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THE HODGE CONJECTURE FOR CM ABELIAN VARIETIES

POL VAN HOFTEN AND JACK SEMPLINER

Abstract. Following the recent work [Ope26], we give a proof of the Hodge conjecture for CM abelian varieties.

1. Introduction

1.1. **Overview.** For a smooth projective variety X over $\mathbb{C}$ and an integer $p \geq 0$ there is a cycle class map $\mathrm{CH}^p(X) \to \mathrm{H}^{2p}(X, \mathbb{Q})$ from the space of codimension p cycles on X to the $2p$-th Betti cohomology of X. This cycle class map lands in $\mathrm{Hdg}^p(X, \mathbb{Q}) := \mathrm{H}^{2p}(X, \mathbb{Q}) \cap \mathrm{H}^{p,p}(X, \mathbb{C})$, where $\mathrm{H}^{p,p}(X, \mathbb{C}) \subset \mathrm{H}^{2p}(X, \mathbb{C})$ is part of the Hodge decomposition. The *Hodge conjecture* for X asserts that the cycle class map surjects onto $\mathrm{Hdg}^p(X, \mathbb{Q})$ for all p, see e.g. [Del06], [Hod52]. The goal of this note is to explain the following result of [Ope26].

Theorem I. *If X is an abelian variety admitting complex multiplication, then the Hodge conjecture holds for X.*

For previous work on the Hodge conjecture for abelian varieties, see [Gor97] and the references therein for a survey of work done up to 1997. For subsequent work, see e.g. [Mil99a],[MZ99],[Abd02a],[Abd99],[Abd02b],[Haz03b],[Haz03a], [Abd04],[Koi04],[Haz06],[RM08],[O'S11],[Abd12], [LPS13],[Abd15],[Abd16],[Mil20c], [Anc21],[VV22],[vG22],[vG23],[Mar23], [Flo25], [Agu25],[Mar25], [vGR26], [FF26], [Mar26] for an incomplete list.

1.1.1. *Consequences for CM abelian varieties.* By [Abd05, Main result], Theorem I implies the generalized Hodge conjecture for CM abelian varieties (cf. [Haz03b, Theorem 8.3]). Since the Mumford–Tate conjecture is known for CM abelian varieties by work of Pohlmann [Poh68], Theorem I implies the Tate conjecture for CM abelian varieties over finitely generated fields of characteristic zero.

1.1.2. *Consequences for all abelian varieties.* Let k be an algebraically closed field. By [Mil02, Theorem 4.6.(a)], Theorem I implies that numerical equivalence coincides with ℓ-adic homological equivalence for abelian varieties over k with ℓ coprime to the characteristic of k. By [Mil02, Theorem 4.6.(b)], Theorem I implies that the Hodge standard conjecture holds for abelian varieties over k with respect to any good (in the sense of [Mil02, §4]) Weil cohomology theory.

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1.1.3. *Consequences for abelian varieties over finite fields.* By [Mil99b, Theorem 7.1], Theorem I implies the Tate conjecture for abelian varieties over finite fields. This in turn implies, by [Kah03, Corollary 3.10], the Lichtenbaum conjectures [Lic84] for special values of their zeta functions. Moreover, [Kah03, Theorem 1] also proves that numerical equivalence coincides with ℓ-adic homological equivalence for abelian varieties over finite fields.

1.2. **Overview of the proof.** The overall structure of our proof is the same as that of [Ope26], see Section 1.3 for a more detailed comparison. It follows from work of Hazama, [Haz03b], that it is enough to prove the conjecture in codimension two, i.e., for $p = 2$.

Given a CM abelian variety A over $\mathbb{C}$ with CM by a CM algebra L, we can write $L = L_1 \times \cdots \times L_n$ as a product of CM fields. We now choose a Galois CM field $E \subset \mathbb{C}$ containing L_i for all i with totally real subfield F and c the nontrivial F-automorphism of E. Let $\Sigma = \mathrm{Hom}(L, E)$ be the set of ring homomorphisms from L to E, equipped with its action of $G = \mathrm{Gal}(E/\mathbb{Q})$, and let $\Phi \subset \Sigma$ denote the CM type of A. We may write

$$\mathrm{H}^1(A, E) = \bigoplus_{s \in \Sigma} \mathrm{H}^1(A, E)_s$$

as a direct sum (one-dimensional) of s-isotypic components. We may thus identify

$$\mathrm{H}^r(A, E) = \bigoplus_{\Delta} \mathrm{H}^r(A, E)_\Delta,$$

where Δ runs over size r subsets of Σ. A result of Pohlmann, see [Poh68, Theorem 1] and [GU25, Theorem 3.1] for the nonsimple case, describes the Hodge classes as

$$\mathrm{Hdg}^2(A, \mathbb{Q}) \otimes E = \bigoplus_{\substack{\Delta \subset \Sigma, |\Delta| = 4 \\ \forall g \in G | (g\Delta \cap \Phi)| = 2}} \mathrm{H}^4(A, E)_\Delta.$$

For Δ as above, we can thus ponder the Δ-part of the (codimension 2) Hodge conjecture for A. A key new idea of [Ope26] is to give a *separate* argument for the Δ-part of the Hodge conjecture for each Δ.

1.2.1. *Auxiliary abelian varieties with CM.* Writing $\Delta = (s_1, s_2, c \circ s_3, c \circ s_4)$, we construct CM types $\Phi_1, \cdots, \Phi_4 \subset \mathrm{Hom}(E, E)$ for the CM field E, defined by

$$\Phi_i := \{g \in G = \mathrm{Hom}(E, E) : g \circ s_i \in \Phi\}.$$

Let $B_1, \cdots, B_4$ be abelian varieties over $\mathbb{C}$ with CM by E and CM types $\Phi_1, \cdots, \Phi_4$, and let $B = B_1 \times \cdots \times B_4$. This construction of the auxiliary CM abelian varieties B_i is a special case of a construction occuring in Deligne's famous proof that Hodge cycles are absolute Hodge [DMOS82]; it also occurs in [And92] where it is used to prove that every Hodge class on A is a sum of pullbacks of Weil classes on auxiliary abelian varieties.

In [Ope26], the auxiliary abelian varieties B_i are used to make the elementary observation that the Δ-part of the Hodge conjecture for A follows for the Hodge conjecture for

$$\bigotimes_i H^1(B_i, E)_{\lambda_i} \subset H^4(B, E), \tag{1.2.1}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\mathrm{Id}, \mathrm{Id}, c, c)$, see Lemma 2.3.2. The benefit of having four CM abelian varieties instead of one, is that we now have an action of four copies of E. We then have the following proposition.

Proposition 1.2.2. *If there is a smooth projective surface S over $\mathbb{C}$ together with maps of $\mathbb{Q}$-Hodge structures*

$$h_i : \mathrm{H}^1(B_i, \mathbb{Q}) \to \mathrm{H}^1(S, \mathbb{Q})$$

for $i = 1, 2, 3, 4$, and nonzero vectors $\alpha_i \in \mathrm{H}^1(B_i, E)_{\lambda_i}$ such that

$$h_1(\alpha_1) \cup h_2(\alpha_2) \cup h_3(\alpha_3) \cup h_4(\alpha_4) \neq 0$$

then the line (1.2.1) *lies in the image of the E-linear cycle class map*

To prove Proposition 1.2.2, we note that the h_i induce a morphism $h : S \to B$ and $h_*[S]$ gives a cycle Z on B in dimension two which pairs nontrivially with the line (1.2.1). Next, we use the action of E on B_i and the algebraicity of the Künneth projectors for each B_i to get an idempotent endomorphism e of the E-linear motive associated to B projecting onto the line (1.2.1). Then $e^t Z$ is a cycle pairing nontrivially with the line (1.2.1), and so by equality of homological and numerical equivalence on B, there must be an E-linear algebraic cycle in codimension two whose cycle class map spans (1.2.1).

1.2.3. Of course it is not at all clear how Proposition 1.2.2 is useful. Another key idea of [Ope26] is to take for S a Picard modular surface associated to E. More precisely, we take a Shimura variety associated to a Hermitian space V over E with signature $(2, 1)$ at the distinguished embedding $E \subset \mathbb{C}$ and $(3, 0)$ at all other places. The first cohomology of these two-dimensional Shimura varieties can be completely understood using theta lifts, see Theorem 3.1.11, which we explain following [Liu21]. It turns out to be essentially immediate from this that the Hodge structures of CM abelian varieties with CM by E appear in this first cohomology, see Corollary 3.1.13.

1.2.4. To carry out the cup product computation, we produce explicit cohomology classes with complex coefficients. More precisely, the work of Bergeron–Millson–Moeglin [BMM16] shows that the first cohomology of S is spanned by theta lifts, see Proposition 4.1.8. Theorem I thus reduces to showing that there are theta lifts whose cup-product is nonzero. To do this, we consider the fourfold cup-product as a functional on a Schwartz space, and then argue that it is enough to show that this functional is nonzero. This we finally do using the doubling method in Section 4.2.

Remark 1.2.5. We recently learned that Wei Zhang had considered the rough strategy followed in this paper and in [Ope26] many years ago, and also tried to use

this to prove the Hodge conjecture for CM abelian varieties. A key point missing to him was a nonvanishing result for the surface period analogous to Theorem 4.0.2.

1.3. **Comparison to [Ope26].** Our proof of Theorem I is heavily inspired by the proof given in [Ope26]. Nevertheless, we do not claim to understand (or even to have read) the details of the arguments written in [Ope26]. The goal of this note is to give a variant of the proofs in [Ope26] that relies as much as possible on the written literature. This has the benefit of making the proofs shorter, and possibly also easier to understand. To the best of the authors' knowledge, the argument for Theorem 4.0.2 is a departure from the argument for the analogous statement in [Ope26]. We leave the task of a careful exegesis of the original proof to others.

1.4. **Tool use and methodology.** This article is entirely human written. That said, we made extensive use of ChatGPT Pro during the preparation of this argument.

We asked the model to give a version of the arguments in [Ope26, Section 2] that relies on [Haz03b], and this formed the starting point of our Section 2. Moreover, Lemma 2.3.2 and its proof were suggested to us by that model. We asked the model for a more direct proof of [Ope26, Proposition 3.1], which lead our proof of Theorem 2.3.4. When we asked the model for a reference to compute the first Betti cohomology of Picard modular surfaces, it suggested [Liu21], and we wrote Section 3 without further assistance except for the proof of Lemma 3.1.5 and Corollary 3.1.13, which were also suggested to us by the model. Section 4 was written after extensive back-and-forth with the model, in which the authors insisted that the corresponding calculation in [Ope26] should be done instead using the doubling method and recourse to the Siegel–Weil formula. In particular, Lemma 4.2.7 was eventually output after this back and forth with the model, the current proof was then worked out after a further extensive back and forth. The basic idea of varying the test function as is carried out in Lemma 4.2.9 was also suggested by the model. In addition we used the model in more quotidian ways: to proofread the document and check the arguments once the paper was assembled, and to do a literature search for the introduction.

1.5. **Acknowledgments.** We thank Yifeng Liu for very helpful discussions and comments on a draft.

2. Reduction to codimension 2 and a surface statement

The goal of this section is to explain a reduction for the Hodge conjecture for abelian varieties of CM type.

2.0.1. *Recollections on the Hodge conjecture.* For a smooth projective variety X over $\mathbb{C}$ we let $\mathrm{CH}^*(X)$ denote the $\mathbb{Q}$-linear Chow ring of cycles on X modulo rational equivalence, with $\mathrm{CH}^i(X)$ the codimension i cycles. We similarly let $\mathrm{H}^*(X, \mathbb{Q})$

denote the rational Betti (or singular) cohomology ring of the topological space $X(\mathbb{C})^{\mathrm{an}}$. This comes equipped with a Hodge decomposition

$$\mathrm{H}^i(X,\mathbb{C}) = \bigoplus_{p+q=i} \mathrm{H}^{p,q}(X,\mathbb{C}).$$

There is a multiplicative cycle class map $\mathrm{CH}^*(X) \to \mathrm{H}^{2*}(X,\mathbb{Q})$ which takes $\mathrm{CH}^p(X)$ to $\mathrm{Hdg}^p(X,\mathbb{Q}) := \mathrm{H}^{2p}(X,\mathbb{Q}) \cap \mathrm{H}^{p,p}(X,\mathbb{C})$. We let $\mathrm{Alg}^p(X,\mathbb{Q}) \subset \mathrm{Hdg}^p(X,\mathbb{Q})$ denote the image of the cycle class map. The *Hodge conjecture* asserts that $\mathrm{Alg}^p(X,\mathbb{Q}) = \mathrm{Hdg}^p(X,\mathbb{Q})$, see e.g. [Del06], [Hod52].

2.1. **Recollections on CM abelian varieties.** We first recall some basics regarding CM abelian varieties following [Mil20c, Section 1.2].

2.1.1. For abelian varieties A, B over a field k we will write $\mathrm{Hom}^0(A,B) = \mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$, where Hom denotes homomorphisms of abelian varieties over k. If $A = B$ we will denote this by $\mathrm{End}^0(A)$.

2.1.2. Let L be a CM algebra and let M be a field of characteristic zero. An *abelian variety with CM by* L over M is a pair (A,ι), where A is an abelian variety over M and ι is an injective ring homomorphism $\iota : L \to \mathrm{End}^0(A)$ such that $2\,\mathrm{Dim}\,A = [L : \mathbb{Q}]$.

2.1.3. Given a CM abelian variety A over $\mathbb{C}$ with CM by a CM algebra L, we can write $L = L_1 \times \cdots \times L_n$ as a product of CM fields. We now choose a Galois CM field $E \subset \mathbb{C}$ containing L_i for all i. Let $\Sigma = \mathrm{Hom}(L,E)$ be the set of ring homomorphisms from L to E, equipped with its action of $G = \mathrm{Gal}(E/\mathbb{Q})$. As explained in [Haz03b, Section (4.A)], the CM abelian variety A gives rise to a CM type $\Phi \subset \Sigma$ such that $\Sigma = \Phi \coprod \rho\Phi$ for a complex conjugation ρ.

2.1.4. We may write (where we are using singular cohomology with E coefficients)

$$\mathrm{H}^1(A,E) = \bigoplus_{s\in\Sigma} \mathrm{H}^1(A,E)_s$$

as a direct sum of s-isotypic components. Because A has CM by L, the space $\mathrm{H}^1(A,E)_s$ is a one-dimensional E-vector space. Identifying

$$\mathrm{H}^r(A,E) = \bigwedge^r \mathrm{H}^1(A,E),$$

we may thus identify

$$\mathrm{H}^r(A,E) = \bigoplus_{\Delta} \mathrm{H}^r(A,E)_\Delta,$$

where Δ runs over size r subsets of Σ and $L^\times$ acts on $\mathrm{H}^r(A,E)_\Delta$ via $\prod_{s\in\Delta} s$. We may use this to describe the Hodge decomposition $p+q=r$

$$\mathrm{H}^{p,q}(A,E) := \bigoplus_{\substack{\Delta \\ |(\Delta\cap\Phi)|=p,\,|(\Delta\cap\overline{\Phi})|=q}} \mathrm{H}^r(A,E)_\Delta.$$

2.1.5. A result of Pohlmann, see [Poh68, Theorem 1] and [GU25, Theorem 3.1] for the nonsimple case, describes the Hodge classes

$$\mathrm{Hdg}^p(A,\mathbb{Q})\otimes E=\bigoplus_{\Delta}\mathrm{H}^{2p}(A,E)_\Delta,$$

where Δ runs over subsets of Σ of size $2p$ with the property that for all $g\in G$ we have

$$|g\Delta\cap\Phi|=p=|g\Delta\cap\overline{\Phi}|.$$

So in degree 4 we get

$$\mathrm{Hdg}^2(A,\mathbb{Q})\otimes E=\bigoplus_{\substack{\Delta\subset\Sigma,|\Delta|=4\\ \forall g\in G|(g\Delta\cap\Phi)|=2}}\mathrm{H}^4(A,E)_\Delta. \tag{2.1.1}$$

For Δ satisfying the condition of (2.1.1) as above, we say that the *Hodge conjecture holds for* (A,Δ) if $\mathrm{H}^4(A,E)_\Delta$ is in the image of the cycle class map tensored with E. Obviously the Hodge conjecture in codimension two for A follows from the Hodge conjecture for (A,Δ) for all Δ.

2.2. **Results of Hazama.** We first recall the following result of Hazama [Haz03b].

Theorem 2.2.1. *If the Hodge conjecture holds in codimension two for all CM abelian varieties over $\mathbb{C}$, then the Hodge conjecture holds for all CM abelian varieties.*

Proof. This is a special case of [Haz03b, Theorem 8.3], see also [Mil08, Theorem 8.5].[1] □

Remark 2.2.2. In fact, Hazama's result is much stronger. He defines for a Galois CM field E of degree $2n$ with Galois group G, and a hyperplane arrangement $A_{\mathbb{A}(2^n)}$ in $\mathbb{R}^n$ a CM abelian variety $A_{\mathbb{A}(2^n)}(G;E)$. Then [Haz03b, Theorem 8.3] states that the Hodge conjecture for $\mathrm{H}^4(A_{\mathbb{A}(2^n)}(G;E))$ for all $G,E,A_{\mathbb{A}(2^n)}$ implies the generalized Hodge conjecture for all CM abelian varieties.

2.2.3. We could now state a version of Theorem 2.2.1 in terms of the decomposition (2.1.1), and this is exactly what [Ope26, Proposition 2.6] is.

2.3. **A further reduction.** We will now explain how to state [Ope26, Proposition 3.1] in our language.

2.3.1. Continuing with the notation from above, in particular the fixed abelian varieties A with CM by L and CM type Φ, we choose $\Delta\subset\Sigma$ of size 4 satisfying the condition in (2.1.1). Write $\Delta=(s_1,s_2,c\circ s_3,c\circ s_4)$ with $s_1,s_2,s_3,s_4\in\Phi$. First, we associate to $i=1,\cdots,4$ an auxiliary abelian variety B_i which has CM by E (not by L!) and whose CM type is given by

$$\Phi_i:=\ \{g\in G=\mathrm{Hom}(E,E):g\circ s_i\in\Phi\}.$$

[1] Note that [Haz03b, Lemmas 7.11, 7.12] are false as stated, but nonetheless [Haz03b, Theorem 7.13] is still true.

Let $B = B_1 \times \cdots \times B_4$. Note that $\mathrm{Id} \in \Phi_i$ for all i, while $c \in \overline{\Phi}_i$ for all i. Moreover, condition (2.1.1) for Δ translates into the following identity of indicator functions (this is the analogue of the relation $v_1 + v_2 = v_3 + v_4$ occuring in [Ope26])

$$\mathbf{1}_{\Phi_1} - \mathbf{1}_{\overline{\Phi}_1} + \mathbf{1}_{\Phi_2} - \mathbf{1}_{\overline{\Phi}_2} = \mathbf{1}_{\Phi_3} - \mathbf{1}_{\overline{\Phi}_3} + \mathbf{1}_{\Phi_4} - \mathbf{1}_{\overline{\Phi}_4}. \tag{2.3.1}$$

Define $U(i) = H^1(B_i, \mathbb{Q})$ for $i = 1, \cdots, 4$ and let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\mathrm{Id}, \mathrm{Id}, c, c)$.

Lemma 2.3.2. *There is $t \in \mathrm{Hom}^0(A, B)$ such that the pullback $\otimes_i U(i)_{E,\lambda_i} \subset \mathrm{H}^4(B, E)$ is given by $\mathrm{H}^4(A, E)_\Delta \subset \mathrm{H}^4(A, E)$. In particular if the E-vector space $\otimes_i U(i)_{E,\lambda_i}$ is contained in the image of the E-linear cycle class map, then the Hodge conjecture holds for (A, Δ).*

Proof. For $i = 1, \cdots, 4$, the morphism $s_i : L \to E$ factors through a simple factor of L; let us call these factors L_{j_i} (they do not have to be distinct) and let us write $\Sigma_{j_i} = \mathrm{Hom}(L_{j_i}, E)$. Each i determines an isogeny factor A_{j_i} of A, and we will construct t by constructing $t_i \in \mathrm{Hom}(A_{j_i}, B_i)$ for $i = 1, \cdots, 4$.

The morphism t_i is constructed as follows. After choosing an L_{j_i}-linear isomorphism $\mathrm{H}^1(A_{j_i}, \mathbb{Q}) \simeq L_{j_i}$ and an E-linear isomorphism $\mathrm{H}^1(B_i, \mathbb{Q}) \simeq E$, we consider the map (induced by the trace map)

$$\mathrm{H}^1(B_i, \mathbb{Q}) = E \xrightarrow{\mathrm{Tr}_{E/s_i(L_{j_i})}} s_i(L_{j_i}) \xrightarrow{s_i^{-1}} L_{j_i} = \mathrm{H}^1(A_{j_i}, \mathbb{Q}).$$

The lemma follows if we can show that a) this is a map of rational Hodge structures and b) that this takes $\mathrm{H}^1(B_i, E)_g$ to $\mathrm{H}^1(A_{j_i}, E)_{g \circ s_i}$ after extending scalars to E. After extending scalars to E, we can identify the map with the E-linear map

$$\bigoplus_G E \to \bigoplus_{\Sigma_{j_i}} E$$

which is the direct sum of the maps for $s \in \Sigma_{j_i}$

$$\bigoplus_{\substack{g \in G \\ g \circ s_i = s}} E \to E$$

given by $(e_g) \mapsto \sum e_g$. Claim b) follows directly from this description, and Claim a) follows from the definition of the CM type Φ_i. □

2.3.3. Finally, we can state a version of [Ope26, Proposition 3.1].

Theorem 2.3.4. *If there is a smooth projective surface S over $\mathbb{C}$ together with maps of $\mathbb{Q}$-Hodge structures*

$$h_i : U(i) \to \mathrm{H}^1(S, \mathbb{Q})$$

for $i = 1, 2, 3, 4$, and nonzero vectors $\alpha_i \in U(i)_{E,\lambda_i}$ such that

$$h_1(\alpha_1) \cup h_2(\alpha_2) \cup h_3(\alpha_3) \cup h_4(\alpha_4) \neq 0$$

then the Hodge conjecture holds for (A, Δ).

Proof. Assume without loss of generality that S is connected. By Lemma 2.3.2, it suffices to prove that the one-dimensional E vector space

$$\bigotimes_i U(i)_{E,\lambda_i} \subset \mathrm{H}^4(B,E)$$

is in the image of the cycle class map tensored with E. By the Hodge conjecture for homomorphisms of abelian varieties and the theory of the Albanese variety, there are morphisms $f_i : S \to B_i$ such that $h_i \cdot m_i = f_i^*$ for some nonzero integer m_i (this is explained in [Ope26, Lemma 2.2]). Let $f = (f_1, \cdots, f_4)$ and for $n = \operatorname{Dim} B$, we let $Z = f_*[S] \in \mathrm{CH}^{n-2}(B)$. If $x = \alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \alpha_4$, then the projection formula tells us that

$$\int_B Z \cup x = m_1 h_1(\alpha_1) \cup \cdots \cup m_4 h_4(\alpha_4) \neq 0,$$

showing that $Z \neq 0$. Next, let e_i be the idempotent element of $E \otimes_{\mathbb{Q}} E \subset \mathrm{End}^0(B_i) \otimes_{\mathbb{Q}} E$ whose induced map on $U(i)_E = \mathrm{H}^1(B_i, E)$ is projection onto $U(i)_{E,\lambda_i}$. Let $p_i = p_{i,1}$ be the Chow–Künneth idempotent inducing the projection $\mathrm{H}^*(B_i, \mathbb{Q}) \to \mathrm{H}^1(B_i, \mathbb{Q})$; this is algebraic by [DM91, Main Theorem]. Then the composition $e_i p_i$ acts on cohomology by projection $\mathrm{H}^*(B_i, E) \to \mathrm{H}^1(B_i, E) \to U(i)_{E,\lambda_i}$. We can now consider $e = \otimes_i e_i p_i$, which induces the projection $\mathrm{H}^4(B, E) \to \otimes_i U(i)_{E,\lambda_i}$.

Consider $x = \alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \alpha_4$ and note that $ex = x$. Under the E-linear intersection product

$$\mathrm{H}^{2n-4}(B,E) \times \mathrm{H}^4(B,E) \to E$$

we have seen that Z pairs nontrivially with $x = ex$. This implies that $e^t Z$ also pairs nontrivially with x, and thus $e^t \mathrm{H}^{2n-4}(B,E)$ is nonzero. It is moreover one-dimensional since $e\mathrm{H}^4(B,E)$ is. Now, because numerical and homological equivalence agree for cycles on CM abelian varieties, see [Lie68], the pairing

$$\mathrm{Alg}^{n-2}(B,E) \times \mathrm{Alg}^2(B,E) \to E$$

is perfect. After applying the projector e and its transpose e^t, the pairing

$$e^t \,\mathrm{Alg}^{n-2}(B,E) \times e\,\mathrm{Alg}^2(B,E) \to E$$

is also perfect. Because $e^t \,\mathrm{Alg}^{n-2}(B,E)$ is nonzero and spanned by $e^t Z$, it follows that $e\,\mathrm{Alg}^2(B,E)$ is nonzero and one-dimensional. Since $e\mathrm{H}^4(B,E)$ is one-dimensional, it follows that $e\,\mathrm{Alg}^2(B,E) = e\mathrm{H}^4(B,E)$. Thus $x = ex$ is in the image of the E-linear cycle class map. □

3. Picard modular surfaces

To apply Theorem 2.3.4, we need to produce an algebraic surface S whose $\mathrm{H}^1(S, \mathbb{Q})$ contains the Hodge structures of CM abelian varieties with CM by a given CM field E. Fortunately, for us, such surfaces are known to exist by work of Liu, see [Liu21, Section 4], as we will explain below.

3.1. **Picard modular surfaces and their Albanese varieties.**

3.1.1. Let E be a CM field with totally real subfield $F \neq \mathbb{Q}$ and fix an embedding $\tau_0 : E \to \mathbb{C}$. Let V be a Hermitian E vector space of dimension 3 such that V has signature $(2,1)$ at $\tau_0|_F$ and has signature $(3,0)$ at all other infinite places of F. Let $\mathsf{G} = \mathrm{U}(V)$ be the algebraic group over F of isometries of V, and let $\mathsf{H} = \mathrm{Res}_{F/\mathbb{Q}}\mathsf{G}$ equipped with its natural Shimura datum (H, Y). Note that the reflex field of (H, Y) is equal to $E = \tau_0(E) \subset \mathbb{C}$.

3.1.2. For $K \subset \mathsf{H}(\mathbb{A}_f)$ a sufficiently small compact open subgroup, there is a smooth projective algebraic surface $\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})$ over $\operatorname{Spec} E$, see e.g. [Liu21, Proposition C.5]. Being a Shimura variety, it will typically not be geometrically connected. The goal of this section is to compute the rational Hodge structures occuring in $\mathrm{H}^1(\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})_{\mathbb{C}}, \mathbb{Q})$ up to isogeny. We observe that

$$\mathrm{H}^1(\mathbf{Sh}(\mathsf{H}, \mathsf{Y})_{\mathbb{C}}, \mathbb{C}) := \varinjlim_K \mathrm{H}^1(\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})_{\mathbb{C}}, \mathbb{C})$$

is equipped with a smooth action of $\mathsf{H}(\mathbb{A}_f)$, which acts via isomorphisms of rational Hodge structures. Our goal is now to compute this space, following [BMM16] and [Liu21].

3.1.3. Let $\mathbb{A}_E^\times$ denotes the ideles of E, a locally compact topological group containing $E^\times$. Recall from [Liu21, Definition 4.1, Definition 4.3] that a *conjugate symplectic automorphic character of weight one* is a continuous homomorphism

$$\mu : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$$

such that its restriction to $\mathbb{A}_F^\times$ is given by the quadratic character $\mu_{E/F}$ associated to the quadratic extension E/F via global class field theory, and such that there is a CM type Φ_μ of E such that for every $\tau \in \Phi_\mu$ we have that

$$\mu|_{E \otimes_{\tau,F} \mathbb{R}}$$

is given by $z \mapsto \frac{|z|}{z}$. We call Φ_μ the *CM type* of μ, which is unique. Associated to Φ_μ is a reflex field $M'_\mu \subset \mathbb{C}$, and a CM type Ψ_μ for M'_μ. We let $M_\mu \subset \mathbb{C}$ be as in [Liu21, Definition 4.3], which is a number field containing M'_μ.

3.1.4. Recall from [Liu21, Definition 4.11] that an *adelic oscillator triple* is a triple (μ, ϵ, χ), where μ is a conjugate symplectic automorphic character of weight one, where $\epsilon \in \prod'_{v \text{finite}} E_v^{-\times}/N_{E_v/F_v}E_v^\times$ and where χ is an auxiliary character that will not matter for our discussion.[2] We say that (μ, ϵ, χ) is *admissible*, see [Liu21, Definition 4.12], if there is an $e \in E^\times$ such that $e^c = -e$ and $\operatorname{Im}\tau(e) < 0$ for $\tau \in \Phi_\mu$ with $\epsilon = (eN_{E_v/F_v}E_v^\times)_v$.

Lemma 3.1.5. *For each CM type Φ of E there is a conjugate symplectic automorphic character μ of weight one with $\Phi_\mu = \Phi$ and an admissible adelic oscillator triple (μ, ϵ, χ).*

[2] We only consider adelic oscillator triples with μ of weight one, whereas [Liu21, Definition 4.11] allows for general μ.

Proof. Applying [CHT08, Lemma 4.1.4] with $I = \Phi, S = \emptyset, a = \|-\|_F^{-1}$ and with $m_\tau = -1$, we get a character[3]

$$\psi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$$

such that, for $N_{E/F} : \mathbb{A}_E^\times \to \mathbb{A}_F^\times$ the norm map, we have

$$\psi|_{\mathbb{A}_F^\times} \circ N_{E/F} = a \circ N_{E/F}$$

and such that

$$\psi|_{(E\otimes\mathbb{R})^\times} = \prod_{\tau\in I} \tau^{-1} \cdot (c\tau)^{w+1}.$$

for some w. In fact it follows from [CHT08, proof of Lemma 4.1.4] that $w = -1$ for our a, which implies that for $\mu = \psi\|-\|_E^{1/2}$ and for $\tau \in \Phi$ we have

$$\left(\mu|_{E\otimes_{\tau,F}\mathbb{R}}\right)(z) = \frac{|z|}{z}.$$

We now claim that μ is a conjugate symplectic automorphic character of weight one. To show that the restriction of μ to $\mathbb{A}_F^\times$ is $\mu_{E/F}$, we note that

$$\begin{aligned}(\mu \circ N_{E/F})(x) &= \psi(N_{E/F}(x)) \cdot \|N_{E/F}(x)\|_E^{1/2} \\ &= \|N_{E/F}(x)\|_F^{-1} \cdot \|N_{E/F}(x)\|_F \\ &= 1\end{aligned}$$

so that $\mu|_{\mathbb{A}_F^\times}$ factors through $F^\times\backslash\mathbb{A}_F^\times/N_{E/F}(\mathbb{A}_E^\times) \simeq \mathrm{Gal}(E/F)$ (see [Mil20a, Theorem] for the last identification). Thus $\mu|_{\mathbb{A}_F^\times}$ is either trivial or equals $\mu_{E/F}$, and the former possibility can be excluded by examining the further restriction of μ to $(F\otimes\mathbb{R})^\times$.

To find a μ-admissible adelic oscillator triple for the given μ, we may take $\chi = 1$, and $\epsilon = (eN_{E_v/F_v}E_v^\times)_v$ for an element $e \in E^\times$ satisfying (where c is the nontrivial F-linear automorphisms of E).

$$e^c = -e$$
$$\forall \tau \in \Phi_\mu \quad \mathrm{Im}\,\tau(e) < 0.$$

This is clearly a μ-admissible adelic oscillator triple. The existence of such e is a routine application of weak approximation. □

3.1.6. Associated to an adelic oscillator triple (μ, ϵ, χ) there is an admissible representation $\omega(\mu, \epsilon, \chi)$ of $\mathsf{H}(\mathbb{A}_f)$. We have the following result, see [Liu21, Proposition 4.13]

[3]To be precise, we use (in the notation of [CHT08]) $\chi = a = \|-\|_F^{-1}$. This notation unfortunately conflicts with our notation.

Proposition 3.1.7. *There is an isomorphism of* $\mathsf{H}(\mathbb{A}_f)$*-representations*

$$\mathrm{H}^1(\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathbb{C}},\mathbb{C}) \xrightarrow{\sim} \bigoplus_{(\mu,\epsilon,\chi)} \omega(\mu,\epsilon,\chi),$$

where the direct sums runs over admissible adelic oscillator triples (μ,ϵ,χ) *with* μ *of weight one.*

3.1.8. We now study the Hodge structure on the cohomology. Let A_K denote the Albanese variety of $\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})$ over the field E, in the sense of [Liu21, Definition 2.3]. Consider the pro-(abelian variety)

$$A_\infty := \varprojlim_K A_K$$

over E together with its action of $\mathsf{G}(\mathbb{A}_{F,f})$.

3.1.9. Associated to a conjugate symplectic automorphic character of weight one μ is a notion of a *CM datum*, see [Liu21, Definition 4.5], which is a tuple $(A_\mu, i_\mu, \lambda_\mu, r_\mu)$. It is not so important what the last two entries are, but A_μ is an abelian variety over E and $i_\mu : M_\mu \to \mathrm{End}(A_\mu)\otimes\mathbb{Q}$ is a CM structure with CM type determined by Ψ_μ. By [Liu21, Proposition 4.6], CM data always exist and are, roughly speaking, unique up to isogeny. In particular the isogeny class of A_μ is a well defined function of μ.

3.1.10. For a fixed conjugate symplectic automorphic character of weight one μ we consider

$$\Omega(\mu) = \mathrm{Hom}(A_\infty, A_\mu)\otimes\mathbb{Q},$$

where we are taking homomorphisms of group schemes over E. The field M_μ acts on A_μ and $\mathsf{H}(\mathbb{A}_f)$ acts on A_∞, and so $\Omega(\mu)$ is an M_μ-linear representation of $\mathsf{H}(\mathbb{A}_f)$. For every ϵ which is part of a μ-admissible adelic oscillator triple (μ,ϵ,χ), there is a $\mathsf{H}(\mathbb{A}_f)$-subrepresentation $\Omega(\mu,\epsilon)\subset\Omega(\mu)$, see [Liu21, Definition 4.19]. We have the following result, see [Liu21, Theorem 4.18], recalling that M_μ is a subfield of $\mathbb{C}$.

Theorem 3.1.11. *There is an isomorphism*

$$\Omega(\mu,\epsilon)\otimes_{M_\mu}\mathbb{C} = \bigoplus_\chi \omega(\mu,\epsilon,\chi),$$

where the direct sum runs over those χ *such that* (μ,ϵ,χ) *is an admissible adelic oscillator triple.*

3.1.12. Theorem 3.1.11 tells us which abelian varieties up to isogeny occur in A_∞, and thus in A_K. Basechanging to $\mathbb{C}$, this therefore tells us which Hodge structure occur in $\mathrm{H}^1(\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathbb{C}},\mathbb{C})$ and with which level. In particular, we can deduce

Corollary 3.1.13. *Suppose that* E *is Galois. Given a CM abelian variety* B *over* $\mathbb{C}$ *with CM by* E*, there is a surjective homomorphism* $A_{\infty,\mathbb{C}}\to B$ *of group schemes.*

Proof. Let Φ be the CM type of B. The CM type Φ is the extension from a primitive CM type Φ_0 on a subfield $E_0 \subset E$, see [Mil20b, Proposition 1.9]. If B_0 is an abelian variety over $\mathbb{C}$ with CM by E_0 and CM type Φ_0, then $B \sim B_0^{[E:E_0]}$, see [Mil20b, Proof of Proposition 3.13]. On the other hand, for a CM type Φ_μ with reflex field (M'_μ, Ψ_μ), we have by construction that

$$A_{\mu,\mathbb{C}} \sim (A'_{\mu,\mathbb{C}})^{[M_\mu : M'_\mu]},$$

where A'_μ is an abelian variety with CM by M'_μ and CM type Ψ_μ. To relate B_0 and A'_μ, we need to relate (E_0, Φ_0) and (M'_μ, Ψ_μ). Because E is Galois, by [Mil20b, Example 1.19], we have $(E_0, \Phi_0) = (M'_\mu, \Psi_\mu)$ if we choose $\Phi_\mu = \Phi^{-1}$ (here we consider $\Phi \subset \mathrm{Gal}_E$ and take its inverse as a subset of the group).

Now choose an adelic oscillator triple (μ, ϵ, χ) with $\Phi_\mu = \Phi^{-1}$. Then by Theorem 3.1.11 A_∞ admits a nonzero map to A_μ. Since $\omega(\mu, \epsilon, \chi)$ is infinite dimensional, it in fact admits infinitely many linearly independent maps to A_μ. The group scheme $A_{\infty,\mathbb{C}}$ therefore surjects onto any power of $A'_{\mu,\mathbb{C}}$. From the discussion above, it follows that $A_{\infty,\mathbb{C}}$ surjects onto any power of $A'_{\mu,\mathbb{C}} \sim B_0$ and thus surjects onto B. □

4. Existence of classes with a nonzero cup product

4.0.1. We continue with the notation from Section 2. In particular we have a fixed CM algebra $L = L_1 \times \cdots \times L_n$, a Galois CM field $E \subset \mathbb{C}$ in which each L_i embeds, a CM type $\Phi \subset \mathrm{Hom}(L, E)$ and an abelian variety A over $\mathbb{C}$ with CM by E and CM type Φ. We choose $\Delta = \{s_1, s_2, c \circ s_3, c \circ s_4\} \subset \Sigma$ satisfying the condition in (2.1.1), so that Δ corresponds to a codimension 2 Hodge class on A, and define auxiliary abelian varieties $B_1, \cdots, B_4$ as in Section 2.3.1. Write $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\mathrm{Id}, \mathrm{Id}, c, c)$ as before.

Choose a Hermitian E vector space V of dimension three with signature $(2, 1)$ at the distinguished embedding of E into $\mathbb{C}$ and signature $(3, 0)$ at the other infinite places, as in Section 3. We let $\mathsf{G} = U(V)$ and $\mathsf{H} = \mathrm{Res}_{F/\mathbb{Q}} \mathsf{G}$, where F is the totally real subfield of E, and we equip H with its natural Shimura datum (H, Y). We have the following theorem, which together with Theorem 2.3.4 and Theorem 2.2.1 implies Theorem I.

Theorem 4.0.2. *There is a compact open subgroup $K \subset \mathsf{H}(\mathbb{A}_f)$, morphisms $h_i : \mathrm{H}^1(B_i, \mathbb{Q}) \to \mathrm{H}^1(\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})_{\mathbb{C}}, \mathbb{Q})$ of rational Hodge structures for $i = 1, \cdots, 4$ and classes $\alpha_i \in \mathrm{H}^1(B_i, E)_{\lambda_i}$ for $i = 1, \cdots, 4$ such that*

$$h_1(\alpha_1) \cup h_2(\alpha_2) \cup h_3(\alpha_3) \cup h_4(\alpha_4) \neq 0.$$

Our proof of Theorem 4.0.2 relies crucially on a very special case of the results of [BMM16], [Liu21] as we now explain.

4.1. **Cohomological theta lifting.** We will need to consider several integrals over adelic class sets $\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A}_F)$ where F is some number field and G is a connected algebraic group, in all such cases we will take this integral with respect to a probability measure on $\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A}_F)$, which will always exist in the examples we consider.

4.1.1. Let V be our Hermitian space over E as above, and let $v_0 = \tau_0|_F$ denote the distinguished real place at which V has non-definite signature. We choose an auxiliary one-dimensional skew-Hermitian space W over E, such that at the place v_0 it has signature (b, a). We equip $\mathbb{W} = V \otimes_E W$ with the structure of a symplectic space over F as in [BMM16, Section 7.1]. We choose a so-called polarization $\mathbb{W} = X \oplus Y$, where X is a Lagrangian F-subspace in $\mathbb{W}$ and Y is a complementary Lagrangian subspace. We consider the symplectic group $\mathrm{Sp}(\mathbb{W})$ over F and the metaplectic cover

$$\mathrm{Mp}(\mathbb{W})(\mathbb{A}_F) \to \mathrm{Sp}(\mathbb{W})(\mathbb{A}_F).$$

4.1.2. Let $\mu_{E/F}$ be as above. By Kudla's splitting construction in [Kud94], in the adelic formulation recalled in [Liu21, B.2], if we choose a pair of automorphic splitting characters $\nu = (\nu_6, \nu_5)$ of $\mathbb{A}_E^\times$ such that $\nu_i|_{\mathbb{A}_F^\times} = \mu_{E/F}$, then we have a homomorphism

$$\widetilde{\iota}_\nu : U(V)(\mathbb{A}_F) \times U(W)(\mathbb{A}_F) \to \mathrm{Mp}(\mathbb{W})(\mathbb{A}_F)$$

which lifts the natural map $\iota : U(V)(\mathbb{A}_F) \times U(W)(\mathbb{A}_F) \to \mathrm{Sp}(\mathbb{W})(\mathbb{A}_F)$. Note that we follow the convention of [Liu21] for the ordering of $\nu = (\nu_6, \nu_5)$; we use ν_6 for the character ν_V and ν_5 for the character ν_W as in [Liu21, §B.2].

4.1.3. For $\psi = \psi_F$ the standard additive character of [Liu21, §1.7], we can form the automorphic Weil representation ω_ψ of the metaplectic group, see [BMM16, §7.1]. Thus the metaplectic group $\mathrm{Mp}(\mathbb{W})(\mathbb{A}_F)$ acts on the Schwartz space $\mathcal{S}(X(\mathbb{A}_F))$, we let $\mathbf{S}(X(\mathbb{A}_F))$ denote the Schrödinger realization of the polynomial Fock space as defined in [BMM16, §7.1], which is dense in $\mathcal{S}(X(\mathbb{A}_F))$. We let $(U(V) \times U(W))(\mathbb{A}_F)$ act on $\mathcal{S}(X(\mathbb{A}_F))$ via the embedding $\widetilde{\iota}_\nu$. We finally consider $\mathbf{S}_{v_0} = \mathbf{S}(X(F_{v_0}))$ and

$$\mathbf{S}_\infty^{v_0} := \bigotimes_{\substack{v:F\to\mathbb{R} \\ v\neq v_0}} \mathbf{S}(X(F_v)),$$

and $\mathcal{S}_f = \mathcal{S}(X(\mathbb{A}_{F,f}))$, and define $\mathcal{S}_\infty^{v_0}, \mathcal{S}_{v_0}$ analogously to the above.

4.1.4. For each $\phi \in \mathbf{S}(X(\mathbb{A}_F))$, we can form the theta function

$$\theta_{\psi,\nu,\phi}(x) = \sum_{\xi \in X(F)} \omega_\psi(x)(\phi)(\xi),$$

which is a function on $\mathrm{Mp}(\mathbb{W})(\mathbb{A}_F)$. Pulling back along $\widetilde{\iota}_\nu$, we get a theta function

$$\theta_{\psi,\nu,\phi}$$

on $U(V)(F)\backslash U(V)(\mathbb{A}_F) \times U(W)(F)\backslash U(W)(\mathbb{A}_F)$. From now on, we will always assume that ν_6 is a conjugate symplectic automorphic character of weight one such that the CM type Φ_{ν_6} contains the distinguished embedding $\tau_0 : E \to \mathbb{C}$ (that we usually write as Id).

4.1.5. Letting $\mathfrak{h}_0$ denote the Lie algebra of $G = \mathsf{G} \otimes_{F,v_0} \mathbb{R} = \mathrm{U}(2,1)$, we have $\mathfrak{h}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where $\mathfrak{k}_0$ is the Lie algebra of the maximal compact subgroup $K_\infty = U(2) \times U(1)$ of G.[4] We denote by $\mathfrak{p}$ the complexification of $\mathfrak{p}_0$. Denote by (where the notation $\wedge^{1,0}$ is explained in [BMM16, §1.2])

$$\varphi_{1,0} \in \mathrm{Hom}_{K_\infty}(\wedge^{1,0}\mathfrak{p}, \mathcal{S}_{v_0})$$

the cocycle constructed in [BMM16, §5, specifically §5.6]. As in [BMM16, Section 8.6], we can think of $\varphi_{1,0}$ as an element of

$$\left[\mathcal{S}_{v_0} \otimes \Omega^{1,0}_{\mathsf{Y}}\right]^{G(\mathbb{R})}.$$

For $\varphi \in \mathcal{S}_f$, we then consider (where φ_0 is as in [BMM16, §8.7])

$$\phi = \varphi_{1,0} \otimes_v (\varphi_0) \otimes \varphi \in \left[(\mathcal{S}_{v_0} \otimes \mathcal{S}^{v_0}_\infty \otimes \mathcal{S}_f) \otimes \Omega^{1,0}_{\mathsf{Y}}\right]^{\mathsf{H}(\mathbb{R})}.$$

4.1.6. Now for $g' \in U(W)(\mathbb{A}_F)$ we can consider the theta function

$$\theta_{\psi,\nu,\phi}(g, g')$$

as a function of g. If ϕ is K-invariant for $K \subset \mathsf{H}(\mathbb{A}_f)$ a compact open subgroup, then as in [BMM16, §8.8], the function $\theta_{\psi,\nu,\phi}(-, g')$ can be interpreted as an element

$$\theta_{\psi,\nu,\phi}(-, g') \in H^0(\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})_{\mathbb{C}}, \Omega^{1,0}_{\mathsf{Y}}).$$

Altogether, this defines a map

$$\begin{aligned}\Theta^{1,0}_{\psi,\nu} : \mathcal{S}_f \times U(W)(F)\backslash U(W)(\mathbb{A}) &\to \varinjlim_K H^0(\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})_{\mathbb{C}}, \Omega^{1,0}_{\mathsf{Y}}) \\ (\phi, g') &\mapsto (g \mapsto \theta_{\psi,\nu,\phi}(g, g'))\,.\end{aligned}$$

Now each $\theta_{\psi,\nu,\phi}(-, g')$ is in fact a closed $(1,0)$-form, see [BMM16, Lemmas 5.7 and 5.13]. Thus we can consider the induced map

$$[\Theta^{1,0}_{\psi,\nu}] : \mathcal{S}_f \times U(W)(F)\backslash U(W)(\mathbb{A}) \to \varinjlim_K H^{1,0}(\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})_{\mathbb{C}}, \mathbb{C}).$$

For $f \in L^2(U(W)(F)\backslash U(W)(\mathbb{A}_F))$ we also consider the integrated θ map

$$\Theta^{1,0}_{\psi,\nu,f} : \mathcal{S}_f \to \varinjlim_K H^0(\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})_{\mathbb{C}}, \Omega^{1,0}_{\mathsf{Y}}).$$

and its cohomological realization $[\Theta^{1,0}_{\psi,\nu,f}]$. We remark that $U(W) \simeq U(1)$ so that $U(W)(F)\backslash U(W)(\mathbb{A})$ is compact.

[4] The notation G here should not be confused with the use of $G = \mathrm{Hom}(E, E)$ in Section 2.

4.1.7. Our goal is now to understand the image of $[\Theta^{1,0}_{\psi,\nu,f}]$. We recall the theta transfer notation $\Theta^V_{\psi,\nu,W}$ from [BMM16, §7.3]. Given an automorphic representation $\pi \subset L^2(U(W)(F)\backslash U(W)(\mathbb{A}_F))$ of $U(W)(\mathbb{A}_F)$ we consider the subspace

$$\Theta^V_{\psi,\nu,W}(\pi) \subset C^\infty(\mathsf{G}(F)\backslash \mathsf{G}(\mathbb{A}_F), \mathbb{C})$$

generated by the functions

$$g \mapsto \int_{U(W)(F)\backslash U(W)(\mathbb{A}_F)} \theta_{\psi,\nu,\phi}(g,g')f(g')\mathrm{d}g'$$

as f runs over all vectors in π and as ϕ varies over $\mathbf{S}(X(\mathbb{A}_F))$. It is explained in [Liu21, Proof of Proposition 4.13, last paragraph], that for every admissible adelic oscillator triple (μ,ϵ,χ) there is a skew-Hermitian space W over E of dimension one, an automorphic character π of $U(W)(\mathbb{A})$ such that (where $A(1,0)$ is as in [BMM16, §6.7])

$$\Theta^V_{\psi,(\mu,\nu_5),W}(\pi) \simeq A(1,0) \otimes \omega(\mu,\epsilon,\chi)$$

for some automorphic character ν_5. Moreover, (W,π) is unique up to isomorphism and W has signature $(0,1)$ at v_0 (see the proof of [BMM16, Theorem 9.1]). Finally, we note that we can use the same ν_5 for different (μ,ϵ,χ) since changing ν_5 just amounts to twisting π by a character. So let us fix ν_5 from now on.

Proposition 4.1.8. *Consider an admissible adelic oscillator triple (μ,ϵ,χ) with corresponding skew-Hermitian space W of signature $(0,1)$ and let $\nu = (\mu,\nu_5)$ and π be as above. If $\Phi_\mu = \Phi_i^{-1}$ with Φ_i as in Section 4.0.1, then for every $f \in \pi$ the image of $[\Theta^{1,0}_{\psi,\nu,f}]$ is contained in the image of the realization map*

$$\mathrm{Hom}^0(A_{\infty,\mathbb{C}}, B_i) \otimes_{\mathbb{Q}} \mathrm{H}^1(B_i,\mathbb{C})_{\mathrm{Id}} \to \varinjlim_K H^{1,0}(\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})_{\mathbb{C}},\mathbb{C}).$$

Furthermore, there exists $f \in \pi$ such that $[\Theta^{1,0}_{\psi,\nu,f}]$ is nonzero.

Proof. This is a direct consequence of [BMM16, Theorem 9.1] and its proof. To elaborate, it is established there that the subspace spanned by the images of $[\Theta^{1,0}_{\psi,\nu,f}]$ as we vary over all (W,ν) and all f is surjective onto $\mathrm{H}^{1,0}(\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathbb{C}},\mathbb{C})$. Since $\Theta^V_{\psi,(\mu,\nu_5),W}(\pi) \simeq A(1,0)\otimes\omega(\mu,\epsilon,\chi)$ and (W,π) is unique with this property, we see that $[\Theta^{1,0}_{\psi,\nu,f}]$ must land in $\omega(\mu,\epsilon,\chi)$, and moreover we can find $f \in \pi$ for which $[\Theta^{1,0}_{\psi,\nu,f}]$ is nonzero.

For the first claim, we recall $\Omega(\mu) = \mathrm{Hom}^0(A_\infty, A_\mu)$, and recall that $M_\mu \subset \mathbb{C}$. If we consider the realization map

$$\Omega(\mu) \otimes_{M_\mu} H^1(A_\mu,\mathbb{C})_{\mathrm{Id}} \to \varinjlim_K H^{1,0}(\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})_{\mathbb{C}},\mathbb{C}),$$

then this has image $\bigoplus_{\epsilon,\chi}\omega(\mu,\epsilon,\chi)$ by the proof of [Liu21, Theorem 4.18], where the direct sum runs over all ϵ,χ such that (μ,ϵ,χ) is a μ-admissible adelic oscillator triple. Now as in the proof of Corollary 3.1.13 we have

$$\mathrm{Hom}^0(A_{\infty,\mathbb{C}}, A_{\mu,\mathbb{C}}) = \mathrm{Hom}^0(A_{\infty,\mathbb{C}}, A'_{\mu,\mathbb{C}}) \otimes_{M'_\mu} M_\mu,$$

and so the natural evaluation map

$$\mathrm{Hom}^0(A_{\infty,\mathbb{C}}, A'_{\mu,\mathbb{C}}) \otimes_{M'_\mu} H^1(A'_{\mu,\mathbb{C}}, \mathbb{C})_{\mathrm{Id}} \to \varinjlim_K H^{1,0}(\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})_{\mathbb{C}}, \mathbb{C}).$$

has image containing $\bigoplus_{\epsilon,\chi} \omega(\mu,\epsilon,\chi)$. Similarly $\mathrm{Hom}^0(A_{\infty,\mathbb{C}}, B_i) \otimes_E \mathbb{C} = \mathrm{Hom}^0(A_{\infty,\mathbb{C}}, A'_{\mu,\mathbb{C}}) \otimes_{M'_\mu} \mathbb{C}$ as the proof of Corollary 3.1.13. Therefore, the realization map of the lemma indeed contains $\bigoplus_{\epsilon,\chi} \omega(\mu,\epsilon,\chi)$ in its image. □

4.2. **A doubling identity.** For $i = 1, \dots, 4$ we let W_i denote the unitary space isomorphic to E with CM type given by Φ_i^{-1}, where Φ_i is the CM type of B_i as in Section 4.0.1. For each $i = 1, 2, 3$ we choose a conjugate symplectic automorphic character of weight one

$$\mu_i : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$$

with CM type given by Φ_i^{-1}; this is possible by Lemma 3.1.5. We choose μ_4 so that $\mu_1\mu_2 = \mu_3\mu_4$, and note that μ_4 is a conjugate symplectic automorphic character of weight one and CM type Φ_4^{-1} by (2.3.1). For now we fix d a nonzero element of E such that $d^c = -d$. For $e = da$, $a \in F^\times$, we can consider the skew-Hermitian line associated to the element e.

Lemma 4.2.1. *We may choose the skew-Hermitian lines $W_i = E \cdot e_i$ such that there is an F-rational isometry*

$$\Psi : W_1 \perp W_2 \xrightarrow{\sim} W_3 \perp W_4$$

and the such that the corresponding tuples $(\mu_i, \epsilon_i, \chi_i)$ is a μ_i-admissible adelic oscillator triple as in Section 3.1.4.

Proof. Weak approximation allows us to choose e_1, e_2 with the right signs at infinity. Then the condition $\mu_1\mu_2 = \mu_3\mu_4$ forces the angular part of the character μ_3 to match with either that of μ_1 or μ_2 at each place above infinity. So we can choose $e_3 \in (W_1 \perp W_2)(E)$ close enough to either e_1 or e_2 depending on the place using weak approximation, so that the signs agree with those of μ_3. Finally we can just take $W_4 = W_3^\perp$. □

4.2.2. Now we fix skew-Hermitian spaces $W_1, \cdots, W_4$ satisfying the conclusion of Lemma 4.2.1 and an F-rational isometry Ψ, define $\mathbb{W}_i = V \otimes_E W_i$ as a symplectic space and choose polarizations $\mathbb{W}_i = X_i \oplus Y_i$ as in Section 4.1. We define $W = W_1 \perp W_2$ which we identify using Ψ with $W_3 \perp W_4$ and $\mathbb{W} = V \otimes_E W$, and $X = X_1 \oplus X_2, Y = Y_1 \oplus Y_2$.

4.2.3. We let $-W$ denote the skew Hermitian space which is isomorphic to W but with the skew-Hermitian form $-\Psi_W$. We define the skew-Hermitian space $W^\square := W \perp -W$, and after tensoring $V \otimes_E W^\square$ we obtain a symplectic F-vector space $\mathbb{S}$ as in [BMM16, Section 7.1]. Taking the diagonal W^Δ and anti-diagonal W^∇ totally isotropic E-subspaces of $W^\square$, we get an F-polarization

$$\mathbb{S} = \left(V \otimes_E W^\Delta\right) \oplus \left(V \otimes_E W^\nabla\right).$$

4.2.4. From now on, we will take ψ to be as above, and take $\nu_i = (\mu_i, \nu_5)$ to form our oscillator representations. Fixing a basis for W^∇ we obtain an identification $V^2(\mathbb{A}_E) \simeq (V \otimes W^\nabla)(\mathbb{A}_F)$, and thus a Weil representation ω of $\mathsf{G} \times U(W^\square)$ on $\mathcal{S}(V^2(\mathbb{A}_E))$ with splitting character $(1, \nu_5)$.

Remark 4.2.5. Since we take the splitting characters μ_i to be unitary, and the Weil representation ω_{-W} to be the conjugate of ω_W, the oscillator representation of $\mathsf{G}(\mathbb{A}_F)$ on $\mathcal{S}(V(\mathbb{A}_E)^2)$ arises simply from the diagonal pullback action on Schwartz functions, as the splitting characters cancel. In addition, the doubling intertwiner below will intertwine the diagonal action of $\mathsf{G}(\mathbb{A}_F)$ on the Schrödinger models. These are small but crucial ways in which we use the hypothesis that $\mu_1\mu_2 = \mu_3\mu_4$ in the proof of Theorem 4.0.2.

As in the discussion of [Li92, §2] on Rallis' method, there is an intertwiner

$$\delta : \mathcal{S}(X(\mathbb{A}_F)^2) \to \mathcal{S}((V \otimes W^\nabla)(\mathbb{A}_F)),$$

given by

$$\delta(\Phi)((x,y),(-x,-y)) = \int_{X(\mathbb{A}_F)} \psi(2\langle u, y\rangle)\Phi(u+x, u-x)\, du,$$

and this doubling intertwiner is equivariant for the actions of the metaplectic group $\mathrm{Mp}(\mathbb{S})(\mathbb{A}_F)$. It will be quite important later that under this identification we have, also as in [Li92, §2],

$$\delta(\phi_1 \otimes \bar{\phi}_2)(0) = \langle \phi_1, \phi_2 \rangle.$$

For the benefit of the reader, we recall a standard result in the theory of doubling integrals which we will invoke below.

Proposition 4.2.6. *Let $\Phi = \phi_1 \otimes \phi_2 \in \mathcal{S}(X(\mathbb{A}_F)) \otimes \mathcal{S}(X(\mathbb{A}_F)) \subset \mathcal{S}(X(\mathbb{A}_F)^2)$, then*

$$\sum_{x \in (V \otimes_E W^\nabla)(F)} \delta(\Phi)(x) = \sum_{y \in X(F)^2} \Phi(y).$$

Proof. This is, for instance, a special case (in his notation $u_1' = u_2' = 1$) of the last claim of [Liu14, Lemma 5.3]. The proof is a standard application of adelic Poisson summation. □

We first prove a general fact about diagonal periods.

Lemma 4.2.7. *Let $\Phi_\infty \in \mathcal{S}(V(E_\infty)^2)$ satisfy $\Phi_\infty(0,0) \neq 0$. Then we may define the functional*

$$I_{\Phi_\infty} : \mathcal{S}(V(\mathbb{A}_{E,f})^2) \to \mathbb{C}$$

by

$$I_{\Phi_\infty}(\Phi_f) = \int_{\mathsf{G}(F)\backslash \mathsf{G}(\mathbb{A}_F)} \sum_{x \in V(E)^2} \omega(g)(\Phi_\infty \otimes \Phi_f)(x)\, dg,$$

here by abuse of notation we also denote the obvious pullback action of G on $\mathcal{S}(V^2(\mathbb{A}_E))$ by $\omega(g)$. This functional is nonzero.

Proof. We pick a Φ_f such that $\Phi_f(0,0) = 1$, and denote $\Phi = \Phi_\infty \cdot \Phi_f$. We define the theta integral

$$T(h) \coloneqq \int_{\mathsf{G}(F)\backslash \mathsf{G}(\mathbb{A}_F)} \sum_{x \in V(E)^2} \omega(g,h)(\Phi_\infty \otimes \Phi_f)(x)\, dg,$$

where here $h \in U(W^\square)(\mathbb{A}_F)$. Picking $B \subset U(W^\square)$ a stabilizer of a complete isotropic flag, and letting N denote its unipotent radical, we will show that the constant term along B satisfies

$$T_N(1) = \int_{N(F)\backslash N(\mathbb{A}_F)} T(n)\, dn = \Phi_\infty(0,0) \cdot \Phi_f(0,0) = \Phi_\infty(0,0).$$

We can write $W^\square = Ey_1 \oplus Ey_2 \oplus Ey_2^* \oplus Ey_1^*$ and take $H_1 = U(Ey_2 \oplus Ey_2^*)$. Now using [GQT14, Proposition 7.2] we can compute this integral in stages. First taking the constant term along the maximal parabolic P_1 which stabilizes the line $E \cdot y_1$ with unipotent radical N_1, we obtain

$$T_{N_1}(a,h') = \int_{N_1(F)\backslash N_1(\mathbb{A}_F)} T(n_1 h)\, dn_1 = \chi(a) \cdot |a|_E^{3/2} \cdot \int_{\mathsf{G}(F)\backslash \mathsf{G}(\mathbb{A}_F)} \sum_{x \in V(E)} (\omega_1(g,h')\Phi^1)(x)\, dg$$

where $h = (a,h') \in \mathbb{A}_E^\times \times H_1(\mathbb{A}_F)$ is an element of the Levi of P_1, and the function $\Phi^1 \in \mathcal{S}(V(\mathbb{A}_E))$ is simply $\Phi^1(x) = \Phi(0,x)$, and χ is a character which will play no role in the proof. If we let $a = 1$ we find that

$$\int_{N_1(F)\backslash N_1(\mathbb{A}_F)} T(n_1 \cdot (1,h'))\, dn_1 = \int_{\mathsf{G}(F)\backslash \mathsf{G}(\mathbb{A}_F)} \sum_{x \in V(E)} (\omega_1(g,h')\Phi^1)(x)\, dg =: T_1(h').$$

Now we are reduced to the smaller unitary group H_1, and we can apply [GQT14, Proposition 7.2] to the maximal parabolic P_2, the stabilizer of $E \cdot y_2$ in H_1 (which has unipotent radical N_2), to obtain

$$\int_{N_2(F)\backslash N_2(\mathbb{A}_F)} T_1(n_2)\, dn_2 = \int_{\mathsf{G}(F)\backslash \mathsf{G}(\mathbb{A}_F)} \Phi(0,0)\, dg = \Phi(0,0).$$

Since $N = N_1 \rtimes N_2$ we have established the claimed result that $T_N(1) = \Phi(0,0) = \Phi_\infty(0,0)$. On the other hand, by the discussion in [GQT14, §3.1] and our assumption that $F \neq \mathbb{Q}$ and thus that V is definite at at least one infinite place (and thus G is anisotropic over F), the original integral converges and the function T is a smooth automorphic function on $U(W^\square)$, in particular if $\gamma \in N(F), h_f \in U(W^\square)(\mathbb{A}_{F,f})$ we see that

$$T(\gamma_\infty, h_f) = T(1, \gamma_f^{-1} h_f).$$

But then on the other hand if the claimed functional vanished identically in Φ_f, then by equivariance under $h_f \in U(W^\square)(\mathbb{A}_{F,f})$ we would have $T(\gamma_\infty, h_f) = T(1, \gamma_f^{-1} h_f) = 0$ for any such h_f, and any $\gamma \in N(F)$. But by weak approximation $N(F) \subset N(F_\infty)$ is dense, so this would imply that T vanishes identically on $N(\mathbb{A}_F)$, which would contradict our constant term calculation. $\square$

4.2.8. *Proof of Theorem 4.0.2.* In this proof, when suppressing the second argument of an unintegrated cohomological theta map, we denote $\Theta^{1,0}_{\psi,\nu}(\varphi) := \Theta^{1,0}_{\psi,\nu}(\varphi, 1)$. Using Proposition 4.1.8, it will turn out to suffice to find finite Schwartz functions $\varphi_i \in \mathcal{S}(X_i(\mathbb{A}_{F,f}))^K$, where each X_i is a Lagrangian subspace of $\mathbb{W}_i$, such that the integral

$$\int_{\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})} \Theta^{1,0}_{\psi,\nu_1}(\varphi_1) \wedge \Theta^{1,0}_{\psi,\nu_2}(\varphi_2) \wedge \overline{\Theta^{1,0}_{\psi,\nu_3}(\varphi_3)} \wedge \overline{\Theta^{1,0}_{\psi,\nu_4}(\varphi_4)}.$$

is nonzero, but we put this aside for now. At infinity, we can fix a $\mathsf{H}(\mathbb{R})$-invariant volume form η on the symmetric space Y, which we take to be normalized. If c_1, c_2, c_3, c_4 are the cocycles at infinity provided in the theta lifting of [BMM16, Proof of Theorem 9.1], then

$$c_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \in (\mathcal{S}_{v_0}(X_1) \otimes \mathcal{S}_{v_0}(X_2) \otimes \mathcal{S}_{v_0}(-X_3) \otimes \mathcal{S}_{v_0}(-X_4) \otimes \wedge^4 \mathfrak{p}^*)^{K_\infty},$$

in particular $c_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 = \phi_{v_0} \otimes \eta$ for some $\phi_{v_0} \in \bigotimes_{i=1}^2 \mathcal{S}_{v_0}(X_i) \otimes \bigotimes_{i=3}^4 \mathcal{S}_{v_0}(-X_i)$.

Given Ψ as in Lemma 4.2.1, we can pick a corresponding unitary intertwiner

$$I_0 : \mathcal{S}((-X_3 \perp -X_4)(\mathbb{A}_F)) \xrightarrow{\sim} \mathcal{S}((-X_1 \perp -X_2)(\mathbb{A}_F)),$$

which is unique up to unit scalar, and which we always normalize to preserve the theta distribution. We can thus define a doubling intertwiner

$$\delta_\infty : ((\bigotimes_{i=1}^2 \mathcal{S}_{v_0}(X_i) \otimes \bigotimes_{i=3}^4 \mathcal{S}_{v_0}(-X_i)) \otimes \wedge^4 \mathfrak{p}^*) \to \mathcal{S}(V(E_{\tau_0})^2) \otimes \wedge^4 \mathfrak{p}^*$$

by the formula $\delta_\infty(\phi_\infty \otimes \eta) = \delta_{\infty,\Psi}(\phi_\infty) \otimes \eta$. Here $\delta_{\infty,\Psi}$ is the usual doubling intertwiner

$$\delta_\infty : \mathcal{S}((X_1 \perp X_2)(\mathbb{A}_F)) \otimes \mathcal{S}((-X_1 \perp -X_2)(\mathbb{A}_F)) \to \mathcal{S}(V(\mathbb{A}_E)^2)$$

composed with I_0 and the inclusion $\mathcal{S}(-X_3) \otimes \mathcal{S}(-X_4) \to \mathcal{S}(-X_3 \perp -X_4)$ in the second factor.

Extending this to the finite places and definite infinite places is unproblematic, so we get a map

$$\delta : \left(\bigotimes_{i=1}^2 \mathcal{S}(X_i(\mathbb{A}_F)) \otimes \bigotimes_{i=3}^4 \mathcal{S}(-X_i(\mathbb{A}_F)) \otimes \wedge^4 \mathfrak{p}^*\right)^{K_\infty} \longrightarrow (\mathcal{S}(V(\mathbb{A}_E)^2) \otimes \wedge^4 \mathfrak{p}^*)^{K_\infty}.$$

We let ϕ_∞ denote the tensor product of ϕ_{v_0} with $\bigotimes_{v|\infty, v\neq v_0} \phi_{v,i}$ and $\phi_{v,i}$ is a Gaussian if $i = 1, 2$ or its conjugate if $i = 3, 4$. Now applying Proposition 4.2.6, we let $g = (g_\infty, g_f) \in \mathsf{H}(\mathbb{A})$, and $z = g_\infty \cdot z_0$ is a point on the locally symmetric space associated to (H, Y) where z_0 is a chosen basepoint then. If we let

$$\Phi := \delta(\phi_\infty \otimes \varphi_1 \otimes \varphi_2 \otimes \overline{\varphi}_3 \otimes \overline{\varphi}_4)\,.$$

Then

$$\left(\Theta^{1,0}_{\psi,\nu_1}(\varphi_1)\wedge\Theta^{1,0}_{\psi,\nu_2}(\varphi_2)\wedge\overline{\Theta^{1,0}_{\psi,\nu_3}(\varphi_3)}\wedge\overline{\Theta^{1,0}_{\psi,\nu_4}(\varphi_4)}\right)(z,g_f)=\left(\sum_{x\in V(E)^2}(\omega(g)\Phi)(x)\right)\eta_z$$
$$=\left(\sum_{x\in V(E)^2}\Phi(g^{-1}x)\right)\eta_z.$$

Where in the last line we use the content of Remark 4.2.5. We note that $F_\Phi(g):=\left(\sum_{x\in V(E)^2}\Phi(g^{-1}x)\right)$ is invariant under $K_\infty\times K$, so we find that

$$\int_{\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})}F_\Phi\cdot\eta=C\int_{\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A})}F_\Phi(g)\,dg$$

where the constant C is the product of the reciprocals of the masses of the compact subgroup K_∞ with respect to a Haar measure compatible with η and the probability measure on $\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A}_F)$, and K with respect to the Haar measure on $\mathsf{G}(\mathbb{A}_{F,f})$.

The upshot of this discussion is that, we can identify the integral of the cup-product of our theta series, up to a nonzero scalar and considered as a functional on $\mathcal{S}(V(\mathbb{A}_{E,f})^2)$ via the doubling intertwiner, with a functional of the form studied in Lemma 4.2.7. The following lemma will provide us with the conditions under which we can show the desired nonvanishing hypothesis in Lemma 4.2.7.

Lemma 4.2.9. *There exists an F-rational isometry $\Psi: W_1\perp W_2\xrightarrow{\sim}W_3\perp W_4$ such that under the doubling map $\delta_{\infty,\Psi}(\phi_\infty)=\Phi_\infty$ is nonvanishing at* 0.

Proof. In general, the set of F-rational isometries between two such skew-Hermitian spaces is a torsor for $U(W_1\perp W_2)$, and we know that this torsor is split by the choice of the W_i as in Lemma 4.2.1. If we fix an isometry $\Psi_{\infty,\mathrm{spl}}$ over $F\otimes\mathbb{R}$ such that $\Psi_{\infty,\mathrm{spl}}$ preserves the splitting at each place, i.e. at each place maps W_1 to W_3 or W_4, and W_2 the other factor, then the doubled function identifies with a nonzero multiple of the Kudla–Millson form $\delta_{\infty,\Psi_{\infty,\mathrm{spl}}}(\phi_\infty\otimes\eta)(0)=D\cdot\varphi_{2,2}(0)$ by [BMM16, Proposition 5.17(5), Proposition C.10], which is nonvanishing by [Ber06, Proposition 3.6, Theorem 3.7(3)]. The term D includes the contributions from other infinite places which, since $\Psi_{\infty,\mathrm{spl}}$ preserves the splitting, just contribute a product of L^2-norms of Gaussians. Now use real approximation for $U(W_1\perp W_2)$ to find a rational isometry Ψ such that Ψ_∞ is very close to $\Psi_{\infty,\mathrm{spl}}$. The doubling map associated with this isometry $\delta_{\infty,\Psi}(\phi_1\otimes\phi_2)(0)$ can be shown to have a continuous dependence on Ψ_∞, as we now explain.

Indeed, backing up a bit if W' is a skew-Hermitian space, X' is a Lagrangian of $V\otimes_E W'$ and $\Psi_{\infty,0}: W'\to -W$ is an isometry, then if $\phi_1\in\mathcal{S}(X(F_\infty)),\phi_2\in\mathcal{S}(X'(F_\infty))$ we can define a corresponding unitary intertwiner

$$I_0:\mathcal{S}(X'(F_\infty))\xrightarrow{\sim}\mathcal{S}(-X(F_\infty))$$

of Schwartz spaces, unique up to a unit scalar. Let δ_∞ denote the standard doubling map. By equation 4.2.4

$$\Phi_{\infty,\Psi_{\infty,0}}(0)=\delta_\infty(\phi_1\otimes I_0(\phi_2)(0)=\langle\phi_1,\overline{I_0(\phi_2)}\rangle.$$

If we switch to $\Psi_\infty = k \cdot \Psi_{\infty,0}$ for some $k \in U(-W)$, we can normalize the resulting unitary intertwiner

$$I' : \mathcal{S}(X'(F_\infty)) \xrightarrow{\sim} \mathcal{S}(-X(F_\infty))$$

to be $I' = \omega_{-W}(k) \cdot I_0$ Then

$$\Phi_{\infty,\Psi_\infty}(0) = \delta_\infty(\phi_1 \otimes I'(\phi_2))(0) = \langle \phi_1, \omega_W(k) \cdot \overline{I_0(\phi_2)} \rangle,$$

using the identification $\overline{\omega_{-W}(k) \cdot \varphi} = \omega_W(k) \cdot \overline{\varphi}$ for $\varphi \in \mathcal{S}(-X(F_\infty))$. So the desired continuity of $\Phi_{\infty,\Psi_\infty}(0)$ in Ψ_∞ follows from (strong) continuity for the oscillator representation ω_W. □

Lemma 4.2.7 tells us that for $\Phi = \Phi_\infty \otimes \Phi_f$ for some Φ_f, the integral in equation 4.2.8 is nonvanishing. On the other hand $\delta_f^{-1}(\Phi_f) = \sum_{i,j,k,l} \varphi_i \otimes \varphi_j \otimes \varphi_k \otimes \varphi_l$, so there must be some pure tensor for which the integral is nonvanishing as well, by linearity. This in hand, we must deduce that the conclusion of Theorem 4.0.2 is true. We make the following elementary deduction.

Lemma 4.2.10. *Let $\Theta^{1,0}_{\psi,\nu}$ be the cohomological theta lift constructed in Subsection 4.1.6 corresponding to a pair (μ, W) where W is a one-dimensional skew-Hermitian space. Let φ_f be a finite Schwartz function, then there are finitely many automorphic characters $\{\chi_1, \ldots, \chi_n\}$ of the torus $U(W)$ such that*

$$\Theta^{1,0}_{\psi,\nu}(\varphi_f, t) = \sum_{i=1}^{n} \chi_i(t) \cdot \theta_{\chi_i}$$

where the $\theta_\chi \in \mathrm{H}^0(\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})_{\mathbb{C}}, \Omega^{1,0}_{\mathsf{Y}})$.

Proof. As a consequence of [BMM16, Proposition 5.17(8)] applied in our special case, and the definition of $\varphi_{a,b}$ in [BMM16, §5.8], we find that $\varphi_{1,0}$ transforms under a character χ_{v_0} under the action of $U(W)(F_{v_0})$. The same is clearly also true for the Gaussian factor φ_0 [BMM16, §8.7]. We refer to by χ_∞ the product of these characters, which defines a character of $U(W)(F_\infty)$. On the other hand φ_f is fixed under the action of some open compact $K_f \subset U(W)(\mathbb{A}_f)$. So we find that the Fourier support of the function

$$\Theta^{1,0}_{\psi,\nu}(\varphi_f, -)$$

on $U(W)(F)\backslash U(W)(\mathbb{A}_F)$ must be supported on the set of automorphic characters of $U(W)(\mathbb{A}_F)$ such that $\chi|_{K_f} = 1$ and $\chi|_{U(W)(F_\infty)} = \chi_\infty$, which defines a finite set since

$$U(W)(F)\backslash U(W)(\mathbb{A}_{F,f})/K_f$$

is finite. □

As a result of the above Lemma, we see that

$$\Theta^{1,0}_{\psi,\nu}(\varphi_f, 1) = \sum_{i=1}^{n} \theta_{\chi_i}$$

where the θ_{χ_i} are Fourier coefficients of $\Theta^{1,0}_{\psi,\nu}$, which are of course integrated theta classes as defined in Section 4.1.6. Applying this to the above nonvanishing statement, and using that $(\sum_i \alpha_i) \cup (\sum_j \beta_j) \neq 0$ implies that $\alpha_i \cup \beta_j \neq 0$ for some pair (i,j), we see that there are $f_i \in \chi_i^{-1}$ such that

$$\int \Theta^{1,0}_{\psi,\nu_1,f_1}(\varphi_1) \wedge \Theta^{1,0}_{\psi,\nu_2,f_2}(\varphi_2) \wedge \overline{\Theta^{1,0}_{\psi,\nu_3,f_3}(\varphi_3)} \wedge \overline{\Theta^{1,0}_{\psi,\nu_4,f_4}(\varphi_4)} \neq 0.$$

At this point we have essentially proved the result, indeed with (μ_i, W_i) fixed, if the Fourier component associated to some character χ_i is nonzero and $f_i \in \chi_i^{-1}$, then the corresponding classes $\Theta^{1,0}_{\psi,\nu_i,f_i}(\varphi_i)$ are integrated theta lifts whose associated representation is $\omega(\mu_i, \epsilon_i, \kappa_i)$ for ϵ_i determined by W_i, and κ_i determined by χ_i. By Proposition 4.1.8 each of the classes $\Theta^{1,0}_{\psi,\nu_i,f_i}(\varphi_i)$ for $i = 1,2$, $\overline{\Theta^{1,0}_{\psi,\nu_i,f_i}(\varphi_i)}$ for $i = 3,4$ arise from a linear combination of classes in the image of $\mathrm{H}^1(B_i, E)_{\lambda_i}$, applying the same trick as was used in the previous paragraph we obtain the statement of Theorem 4.0.2. □

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School of Mathematical Sciences, Zhejiang University, 866 Yuhangtang Rd, Hangzhou, 310058, P. R. China

Email address: `pvhoften@zju.edu.cn`

Mathematical Sciences Building, 520 Portola Plaza, Los Angeles, CA 90095, United States of America

Email address: `jsempliner@math.ucla.edu`