

THE PÓLYA–VINOGRADOV INEQUALITY WITH $\log \log q$

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ABSTRACT. We prove unconditionally that every primitive Dirichlet character χ modulo $q \geq 3$ has character sums of size at most $C\sqrt{q} \log \log q$. Montgomery and Vaughan proved this bound under the generalised Riemann hypothesis, and Paley showed that it is optimal. The proof uses the zero-free half-plane $\operatorname{Re} s > 7/8$, which also gives $|L'/L(1, \chi)| \ll \log \log q$ and two-sided bounds for the Euler–Kronecker constants of the cyclotomic fields $\mathbb{Q}(\zeta_q)$.

1. INTRODUCTION

Let χ be a primitive Dirichlet character modulo $q \geq 3$. The Pólya–Vinogradov inequality states that $|\sum_{n \leq x} \chi(n)| \ll \sqrt{q} \log q$ uniformly in x .¹ Paley proved that the factor $\log q$ cannot be replaced by anything smaller than $\log \log q$: for infinitely many primitive quadratic characters the maximum over x exceeds $c\sqrt{q} \log \log q$.² Montgomery and Vaughan [2] proved the upper bound $\sqrt{q} \log \log q$ under the generalised Riemann hypothesis. Without hypothesis, Granville and Soundararajan lowered the exponent of $\log q$ for characters of fixed odd order.³ We prove the bound of Montgomery and Vaughan for every primitive character.

Theorem 1.1. *There is an absolute constant C such that for every primitive character χ modulo $q \geq 3$ and every $x \geq 1$,*

$$\left| \sum_{n \leq x} \chi(n) \right| \leq C\sqrt{q} \log \log q.$$

By Paley’s theorem, Theorem 1.1 is optimal up to the value of C . The input that replaces the Riemann hypothesis is the following theorem.

2020 *Mathematics Subject Classification.* Primary 11L40; Secondary 11M06, 11M20, 11L07, 11N13, 11N36, 11R42.

Key words and phrases. Pólya–Vinogradov inequality, character sums, zero-free regions, Selberg sieve, Euler–Kronecker constants.

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¹See H. Davenport, *Multiplicative Number Theory*, 3rd ed., Grad. Texts in Math. 74, Springer, 2000, Chapter 23.

²R. E. A. C. Paley, *A theorem on characters*, J. London Math. Soc. **7** (1932), 28–32.

³A. Granville and K. Soundararajan, *Large character sums: pretentious characters and the Pólya–Vinogradov theorem*, J. Amer. Math. Soc. **20** (2007), 357–384.

Theorem 1.2 ([3]). *If χ is a nonprincipal Dirichlet character, then $L(s, \chi) \neq 0$ for $\text{Re } s > 7/8$.*

Theorem 1.2 is proved, and formalised in Lean 4, in [3]. It is the only result we use without proof; everything else is proved below from first principles, and the constants are effective.

Montgomery and Vaughan use the Riemann hypothesis in one place: on the major arcs, for prime sums of the characters $\chi\psi$ twisted by additive characters, where ψ runs over the characters of a small modulus r . A power saving in these sums is enough there, and Theorem 1.2 provides one (Corollary 3.8). On the minor arcs they use their unconditional bound for exponential sums with multiplicative coefficients. A weaker form of it suffices, and we prove that form in Section 4 with Selberg's sieve as the only arithmetic input. Figure 1 shows how the parts fit together.

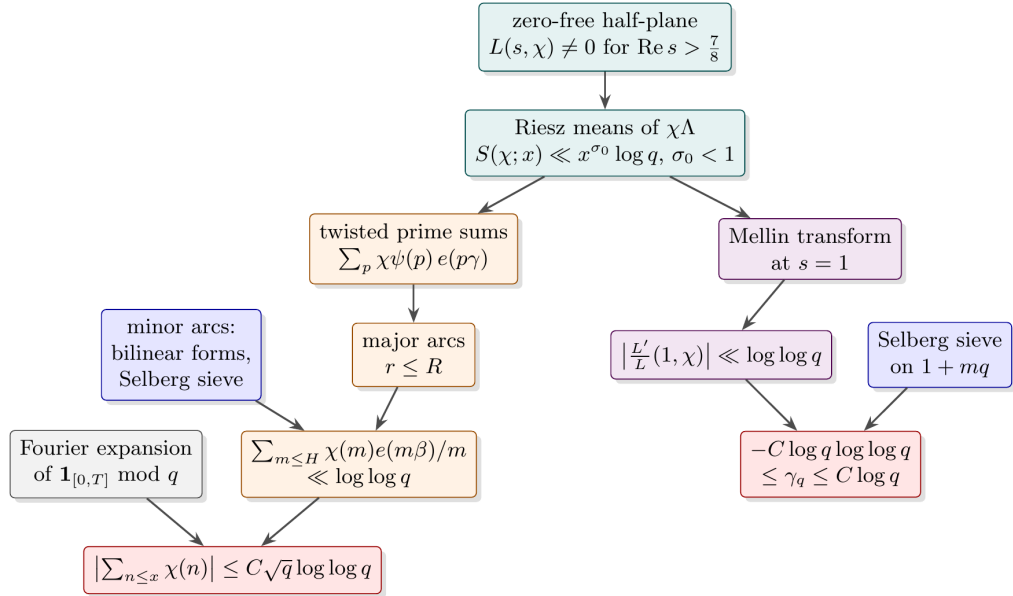


FIGURE 1. The structure of the proofs. The left branch proves Theorem 1.1: the Fourier expansion of a character sum reduces it to the sums $\sum_{m \leq H} \chi(m) e(m\beta)/m$, which are bounded on the major arcs by the half-plane and on the minor arcs by bilinear forms and the sieve. The right branch proves Theorem 1.3.

The same half-plane controls the logarithmic derivative of $L(s, \chi)$ at $s = 1$. For a number field K write $\zeta_K(s) = c_{-1}(s-1)^{-1} + c_0 + O(s-1)$ near $s = 1$; the Euler–Kronecker constant of K is $\gamma_K = c_0/c_{-1}$. Let γ_q be the Euler–Kronecker constant of $\mathbb{Q}(\zeta_q)$. For a prime q the factorisation

$\zeta_{\mathbb{Q}(\zeta_q)}(s) = \zeta(s) \prod_{\chi \neq \chi_0} L(s, \chi)$, the product over the nonprincipal characters modulo q ,⁴ gives

$$(1) \quad \gamma_q = \gamma + \sum_{\chi \neq \chi_0} \frac{L'}{L}(1, \chi),$$

where γ is Euler's constant.

Theorem 1.3. *There is an absolute constant C with the following properties.*

(i) *For every nonprincipal character χ modulo q ,*

$$\left| \frac{L'}{L}(1, \chi) \right| \leq C \log \log q.$$

(ii) *For every prime $q \geq 3$, $-C \log q \log \log q \leq \gamma_q \leq C \log q$.*

Under the generalised Riemann hypothesis the bound (i) is classical, Ihara proved $\gamma_q \leq (2 + o(1)) \log q$ and Badzyan proved a lower bound of the order of (ii);⁵ Ford, Luca and Moree proved the two bounds of (ii) without hypothesis for a set of primes q of density one.⁶ For all primes the best unconditional bound we know is $|\gamma_q| \ll_{\varepsilon} q^{\varepsilon}$, due to Ihara, Murty and Shimura [1]. The upper bound in (ii) needs only Chebyshev's bound. The lower bound needs an upper bound for the prime powers $n \leq q^{17}$ with $n \equiv 1 \pmod{q}$, which we obtain from Selberg's sieve applied in dyadic blocks (Lemma 6.3).

We have not found Theorem 1.1 or Theorem 1.3 among the consequences of Theorem 1.2 drawn in [3] or in the work that followed it. The arguments work, with other constants, whenever $L(s, \chi) \neq 0$ on $\operatorname{Re} s > \theta$ for all nonprincipal χ and some fixed $\theta < 1$.

The paper is organised as follows. Section 2 collects Chebyshev's bounds and Selberg's sieve. Section 3 bounds Riesz means of $\chi\Lambda$ by a contour shift. Section 4 proves the bound for exponential sums with multiplicative coefficients, Section 5 proves Theorem 1.1 and Section 6 proves Theorem 1.3.

⁴L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Grad. Texts in Math. 83, Springer, 1997, Theorem 4.3. All nonprincipal characters modulo a prime are primitive.

⁵Y. Ihara, *On the Euler–Kronecker constants of global fields and primes with small norms*, in: Algebraic Geometry and Number Theory, Progr. Math. 253, Birkhäuser, 2006, 407–451; A. I. Badzyan, *The Euler–Kronecker constant*, Math. Notes **87** (2010), 31–42.

⁶K. Ford, F. Luca and P. Moree, *Values of the Euler ϕ -function not divisible by a given odd prime, and the distribution of Euler–Kronecker constants for cyclotomic fields*, Math. Comp. **83** (2014), 1447–1476; see also P. Moree, *Irregular behaviour of class numbers and Euler–Kronecker constants of cyclotomic fields: the log log log devil at play*, in: Irregularities in the Distribution of Prime Numbers, Springer, 2018, 143–163.

2. PRELIMINARIES

Throughout, p and ℓ denote primes, Λ is the von Mangoldt function, $\theta(x) = \sum_{p \leq x} \log p$, $\psi(x) = \sum_{n \leq x} \Lambda(n)$, $e(x) = e^{2\pi i x}$, and $\|x\|$ is the distance from x to the nearest integer. We write $\omega(n)$ for the number of prime factors of n , $\tau_3(n)$ for the number of ordered factorisations $n = abc$, and $P(n)$ for the largest prime factor of $n \geq 2$, with $P(1) = 1$.

Lemma 2.1. *For $x \geq 1$ and integers $N \geq 2$,*

- (i) $\theta(x) \leq x \log 4$;
- (ii) $\psi(x) - \theta(x) \leq 2\sqrt{x} \log x$ and $\psi(x) \leq 4x$;
- (iii) $\sum_{p \leq N} \frac{\log p}{p} \leq \log N + \log 4$;
- (iv) $\sum_{p \leq N} \frac{1}{p} \leq \log \log N + 10$.

Proof. (i) It suffices to treat integers $m \geq 1$, and we use induction on m . The cases $m = 1, 2$ are clear, and for even $m > 2$ we have $\theta(m) = \theta(m-1)$. Let $m = 2n+1$. The primes in $(n+1, 2n+1]$ divide $\binom{2n+1}{n}$, and $2\binom{2n+1}{n} \leq 2^{2n+1}$ because $\binom{2n+1}{n}$ and $\binom{2n+1}{n+1}$ are equal terms of the binomial expansion of $(1+1)^{2n+1}$. Hence $\theta(2n+1) \leq \theta(n+1) + n \log 4 \leq (2n+1) \log 4$.

(ii) We have $\psi(x) - \theta(x) = \sum_{k \geq 2} \theta(x^{1/k})$, where only $k \leq \log x / \log 2$ contribute. Each term is at most $\sqrt{x} \log 4$ by (i), so the sum is at most $(\log x / \log 2) \sqrt{x} \log 4 = 2\sqrt{x} \log x$. Since $\log x \leq \sqrt{x}$, $\psi(x) \leq (\log 4 + 2)x \leq 4x$.

(iii) By Legendre's formula and (i),

$$N \log N \geq \log N! \geq \sum_{p \leq N} \left\lfloor \frac{N}{p} \right\rfloor \log p \geq N \sum_{p \leq N} \frac{\log p}{p} - N \log 4.$$

(iv) Let $A(u) = \sum_{p \leq u} \log p / p$. Partial summation and (iii) give

$$\sum_{p \leq N} \frac{1}{p} = \frac{A(N)}{\log N} + \int_2^N \frac{A(u)}{u \log^2 u} du \leq 1 + \frac{\log 4}{\log 2} + \int_2^N \left(\frac{1}{u \log u} + \frac{\log 4}{u \log^2 u} \right) du,$$

and the integral is at most $\log \log N - \log \log 2 + 2$. $\square$

We use Selberg's upper bound sieve in the following form.⁷

Lemma 2.2. *Let $a_1, \dots, a_X$ be integers, P a squarefree integer, ν a multiplicative function with $0 < \nu(p) < 1$ for $p \mid P$, and $y \geq 1$. For $d \mid P$ put $R_d = \#\{n \leq X : d \mid a_n\} - \nu(d)X$, and for $l \mid P$ put $g(l) = \prod_{p \mid l} \nu(p)/(1 - \nu(p))$.*

⁷For the method see H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, 1974, Chapter 3. We give the proof of the form we need.

$\nu(p)$). Then

$$\#\{n \leq X : (a_n, P) = 1\} \leq \frac{X}{S} + \sum_{\substack{d|P \\ d \leq y}} 3^{\omega(d)} |R_d|, \quad S = \sum_{\substack{l|P \\ l \leq \sqrt{y}}} g(l).$$

Proof. Let D be the set of divisors d of P with $d \leq \sqrt{y}$, and let $(\lambda_d)_{d \in D}$ be real numbers with $\lambda_1 = 1$. If $(a_n, P) = 1$ then $\sum_{d \in D, d|a_n} \lambda_d = 1$, and otherwise this sum is real, so

$$\begin{aligned} \#\{n : (a_n, P) = 1\} &\leq \sum_{n \leq X} \left(\sum_{\substack{d \in D \\ d|a_n}} \lambda_d \right)^2 \\ &= X \sum_{d_1, d_2 \in D} \lambda_{d_1} \lambda_{d_2} \nu([d_1, d_2]) + \sum_{d_1, d_2 \in D} \lambda_{d_1} \lambda_{d_2} R_{[d_1, d_2]}. \end{aligned}$$

Write Q for the double sum in the first term. Since ν is multiplicative and d_1, d_2 are squarefree, $\nu([d_1, d_2]) = \nu(d_1)\nu(d_2)/\nu((d_1, d_2))$. Let h be the multiplicative function on the divisors of P with $h(p) = 1/\nu(p) - 1 = 1/g(p)$; then $1/\nu(m) = \sum_{l|m} h(l)$ for $m | P$, and

$$Q = \sum_{l|P} h(l) \left(\sum_{\substack{d \in D \\ l|d}} \lambda_d \nu(d) \right)^2 = \sum_{l \in D} \frac{y_l^2}{g(l)}, \quad y_l = \sum_{\substack{d \in D \\ l|d}} \lambda_d \nu(d).$$

Möbius inversion gives $\lambda_d \nu(d) = \sum_{l \in D, d|l} \mu(l/d) y_l$, so the condition $\lambda_1 = 1$ reads $\sum_{l \in D} \mu(l) y_l = 1$. By Cauchy's inequality

$$1 \leq \left(\sum_{l \in D} \frac{y_l^2}{g(l)} \right) \left(\sum_{l \in D} g(l) \right) = QS,$$

with equality for $y_l = \mu(l)g(l)/S$. We make this choice, so that $Q = 1/S$. Then, writing $l = de$,

$$\lambda_d = \frac{1}{\nu(d)S} \sum_{\substack{l \in D \\ d|l}} \mu(l/d) \mu(l) g(l) = \frac{\mu(d)g(d)}{\nu(d)S} \sum_{\substack{e \in D, (e,d)=1 \\ de \leq \sqrt{y}}} g(e).$$

Since $g(d)/\nu(d) = \prod_{p|d} (1 - \nu(p))^{-1} = \prod_{p|d} (1 + g(p)) = \sum_{f|d} g(f)$, we get $|\lambda_d| \leq S^{-1} \sum_{f|d} \sum_e g(fe)$, and the products fe are distinct elements of D ; hence $|\lambda_d| \leq 1$. Finally $[d_1, d_2] \leq d_1 d_2 \leq y$, and every $d | P$ is the least common multiple of at most $3^{\omega(d)}$ pairs (d_1, d_2) , so the second term is at most $\sum_{d|P, d \leq y} 3^{\omega(d)} |R_d|$. $\square$

3. RIESZ MEANS OF PRIME SUMS

For an arithmetic function a and $x \geq 1$ put

$$S(a; x) = \sum_{n \leq x} a(n) \left(1 - \frac{n}{x}\right), \quad S(\chi; x) = S(\chi \Lambda; x) = \sum_{n \leq x} \chi(n) \Lambda(n) \left(1 - \frac{n}{x}\right).$$

The weight $1 - n/x$ makes the Mellin kernel absolutely integrable, which keeps the contour shift below free of truncation errors. The aim of this section is Proposition 3.7.

Lemma 3.1. *Let $a: \mathbb{N} \rightarrow \mathbb{C}$ satisfy $\sum_n |a(n)| n^{-\sigma} < \infty$ for $\sigma > 1$, and let $F(s) = \sum_n a(n) n^{-s}$ for $\operatorname{Re} s > 1$.*

- (i) *For $\operatorname{Re} s > 1$, $\int_1^\infty S(a; x) x^{-s-1} dx = \frac{F(s)}{s(s+1)}$.*
- (ii) *For $x > 0$, $S(a; x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(2+it) \frac{x^{2+it}}{(2+it)(3+it)} dt$.*

In other words the following square commutes, and (ii) inverts its right column on the line $\operatorname{Re} s = 2$:

$$\begin{array}{ccc} a & \xrightarrow{\text{Riesz mean}} & S(a; \cdot) \\ \text{Dirichlet series} \downarrow & & \downarrow \text{Mellin transform} \\ F & \xrightarrow{\times \frac{1}{s(s+1)}} & \frac{F(s)}{s(s+1)} \end{array}$$

Proof. (i) For $n \geq 1$, $\int_n^\infty (1 - n/x) x^{-s-1} dx = n^{-s}/s - n^{-s}/(s+1) = n^{-s}/(s(s+1))$. Since $\sum_n |a(n)| \int_n^\infty x^{-\sigma-1} dx = \sigma^{-1} \sum_n |a(n)| n^{-\sigma} < \infty$, the sum and the integral may be exchanged.

(ii) For $y > 0$ let $k(y) = \frac{1}{2\pi i} \int_{(2)} y^s \frac{ds}{s(s+1)}$. If $y \geq 1$, moving the line of integration to $\operatorname{Re} s = -U$ and letting $U \rightarrow \infty$ picks up the residues 1 at $s = 0$ and $-1/y$ at $s = -1$, so $k(y) = 1 - 1/y$; if $y < 1$, moving it to the right gives $k(y) = 0$. As $|s(s+1)|^{-1}$ is integrable on $\operatorname{Re} s = 2$ and $\sum_n |a(n)| n^{-2} < \infty$, summation and integration may be exchanged, and the right side equals $\sum_n a(n) k(x/n) = S(a; x)$. $\square$

Lemma 3.2. *Let χ be nonprincipal modulo q . For $\operatorname{Re} s = \sigma > 0$, $|L(s, \chi)| \leq q|s|/\sigma$.*

Proof. Put $A(t) = \sum_{n \leq t} \chi(n)$; then $|A(t)| \leq q$, and partial summation gives $L(s, \chi) = s \int_1^\infty A(t) t^{-s-1} dt$ for $\sigma > 0$. Hence $|L(s, \chi)| \leq q|s| \int_1^\infty t^{-\sigma-1} dt = q|s|/\sigma$. $\square$

Lemma 3.3. *Let $A_0 = \frac{3}{2} \sum_\ell \ell^{-3/2}$. For every Dirichlet character χ the series $E(s) = -\sum_\ell \log(1 - \chi(\ell) \ell^{-s})$ is holomorphic on $\operatorname{Re} s > 1$, $e^{E(s)} = L(s, \chi)$ there, and $|E(s)| \leq A_0$ for $\operatorname{Re} s \geq 3/2$.*

Proof. For $\operatorname{Re} s \geq 1$ we have $|\chi(\ell)\ell^{-s}| \leq 1/2$, and $|\log(1-w)| \leq \sum_k |w|^k/k \leq \frac{3}{2}|w|$ for $|w| \leq 1/2$. The series therefore converges locally uniformly on $\operatorname{Re} s > 1$ and is bounded by A_0 on $\operatorname{Re} s \geq 3/2$, and the Euler product gives $e^E = L$. $\square$

Lemma 3.4 (Borel–Carathéodory). *Let h be holomorphic on a neighbourhood of the disc $|z - c| \leq R$ and let $0 < r < R$. Then*

$$\max_{|z-c| \leq r} |h(z) - h(c)| \leq \frac{2r}{R-r} \max_{|w-c|=R} \operatorname{Re}(h(w) - h(c)).$$

Proof. Put $g(z) = h(c+z) - h(c)$ and $A = \max_{|w|=R} \operatorname{Re} g(w)$. The mean of $\operatorname{Re} g$ over the circle $|w| = R$ is $\operatorname{Re} g(0) = 0$, so $A \geq 0$. If $A = 0$, then $\operatorname{Re} g$ vanishes on the circle, hence on the disc, and g is an imaginary constant equal to $g(0) = 0$. Let $A > 0$. By the maximum principle $\operatorname{Re} g \leq A$ on $|z| \leq R$, so $|2A - g| \geq A$ there and $\phi = g/(2A - g)$ is holomorphic. Since $|g|^2 - |2A - g|^2 = 4A(\operatorname{Re} g - A) \leq 0$, we have $|\phi| \leq 1$, and $\phi(0) = 0$. Schwarz’s lemma gives $|\phi(z)| \leq |z|/R$, and from $g = 2A\phi/(1 + \phi)$ we get $|g(z)| \leq 2A|z|/(R - |z|)$. $\square$

Lemma 3.5. *Let $0 < \delta < 1/1000$, $1/2 \leq a \leq 1$, and let χ be nonprincipal modulo q with $L(s, \chi) \neq 0$ for $\operatorname{Re} s > a + 2\delta$. Put $K_1 = (1 + 2A_0)/(2\delta^2)$. For $a + 8\delta \leq \sigma \leq 2$ and $t \in \mathbb{R}$,*

$$\left| \frac{L'}{L}(\sigma + it, \chi) \right| \leq K_1 \log(2q(3 + |t|)^2).$$

Proof. Let $c = 2 + it$, $R = 2 - a - 3\delta$, $r = 2 - a - 6\delta$ and $\Lambda_0 = \log(2q(3 + |t|)^2) \geq \log 18 > 1$. On the disc $|z - c| < 2 - a - 2\delta$ we have $\operatorname{Re} z > a + 2\delta$, so $L(z, \chi)$ has no zero there and has a holomorphic logarithm h . On $|z - c| < 1/2$ both h and the function E of Lemma 3.3 are logarithms of L , so they differ by a constant in $2\pi i\mathbb{Z}$; we choose $h = E$ there. Then $|h(c)| \leq A_0$.

If $|z - c| \leq R$, then $\operatorname{Re} z \geq a + 3\delta \geq 1/2$ and $|z| \leq |c| + 3/2 \leq 4 + |t|$, so Lemma 3.2 gives $\operatorname{Re} h(z) = \log |L(z, \chi)| \leq \log(2q(4 + |t|)) \leq \Lambda_0$. Lemma 3.4 gives, for $|z - c| \leq r$,

$$|h(z) - h(c)| \leq \frac{2r}{R-r} (\Lambda_0 + A_0) \leq \frac{\Lambda_0 + A_0}{\delta},$$

as $r \leq 3/2$ and $R - r = 3\delta$. Hence $|h(z)| \leq A_0 + (\Lambda_0 + A_0)/\delta \leq (1 + 2A_0)\Lambda_0/\delta$ on $|z - c| \leq r$. For $w = \sigma + it$ with $a + 8\delta \leq \sigma \leq 2$ we have $|w - c| = 2 - \sigma \leq r - 2\delta$, so the circle $|z - w| = 2\delta$ lies in $|z - c| \leq r$, and Cauchy’s estimate gives $|L'/L(w, \chi)| = |h'(w)| \leq (1 + 2A_0)\Lambda_0/(2\delta^2)$. $\square$

Lemma 3.6. *For $q \geq 1$, $t \in \mathbb{R}$ and $s = \sigma + it$ with $\sigma \geq 1/2$,*

$$\log(2q(3 + |t|)^2) \leq 10 \log(q + 2)(1 + |t|)^{1/2}, \quad |s(s + 1)| \geq \frac{1}{8}(1 + |t|)^2.$$

Proof. We have $\log(2q) \leq 2\log(q+2)$, and $\log y \leq 2y^{1/2}$ with $3 + |t| \leq 4(1 + |t|)$ gives $\log(3 + |t|) \leq 4(1 + |t|)^{1/2}$; the first bound follows from $\log(q+2) \geq 1$. For the second, $|s| \geq \max(\sigma, |t|) \geq (1 + |t|)/4$ and $|s+1| \geq \max(\sigma+1, |t|) \geq (1 + |t|)/2$. $\square$

Proposition 3.7. *Let $0 < \delta < 1/1000$. There is $K = K(\delta)$ such that the following holds. Let $1/2 \leq a \leq 1$ and let χ be nonprincipal modulo q with $L(s, \chi) \neq 0$ for $\operatorname{Re} s > a + 2\delta$. Then for $x \geq 1$,*

$$|S(\chi; x)| \leq K x^{a+8\delta} \log(q+2).$$

Proof. Put $\sigma_0 = a + 8\delta$ and

$$\Phi(s) = -\frac{L'}{L}(s, \chi) \frac{x^s}{s(s+1)}.$$

For $\operatorname{Re} s > 1$, $-L'/L(s, \chi) = \sum_n \chi(n) \Lambda(n) n^{-s}$, so Lemma 3.1(ii) gives $S(\chi; x) = \frac{1}{2\pi} \int \Phi(2+it) dt$. The function Φ is holomorphic on $\sigma_0 \leq \operatorname{Re} s \leq 2$, and by Lemmas 3.5 and 3.6,

$$|\Phi(\sigma + it)| \leq 80K_1 \log(q+2) x^\sigma (1 + |t|)^{-3/2} \quad (\sigma_0 \leq \sigma \leq 2).$$

This tends to 0 as $|t| \rightarrow \infty$ uniformly in σ , so Cauchy's theorem on the rectangles $[\sigma_0, 2] \times [-T, T]$, $T \rightarrow \infty$, moves the integral to $\operatorname{Re} s = \sigma_0$ (Figure 2). Since $\int_{\mathbb{R}} (1 + |t|)^{-3/2} dt = 4$, we get $|S(\chi; x)| \leq \frac{160K_1}{\pi} x^{\sigma_0} \log(q+2)$. $\square$

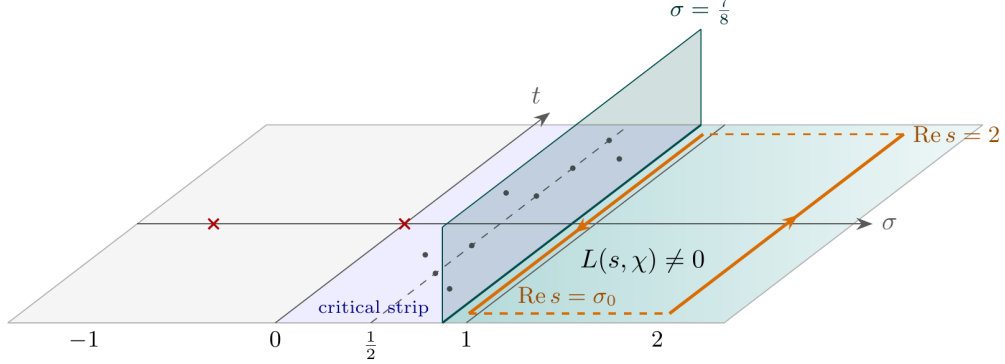


FIGURE 2. The contour shift in the proof of Proposition 3.7. The zeros of $L(s, \chi)$ lie in the strip $0 \leq \operatorname{Re} s \leq 7/8$ (dots), the kernel $x^s/(s(s+1))$ has poles at 0 and -1 (crosses), and the line of integration moves from $\operatorname{Re} s = 2$ to $\operatorname{Re} s = \sigma_0$ inside the zero-free half-plane.

From now on fix $\delta = 1/2000$ and $a = 7/8 - 1/1000$, so that $a + 2\delta = 7/8$ and

$$\sigma_0 = a + 8\delta = \frac{7}{8} + \frac{3}{1000},$$

and let K be the constant of Proposition 3.7 for this δ . By Theorem 1.2, Proposition 3.7 applies to every nonprincipal character.

Corollary 3.8. *There is an absolute constant C_3 such that for every non-principal character ξ modulo k :*

- (i) $|S(\xi; x)| \leq Kx^{\sigma_0} \log(k+2)$ for $x \geq 1$;
 - (ii) $|\sum_{p \leq x} \xi(p) \log p| \leq C_3 x^{19/20} \log(k+2)$ for $x \geq 1$;
 - (iii) for integers $2 \leq A \leq B$ and $\gamma \in \mathbb{R}$,
- $$(2) \quad \left| \sum_{A < p \leq B} \xi(p) e(p\gamma) \right| \leq 2C_3 \frac{B^{19/20} \log(k+2)(2 + 2\pi|\gamma|B)}{\log A}.$$

Proof. (i) is Proposition 3.7. For (ii) put $T(y) = yS(\xi; y)$, that is, $T(y) = \sum_{n \leq y} \xi(n) \Lambda(n)(y - n)$, and let $0 < h \leq x$. Then

$$T(x+h) - T(x) = h \sum_{n \leq x} \xi(n) \Lambda(n) + \sum_{x < n \leq x+h} \xi(n) \Lambda(n)(x+h-n),$$

and the last sum has at most $h+1$ terms, each of modulus at most $h \log(2x)$. With $|T(y)| \leq Ky^{1+\sigma_0} \log(k+2)$ this gives

$$\left| \sum_{n \leq x} \xi(n) \Lambda(n) \right| \leq \frac{2K(2x)^{1+\sigma_0} \log(k+2)}{h} + (h+1) \log(2x).$$

Take $h = x^{47/50}$. Since $1 + \sigma_0 - 47/50 < 19/20$ and $\log(2x) \leq 1 + 100x^{1/100}$, the right side is at most $(8K + 202)x^{19/20} \log(k+2)$. The prime powers contribute at most $\psi(x) - \theta(x) \leq 2\sqrt{x} \log x \leq 200x^{19/20}$ by Lemma 2.1(ii), and (ii) follows with $C_3 = 8K + 402$.

(iii) Let $\Pi(u) = \sum_{p \leq u} \xi(p) \log p$ and $f(u) = e(u\gamma)/\log u$. Partial summation gives

$$\sum_{A < p \leq B} \xi(p) e(p\gamma) = f(B)\Pi(B) - f(A)\Pi(A) - \int_A^B \Pi(u) f'(u) du,$$

and $|f'(u)| \leq 2\pi|\gamma|/\log A + 1/(u \log^2 u)$ for $u \geq A$. By (ii) the right side is at most $C_3 B^{19/20} \log(k+2)(3 + 2\pi|\gamma|B)/\log A$, which gives (2). $\square$

4. EXPONENTIAL SUMS WITH MULTIPLICATIVE COEFFICIENTS

For $M > 0$ put $w_M(\theta) = M$ if $\|\theta\| < 1/M$ and $w_M(\theta) = 1/\|\theta\|$ otherwise; then $0 \leq w_M(\theta) \leq M$ and $w_M(-\theta) = w_M(\theta)$.

Theorem 4.1. *There is an absolute constant C_1 with the following property. Let f be completely multiplicative with $|f(n)| \leq 1$, let $\alpha \in \mathbb{R}$, and let a, r be coprime integers with $r \geq 1$, $|\alpha - a/r| \leq r^{-2}$ and $2 \leq R \leq r \leq N/R$. Then*

$$\left| \sum_{n \leq N} f(n) e(n\alpha) \right| \leq C_1 \left(\frac{N}{\log N} + \frac{N(\log N)^2}{R^{1/4}} \right).$$

Montgomery and Vaughan [2, Corollary 1] proved this for multiplicative f with $N(\log R)^{3/2}R^{-1/2}$ in place of $N(\log N)^2R^{-1/4}$. The proof below follows their approach, a bilinear decomposition together with the spacing of the multiples of α and an upper bound for prime pairs, with cruder estimates, which suffice for Theorem 1.1.

Lemma 4.2. *Let a, r be coprime with $r \geq 2$ and $|\alpha - a/r| \leq r^{-2}$, and let $H \geq 0$ be an integer.*

- (i) $\|k\alpha\| \geq 1/(2r)$ for every integer k with $0 < |k| \leq r/2$.
- (ii) $\#\{1 \leq h \leq H : \|h\alpha\| < t\} \leq (4H/r + 1)(4rt + 1)$ for every $t \geq 0$.
- (iii) For $M > 0$ and every integer $K \geq 0$ with $M \leq 2^K$,

$$\sum_{h \leq H} w_M(h\alpha) \leq \left(\frac{4H}{r} + 1\right)(3M + 4r + 8rK),$$

$$\sum_{h \leq H} w_M(h\alpha)^2 \leq \left(\frac{4H}{r} + 1\right)(20rM + 3M^2).$$

Proof. (i) Let z be the integer nearest to $k\alpha$. Since $0 < |k| < r$ and $(a, r) = 1$, $r \nmid ka$, so $|ka/r - z| \geq 1/r$ and $\|k\alpha\| \geq 1/r - |k| |\alpha - a/r| \geq 1/(2r)$.

(ii) Let $L = \lfloor r/2 \rfloor$, so $L \geq r/4$. For k in a block of L consecutive integers let $y_k = k\alpha - z_k$ with z_k the integer nearest to $k\alpha$. For $k \neq k'$ in the block, $0 < |k - k'| < L \leq r/2$ and $y_k - y_{k'} \equiv (k - k')\alpha \pmod{1}$, so $|y_k - y_{k'}| \geq 1/(2r)$ by (i). Points of $(-t, t)$ at mutual distances at least $1/(2r)$ number at most $4rt + 1$, and $[1, H]$ is covered by at most $H/L + 1 \leq 4H/r + 1$ blocks.

(iii) If $\|\theta\| \geq 1/M$, let $k \geq 1$ be least with $\|\theta\| < 2^k/M$; then $k \leq K$ because $\|\theta\| \leq 1/2 < 2^K/M$, and $\|\theta\| \geq 2^{k-1}/M$, so $w_M(\theta) \leq 2^{1-k}M$. Hence

$$w_M(\theta)^j \leq M^j [\|\theta\| < 1/M] + \sum_{k=1}^K (2^{1-k}M)^j [\|\theta\| < 2^k/M] \quad (j = 1, 2).$$

Summing over $h \leq H$ with (ii), the first sum is at most $(4H/r + 1)(4r + M + \sum_{k \leq K} (8r + 2^{1-k}M))$ and the second at most $(4H/r + 1)(4rM + M^2 + \sum_{k \geq 1} (16rM2^{-k} + 4M^24^{-k}))$. $\square$

Lemma 4.3. *Let $\theta, \alpha \in \mathbb{R}$, $U > 0$, and let A, B, N be nonnegative integers with $B \leq A + U$.*

- (i) $|\sum_{A < x \leq B} e(x\theta)| \leq w_U(\theta)$.
- (ii) For a finite set V of positive integers and $b : V \rightarrow \mathbb{C}$,

$$\sum_{A < u \leq B} \left| \sum_{\substack{v \in V \\ uv \leq N}} b(v)e(uv\alpha) \right|^2 \leq \sum_{v, v' \in V} |b(v)| |b(v')| w_U((v - v')\alpha).$$

(iii) For $M > 0$ and an integer $H \geq 0$, $\sum_{1 \leq m, m' \leq H} w_M((m - m')\alpha) \leq H(M + 2 \sum_{h \leq H} w_M(h\alpha))$.

Proof. (i) If $\|\theta\| < 1/U$, the sum has at most U terms of modulus 1. Otherwise $\theta \notin \mathbb{Z}$, the sum has modulus at most $|e((B - A)\theta) - 1|/|e(\theta) - 1|$, and $|e(\theta) - 1| = 2|\sin \pi\theta| \geq 4\|\theta\|$, so the modulus is at most $1/(2\|\theta\|)$. (ii) Expanding the square gives $\sum_{v, v'} b(v)\overline{b(v')} \sum_u e(u(v - v')\alpha)$, where u runs over the integers of $(A, \min(B, N/v, N/v'))]$, an interval of length at most U ; apply (i). (iii) For fixed m , the term $m' = m$ is $w_M(0) = M$, and each $h = |m - m'| \in [1, H]$ arises from at most two m' . $\square$

Lemma 4.4. Let $C_5 = 1 + 4\zeta(3/2)$. For every completely multiplicative f with $|f(n)| \leq 1$, every $\alpha \in \mathbb{R}$ and every integer $N \geq 1$, the sum $S = \sum_{n \leq N} f(n)e(n\alpha)$ satisfies

$$\left| S \log N - \sum_{p \leq N} \log p f(p) \sum_{m \leq N/p} f(m)e(pm\alpha) \right| \leq C_5 N.$$

Proof. Write $S \log N = S_1 + S_2$ with $S_1 = \sum_{n \leq N} f(n)e(n\alpha) \log n$ and $S_2 = \sum_{n \leq N} f(n)e(n\alpha) \log(N/n)$. Then $|S_2| \leq N \log N - \log N! \leq N$, since $N^N/N! \leq e^N$. In S_1 write $\log n = \sum_{d|n} \Lambda(d)$ and $n = dm$; complete multiplicativity gives $\sum_{d \leq N} \Lambda(d)f(d) \sum_{m \leq N/d} f(m)e(dm\alpha)$. The terms with d prime form the sum in the statement, and the others have modulus at most $N \sum_p \sum_{k \geq 2} \log p/p^k \leq 2N \sum_p \log p/p^2 \leq 4\zeta(3/2)N$, since $\log p \leq 2\sqrt{p}$. $\square$

Let $G(h) = \prod_{p|h} p/(p - 1)$.

Lemma 4.5. (i) For every integer $X \geq 2$ and every even $h \geq 2$,

$$\#\{0 \leq n < X : n \text{ and } n + h \text{ prime}\} \leq \frac{400 G(h)^2 X}{(\log X)^2} + 3X^{3/5}.$$

(ii) $\sum_{h \leq X} G(h)^4 \leq e^{30} X$ for every $X \geq 1$.

Proof. (i) Apply Lemma 2.2 to $a_n = n(n + h)$, $0 \leq n < X$, with $y = X^{1/5}$, $z = \lfloor y \rfloor$, P the product of the primes $p \leq z$ with $p \nmid h$, and $\nu(d) = 2^{\omega(d)}/d$. Every $p \mid P$ is odd, so $\nu(p) = 2/p < 1$. For $d \mid P$ the residues n with $d \mid n(n + h)$ form $2^{\omega(d)}$ classes modulo d (namely $n \equiv 0$ or $-h \pmod{p}$ for each $p \mid d$, distinct as $p \nmid h$), and each class contains between $X/d - 1$ and $X/d + 1$ of the integers $0 \leq n < X$; hence $|R_d| \leq 2^{\omega(d)}$. Since the prime factors of d are at least 3, $3^{\omega(d)}|R_d| \leq 6^{\omega(d)} \leq d^2 \leq y^2$, and the error term is at most $y^3 = X^{3/5}$.

Let $\mathcal{M}$ be the set of $m \leq \sqrt{y}$ whose prime factors divide P . Since $\nu(p)/(1 - \nu(p)) = \sum_{k \geq 1} \nu(p)^k$ and $\nu(m) = 2^{\Omega(m)}/m$ for these m , $g(l)$ is the sum of $2^{\Omega(m)}/m$ over the m whose set of prime factors is that of l .

An $m \in \mathcal{M}$ is counted in $g(l)$ for l the product of its prime factors, and $l \leq m \leq \sqrt{y}$. Hence, with $\tau(m) \leq 2^{\Omega(m)}$ the number of divisors of m ,

$$S \geq \sum_{m \in \mathcal{M}} \frac{2^{\Omega(m)}}{m} \geq \sum_{m \in \mathcal{M}} \frac{\tau(m)}{m}.$$

Let $T = \lfloor y^{1/4} \rfloor$ and let B be the set of $b \leq T$ that are prime to h ; these b are odd. For $b, b' \in B$ the product bb' lies in $\mathcal{M}$, and a given m is a product bb' in at most $\tau(m)$ ways, so $S \geq (\sum_{b \in B} 1/b)^2$. If E is a set of primes and $p \notin E$, every $b \leq T$ prime to E is uniquely $p^k b'$ with b' prime to $E \cup \{p\}$, so the sum of $1/b$ over the $b \leq T$ prime to E is at most $p/(p-1)$ times the same sum for $E \cup \{p\}$. Adding the prime factors of h one at a time gives

$$\sum_{b \in B} \frac{1}{b} \geq \frac{1}{G(h)} \sum_{b \leq T} \frac{1}{b} \geq \frac{\log(T+1)}{G(h)}.$$

As $T+1 > y^{1/4} = X^{1/20}$, we get $S \geq (\log X)^2/(400G(h)^2)$, that is, $X/S \leq 400G(h)^2 X/(\log X)^2$. Finally, if $n < X$ and $n, n+h$ are prime, then $n \leq z$ or $(n(n+h), P) = 1$, so the count is at most $z+1+X/S+y^3 \leq 400G(h)^2 X/(\log X)^2 + 3X^{3/5}$.

(ii) For $t = p-1 \geq 1$, $(t+1)^5 \leq (t+31)t^4$, that is, $(p/(p-1))^4 \leq 1+30/p$. Hence $G(h)^4 \leq \sum_d 30^{\omega(d)}/d$ over the squarefree $d \mid h$, and

$$\begin{aligned} \sum_{h \leq X} G(h)^4 &\leq \sum_{d \leq X} \mu(d)^2 \frac{30^{\omega(d)}}{d} \cdot \frac{X}{d} \leq X \prod_p \left(1 + \frac{30}{p^2}\right) \\ &\leq X \exp\left(30 \sum_{n \geq 2} \frac{1}{n^2}\right) \leq e^{30} X. \end{aligned} \quad \square$$

Proof of Theorem 4.1. Let $S = \sum_{n \leq N} f(n)e(n\alpha)$, $L = \log N$ and $\ell = \lfloor \log_2 N \rfloor$. Since $N \geq rR \geq 4$, $L \geq 1$. If $\ell < 64$, then $L < 64 \log 2 < 45$ and $|S| \leq N \leq 45N/L$. Let $\ell \geq 64$, $I = \lfloor \log_2 \ell \rfloor + 1$ and $J = \ell + 1 - 4I$. Then $16^I > \ell^4$, $1 \leq J \leq \ell \leq 2L$, $I \leq \ell \leq 2L$, $2^I \leq 2\ell$, $N < 2^{\ell+1} = 2^J 16^I$ and $J^2 2^J \leq (\ell+1)^2 2^{\ell+1} \ell^{-4} \leq 2N$. By Lemma 4.4, $|S|L \leq C_5 N + |T_1| + |T_2|$, where T_1 and T_2 are the parts of $\sum_p \log p f(p) \sum_{m \leq N/p} f(m)e(pm\alpha)$ with $p \leq 2^J$ and with $p > 2^J$.

The small primes. Let $0 \leq j < J$, $U = 2^j$, $H = \lfloor N/U \rfloor$, and for $U < x \leq 2U$ let $B(x) = \sum_{m \leq H, xm \leq N} f(m)e(xm\alpha)$; the inner sum for a prime $p \in (U, 2U]$ is $B(p)$. By Cauchy's inequality, Lemma 4.3(ii), (iii) and Lemma 4.2(iii) with $M = U$, $K = j$,

$$\sum_{U < p \leq 2U} \log p |B(p)| \leq (U \log^2(2U))^{1/2} Q_j^{1/2},$$

where $Q_j = H(U + 2(4H/r + 1)(3U + 4r + 8rj))$. Since $HU \leq N$, $UQ_j \leq 7NU + (8 + 16j)Nr + 24N^2/r + (32 + 64j)N^2/U$, and $\log(2U) \leq j + 1$. Summing over $j < J$ with $j + 1 \leq J$, $\sum_{j < J} 2^{j/2} \leq 3 \cdot 2^{J/2}$, $\sqrt{(32 + 64j)2^{-j}} \leq 8(j + 1)(3/4)^j$ and $\sum_{j \geq 0} (j + 1)^2(3/4)^j = 112$,

$$|T_1| \leq 3\sqrt{7}J\sqrt{N2^J} + J^2N\sqrt{24/r} + 896N + J^2\sqrt{24JNr}.$$

Now $J^22^J \leq 2N$ gives $3\sqrt{7}J\sqrt{N2^J} \leq 12N$; $r \geq R \geq \sqrt{R}$ gives $\sqrt{24/r} \leq 5R^{-1/4}$; and $r\sqrt{R} \leq rR \leq N$ gives $\sqrt{24JNr} \leq 5JNR^{-1/4}$. With $J \leq 2L$ and $L \geq 1$, $|T_1| \leq 908N + 60L^3NR^{-1/4}$.

The large primes. For $p > 2^J$ we have $N/p < 16^I$, so the m in T_2 lie in $(0, 16^I - 1]$, the union of the blocks $(16^i - 1, 16^{i+1} - 1]$ with $0 \leq i < I$. Fix i , let $A = 16^i - 1$, $U = 15 \cdot 16^i$ and $X = \lfloor N/16^i \rfloor$, and assume $X \geq 1$, the block being empty otherwise. Only the primes $p \in (2^J, X]$ contribute, since $p \geq X + 1$ and $m \geq 16^i$ give $pm > N$. Let V be the set of these primes, which are odd, and $B(m) = \sum_{p \in V, pm \leq N} \log p f(p) e(pm\alpha)$. The block is $\sum_{A < m \leq A+U} f(m)B(m)$, and by Cauchy's inequality, Lemma 4.3(ii) and $\log p \leq \lambda = \log(X + 1)$, its square is at most $U\lambda^2W$, where

$$W = \sum_{p, p' \in V} w_U((p - p')\alpha) = |V|U + 2 \sum_{h=1}^X w_U(h\alpha) \#\{p \in V : p + h \in V\}.$$

The count vanishes for odd h , and for even h Lemma 4.5(i), applied to $X + 1$, bounds it by $400G(h)^2(X + 1)\lambda^{-2} + 3(X + 1)^{3/5}$. By Cauchy's inequality, Lemma 4.5(ii) and Lemma 4.2(iii), $\sum_{h \leq X} G(h)^2 w_U(h\alpha) \leq e^{15}\sqrt{X}Q^{1/2}$ with $Q = (4X/r + 1)(20rU + 3U^2)$, and $\sum_{h \leq X} w_U(h\alpha) \leq XU$. Hence $U\lambda^2W \leq A_1 + 800e^{15}Z + A_3$, where

$$A_1 = U^2\lambda^2X, \quad Z = U(X + 1)\sqrt{X}Q^{1/2}, \quad A_3 = 6U^2\lambda^2(X + 1)^{3/5}X.$$

Let $\kappa = \log(N + 1) \geq \lambda$ and $T_i = 4^i\kappa(N(2N)^{3/5})^{1/2}$. Since $U^2X = 225 \cdot 16^i(16^iX) \leq 225 \cdot 16^iN$ and $X + 1 \leq 2N$, we get $A_1 \leq 225T_i^2$ and $A_3 \leq 1350T_i^2 \leq 37^2T_i^2$. With $w = UX \leq 15N$, $X \leq N/16^i$ and $R \leq r \leq N/R$,

$$\begin{aligned} Z^2 &\leq 4U^2X^3Q = 320w^3X + \frac{48w^4}{r} + 80rw^3 + 12Uw^3 \\ &\leq \frac{320 \cdot 15^3N^4}{16^i} + \frac{48 \cdot 15^4N^4}{R} + \frac{80 \cdot 15^3N^4}{R} \\ &\quad + 180 \cdot 15^3 \cdot 16^iN^3, \end{aligned}$$

and every coefficient is at most 40^4 , so $Z^{1/2} \leq 40(N2^{-i} + 2NR^{-1/4} + 2^iN^{3/4})$. The block is therefore at most

$$\begin{aligned} &15T_i + 40\sqrt{800e^{15}}(N2^{-i} + 2NR^{-1/4} + 2^iN^{3/4}) + 37T_i \\ &\leq K_b(N2^{-i} + NR^{-1/4} + 2^iN^{3/4} + T_i), \end{aligned}$$

with $K_b = 80\sqrt{800e^{15}} + 52$. Summing over $i < I$,

$$|T_2| \leq K_b(2N + INR^{-1/4} + 2^I N^{3/4} + 4^I \kappa(N(2N)^{3/5})^{1/2}).$$

Here $I \leq 2L^3$; $2^I \leq 4L \leq 16N^{1/4}$; $4^I \leq 16L^2$ and $\kappa \leq 2L$; and $\log x \leq 15x^{1/15}$ gives $L^3(N(2N)^{3/5})^{1/2} \leq 2 \cdot 15^3 N$. Hence $|T_2| \leq K_b(216018N + 2L^3NR^{-1/4})$.

Altogether $|S|L \leq (C_5 + 908 + 216018K_b)N + (60 + 2K_b)L^3NR^{-1/4}$, and dividing by L gives the theorem with $C_1 = 45 + C_5 + 908 + 216018K_b + 60 + 2K_b$. $\square$

5. PROOF OF THEOREM 1.1

In this section χ is a primitive character modulo $q \geq 3$.

Lemma 5.1. *Let $B \geq 0$ be such that $|\sum_{m \leq H} \bar{\chi}(m)e(m\beta)/m| \leq B$ for every integer $H \leq q/2$ and every $\beta \in \mathbb{R}$. Then $|\sum_{n \leq x} \chi(n)| \leq \sqrt{q}(B + 6)$ for every $x \geq 0$.*

Proof. We may take $x = T$ to be an integer. Let $\tau = \sum_{a \bmod q} \bar{\chi}(a)e(a/q)$. Since $\bar{\chi}$ is primitive, $\sum_a \bar{\chi}(a)e(an/q) = \chi(n)\tau$ for every n , and summing the squares of the moduli of these identities over $n \bmod q$ gives $q\varphi(q) = |\tau|^2\varphi(q)$, so $|\tau| = \sqrt{q}$. Summing over $1 \leq n \leq T$,

$$\tau \sum_{n \leq T} \chi(n) = \sum_{0 < |a| \leq q/2} \bar{\chi}(a) \frac{e(aT/q) - 1}{1 - e(-a/q)},$$

where a runs over the integers with $-q/2 < a \leq q/2$, $a \neq 0$. For $0 < |\theta| \leq 1/2$ put $r(\theta) = (1 - e(-\theta))^{-1} - (2\pi i\theta)^{-1}$; then $|1 - e(-\theta)| = 2|\sin \pi\theta| \geq 4|\theta|$. If $2\pi|\theta| \leq 1$, then $|e(-\theta) - 1 + 2\pi i\theta| \leq (2\pi\theta)^2$ and $|r(\theta)| \leq (2\pi\theta)^2/(4|\theta| \cdot 2\pi|\theta|) < 3$; otherwise $|r(\theta)| \leq 1/(4|\theta|) + 1/(2\pi|\theta|) < 3$. The terms with $r(a/q)$ contribute at most $6q$. Writing $a = \pm m$ with $1 \leq m \leq q/2$, the remaining terms are $q/(2\pi i)$ times a combination, with coefficients of modulus at most 1, of four sums $\sum_{m \leq H} \bar{\chi}(m)e(m\beta)/m$ with $H \leq q/2$, so they contribute at most $2qB/\pi$. Dividing by $|\tau|$ gives the claim. $\square$

Lemma 5.2. *For $Y \geq 2$ and $H \geq 1$,*

$$\sum_{\substack{n \leq H \\ P(n) \leq Y}} \frac{1}{n} \leq e^{11} \log Y.$$

Proof. The sum is at most $\prod_{p \leq Y} (1 - 1/p)^{-1} \leq \exp(\sum_{p \leq Y} \frac{1}{p-1})$, and $\sum_p (\frac{1}{p-1} - \frac{1}{p}) \leq 1$; apply Lemma 2.1(iv). $\square$

Lemma 5.3. *There is an absolute constant C_2 such that for $0 < r < q$, $b \in \mathbb{Z}$, $\beta \in \mathbb{R}$, integers $Y \geq \max(2, r)$ and $t \geq 1$,*

$$\left| \sum_{\substack{n \leq t \\ P(n) > Y}} \chi(n) e\left(n\left(\frac{b}{r} + \beta\right)\right) \right| \leq C_2 \frac{r \log(qr + 2) (2 + 2\pi|\beta|t) t}{Y^{1/20} \log Y}.$$

Proof. Every $n \leq t$ with $P(n) > Y$ is uniquely $n = pm$ with $p = P(n)$ prime, $Y < p \leq t/m$ and $P(m) \leq p$, and then $m \leq t/Y$. For such p we have $p > r$, so $p \nmid r$, and on the units modulo r the function $u \mapsto e(umb/r)$ equals $\sum_{\psi} c_{\psi} \psi(u)$, the sum over the characters ψ modulo r , with $|c_{\psi}| \leq 1$.

The character $\chi\psi$ modulo qr is nonprincipal. Otherwise $\chi(n) = \bar{\psi}(n)$ for every n prime to qr . Let $d = (q, r)$ and let n be prime to q with $n \equiv 1 \pmod{d}$. By the Chinese remainder theorem there is n' prime to qr with $n' \equiv n \pmod{q}$ and $n' \equiv 1 \pmod{r}$, and then $\chi(n) = \chi(n') = \bar{\psi}(n') = 1$. So χ is trivial on the units congruent to 1 modulo d , and is induced by a character modulo $d \leq r < q$, which contradicts primitivity.

Applying (2) to $\xi = \chi\psi$ with $\gamma = m\beta$, $A = \max(Y, P(m) - 1)$ and $B = \lfloor t/m \rfloor$ (the inner sum is empty if $A \geq B$) gives

$$\left| \sum_{\substack{n \leq t \\ P(n) > Y}} \chi(n) e\left(n\left(\frac{b}{r} + \beta\right)\right) \right| \leq 2C_3 r \frac{\log(qr + 2)(2 + 2\pi|\beta|t)}{\log Y} \sum_{m \leq t/Y} \left(\frac{t}{m}\right)^{19/20},$$

and $\sum_{m \leq M} m^{-19/20} \leq 20M^{1/20}$. Hence $C_2 = 40C_3$ will do. $\square$

Proposition 5.4. *There is an absolute constant C_4 such that if $\log q \geq 200$, then $\left| \sum_{m \leq H} \chi(m) e(m\beta) / m \right| \leq C_4 \log \log q$ for every integer $H \leq q$ and every $\beta \in \mathbb{R}$.*

Proof. Let $L = \log q$, $M = \lceil L \rceil$, $R = M^{16}$ and $Y = M^{700}$. Since $e^{L/16} \geq (L/16)^4/24 > L + 1 \geq M$, we have $2 \leq R < q$ and $R \leq Y$. Split the sum according to $P(m) \leq Y$ or $P(m) > Y$. By Lemma 5.2 the first part is at most $e^{11} \log Y$. For the second put $A(t) = \sum_{n \leq t, P(n) > Y} \chi(n) e(n\beta)$ and $w(t) = 1/t - 1/(t+1)$. Partial summation and $|A(H)| \leq H$ give

$$\left| \sum_{\substack{m \leq H \\ P(m) > Y}} \frac{\chi(m) e(m\beta)}{m} \right| \leq 1 + \sum_{1 \leq t < H} |A(t)| w(t).$$

If $t \leq Y$, then $A(t) = 0$. Let $t > Y$ and $Q = \lfloor t/R \rfloor \geq 1$. By Dirichlet's theorem there are coprime a, r with $1 \leq r \leq Q$ and $|\beta - a/r| \leq 1/(r(Q+1))$; then $|\beta - a/r| t \leq R$ (Figure 3). If $r \leq R$, then Lemma 5.3 with $b = a$ gives $|A(t)| \leq C_2 k t$, where

$$k = \frac{R \log(qR + 2)(2 + 2\pi R)}{Y^{1/20} \log Y}.$$

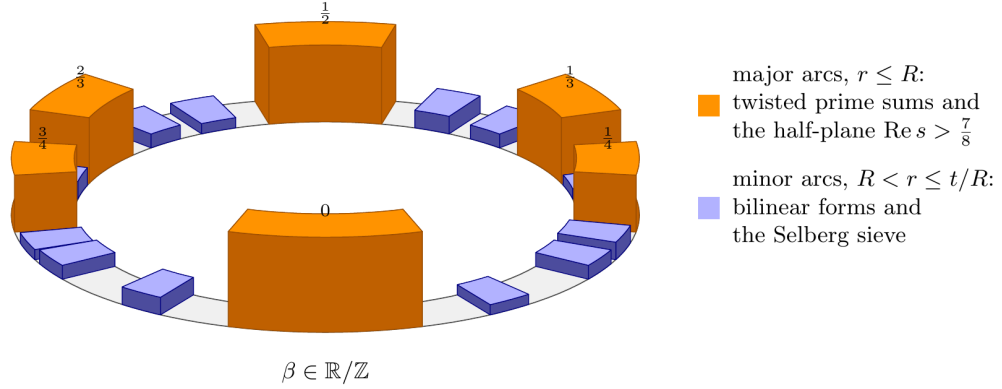
If $r > R$, then $R \leq r \leq t/R$ and $|\beta - a/r| \leq r^{-2}$, so Theorem 4.1 for $f = \chi$ bounds the full sum $\sum_{n \leq t} \chi(n)e(n\beta)$, and since $\log t \leq L$, removing the $\Psi(t) = \#\{n \leq t : P(n) \leq Y\}$ smooth terms gives

$$|A(t)| \leq C_1 \left(\frac{t}{\log t} + t\rho \right) + \Psi(t), \quad \rho = \frac{L^2}{R^{1/4}} = \frac{L^2}{M^4}.$$

Since $tw(t) \leq 1/t$ and $w(t) \leq 1/t^2$,

$$\sum_{1 \leq t < H} |A(t)| w(t) \leq C_1 \sum_{2 \leq t \leq H} \frac{1}{t \log t} + (C_1 \rho + C_2 k) \sum_{t \leq H} \frac{1}{t} + \sum_{1 \leq t < H} \Psi(t) w(t).$$

The first sum is at most $3 + \log L$, the second at most $1 + L$, and by partial summation the third is at most $\sum_{n \leq H, P(n) \leq Y} 1/n \leq e^{11} \log Y$. Finally $L \leq M \leq 2L$ gives $\rho(1 + L) \leq 2/M \leq 1$; $\log(qR + 2) \leq 18M$, $2 + 2\pi R \leq 8M^{16}$, $Y^{1/20} = M^{35}$ and $\log Y = 700 \log M \geq 700$ give $k(1 + L) \leq 288M^{34}/(700M^{35}) \leq 1$; and $\log Y = 700 \log M \leq 1400 \log L$. Collecting terms, the sum is at most $1 + 4C_1 + C_2 + (2800e^{11} + C_1) \log L \leq (2800e^{11} + 1 + 5C_1 + C_2) \log L$. $\square$



If $\log q < 200$, then every sum $\sum_{m \leq H} \bar{\chi}(m)e(m\beta)/m$ with $H \leq q$ has modulus at most $1 + \log q < 201$, and Lemma 5.1 gives

$$\left| \sum_{n \leq x} \chi(n) \right| \leq 207\sqrt{q} \leq \frac{207}{\log \log 3} \sqrt{q} \log \log q. \quad \square$$

6. THE LOGARITHMIC DERIVATIVE AT $s = 1$

Recall that $\sigma_0 = 7/8 + 3/1000$ and that K is the constant of Corollary 3.8.

Lemma 6.1. *For every nonprincipal character χ modulo q ,*

$$-\frac{L'}{L}(1, \chi) = 2 \int_1^\infty \frac{S(\chi; t)}{t^2} dt,$$

and the integral converges absolutely.

Proof. By Lemma 3.1(i) with $a = \chi\Lambda$, for real $s > 1$,

$$\int_1^\infty S(\chi; t) t^{-s-1} dt = -\frac{L'}{L}(s, \chi) \frac{1}{s(s+1)}.$$

As $s \rightarrow 1^+$ the left side tends to $\int_1^\infty S(\chi; t) t^{-2} dt$ by dominated convergence, since $|S(\chi; t)| \leq K t^{\sigma_0} \log(q+2)$ and $\sigma_0 < 1$. The function $L(s, \chi)$ is entire and does not vanish at $s = 1$ by Theorem 1.2, so the right side tends to $-\frac{1}{2} \frac{L'}{L}(1, \chi)$. $\square$

Proof of Theorem 1.3(i). Let χ be nonprincipal modulo q , so $q \geq 3$, and put $Y = (\log(q+2))^{1/(1-\sigma_0)} \geq 1$. Since $|S(\chi; t)| \leq \psi(t) \leq 4t$ by Lemma 2.1(ii), Lemma 6.1 gives

$$\left| \frac{L'}{L}(1, \chi) \right| \leq 2 \int_1^Y \frac{4 dt}{t} + 2K \log(q+2) \int_Y^\infty t^{\sigma_0-2} dt = \frac{8 \log \log(q+2) + 2K}{1 - \sigma_0}.$$

As $q+2 \leq q^2$, $\log \log(q+2) \leq \log 2 + \log \log q$, and since $\log \log q \geq \log \log 3 > 0$ the right side is at most $C \log \log q$. $\square$

For the lower bound in Theorem 1.3(ii) we need to control the prime powers $n \leq q^{17}$ with $n \equiv 1 \pmod{q}$. We count them in dyadic blocks with Selberg's sieve.

Lemma 6.2. *Let q be prime, let $M \geq 1$ be an integer and let $1 < z < q$. The number of m with $M \leq m < 2M$ for which $1 + mq$ is a prime power is at most*

$$\frac{2M}{\log z} + z(1 + \log z)^2 + z.$$

Proof. If $1 + mq = p^k$ with $p \leq z$, then $p^k \in (Mq, 2Mq)$, and for each prime p at most one power of p lies in this interval, since $p^{k+1} \geq 2p^k$. These m number at most z . If $p > z$, then $1 + mq$ has no prime factor up to z , and we apply Lemma 2.2 to the M integers $a_m = 1 + mq$, $M \leq m < 2M$, with

P the product of the primes up to z , $\nu(d) = 1/d$ and $y = z$. Every prime $p \leq z$ differs from q , so for $d \mid P$ the condition $d \mid 1 + mq$ fixes m in one residue class modulo d , and $|R_d| \leq 1$. Here $g(l) = \prod_{p \mid l} (p-1)^{-1}$, which is the sum of $1/n$ over the n whose set of prime factors is that of l . Every $n \leq \sqrt{z}$ has all its prime factors at most z and the product of its distinct prime factors is at most $\sqrt{z}$, so

$$S = \sum_{\substack{l \mid P \\ l \leq \sqrt{z}}} g(l) \geq \sum_{n \leq \sqrt{z}} \frac{1}{n} \geq \frac{\log z}{2}.$$

For squarefree d , $3^{\omega(d)} = \tau_3(d)$, and $\sum_{d \leq z} \tau_3(d) = \#\{(a, b, c) : abc \leq z\} \leq z(\sum_{a \leq z} 1/a)^2 \leq z(1 + \log z)^2$. Lemma 2.2 therefore bounds the number of these m by $2M/\log z + z(1 + \log z)^2$. $\square$

Lemma 6.3. *There is an absolute constant C_0 such that for every prime $q \geq 3$,*

$$\sum_{\substack{1 < n \leq q^{17} \\ n \equiv 1 \pmod{q}}} \frac{\Lambda(n)}{n} \leq C_0 \frac{\log q \log \log q}{q}.$$

Proof. Write $n = 1 + mq$ with $1 \leq m < q^{16}$, and let $J = \lfloor 16 \log_2 q \rfloor + 1$, so that $2^J > q^{16}$ and $J \leq 25 \log q$. We split the range of m into the blocks $2^j \leq m < 2^{j+1}$, $0 \leq j < J$ (Figure 4), and let N_j be the number of m in the j -th block with $1 + mq$ a prime power. We claim $N_j \leq 393 \cdot 2^j/(j+1)$. For $j < 32$ this holds trivially. For $j \geq 32$ apply Lemma 6.2 with $M = 2^j$ and $z = 2^u$, $u = j/32 \geq 1$. Since $2^j < 2^J \leq 2q^{16}$, we have $z < 2^{1/32} q^{1/2} < q$. Then $2M/\log z = 64 \cdot 2^j/(j \log 2) \leq 96 \cdot 2^j/(j+1)$, and since $\log z \leq u$,

$$z(1 + \log z)^2 + z \leq 2^u(2 + u)^2 \leq \frac{297 \cdot 2^{4u}}{32u + 1} \leq \frac{297 \cdot 2^j}{j + 1},$$

because $(2 + u)^2(32u + 1) \leq 9u^2 \cdot 33u = 297u^3$ and $u^3 \leq 2^{3u}$. This proves the claim.

A term of the j -th block has $\Lambda(n) \leq \log(1 + 2^{j+1}q) \leq (j+2) \log 2 + \log q$ and $n > 2^j q$, so the block contributes at most

$$\frac{393}{(j+1)q} ((j+2) \log 2 + \log q) \leq \frac{393}{q} \left(2 \log 2 + \frac{\log q}{j+1} \right).$$

Summing over $j < J$ and using $\sum_{j < J} 1/(j+1) \leq 1 + \log J$, $J \leq 25 \log q$ and $\log J \leq 4 + \log \log q$, the sum in the lemma is at most $\frac{393}{q} (35 \log q + 5 \log q + \log q \log \log q)$. Since $\log \log q \geq \log \log 3 > 1/11$, this is at most $C_0 \log q \log \log q/q$ with $C_0 = 393 \cdot 441$. $\square$

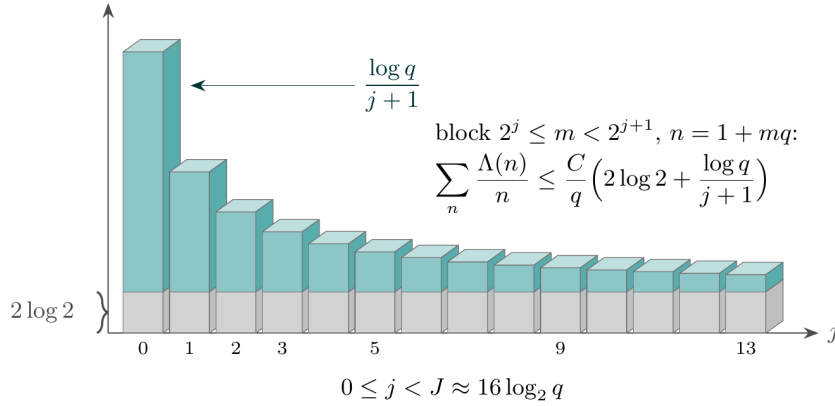


FIGURE 4. The dyadic blocks in the proof of Lemma 6.3. The block $2^j \leq m < 2^{j+1}$ contributes a constant part, of total size $\ll J \ll \log q$, and a part of size $\log q/(j+1)$, whose total is $\ll \log q \log J$. The second part produces the factor $\log \log q$.

Proof of Theorem 1.3(ii). Let $q \geq 3$ be prime and put $c(n) = \sum_{\chi \neq \chi_0} \chi(n)$, the sum over the nonprincipal characters modulo q . By orthogonality $c(n) = (q-1)\mathbf{1}[n \equiv 1 \pmod{q}] - \mathbf{1}[q \nmid n]$. Put

$$T(t) = \sum_{\chi \neq \chi_0} S(\chi; t) = \sum_{n \leq t} c(n) \Lambda(n) \left(1 - \frac{n}{t} \right), \quad I = \int_1^\infty \frac{T(t)}{t^2} dt.$$

By Lemma 6.1, $\sum_{\chi \neq \chi_0} \frac{L'}{L}(1, \chi) = -2I$, so by (1) it suffices to show $-C \log q \leq I \leq C \log q \log \log q$.

By Corollary 3.8(i), $|T(t)| \leq qK \log(q+2) t^{\sigma_0}$. Let $Y = q^{17}$. Since $17(\sigma_0 - 1) < -2$ and $\log(q+2) \leq q$,

$$\left| \int_Y^\infty \frac{T(t)}{t^2} dt \right| \leq \frac{qK \log(q+2) Y^{\sigma_0-1}}{1 - \sigma_0} \leq \frac{K \log(q+2)}{q(1 - \sigma_0)} \leq \frac{K}{1 - \sigma_0}.$$

For the lower bound, $c(n) \geq -1$ gives $-T(t) \leq \psi(t) \leq 4t$, so

$$-I \leq \int_1^Y \frac{4dt}{t} + \frac{K}{1 - \sigma_0} = 68 \log q + \frac{K}{1 - \sigma_0}.$$

For the upper bound, $c(n) \leq (q-1)\mathbf{1}[n \equiv 1 \pmod{q}]$ gives

$$\begin{aligned} \int_1^Y \frac{T(t)}{t^2} dt &\leq (q-1) \sum_{\substack{n \leq Y \\ n \equiv 1 \pmod{q}}} \Lambda(n) \int_n^\infty \left(1 - \frac{n}{t} \right) \frac{dt}{t^2} \\ &= \frac{q-1}{2} \sum_{\substack{1 < n \leq q^{17} \\ n \equiv 1 \pmod{q}}} \frac{\Lambda(n)}{n}, \end{aligned}$$

which is at most $\frac{C_0}{2} \log q \log \log q$ by Lemma 6.3. Hence

$$I \leq \frac{C_0}{2} \log q \log \log q + \frac{K}{1 - \sigma_0},$$

and both bounds follow since $\log q > 1$ and $\log \log q > 1/11$. $\square$

DATA AVAILABILITY

Deep Bhattacharjee computed the Lean formalisation and the Julia, C, C++, shell and Python computations with the help of AI assistance (Claude, Anthropic). The Lean formalisation of the proofs in this paper is archived at Zenodo, doi:10.5281/zenodo.23383716. The paper is available at <https://github.com/creelie/Breakthrough-problems-3/blob/main/preprintPolyaVinogradovLogLog/PolyaVinogradovLogLog.pdf>.

COMPETING INTERESTS

None.

FUNDING

None.

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