

Separated distances in fixed dimension

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Abstract

A finite Euclidean point set is distance-separated if the lengths of any two distinct unordered edges differ by at least one. Counting gives a diameter lower bound of $\binom{n}{2}$, whereas Erdős asked for a lower bound with leading term n^2 . We give two constructions in fixed dimension. For every $0 < \varepsilon \leq 1/2$, put $m_0 = \lceil 8192/\varepsilon \rceil$. For every $n \geq 4m_0$ there is a distance-separated n -point set in $\mathbb{R}^{10m_0-1}$ with diameter $< (1/2 + \varepsilon)n^2$. A six-cluster construction gives arbitrarily large configurations in $\mathbb{R}^{29}$ with diameter $< 0.998n^2$, and an explicit prime-interval bound gives diameter $< 0.999n^2$ for every $n \geq 400000$. The constructions use two orthogonal circle families to control distances within and between clusters separately. Their common geometric argument reduces to a uniform operator estimate for the first result and three rational positive-definiteness certificates for the second. Lean 4 formalizations accompany the tolerance family and the dimension-29 result for arbitrarily large sizes; the all-size corollary is a written consequence.

Keywords: separated distances, Euclidean diameter, Gram completion, quadratic-residue rulers.

1 The problem and the results

How small can the diameter of a point configuration be if all of its edge lengths must stay at least one unit apart? The number of edges gives a quadratic lower bound. In a fixed Euclidean dimension, however, the geometry may impose a further loss. This paper gives two constructions that separate these questions: a uniform family approaching the counting coefficient as the dimension increases, and a counterexample to the coefficient-one bound in dimension 29.

Definition 1.1. An *edge* of a finite set $X \subset \mathbb{R}^d$ is an unordered pair of distinct points. We call X *distance-separated* if

$$|||x - y|| - ||u - v||| \geq 1 \quad \text{for any two different edges } \{x, y\}, \{u, v\}. \quad (1.1)$$

For $n \geq 3$, let $\Delta_d(n)$ be the infimum of the diameters of all distance-separated n -point subsets of $\mathbb{R}^d$.

The comparison in (1.1) is between edges, not merely between distinct values in a distance set: repeated edge lengths are forbidden. Every edge has length at least one. Indeed, if z is a third point, then $1 \leq |||x - z|| - ||y - z||| \leq ||x - y||$. Sorting the $\binom{n}{2}$ edge lengths therefore gives

$$\Delta_d(n) \geq \binom{n}{2} = \left(\frac{1}{2} - o(1)\right) n^2. \quad (1.2)$$

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The class defining the infimum is nonempty: the positive differences between distinct powers of two are distinct integers, so a suitable set on a line already supplies an example.

Erdős asked whether the stronger lower bound $(1 + o(1))n^2$ holds; see [1, Problem 20]. His wording does not specify the quantifier order in the dimension. The fixed-dimensional question asks whether

$$\forall d \geq 1 \forall \varepsilon > 0 \exists N \forall n \geq N : \Delta_d(n) \geq (1 - \varepsilon)n^2. \quad (1.3)$$

The threshold N may depend on both d and ε .

Ho's earlier construction [3] refutes a bound uniform in dimension: for infinitely many n , it gives configurations in $\mathbb{R}^{n^2-n}$ with diameter $(1 - 1/\pi^2 + o(1))n^2$. His Remark 10 proves the improved coefficient $(\pi + \sqrt{\pi^2 - 4})/(2\pi) = 0.885589\dots$ in that construction, while Remark 9 leaves the fixed-dimensional question open. His proof also has a Lean formalization. The distinction addressed here is the dimension's independence from the number of points.

The earlier unrestricted-dimensional construction [11] achieves diameter $\binom{n}{2} + O(\sqrt{\log n})$ for every $n \geq 3$, with dimension allowed to grow with n . Here the aim is to keep the dimension fixed while the number of points grows.

1.1 Dimension linear in the inverse tolerance

Theorem 1.2 (The two-circle family). *Let $0 < \varepsilon \leq \frac{1}{2}$ and put $m_0 = \lceil 8192/\varepsilon \rceil$. For every integer $n \geq 4m_0$ there is an n -point set $X \subset \mathbb{R}^{10m_0-1}$ with minimum point distance at least one, with the lengths of different unordered edges strictly more than one apart, and with*

$$\text{diam } X < \left(\frac{1}{2} + \varepsilon \right) n^2.$$

For each fixed tolerance, both the ambient dimension and the size threshold are $O(\varepsilon^{-1})$. The same construction gives a useful statement when dimension and point count are specified in advance.

Corollary 1.3 (Uniform dimension and size). *If $d \geq 163840$ and $n \geq 65536$, there is a configuration in $\mathbb{R}^d$ with the same minimum-distance and strict edge-separation properties and*

$$\text{diam } X < \left(\frac{1}{2} + \frac{102400}{d} + \frac{40960}{n} \right) n^2.$$

The dimension depends on ε in Theorem 1.2. Consequently, neither statement constructs diameter $(1/2 + o(1))n^2$ in one fixed dimension. An earlier unpublished deterministic-incidence construction [12] also gave an $O(1/d + 1/n)$ error. The two-circle construction gives this order with explicit constants and a simpler geometric proof. We do not claim that the dimension order is optimal.

1.2 A fixed witness in dimension 29

Theorem 1.4 (Dimension 29). *For every natural number N , there exist an integer $n \geq \max\{N, 600\}$ and a set $X \subset \mathbb{R}^{29}$ of n points such that the minimum point distance is at least one, different unordered edge lengths differ by at least one, and*

$$\text{diam } X < \frac{499}{500} n^2 = 0.998n^2.$$

The sizes are obtained by first choosing a prime and then choosing the number of points from it.

Corollary 1.5 (All sufficiently large sizes in dimension 29). *For every integer $n \geq 400000$, there is an n -point set in $\mathbb{R}^{29}$ with minimum point distance at least one, distinct unordered edge lengths strictly more than one apart, and diameter $< 0.999n^2$.*

This corollary uses an explicit prime-interval theorem in addition to the same geometry. Its written proof is given at the end of Section 5; it is not one of the Lean endpoints listed in Section 6.

Corollary 1.6 (Negation of the fixed-dimensional statement). *The proposition (1.3) is false. In fact, $\liminf_{n \rightarrow \infty} \Delta_{29}(n)/n^2 \leq 499/500 < 1$.*

Proof. If (1.3) held, choose $d = 29$ and $\varepsilon = 1/1000$. Its eventual lower bound $0.999n^2$ contradicts Theorem 1.4 at a selected size beyond that threshold. $\square$

The dimension-29 theorem trades a weaker diameter coefficient for a much smaller dimension and different size quantifiers. The earlier published orthogonal-circle theorem [10] gives diameter below $0.735n^2$ in $\mathbb{R}^{81794}$ for every $n \geq 36354$. That theorem remains valid; it is not superseded in every parameter by Theorem 1.4.

Construction	Ambient dimension	Diameter and size quantifier
Published finite version	81794	$< 0.735n^2$ for every $n \geq 36354$
Theorem 1.4	29	$< 0.998n^2$ for arbitrarily large selected $n \geq 600$
Corollary 1.5	29	$< 0.999n^2$ for every $n \geq 400000$ (written corollary)
Theorem 1.2	$10\lceil 8192/\varepsilon \rceil - 1$	$< (1/2 + \varepsilon)n^2$ for every $n \geq 4\lceil 8192/\varepsilon \rceil$

Table 1: The results have different quantifiers. The dimension in the last row is fixed for each chosen tolerance, independently of n .

1.3 Why two circles help

Replace each of m cluster centres, called *macro vertices*, by k nearby points. Call an edge *local* if its endpoints lie in the same cluster and *cross* otherwise. The local lengths must all be distinct, while each pair of clusters contributes a block of k^2 cross lengths. A tilted circle can encode the coarse and fine digits of those blocks. A second circle, in independent coordinates, can encode the local ruler without constraining its angular span to the digit circle’s small arc. A whole-matrix Gram completion makes the perturbation spaces of different clusters orthogonal. The resulting cross lengths are *exactly*

$$\Phi_\nu(T) = \sqrt{\nu^2 + 2\nu T}.$$

Here ν is the central distance of a cross block and T orders the k^2 lengths within it. A derivative bound controls their gaps directly; the error relative to $\nu + T$ need not remain bounded as the clusters grow.

The general family labels macro edges in colexicographic order: first by the larger endpoint and then by the smaller one. Uniform operator estimates control the resulting simplex. The dimension-29 construction replaces that large simplex by six vertices with an explicit label pattern. Three exact 5×5 positivity checks then control *all* cluster sizes in its admissible range. The proofs below retain these two distinct macro inputs.

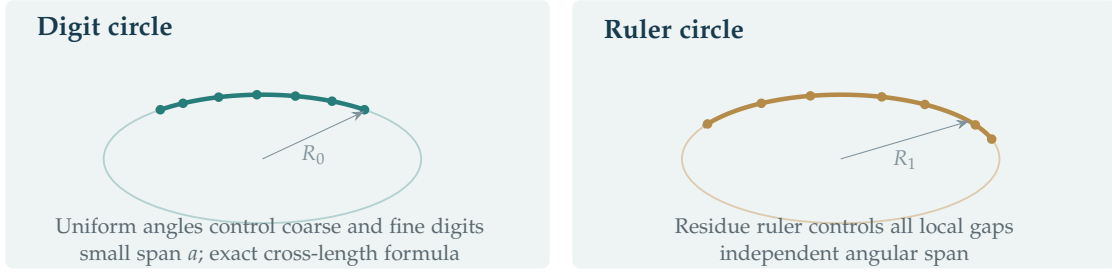


Figure 1: Two independent coordinate channels within each cluster, shown schematically. The ellipses depict circle planes; they are not a literal planar realization of the high-dimensional construction. Whole-matrix completion makes different clusters' perturbation spaces orthogonal.

The common geometry and separation lemmas are proved in Section 2. The general family uses the colex input from Section 3; its proof, including the passage to every sufficiently large size, is in Section 4. The dimension-29 proof in Section 5 replaces the macro input and numerical parameters.

2 The common geometric mechanism

We use real Euclidean inner products, transpose $\top$, and the operator norm $\|\cdot\|_{\text{op}}$. Matrices act on column vectors. For a prime p , write $[v]_p \in \{0, \dots, p-1\}$ for the least nonnegative residue.

Lemma 2.1 (Two-circle completion). *Let $R_0, R_1 > 0$, let $q_A \in \mathbb{R}^{m-1}$, $0 \leq A < m$, and let U be an $(m-1) \times m$ matrix with $\|U\|_{\text{op}} \leq R_0$. Put $W = (R_0^2 I - U^\top U)^{1/2}$. For real z, w , define*

$$\begin{aligned}\eta_A(z) &= (U_A \sin z, W e_A \sin z, R_0 e_A \cos z), \\ \zeta_A(w) &= (R_1 e_A \sin w, R_1 e_A \cos w).\end{aligned}$$

The vector $\eta_A(z)$ lies in $\mathbb{R}^{m-1} \oplus \mathbb{R}^m \oplus \mathbb{R}^m$ and has norm R_0 ; $\zeta_A(w)$ lies in two further m -dimensional blocks and has norm R_1 . The perturbation spaces for different A are orthogonal. Within one cluster the two chord lengths are respectively $2R_0 |\sin((z - z')/2)|$ and $2R_1 |\sin((w - w')/2)|$.

Proof. The identity $U^\top U + W^\top W = R_0^2 I$ gives both the norms of the sine directions and their mutual orthogonality; the cosine directions are orthogonal unit-coordinate directions in a separate block. The remaining claims are the elementary chord identity, applied in orthogonal blocks. The square root is the usual positive-semidefinite matrix square root. $\square$

This elementary completion is a Gram-deficit construction, related to standard dilation arguments [8]. The required hypothesis is a bound on the *whole* matrix U , not separate bounds on its columns.

Write $R^2 = R_0^2 + R_1^2$, choose a scale $S > 0$, and suppose that

$$\|q_A - q_B\|^2 = D_{AB}^2 - 2(R/S)^2 \quad (A \neq B). \quad (2.1)$$

For $\theta > 0$ and $0 \leq i < k$, put $z_i = \theta(i - (k-1)/2)$ and $f_i = \sin z_i / \theta$, and set

$$x_{Ai} = (Sq_A, 0, 0, 0, 0) + (\eta_A(z_i), \zeta_A(w_{Ai})). \quad (2.2)$$

There are $5m - 1$ coordinates. Suppose also that, whenever $A < B$,

$$\langle q_A - q_B, U_A \rangle = \frac{\alpha}{\theta} D_{AB}, \quad \langle q_B - q_A, U_B \rangle = \frac{\beta}{\theta} D_{AB}.$$

Expanding the square, the constant $2R^2$ from the perturbations cancels the shortening in (2.1), leaving

$$\|x_{Ai} - x_{Bj}\| = \Phi_{SD_{AB}}(\alpha f_i + \beta f_j), \quad \Phi_\nu(T) = \sqrt{\nu^2 + 2\nu T}. \quad (2.3)$$

The ruler-circle blocks have zero macro projection and hence do not enter the digit sum. This decoupling is the central feature of the construction.

Lemma 2.2 (An ordered cross block). *Let $k \geq 2$ be an integer, $\alpha, \beta, \gamma > 0$, and $0 < \tau \leq 1$, with $\alpha\gamma > \beta(k-1)$. Suppose $\gamma \leq f_{i+1} - f_i \leq 1$, $f_{k-1} - f_0 \leq k-1$, and $\tau \leq \Phi'_\nu(T) \leq 1/\tau$ on the interval containing zero and all digit sums $\alpha f_i + \beta f_j$ in each block. Within one macro block, the lexicographic order of (i, j) has fine and carry gaps at least*

$$\tau\beta\gamma, \quad \tau(\alpha\gamma - \beta(k-1)),$$

respectively. If $|T| \leq T_{\max}$ in every block and successive macro baselines are at least h apart, their lengths are separated by at least $h - 2T_{\max}/\tau$.

Proof. A fine step changes only j . A carry replaces $(i, k-1)$ by $(i+1, 0)$. Apply the mean value theorem to the resulting changes in T . Finally, $\Phi_\nu(0) = \nu$ and the derivative bound imply $|\Phi_\nu(T) - \nu| \leq T_{\max}/\tau$, which gives the inter-block bound. $\square$

The derivative condition is easy to check uniformly. For $\nu > 0$,

$$\Phi'_\nu(T) = (1 + 2T/\nu)^{-1/2}.$$

If $|2T/\nu| \leq 1 - \tau^2$, its upper bound is $1/\tau$ and its lower bound is at least τ , since $\tau^2(2 - \tau^2) \leq 1$.

Lemma 2.3 (A quadratic-residue ruler). *Let $m, k, L \geq 2$ be integers, and let $p > mk$ be prime. For $0 \leq A < m$ and $0 \leq i < k$, put*

$$y_{Ai} = Lpi + [i^2 + 2kAi]_p.$$

Every local edge $(A, i, i+d)$, where $1 \leq d < k$ and $0 \leq i < k-d$, has a different positive integer difference $g = y_{A, i+d} - y_{Ai}$. Moreover,

$$(Ld-1)p < g < (Ld+1)p, \quad (2.4)$$

so ordering these differences also orders their lags nondecreasingly.

Proof. The residue terms differ by less than p in absolute value, giving (2.4). The open windows for different lags are disjoint and ordered. At a fixed lag,

$$g \equiv d(2i + d + 2kA) \pmod{p}.$$

Equal differences would imply $2d[(i+kA) - (i'+kA')] \equiv 0 \pmod{p}$. Since $p > mk \geq 2k$, $2d$ is invertible modulo p . The integers $i+kA$ lie in $[0, mk)$, so they are distinct modulo p unless $(A, i) = (A', i')$. $\square$

Lemma 2.4 (A monotone chord comparison). *Let $H_d \geq 0$ be nondecreasing in d , and let $B(g) > 0$ be differentiable. Consider $d_1 \leq d_2$ and $g_1 < g_2$. If, throughout $[g_1, g_2]$,*

$$H_{d_1} \leq cB(g), \quad B'(g) \geq v > 0,$$

where $c \geq 0$, then for $L(d, g) = \sqrt{H_d^2 + B(g)^2}$,

$$L(d_2, g_2) - L(d_1, g_1) \geq \frac{v}{\sqrt{1+c^2}}(g_2 - g_1).$$

In particular, distinct integer ruler differences give length gaps greater than one if $v/\sqrt{1+c^2} > 1$.

Proof. Keep the smaller lag fixed. The function $F(g) = \sqrt{H_{d_1}^2 + B(g)^2}$ has derivative $F'(g) = B'(g)/\sqrt{1 + (H_{d_1}/B(g))^2} \geq v/\sqrt{1+c^2}$. Apply the mean value theorem on $[g_1, g_2]$, then use $L(d_2, g_2) \geq L(d_1, g_2)$. $\square$

3 The colex macro input

For $0 \leq A < B < m$, put

$$D_{AB} = K + \binom{B}{2} + A + 1, \quad D_{BA} = D_{AB}. \quad (3.1)$$

The edge labels run through $1, \dots, \binom{m}{2}$ exactly once. Write $H_m = \sum_{j=1}^m 1/j$.

Lemma 3.1 (Colex macro skeleton). *Let $m \geq 4$ and suppose*

$$K \geq m, \quad K \geq 16384H_m^2, \quad 0 \leq \rho \leq K/4. \quad (3.2)$$

There are affinely independent $q_0, \dots, q_{m-1} \in \mathbb{R}^{m-1}$ satisfying $\|q_A - q_B\|^2 = D_{AB}^2 - 2\rho^2$ for $A \neq B$. There are unique vectors F_A, G_A satisfying

$$\langle q_A - q_J, F_A \rangle = D_{AJ} \mathbf{1}_{J < A}, \quad \langle q_A - q_J, G_A \rangle = D_{AJ} \mathbf{1}_{J > A} \quad (J \neq A). \quad (3.3)$$

Their whole column matrices obey

$$\|F\|_{\text{op}} \leq 24\sqrt{K+m}, \quad \|G\|_{\text{op}} \leq 60\sqrt{K+m}. \quad (3.4)$$

This input is inherited from the earlier fixed-dimensional construction [4, 10]. Its complete proof is reproduced in Section A. It uses an increment Gram matrix, a weighted diagonal reserve, and a finite Hardy inequality; neither cluster construction is used in that proof.

4 Proof of the inverse-tolerance dimension bound

4.1 Parameters and the whole-matrix budget

We first work with an auxiliary tolerance $0 < \xi \leq 1/8$; at the end we choose $\xi = \varepsilon/8$. Fix integers $m \geq 1024/\xi$ and $k \geq 2$. Define

$$\begin{aligned} s &= 1 + \xi, \quad a = \sqrt{\xi}, \quad \theta = a/k, \quad K = \lceil \xi m^2 \rceil, \quad S = s^3 k^2, \\ R_0 &= R_1 = 128mk^2, \quad R = \sqrt{2}R_0, \quad \rho = R/S, \quad \tau = \sqrt{1 - \frac{1}{s(K+1)}}, \quad \gamma = 1 - \xi/8. \end{aligned} \quad (4.1)$$

Then $m \geq 8192$, $\xi m \geq 1024$, and

$$K \geq 1024m, \quad K + m \leq \xi m^2 + 2m \leq \frac{9}{8}\xi m^2.$$

To check the harmonic gate, choose $r \geq 13$ with $2^r \leq m < 2^{r+1}$. Dyadic grouping gives $H_m \leq r + 2$. The inequality $(r+2)^2 \leq 2^r/16$ holds at $r = 13$ and is preserved by induction since $(r+3)^2 \leq 2(r+2)^2$. Thus $H_m^2 \leq m/16$ and $16384H_m^2 \leq 1024m \leq K$. Also $\rho = 128\sqrt{2}m/s^3 < K/4$. Hence Lemma 3.1 applies.

Set

$$U = \frac{s}{\theta}(F + skG).$$

By (3.4),

$$\|U\|_{\text{op}} \leq \frac{84s^2k}{\theta} \sqrt{K+m} \leq 84(9/8)^{5/2}mk^2 < 128mk^2 = R_0.$$

The last comparison is the rational inequality $84^2(9/8)^5 < 128^2$. Moreover $1/[s(K+1)] < \xi/8$, so $\tau > 1 - \xi/8$ and

$$\tau s \gamma > (1 - \xi/8)^2(1 + \xi) > 1. \quad (4.2)$$

Indeed the last expression minus one is $\xi(3/4 - 15\xi/64 + \xi^2/64) > 0$.

4.2 Independent digits and ruler marks

The digits are uniform sine marks:

$$z_i = \theta \left(i - \frac{k-1}{2} \right), \quad f_i = \sin z_i / \theta.$$

They satisfy $|f_i| \leq (k-1)/2$, $\gamma \leq f_{i+1} - f_i \leq 1$, and $f_{k-1} - f_0 \leq k-1$, since $|z_i| < a/2$ and $\cos(a/2) \geq 1 - a^2/8 = \gamma$.

By Bertrand's postulate choose a prime $2mk < p < 4mk$. Independently of the digits, put

$$y_{Ai} = 3pi + [i^2 + 2kAi]_p, \quad w_{Ai} = y_{Ai} / (3pk). \quad (4.3)$$

Use these angles in (2.2). The resulting cross identity is

$$\|x_{Ai} - x_{Bj}\| = \Phi_v(s^2kf_i + sf_j), \quad v = SD_{AB}, \quad A < B. \quad (4.4)$$

This follows from (3.3) and the common completion in Lemma 2.1.

4.3 All local lengths

For a local edge with $d = j - i \in \{1, \dots, k-1\}$, let $g = y_{Aj} - y_{Ai}$. By Lemma 2.3, with $L = 3$,

$$(3d-1)p < g < (3d+1)p, \quad g \equiv d(2i + d + 2kA) \pmod{p}. \quad (4.5)$$

These differences are distinct positive integers, and ordering them also orders their lags nondecreasingly.

The exact local length is

$$L(d, g) = \sqrt{H_d^2 + B(g)^2}, \quad H_d = 2R_0 \sin \frac{ad}{2k}, \quad B(g) = 2R_0 \sin \frac{g}{6pk}.$$

Order two local edges by $g_1 < g_2$; then $d_1 \leq d_2$. Throughout their intervening g -interval, $g > (3d_1-1)p \geq 2pd_1$ and $g < 3pk$. For $0 \leq u \leq 1/2$, the elementary bounds $\sin u \geq (7/8)u$ and $\cos u \geq 7/8$ give

$$B(g) \geq \frac{7}{12} \frac{R_0 d_1}{k}, \quad B'(g) \geq \frac{7}{8} \frac{R_0}{3pk}, \quad H_{d_1} \leq \frac{R_0 a d_1}{k} \leq \frac{12a}{7} B(g).$$

Since $a^2 = \zeta \leq 1/8$, we have $12a/7 < 1$. Also $p < 4mk$ gives

$$B'(g) > \frac{28}{3}, \quad \frac{28/3}{\sqrt{1 + (12a/7)^2}} > \frac{14}{3} > 1.$$

The digit chords H_d increase with d , so Lemma 2.4 proves strict unit separation of all local lengths. The smallest local length is greater than $7R_0/(12k) > 1$, while every local length is less than $R_0\sqrt{1 + a^2} < 144mk^2$.

4.4 Cross lengths and the local–cross gap

Put $h = (k - 1)/2$ and $T_{\max} = s(sk + 1)h$. Then

$$2T_{\max} = s^2k^2 - s(s - 1)k - s < s^2k^2.$$

Since $\nu \geq S(K + 1)$, we have $2|T|/\nu < 1/[s(K + 1)] = 1 - \tau^2$. Consequently $\tau \leq \Phi'_\nu(T) \leq 1/\tau$ and $|\Phi_\nu(T) - \nu| \leq T_{\max}/\tau$.

In Lemma 2.2, fine steps have gap at least $\tau s\gamma > 1$. At a carry the digit-sum increment is at least

$$s^2k\gamma - s(k - 1) = s[(s\gamma - 1)k + 1] > s,$$

so its length gap is also greater than one. Consecutive macro baselines are S apart, and

$$S - \frac{2T_{\max}}{\tau} = s^2(s - 1/\tau)k^2 + \frac{s(s - 1)k}{\tau} + \frac{s}{\tau} > 1$$

by (4.2). Thus all cross lengths are mutually more than one apart.

Every cross length is greater than $s^3Kk^2 \geq 1024mk^2$: indeed its lower bound $S(K + 1) - T_{\max}/\tau$ exceeds SK because $T_{\max}/\tau < s^2k^2/(2\tau) < S$. This leaves a mixed gap greater than $880mk^2 > 1$ above all local lengths. These estimates also prove injectivity and the minimum-distance assertion. The actual diameter is bounded by

$$\text{diam}\{x_{Ai}\} \leq \left[s^3 \left(K + \binom{m}{2} \right) + \frac{s^2}{2\tau} \right] k^2. \quad (4.6)$$

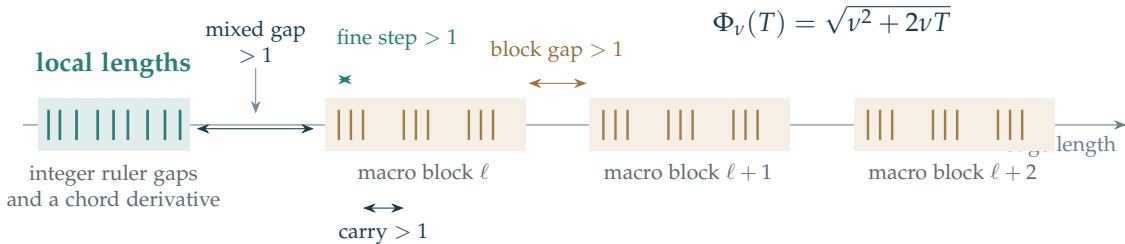


Figure 2: The proof accounts for every pair of different edges: two local edges, a local and a cross edge, or two cross edges. Within a cross block, lexicographic order reduces the checks to fine steps and carries; between blocks, the macro baseline pays for the full digit excursion. Schematic, not to scale.

4.5 The coefficient and the arbitrary-size passage

We have $\tau > 63/64$, $s \leq 9/8$, $K \leq \xi m^2 + 1$, and $s^3 \leq 1 + 4\xi$. Dividing (4.6) by $(mk)^2$ yields

$$\begin{aligned} \frac{\text{diam}\{x_{Ai}\}}{(mk)^2} &\leq (1 + 4\xi)(1/2 + \xi) + 3/m^2 \\ &\leq 1/2 + (7/2)\xi + 3/m^2 \leq 1/2 + 4\xi. \end{aligned}$$

For the last step use $m \geq 1024/\xi$, which implies $3/m^2 \leq \xi/2$.

Put $m_0 = \lceil 1024/\xi \rceil$. Given $n \geq 4m_0$, choose $k = \lfloor n/m_0 \rfloor$ and $m = \lceil n/k \rceil$. Then $k \geq 4$. The inequalities $km_0 \leq n < (k+1)m_0$ give $m_0 \leq m \leq m_0 + \lceil m_0/k \rceil \leq 2m_0$, and

$$n \leq mk < n + k \leq (1 + 1/m_0)n.$$

Delete points to obtain exactly n and pad coordinates to dimension $10m_0 - 1$. Neither operation worsens the required estimates. Since $1/m_0 \leq \xi/1024$,

$$\text{diam } X < (1/2 + 4\xi)(1 + \xi/1024)^2 n^2 < (1/2 + 5\xi)n^2. \quad (4.7)$$

For example, the last inequality follows from $1/2 + 4\xi \leq 1$ and $(1 + \xi/1024)^2 - 1 < \xi$. Choosing $\xi = \varepsilon/8$ proves Theorem 1.2, with strictly positive coefficient slack.

For Corollary 1.3, choose $\xi = \max\{20480/d, 8192/n\}$. The hypotheses imply $0 < \xi \leq 1/8$. Moreover $10\lceil 1024/\xi \rceil - 1 \leq d$ and $4\lceil 1024/\xi \rceil \leq n$: the respective left sides are at most $d/2 + 9$ and $n/2 + 4$. Apply (4.7), use $\max\{u, v\} \leq u + v$, and pad to dimension d .

5 Six macro vertices and dimension 29

The preceding uniform estimates favor a large number of clusters. To obtain a small fixed dimension, we instead keep $m = 6$ and certify the entire macro completion directly. All parameters are rational:

$$\begin{aligned} K &= 457/25, & H &= 569/500, & \eta &= 533/500, & \sigma &= 138/125, \\ a &= 519/1000, & b &= 879/500, & r_0 &= 507/100, & r_1 &= 1131/100, \\ k_0 &= 100, & t &= 971/1000, & C &= 101/100. \end{aligned} \quad (5.1)$$

We work for every integer $k \geq 100$ and every prime p such that

$$6k < p \leq 6Ck. \quad (5.2)$$

5.1 A finite macro certificate

Use the symmetric label matrix

$$\ell = \begin{pmatrix} 0 & 3 & 5 & 7 & 14 & 15 \\ 3 & 0 & 2 & 4 & 13 & 12 \\ 5 & 2 & 0 & 1 & 11 & 10 \\ 7 & 4 & 1 & 0 & 9 & 8 \\ 14 & 13 & 11 & 9 & 0 & 6 \\ 15 & 12 & 10 & 8 & 6 & 0 \end{pmatrix}. \quad (5.3)$$

Its upper-triangular labels are exactly $1, \dots, 15$. For $A \neq B$, put $D_{AB} = K + H\ell_{AB}$ and $E_{AB} = D_{AB}^2 - 2(r_0^2 + r_1^2)$; put both diagonal entries equal to zero. Define the 5×5 anchored Gram matrix

$$B_{ij} = \frac{E_{0i} + E_{0j} - E_{ij}}{2} \quad (1 \leq i, j \leq 5).$$

Let $P_{AJ}^G = D_{AJ}$ for $A < J$ and zero otherwise, and set $P^F = D - P^G$. For $\star \in \{F, G\}$ define the 5×6 matrix

$$T_{jA}^\star = P_{A0}^\star - P_{Aj}^\star \quad (1 \leq j \leq 5, 0 \leq A \leq 5).$$

The three rational matrices to be checked are

$$B, \quad M(x) = B - \frac{(\sigma T^G + \eta x T^F)(\sigma T^G + \eta x T^F)^\top}{(ar_0)^2}, \quad x \in \{0, 1/100\}. \quad (5.4)$$

Lemma 5.1 (Exact whole-matrix certificate). *All three matrices in (5.4) are positive definite.*

Proof. Substitute the rational parameters (5.1) and the integer matrix (5.3), then eliminate successively without pivoting. Every diagonal pivot in the resulting $LDL^\top$ factorization is a strictly positive rational. Table 2 gives certified rational lower bounds for all fifteen pivots; the exact recurrence and a complete stand-alone reconstruction program are in Section B. No floating-point positivity decision is used. $\square$

Matrix	d_1	d_2	d_3	d_4	d_5
B	> 1	> 1	> 1	> 1	> 1
$M(0)$	> 1	> 1	$> \frac{1}{2}$	> 1	> 1
$M(1/100)$	> 1	> 1	$> \frac{1}{2}$	> 1	> 1

Table 2: Strict rational lower bounds for the five successive $LDL^\top$ pivots of each matrix. These are lower bounds, not rounded pivot values. Every entry is obtained by exact fraction arithmetic.

Choose $q_0 = 0$ and $q_1, \dots, q_5 \in \mathbb{R}^5$ with Gram matrix B . The polarization identity then gives $\|q_A - q_B\|^2 = E_{AB}$. Let Q be the square matrix with columns $q_1, \dots, q_5$ and define $F = Q^{-\top} T^F$, $G = Q^{-\top} T^G$. Subtracting their anchored inner products gives

$$\langle q_A - q_J, F_A \rangle = P_{AJ}^F, \quad \langle q_A - q_J, G_A \rangle = P_{AJ}^G.$$

By Lemma 5.1, both endpoint matrices $Q^{-\top}(\sigma T^G + \eta x T^F)$ have norm less than ar_0 . Indeed, writing $A(x) = \sigma T^G + \eta x T^F$, congruence gives

$$Q^{-\top} M(x) Q^{-1} = I - \frac{[Q^{-\top} A(x)][Q^{-\top} A(x)]^\top}{(ar_0)^2} \succ 0.$$

The matrix depends affinely on x , and the operator norm is convex. Thus the same strict bound holds for every $x \in [0, 1/100]$. In particular,

$$U = \frac{k}{a}(\eta F + \sigma k G) = \frac{k^2}{a} Q^{-\top} A(1/k) \quad \text{satisfies} \quad \|U\|_{\text{op}} < r_0 k^2 \quad (k \geq 100). \quad (5.5)$$

This is why the two endpoint checks control every finite $k \geq 100$.

5.2 Coordinates and cross lengths

Put $S = k^2$, $R_0 = r_0 k^2$, $R_1 = r_1 k^2$, and $\theta = a/k$. Use the same uniform digit marks f_i as before, and the independent ruler marks

$$y_{Ai} = 2pi + [i^2 + 2kAi]_p, \quad w_{Ai} = \frac{b y_{Ai}}{2pk}. \quad (5.6)$$

With U from (5.5), construct the actual points by (2.2). Their coordinate blocks have dimensions 5, 6, 6, 6, 6, so the ambient dimension is exactly 29. The cross identity becomes

$$\|x_{Ai} - x_{Bj}\| = \Phi_v(\sigma k f_i + \eta f_j), \quad v = k^2 D_{AB}, \quad A < B. \quad (5.7)$$

Let $\gamma = 1 - a^2/8$ and $T_{\max} = (\sigma k + \eta)(k - 1)/2$. Since $\sigma \geq \eta$, $2T_{\max} < \sigma k^2$. The rational inequalities

$$\begin{aligned} t^2 &< 1 - \frac{\sigma}{K+H}, \quad t\eta\gamma > 1, \quad \sigma\gamma - \eta > 0, \quad t\eta > 1, \\ H - \sigma/t &> 0, \quad \sigma - \eta > 0, \quad \eta/t > 1 \end{aligned} \quad (5.8)$$

imply $t \leq \Phi'_v(T) \leq 1/t$ on every digit interval. They are verified by direct rational substitution in the certificate accompanying Lemma 5.1.

Fine steps have length gap at least $t\eta\gamma > 1$. Carries have digit increment at least $(\sigma\gamma - \eta)k + \eta$, giving a length gap greater than one. Successive macro labels differ by one, so their baseline gap is Hk^2 , and the separation between blocks is at least

$$Hk^2 - 2T_{\max}/t = (H - \sigma/t)k^2 + (\sigma - \eta)k/t + \eta/t > 1.$$

Thus all cross-edge lengths are mutually strictly separated.

5.3 Local lengths and the mixed gap

For a local edge, write $d = j - i$ and $g = y_{Aj} - y_{Ai}$. Then

$$(2d - 1)p < g < (2d + 1)p, \quad g \equiv d(2i + d + 2kA) \pmod{p}.$$

By Lemma 2.3, now with $m = 6$ and $L = 2$, all these differences are distinct positive integers and their lags are nondecreasing in g .

The local lengths are now

$$L(d, g) = \sqrt{H_d^2 + B(g)^2}, \quad H_d = 2R_0 \sin \frac{ad}{2k}, \quad B(g) = 2R_1 \sin \frac{bg}{4pk}.$$

Both factors increase over their relevant ranges. In comparing two ordered ruler differences, keep the smaller lag d_1 fixed. Since $g \geq pd_1$ and the angle in B lies in $(0, b/2)$, sine concavity gives

$$B(g) \geq r_1 k \sin(b/2) d_1, \quad B'(g) \geq \frac{r_1 b \cos(b/2)}{12C}, \quad H_{d_1} \leq r_0 a k d_1.$$

It follows that

$$\frac{\partial L(d_1, g)}{\partial g} \geq \frac{r_1 b \cos(b/2) / (12C)}{\sqrt{1 + [r_0 a / (r_1 \sin(b/2))]^2}} > 1. \quad (5.9)$$

The last inequality is an exact rational consequence of the alternating Taylor bounds, with $z = b/2 \in (0, 1)$,

$$z - z^3/6 \leq \sin z \leq z - z^3/6 + z^5/120, \quad \cos z \geq 1 - z^2/2 + z^4/24 - z^6/720.$$

The square of the lower bound in (5.9) exceeds one by a strictly positive rational, checked in the same certificate. Apply Lemma 2.4 with $c = r_0 a / [r_1 \sin(b/2)]$ and $v = r_1 b \cos(b/2) / (12C)$. The integer gap in g proves strict separation of all local lengths.

The minimum local distance is at least $r_1 k \sin(b/2) > 1$. Using the same Taylor upper bound and $\sin u \leq u$, every local length is less than $18k^2$. For cross lengths, $2|T|/v < \sigma/(K+H) < 1 - t^2$ gives $\Phi_v(T) > tv \geq t(K+H)k^2 > (37/2)k^2$. Thus the mixed gap is greater than $k^2/2 > 1$. All pair distances are positive, so the indexing is injective.

5.4 Diameter and the prime-first sizes

The cross lengths, and hence all lengths, are less than

$$[K + 15H + \sigma/(2t)]k^2.$$

For $n = 6k$, the corresponding coefficient is

$$\frac{K + 15H + \sigma/(2t)}{36} = \frac{697537}{699120} < \frac{499}{500}. \quad (5.10)$$

The diameter is the largest of these finitely many edge lengths.

For every sufficiently large prime p , let $k = \lfloor (p-1)/6 \rfloor$. Then $6k < p \leq 6(k+1) \leq 6Ck$ whenever $k \geq 100$. Infinitude of primes gives unbounded such k . Choosing a large enough prime makes $6k \geq N$ and proves Theorem 1.4. No prime number theorem or short-interval prime estimate is required for this theorem.

5.5 Every sufficiently large size

To prove Corollary 1.5, set $k = \lceil n/6 \rceil$ and $x = 6k$. Dusart's explicit bound [14, Proposition 6.8] supplies a prime

$$x < p \leq x \left(1 + \frac{1}{25(\log x)^2} \right) \quad (x \geq 396738).$$

Here $x \geq n \geq 400000$ and $\log x > 2$, so $6k < p < 1.01(6k)$. Thus (5.2) holds and $k \geq 100$. The preceding construction gives $6k$ points, with strict edge-length separation and diameter $< 0.998(6k)^2$. Delete points until exactly n remain. All separation properties persist and the diameter cannot increase. Since $6k \leq n + 5$ and

$$\frac{999}{1000} - \frac{499}{500} \left(1 + \frac{5}{400000} \right)^2 = \frac{3120159501}{3200000000000} > 0,$$

the diameter is $< 0.999n^2$. The threshold is deliberately conservative.

6 Formalization and provenance

The two-circle family, its uniform form, and the prime-first dimension-29 theorem have corresponding Lean 4 declarations, listed in Table 3. Both formalizations use Lean 4.33.1 and the same locked Mathlib revision; the exact source identities are in Section C. Their recorded transitive axiom reports contain only `propext`, `Classical.choice`, and `Quot.sound`. The epsilon-family endpoints have only the numerical hypotheses of Theorem 1.2 and Corollary 1.3; the geometric and operator estimates are proved within the development.

Mathematical conclusion	Exact Lean declaration
Two-circle family	<code>Erdos670EpsilonFamily.epsilon_family</code>
Uniform dimension bound	<code>Erdos670EpsilonFamily.uniform_family</code>
R29 indexed configuration	<code>Erdos670R29.r29_unbounded_counterexamples</code>
R29 finite-set statement	<code>Erdos670R29Minimal.finite_set_counterexamples</code>
R29 conjecture negation	<code>Erdos670R29Minimal.not_fixedDimensionStatement_r29</code>

Table 3: The exact endpoint names. The finite-set translation and the indexed geometric theorem are distinct declarations.

The indexed R29 theorem and its finite-set adapter have different roles: the first constructs the points, and the second transports that result to the fixed-dimensional proposition (1.3). The adapter uses the exact statement extracted from the pinned Formal Conjectures source. Its unit edge-separation inequality is weak, as in Theorem 1.4, although the written estimates above give strict separation. The earlier public $\mathbb{R}^{81794}$ theorem has a separate proof and statement adapter [10, 13].

The public repository records the statement headers, axiom audits and source-specific build receipts; the complete newer Lean proof closures remain private. This paper reports those existing records and does not claim a new Lean execution or independent mathematical review. The Palomar and Prove2Me records apply only to the archived 0.74 theorem [13].

The quadratic-residue ruler is classical in the Erdős–Turán tradition [2]. Gram realization, positive-semidefinite completion and the operator estimates use classical methods [5, 6, 7, 8]; orthogonal-circle geometry also appears in Lenz-type constructions [9]. The colex estimates are inherited from the earlier paper. The present construction separates the digit and ruler circles and uses an explicit six-vertex label pattern for the dimension-29 result.

7 What remains open

The coefficient-one fixed-dimensional proposition is refuted by a single fixed dimension. Its failure does not determine the exact asymptotic diameter in each dimension. Three concrete questions remain:

1. Does the coefficient-one bound fail in the plane, or in another prescribed dimension below 29?
2. Can one fixed dimension support unit-separated edge lengths with diameter $(1/2 + o(1))n^2$?
The dimension-dependent tolerance family does not answer this question.
3. What is the best tradeoff between dimension, the all-size threshold, and the diameter coefficient?

The finite macro certificate and uniform colex estimates optimize different aspects of that tradeoff.

No lower bound on the dimension required by an arbitrary construction is claimed here. Nor do the displayed constants establish optimality within all two-circle constructions.

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A Proof of the macro input

We prove Lemma 3.1, following the first version's macro estimates [4, Lemmas 2–3]. None of the cluster geometry of the main text is used here. The plan is as follows.

1. Compute the Gram matrix Γ of the *increments* $q_j - q_{j-1}$ that the colex distances would require (Section A.1).
2. Show that Γ is dominated by its tridiagonal part, and that this part contains two “reserves” of positivity: one large enough to pay for the shortening by $2\rho^2$, and one diagonal reserve (Section A.2). This proves the Euclidean realization.
3. Use the diagonal reserve to bound the vectors F_A, G_A all at once, through their projections on the increments (Section A.3). This proves the whole-matrix estimates.

Throughout, $m \geq 4$ and K, ρ satisfy (3.2). Since $H_m \geq 1$, the harmonic gate implies $K \geq 16384 > 10$, as used below. Vectors $x \in \mathbb{R}^{m-1}$ are indexed $x_1, \dots, x_{m-1}$, and e_A ($0 \leq A < m$) denotes the standard basis of $\mathbb{R}^m$.

A.1 The increment Gram matrix

Let ∂ be the $(m-1) \times m$ matrix whose j -th row is $e_{j-1}^\top - e_j^\top$ ($1 \leq j < m$). For a symmetric $m \times m$ matrix $\mathbf{D}$ with zero diagonal, the matrix $-\frac{1}{2}\partial\mathbf{D}\partial^\top$ is the Gram matrix that the increments must have: if points q_A satisfy $\|q_A - q_B\|^2 = \mathbf{D}_{AB}$, then $\langle q_i - q_{i-1}, q_j - q_{j-1} \rangle = (-\frac{1}{2}\partial\mathbf{D}\partial^\top)_{ij}$, by the polarization identity. This is Schoenberg's Euclidean-distance criterion in consecutive-increment coordinates [5, Theorem 1].

Let $\mathbf{D}$ have entries D_{AB}^2 off the diagonal, put $\Gamma = -\frac{1}{2}\partial\mathbf{D}\partial^\top$, and write

$$D_j = D_{j-1,j} = K + \frac{j(j+1)}{2} \quad (1 \leq j < m)$$

for the lengths of the consecutive macro edges. Since $D_{j-1,j+1} = D_j + j$ and $D_{j,j+1} = D_j + j + 1$, a direct computation gives

$$\Gamma_{jj} = D_j^2, \quad \Gamma_{j,j+1} = -c_j, \quad \text{where } c_j = \frac{1}{2}(D_j + 1)^2 + j. \quad (\text{A.1})$$

If $i \leq j-2$, the four lengths $D_{i-1,j-1}, D_{i-1,j}, D_{i,j-1}, D_{ij}$ have the form $\alpha, \beta, \alpha+1, \beta+1$ with $\beta - \alpha = j-1$, and therefore $\Gamma_{ij} = -(j-1)$. In other words, $\Gamma_{ij} = -(\max\{i, j\} - 1)$ whenever $|i - j| \geq 2$.

A.2 Two reserves of positivity

Split $\Gamma = B - N$, where the *band* B keeps the diagonal and the two neighbouring diagonals of Γ , and the *far part* N has entries $\max\{i, j\} - 1$ for $|i - j| \geq 2$ and zero otherwise. Completing squares in $-2c_j x_j x_{j+1}$ gives

$$B[x] = \sum_{j=1}^{m-2} c_j (x_j - x_{j+1})^2 + \sum_{j=1}^{m-1} \alpha_j x_j^2 + c_{m-1} x_{m-1}^2, \quad (\text{A.2})$$

where c_{m-1} is given by the same formula as the other c_j , $\alpha_j = D_j^2 - c_j - c_{j-1}$ with $c_0 = 0$, and explicitly

$$\alpha_1 = \frac{K^2}{2} - 2, \quad \alpha_2 = -4, \quad \alpha_j = (j-2)K + \frac{j^3}{2} - j^2 - 2j \quad (j \geq 3).$$

The band dominates a diagonal form. Define $\Omega = \text{diag}(\omega_1, \dots, \omega_{m-1})$ by

$$\omega_1 = \omega_2 = \frac{K^2}{24}, \quad \omega_j = \frac{Kj + j^3}{24} \quad (j \geq 3).$$

We claim $B \succeq \Omega$. The terms of (A.2) involving only x_1, x_2 are $c_1(u-v)^2 + \alpha_1 u^2 - 4v^2$, with $u = x_1$, $v = x_2$. Since $c_1, \alpha_1 \geq K^2/4$ and $(u-v)^2 + u^2 \geq (u^2 + v^2)/3$,

$$c_1(u-v)^2 + \alpha_1 u^2 - 4v^2 \geq \frac{K^2}{12}(u^2 + v^2) - 4v^2 \geq \frac{K^2}{24}(u^2 + v^2)$$

for $K \geq 10$. For $j \geq 3$ we have $\alpha_j \geq \omega_j$, because

$$24(\alpha_j - \omega_j) = (23j - 48)K + 11j^3 - 24j^2 - 48j,$$

which is positive at $j = 3$ (it equals $21K - 63$), and has both terms nonnegative for $j \geq 4$. All remaining terms of (A.2) are nonnegative.

The far part is small. Put $\sigma_0 = \sum_j \omega_j^{-1}$ and $\sigma_2 = \sum_j j^2 \omega_j^{-1}$. For $K \geq 10$,

$$\sigma_0 \leq \frac{48}{K^2} + \frac{24H_m}{K} \leq \frac{29H_m}{K}, \quad \sigma_2 \leq \frac{120}{K^2} + 24H_m \leq 26H_m.$$

With $b_0 = \sum_j |x_j|$ and $b_1 = \sum_j j|x_j|$ we have $|N[x]| \leq \sum_{i,j} \max\{i, j\} |x_i| |x_j| \leq 2b_0 b_1$, while weighted Cauchy-Schwarz gives $b_0^2 \leq \sigma_0 \Omega[x]$ and $b_1^2 \leq \sigma_2 \Omega[x]$. Therefore

$$|N[x]| \leq 2\sqrt{\sigma_0 \sigma_2} \Omega[x] \leq \frac{\sqrt{754}}{64} \Omega[x] \leq \frac{1}{2} \Omega[x] \leq \frac{1}{2} B[x], \quad (\text{A.3})$$

using $K \geq 16384H_m^2$ (so that $\sqrt{29 \cdot 26} H_m / \sqrt{K} \leq \sqrt{754}/128$) and $754 < 1024$. Hence

$$\Gamma \succeq \frac{1}{2} B.$$

First reserve: a path energy. Let Π be the quadratic form

$$\Pi[x] = D_1^2 x_1^2 + \sum_{j=1}^{m-2} D_j^2 (x_j - x_{j+1})^2 + D_{m-1}^2 x_{m-1}^2.$$

In (A.2), bound $-4x_2^2 \geq -8x_1^2 - 8(x_2 - x_1)^2$. Since $\alpha_1 - 8 \geq D_1^2/4$, $c_1 - 8 \geq D_1^2/4$, $c_j \geq D_j^2/4$ for all j , and $\alpha_j \geq 0$ for $j \geq 3$, this proves

$$B \succeq \frac{1}{4} \Pi. \quad (\text{A.4})$$

Second reserve: a diagonal. For every j ,

$$D_j^2 \leq 36(K+m)\omega_j, \quad \text{hence} \quad B \succeq \Omega \succeq \frac{\text{diag}(D_1^2, \dots, D_{m-1}^2)}{36(K+m)}. \quad (\text{A.5})$$

For $j = 1, 2$ the first inequality is immediate. For $j \geq 3$, use $D_j \leq K + j^2 \leq (K+m)j$ to get the stronger bound $D_j^2 \leq (K+j^2)(K+m)j = (K+m)(Kj+j^3)$.

Paying for the shortening. Shortening every squared distance by $2\rho^2$ replaces $\mathbf{D}$ by $\mathbf{D}_\rho$, with entries $D_{AB}^2 - 2\rho^2$ off the diagonal. Since the rows of ∂ sum to zero, the new increment Gram matrix is

$$\Gamma_\rho = -\frac{1}{2}\partial\mathbf{D}_\rho\partial^\top = \Gamma - \rho^2\partial\partial^\top.$$

Now $\partial\partial^\top[x] = x_1^2 + \sum_j (x_j - x_{j+1})^2 + x_{m-1}^2 \leq \Pi[x]/K^2$, because every $D_j \geq K$. As $\rho \leq K/4$, the subtraction costs at most $\Pi/16 \preceq B/4$, by (A.4). Combining this with $\Gamma \succeq B/2$ and the two reserves,

$$\Gamma_\rho \succeq \frac{1}{4}B \succeq \frac{1}{16}\Pi \succ 0, \quad \Gamma_\rho \succeq \frac{\text{diag}(D_1^2, \dots, D_{m-1}^2)}{144(K+m)}. \quad (\text{A.6})$$

Here $\Pi \succ 0$ because $\Pi \succeq K^2\partial\partial^\top$ and ∂ has full row rank.

Euclidean realization. Since $\Gamma_\rho \succ 0$, we may write $\Gamma_\rho = \mathbf{V}^\top \mathbf{V}$ for an invertible $(m-1) \times (m-1)$ matrix $\mathbf{V}$ with columns $v_1, \dots, v_{m-1}$ (for instance, by Cholesky factorization). Put $q_0 = 0$ and $q_A = v_1 + \dots + v_A$. For $A < B$ let $u_{AB} = \sum_{j=A+1}^B e_j \in \mathbb{R}^{m-1}$; then $\partial^\top u_{AB} = e_A - e_B$, and so

$$\|q_B - q_A\|^2 = \Gamma_\rho[u_{AB}] = -\frac{1}{2}(e_A - e_B)^\top \mathbf{D}_\rho(e_A - e_B) = D_{AB}^2 - 2\rho^2.$$

The increments $v_j = q_j - q_{j-1}$ are linearly independent, so the points q_A are affinely independent.

A.3 Bounding all projections at once

Let $Y_0, \dots, Y_{m-1} \in \mathbb{R}^{m-1}$ be vectors, and record their projections on the increments in the $(m-1) \times m$ matrix

$$Z_{jA} = \langle q_j - q_{j-1}, Y_A \rangle, \quad \text{that is,} \quad Z = \mathbf{V}^\top Y.$$

As $\mathbf{V}$ is invertible, Z determines Y , and $Y^\top Y = Z^\top (\mathbf{V}^\top \mathbf{V})^{-1} Z = Z^\top \Gamma_\rho^{-1} Z$. Writing $\mathbf{P} = \text{diag}(D_1, \dots, D_{m-1})$, the second inequality in (A.6) gives $\Gamma_\rho^{-1} \preceq 144(K+m)\mathbf{P}^{-2}$, and hence

$$\|Y\|_{\text{op}} \leq 12\sqrt{K+m} \|\mathbf{P}^{-1}Z\|_{\text{op}}. \quad (\text{A.7})$$

So it suffices to show that the projection matrices of F and G , normalized by $\mathbf{P}^{-1}$, have bounded operator norm.

Taking consecutive differences of the defining equations (3.3) (with the convention that the right-hand sides vanish for $J = A$) gives, for $1 \leq j < m$ and $0 \leq A < m$,

$$\begin{aligned} Z_{jA}^F &= D_j \mathbf{1}_{A=j} - \mathbf{1}_{A>j}, \\ Z_{jA}^G &= -D_j \mathbf{1}_{A=j-1} - (j-1) \mathbf{1}_{A<j-1}. \end{aligned}$$

For example, for $A > j$ we get $Z_{jA}^F = D_{j-1,A} - D_{jA} = -1$, because the colex labels of $\{j-1, A\}$ and $\{j, A\}$ differ by one. The columns F_0 and G_{m-1} vanish. Delete them, and index the remaining

columns by $1, \dots, m-1$ (column A of Z^F , and column $A-1$ of Z^G , become column A). The two projection matrices become

$$Z_{\text{fine}} = P - E_+, \quad Z_{\text{coarse}} = -(P + E_-),$$

where $(E_+)_{ij} = \mathbf{1}_{i < j}$ and $(E_-)_{ij} = (i-1)\mathbf{1}_{j < i}$.

For the first matrix, the Frobenius norm gives $\|P^{-1}E_+\|_{\text{op}} \leq m/K \leq 1$, as all entries of $P^{-1}E_+$ are at most $1/K$. For the second,

$$0 \leq (P^{-1}E_-)_{ij} = \frac{i-1}{D_i} \mathbf{1}_{j < i} \leq \frac{2}{i} \mathbf{1}_{j < i}$$

since $D_i \geq i(i+1)/2$. The finite Hardy matrix $\mathcal{H}_{ij} = i^{-1}\mathbf{1}_{j < i}$ has operator norm at most 2 by the weighted Schur test [6, 7]. For completeness, take the weights $w_j = j^{-1/2}$; by comparison with integrals,

$$\sum_{j < i} \frac{w_j}{i} \leq 2w_i, \quad \sum_{i > j} \frac{w_i}{i} \leq 2w_j.$$

For any real vector x , Cauchy–Schwarz with these weights then gives

$$\sum_i \left(\sum_j \mathcal{H}_{ij} x_j \right)^2 \leq \sum_i \left(\sum_j \mathcal{H}_{ij} w_j \right) \left(\sum_j \mathcal{H}_{ij} \frac{x_j^2}{w_j} \right) \leq \sum_j \frac{x_j^2}{w_j} \sum_i 2w_i \mathcal{H}_{ij} \leq 4 \sum_j x_j^2.$$

A matrix with nonnegative entries bounded by those of $2\mathcal{H}$ has operator norm at most $2\|\mathcal{H}\|_{\text{op}}$ (apply it to $|x|$), so $\|P^{-1}E_-\|_{\text{op}} \leq 4$.

Thus $\|P^{-1}Z_{\text{fine}}\|_{\text{op}} \leq 2$ and $\|P^{-1}Z_{\text{coarse}}\|_{\text{op}} \leq 5$, and (A.7) gives $\|F\|_{\text{op}} \leq 24\sqrt{K+m}$ and $\|G\|_{\text{op}} \leq 60\sqrt{K+m}$; restoring the zero columns does not change the norms. This proves (3.4) and completes the proof of Lemma 3.1. $\square$

B Reconstructing the rational certificate

The matrices in (5.4) are determined entirely by the displayed rational parameters and label matrix. Starting with a symmetric matrix $A^{(0)}$, successive elimination uses

$$d_k = A_{kk}^{(k-1)}, \quad A_{ij}^{(k)} = A_{ij}^{(k-1)} - \frac{A_{ik}^{(k-1)} A_{jk}^{(k-1)}}{d_k} \quad (i, j > k).$$

If every d_k is positive, this is an LDL^T decomposition with positive diagonal, so the matrix is positive definite. All inputs and operations are rational. The following complete Python program reconstructs all three matrices, checks the fifteen positive pivots and the scalar guards, and prints exact fractions and the lower bounds in Table 2. It uses only the Python standard library; checks are not removable assert statements. Running under python3 -0 gives the same output. No finite sampling in k or floating-point estimate enters this certificate. The passage to every $k \geq 100$ is the operator-norm convexity argument in (5.5).

```
"""Exact reconstruction of the three matrices in the dimension-29 proof."""
from fractions import Fraction as F
import json

K, H, eta, sigma = F(457,25), F(569,500), F(533,500), F(138,125)
```

```

a, b, r0, r1 = F(519,1000), F(879,500), F(507,100), F(1131,100)
t, C, m = F(971,1000), F(101,100), 6
labels = [[0,3,5,7,14,15], [3,0,2,4,13,12], [5,2,0,1,11,10],
          [7,4,1,0,9,8], [14,13,11,9,0,6], [15,12,10,8,6,0]]
D = [[K+H*labels[i][j] if i != j else F(0)
      for j in range(m)] for i in range(m)]
E = [[D[i][j]**2-2*(r0*r0+r1*r1) if i != j else F(0)
      for j in range(m)] for i in range(m)]
B = [[(E[0][i]+E[0][j]-E[i][j])/2 for j in range(1,m)]
      for i in range(1,m)]
PG = [[D[i][j] if i<j else F(0) for j in range(m)]
       for i in range(m)]
PF = [[D[i][j]-PG[i][j] for j in range(m)] for i in range(m)]
TG = [[PG[A][0]-PG[A][j] for A in range(m)] for j in range(1,m)]
TF = [[PF[A][0]-PF[A][j] for A in range(m)] for j in range(1,m)]

def pivots(matrix):
    R = [row[:] for row in matrix]
    result = []
    for k in range(len(R)):
        p = R[k][k]
        if p <= 0:
            raise ValueError("nonpositive pivot")
        result.append(p)
        for i in range(k+1,len(R)):
            for j in range(i,len(R)):
                R[j][i] -= R[i][k]*R[j][k]/p
                R[i][j] = R[j][i]
    return result

matrices = {"B": B}
for name,x in [(("M(0)",F(0)), ("M(1/100)",F(1,100)))]:
    T = [[sigma*TG[i][j]+eta*x*TF[i][j] for j in range(m)]
          for i in range(m-1)]
    matrices[name] = [[B[i][j]-sum(T[i][h]*T[j][h] for h in range(m))/(a*r0)**2
                              for j in range(m-1)] for i in range(m-1)]

def simple_lower_bound(v):
    q = F(1)
    while q >= v:
        q /= 2
    return q

rows = {name: {"pivots": list(map(str,pivots(A))),
              "lower_bounds": list(map(str,map(simple_lower_bound,pivots(A))))}
        for name,A in matrices.items()}
z, gamma = b/2, 1-a*a/8
sinlo, sinhi = z-z**3/6, z-z**3/6+z**5/120
coslo = 1-z*z/2+z**4/24-z**6/720
margins = [1-sigma/(K+H)-t*t, t*eta*gamma-1,
            sigma*gamma-eta, t*eta-1, H-sigma/t,
            (sigma-eta)/t, eta/t-1,
            (r1*b*coslo/(2*C*m))**2/(1+(r0*a/(r1*sinlo))**2)-1,
            r1*sinlo*100-1,
            F(18)**2-((r0*a)**2+(2*r1*sinhi)**2),
            (K+H)*t-F(37,2),
            F(499,500)-(K+15*H+sigma/(2*t))/36]
if any(v <= 0 for v in margins):
    raise ValueError("nonpositive scalar margin")
if sorted(labels[i][j] for i in range(m) for j in range(i+1,m)) != list(range(1,16)):
    raise ValueError("wrong macro labels")
print(json.dumps({"status":"PASS", "matrices":rows,
                  "positive_scalar_margins":list(map(str,margins))},indent=2))

```

C Source correspondence

The following source identities specify the written construction and formal declarations used in this paper; their mathematical correspondence is described in [Section 6](#).

Source object	Immutable Git blob identity
Two-circle written proof	63fdaf3adb4357e39781a9020d8815a4d0de4dca
Epsilon-family formal source	0670e6c3af874912519aaca79dd6a9b77f16e22
R29 indexed formal source	8d09609eef219795b3dce076f3dc10c50a901ee0
R29 minimal exact adapter	8ac23307f1491887b81d8a6edf67e80a709929fb
R29 exact matrix inputs	0eea70459e28e7d23ea73f22edf202f37d6eea5c

Table 4: Selected exact source identities, retained for version matching. A blob identity identifies source bytes; it is not an execution receipt.

The locked Mathlib revision for the newer endpoints is

0df444a360eaa60ab8c11dca51a86af692955474.

The extracted Formal Conjectures statement is pinned at project revision

c77fdd446d23539302aa73bcfae4a9fa962b8382.

Its source blob is 1ae473043edcd960c0b9eed81ccd6e82994f2a22. The public repository preserves the corresponding statement headers, axiom audits and build receipts separately from the paper source.

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