

The Pólya–Vinogradov inequality with $\log \log q$

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Abstract

We prove that every primitive Dirichlet character χ modulo $q \geq 3$ has character sums $\sum_{n \leq x} \chi(n)$ of size at most $C\sqrt{q} \log \log q$, with an absolute constant C . Montgomery and Vaughan proved this bound under the generalised Riemann hypothesis, and Paley showed that it is optimal. The only input from outside the paper is the zero-free half-plane $\operatorname{Re} s > 7/8$ for Dirichlet L -functions. On the major arcs this half-plane gives a power saving for twisted prime sums; on the minor arcs we prove a weak form of the Montgomery–Vaughan bound for exponential sums with multiplicative coefficients, with Selberg’s sieve as the only arithmetic input. The same half-plane gives $|L'/L(1, \chi)| \ll \log \log q$ for every nonprincipal χ and, for every prime q , the two-sided bound $-C \log q \log \log q \leq \gamma_q \leq C \log q$ for the Euler–Kronecker constant of the cyclotomic field $\mathbb{Q}(\zeta_q)$.

1. Introduction

Let χ be a primitive Dirichlet character modulo $q \geq 3$. The Pólya–Vinogradov inequality states that $|\sum_{n \leq x} \chi(n)| \ll \sqrt{q} \log q$ uniformly in x [3, Chapter 23].¹ Paley proved that the factor $\log q$ cannot be replaced by anything smaller than $\log \log q$: for infinitely many primitive quadratic characters the maximum over x exceeds $c\sqrt{q} \log \log q$ [12]. Montgomery and Vaughan [9] proved the upper bound $\sqrt{q} \log \log q$ under the generalised Riemann hypothesis. Without hypothesis, Granville and Soundararajan lowered the exponent of

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¹The proof expands χ in additive characters through the Gauss sum $\tau(\bar{\chi})$, whose modulus is $\sqrt{q}$ for primitive χ , and bounds the resulting geometric series trivially; Lemma 5.1 below is the same argument with the harmonic sums kept.

$\log q$ for characters of fixed odd order [5].² We prove the bound of Montgomery and Vaughan for every primitive character.

THEOREM 1.1. *There is an absolute constant C such that for every primitive character χ modulo $q \geq 3$ and every $x \geq 1$,*

$$\left| \sum_{n \leq x} \chi(n) \right| \leq C \sqrt{q} \log \log q.$$

By Paley's theorem, Theorem 1.1 is optimal up to the value of C . The input that replaces the Riemann hypothesis is the following theorem.

THEOREM 1.2 ([11]). *If χ is a nonprincipal Dirichlet character, then $L(s, \chi) \neq 0$ for $\operatorname{Re} s > 7/8$.*

Theorem 1.2 is proved, and formalised in Lean 4, in [11]. It is the only result we use without proof; everything else is proved below from first principles, and the constants are effective.

Montgomery and Vaughan use the Riemann hypothesis in one place: on the major arcs, for prime sums of the characters $\chi\psi$ twisted by additive characters, where ψ runs over the characters of a small modulus r . A power saving in these sums is enough there, and Theorem 1.2 provides one (Corollary 3.8). On the minor arcs they use their unconditional bound for exponential sums with multiplicative coefficients. A weaker form of it suffices, and we prove that form in Section 4 with Selberg's sieve as the only arithmetic input. Figure 1 shows how the parts fit together.

The same half-plane controls the logarithmic derivative of $L(s, \chi)$ at $s = 1$. For a number field K write $\zeta_K(s) = c_{-1}(s-1)^{-1} + c_0 + O(s-1)$ near $s = 1$; the Euler–Kronecker constant of K is $\gamma_K = c_0/c_{-1}$. Let γ_q be the Euler–Kronecker constant of $\mathbb{Q}(\zeta_q)$. For a prime q the factorisation $\zeta_{\mathbb{Q}(\zeta_q)}(s) = \zeta(s) \prod_{\chi \neq \chi_0} L(s, \chi)$, the product over the nonprincipal characters modulo q [13, Theorem 4.3],³ gives

$$(1.1) \quad \gamma_q = \gamma + \sum_{\chi \neq \chi_0} \frac{L'}{L}(1, \chi),$$

where γ is Euler's constant.

THEOREM 1.3. *There is an absolute constant C with the following properties.*

²For a primitive character of odd order g they obtain $\sqrt{q}(\log q)^{1-\delta_g/2+o(1)}$ with $\delta_g = 1 - \frac{g}{\pi} \sin \frac{\pi}{g}$, and under the generalised Riemann hypothesis the same exponent of $\log \log q$.

³All nonprincipal characters modulo a prime are primitive, so no Euler factors have to be removed: the conductor of each nonprincipal χ is q itself, and $L(s, \chi)$ is the L -function of the corresponding primitive character.

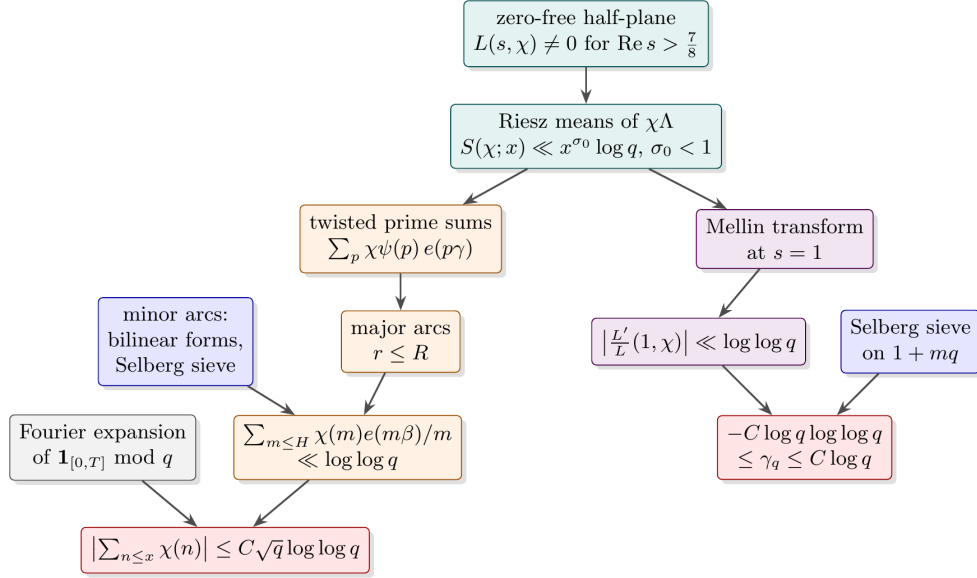


Figure 1. The structure of the proofs. The left branch proves Theorem 1.1: the Fourier expansion of a character sum reduces it to the sums $\sum_{m \leq H} \chi(m) e(m\beta)/m$, which are bounded on the major arcs by the half-plane and on the minor arcs by bilinear forms and the sieve. The right branch proves Theorem 1.3.

(i) For every nonprincipal character χ modulo q ,

$$\left| \frac{L'}{L}(1, \chi) \right| \leq C \log \log q.$$

(ii) For every prime $q \geq 3$, $-C \log q \log \log q \leq \gamma_q \leq C \log q$.

Under the generalised Riemann hypothesis the bound (i) is classical, Ihara proved $\gamma_q \leq (2 + o(1)) \log q$ and Badzayan proved a lower bound of the order of (ii) [7] and [1]. Ford, Luca and Moree proved the two bounds of (ii) without hypothesis for a set of primes q of density one [4] and [10].⁴ For all primes the best unconditional bound we know is $|\gamma_q| \ll_{\varepsilon} q^{\varepsilon}$, due to Ihara, Murty and Shimura [8]. The upper bound in (ii) needs only Chebyshev's bound. The lower bound needs an upper bound for the prime powers $n \leq q^{17}$ with $n \equiv 1 \pmod{q}$, which we obtain from Selberg's sieve applied in dyadic blocks (Lemma 6.3).

⁴The difficulty for an individual prime q lies in the prime powers $n \equiv 1 \pmod{q}$, which enter γ_q with the weight $(q-1)\Lambda(n)/n$. Below q^{17} we bound them for each prime q separately by Lemma 6.3; beyond q^{17} the zero-free half-plane makes their contribution bounded.

Theorems 1.1 and 1.3 are not among the consequences of Theorem 1.2 drawn in [11]. The arguments work, with other constants, whenever $L(s, \chi) \neq 0$ on $\operatorname{Re} s > \theta$ for all nonprincipal χ and some fixed $\theta < 1$. The paper is organised as follows. Section 2 collects Chebyshev's bounds and Selberg's sieve. Section 3 bounds Riesz means of $\chi\Lambda$ by a contour shift. Section 4 proves the bound for exponential sums with multiplicative coefficients, Section 5 proves Theorem 1.1 and Section 6 proves Theorem 1.3.

2. Preliminaries

Throughout, p and ℓ denote primes, Λ is the von Mangoldt function, $\theta(x) = \sum_{p \leq x} \log p$, $\psi(x) = \sum_{n \leq x} \Lambda(n)$, $e(x) = e^{2\pi i x}$, and $\|x\|$ is the distance from x to the nearest integer. We write $\omega(n)$ for the number of prime factors of n , $\tau_3(n)$ for the number of ordered factorisations $n = abc$, and $P(n)$ for the largest prime factor of $n \geq 2$, with $P(1) = 1$.

LEMMA 2.1. *For $x \geq 1$ and integers $N \geq 2$,*

- (i) $\theta(x) \leq x \log 4$;
- (ii) $\psi(x) - \theta(x) \leq 2\sqrt{x} \log x$ and $\psi(x) \leq 4x$;
- (iii) $\sum_{p \leq N} \frac{\log p}{p} \leq \log N + \log 4$;
- (iv) $\sum_{p \leq N} \frac{1}{p} \leq \log \log N + 10$.

Proof. (i) It suffices to treat integers $m \geq 1$, and we use induction on m . The cases $m = 1, 2$ are clear, and for even $m > 2$ we have $\theta(m) = \theta(m-1)$. Let $m = 2n+1$. The primes in $(n+1, 2n+1]$ divide $\binom{2n+1}{n}$, and $2\binom{2n+1}{n} \leq 2^{2n+1}$ because $\binom{2n+1}{n}$ and $\binom{2n+1}{n+1}$ are equal terms of the binomial expansion of $(1+1)^{2n+1}$. Hence $\theta(2n+1) \leq \theta(n+1) + n \log 4 \leq (2n+1) \log 4$.

(ii) We have $\psi(x) - \theta(x) = \sum_{k \geq 2} \theta(x^{1/k})$, where only $k \leq \log x / \log 2$ contribute. Each term is at most $\sqrt{x} \log 4$ by (i), so the sum is at most $(\log x / \log 2) \sqrt{x} \log 4 = 2\sqrt{x} \log x$. Since $\log x \leq \sqrt{x}$, $\psi(x) \leq (\log 4 + 2)x \leq 4x$.

(iii) By Legendre's formula and (i),

$$N \log N \geq \log N! \geq \sum_{p \leq N} \left\lfloor \frac{N}{p} \right\rfloor \log p \geq N \sum_{p \leq N} \frac{\log p}{p} - N \log 4.$$

(iv) Let $A(u) = \sum_{p \leq u} \log p / p$. Partial summation and (iii) give

$$\sum_{p \leq N} \frac{1}{p} = \frac{A(N)}{\log N} + \int_2^N \frac{A(u)}{u \log^2 u} du \leq 1 + \frac{\log 4}{\log 2} + \int_2^N \left(\frac{1}{u \log u} + \frac{\log 4}{u \log^2 u} \right) du,$$

and the integral is at most $\log \log N - \log \log 2 + 2$. $\square$

We use Selberg's upper bound sieve in the following form; for the method see [6, Chapter 3], and we give the proof of the form we need.⁵

LEMMA 2.2. *Let $a_1, \dots, a_X$ be integers, P a squarefree integer, ν a multiplicative function with $0 < \nu(p) < 1$ for $p \mid P$, and $y \geq 1$. For $d \mid P$ put $R_d = \#\{n \leq X : d \mid a_n\} - \nu(d)X$, and for $l \mid P$ put $g(l) = \prod_{p \mid l} \nu(p)/(1 - \nu(p))$. Then*

$$\#\{n \leq X : (a_n, P) = 1\} \leq \frac{X}{S} + \sum_{\substack{d \mid P \\ d \leq y}} 3^{\omega(d)} |R_d|, \quad S = \sum_{\substack{l \mid P \\ l \leq \sqrt{y}}} g(l).$$

Proof. Let D be the set of divisors d of P with $d \leq \sqrt{y}$, and let $(\lambda_d)_{d \in D}$ be real numbers with $\lambda_1 = 1$. If $(a_n, P) = 1$ then $\sum_{d \in D, d \mid a_n} \lambda_d = 1$, and otherwise this sum is real, so

$$\begin{aligned} \#\{n : (a_n, P) = 1\} &\leq \sum_{n \leq X} \left(\sum_{\substack{d \in D \\ d \mid a_n}} \lambda_d \right)^2 \\ &= X \sum_{d_1, d_2 \in D} \lambda_{d_1} \lambda_{d_2} \nu([d_1, d_2]) + \sum_{d_1, d_2 \in D} \lambda_{d_1} \lambda_{d_2} R_{[d_1, d_2]}. \end{aligned}$$

Write Q for the double sum in the first term. Since ν is multiplicative and d_1, d_2 are squarefree, $\nu([d_1, d_2]) = \nu(d_1)\nu(d_2)/\nu((d_1, d_2))$. Let h be the multiplicative function on the divisors of P with $h(p) = 1/\nu(p) - 1 = 1/g(p)$; then $1/\nu(m) = \sum_{l \mid m} h(l)$ for $m \mid P$, and

$$Q = \sum_{l \mid P} h(l) \left(\sum_{\substack{d \in D \\ l \mid d}} \lambda_d \nu(d) \right)^2 = \sum_{l \in D} \frac{y_l^2}{g(l)}, \quad y_l = \sum_{\substack{d \in D \\ l \mid d}} \lambda_d \nu(d).$$

Möbius inversion gives $\lambda_d \nu(d) = \sum_{l \in D, d \mid l} \mu(l/d) y_l$, so the condition $\lambda_1 = 1$ reads $\sum_{l \in D} \mu(l) y_l = 1$. By Cauchy's inequality

$$1 \leq \left(\sum_{l \in D} \frac{y_l^2}{g(l)} \right) \left(\sum_{l \in D} g(l) \right) = QS,$$

with equality for $y_l = \mu(l)g(l)/S$. We make this choice, so that $Q = 1/S$. Then, writing $l = de$,

$$\lambda_d = \frac{1}{\nu(d)S} \sum_{\substack{l \in D \\ d \mid l}} \mu(l/d) \mu(l) g(l) = \frac{\mu(d)g(d)}{\nu(d)S} \sum_{\substack{e \in D, (e,d)=1 \\ de \leq \sqrt{y}}} g(e).$$

⁵The weights λ_d are those of Selberg's quadratic form, and the only feature used later is the bound $|\lambda_d| \leq 1$, which makes the remainder term at most a sum of $3^{\omega(d)} |R_d|$ over $d \leq y$.

Since $g(d)/\nu(d) = \prod_{p|d}(1 - \nu(p))^{-1} = \prod_{p|d}(1 + g(p)) = \sum_{f|d} g(f)$, we get $|\lambda_d| \leq S^{-1} \sum_{f|d} \sum_e g(fe)$, and the products fe are distinct elements of D ; hence $|\lambda_d| \leq 1$. Finally $[d_1, d_2] \leq d_1 d_2 \leq y$, and every $d \mid P$ is the least common multiple of at most $3^{\omega(d)}$ pairs (d_1, d_2) , so the second term is at most $\sum_{d|P, d \leq y} 3^{\omega(d)} |R_d|$. $\square$

3. Riesz means of prime sums

For an arithmetic function a and $x \geq 1$ put

$$S(a; x) = \sum_{n \leq x} a(n) \left(1 - \frac{n}{x}\right), \quad S(\chi; x) = S(\chi \Lambda; x) = \sum_{n \leq x} \chi(n) \Lambda(n) \left(1 - \frac{n}{x}\right).$$

The weight $1 - n/x$ makes the Mellin kernel absolutely integrable, which keeps the contour shift below free of truncation errors. The aim of this section is Proposition 3.7.

LEMMA 3.1. *Let $a: \mathbb{N} \rightarrow \mathbb{C}$ satisfy $\sum_n |a(n)| n^{-\sigma} < \infty$ for $\sigma > 1$, and let $F(s) = \sum_n a(n) n^{-s}$ for $\operatorname{Re} s > 1$.*

- (i) *For $\operatorname{Re} s > 1$, $\int_1^\infty S(a; x) x^{-s-1} dx = \frac{F(s)}{s(s+1)}$.*
- (ii) *For $x > 0$, $S(a; x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(2+it) \frac{x^{2+it}}{(2+it)(3+it)} dt$.*

In other words the following square commutes, and (ii) inverts its right column on the line $\operatorname{Re} s = 2$:

$$\begin{array}{ccc} a & \xrightarrow{\text{Riesz mean}} & S(a; \cdot) \\ \text{Dirichlet series} \downarrow & & \downarrow \text{Mellin transform} \\ F & \xrightarrow{\times \frac{1}{s(s+1)}} & \frac{F(s)}{s(s+1)} \end{array}$$

Proof. (i) For $n \geq 1$, $\int_n^\infty (1 - n/x) x^{-s-1} dx = n^{-s}/s - n^{-s}/(s+1) = n^{-s}/(s(s+1))$. Since $\sum_n |a(n)| \int_n^\infty x^{-\sigma-1} dx = \sigma^{-1} \sum_n |a(n)| n^{-\sigma} < \infty$, the sum and the integral may be exchanged.

(ii) For $y > 0$ let $k(y) = \frac{1}{2\pi i} \int_{(2)} y^s \frac{ds}{s(s+1)}$. If $y \geq 1$, moving the line of integration to $\operatorname{Re} s = -U$ and letting $U \rightarrow \infty$ picks up the residues 1 at $s = 0$ and $-1/y$ at $s = -1$, so $k(y) = 1 - 1/y$; if $y < 1$, moving it to the right gives $k(y) = 0$. As $|s(s+1)|^{-1}$ is integrable on $\operatorname{Re} s = 2$ and $\sum_n |a(n)| n^{-2} < \infty$, summation and integration may be exchanged, and the right side equals $\sum_n a(n) k(x/n) = S(a; x)$. $\square$

LEMMA 3.2. *Let χ be nonprincipal modulo q . For $\operatorname{Re} s = \sigma > 0$, $|L(s, \chi)| \leq q|s|/\sigma$.*

Proof. Put $A(t) = \sum_{n \leq t} \chi(n)$; then $|A(t)| \leq q$, and partial summation gives $L(s, \chi) = s \int_1^\infty A(t) t^{-s-1} dt$ for $\sigma > 0$. Hence $|L(s, \chi)| \leq q|s| \int_1^\infty t^{-\sigma-1} dt = q|s|/\sigma$. $\square$

LEMMA 3.3. *Let $A_0 = \frac{3}{2} \sum_\ell \ell^{-3/2}$. For every Dirichlet character χ the series $E(s) = -\sum_\ell \log(1 - \chi(\ell)\ell^{-s})$ is holomorphic on $\operatorname{Re} s > 1$, $e^{E(s)} = L(s, \chi)$ there, and $|E(s)| \leq A_0$ for $\operatorname{Re} s \geq 3/2$.*

Proof. For $\operatorname{Re} s \geq 1$ we have $|\chi(\ell)\ell^{-s}| \leq 1/2$, and $|\log(1 - w)| \leq \sum_k |w|^k/k \leq \frac{3}{2}|w|$ for $|w| \leq 1/2$. The series therefore converges locally uniformly on $\operatorname{Re} s > 1$ and is bounded by A_0 on $\operatorname{Re} s \geq 3/2$, and the Euler product gives $e^E = L$. $\square$

LEMMA 3.4 (Borel–Carathéodory). *Let h be holomorphic on a neighbourhood of the disc $|z - c| \leq R$ and let $0 < r < R$. Then*

$$\max_{|z-c| \leq r} |h(z) - h(c)| \leq \frac{2r}{R-r} \max_{|w-c|=R} \operatorname{Re}(h(w) - h(c)).$$

Proof. Put $g(z) = h(c+z) - h(c)$ and $A = \max_{|w|=R} \operatorname{Re} g(w)$. The mean of $\operatorname{Re} g$ over the circle $|w| = R$ is $\operatorname{Re} g(0) = 0$, so $A \geq 0$. If $A = 0$, then $\operatorname{Re} g$ vanishes on the circle, hence on the disc, and g is an imaginary constant equal to $g(0) = 0$. Let $A > 0$. By the maximum principle $\operatorname{Re} g \leq A$ on $|z| \leq R$, so $|2A - g| \geq A$ there and $\phi = g/(2A - g)$ is holomorphic. Since $|g|^2 - |2A - g|^2 = 4A(\operatorname{Re} g - A) \leq 0$, we have $|\phi| \leq 1$, and $\phi(0) = 0$. Schwarz's lemma gives $|\phi(z)| \leq |z|/R$, and from $g = 2A\phi/(1 + \phi)$ we get $|g(z)| \leq 2A|z|/(R - |z|)$. $\square$

LEMMA 3.5. *Let $0 < \delta < 1/1000$, $1/2 \leq a \leq 1$, and let χ be nonprincipal modulo q with $L(s, \chi) \neq 0$ for $\operatorname{Re} s > a + 2\delta$. Put $K_1 = (1 + 2A_0)/(2\delta^2)$. For $a + 8\delta \leq \sigma \leq 2$ and $t \in \mathbb{R}$,*

$$\left| \frac{L'}{L}(\sigma + it, \chi) \right| \leq K_1 \log(2q(3 + |t|)^2).$$

Proof. Let $c = 2 + it$, $R = 2 - a - 3\delta$, $r = 2 - a - 6\delta$ and $\Lambda_0 = \log(2q(3 + |t|)^2) \geq \log 18 > 1$. On the disc $|z - c| < 2 - a - 2\delta$ we have $\operatorname{Re} z > a + 2\delta$, so $L(z, \chi)$ has no zero there and has a holomorphic logarithm h . On $|z - c| < 1/2$ both h and the function E of Lemma 3.3 are logarithms of L , so they differ by a constant in $2\pi i\mathbb{Z}$; we choose $h = E$ there. Then $|h(c)| \leq A_0$.

If $|z - c| \leq R$, then $\operatorname{Re} z \geq a + 3\delta \geq 1/2$ and $|z| \leq |c| + 3/2 \leq 4 + |t|$, so Lemma 3.2 gives $\operatorname{Re} h(z) = \log |L(z, \chi)| \leq \log(2q(4 + |t|)) \leq \Lambda_0$. Lemma 3.4 gives, for $|z - c| \leq r$,

$$|h(z) - h(c)| \leq \frac{2r}{R-r} (\Lambda_0 + A_0) \leq \frac{\Lambda_0 + A_0}{\delta},$$

as $r \leq 3/2$ and $R-r = 3\delta$. Hence $|h(z)| \leq A_0 + (\Lambda_0 + A_0)/\delta \leq (1+2A_0)\Lambda_0/\delta$ on $|z-c| \leq r$. For $w = \sigma + it$ with $a+8\delta \leq \sigma \leq 2$ we have $|w-c| = 2-\sigma \leq r-2\delta$, so the circle $|z-w| = 2\delta$ lies in $|z-c| \leq r$, and Cauchy's estimate gives $|L'/L(w, \chi)| = |h'(w)| \leq (1+2A_0)\Lambda_0/(2\delta^2)$. $\square$

LEMMA 3.6. *For $q \geq 1$, $t \in \mathbb{R}$ and $s = \sigma + it$ with $\sigma \geq 1/2$,*

$$\log(2q(3+|t|)^2) \leq 10 \log(q+2)(1+|t|)^{1/2}, \quad |s(s+1)| \geq \frac{1}{8}(1+|t|)^2.$$

Proof. We have $\log(2q) \leq 2 \log(q+2)$, and $\log y \leq 2y^{1/2}$ with $3+|t| \leq 4(1+|t|)$ gives $\log(3+|t|) \leq 4(1+|t|)^{1/2}$; the first bound follows from $\log(q+2) \geq 1$. For the second, $|s| \geq \max(\sigma, |t|) \geq (1+|t|)/4$ and $|s+1| \geq \max(\sigma+1, |t|) \geq (1+|t|)/2$. $\square$

PROPOSITION 3.7. *Let $0 < \delta < 1/1000$. There is $K = K(\delta)$ such that the following holds. Let $1/2 \leq a \leq 1$ and let χ be nonprincipal modulo q with $L(s, \chi) \neq 0$ for $\operatorname{Re} s > a + 2\delta$. Then for $x \geq 1$,*

$$|S(\chi; x)| \leq K x^{a+8\delta} \log(q+2).$$

Proof. Put $\sigma_0 = a + 8\delta$ and

$$\Phi(s) = -\frac{L'}{L}(s, \chi) \frac{x^s}{s(s+1)}.$$

For $\operatorname{Re} s > 1$, $-L'/L(s, \chi) = \sum_n \chi(n) \Lambda(n) n^{-s}$, so Lemma 3.1(ii) gives $S(\chi; x) = \frac{1}{2\pi} \int \Phi(2+it) dt$. The function Φ is holomorphic on $\sigma_0 \leq \operatorname{Re} s \leq 2$, and by Lemmas 3.5 and 3.6,

$$|\Phi(\sigma + it)| \leq 80K_1 \log(q+2) x^\sigma (1+|t|)^{-3/2} \quad (\sigma_0 \leq \sigma \leq 2).$$

This tends to 0 as $|t| \rightarrow \infty$ uniformly in σ , so Cauchy's theorem on the rectangles $[\sigma_0, 2] \times [-T, T]$, $T \rightarrow \infty$, moves the integral to $\operatorname{Re} s = \sigma_0$ (Figure 2). Since $\int_{\mathbb{R}} (1+|t|)^{-3/2} dt = 4$, we get $|S(\chi; x)| \leq \frac{160K_1}{\pi} x^{\sigma_0} \log(q+2)$. $\square$

From now on fix $\delta = 1/2000$ and $a = 7/8 - 1/1000$, so that $a + 2\delta = 7/8$ and

$$\sigma_0 = a + 8\delta = \frac{7}{8} + \frac{3}{1000},$$

and let K be the constant of Proposition 3.7 for this δ . By Theorem 1.2, Proposition 3.7 applies to every nonprincipal character.

COROLLARY 3.8. *There is an absolute constant C_3 such that for every nonprincipal character ξ modulo k :*

- (i) $|S(\xi; x)| \leq K x^{\sigma_0} \log(k+2)$ for $x \geq 1$;
- (ii) $|\sum_{p \leq x} \xi(p) \log p| \leq C_3 x^{19/20} \log(k+2)$ for $x \geq 1$;

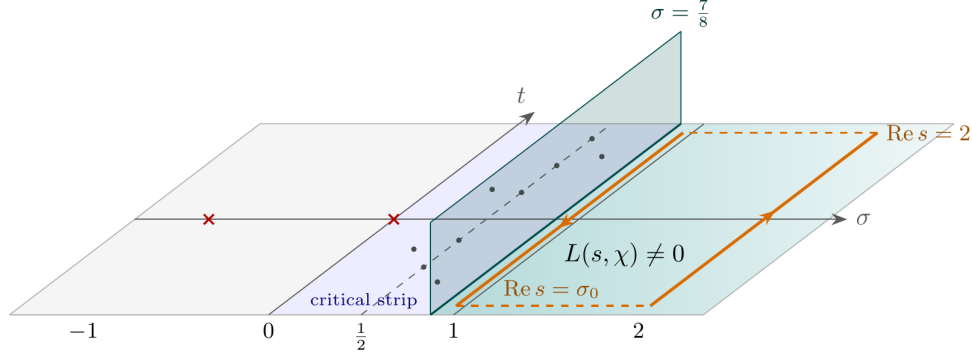


Figure 2. The contour shift in the proof of Proposition 3.7. The zeros of $L(s, \chi)$ lie in the strip $0 \leq \operatorname{Re} s \leq 7/8$ (dots), the kernel $x^s/(s(s+1))$ has poles at 0 and -1 (crosses), and the line of integration moves from $\operatorname{Re} s = 2$ to $\operatorname{Re} s = \sigma_0$ inside the zero-free half-plane.

(iii) for integers $2 \leq A \leq B$ and $\gamma \in \mathbb{R}$,

$$(3.1) \quad \left| \sum_{A < p \leq B} \xi(p) e(p\gamma) \right| \leq 2C_3 \frac{B^{19/20} \log(k+2)(2 + 2\pi|\gamma|B)}{\log A}.$$

Proof. (i) is Proposition 3.7. For (ii) put $T(y) = yS(\xi; y)$, that is, $T(y) = \sum_{n \leq y} \xi(n) \Lambda(n)(y - n)$, and let $0 < h \leq x$. Then

$$T(x+h) - T(x) = h \sum_{n \leq x} \xi(n) \Lambda(n) + \sum_{x < n \leq x+h} \xi(n) \Lambda(n)(x+h-n),$$

and the last sum has at most $h+1$ terms, each of modulus at most $h \log(2x)$. With $|T(y)| \leq Ky^{1+\sigma_0} \log(k+2)$ this gives

$$\left| \sum_{n \leq x} \xi(n) \Lambda(n) \right| \leq \frac{2K(2x)^{1+\sigma_0} \log(k+2)}{h} + (h+1) \log(2x).$$

Take $h = x^{47/50}$. Since $1 + \sigma_0 - 47/50 < 19/20$ and $\log(2x) \leq 1 + 100x^{1/100}$, the right side is at most $(8K + 202)x^{19/20} \log(k+2)$. The prime powers contribute at most $\psi(x) - \theta(x) \leq 2\sqrt{x} \log x \leq 200x^{19/20}$ by Lemma 2.1(ii), and (ii) follows with $C_3 = 8K + 402$.

(iii) Let $\Pi(u) = \sum_{p \leq u} \xi(p) \log p$ and $f(u) = e(u\gamma)/\log u$. Partial summation gives

$$\sum_{A < p \leq B} \xi(p) e(p\gamma) = f(B)\Pi(B) - f(A)\Pi(A) - \int_A^B \Pi(u) f'(u) du,$$

and $|f'(u)| \leq 2\pi|\gamma|/\log A + 1/(u \log^2 u)$ for $u \geq A$. By (ii) the right side is at most $C_3 B^{19/20} \log(k+2)(3 + 2\pi|\gamma|B)/\log A$, which gives (3.1). $\square$

4. Exponential sums with multiplicative coefficients

For $M > 0$ put $w_M(\theta) = M$ if $\|\theta\| < 1/M$ and $w_M(\theta) = 1/\|\theta\|$ otherwise; then $0 \leq w_M(\theta) \leq M$ and $w_M(-\theta) = w_M(\theta)$.

THEOREM 4.1. *There is an absolute constant C_1 with the following property. Let f be completely multiplicative with $|f(n)| \leq 1$, let $\alpha \in \mathbb{R}$, and let a, r be coprime integers with $r \geq 1$, $|\alpha - a/r| \leq r^{-2}$ and $2 \leq R \leq r \leq N/R$. Then*

$$\left| \sum_{n \leq N} f(n) e(n\alpha) \right| \leq C_1 \left(\frac{N}{\log N} + \frac{N(\log N)^2}{R^{1/4}} \right).$$

Montgomery and Vaughan [9, Corollary 1] proved this for multiplicative f with $N(\log R)^{3/2} R^{-1/2}$ in place of $N(\log N)^2 R^{-1/4}$. The proof below follows their approach, a bilinear decomposition together with the spacing of the multiples of α and an upper bound for prime pairs, with cruder estimates, which suffice for Theorem 1.1.

LEMMA 4.2. *Let a, r be coprime with $r \geq 2$ and $|\alpha - a/r| \leq r^{-2}$, and let $H \geq 0$ be an integer.*

- (i) $\|k\alpha\| \geq 1/(2r)$ for every integer k with $0 < |k| \leq r/2$.
- (ii) $\#\{1 \leq h \leq H : \|h\alpha\| < t\} \leq (4H/r + 1)(4rt + 1)$ for every $t \geq 0$.
- (iii) For $M > 0$ and every integer $K \geq 0$ with $M \leq 2^K$,

$$\begin{aligned} \sum_{h \leq H} w_M(h\alpha) &\leq \left(\frac{4H}{r} + 1 \right) (3M + 4r + 8rK), \\ \sum_{h \leq H} w_M(h\alpha)^2 &\leq \left(\frac{4H}{r} + 1 \right) (20rM + 3M^2). \end{aligned}$$

Proof. (i) Let z be the integer nearest to $k\alpha$. Since $0 < |k| < r$ and $(a, r) = 1$, $r \nmid ka$, so $|ka/r - z| \geq 1/r$ and $\|k\alpha\| \geq 1/r - |k| |\alpha - a/r| \geq 1/(2r)$.

(ii) Let $L = \lfloor r/2 \rfloor$, so $L \geq r/4$. For k in a block of L consecutive integers let $y_k = k\alpha - z_k$ with z_k the integer nearest to $k\alpha$. For $k \neq k'$ in the block, $0 < |k - k'| < L \leq r/2$ and $y_k - y_{k'} \equiv (k - k')\alpha \pmod{1}$, so $|y_k - y_{k'}| \geq 1/(2r)$ by (i). Points of $(-t, t)$ at mutual distances at least $1/(2r)$ number at most $4rt + 1$, and $[1, H]$ is covered by at most $H/L + 1 \leq 4H/r + 1$ blocks.

(iii) If $\|\theta\| \geq 1/M$, let $k \geq 1$ be least with $\|\theta\| < 2^k/M$; then $k \leq K$ because $\|\theta\| \leq 1/2 < 2^K/M$, and $\|\theta\| \geq 2^{k-1}/M$, so $w_M(\theta) \leq 2^{1-k}M$. Hence

$$w_M(\theta)^j \leq M^j [\|\theta\| < 1/M] + \sum_{k=1}^K (2^{1-k}M)^j [\|\theta\| < 2^k/M] \quad (j = 1, 2).$$

Summing over $h \leq H$ with (ii), the first sum is at most $(4H/r + 1)(4r + M + \sum_{k \leq K}(8r + 2^{1-k}M))$ and the second at most $(4H/r + 1)(4rM + M^2 + \sum_{k \geq 1}(16rM2^{-k} + 4M^24^{-k}))$. $\square$

LEMMA 4.3. *Let $\theta, \alpha \in \mathbb{R}$, $U > 0$, and let A, B, N be nonnegative integers with $B \leq A + U$.*

- (i) $|\sum_{A < x \leq B} e(x\theta)| \leq w_U(\theta)$.
(ii) *For a finite set V of positive integers and $b : V \rightarrow \mathbb{C}$,*

$$\sum_{A < u \leq B} \left| \sum_{\substack{v \in V \\ uv \leq N}} b(v)e(uv\alpha) \right|^2 \leq \sum_{v, v' \in V} |b(v)| |b(v')| w_U((v - v')\alpha).$$

- (iii) *For $M > 0$ and an integer $H \geq 0$, $\sum_{1 \leq m, m' \leq H} w_M((m - m')\alpha) \leq H(M + 2 \sum_{h \leq H} w_M(h\alpha))$.*

Proof. (i) If $\|\theta\| < 1/U$, the sum has at most U terms of modulus 1. Otherwise $\theta \notin \mathbb{Z}$, the sum has modulus at most $|e((B - A)\theta) - 1|/|e(\theta) - 1|$, and $|e(\theta) - 1| = 2|\sin \pi\theta| \geq 4\|\theta\|$, so the modulus is at most $1/(2\|\theta\|)$. (ii) Expanding the square gives $\sum_{v, v'} b(v)\overline{b(v')} \sum_u e(u(v - v')\alpha)$, where u runs over the integers of $(A, \min(B, N/v, N/v'))$, an interval of length at most U ; apply (i). (iii) For fixed m , the term $m' = m$ is $w_M(0) = M$, and each $h = |m - m'| \in [1, H]$ arises from at most two m' . $\square$

LEMMA 4.4. *Let $C_5 = 1 + 4\zeta(3/2)$. For every completely multiplicative f with $|f(n)| \leq 1$, every $\alpha \in \mathbb{R}$ and every integer $N \geq 1$, the sum $S = \sum_{n \leq N} f(n)e(n\alpha)$ satisfies*

$$\left| S \log N - \sum_{p \leq N} \log p f(p) \sum_{m \leq N/p} f(m)e(pm\alpha) \right| \leq C_5 N.$$

Proof. Write $S \log N = S_1 + S_2$ with $S_1 = \sum_{n \leq N} f(n)e(n\alpha) \log n$ and $S_2 = \sum_{n \leq N} f(n)e(n\alpha) \log(N/n)$. Then $|S_2| \leq N \log N - \log N! \leq N$, since $N^N/N! \leq e^N$. In S_1 write $\log n = \sum_{d|n} \Lambda(d)$ and $n = dm$; complete multiplicativity gives $\sum_{d \leq N} \Lambda(d)f(d) \sum_{m \leq N/d} f(m)e(dm\alpha)$. The terms with d prime form the sum in the statement, and the others have modulus at most $N \sum_p \sum_{k \geq 2} \log p/p^k \leq 2N \sum_p \log p/p^2 \leq 4\zeta(3/2)N$, since $\log p \leq 2\sqrt{p}$. $\square$

Let $G(h) = \prod_{p|h} p/(p - 1)$.

LEMMA 4.5. (i) *For every integer $X \geq 2$ and every even $h \geq 2$,*

$$\#\{0 \leq n < X : n \text{ and } n + h \text{ prime}\} \leq \frac{400 G(h)^2 X}{(\log X)^2} + 3X^{3/5}.$$

- (ii) $\sum_{h \leq X} G(h)^4 \leq e^{30} X$ for every $X \geq 1$.

Proof. (i) Apply Lemma 2.2 to $a_n = n(n+h)$, $0 \leq n < X$, with $y = X^{1/5}$, $z = \lfloor y \rfloor$, P the product of the primes $p \leq z$ with $p \nmid h$, and $\nu(d) = 2^{\omega(d)}/d$. Every $p \mid P$ is odd, so $\nu(p) = 2/p < 1$. For $d \mid P$ the residues n with $d \mid n(n+h)$ form $2^{\omega(d)}$ classes modulo d (namely $n \equiv 0$ or $-h \pmod{p}$ for each $p \mid d$, distinct as $p \nmid h$), and each class contains between $X/d-1$ and $X/d+1$ of the integers $0 \leq n < X$; hence $|R_d| \leq 2^{\omega(d)}$. Since the prime factors of d are at least 3, $3^{\omega(d)}|R_d| \leq 6^{\omega(d)} \leq d^2 \leq y^2$, and the error term is at most $y^3 = X^{3/5}$.

Let $\mathcal{M}$ be the set of $m \leq \sqrt{y}$ whose prime factors divide P . Since $\nu(p)/(1-\nu(p)) = \sum_{k \geq 1} \nu(p)^k$ and $\nu(m) = 2^{\Omega(m)}/m$ for these m , $g(l)$ is the sum of $2^{\Omega(m)}/m$ over the m whose set of prime factors is that of l . An $m \in \mathcal{M}$ is counted in $g(l)$ for l the product of its prime factors, and $l \leq m \leq \sqrt{y}$. Hence, with $\tau(m) \leq 2^{\Omega(m)}$ the number of divisors of m ,

$$S \geq \sum_{m \in \mathcal{M}} \frac{2^{\Omega(m)}}{m} \geq \sum_{m \in \mathcal{M}} \frac{\tau(m)}{m}.$$

Let $T = \lfloor y^{1/4} \rfloor$ and let B be the set of $b \leq T$ that are prime to h ; these b are odd. For $b, b' \in B$ the product bb' lies in $\mathcal{M}$, and a given m is a product bb' in at most $\tau(m)$ ways, so $S \geq (\sum_{b \in B} 1/b)^2$. If E is a set of primes and $p \notin E$, every $b \leq T$ prime to E is uniquely $p^k b'$ with b' prime to $E \cup \{p\}$, so the sum of $1/b$ over the $b \leq T$ prime to E is at most $p/(p-1)$ times the same sum for $E \cup \{p\}$. Adding the prime factors of h one at a time gives

$$\sum_{b \in B} \frac{1}{b} \geq \frac{1}{G(h)} \sum_{b \leq T} \frac{1}{b} \geq \frac{\log(T+1)}{G(h)}.$$

As $T+1 > y^{1/4} = X^{1/20}$, we get $S \geq (\log X)^2/(400G(h)^2)$, that is, $X/S \leq 400G(h)^2 X/(\log X)^2$. Finally, if $n < X$ and $n, n+h$ are prime, then $n \leq z$ or $(n(n+h), P) = 1$, so the count is at most $z+1 + X/S + y^3 \leq 400G(h)^2 X/(\log X)^2 + 3X^{3/5}$.

(ii) For $t = p-1 \geq 1$, $(t+1)^5 \leq (t+31)t^4$, that is, $(p/(p-1))^4 \leq 1+30/p$. Hence $G(h)^4 \leq \sum_d 30^{\omega(d)}/d$ over the squarefree $d \mid h$, and

$$\begin{aligned} \sum_{h \leq X} G(h)^4 &\leq \sum_{d \leq X} \mu(d)^2 \frac{30^{\omega(d)}}{d} \cdot \frac{X}{d} \leq X \prod_p \left(1 + \frac{30}{p^2}\right) \\ &\leq X \exp\left(30 \sum_{n \geq 2} \frac{1}{n^2}\right) \leq e^{30} X. \end{aligned} \quad \square$$

Proof of Theorem 4.1. Let S be the sum in the theorem, $L = \log N$ and $\ell = \lfloor \log_2 N \rfloor$. Since $N \geq rR \geq 4$, we have $L \geq 1$. If $\ell < 64$, then $L < 64 \log 2 < 45$ and $|S| \leq N \leq 45N/L$. Let now $\ell \geq 64$, and put $I = \lfloor \log_2 \ell \rfloor + 1$ and $J = \ell + 1 - 4I$. Then $16^I > \ell^4$, $1 \leq J \leq \ell \leq 2L$, $I \leq \ell \leq 2L$, $2^I \leq 2\ell$, $N < 2^{\ell+1} = 2^J 16^I$ and $J^2 2^J \leq (\ell+1)^2 2^{\ell+1} \ell^{-4} \leq 2N$. By Lemma 4.4, $|S|L \leq C_5 N +$

$|T_1| + |T_2|$, where T_1 and T_2 are the parts of $\sum_p \log p f(p) \sum_{m \leq N/p} f(m) e(pm\alpha)$ with $p \leq 2^J$ and with $p > 2^J$.

The small primes. Let $0 \leq j < J$, $U = 2^j$, $H = \lfloor N/U \rfloor$, and for $U < x \leq 2U$ let $B(x) = \sum_{m \leq H, xm \leq N} f(m) e(xm\alpha)$; the inner sum for a prime $p \in (U, 2U]$ is $B(p)$. By Cauchy's inequality, Lemma 4.3(ii), (iii) and Lemma 4.2(iii) with $M = U$, $K = j$,

$$\sum_{U < p \leq 2U} \log p |B(p)| \leq (U \log^2(2U))^{1/2} Q_j^{1/2},$$

where $Q_j = H(U + 2(4H/r + 1)(3U + 4r + 8rj))$. Since $HU \leq N$, $UQ_j \leq 7NU + (8 + 16j)Nr + 24N^2/r + (32 + 64j)N^2/U$, and $\log(2U) \leq j + 1$. Summing over $j < J$ with $j + 1 \leq J$, $\sum_{j < J} 2^{j/2} \leq 3 \cdot 2^{J/2}$, $\sqrt{(32 + 64j)2^{-j}} \leq 8(j + 1)(3/4)^j$ and $\sum_{j \geq 0} (j + 1)^2 (3/4)^j = 112$,

$$|T_1| \leq 3\sqrt{7}J\sqrt{N2^J} + J^2N\sqrt{24/r} + 896N + J^2\sqrt{24JNr}.$$

Now $J^2 2^J \leq 2N$ gives $3\sqrt{7}J\sqrt{N2^J} \leq 12N$; $r \geq R \geq \sqrt{R}$ gives $\sqrt{24/r} \leq 5R^{-1/4}$; and $r\sqrt{R} \leq rR \leq N$ gives $\sqrt{24JNr} \leq 5JNR^{-1/4}$. With $J \leq 2L$ and $L \geq 1$, $|T_1| \leq 908N + 60L^3NR^{-1/4}$.

The large primes. For $p > 2^J$ we have $N/p < 16^I$, so the m in T_2 lie in $(0, 16^I - 1]$, the union of the blocks $(16^i - 1, 16^{i+1} - 1]$ with $0 \leq i < I$. Fix i , let $A = 16^i - 1$, $U = 15 \cdot 16^i$ and $X = \lfloor N/16^i \rfloor$, and assume $X \geq 1$, the block being empty otherwise. Only the primes $p \in (2^J, X]$ contribute, since $p \geq X + 1$ and $m \geq 16^i$ give $pm > N$. Let V be the set of these primes, which are odd, and $B(m) = \sum_{p \in V, pm \leq N} \log p f(p) e(pm\alpha)$. The block is $\sum_{A < m \leq A+U} f(m) B(m)$, and by Cauchy's inequality, Lemma 4.3(ii) and $\log p \leq \lambda = \log(X + 1)$, its square is at most $U\lambda^2 W$, where

$$W = \sum_{p, p' \in V} w_U((p - p')\alpha) = |V|U + 2 \sum_{h=1}^X w_U(h\alpha) \#\{p \in V : p + h \in V\}.$$

The count vanishes for odd h , and for even h Lemma 4.5(i), applied to $X + 1$, bounds it by $400G(h)^2(X + 1)\lambda^{-2} + 3(X + 1)^{3/5}$. By Cauchy's inequality, Lemma 4.5(ii) and Lemma 4.2(iii), $\sum_{h \leq X} G(h)^2 w_U(h\alpha) \leq e^{15} \sqrt{X} Q^{1/2}$ with $Q = (4X/r + 1)(20rU + 3U^2)$, and $\sum_{h \leq X} w_U(h\alpha) \leq XU$. Hence $U\lambda^2 W \leq A_1 + 800e^{15}Z + A_3$, where

$$A_1 = U^2 \lambda^2 X, \quad Z = U(X + 1) \sqrt{X} Q^{1/2}, \quad A_3 = 6U^2 \lambda^2 (X + 1)^{3/5} X.$$

Let $\kappa = \log(N + 1) \geq \lambda$ and $T_i = 4^i \kappa (N(2N)^{3/5})^{1/2}$. Since $U^2 X = 225 \cdot 16^i (16^i X) \leq 225 \cdot 16^i N$ and $X + 1 \leq 2N$, we get $A_1 \leq 225T_i^2$ and $A_3 \leq$

$1350T_i^2 \leq 37^2T_i^2$. With $w = UX \leq 15N$, $X \leq N/16^i$ and $R \leq r \leq N/R$,

$$\begin{aligned} Z^2 &\leq 4U^2X^3Q = 320w^3X + \frac{48w^4}{r} + 80rw^3 + 12Uw^3 \\ &\leq \frac{320 \cdot 15^3N^4}{16^i} + \frac{48 \cdot 15^4N^4}{R} + \frac{80 \cdot 15^3N^4}{R} \\ &\quad + 180 \cdot 15^3 \cdot 16^iN^3, \end{aligned}$$

and every coefficient is at most 40^4 , so $Z^{1/2} \leq 40(N2^{-i} + 2NR^{-1/4} + 2^iN^{3/4})$. The block is therefore at most

$$\begin{aligned} &15T_i + 40\sqrt{800e^{15}}(N2^{-i} + 2NR^{-1/4} + 2^iN^{3/4}) + 37T_i \\ &\leq K_b(N2^{-i} + NR^{-1/4} + 2^iN^{3/4} + T_i), \end{aligned}$$

with $K_b = 80\sqrt{800e^{15}} + 52$. Summing over $i < I$,

$$|T_2| \leq K_b(2N + INR^{-1/4} + 2^I N^{3/4} + 4^I \kappa(N(2N)^{3/5})^{1/2}).$$

Here $I \leq 2L^3$; $2^I \leq 4L \leq 16N^{1/4}$; $4^I \leq 16L^2$ and $\kappa \leq 2L$; and $\log x \leq 15x^{1/15}$ gives $L^3(N(2N)^{3/5})^{1/2} \leq 2 \cdot 15^3N$. Hence $|T_2| \leq K_b(216018N + 2L^3NR^{-1/4})$.

Altogether $|S|L \leq (C_5 + 908 + 216018K_b)N + (60 + 2K_b)L^3NR^{-1/4}$, and dividing by L gives the theorem with $C_1 = 45 + C_5 + 908 + 216018K_b + 60 + 2K_b$. $\square$

5. Proof of Theorem 1.1

In this section χ is a primitive character modulo $q \geq 3$.

LEMMA 5.1. *Let $B \geq 0$ be such that $|\sum_{m \leq H} \bar{\chi}(m)e(m\beta)/m| \leq B$ for every integer $H \leq q/2$ and every $\beta \in \mathbb{R}$. Then $|\sum_{n \leq x} \chi(n)| \leq \sqrt{q}(B + 6)$ for every $x \geq 0$.*

Proof. We may take $x = T$ to be an integer. Let $\tau = \sum_{a \bmod q} \bar{\chi}(a)e(a/q)$. Since $\bar{\chi}$ is primitive, $\sum_a \bar{\chi}(a)e(an/q) = \chi(n)\tau$ for every n , and summing the squares of the moduli of these identities over $n \bmod q$ gives $q\varphi(q) = |\tau|^2\varphi(q)$, so $|\tau| = \sqrt{q}$. Summing over $1 \leq n \leq T$,

$$\tau \sum_{n \leq T} \chi(n) = \sum_{0 < |a| \leq q/2} \bar{\chi}(a) \frac{e(aT/q) - 1}{1 - e(-a/q)},$$

where a runs over the integers with $-q/2 < a \leq q/2$, $a \neq 0$. For $0 < |\theta| \leq 1/2$ put $r(\theta) = (1 - e(-\theta))^{-1} - (2\pi i\theta)^{-1}$; then $|1 - e(-\theta)| = 2|\sin \pi\theta| \geq 4|\theta|$. If $2\pi|\theta| \leq 1$, then $|e(-\theta) - 1 + 2\pi i\theta| \leq (2\pi\theta)^2$ and $|r(\theta)| \leq (2\pi\theta)^2/(4|\theta| \cdot 2\pi|\theta|) < 3$; otherwise $|r(\theta)| \leq 1/(4|\theta|) + 1/(2\pi|\theta|) < 3$. The terms with $r(a/q)$ contribute at most $6q$. Writing $a = \pm m$ with $1 \leq m \leq q/2$, the remaining terms are $q/(2\pi i)$ times a combination, with coefficients of modulus at most 1, of four

sums $\sum_{m \leq H} \bar{\chi}(m)e(m\beta)/m$ with $H \leq q/2$, so they contribute at most $2qB/\pi$. Dividing by $|\tau|$ gives the claim. $\square$

LEMMA 5.2. *For $Y \geq 2$ and $H \geq 1$,*

$$\sum_{\substack{n \leq H \\ P(n) \leq Y}} \frac{1}{n} \leq e^{11} \log Y.$$

Proof. The sum is at most $\prod_{p \leq Y} (1 - 1/p)^{-1} \leq \exp(\sum_{p \leq Y} \frac{1}{p-1})$, and $\sum_p (\frac{1}{p-1} - \frac{1}{p}) \leq 1$; apply Lemma 2.1(iv). $\square$

LEMMA 5.3. *There is an absolute constant C_2 such that for $0 < r < q$, $b \in \mathbb{Z}$, $\beta \in \mathbb{R}$, integers $Y \geq \max(2, r)$ and $t \geq 1$,*

$$\left| \sum_{\substack{n \leq t \\ P(n) > Y}} \chi(n)e\left(n\left(\frac{b}{r} + \beta\right)\right) \right| \leq C_2 \frac{r \log(qr + 2)(2 + 2\pi|\beta|t)t}{Y^{1/20} \log Y}.$$

Proof. Every $n \leq t$ with $P(n) > Y$ is uniquely $n = pm$ with $p = P(n)$ prime, $Y < p \leq t/m$ and $P(m) \leq p$, and then $m \leq t/Y$. For such p we have $p > r$, so $p \nmid r$, and on the units modulo r the function $u \mapsto e(umb/r)$ equals $\sum_{\psi} c_{\psi} \psi(u)$, the sum over the characters ψ modulo r , with $|c_{\psi}| \leq 1$.

The character $\chi\psi$ modulo qr is nonprincipal. Otherwise $\chi(n) = \bar{\psi}(n)$ for every n prime to qr . Let $d = (q, r)$ and let n be prime to q with $n \equiv 1 \pmod{d}$. By the Chinese remainder theorem there is n' prime to qr with $n' \equiv n \pmod{q}$ and $n' \equiv 1 \pmod{r}$, and then $\chi(n) = \chi(n') = \bar{\psi}(n') = 1$. So χ is trivial on the units congruent to 1 modulo d , and is induced by a character modulo $d \leq r < q$, which contradicts primitivity.

Applying (3.1) to $\xi = \chi\psi$ with $\gamma = m\beta$, $A = \max(Y, P(m) - 1)$ and $B = \lfloor t/m \rfloor$ (the inner sum is empty if $A \geq B$) gives

$$\left| \sum_{\substack{n \leq t \\ P(n) > Y}} \chi(n)e\left(n\left(\frac{b}{r} + \beta\right)\right) \right| \leq 2C_3 r \frac{\log(qr + 2)(2 + 2\pi|\beta|t)}{\log Y} \sum_{m \leq t/Y} \left(\frac{t}{m}\right)^{19/20},$$

and $\sum_{m \leq M} m^{-19/20} \leq 20M^{1/20}$. Hence $C_2 = 40C_3$ will do. $\square$

PROPOSITION 5.4. *There is an absolute constant C_4 such that if $\log q \geq 200$, then $|\sum_{m \leq H} \chi(m)e(m\beta)/m| \leq C_4 \log \log q$ for every integer $H \leq q$ and every $\beta \in \mathbb{R}$.*

Proof. Let $L = \log q$, $M = \lceil L \rceil$, $R = M^{16}$ and $Y = M^{700}$. Since $e^{L/16} \geq (L/16)^4/24 > L + 1 \geq M$, we have $2 \leq R < q$ and $R \leq Y$. Split the sum according to $P(m) \leq Y$ or $P(m) > Y$. By Lemma 5.2 the first part is at most $e^{11} \log Y$. For the second put $A(t) = \sum_{n \leq t, P(n) > Y} \chi(n)e(n\beta)$ and

$w(t) = 1/t - 1/(t+1)$. Partial summation and $|A(H)| \leq H$ give

$$\left| \sum_{\substack{m \leq H \\ P(m) > Y}} \frac{\chi(m)e(m\beta)}{m} \right| \leq 1 + \sum_{1 \leq t < H} |A(t)| w(t).$$

If $t \leq Y$, then $A(t) = 0$. Let $t > Y$ and $Q = \lfloor t/R \rfloor \geq 1$. By Dirichlet's theorem there are coprime a, r with $1 \leq r \leq Q$ and $|\beta - a/r| \leq 1/(r(Q+1))$; then $|\beta - a/r|t \leq R$ (Figure 3). If $r \leq R$, then Lemma 5.3 with $b = a$ gives $|A(t)| \leq C_2 kt$, where

$$k = \frac{R \log(qR+2)(2+2\pi R)}{Y^{1/20} \log Y}.$$

If $r > R$, then $R \leq r \leq t/R$ and $|\beta - a/r| \leq r^{-2}$, so Theorem 4.1 for $f = \chi$ bounds the full sum $\sum_{n \leq t} \chi(n)e(n\beta)$, and since $\log t \leq L$, removing the $\Psi(t) = \#\{n \leq t : P(n) \leq Y\}$ smooth terms gives

$$|A(t)| \leq C_1 \left(\frac{t}{\log t} + t\rho \right) + \Psi(t), \quad \rho = \frac{L^2}{R^{1/4}} = \frac{L^2}{M^4}.$$

Since $tw(t) \leq 1/t$ and $w(t) \leq 1/t^2$,

$$\sum_{1 \leq t < H} |A(t)| w(t) \leq C_1 \sum_{2 \leq t \leq H} \frac{1}{t \log t} + (C_1 \rho + C_2 k) \sum_{t \leq H} \frac{1}{t} + \sum_{1 \leq t < H} \Psi(t) w(t).$$

The first sum is at most $3 + \log L$, the second at most $1 + L$, and by partial summation the third is at most $\sum_{n \leq H, P(n) \leq Y} 1/n \leq e^{11} \log Y$. Finally $L \leq M \leq 2L$ gives $\rho(1+L) \leq 2/M \leq 1$; $\log(qR+2) \leq 18M$, $2+2\pi R \leq 8M^{16}$, $Y^{1/20} = M^{35}$ and $\log Y = 700 \log M \geq 700$ give $k(1+L) \leq 288M^{34}/(700M^{35}) \leq 1$; and $\log Y = 700 \log M \leq 1400 \log L$. Collecting terms, the sum is at most $1 + 4C_1 + C_2 + (2800e^{11} + C_1) \log L \leq (2800e^{11} + 1 + 5C_1 + C_2) \log L$. $\square$

Proof of Theorem 1.1. The character $\bar{\chi}$ is primitive modulo q . If $\log q \geq 200$, then Proposition 5.4 for $\bar{\chi}$ and Lemma 5.1 give

$$\left| \sum_{n \leq x} \chi(n) \right| \leq \sqrt{q} (C_4 \log \log q + 6) \leq (C_4 + 6) \sqrt{q} \log \log q.$$

If $\log q < 200$, then every sum $\sum_{m \leq H} \bar{\chi}(m)e(m\beta)/m$ with $H \leq q$ has modulus at most $1 + \log q < 201$, and Lemma 5.1 gives

$$\left| \sum_{n \leq x} \chi(n) \right| \leq 207 \sqrt{q} \leq \frac{207}{\log \log 3} \sqrt{q} \log \log q. \quad \square$$

For the lower bound in Theorem 1.3(ii) we need to control the prime powers $n \leq q^{17}$ with $n \equiv 1 \pmod{q}$. We count them in dyadic blocks with Selberg's sieve.

LEMMA 6.2. *Let q be prime, let $M \geq 1$ be an integer and let $1 < z < q$. The number of m with $M \leq m < 2M$ for which $1 + mq$ is a prime power is at most*

$$\frac{2M}{\log z} + z(1 + \log z)^2 + z.$$

Proof. If $1 + mq = p^k$ with $p \leq z$, then $p^k \in (Mq, 2Mq)$, and for each prime p at most one power of p lies in this interval, since $p^{k+1} \geq 2p^k$. These m number at most z . If $p > z$, then $1 + mq$ has no prime factor up to z , and we apply Lemma 2.2 to the M integers $a_m = 1 + mq$, $M \leq m < 2M$, with P the product of the primes up to z , $\nu(d) = 1/d$ and $y = z$. Every prime $p \leq z$ differs from q , so for $d \mid P$ the condition $d \mid 1 + mq$ fixes m in one residue class modulo d , and $|R_d| \leq 1$. Here $g(l) = \prod_{p \mid l} (p-1)^{-1}$, which is the sum of $1/n$ over the n whose set of prime factors is that of l . Every $n \leq \sqrt{z}$ has all its prime factors at most z and the product of its distinct prime factors is at most $\sqrt{z}$, so

$$S = \sum_{\substack{l \mid P \\ l \leq \sqrt{z}}} g(l) \geq \sum_{n \leq \sqrt{z}} \frac{1}{n} \geq \frac{\log z}{2}.$$

For squarefree d , $3^{\omega(d)} = \tau_3(d)$, and $\sum_{d \leq z} \tau_3(d) = \#\{(a, b, c) : abc \leq z\} \leq z(\sum_{a \leq z} 1/a)^2 \leq z(1 + \log z)^2$. Lemma 2.2 therefore bounds the number of these m by $2M/\log z + z(1 + \log z)^2$. $\square$

LEMMA 6.3. *There is an absolute constant C_0 such that for every prime $q \geq 3$,*

$$\sum_{\substack{1 < n \leq q^{17} \\ n \equiv 1 \pmod{q}}} \frac{\Lambda(n)}{n} \leq C_0 \frac{\log q \log \log q}{q}.$$

Proof. Write $n = 1 + mq$ with $1 \leq m < q^{16}$, and let $J = \lfloor 16 \log_2 q \rfloor + 1$, so that $2^J > q^{16}$ and $J \leq 25 \log q$. We split the range of m into the blocks $2^j \leq m < 2^{j+1}$, $0 \leq j < J$ (Figure 4), and let N_j be the number of m in the j -th block with $1 + mq$ a prime power. We claim $N_j \leq 393 \cdot 2^j/(j+1)$. For $j < 32$ this holds trivially. For $j \geq 32$ apply Lemma 6.2 with $M = 2^j$ and $z = 2^u$, $u = j/32 \geq 1$. Since $2^j < 2^J \leq 2q^{16}$, we have $z < 2^{1/32} q^{1/2} < q$. Then $2M/\log z = 64 \cdot 2^j/(j \log 2) \leq 96 \cdot 2^j/(j+1)$, and since $\log z \leq u$,

$$z(1 + \log z)^2 + z \leq 2^u(2 + u)^2 \leq \frac{297 \cdot 2^{4u}}{32u + 1} \leq \frac{297 \cdot 2^j}{j + 1},$$

because $(2+u)^2(32u+1) \leq 9u^2 \cdot 33u = 297u^3$ and $u^3 \leq 2^{3u}$. This proves the claim.

A term of the j -th block has $\Lambda(n) \leq \log(1+2^{j+1}q) \leq (j+2)\log 2 + \log q$ and $n > 2^j q$, so the block contributes at most

$$\frac{393}{(j+1)q} ((j+2)\log 2 + \log q) \leq \frac{393}{q} \left(2\log 2 + \frac{\log q}{j+1} \right).$$

Summing over $j < J$ and using $\sum_{j < J} 1/(j+1) \leq 1 + \log J$, $J \leq 25 \log q$ and $\log J \leq 4 + \log \log q$, the sum in the lemma is at most $\frac{393}{q} (35 \log q + 5 \log q + \log q \log \log q)$. Since $\log \log q \geq \log \log 3 > 1/11$, this is at most $C_0 \log q \log \log q / q$ with $C_0 = 393 \cdot 441$. $\square$

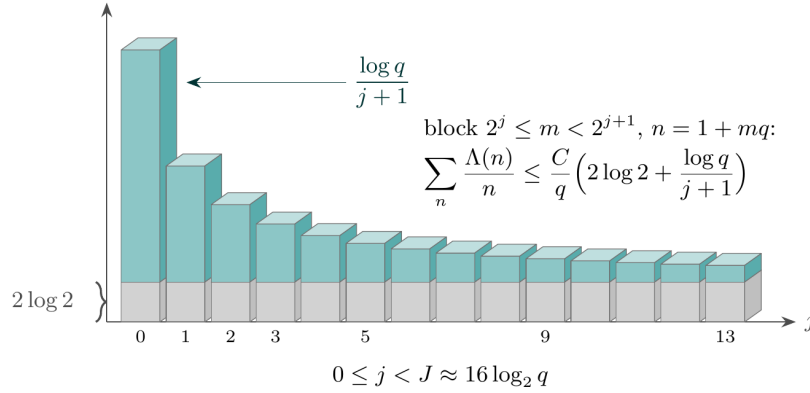


Figure 4. The dyadic blocks in the proof of Lemma 6.3. The block $2^j \leq m < 2^{j+1}$ contributes a constant part, of total size $\ll J \ll \log q$, and a part of size $\log q/(j+1)$, whose total is $\ll \log q \log J$. The second part produces the factor $\log \log q$.

Proof of Theorem 1.3(ii). Let $q \geq 3$ be prime and put $c(n) = \sum_{\chi \neq \chi_0} \chi(n)$, the sum over the nonprincipal characters modulo q . By orthogonality $c(n) = (q-1)\mathbf{1}[n \equiv 1 \pmod{q}] - \mathbf{1}[q \nmid n]$. Put

$$T(t) = \sum_{\chi \neq \chi_0} S(\chi; t) = \sum_{n \leq t} c(n) \Lambda(n) \left(1 - \frac{n}{t} \right), \quad I = \int_1^\infty \frac{T(t)}{t^2} dt.$$

By Lemma 6.1, $\sum_{\chi \neq \chi_0} \frac{L'}{L}(1, \chi) = -2I$, so by (1.1) it suffices to show $-C \log q \leq I \leq C \log q \log \log q$.

By Corollary 3.8(i), $|T(t)| \leq qK \log(q+2) t^{\sigma_0}$. Let $Y = q^{17}$. Since $17(\sigma_0 - 1) < -2$ and $\log(q+2) \leq q$,

$$\left| \int_Y^\infty \frac{T(t)}{t^2} dt \right| \leq \frac{qK \log(q+2) Y^{\sigma_0-1}}{1 - \sigma_0} \leq \frac{K \log(q+2)}{q(1 - \sigma_0)} \leq \frac{K}{1 - \sigma_0}.$$

For the lower bound, $c(n) \geq -1$ gives $-T(t) \leq \psi(t) \leq 4t$, so

$$-I \leq \int_1^Y \frac{4dt}{t} + \frac{K}{1-\sigma_0} = 68 \log q + \frac{K}{1-\sigma_0}.$$

For the upper bound, $c(n) \leq (q-1)\mathbf{1}[n \equiv 1 \pmod{q}]$ gives

$$\begin{aligned} \int_1^Y \frac{T(t)}{t^2} dt &\leq (q-1) \sum_{\substack{n \leq Y \\ n \equiv 1 \pmod{q}}} \Lambda(n) \int_n^\infty \left(1 - \frac{n}{t}\right) \frac{dt}{t^2} \\ &= \frac{q-1}{2} \sum_{\substack{1 < n \leq q^{17} \\ n \equiv 1 \pmod{q}}} \frac{\Lambda(n)}{n}, \end{aligned}$$

which is at most $\frac{C_0}{2} \log q \log \log q$ by Lemma 6.3. Hence

$$I \leq \frac{C_0}{2} \log q \log \log q + \frac{K}{1-\sigma_0},$$

and both bounds follow since $\log q > 1$ and $\log \log q > 1/11$. $\square$

Data and code availability. The Lean 4 formalisation of every proof in this paper, compiled together with the unchanged Lean sources of the zero-free half-plane [11], and the programs that check the numerical steps are in the repository <https://github.com/creelie/Breakthrough-problems-3>, release v1.1.2, <https://github.com/creelie/Breakthrough-problems-3/releases/tag/v1.1.2> [2]. The main theorems are `Vinogradov.PV.polya_vinogradov_loglog` (Theorem 1.1), `Vinogradov.logDeriv_one_le` and `Vinogradov.sum_logDeriv_one` (Theorem 1.3) and `Vinogradov.MV.weak_mv` (Theorem 4.1); they depend only on the axioms `propext`, `Classical.choice` and `Quot.sound`.

Use of AI tools. The proofs, the Lean 4 formalisation, the checking programs and the text of this paper were drafted with Claude (Anthropic), used under the direction of the authors, who checked every statement and take full responsibility for the content of the paper, including its correctness and its citations. The ideas that came from this work are: replacing the generalised Riemann hypothesis on the major arcs by the half-plane of Theorem 1.2 (Section 3, Corollary 3.8 and Lemma 5.3); the weak form of the Montgomery–Vaughan bound with Selberg’s sieve as its only arithmetic input (Section 4); and the dyadic application of Selberg’s sieve to prime powers in progressions (Lemma 6.3). Theorem 1.2 is the work of OpenAI [11] and is not claimed here.

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