\mathcal{Z} \dashrightarrow A \times S$ and shows that $(\mathcal{Z}, F_{\mathcal{Z}})$ is isotrivial. After trivializing this family, Proposition 13.12 applies to its finite equivariant map p and makes $(\mathcal{X}, F)$ isotrivial. Finally Corollary 4.2 removes the initial iterate. □

Corollary 15.8 (Bounded varying covers of nonisotrivial exceptional families)**.** *Let F be a nonisotrivial algebraic family of polarized endomorphisms of normal projective m-folds, and suppose its geometric generic system is exceptional. After a finite parameter extension, it has a connected finite Galois cover of degree at most*

$$M_m = |\operatorname{GL}_{2m}(\mathbf{F}_3)|$$

whose birational Albanese variety is nonisotrivial. A fixed iterate of F lifts to this cover and its normalized full pullback is connected and isomorphic to the original cover.

Proof. Propositions 11.7 and 11.8 give a full Galois AT presentation of degree at most M_m. Its construction gives $L\sigma(E) = E$ for the corresponding function fields E/L and the lifted dynamical embedding σ; this identity is preserved when the translation kernel is removed, since that kernel is preserved by the deck-group automorphism. Thus its normalization over the projective source has the asserted connected normalized pullback property.

If the abelian base of this presentation were isotrivial, Theorem 15.7 would make F isotrivial. Its abelian base therefore varies. By Lemma 15.6 that base is the birational Albanese of the normal projective cover. □

Proposition 15.9 (Pulling back a varying exceptional factor)**.** *Let F be an algebraic family of polarized endomorphisms of normal projective varieties. Suppose it has a dominant rational semiconjugacy to a nonisotrivial exceptional family (B, g) of dimension $b > 0$. After finite parameter extension, a fixed iterate of F has a connected finite Galois cover of degree at most M_b with connected normalized full pullback and nonisotrivial birational Albanese. Geometric connectedness of the rational factor is not required.*

Proof. Use Corollary 15.8 on (B, g). Write K for the parameter function field, put $L = K(X)$ and $L_0 = K(B)$, and let E/L_0 be the resulting Galois extension. After the indicated iterate, its full normalized

pullback is itself. The finite reduced L-algebra

$$R = L \otimes_{L_0} E$$

therefore satisfies

$$L \otimes_{L,F^*} R \simeq R.$$

After finite parameter extension, its connected geometric components are defined over K. Write $R = \prod_{i=1}^{r} L_i$. Each algebra $L \otimes_{L,F^*} L_i$ has at least one field component. Since their total number is r, each has exactly one and pullback permutes the L_i. An iterate fixes a chosen component and its pullback is connected. Base change of a finite Galois extension is a product of finite Galois extensions, so L_i/L is Galois of degree at most $[E : L_0] \leq M_b$.

Normalize X in L_i. The dynamical field embedding extends to a finite polarized endomorphism on this normal projective cover. The inclusion $E \hookrightarrow L_i$ gives a dominant rational map from the cover to the AT compactification associated with E. Consequently its birational Albanese maps surjectively to the varying abelian base of that presentation. A quotient family of a constant abelian variety is constant after finite parameter extension: its connected kernel is rigid, and its finite kernel lies in one fixed torsion group. The birational Albanese of the cover therefore cannot be isotrivial. □

Corollary 15.10 (A varying exceptional system has a trace-zero abelian factor)**.** *Every nonisotrivial exceptional family has a positive-dimensional rational factor with a finite presentation by a trace-zero abelian variety over* $\overline{\mathbf{C}(S)}$*. This factor is itself nonisotrivial.*

Proof. Choose a full Galois AT presentation $P \to X^\circ$ with abelian base A. Its finite deck group G acts affinely on A, as in Proposition 11.8. The abelian variety A is nonisotrivial by Theorem 15.7. Therefore its quotient A_{var} by the image of its $\mathbf{C}$-trace has positive dimension. Every group homomorphism of A preserves the trace, so the deck action and the polarized isogeny descend to affine actions on A_{var}. The map

$$P \longrightarrow A_{\mathrm{var}}/G$$

is deck invariant and descends to a dominant rational factor of X. The abelian quotient is polarized, and its finite quotient is exceptional. If this quotient family were isotrivial, Lemma 15.2 would make its finite abelian cover isotrivial. This contradicts positive dimension and zero trace. □

15.6. **A closed induction for the varying exceptional factor.** All trace-zero assertions below are geometric: the constant trace is taken after passage to the algebraic closure of the parameter function field. In particular a positive-dimensional geometrically trace-zero abelian variety is not isotrivial.

Lemma 15.11 (Descending a varying abelian-quotient factor through a finite cover)**.** *Let $p : (Y, F) \to (X, f)$ be a finite Galois compatible cover of polarized one-parameter families. Suppose an iterate of F commutes with the deck group and has a positive-dimensional rational factor with a finite presentation by a trace-zero abelian variety. Then an iterate of f has a factor of the same kind. This descended factor is nonisotrivial.*

Proof. Write Γ for the deck group and $\rho : Y \dashrightarrow E$ for the given factor. Replace the dynamics by the indicated iterate. Form all conjugate factors $\rho\gamma$, with $\gamma \in \Gamma$, using the same projective target E. The normalization D of their joint image in E^Γ is a polarized factor. The deck group acts on D regularly by permuting the factors, and its action commutes with the dynamics. Lemma 13.9 gives a finite abelian presentation $A \to D$, with A of positive dimension and zero constant trace.

The finite quotient $D \to D/\Gamma$ is equivariant and polarized; for example, average a polarization over Γ and take a power descending to the quotient. The composite $A \to D/\Gamma$ is a finite abelian presentation of the descended factor, whose function field is contained in $k(X)$. The factor has the same positive dimension as D. If it were isotrivial, Lemma 15.2 would make its finite abelian cover A isotrivial, contradicting its positive dimension and zero trace. □

Theorem 15.12 (The varying exceptional-factor theorem)**.** *Every nonisotrivial normal projective polarized one-parameter family in the pointed reference spectral class of Definition 5.1 has a positive-dimensional rational factor with a finite presentation by a trace-zero abelian variety. In particular it has a nonisotrivial exceptional rational factor.*

Proof. We prove the statement simultaneously in all dimensions. The one-dimensional case is Corollary 5.6, followed by Corollary 15.10. Assume the statement in dimensions less than N and consider a nonisotrivial family of dimension N.

The finite algebraic alternatives. Proposition 5.3 applies to the exact pointed spectral hypothesis. The local alternative and its global data, in Theorem 9.9 and Corollary 9.10, therefore apply. The conjugacy

alternative makes the family isotrivial and is excluded. The remaining alternatives supply one of the following finite algebraic presentations at the reference parameter:

(1) an exceptional presentation of the source;
(2) a proper positive-dimensional connected rational polarized factor of the source;
(3) a finite Galois dynamical cover $Y \to X$ and a generically finite rational map
$$Y \dashrightarrow T \subset \prod_{i=1}^{r} B_i,$$
where each B_i is a positive-dimensional proper connected polarized rational factor of Y and T is invariant.

Here is the reduction of the geometric alternatives to this list. For a rational Killing connection, Corollary 14.8 and the deck-stable flow construction give a positive-dimensional semiabelian action on a finite cover. Its open-orbit case is exceptional; otherwise its invariant field, followed by deck invariants, gives the second presentation. For a finite web, Lemma 14.12 selects its members on a finite Galois dynamical cover. The purely transcendental dense-leaf branch descends by Proposition 14.13. The other branch supplies proper factors of this cover by Theorem 14.11. Lemma 14.14 then gives presentation (2) or (3), according to the dimension of the join of their deck conjugates.

Corollary 14.16 spreads the selected presentation over a finite parameter cover defined over the fixed countable field k_0. It retains a lift of the same geometrically generic reference parameter. Every cover, factor and periodic return used below remains in the reference spectral class by Proposition 5.2. All dynamics may temporarily be replaced by one common iterate; the removal of this iterate is described at the end of the proof.

In case (1), the source is nonisotrivial and exceptional, so Corollary 15.10 gives the assertion.

A proper factor. In case (2), write $\pi : X \dashrightarrow B$ for the connected factor. If B is nonisotrivial, its dimension is less than N, and the induction hypothesis supplies its trace-zero abelian- quotient factor. Composing with π proves the assertion for X. Suppose instead that B is isotrivial. Trivialize it after a finite parameter extension, giving one fixed polarized system (B, g). Use the normalized equivariant graph to make π a morphism. Its generic fiber is geometrically integral, of dimension
$$m = N - \dim B < N.$$

At the reference parameter, smooth repelling Julia periodic points of g in the flat normal-fiber good open form a Zariski- dense set $\mathcal{P}$. For each such constant base point, its return family has the reference spectral property by Proposition 5.2. Let $\mathcal{I} \subset \mathcal{P}$ be the points whose return family is isotrivial. Theorem 15.5 shows that $\mathcal{I}$ cannot be Zariski dense, since the whole source family is nonisotrivial. Thus $\mathcal{P} \backslash \overline{\mathcal{I}}$ is dense and consists of nonisotrivial return families of dimension m.

The induction hypothesis gives a nonisotrivial trace-zero abelian-quotient factor of each such return. Proposition 15.9 gives a connected Galois Cartesian cover of its source, of degree at most M_m, with nonisotrivial birational Albanese. The periods and parameter covers may depend on the selected point; this degree bound does not. The relative gluing theorem 12.9, together with Remark 13.8 and Lemma 13.7, gives a positive-dimensional trace-zero abelian-quotient factor of X, nonconstant on the generic π-fiber. The gluing theorem requires only polarization of the fixed base, and does not require that (B, g) itself be exceptional. This completes case (2).

A full-dimensional join upstairs. Consider case (3). Every factor family B_i lies in the reference spectral class and has dimension less than N. If one B_i is nonisotrivial, the induction hypothesis gives a trace-zero abelian-quotient factor of B_i, hence of Y. Lemma 15.11 descends such a factor to X.

It remains to consider the case where all the B_i are isotrivial. After one common finite parameter extension, trivialize them as fixed polarized systems. Their product is fixed. The invariant image family T is then fixed by Lemma 14.17. Normalize this fixed image; it is again a fixed normal projective polarized system. The generically finite rational semiconjugacy from Y to it makes Y isotrivial by Corollary 15.3. Its actual finite equivariant map to X makes X isotrivial by Proposition 13.12. This contradicts the standing assumption, so this last subcase does not occur.

Returning to the original iterate. If a trace-zero abelian-quotient factor ρ was obtained for f^r, join the factor maps

$$\rho, \rho f, \ldots, \rho f^{r-1}.$$

Their normalized joint image is preserved by the cyclic-shift map induced by f. Proposition 11.3 gives its normal projective q-polarized model. Its abelian presentation still has zero trace by Lemma 13.9, applied to the common return f^r. Thus it is the required factor for f itself. If any argument above instead made f^r isotrivial, Corollary 4.2 makes f isotrivial as well. This finishes the induction. □

Corollary 15.13 (Uniform varying covers without a trichotomy hypothesis)**.** *Every nonisotrivial m-dimensional family in the reference spectral class admits, after finite parameter extension and an iterate, a connected Galois Cartesian dynamical cover of degree at most M_m with nonisotrivial birational Albanese. The same bound applies to every nonisotrivial good periodic-fiber return of an N-dimensional family, with $M_m \leq M_{N-1}$.*

Proof. Apply Theorem 15.12 and Proposition 15.9. Periodic returns belong to the same class by its closure proposition, and their dimension is less than N. □

16. Simultaneous coordinates for the generic fiber arrows

Lemma 16.1 (A stationary representative of constant path classes)**.** *Consider an algebraic family of finite polarized fiber arrows over a dominant phase map. Suppose that every finite path has one constant geometric polarized isomorphism class defined over $\mathbf{C}$. Then its generic fiber arrows have simultaneous local product coordinates in the sense of Definition 1.5.*

More precisely, there are a fixed polarized system (W, L_W, h) over $\mathbf{C}$, an algebraic subgroup $H \subset \mathrm{Aut}(W, L_W)$, and an integer n_0 such that the length-n_0 generic frame scheme is an H-torsor representing all forward frames. Forward transport is equivariant for an algebraic endomorphism $\tau : H \to H$ satisfying

$$h\alpha = \tau(\alpha)h.$$

Proof. Choose compatible constant representatives recursively:

$$C_n = (W_0 \xrightarrow{h_0} W_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{n-1}} W_n).$$

Indeed, a representative of length $n+1$ has prefix isomorphic to the chosen C_n, and a constant isomorphism transports this prefix to C_n. Dominance of the phase map implies that the shifted length-n chain has the same class as C_n.

Let J_n be the space of polarized isomorphisms $u_0 : W_0 \to W_1$ which extend to an isomorphism from C_n to its shift. Such an extension is unique once u_0 is fixed: the surjectivity of h_i determines u_{i+1} from

$$u_{i+1}h_i = h_{i+1}u_i.$$

The path-isomorphism scheme is a torsor under the algebraic path-automorphism group. Its image at the first vertex is therefore a coset of an algebraic subgroup in $\mathrm{Isom}((W_0, L_0), (W_1, L_1))$, and is closed.

The spaces J_n are nonempty and nested. The ambient isomorphism scheme is of finite type: polarized isomorphisms are induced by projective linear transformations after one sufficiently high power of the polarization. Noetherianity thus makes (J_n) stationary. Choose u_0 in their intersection and denote its compatible extensions by u_i.

Set $r_0 = \mathrm{id}_{W_0}$ and $r_{i+1} = r_i u_i^{-1} : W_{i+1} \to W_0$. Then

$$r_{i+2} h_{i+1} r_{i+1}^{-1} = r_{i+1} h_i r_i^{-1}.$$

Every transported arrow is consequently the same map $h = u_0^{-1} h_0$ on $W = W_0$. The transported polarizations are $L_W = L_0$, and $h^* L_W \simeq L_W^{\otimes q}$.

Let $H_n \subset \mathrm{Aut}(W, L_W)$ consist of the first-vertex automorphisms extending through the first n arrows of the stationary chain. These are descending closed algebraic subgroups. They stabilize at some $H_{n_0} = H$. Let I_n be the scheme of polarized isomorphisms from this constant length-n chain to the actual generic path. It is a geometrically nonempty H_n-torsor. Here $\mathrm{Aut}(W, L_W)$ denotes the algebraic group of automorphisms preserving the isomorphism class of L_W. For $n \geq n_0$, prefix restriction $I_{n+1} \to I_n$ is equivariant for the identity of H and is therefore an isomorphism. Thus I_{n_0} represents the compatible frames of every forward length.

Extend a frame by one arrow and discard its first vertex. This gives the rational forward transport of I_{n_0} over the phase map. Shifting an infinite chain automorphism gives an algebraic endomorphism τ of H; the first commuting square is precisely $h\alpha = \tau(\alpha)h$.

The torsor, its transport, and the universal fiber isomorphisms spread over dense phase opens. Algebraic groups in characteristic zero are smooth, so these torsors have sections locally in the *étale* topology. For a finite path, successive transport of one such section gives all its vertex frames. This proves the coordinate description in Definition 1.5. □

Proposition 16.2 (Constancy of every finite path class)**.** *Let $\pi : X \dashrightarrow Z$ be a connected polarized rational factor of a one-parameter family. Suppose that the factor has a finite dynamical presentation by a geometrically trace-zero abelian variety. Suppose further that all good marked periodic fiber returns outside a proper closed phase subset are isotrivial. After replacing the phase polarization by one fixed positive tensor power, every finite polarized path of generic fiber arrows has one constant geometric isomorphism class defined over* $\mathbf{C}$.

Proof. Work on an equivariant normal graph model and choose one high power of its polarization which is relatively very ample over a dense

normal-fiber open. Fix a length n. Record its $n+1$ embedded fibers and its n arrow graphs in the product of the corresponding Hilbert schemes. Applying the Rosenlicht construction of Proposition 13.6 with one projective linear group at each vertex gives a rational map μ_n separating the geometric polarized path classes.

Pass to the finite abelian presentation $A \to Z$. Along each retained marked periodic cycle, the return family is isotrivial. Lemmas 13.4 and 13.5 make its entire polarized cyclic chain constant over the parameter. Every length-n subpath of the repeated cycle consequently has constant class over $\mathbf{C}$. Thus μ_n is constant along its periodic Betti section on A.

The good periodic Betti labels outside the prescribed proper closed subset are dense on the locus of definition of μ_n, by Lemma 13.2 and the good-cycle restriction. Continuity in Betti coordinates makes every local coordinate of μ_n constant along Betti leaves. Apply Lemma 13.1 to its rational affine coordinates. The geometric generic abelian variety has zero $\mathbf{C}$-trace, so each coordinate is constant. Hence the pulled-back path-moduli map has one value over $\mathbf{C}$. The same holds downstairs because $A \to Z$ is dominant and μ_n separates geometric path classes. This argument applies to every n with the same fiber polarization. □

Corollary 16.3 (Maximal trace-zero reduction)**.** *Let f be a nonisotrivial family in the pointed reference spectral class. Choose a maximal connected rational factor $\pi : X \dashrightarrow Z$ with a finite presentation by a geometrically trace-zero abelian variety. Then $\dim Z > 0$, and the generic fiber arrows of π have simultaneous local product coordinates on dense phase opens, with one fixed polarized fiber system defined over $\mathbf{C}$.*

Proof. Theorem 15.12 gives a positive-dimensional factor, and Lemma 13.9 gives its maximal connected join. If $\dim Z = \dim X$, its geometrically integral generic fiber is a point, and the coordinate assertion follows with the point system.

Suppose that $\dim Z < \dim X$. If the nonisotrivial good periodic fiber returns were Zariski dense, then Corollary 15.13 would supply connected Galois Cartesian covers of uniformly bounded degree with nonisotrivial Albanese varieties. Relative gluing in Theorem 12.9, together with Remark 13.8 and Lemma 13.7, would give a trace-zero abelian-quotient factor nonconstant on the geometric generic fiber of π. Its join with Z has larger dimension, contrary to maximality.

The closure of the nonisotrivial good returns is therefore a proper closed phase subset. Proposition 16.2 gives a constant polarized class for every finite generic fiber path. Lemma 16.1 supplies the fixed system (W, L_W, h) and the finite-length frame torsor. Both are constructed

for the original factor $X \dashrightarrow Z$; their local sections give the asserted coordinates. $\square$

17. The structural implication

Proposition 17.1. *Let S be a smooth connected complex algebraic curve and let f be a nonisotrivial J-stable polarized family $p : \mathcal{X} \to S$ with geometrically integral normal fibers. Put $K = \mathbf{C}(S)$ and denote its generic fiber by X_η. Then the second alternative of Theorem 1.6 holds. The factor can be chosen maximal among connected factors with geometrically trace-zero abelian presentations.*

Proof. Choose an algebraically closed countable field $k_0 \subset \mathbf{C}$ over which the family is defined, and a parameter s_0 generic over k_0. Proposition 3.1 makes the marked repelling spectra constant. Thus the family belongs to the pointed reference spectral class of Definition 5.1.

Theorem 15.12 gives a positive-dimensional rational factor with a trace-zero abelian presentation. By Lemma 13.9, take its maximal connected join and write it as

$$\pi : X_{\eta,\overline{K}} \dashrightarrow Z, \qquad \pi f = g\pi.$$

Its generic fiber is geometrically integral. The maximal-factor construction gives Z normal and projective and g regular; its polarization has weight q by [MZ18, Theorem 1.3]. Its abelian presentation consists of a finite surjective map $\eta : A \to Z$ and a polarized affine isogeny α satisfying $\eta\alpha = g^e\eta$ for some $e \geq 1$.

Write $\alpha = t_a \circ \beta$, where β is a group isogeny, and choose an ample line bundle L_A polarizing α. Translations act trivially on the Néron–Severi group, so $\beta^* c_1(L_A) = q^e c_1(L_A)$. If $(\ker(1-\beta))^0$ had positive dimension, restriction to this abelian subvariety would contradict ampleness, since β is the identity there. Thus $1-\beta$ is an isogeny. Choose b satisfying $(1-\beta)b = a$. The conjugate $t_{-b}\alpha t_b = \beta$ is q^e-polarized by $t_b^* L_A$. Equivalently, take b as the origin of A. This proves that g is exceptional of abelian variety type as defined in Definition 1.4.

Corollary 16.3 applies to this maximal factor. It supplies a single polarized system (W, L_W, h) over $\mathbf{C}$ and simultaneous local product coordinates for all generic fiber paths. Their finite-length frame torsor and forward transport give the coordinate data in Definition 1.5. Since $N = \dim X_\eta$ and $r = \dim Z$ satisfy $1 \leq r \leq N$, the fiber dimension is $\dim W = N - r \in \{0, \ldots, N-1\}$.

The varieties, morphisms, rational factor, line bundles, abelian presentation, and finite-length frame torsor are all of finite presentation. They descend from $\overline{K}$ to one finite extension K'/K. Their finitely

many defining identities spread over a nonempty open subset of the corresponding finite cover of S. The rational transport then supplies each finite path on its dense phase domain. This proves the proposition. □

18. Stability of the two structural alternatives

This section proves the converse implication in the main theorem. We construct periodic sections whose empirical measures converge to the equilibrium measure in every fiber, and then apply Proposition 2.24.

Proposition 18.1 (Generic stability of isotrivial families)**.** *An isotrivial algebraic family of polarized endomorphisms with geometrically integral normal projective fibers is J-stable on a nonempty Zariski open subset of its parameter space.*

Proof. Let S be the parameter space. By isotriviality, a finite extension of $\mathbf{C}(S)$ identifies the generic dynamical system with a fixed complex system (X_0, h). Spread this isomorphism and its inverse, and then shrink S. Since the extension is separable, this gives a finite surjective étale cover $S' \to S$ and an equivariant isomorphism

$$\mathcal{X} \times_S S' \simeq X_0 \times S'.$$

The map h is polarized: transport the polarization from any complex fiber of this isomorphism.

A dynamical isomorphism carries equilibrium measures and their supports to one another. This follows from normalized inverse-image equidistribution, as used in the proof of Proposition 2.4; see [DNT15, Theorem 1.2]. Consequently, the displayed isomorphism identifies the smooth repelling Julia cycles with the constant cycles of h, for every period simultaneously. Local analytic inverse sections of $S' \to S$ give the required motions on S. This proves J-stability on the chosen parameter open. □

Lemma 18.2 (The connected frame group)**.** *Let (W, L_W, h) be a polarized complex system and let H be its algebraic group of compatible forward frames. Write*

$$ha = \tau(a)h \qquad (a \in H).$$

Then $T = H^0$ is a torus, and $\tau|_T$ is an isogeny. For every $n \geq 1$ and every $a \in H$, the morphism

$$T \longrightarrow T, \qquad b \longmapsto b^{-1}a\tau^n(b)a^{-1}$$

is an isogeny. In particular, the maps ah^n, with a in one connected component of H, are conjugate by elements of T.

Proof. The kernel of dynamical transport is contained in the finite deck group of h. Its restriction to T is therefore an isogeny. The orbit-degree argument in Proposition 15.4 excludes additive one-parameter subgroups: a nontrivial additive orbit has degree bounded by a fixed projective embedding dimension, whereas its images under h^n are birational additive orbits of degree multiplied by q^n. Since $T \subset \mathrm{Aut}(W, L_W)$ is linear, absence of additive one-parameter subgroups makes T a torus.

The displayed morphism is a homomorphism of tori. Its kernel consists of the elements of T commuting with ah^n. This kernel is finite: the identity component of the automorphism group of a polarized map fixes every finite set of periodic points, and periodic points are Zariski dense by [Fak03]. It therefore acts trivially. A homomorphism of tori of equal dimension with finite kernel is an isogeny. The final assertion follows from

$$b^{-1}(ah^n)b = \big(b^{-1}a\tau^n(b)\big)h^n.$$

□

Lemma 18.3 (Finite covers of an abelian base)**.** *Let A be a positive-dimensional abelian variety over an algebraically closed field of characteristic zero. Suppose $\alpha : A \to A$ is Q-polarized and $p : B \to A$ is finite and surjective, with B normal. If a surjective endomorphism $\beta : B \to B$ satisfies $p\beta = \alpha p$, then p is étale. After choosing an origin, B is an abelian variety and β is an affine isogeny.*

Proof. Let $L = p^*L_A$, where L_A polarizes α. Then L is ample and $\beta^*L \simeq L^{\otimes Q}$. Pulling back a nonvanishing rational top form on A identifies the canonical Weil divisor of B with the effective ramification divisor R_p. The ramification formula for β, interpreted at codimension-one points, gives

$$R_p = \beta^*R_p + R_\beta.$$

If $r = \dim B$, the finite projection formula for Weil divisors and Cartier intersections gives

$$(R_p \cdot L^{r-1}) = Q(R_p \cdot L^{r-1}) + (R_\beta \cdot L^{r-1}).$$

Ample positivity forces $R_p = R_\beta = 0$. Purity of the branch locus over the smooth variety A [Stacks, Tag 0BMB] makes p étale. A connected finite étale cover of an abelian variety is an abelian variety after choosing a point above the origin; every finite self-map of an abelian variety is a translation followed by an isogeny. □

Lemma 18.4 (Extension of the frame bundle over the abelian base)**.** *Let $\rho : Y \to A$ be a projective morphism from a normal variety to an*

abelian variety. Let $F : Y \to Y$ and $\alpha : A \to A$ be finite polarized endomorphisms satisfying $\rho F = \alpha\rho$. Suppose that the generic fiber arrows have the simultaneous coordinates of Definition 1.5, with fixed system (W, L_W, h). After a finite étale cover of A and an iterate, the normalization of the pullback of Y is a bundle associated to a torus torsor and the polarized T-variety W. Its forward transport is induced by h and the isogeny τ of Lemma 18.2.

Proof. Let I be the generic H-torsor of all forward frames, and put $T = H^0$. The finite scheme I/T gives a finite cover of A. Normalize A in its component fields. Forward transport extends to a finite map of this normalization over α: integrality over the coordinate rings of A gives the extension. Its image on the finite set of irreducible components has a periodic component. Choose such a component and replace the maps by the corresponding iterate. Its self-map is polarized by the pullback of a polarization on A. Lemma 18.3 makes the resulting connected cover $A' \to A$ étale.

Normalize the pullback of Y to A'. The chosen component of I/T leaves a generic torsor under the split torus T, by reduction of structure group [Ser58, Section 3.5, Proposition 9]. Hilbert's Theorem 90 [Stacks, Tag 03P7] gives it a rational point. Thus a rational compatible forward frame exists over A'. Replace notation by Y, A, F, α, and replace h by the same iterate throughout the stationary fiber chain.

Call a phase point good if, on an étale neighborhood of that point, the source isomorphism of a frame in the chosen generic T-torsor extends to an isomorphism from the constant polarized family with fiber W. These neighborhoods form a nonempty open set $V \subset A$. We show that $\alpha(V) \subset V$.

Let $U \to V$ be one such neighborhood with source frame $c_0 : U \times W \xrightarrow{\sim} Y_U$. Since α is étale, the target $Y_{\alpha,U} := Y \times_{A,\alpha} U$ is normal. The maps

$$Fc_0 : U \times W \longrightarrow Y_{\alpha,U}, \qquad \mathrm{id}_U \times h : U \times W \longrightarrow U \times W$$

are finite. Their joint image is finite over both targets. Over the generic point it is the graph of the next frame, and both projections are birational. Normality makes both projections isomorphisms. This extends the next frame over U. The extended frame is polarized: its relative Picard class agrees generically with L_W, and the equality persists by separatedness of the relative Picard functor. Hence $\alpha(V) \subset V$.

The complement $B = A \setminus V$ is closed and satisfies $\alpha^{-1}B \subset B$. It must be empty. Indeed, take its components of largest dimension. Finiteness of their set and the backward-invariance inclusion imply that

their reduced cycle C satisfies $\alpha^* C = C$. If its codimension is $c > 0$, intersecting with an ample polarization gives

$$(C \cdot L_A^{r-c}) = Q^c (C \cdot L_A^{r-c}),$$

contrary to positivity. The transition maps between the extending frames lie generically in the closed subgroup T of $\mathrm{Aut}(W, L_W)$, and therefore lie in T everywhere. They define a torus torsor $I \to A$, and the associated-bundle construction [Ser58, Section 3.2, Proposition 4] identifies Y with $I \times^T W$. The forward compatibility identities extend with the frames. □

Proposition 18.5 (The structural converse)**.** *Let f be an algebraic family of polarized endomorphisms with geometrically integral normal projective fibers. Suppose that, after a finite parameter extension, it has a rational factor of abelian variety type whose generic fiber arrows have simultaneous local product coordinates with one fixed polarized complex fiber system. Then f is J-stable on a nonempty parameter open.*

Proof. Work over the parameter function field and write $X = X_\eta$. Choose an equivariant normal projective graph model and pull it back to the finite abelian presentation of the factor. Normalize this pullback and replace the dynamics by the presentation iterate. We obtain a polarized map F on a normal projective variety Y, a morphism $\rho : Y \to A$ over a polarized abelian isogeny, and a proper generically finite dynamical morphism $p : Y \to X$. Lemma 18.4 gives, after another finite parameter extension and an iterate, a global bundle of copies of W over an abelian variety A'. All these finite-type data spread over a parameter open. Shrink that open so that A' is an abelian scheme, the bundle and its dynamics are defined everywhere over it, and p remains a proper generically finite dynamical morphism. Denote the polarization weight of the resulting iterate by Q and its fixed fiber map by u.

Work over an analytic coordinate disk D in this parameter open. The fixed points of each iterate of the abelian base map are finite étale over D. They therefore give holomorphic sections on the same disk, for all periods. Over a section of period dividing n, the torus torsor is holomorphically trivial. The fiber return in such a trivialization has the form

$$a(s)u^n, \qquad a : D \longrightarrow T.$$

The torus Lang isogeny in Lemma 18.2 and simple connectivity of D conjugate this return holomorphically to u^n on all of D.

Let $r = \dim A'$ and $m = \dim W$. The base has $Q^{rn} + o(Q^{rn})$ fixed points of its nth iterate. The fixed system u has $Q^{mn} + o(Q^{mn})$ smooth repelling Julia points fixed by u^n, by periodic equidistribution (and

exactly one when $m = 0$). Choosing these points on every base section gives

$$Q^{(r+m)n} + o(Q^{(r+m)n})$$

distinct holomorphic periodic sections on D. Each section is smooth and repelling: the horizontal eigenvalues are those of an abelian isogeny and the vertical ones are those of the chosen return of u. Each lies in the ambient Julia set by Lemma 15.1, since the equilibrium measure of an abelian isogeny has full support. In every fiber these sections form an asymptotically full set of repelling periodic points. Their normalized counting measures converge to μ_{F_s}. The collection is permuted by F, because it contains all the smooth repelling Julia points of the returns over every base point fixed by the nth iterate. Its counting measure is therefore F-invariant.

Compose the sections with the spread of p. They are holomorphic periodic sections of the original phase family, and their empirical measures converge to

$$p_*\mu_{F_s} = \mu_{f_s}.$$

For this identity, choose a generic point of X_s whose finite inverse image under p avoids the exceptional sets for backward equidistribution. Normalized inverse images of this finite fiber, pushed down by p, are the normalized inverse images of that point. Apply [DNT15, Theorem 1.2] on resolutions of both varieties; the exceptional sets have equilibrium measure zero. This gives $p_*\mu_{F_s} = \mu_{f_s}$. After restricting the parameter open as in Proposition 2.24, its relative periodic-section criterion produces an equilibrium web and hence J-stability. Stability of an iterate is equivalent to stability of the original map, because they have the same smooth repelling Julia periodic points.

Finally, delete finitely many points of S so that the finite parameter cover is étale. Stability descends through its local analytic inverse sections. Thus the original family is stable on a nonempty parameter open. □

19. Proof of the equivalence and the projective-space version

Proof of Theorem 1.6. If f is generically J-stable, restrict to a nonempty open of S on which it is stable. The function field remains K. If f is nonisotrivial, Proposition 17.1 gives the second alternative. Conversely, the first alternative gives generic stability by Proposition 18.1, and the second does so by Proposition 18.5. □

Proof of Corollary 1.7. After restricting to a nonempty parameter open, the family is polarized by $\mathcal{O}(1)$ with weight d. Every automorphism of $\mathbf{P}^N$ belongs to PGL_{N+1}. The equivalence follows from the main theorem and Corollary 2.21, which extends stability from a nonempty Zariski open to the whole parameter space for a fixed phase variety. Dominance of π gives geometric unirationality of Z. If $\dim Z = 1$, its geometric genus is zero, so $Z_{\overline{K}} \simeq \mathbf{P}^1_{\overline{K}}$. Its abelian presentation makes an iterate Lattès; since the equilibrium measure is unchanged by iteration, the characterization in [BD05, Theorem 1] makes g itself Lattès.

Suppose $\dim Z = N$. The connected rational fibration π is then birational. Composing its inverse with an abelian presentation gives a generically finite rational map

$$\varphi : A \dashrightarrow \mathbf{P}^N, \qquad \varphi\alpha = f^e\varphi.$$

Finite postcomposition does not remove indeterminacy: the normalized graph has connected fibers over its normal source, and a positive-dimensional fiber cannot be contracted by a finite map of the target. Since α is étale, it follows that its indeterminacy locus I satisfies $\alpha^{-1}I = I$. The intersection argument in Lemma 18.4 gives $I = \varnothing$. Thus φ is a morphism. The line bundle $\varphi^*\mathcal{O}(1)$ is nef and big on A, hence ample; therefore φ is finite. Its pushforward of Haar measure is the equilibrium measure of f^e, and is absolutely continuous. Again $\mu_f = \mu_{f^e}$ and [BD05, Theorem 1] show that f is Lattès. For a generic system over $\overline{K}$, apply the analytic characterization after spreading over a complex parameter open. Finite abelian presentations are encoded by countably many finite-type incidence spaces, as in Lemma 14.15. Since their images contain every complex point of that open, one image is dominant: a countable union of finite subsets cannot cover a nonempty complex algebraic curve. A closed point of its generic fiber gives a presentation after a finite extension of K. The case $\dim Z = 1$ descends in the same way. This proves both assertions over $\overline{K}$. □

References

[And07] Yves André. An algebraic proof of Deligne's regularity criterion for integrable connections. *RIMS Kôkyûroku Bessatsu*, B2:1–13, 2007. arXiv:math/0701895.

[BDM08] François Berteloot, Christophe Dupont, and Laura Molino. Normalization of bundle holomorphic contractions and applications to dynamics. *Annales de l'Institut Fourier*, 58(6):2137–2168, 2008.

[BJ08] Oscar F. Bandtlow and Oliver Jenkinson. On the Ruelle eigenvalue sequence. *Ergodic Theory and Dynamical Systems*, 28(6):1701–1711, 2008.

[Bis64] Errett Bishop. Conditions for the analyticity of certain sets. *Michigan Mathematical Journal*, 11:289–304, 1964.

[Bon06] Philippe Bonnet. Minimal invariant varieties and first integrals for algebraic foliations. *Bulletin of the Brazilian Mathematical Society (N.S.)*, 37(1):1–17, 2006.

[CDM26] Junyan Cao, Ya Deng, and Shin-ichi Matsumura. A flatness criterion for pseudo-effective sheaves on compact Kähler spaces, 2026. arXiv:2609.05154.

[CDY25] Benoît Cadorel, Ya Deng, and Katsutoshi Yamanoi. Hyperbolicity and fundamental groups of complex quasi-projective varieties (III): applications, 2025. arXiv:2512.20360.

[CP19] Frédéric Campana and Mihai Păun. Foliations with positive slopes and birational stability of orbifold cotangent bundles. *Publications Mathématiques de l'IHÉS*, 129:1–49, 2019.

[Cas06] Guy Casale. Feuilletages singuliers de codimension un, groupoïde de Galois et intégrales premières. *Annales de l'Institut Fourier*, 56(3):735–779, 2006.

[DNT15] Tien-Cuong Dinh, Viêt-Anh Nguyên, and Tuyen Trung Truong. Equidistribution for meromorphic maps with dominant topological degree. *Indiana University Mathematics Journal*, 64(6):1805–1828, 2015.

[DT21] Christophe Dupont and Johan Taflin. Dynamics of fibered endomorphisms of $\mathbf{P}^k$. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (5)*, 22(1):53–78, 2021.

[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*, volume 163 of *Lecture Notes in Mathematics*. Springer, 1970.

[Fak03] Najmuddin Fakhruddin. Questions on self maps of algebraic varieties. *Journal of the Ramanujan Mathematical Society*, 18(2):109–122, 2003.

[GQS66] Victor Guillemin, Daniel Quillen, and Shlomo Sternberg. The classification of the complex primitive infinite pseudogroups. *Proceedings of the National Academy of Sciences of the United States of America*, 55(4):687–690, 1966.

[Gue05] Vincent Guedj. Ergodic properties of rational mappings with large topological degree. *Annals of Mathematics*, 161(3):1589–1607, 2005.

[JSX26] Zhuchao Ji, Jiarui Song, and Junyi Xie. Exceptional maps and abelian points in backward orbits, 2026. Preprint, arXiv:2609.24701v1.

[LBPRT22] Federico Lo Bianco, Jorge Vitório Pereira, Erwan Rousseau, and Frédéric Touzet. Rational endomorphisms of codimension one holomorphic foliations. *Journal für die reine und angewandte Mathematik*, 789:43–101, 2022.

[JX23] Zhuchao Ji and Junyi Xie. Homoclinic orbits, multiplier spectrum and rigidity theorems in complex dynamics. *Forum of Mathematics, Pi*, 11:e11, 1–37, 2023. doi:10.1017/fmp.2023.12.

[LO10] Max Lieblich and Martin Olsson. Generators and relations for the étale fundamental group. *Pure and Applied Mathematics Quarterly*, 6(1):209–244, 2010. arXiv:math/0703139.

[MZ18] Sheng Meng and De-Qi Zhang. Building blocks of polarized endomorphisms of normal projective varieties. *Advances in Mathematics*, 325:243–273, 2018. doi:10.1016/j.aim.2017.11.026.

[MZ22] Sheng Meng and De-Qi Zhang. Kawaguchi–Silverman conjecture for certain surjective endomorphisms. *Documenta Mathematica*, 27:1605–1642, 2022. doi:10.4171/DM/X13.

[MZ24] Sheng Meng and Guolei Zhong. Kawaguchi–Silverman conjecture for int-amplified endomorphism, 2024. arXiv:2408.00566.

[Mal10] Bernard Malgrange. Pseudogroupes de Lie et théorie de Galois différentielle, 2010. Preprint IHES/M/10/11.

[McM87] Curt McMullen. Families of rational maps and iterative root-finding algorithms. *Annals of Mathematics (2)*, 125(3):467–493, 1987. doi:10.2307/1971408.

[Ols06] Martin C. Olsson. Hom-stacks and restriction of scalars. *Duke Mathematical Journal*, 134(1):139–164, 2006.

[RR88] Jean-Pierre Rosay and Walter Rudin. Holomorphic maps from $\mathbf{C}^n$ to $\mathbf{C}^n$. *Transactions of the American Mathematical Society*, 310(1):47–86, 1988.

[Ros56] Maxwell Rosenlicht. Some basic theorems on algebraic groups. *American Journal of Mathematics*, 78(2):401–443, 1956.

[Sch85] Peter Schenzel. Symbolic powers of prime ideals and their topology. *Proceedings of the American Mathematical Society*, 93(1):15–20, 1985.

[Shi87] Mitsuhiro Shishikura. On the quasiconformal surgery of rational functions. *Annales scientifiques de l'École Normale Supérieure (4)*, 20(1):1–29, 1987.

[Wei23] Dario Weissmann. *A functorial approach to the stability of vector bundles*. PhD thesis, Universität Duisburg–Essen, 2023.

[Wei24] Dario Weißmann. A functorial approach to the stability of vector bundles. *Mathematische Zeitschrift*, 308:Article 63, 2024.

[BB26] François Berteloot and Xavier Buff. On the stability of holomorphic families of endomorphisms of $\mathbb{P}^k$. *Rev. Mat. Iberoam.*, 42(4):1521–1530, 2026.

[BBD18] François Berteloot, Fabrizio Bianchi, and Christophe Dupont. Dynamical stability and Lyapunov exponents for holomorphic endomorphisms of $\mathbb{P}^k$. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(1):215–262, 2018.

[BBr23] François Berteloot and Maxence Brévard. Ramification current, post-critical normality and stability of holomorphic endomorphisms of $\mathbb{P}^k$. *Fundamenta Mathematicae*, 263(3):253–270, 2023.

[DNS10] Tien-Cuong Dinh, Viêt-Anh Nguyên, and Nessim Sibony. Exponential estimates for plurisubharmonic functions and stochastic dynamics. *Journal of Differential Geometry*, 84(3):465–488, 2010.

[Dan20] Nguyen-Bac Dang. Degrees of iterates of rational maps on normal projective varieties. *Proc. Lond. Math. Soc. (3)*, 121(5):1268–1310, 2020.

[Laz04] Robert Lazarsfeld. *Positivity in Algebraic Geometry I*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 48. Springer-Verlag, Berlin, 2004.

[BD05] François Berteloot and Christophe Dupont. Une caractérisation des endomorphismes de Lattès par leur mesure de Green. *Commentarii Mathematici Helvetici*, 80(2):433–454, 2005. doi:10.4171/CMH/21.

[BT76] Eric Bedford and B. A. Taylor. The Dirichlet problem for a complex Monge–Ampère equation. *Inventiones Mathematicae*, 37(1):1–44, 1976. doi:10.1007/BF01418826.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics, vol. 52. Springer-Verlag, New York, 1977.

[PR04] Richard Pink and Damian Roessler. On ψ-invariant subvarieties of semiabelian varieties and the Manin–Mumford conjecture. *Journal of Algebraic Geometry*, 13(4):771–798, 2004. doi:10.1090/S1056-3911-04-00368-6.

[Ser58] Jean-Pierre Serre. Espaces fibrés algébriques. *Séminaire Claude Chevalley*, 3, Exposé 1:1–37, 1958. https://numdam.org/item/SCC_1958__3__A1_0/.

[Stacks] The Stacks Project Authors. *The Stacks Project*, 2026. https://stacks.math.columbia.edu.