

An Intrinsic Formula for the Ray–West Correction Term

Preliminary technical report, version 0.1

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August 9, 2026

doi:10.5281/zenodo.21857882

Abstract

For a nonempty permutation $\beta \in \mathfrak{S}_m$, let $g_2(\beta)$ denote the number of distinct permutations of length $m + 2$ containing β . Ray and West showed that

$$g_2(\beta) = \frac{m^4 + 2m^3 + m^2 + 4m + 4 - 2j(\beta)}{2}, \quad 0 \leq j(\beta) \leq m - 1 \quad (m \geq 1),$$

but their correction term was not expressed as a statistic of β . This was posed explicitly as Problem 3.7 by Vatter. We give a polynomial-time, β -internal formula. The statistic is the number of licensed gaps of β , plus the number of anchor-free two-deletion classes containing a collision whose canonical alignment support is a permutation interval. We also derive the Ray–West bound internally and show that equality holds exactly for layered or reverse-layered permutations. The proof is combinatorial. Exact computation is used only for independent corroboration; the accompanying replay package checks the formula on bounded ranges and contains the full proof supplement.

Keywords. permutation patterns; Ray–West statistic; near-diagonal containment; Wilf equivalence; deletion classes; strong intervals.

1 Introduction

Write $\sigma \leq \pi$ when π contains a subsequence order-isomorphic to σ . For a fixed $\beta \in \mathfrak{S}_m$, let

$$g_k(\beta) = |\{\pi \in \mathfrak{S}_{m+k} : \beta \leq \pi\}|.$$

The word *distinct* is important: g_k counts upper objects, not embeddings. Ray and West [1] proved the displayed formula in the abstract and, for $m \geq 1$, established $0 \leq j(\beta) \leq m - 1$. Vatter asked for an expression for j in terms of statistics of β [3, Problem 3.7]; the question was raised at Permutation Patterns 2007 and recorded in its published problem session [2].

Our answer uses only intervals and collisions in the two-deletion shadow of β . In particular, its definition neither constructs a permutation longer than β nor counts insertion histories. Upper-shadow fibres enter only in the proof.

Theorem 1.1 (Main theorem). *For every $\beta \in \mathfrak{S}_m$ with $m \geq 1$,*

$$2g_2(\beta) + 2(L(\beta) + R(\beta)) = m^4 + 2m^3 + m^2 + 4m + 4,$$

where the intrinsic statistics L and R are defined in Section 2. Consequently

$$j(\beta) = L(\beta) + R(\beta).$$

Moreover, for $m \geq 1$,

$$0 \leq L(\beta) + R(\beta) \leq m - 1.$$

For $m \geq 2$, equality holds if and only if β is layered or reverse-layered.

For the empty permutation we separately set $L(\emptyset) = R(\emptyset) = 0$; the main identity then remains valid with $g_2(\emptyset) = 2$, without invoking Ray–West’s range statement at $m = 0$.

The proof has three components. First, a fibre-forest identity rewrites Ray–West’s correction as a count of boundaries between overlap components. Second, one classwise theorem identifies the defect of a two-deletion class: licensed gaps account for its rectangular boundaries, while an active anchor-free class accounts for one nonrectangular boundary. Third, a strong-interval-tree charging theorem bounds the resulting intrinsic objects and determines equality. Proposition 2.4 shows that the side conditions used to orient the boundary geometry are automatic, leaving the formula $J = L + R$.

Scope. This preliminary technical report is intended to establish a citable, auditable proof claim. Several local lemmas are lengthy; their complete versions are supplied in the accompanying proof supplement and are summarized in the appendices. The PDF is a theorem-driven roadmap; the PDF and integral companion supplement together constitute the claimed proof. No machine-checked formalization is included or claimed, and the result is not described as fully machine-checked.

Integral proof supplement and reproducibility archive. The complete companion proof supplement, which is part of the claimed proof, and the independent exact-computation replay are permanently available in the original Zenodo release at <https://doi.org/10.5281/zenodo.21857882>. Download the proof-and-replay archive from that record and consult its `proof_supplement/` directory.

2 The intrinsic statistic

Write $[m] = \{0, \dots, m - 1\}$; positions are zero based. For $D \in \binom{[m]}{2}$, let

$$\partial_D \beta = \text{std}(\beta \setminus D).$$

Partition the deletion pairs into classes

$$\mathcal{C}_\tau = \{D : \partial_D \beta = \tau\}.$$

2.1 Licensed gaps

A consecutive set of positions is a *permutation interval* if its set of values is consecutive. Let δ_r and ι_r denote the decreasing and increasing permutations of length r .

Definition 2.1. *The gap i between positions i and $i + 1$ is licensed if some permutation interval $[r, s]$ crossing the gap induces either*

$$\delta_p \oplus \delta_q \quad \text{or} \quad \iota_p \ominus \iota_q, \quad p = i - r + 1, \quad q = s - i.$$

Singleton blocks are allowed. Let $L(\beta)$ be the number of licensed gaps. For a deletion class $\mathcal{C}$, let $a_\beta(\mathcal{C})$ be the number of licensed gaps whose adjacent deletion pair $\{i, i + 1\}$ lies in $\mathcal{C}$. Call $\mathcal{C}$ anchored if $a_\beta(\mathcal{C}) > 0$ and anchor-free otherwise.

Every licensed gap belongs to one deletion class, so

$$L(\beta) = \sum_{\mathcal{C}} a_\beta(\mathcal{C}). \tag{1}$$

2.2 Bi-interval collisions

Let D, E be distinct pairs in the same deletion class. List the surviving positions

$$[m] \setminus D = \{x_1 < \cdots < x_{m-2}\}, \quad [m] \setminus E = \{y_1 < \cdots < y_{m-2}\}.$$

The *canonical survivor alignment* maps x_k to y_k . Its support is

$$S(D, E) = D \cup E \cup \{x_k, y_k : x_k \neq y_k\}. \tag{2}$$

Definition 2.2. *The pair $\{D, E\}$ is a bi-interval collision if $S(D, E)$ is a consecutive set of positions and $\{\beta_i : i \in S(D, E)\}$ is a consecutive set of values. Let $R(\beta)$ be the number of anchor-free deletion classes that contain a bi-interval collision.*

Thus the answer to Vatter's problem is the particularly short statistic

$$J(\beta) = L(\beta) + R(\beta). \tag{3}$$

The two interval requirements are both necessary. Exhaustive replay from length zero upward finds the first failures at length four. Dropping position intervality changes 1423, 2314, 3241, 4132, while dropping value intervality changes 1342, 2431, 3124, 4213; in every case the true value $J = 2$ becomes 3 under the mutant definition.

Example. For $\beta = 2413$, there are no licensed gaps. The deletion class with core 12 contains the bi-interval collision $\{\{0, 1\}, \{2, 3\}\}$, and the class with core 21 contains $\{\{0, 2\}, \{1, 3\}\}$. Hence $J(2413) = 2$.

2.3 Automatic side conditions

For reference, an *oriented interval shift* is a bi-interval collision whose deletion pairs are disjoint, whose increasing position pairs are componentwise comparable, and whose increasing deleted-value pairs are componentwise comparable. These conditions are convenient in the boundary proof but are unnecessary in the statement.

Lemma 2.3 (Equal one-deletions). *If two distinct positions of a permutation give the same standardized one-deletion, then every position between them gives that same deletion, and the intervening entries form a monotone run of consecutive values.*

Proof. Let the deleted entries have values u and v . For any surviving value z , its rank after deleting u is

$$z - \mathbf{1}_{z>u},$$

and analogously for v . If the deleted positions are $a < b$, then at output coordinate k , for $a \leq k < b$, equality of the two standardized words gives

$$\beta_{k+1} - \mathbf{1}_{\beta_{k+1}>u} = \beta_k - \mathbf{1}_{\beta_k>v}.$$

The indicator difference is in $\{-1, 0, 1\}$ and adjacent entries are distinct, so $\beta_{k+1} - \beta_k = \pm 1$. If $u < v$, a -1 step would require simultaneously $\beta_{k+1} \leq u$ and $\beta_k > v$, which is impossible; hence every step is $+1$. The case $u > v$ is symmetric and every step is -1 . Thus the intervening entries form a monotone run of consecutive values. Conversely, deleting any entry of such a run merely shifts and shortens it, preserving all comparisons and giving the same standardized word. $\square$

Proposition 2.4 (Comparability cleanup). *Let $\mathcal{C}$ be anchor-free. Every bi-interval collision $\{D, E\}$ in $\mathcal{C}$ is an oriented interval shift: D and E are disjoint and the two position pairs and two deleted-value pairs are componentwise comparable.*

Proof. Suppose first that D and E share a position z . Delete z . The two remaining holes give equal one-deletions, so Lemma 2.3 produces a monotone consecutive run. Let t be the number of old run entries before z , and let u be the one-based rank of z among the support values. If $u = t + 1$, z and a neighboring run entry are consecutive in both position and value, so their singleton blocks license the gap. If $u > t + 1$, the singleton z and the run segment of ranks $t + 1, \dots, u - 1$ form an increasing skew cut immediately after z ; if $u < t + 1$, the symmetric segment forms such a cut immediately before z . For a decreasing run, value complementation turns these into decreasing direct cuts. In every case the adjacent pair at the cut remains in $\mathcal{C}$, giving an anchor, a contradiction.

Now assume the pairs are disjoint but not componentwise comparable in position. After exchanging their names, their four holes have order

$$d_1 < e_1 < e_2 < d_2.$$

Restrict to the bi-interval support and standardize. Full support of the canonical alignment forces d_1 and d_2 to be the two extreme positions and $e_2 = e_1 + 1$. The alignment has two directed paths running outward from this central gap. Conjugating the position alignment by the permutation gives the analogous value alignment. Between its two sources, the two path colors occur as $P^r Q^s$ (this follows directly by listing the six relative orders of the two domain and two range holes). Consequently the entries abutting the central gap form either two decreasing consecutive blocks in direct-sum order or two increasing consecutive blocks in skew-sum order. Thus the central pair is a licensed anchor in $\mathcal{C}$, again a contradiction.

The position pairs are therefore comparable. Apply the same argument to β^{-1} . Position comparability there is deleted-value comparability here. The licensed-anchor transport uses the opposite-corner lemma: in a licensing interval, the adjacent pair at the position cut and the adjacent pair at the value cut have the same two-deletion core. After transposition, the value-cut corner is therefore an adjacent licensed pair in the original class. This again contradicts anchor-freeness, so the deleted-value pairs are comparable as well. $\square$

In an anchored class, any additional shift bit is absorbed by $a_\beta(\mathcal{C}) \geq 1$; in an anchor-free class, Proposition 2.4 identifies bi-interval collisions with oriented interval shifts. This proves that the reader-facing formula (3) is exactly the expanded statistic used in the boundary proof.

3 The fibre-defect identity

Put $N = m + 2$. For $\rho \in \mathfrak{S}_N$, define the producing fibre

$$F_\rho = \{X \in \binom{[N]}{2} : \text{std}(\rho \setminus X) = \beta\}.$$

Join two members of F_ρ when their two-sets intersect. When the fibre is nonempty, write $e_\rho = |F_\rho|$ and let c_ρ be its number of overlap components.

Proposition 3.1 (Fibre forest identity). *For every $\beta \in \mathfrak{S}_m$,*

$$\sum_{\rho: F_\rho \neq \emptyset} (e_\rho - c_\rho) = 2m(m+1)(m+2). \quad (4)$$

The proof is an exact split-and-insert count. Its delicate point is not the enumeration but the forest assertion inside each one-deletion run. The full argument is given in Appendix A and in the proof supplement.

There are

$$H_m = 2 \binom{m+2}{2}^2$$

two-point insertion histories: choose two positions, two values, and one of the two bijections. Histories are precisely pairs (ρ, X) with $X \in F_\rho$. Therefore

$$H_m - g_2(\beta) = \sum_{\rho: F_\rho \neq \emptyset} (e_\rho - 1). \quad (5)$$

Write

$$T(\beta) = \sum_{\rho: F_\rho \neq \emptyset} (c_\rho - 1).$$

Since $e_\rho - 1 = (e_\rho - c_\rho) + (c_\rho - 1)$, substitution of Proposition 3.1 into (5) yields

$$g_2(\beta) = H_m - 2m(m+1)(m+2) - T(\beta) \quad (6)$$

$$= \frac{m^4 + 2m^3 + m^2 + 4m + 4}{2} - T(\beta). \quad (7)$$

Thus Ray–West’s correction satisfies

$$j(\beta) = T(\beta). \quad (8)$$

4 Boundary classification

Regard the pairs in F_ρ as edges of a ground graph on the physical points of ρ . Fibre overlap components are precisely the nontrivial connected components of this graph. Distinct such components have disjoint position hulls and disjoint value hulls; the two hull orders either agree or

reverse. Order the components by position hull as $K_1 < \dots < K_{c_\rho}$. The adjacent pairs (K_i, K_{i+1}) are the *boundary units*; there are $c_\rho - 1$.

For a boundary $A < B$, choose $X \in A$ and $Y \in B$. The pairs are disjoint. Delete X , identify $\rho - X$ with β , and transport Y into β . A shielding argument shows that the resulting two-deletion class is independent of X and Y . Denote it by $\mathcal{C}(A, B)$. If $b(\mathcal{C})$ counts boundaries with label $\mathcal{C}$, then

$$T(\beta) = \sum_{\mathcal{C}} b(\mathcal{C}). \quad (9)$$

The two moving tracks in the canonical identification can be encoded, in position and value order, by two monotone lattice paths. Hull separation rules out an A, B, A component pattern and any third component between adjacent A, B . Hence there is a unique transition excursion. It is *rectangular* when its two words are $A^p B^q$ and $B^q A^p$; otherwise it is *nonrectangular*. If either track has size one, the excursion is necessarily rectangular. Appendix B gives the elementary two-letter prolongation proof explicitly.

Proposition 4.1 (Classwise boundary theorem). *For every two-deletion class $\mathcal{C}$:*

- (i) *rectangular boundaries labelled $\mathcal{C}$ are in bijection with licensed gaps whose adjacent pair lies in $\mathcal{C}$;*
- (ii) *if $\mathcal{C}$ is anchored, it labels no nonrectangular boundary;*
- (iii) *if $\mathcal{C}$ is anchor-free, it labels exactly one nonrectangular boundary when it contains a bi-interval collision, and none otherwise.*

Consequently

$$b(\mathcal{C}) = \begin{cases} a_\beta(\mathcal{C}), & a_\beta(\mathcal{C}) > 0, \\ 1, & a_\beta(\mathcal{C}) = 0 \text{ and } \mathcal{C} \text{ has a bi-interval collision,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof sketch; complete proof in the companion supplement. A rectangular excursion exposes a position/value cut whose two sides are a direct sum of decreasing blocks or a skew sum of increasing blocks; this is a licensed gap. Conversely, prolonging a licensed gap produces two corner occurrences. The central position/value cut forbids every mixed producing deletion, so those corners lie in distinct adjacent full-fibre components. The cut and prolongation recover one another, proving (i).

A nonrectangular excursion has a strong simple-root support. If its class had a licensed pair, transport through the strong-interval tree would carry an anchor into that simple support, contradicting simplicity; the terminal two-point-child case is checked separately. This gives (ii).

For (iii), start with an anchor-free collision. By Proposition 2.4 it has the oriented geometry used by the two-track proof. Among all supports in the class there is a common minimum simple support. On a fixed support the track partition and witness are unique. Contracting through same-class linear strips reaches the unique terminal witness, while restoring the strips adds the same constant fringe to both coordinate words. At a maximal support the two endpoint pairs lie in distinct full-fibre components. A third component would force either a nested same-class witness on the terminal support or a strictly larger same-class support, both contradictions. Thus the canonical fibre has exactly the two endpoint components and produces one adjacent boundary. Finally, the literal inverse reads the local position and value words from any nonrectangular boundary and reconstructs the same support, ambient permutation, and ordered component pair.

This proves existence and uniqueness. Appendix C records the precise lemma chain and the proof supplement contains the complete local arguments. $\square$

Combining (9), Proposition 4.1, and (1) gives

$$T(\beta) = L(\beta) + R(\beta).$$

Together with (8), this proves the identity in Theorem 1.1.

5 Range and extremizers

Permutation intervals that are strong form a laminar rooted tree. If an internal node N has k_N maximal proper strong children, its quotient is increasing, decreasing, or simple; a simple quotient has $k_N \geq 4$. Every position gap has a unique owner, namely the lowest strong node whose two consecutive children contain its endpoints. Hence

$$\sum_{N \text{ internal}} (k_N - 1) = m - 1. \quad (10)$$

Proposition 5.1 (Strong-tree charging). *Licensed gaps are owned by linear nodes. Every anchor-free class containing a bi-interval collision has a minimum support equal to a simple strong node, and at most two such classes are assigned to any one simple node.*

Proof sketch. The complete four-case analysis is in the companion supplement. A licensing interval crossing two children of a simple node would induce a nontrivial sum or skew cut in its simple quotient. Thus licensed gaps are owned by linear nodes.

Take a minimum collision support in an anchor-free class. At a direct or skew cut, the four possible distributions of its endpoint deletions either expose a licensed pair or produce a strictly smaller same-class collision. The common-hole boundary case reduces to Lemma 2.3. Therefore a minimum support cannot straddle a linear node and its least strong container is simple; minimality makes the support that whole node.

Orient a full-support collision from left to right. Full support gives exactly two corner possibilities:

	D	E
+	{first position, min}	{last position, max}
-	{first position, max}	{last position, min}.

At a simple quotient these four extreme points occupy four distinct quotient children: identifying a position extreme with a value extreme would leave a proper prefix or suffix quotient interval. Each displayed orientation therefore determines at most one deletion core, so a simple node receives at most two residual classes. $\square$

A linear node uses at most its $k_N - 1$ owned-gap slots. A simple node uses at most two slots, and $2 \leq k_N - 1$ because $k_N \geq 4$. Summing against (10) proves $J(\beta) \leq m - 1$.

If equality holds, no simple node can occur: at a simple node the residual load is at most $2 < k_N - 1$, whereas every linear-node gap must be licensed. At an increasing node, licensing every child boundary forces every child to be decreasing; otherwise a proper licensing suffix would split a maximal sum-indecomposable child. Dually, every child of a decreasing node is increasing. Thus β is a direct sum of decreasing blocks (layered) or a skew sum of increasing

blocks (reverse-layered). Conversely, every gap of a layered permutation is licensed, and value complementation gives the reverse-layered case. This proves the equality characterization.

The usual avoidance descriptions are

$$\text{layered} = \text{Av}(231, 312), \quad \text{reverse-layered} = \text{Av}(213, 132).$$

6 Consequences and computation

The proof yields several immediate corollaries.

Corollary 6.1. *For a simple permutation β , $j(\beta) \leq 2$. The bound is attained by 2413 and 3142.*

Proof. For $m < 4$ the assertion is checked directly. For a simple permutation of length $m \geq 4$, the root of the strong-interval tree is simple. It owns no licensed gap and, by Proposition 5.1, receives at most the two residual corner orientations. The two length-four examples have value two by the explicit calculation in Section 2 and its symmetry image. $\square$

Corollary 6.2. *The statistic j is invariant under reversal, complementation, and inversion. For $m \geq 2$, exactly $2^m - 2$ permutations in $\mathfrak{S}_m$ have $j = m - 1$.*

Proof. Plot symmetries carry equal-deletion classes and bi-interval supports to their images. Reversal and complementation visibly carry licensing intervals to licensing intervals. For inversion, the opposite-corner lemma used in Proposition 2.4 supplies the correctly adjacent image anchor; the detailed transport is in the companion supplement. Thus J is dihedrally invariant. Layered permutations are indexed by the 2^{m-1} compositions of m , as are reverse-layered permutations. Their intersection consists of the increasing and decreasing permutations, so their union has $2^{m-1} + 2^{m-1} - 2 = 2^m - 2$ elements. $\square$

The accompanying clean evaluator uses $O(m^2)$ internal deletions and tests pairs only within equal-core classes. Its straightforward worst-case running time is $O(m^5)$ and it never allocates a permutation longer than m .

Exact computations are corroboration rather than proof. The public replay compares the intrinsic route with a semantically independent direct g_2 route on a frozen exhaustive range; compares the clean and expanded statistics through length eight; checks range, symmetry, and extremizers; and mutation-tests both interval requirements. It also independently enumerates the binary prolongation theorem through target length thirteen, checking the coordinate sets, path-type uniqueness, and constant-fringe commutation. A separate exact normal-path audit checks all 31,200 pairs with individual track counts at most fifteen: 4,691 pairs have only rectangular excursions, 2,744 have multiple raw excursions, and none has more than one nonrectangular excursion. A separate slow hostile replay exhausts all permutations of length nine. It checks 7,979,820 within-class deletion-pair comparisons, including 6,577,712 anchor-free comparisons, with no cleanup, clean/expanded-formula, range, or extremizer failure; its maximizer set is exactly the 510 layered or reverse-layered permutations.

7 Discussion and formalization status

Formula (3) identifies the codimension-two containment term as a combination of local interval cuts and lower-shadow collisions. The strong-interval-tree proof explains the otherwise opaque upper bound $m - 1$: it is the total budget $\sum_N (k_N - 1)$.

No machine-checked formalization accompanies this release, and none is used as evidence. If a formalization is undertaken, the natural order is the equal-one-deletion lemma and cleanup proposition, the strong-tree charging theorem and equality case, and finally the larger fibre and boundary layers.

Acknowledgments and disclosure

This work was carried out as part of the conjecture-prover and autoformalization project led by the author for Stanford’s Lean Club. OpenAI’s GPT-5.6 Sol Ultra was used extensively for conjecture generation, exact computation, proof development, adversarial repair, and preparation of the initial proof artifacts. Codex assisted with independent reconstruction, simplification, and preparation of this preliminary technical report. The author directed the project, selected the claims for release, and takes responsibility for the manuscript. Automated audits and exact replay are not substitutes for external peer review. Stanford University and Stanford Lean Club do not endorse the mathematical claims by virtue of the affiliation or project context. This report is released under the Creative Commons Attribution 4.0 International license.

A The fibre forest theorem

We record the points needed to make Proposition 3.1 uniform in m . Fix an elementary split of a producing occurrence. The producing positions for a fixed one-deletion form the entire interval between any two of them, by Lemma 2.3. In the associated grid of possible second insertions, accept the non-strict-frontier cells and contract each one-deletion run. First-exit induction—including the case where the first point is already the first element of its run—shows that every accepted edge joins the contracted path without closing a cycle. Conversely every non-strict-frontier cell is reached, so the accepted graph is a spanning forest of each overlap component. Strict-frontier cells are rejected.

Index a rejected cell by its first strict exit and the elementary edge crossed there. The first-exit index is in range, and two rejected cells cannot have the same index: reversing the split at the first differing grid coordinate recovers the cell. Thus the rejected cells are counted without collision. The elementary-edge enumeration gives $2m(m+2)^2$ total edges. For each of the two split orientations and each split position $i = 1, \dots, m$, exactly $3 + 2(m-i)$ cells are rejected. Hence the number of rejected edges is

$$2 \sum_{i=1}^m (3 + 2(m-i)) = 2m(m+2),$$

and the accepted forest has

$$2m(m+2)^2 - 2m(m+2) = 2m(m+1)(m+2)$$

edges. Accepted edges contribute exactly $e_\rho - c_\rho$ in each fibre, proving (4). The proof supplement gives the literal grid notation, frontier inequalities, and inverse indexing map.

B Binary-word prolongations

We give the word theorem used in the two-track argument, including the indexing that is easy to obscure in a path picture. If a word U contains $a+1$ copies of A and $b+1$ copies of B , number

the occurrences of each letter from zero. Let $T_f(U)$ delete A_0, B_0 , let $T_\ell(U)$ delete A_a, B_b , and put

$$\mathcal{D}_U(P) = \{(i, j) : U - \{A_i, B_j\} = P\}.$$

If $P = p_1 \cdots p_{a+b}$, its prefix-count path has vertices

$$v_k = (|p_1 \cdots p_k|_A, |p_1 \cdots p_k|_B), \quad 0 \leq k \leq a+b.$$

Thus the path runs from $(0, 0)$ to (a, b) , exactly the zero-based deletion index range.

Lemma B.1 (Binary prolongation). *Suppose $a, b \geq 1$ and*

$$T_f(U) = T_\ell(U) = P.$$

Then $\mathcal{D}_U(P)$ is either the whole rectangle $\{0, \dots, a\} \times \{0, \dots, b\}$ or precisely $\{v_0, \dots, v_{a+b}\}$. The rectangle occurs only for the two block words. For every admissible target P , its nonblock, path-type prolongation is unique.

Proof. Assume first that $a \geq b$ and put $d = a - b$. We claim that the complete list of U is

$$\begin{aligned} F_t &= A^{a+1-t}(BA)^t B^{b+1-t}, \quad 0 \leq t \leq b, \\ G_t &= B^{b+1-t}(AB)^t A^{a+1-t}, \quad 0 \leq t \leq b, \\ H_r &= A^r(BA)^{b+1} A^{d-r}, \quad 1 \leq r \leq d. \end{aligned} \tag{11}$$

The last family is empty when $d = 0$; if $a < b$, exchange the roles of A and B , and hence also their counts.

Here is a direct necessity proof. If U starts in A , ends in B , and is not a block word, write

$$U = A^u B M A B^v, \quad u, v \geq 1,$$

using the first B and last A . Equality of the two trims, after cancelling the common outside runs, is

$$M A B = A B M.$$

Since AB is primitive, a word commuting with it is $(AB)^k$. Directly, the empty word is the $k = 0$ case, a one-letter word cannot commute with AB , and every longer solution begins with AB ; cancelling it proves the claim by induction. Hence $U = A^u (BA)^{k+1} B^v = F_{k+1}$. Exchanging the letters gives G_t .

If U starts and ends in A , write

$$U = A^u B M B A^v.$$

Trim equality becomes $M B A = A B M$. If M were empty this would say $BA = AB$, so M is nonempty. Thus $M = A M'$ and $M' B A = B A M'$, so $M' = (BA)^k$. Counting B 's gives $k = b - 1$, while counting A 's gives $u + v = d + 1$. This is exactly H_u . The start/end B case is its letter-exchanged version and can occur only when $b > a$. This proves that (11) is exhaustive; direct trimming proves sufficiency.

For $t \geq 1$, the three target families are

$$\begin{aligned} T_f(F_t) &= A^{a+1-t}(BA)^{t-1} B^{b+1-t}, \\ T_f(G_t) &= B^{b+1-t}(AB)^{t-1} A^{a+1-t}, \\ T_f(H_r) &= A^r(BA)^b A^{d-r}. \end{aligned}$$

Their first and last letters separate the families, and their initial constant-run lengths determine t or r . Thus each target has exactly one nonblock prolongation, including the two-block targets, whose path prolongations are F_1 and G_1 .

It remains to identify the working coordinates. For $F_t = A^R(BA)^tB^S$, where $R = a + 1 - t$ and $S = b + 1 - t$, direct deletion gives

$$\begin{aligned}\mathcal{D}_{F_t}(P) = & \{(i, 0) : 0 \leq i \leq R\} \\ & \cup \{(R + q - 1, q), (R + q, q) : 1 \leq q \leq t - 1\} \\ & \cup \{(a, j) : t \leq j \leq b\}.\end{aligned}$$

These are exactly the prefix-count vertices of $P = A^R(BA)^{t-1}B^S$. Scanning a putative deletion equality from left to right proves the converse: before the first B , at each successive BA , and after the last alternating pair, any coordinate off the displayed list makes the next surviving letter disagree with P . Letter exchange handles G_t . For H_r the identical scan gives

$$\begin{aligned}\mathcal{D}_{H_r}(P) = & \{(i, 0) : 0 \leq i \leq r\} \\ & \cup \{(r + q - 1, q), (r + q, q) : 1 \leq q \leq b\} \\ & \cup \{(i, b) : r + b \leq i \leq a\},\end{aligned}$$

again exactly the prefix-count vertices. At $t = 0$, deleting any A_i, B_j from a block word gives the same block target, yielding the full rectangle. $\square$

The count-one excursion is a separate elementary corollary. If its increments are $(r, 1)$, every path word has the form

$$W_i = A^iBA^{r-i}.$$

For $i < j$, the two sides of the sole excursion are BA^{j-i} and $A^{j-i}B$, the opposite sides of a rectangle. Exchanging A and B handles increments $(1, s)$. Thus an excursion with either track count one is rectangular. The companion supplement proves the remaining component-labelled bridge statements.

The supplement also proves the exact normal-path statement needed later. For two distinct normal words with the same letter counts, either every excursion is rectangular or exactly one excursion is nonrectangular. The proof orders the normal words by the displayed F, H, G families and compares their prefix-height functions. Consecutive rows replace a factor $(AB)^r$ by $(BA)^r$ and can therefore have several unit-square excursions; any two nonconsecutive rows have one excursion. When both track counts are at least two, that excursion is nonrectangular except for the two block extremes. This is the corrected statement: the proof never infers that an arbitrary pair of normal words has only one raw excursion.

Finally, if normal targets satisfy $P = P_0X^q$ or $P = X^qP_0$, their unique path prolongations satisfy the same identity. The proof supplement checks the four word-level endpoint families explicitly, including transitions in which adding the fringe changes the named F, H, G family. Thus constant fringe restoration preserves both the ambient amalgam and its unique nonrectangular excursion in the anchor-free inverse chain.

C The anchor-free inverse chain

For auditability we list the exact logical order of the residual argument. The proof supplement proves each item without assuming the classwise count.

1. An active anchor-free equal-two-deletion class has a common minimum simple support contained in every oriented witness support.
2. On a fixed support there is at most one witness, and track partitions restrict literally along nested supports.
3. Same-class linear strips contract to the unique terminal witness.
4. Reversing a strip appends the same constant fringe to both coordinate words and therefore preserves the ambient boundary.
5. At maximal support the two canonical endpoint pairs lie in different full-fibre components.
6. A third component would make the three pairwise four-deletion cores equal. Transporting the middle component then gives either a nested same-class terminal witness or a larger same-class support, each impossible. Hence there are exactly two components.
7. From any nonrectangular boundary, track occurrence counts fix every outside label. The local prolongation table then reconstructs the exact position word, value word, ambient permutation, and ordered endpoint components. This is inverse to the canonical construction.

Only this seven-step inverse chain is used; no local-confluence claim is required. The exact source file for every step is listed in the companion supplement's `DEPENDENCY_MAP.md`.

References

- [1] Nigel Ray and Julian West. Posets of matrices, and permutations with forbidden subsequences. *Annals of Combinatorics*, 7(1):55–88, 2003.
- [2] Vincent Vatter. Problems and conjectures. In *Permutation Patterns*, volume 376 of *London Mathematical Society Lecture Note Series*, pages 339–345. Cambridge University Press, 2010.
- [3] Vincent Vatter. An assortment of problems in permutation patterns: Unimodality, equivalence, derangements, and sorting, 2026. arXiv:2602.16355.