

SEQUENTIAL GAMES

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ABSTRACT. We formulate a time-series game in which each player chooses a hidden Markov model (HMM) as a mixed strategy. The HMM generates a finite, variable-length sequence of probability vectors, and the payoff is assigned to the completed game as the sum of its stage rewards. We prove a conditional Nash-equilibrium existence result for games with a common finite bound on output lengths. The proof applies a fixed-point argument to the best-response correspondence when each player's admissible HMM-density set is nonempty, convex, and compact in (L^1) , and the completed-game payoff is bounded and measurable. A common BV bound together with (L^1) -closedness can ensure compactness; closure under suitable disjoint-state mixtures can ensure convexity. These properties do not follow from the HMM representation or BV regularity alone and must be verified for a specified model. Consistent estimation of HMM densities may support computation, but by itself does not establish equilibrium existence or replace the theorem's assumptions.

1. INTRODUCTION

Nash's foundational work on non-cooperative games studies equilibrium in finite strategic-form games, where each player selects a strategy from a fixed set and mixed strategies assign probabilities to those strategies [5]. This paper introduces the concept of a *time-series game* to model strategic choices that unfold over sequences. A player's pure strategy is a finite, potentially variable-length sequence of probability vectors, and a hidden Markov model (HMM) represents a mixed strategy by generating such sequences. Payoffs are assigned to the completed game and may aggregate rewards across its successive stages.

The time-series formulation is intended to extend the strategic-form perspective by making the temporal structure of strategies explicit. It raises an equilibrium-existence question for games whose mixed strategies are HMM-generated laws over sequences, rather than arbitrary probability distributions over a finite pure-strategy set. We formulate a conditional existence result using a fixed-point argument and state the compactness, convexity, payoff, and output-length assumptions required by that argument.

2. LITERATURE SURVEY

Nash’s work provides the classical starting point for non-cooperative equilibrium. In a finite strategic-form game, each player has a finite pure-strategy set and may randomize over it; Nash proves the existence of an equilibrium in mixed strategies [5]. The present paper changes the strategy objects: a pure strategy is a sequence of probability vectors, while an HMM specifies a probability law over such sequences.

For games with more general strategy spaces, Glicksberg extended fixed-point methods to compact strategy spaces and continuous payoff functions [2]. This line of work is relevant to the conditional existence result developed below: compactness and payoff continuity support equilibrium existence beyond finite action sets. It does not, by itself, establish existence for a restricted family of HMM-generated laws or for families allowing arbitrarily large (but always finite) sequence lengths; those properties must be verified for the model at hand.

A related dynamic framework is Shapley’s theory of stochastic games, in which play proceeds through states and stages, with transitions between states and rewards accumulated over time [7]. That work supplies an important precedent for games with temporal evolution. The present formulation is different in its choice of strategy representation: each player selects an HMM law over an entire variable-length sequence, and the payoff is assigned to the completed sequence profile. The extensive-form tradition provides another central foundation for sequential interaction. Kuhn’s treatment of extensive games formalizes the order of decisions and the information available to players at their decision points [4]. Kreps and Wilson later introduced sequential equilibrium, requiring sequential rationality together with consistent beliefs in extensive games with imperfect information [3]. These frameworks focus on how actions and information unfold within a game tree. The present model instead takes each player’s complete finite sequence of probability vectors as the output of an HMM and evaluates the resulting completed profile; it does not, as currently specified, model strategies contingent on the observed history of play.

Repeated-game theory studies strategic interaction across multiple plays of a stage game. For example, Fudenberg and Maskin characterize equilibrium payoff possibilities in infinitely repeated games under discounting or incomplete information [1]. That literature highlights how continuation incentives and history-dependent strategies can shape outcomes. The time-series games considered here have finite outputs almost surely and use an undiscounted sum of stage rewards over the completed game. Thus the repeated-game results provide useful context, but do not directly establish the equilibrium claim for the HMM strategy class studied here.

HMMs are established probabilistic models for sequential observations. Rabiner’s tutorial describes their basic formulation and inference problems [6]. Here the HMM is used as a player’s mixed strategy, so the game-theoretic

question concerns equilibrium among HMM-generated distributions rather than statistical inference alone.

Taken together, these references motivate the ingredients of the present model—mixed strategies, fixed-point existence methods, stage-based dynamic games, and HMM sequence models—while leaving the specific combination studied here to be analyzed under explicit compactness, integrability, and payoff-regularity assumptions.

3. SEQUENTIAL GAMES

3.1. Pure strategies. Let m denote the number of available static actions. Write

$$\Delta_m = \left\{ x \in [0, 1]^m : \sum_{a=1}^m x_a = 1 \right\}$$

for the probability simplex over these actions. A pure strategy of player i is a finite sequence

$$s_i = (x_{i,1}, \dots, x_{i,T_i}), \quad x_{i,q} \in \Delta_m,$$

where the length $T_i \geq 1$ may vary across strategies. Thus each stage of a pure strategy specifies a probability vector over the player's static actions. Infinite sequences are not included in the pure-strategy space.

3.2. Mixed strategies. A mixed strategy for player i is an admissible hidden Markov model (HMM) whose output is a random pure strategy s_i . In particular, the HMM induces a probability law on player i 's set of finite sequences; it does not directly induce a probability distribution on the completed-game space Ω . We require the HMM to terminate almost surely, so its output has finite length with probability one. The players choose their HMMs independently, as specified in the model below.

3.3. Payoff. Let n be the number of players. At stage q , collect the players' probability vectors into a matrix $A_q \in \mathbb{R}^{m \times n}$, with the i -th column representing player i 's vector. The set of feasible stage profiles is

$$\Gamma = \left\{ A \in [0, 1]^{m \times n} : \sum_{a=1}^m A_{ai} = 1 \text{ for every } i = 1, \dots, n \right\}.$$

When players' sequence lengths differ, a specified completion rule determines the stage profiles after a player's output ends. For a realized profile $s = (s_1, \dots, s_n)$, let $\Phi(s) = (A_1, \dots, A_T)$ denote the resulting completed game, where $T = \max_i T_i$. Thus Ω is the space of all finite sequences of feasible stage-profile matrices, with sequences of different lengths treated as distinct games.

For each player i , let $\zeta_i : \Gamma \rightarrow \mathbb{R}$ be the stage-reward function. The payoff is assigned to the completed game and is the sum of the stage rewards:

$$p_i : \Omega \rightarrow \mathbb{R}, \quad p_i(\omega) = \sum_{q=1}^T \zeta_i(A_q), \quad \omega = (A_1, \dots, A_T).$$

We require the realized reward sequence to be square-summable,

$$\sum_{q=1}^T |\zeta_i(A_q)|^2 < \infty.$$

Since every realized game has finite length and each $\zeta_i(A_q)$ is real-valued, this condition holds automatically for each completed game. Finiteness of each realized payoff does not by itself imply that its expectation under every admissible HMM profile is finite; the required integrability conditions are stated in the existence theorem.

3.4. BV sequential Games. BV sequential Games are mixed strategy sequential Games, whose payoff p_i is BV for $i = 1, 2, \dots, n$. Also ζ_i is BV, $i = 1, 2, \dots, n$.

3.5. Scope of the bounded-variation assumption. Bounded variation is a broad and useful regularity condition for payoffs, but it is not the widest possible payoff class and it does not include every game. On compact finite-dimensional action domains, Lipschitz and continuously differentiable payoffs are of bounded variation. The class also allows many nonsmooth payoffs, including finite piecewise-smooth rules and finite threshold or jump penalties, provided their total variation is finite. Thus BV regularity accommodates a wide range of smooth and discontinuous payoff specifications used in applications.

It would nevertheless be incorrect to claim that every continuous payoff is BV. For example, on $[0, 1]$, set $g(0) = 0$ and $g(x) = x \sin(1/x^2)$ for $x > 0$. This function is continuous and bounded but has infinite total variation: along successive extrema approaching zero, the sum of absolute changes dominates a constant multiple of $\sum_{k \geq 1} k^{-1/2}$, which diverges. Hence BV is a substantive restriction, even within continuous payoffs. No single regularity class should be described as the widest class without specifying the domain, the regularity ordering, and the properties needed by the equilibrium proof.

In this paper, BV is imposed as a tractable regularity hypothesis on the stage rewards and whole-game payoff. The equilibrium result additionally requires compactness and convexity of the admissible HMM mixed-strategy sets and continuity and integrability of expected payoffs. Accordingly, BV should be understood as covering many practically relevant payoff forms under the stated conditions, not as covering games of every possible type or as sufficient by itself for equilibrium existence.

Establish sufficient conditions for Nash equilibrium in sequential games. All games are assumed to be non-co-operative.

4. A CONDITIONAL EXISTENCE THEOREM FOR SEQUENTIAL GAMES

We give a sufficient condition for the existence of a Nash equilibrium in a time-series game whose players choose hidden Markov models (HMMs) generating finite, variable-length pure strategies. Each admissible HMM terminates after finitely many stages with probability one. Each pure strategy is a finite sequence of probability vectors, and the payoff of a complete game is the sum of its stage rewards, which may be signed. The result uses a fixed-point argument on compact convex sets of normalized HMM densities. Bounded variation of individual payoffs and densities is not, by itself, enough for this argument: compactness of the feasible density sets and control of output lengths are also required.

4.1. Model. Let $n \geq 2$ be the number of players and m the number of static actions. Define the closed probability simplex

$$\Delta_m = \left\{ x \in [0, 1]^m : \sum_{a=1}^m x_a = 1 \right\}.$$

A pure strategy of player i is a finite sequence of vectors in Δ_m ; infinite sequences are excluded. In the existence theorem, we impose a common finite bound L on the output length. Thus the theorem uses the pure-strategy space

$$S_i = \bigcup_{T=1}^L \Delta_m^T.$$

The finite union is understood as a disjoint union, so sequences of different lengths are distinct strategies. For the density formulation, equip each component Δ_m^T with its normalized Hausdorff measure λ_T on its affine hull and use the finite direct-sum reference measure $\mu_i = L^{-1} \bigoplus_{T=1}^L \lambda_T$. Every HMM law represented by a density is required to be absolutely continuous with respect to μ_i . For a profile $s = (s_1, \dots, s_n)$, a specified measurable rule combines the players' sequences into the completed game $\Phi(s) \in \Omega$. The rule must specify how the game is represented when the players' output lengths differ; the game ends when the longest output ends.

For each player i , let $\zeta_i : \Gamma \rightarrow \mathbb{R}$ be measurable and bounded. The payoff for a completed game $\omega = (A_1, \dots, A_T)$ is

$$p_i(\omega) = \sum_{q=1}^T \zeta_i(A_q).$$

Consequently, p_i is measurable and bounded on the game space under the theorem assumptions. With bounded ζ_i and $T \leq L$, square-summability holds automatically. For finite sequences whose lengths have no common deterministic bound, square-summability alone does not ensure integrability of the total payoff; the model must additionally require that the payoff

is integrable under every admissible HMM profile. We may additionally assume p_i and ζ_i have bounded variation; the existence proof below does not rely on BV alone.

A mixed strategy for player i is a probability density f_i over S_i , with respect to a fixed reference measure μ_i . It is normalized:

$$f_i \geq 0, \quad \int_{S_i} f_i d\mu_i = 1.$$

The players' HMM outputs are independent. The expected payoff at a profile $f = (f_1, \dots, f_n)$ is

$$U_i(f) = \int_{S_1 \times \dots \times S_n} p_i(\Phi(s_1, \dots, s_n)) \prod_{j=1}^n f_j(s_j) d(\mu_1 \times \dots \times \mu_n).$$

A profile f^* is a Nash equilibrium if

$$U_i(f^*) \geq U_i(g_i, f_{-i}^*)$$

for every player i and every admissible alternative density g_i .

4.2. Assumptions for existence. For each player i , let $\mathcal{H}_i \subset L^1(\mu_i)$ be the set of admissible HMM strategy densities. Assume:

- (1) $\mathcal{H}_i$ is nonempty, convex, and compact in the $L^1(\mu_i)$ norm.
- (2) Every $f_i \in \mathcal{H}_i$ is a normalized probability density:

$$f_i \geq 0 \quad \mu_i\text{-almost everywhere,} \quad \int_{S_i} f_i d\mu_i = 1.$$

- (3) The game-completion map $\Phi : S_1 \times \dots \times S_n \rightarrow \Omega$ is measurable.
- (4) For each player i , the whole-game payoff $p_i : \Omega \rightarrow \mathbb{R}$ is bounded and measurable.

These assumptions concern the admissible HMM density sets themselves. The density-set verification subsection below gives sufficient additional conditions for compactness and convexity; absent such a verification, those properties are hypotheses of the theorem.

4.3. Verifying the HMM density-set assumptions. The compactness and convexity requirements do not follow merely from describing strategies as HMMs or from requiring each density to be BV. The following additional conditions provide a sufficient verification for the finite-horizon model.

For each player, require that the admissible HMMs (i) have outputs of length at most L , (ii) generate laws absolutely continuous with respect to the direct-sum reference measure μ_i above, and (iii) allow any finite number of hidden states, with the allowed topology and emission class preserved

when two state spaces are relabelled as disjoint blocks. Impose a common bound $C_i < \infty$ on the componentwise BV norms of all induced densities:

$$\sum_{T=1}^L \left\| f_i|_{\Delta_m^T} \right\|_{\text{BV}(\Delta_m^T)} \leq C_i.$$

Finally, require that the family $\mathcal{H}_i$ of admissible HMM densities is closed in $L^1(\mu_i)$. This last property is architecture-dependent and must be checked for the chosen emission family; it is not implied by the HMM label or the BV bound.

Under these conditions, $\mathcal{H}_i$ is convex: for two admissible HMMs with densities f_i and g_i , relabel their hidden states into disjoint blocks and choose the initial distribution to select the first block with probability θ and the second with probability $1 - \theta$. The resulting HMM generates the density $\theta f_i + (1 - \theta)g_i$. This construction preserves blockwise transition restrictions, including a no-self-transition restriction, when those restrictions hold in each component model. It requires that the admissible class permit the resulting finite state count; convexity is not established for a class with a fixed state-count cap.

The common BV bound gives relative compactness in L^1 on each compact finite-dimensional simplex product Δ_m^T . Since there are only finitely many lengths $T \leq L$, the full family is relatively compact in $L^1(\mu_i)$. Its assumed L^1 -closedness then makes $\mathcal{H}_i$ compact. Nonnegativity and unit integral are preserved under L^1 limits. In the present manuscript, no emission family or HMM parameter class is specified from which this closedness and the common BV bound can be derived. They therefore remain explicit, model-specific hypotheses; the theorem cannot claim unconditional compactness for an unspecified HMM class.

4.4. Generator bounds in a continuous-time extension. A bounded generator controls regularity of the hidden-state marginal trajectory, but this is only one part of the HMM density-set problem. For a continuous-time Markov chain with a fixed number K of hidden states, let $p(t)$ be a row probability vector satisfying

$$\dot{p}(t) = p(t)Q,$$

where Q is a generator (its off-diagonal entries are nonnegative and each row sums to zero). If

$$\|Q\|_\infty := \max_{1 \leq r \leq K} \sum_{s=1}^K |Q_{rs}| \leq M,$$

then

$$\|\dot{p}(t)\|_1 = \|p(t)Q\|_1 \leq \|p(t)\|_1 \|Q\|_\infty \leq M.$$

Consequently,

$$\text{Var}_{[0,T]}(p; \ell^1) \leq \int_0^T \|\dot{p}(t)\|_1 dt \leq MT.$$

Thus a uniform generator bound gives a uniform BV bound (indeed, a uniform Lipschitz bound) on the hidden-state marginal paths. For fixed K , initial distributions in the simplex, and generators in a closed bounded set, these paths form a compact family in $C([0, T], \mathbb{R}^K)$.

This does not by itself imply compactness, L^1 -closedness, or a BV bound for the emitted sequence densities. Those conclusions require additional hypotheses on the emission densities and continuity of the map from the HMM parameters and hidden-state path law to the output law. Also, if output stages are counted at transition times, a bounded-rate continuous-time chain can make an unbounded number of transitions on $[0, T]$, although that number is finite almost surely; the finite- L theorem above would then need a uniform tail-control version. Generator regularization is therefore a useful additional condition for a continuous-time extension, not a replacement for the emission-family and output-length conditions in the theorem.

4.5. Existence theorem.

Theorem 4.1. *Under assumptions 1–4, there exists a Nash equilibrium $f^* \in \prod_{i=1}^n \mathcal{H}_i$.*

Proof. Let

$$X = \prod_{i=1}^n \mathcal{H}_i, \quad G_i(s_1, \dots, s_n) = p_i(\Phi(s_1, \dots, s_n)).$$

By assumptions 3 and 4, G_i is measurable and bounded. For $f = (f_1, \dots, f_n) \in X$, write its joint density as

$$F_f(s_1, \dots, s_n) = \prod_{j=1}^n f_j(s_j).$$

The expected payoff

$$U_i(f) = \int G_i(s) F_f(s) d(\mu_1 \times \dots \times \mu_n)(s)$$

is therefore well-defined and finite.

For any two profiles $f, g \in X$, expand the difference of their product densities as

$$F_f - F_g = \sum_{k=1}^n \left(\prod_{j < k} f_j \right) (f_k - g_k) \left(\prod_{j > k} g_j \right).$$

All component densities are normalized. Integrating the absolute value of each summand and using Fubini's theorem gives

$$\|F_f - F_g\|_{L^1(\mu_1 \times \dots \times \mu_n)} \leq \sum_{j=1}^n \|f_j - g_j\|_{L^1(\mu_j)}.$$

Consequently,

$$|U_i(f) - U_i(g)| \leq \|G_i\|_\infty \sum_{j=1}^n \|f_j - g_j\|_{L^1(\mu_j)}.$$

Thus U_i is jointly continuous on X . For fixed f_{-i} , it is affine in f_i , since $F_{(f_i, f_{-i})}$ is linear in f_i .

For each player i , define the best-response correspondence

$$B_i(f_{-i}) = \arg \max_{g_i \in \mathcal{H}_i} U_i(g_i, f_{-i}).$$

Because $\mathcal{H}_i$ is nonempty and compact and U_i is continuous, the maximum is attained. Hence $B_i(f_{-i})$ is nonempty and compact. Affineness in g_i and convexity of $\mathcal{H}_i$ imply that $B_i(f_{-i})$ is convex.

The graph of B_i is closed: if $f_{-i}^k \rightarrow f_{-i}$, $g_i^k \rightarrow g_i$, and $g_i^k \in B_i(f_{-i}^k)$, then for every $h_i \in \mathcal{H}_i$,

$$U_i(g_i^k, f_{-i}^k) \geq U_i(h_i, f_{-i}^k).$$

Taking limits and using continuity yields $U_i(g_i, f_{-i}) \geq U_i(h_i, f_{-i})$, so $g_i \in B_i(f_{-i})$. Since the domain and range are compact metric spaces, the closed-graph property implies that B_i is upper hemicontinuous.

Now define $B : X \rightrightarrows X$ by

$$B(f) = \prod_{i=1}^n B_i(f_{-i}).$$

This correspondence has nonempty, compact, convex values and is upper hemicontinuous. The set X is nonempty, compact, and convex by assumptions 1–2. Kakutani’s fixed-point theorem therefore gives $f^* \in X$ such that $f^* \in B(f^*)$. By the definition of each best-response set, no player can improve their expected payoff by unilaterally changing their HMM density. Thus f^* is a Nash equilibrium. $\square$

4.6. Scope of the result. The theorem is an existence result under explicit hypotheses on the strategy-density sets and the completed-game payoff. It does not, by itself, establish those hypotheses for every HMM architecture. In particular, the finite-horizon formulation assumes a common deterministic length bound L . Almost-sure termination alone is weaker: it permits arbitrarily long outputs and does not give a compact finite-horizon strategy space. If output lengths are unbounded, a different existence argument must control the tails uniformly and ensure that expected whole-game payoffs are well defined and continuous under changes in strategy.

For a finite-state HMM with a strictly left-to-right transition structure and no self-transitions, the number of emitted stages is bounded by the number of states, provided the model’s termination convention permits only forward moves and termination. This is one way to obtain a common bound L when the state count is itself uniformly bounded. It is not a consequence

of finite state count alone for arbitrary transition topologies, nor does it establish compactness of the induced output-density family.

The theorem’s other substantive strategy-side requirements are that each admissible density set be nonempty, convex, and compact in L^1 . A uniform componentwise BV bound on the finite-horizon density family yields relative compactness, but compactness additionally requires L^1 -closedness. Convexity also needs to be verified: the disjoint-block mixture construction works when the HMM class allows the required increase in state count and preserves its transition and emission restrictions. A fixed state-count cap or a restrictive emission architecture may prevent this construction.

An ideal estimation method does not, on its own, replace these existence hypotheses. Suppose that, as the sample size grows, an estimator consistently recovers every relevant HMM output density and parameter. This is a statistical statement about recovering a data-generating model (subject to the estimator’s assumptions); Nash equilibrium is a strategic statement requiring a profile at which no player can gain by deviating. Consistent estimation does not ensure that such a profile exists, that best responses attain their suprema, or that limits of estimated strategies remain admissible. Compactness, convexity, and payoff continuity address these issues in the fixed-point proof.

The distinction also matters computationally. If equilibrium existence is established by another argument, and estimated payoffs and deviations approximate the true ones uniformly over the relevant strategy sets, consistent density estimation may help construct approximate equilibria. Pointwise convergence of estimated HMM densities or parameters alone does not provide that uniform control and does not establish an exact equilibrium. An ideal estimator may remove a practical obstacle to learning or computing strategies, but it cannot by itself replace the mathematical existence argument or its hypotheses. Finally, bounded measurability of the whole-game payoff and measurability of the completion rule make the expected-payoff expression well defined in the stated finite-horizon model; the L^1 estimate in the proof supplies its continuity in the players’ densities. Consequently, BV regularity—of payoffs, hidden-state paths, or individual output densities—is useful only as part of a verification. It cannot replace the length, closure, convexity, and payoff hypotheses. For a specified HMM family, each of these properties must be checked before the theorem yields an equilibrium.

5. POTENTIAL PRACTICAL APPLICATIONS

The framework can represent finite interactions in which each player commits to an HMM that generates a finite sequence of probability vectors. At each stage, the players’ vectors form the columns of a stage-profile matrix, and a specified stage-reward function assigns each player a reward. The total payoff is the sum of those rewards over the completed game, including any stages supplied by the completion rule when output lengths differ. An

HMM’s hidden states describe the internal evolution of the player’s randomized sequence; they do not, without additional modelling, represent observed opponent actions or guarantee adaptation to public history.

The following examples indicate possible modelling choices. They are illustrations of the framework, not empirical findings or applications for which the paper has established equilibrium existence.

- **Capacity and resource allocation.** Players such as service providers or infrastructure operators may generate sequences of probability vectors over allocation options. Stage rewards can encode delivered service, operating costs, congestion, and unmet demand. A model must specify how the allocation vectors jointly determine outcomes and how play proceeds after a player’s sequence ends.
- **Communication and network access.** Competing devices or providers may generate probability vectors over channels, routes, or access options. The stage-reward functions can represent throughput, delay, interference, and resource use. Latent HMM states can encode a player’s internal operating modes; if external congestion is to affect decisions, the model must specify how that information enters the player’s strategy.
- **Market participation.** Traders or institutions may generate sequences of probability vectors over market actions or asset choices. Stage rewards can account for returns, transaction costs, and risk exposure, provided the market outcome and reward are defined as measurable functions of the stage-profile matrices. The finite-horizon payoff assumptions must be checked for the chosen reward rule.
- **Security and defense.** An attacker and a defender may generate probability vectors over targets, monitoring choices, or protective actions. Stage rewards can combine losses, defensive costs, and successful interventions. A model intended to capture learning from observed moves must add an explicit information and response mechanism; autonomous HMM state evolution alone does not encode that feature.
- **Inventory and supply chains.** Suppliers, carriers, and retailers may generate sequences of probability vectors over production, replenishment, or shipping options. Stage rewards can include fulfillment, holding costs, shortages, and waste. The completion rule should specify the operational meaning of stages after an agent’s output ends.

Across these examples, the dimension m is the number of entries in each player’s stage probability vector, and (n) is the number of players, as in the model. Applying the existence theorem requires a concrete completion rule and measurable, bounded whole-game payoffs on the finite-horizon game space. It also requires verification that the selected HMM density sets are nonempty, convex, and compact in L^1 , with a common deterministic

output-length bound. The examples therefore identify domains where the model may be useful; they do not show that the theorem’s assumptions hold automatically in those domains.

6. CONCLUSION

This paper formulates sequential games in which each player chooses an HMM as a mixed strategy. The HMM induces a probability law on that player’s finite sequence of probability vectors. Given a measurable rule that combines the players’ sequences into a completed game, the payoff is the sum of stage rewards over the entire game. This formulation places variable-length, history-dependent random behavior within a strategic game framework.

The paper proves a conditional equilibrium-existence theorem using the best-response correspondence and a fixed-point argument. In the finite-horizon setting, the result applies when each player’s admissible HMM density set is nonempty, convex, and compact in (L^1) , and when the completed-game payoff is bounded and measurable. The proof then gives continuity of expected payoffs in the strategy densities. A common BV bound and (L^1) -closedness can provide compactness, while closure of the HMM class under disjoint-block mixtures can provide convexity. These properties are not consequences of the HMM representation or BV regularity alone, and must be verified for the chosen emission and transition architecture.

A consistent estimator of HMM densities and parameters addresses a different question from equilibrium existence. Estimation can aid learning and computation, but consistency by itself neither guarantees an equilibrium nor ensures that limits of strategies remain admissible or that deviations are controlled uniformly. To infer approximate equilibria from estimated models requires additional control of payoff and deviation errors; an exact equilibrium still needs a separate existence argument.

The result also relies on a common finite bound on output lengths. If the model permits arbitrarily long finite outputs, almost-sure termination alone does not justify the finite-horizon proof. A corresponding extension would need uniform tail control and conditions ensuring that expected total payoffs are well defined and continuous. Establishing such conditions for concrete HMM families and studying estimation-based approximate equilibrium are natural directions for future work.

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