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Concluding the logarithmic Kazhdan–Lusztig conjecture

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Abstract

In this article, we conclude the proof of the logarithmic Kazhdan–Lusztig conjecture, which was originally posed in the work of Feigin, Gainutdinov, Semikhatov, and Tipunin [FGST06], for the Feigin–Tipunin vertex algebras $\mathcal{W}_{\sqrt{p}Q}$ in all finite Lie types. We establish two missing ingredients on the vertex-algebra side and refine the categorical reconstruction of Creutzig, Lentner, and Rupert [CLR23, Len25] to accommodate quasi-monoidal splitting functors. First, we prove that the category of grading-restricted generalized modules is a factorizable finite ribbon category, by establishing C_2-cofiniteness through a null-cone argument inspired by classical invariant theory and applying McRae's rigidity theorem. Second, we show that quantum dimensions agree with Frobenius–Perron dimensions. Both arguments rely on a distinguished invariant field, analogous to a higher Segal–Sugawara vector, studied through the $\hbar$-deformed Zhu algebra. Its specializations at $\hbar = 0$ and $\hbar = 1$ control C_2-cofiniteness and identify the unique simple module of least conformal weight, respectively. These results complete the categorical reconstruction and yield the desired ribbon equivalence with the relative Drinfeld center of the category of comodules of the Nichols algebra of screening operators.

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Introduction

Background

A *vertex operator algebra* V is, very roughly speaking, a commutative algebra whose multiplication depends analytically on a complex variable z. It is a mathematical axiomatization of the chiral parts of a 2-dimensional conformal quantum field theory. Mathematical textbooks on vertex algebras include [Kac97, FBZ04].

There is a notion of *vertex module* for which, under suitable assumptions, there is a tensor product with a commutativity law, so to a well-behaved vertex operator algebra V there is associated a *braided tensor category* $\mathrm{Rep}(V)$, see [HL94, HLZ06]. The data visible at the categorical level still retain some of the analytic aspects of the vertex operator algebra. For example, the braiding is related to the multivaluedness of certain analytic functions and the $\mathrm{SL}_2(\mathbb{Z})$-action on certain Hom-spaces captures the modular transformation behavior of the q-characters attached to vertex modules. These are particular instances of a topological field theory picture, which becomes more involved in the non-semisimple case.

On the algebra side, a very interesting class of braided tensor categories is given by representations of the *quantum group* $U_q(\mathfrak{g})$, which is the universal enveloping algebra of a semisimple complex finite-dimensional Lie algebra $\mathfrak{g}$, deformed by a generic complex parameter q. Its category of finite-dimensional type 1 representations is a semisimple braided tensor category with infinitely many simple objects, which is equivalent as an abelian category to the category of finite-dimensional representations of $\mathfrak{g}$, but the braiding and associator are nontrivial expressions involving q. They are related to the monodromy of the Knizhnik–Zamolodchikov partial differential equation, which was the original motivation for defining quantum groups in [Dri89].

The *Kazhdan–Lusztig correspondence* [KL93] is an equivalence of braided tensor categories of representations of $U_q(\mathfrak{g})$ and of a vertex operator algebra associated to the affine Lie algebra $\hat{\mathfrak{g}}$ at a level κ related to q, at generic values. In some sense, this gives a physical context to the observations above.

If q is specialized to an ℓ-th root of unity, one obtains the finite-dimensional *small quantum group* $u_q(\mathfrak{g})$ [Lus89]. Its category of representations is a nonsemisimple finite tensor category, related to the representations of $\mathfrak{g}$ in finite characteristic.

The *logarithmic Kazhdan–Lusztig correspondence* [FGST06, FT10, AM14] gives a conjectural link between two representation categories. On one side one has the vertex operator algebra $\mathcal{W}_{\sqrt{p}Q}$, defined inside the lattice vertex algebra $V_{\sqrt{p}Q}$ of the rescaled root lattice Q, as the common kernel of screening operators. On the other side one has a quasi-version of the small quantum group $\tilde{u}_q(\mathfrak{g}^\vee)$ at $q = e^{\frac{\pi \mathrm{i}}{p}}$. The theorem below specifies the latter category by a relative center.

Theorem A (Logarithmic Kazhdan–Lusztig correspondence)**.** *Let Q be the root lattice of $\mathfrak{g}$, with lacing number t. Take $p = tm$ with $m \in \mathbb{Z}$ and $m \geq \max\{2, (\rho^\vee, \theta_s)\}$, where $\rho^\vee$ is the half-sum of the positive coroots and θ_s is the highest short root (the highest root in simply laced type). There is an equivalence of braided tensor categories*

$$\mathrm{Rep}_{\mathrm{gr}}(\mathcal{W}_{\sqrt{p}Q}) \simeq \mathcal{Z}_{\mathcal{C}^{\mathrm{rev}}}\left(\mathrm{Comod}_{\mathcal{C}^{\mathrm{rev}}}(\mathfrak{B}(q))\right).$$

Here $\mathcal{C} = \mathrm{Rep}(V_{\sqrt{p}Q})$ and $\mathfrak{B}(q)$ is the Nichols algebra in $\mathcal{C}^{rev}$ associated to the braiding matrix $q_{ij} = e^{-\pi \mathrm{i}(\alpha_i^\vee, \alpha_j^\vee)/p}$. With the lattice ribbon structure of Theorem 7.1, this is a ribbon equivalence.

In fact, one can remove the restrictions on p from [SL26], which we will address in a forthcoming version. This conjecture has been solved in the case $\mathfrak{sl}_2$ by the cumulative work of many authors [FGST06, AM08, TW13, CGR20, CLR21, GN26, CLR23]. Main contributions to the case of general $\mathfrak{g}$ are [FGST06, FT10, AM14, Len21, Sug21, SL26, CLR23, Len25].

Goal of this article

The present article builds on the approach we developed in [CLR23, Len25], which is outlined below and provides an essentially algebraic proof of the logarithmic Kazhdan–Lusztig conjecture by reconstructing the braided tensor category $\mathcal{D}$ from its free-field realization $V_{\sqrt{p}Q}$ (taking the role of a Cartan part), from the Nichols algebra of screening operators $\mathfrak{B}(q)$ (taking the role of a nilpotent part) and from a Drinfeld center construction (which provides the second nilpotent part and the braiding). This is part of a much more general conjectural picture.

The proof in those works is conditional in that it crucially relies on enough categorical structure being established on the vertex algebra side, starting with the very existence of the tensor product and including rigidity. These are notoriously difficult questions, see for example recent progress in [CY21, CJOH$^+$21, McR24, CMY21, CMY22, CMY23, NORW24]. The third main issue is to prove that the constructed category encompasses the entire representation theory of the vertex algebra in question, that is, there are no additional simple modules, extensions, etc.

The present article is an initiative of A. Riesen to settle these missing points with the use of AI and thereby conclude the proof of the logarithmic Kazhdan–Lusztig conjecture. As it turns out, this uncovers new constructions both on the vertex algebra and the categorical side that we expect are of independent interest to the community.

Content of this article

In Section 1, we recall the necessary preliminaries and establish our notation. In Section 2, we prove the C_2-cofiniteness of the Feigin–Tipunin algebra $\mathcal{W}_{\sqrt{p}Q}$ in the desired parameter range. As a consequence, its category of grading-restricted generalized modules admits the structure of a finite braided vertex tensor category; see, for example, [HLZ06].

Our starting point is the chain of inclusions

$$V_{\sqrt{p}Q} \supset W_{\sqrt{p}Q} \supset U,$$

where $W_{\sqrt{p}Q}$ is the common kernel of the short screening operators, and $U = W^G_{\sqrt{p}Q}$ is the invariant subalgebra under the geometric action of G, identified with the principal W-algebra at level $1/p - h^\vee$. Under this action, we have the following decomposition

$$W_{\sqrt{p}Q} = \bigoplus_{\lambda \in P_+ \cap Q} L(\lambda) \otimes M_\lambda,$$

where $L(\lambda)$ is the irreducible G-module of highest weight λ, and M_λ is a simple U-module. The geometric action and this decomposition are in some sense the initial motivation for defining the vertex algebra. They have been rigorously established in the simply-laced case by Sugimoto [Sug21] Main Theorem (1) and in the non-simply-laced case by Sugimoto and Li [SL26, (3.1), Theorems 3.2 and 4.3].

Recall that the C_2-algebra of the principal W-algebra is $R_U \cong \mathbb{C}[\mathfrak{h}^*]^W$ [Ara15]. The inclusion $U \subset \mathcal{W}_{\sqrt{p}Q}$ induces a homomorphism of C_2-algebras

$$\rho : R_U \longrightarrow R_{\mathcal{W}_{\sqrt{p}Q}},$$

whose kernel plays a central role in our argument. Our proof of C_2-cofiniteness proceeds in two steps.

First, in Section 2.1, we use the Schur–Weyl decomposition of Sugimoto and Li, together with a conformal-weight argument, to show that only finitely many highest weights survive in the C_2-quotient. Consequently, $R_{\mathcal{W}_{\sqrt{p}Q}}$ is a finite R_U-module.

Second, in Section 2.2, we show that the kernel of ρ contains a finite-codimensional ideal. Our argument is motivated by a construction of I. Frenkel for level-one ADE lattice vertex algebras, further studied by Bakalov and Milanov [Fre85, BM13], which we recall as a motivating example. In particular, we construct an invariant field $\mathcal{Q}(z)$ whose evaluation at a character of the Heisenberg algebra, through the Miura map, is an exponential polynomial. The higher coefficients of this field give elements of $\ker \rho$. At any common zero of these coefficients, the exponential polynomial must reduce to an ordinary polynomial. By the linear independence of exponentials with distinct frequencies, together with the nonvanishing of their coefficient polynomials, this forces all frequencies to vanish and hence the character to be zero. Thus the common zero locus is the origin, and the kernel of ρ contains a finite-codimensional ideal. In fact, we formulate the relations of $\mathcal{Q}(z)$ uniformly in the $\hbar$-deformed Zhu algebra. The specialization at $\hbar = 0$ gives the C_2-relations above, while the specialization at $\hbar = 1$ provides the Zhu-algebra relations used in Section 3 to identify the unique simple module of least conformal weight.

This approach may be viewed as an algebraic analogue of the geometric approach of Creutzig, Nakatsuka, and Sugimoto [CNS24], who recover the C_2-cofiniteness of the rank-one triplet algebra through Drinfeld–Sokolov reduction and the intersection of the nilpotent cone with the principal Slodowy slice. We believe that a generalization of this geometric approach should provide an alternative proof of C_2-cofiniteness in higher rank.

The vertex tensor category $\mathrm{Mod}_{gr}(\mathcal{W}_{\sqrt{p}Q})$ is moreover *rigid* by a recent theorem of R. McRae [McR26b, Theorem 4.5], building on a general argument by Etingof and Penneys [EP26]. The requisite Grothendieck–Verdier duality for vertex algebra modules is discussed in [ALSW21]. This implies

> **Corollary 2.6.** The category $\mathcal{D} = \mathrm{Rep}_{\mathrm{gr}}(\mathcal{W}_{\sqrt{p}Q})$ of grading-restricted generalized $\mathcal{W}_{\sqrt{p}Q}$-modules is a factorizable finite ribbon category.

For an alternative approach to rigidity using the free-field extension one could use [CMSY24] together with the statement that every A-module has a composition series of local A-modules proven in [CN24] in a sufficiently unrolled situation. Note that we need rigidity for the entire category to prove bounds via Frobenius–Perron dimensions, not just those with a priori composition series of local A-modules.

In Section 3, we determine the Frobenius–Perron dimensions of the grading-restricted generalized modules of $\mathcal{W}_{\sqrt{p}Q}$ through their characters. A standard approach to studying quantum dimensions is to use modular transformations together with the existence of a simple module of globally minimal conformal weight; see, for example, [Zhu96, Hua08, DJX13, Miy04, AN13, Fio16, BM15, BM17, CM14, CMW17, CG17, GR19, Cre24] and references therein.

To establish the existence of such a minimum, we return to the invariant field $\mathcal{Q}(z)$ used in the proof of C_2-cofiniteness, now evaluated at $\hbar = 1$ in the deformed Zhu algebra. In contrast to the exponential polynomials appearing at $\hbar = 0$, its evaluation at a Heisenberg character involves expressions of the form $(1+z)^a$ multiplied by polynomials. The Zhu-algebra relations force their sum to be a polynomial, and an argument using monodromy around $z = -1$ shows that the exponents must be integers. This gives a real lower bound on the conformal weights of simple modules and allows us to identify the unique simple module attaining it.

We then use the character formulas of Sugimoto and Li [SL26] to compute the quantum dimensions of the lattice modules. To identify these with Frobenius–Perron dimensions, we use Miyamoto's modular invariance theorem for pseudotraces [Miy04] and McRae's open Hopf link formula [McR26a, Theorem 5.2]. These allow us to extract a nonnegative generalized eigenvector of a positive fusion matrix, which Perron–Frobenius theory identifies with the Frobenius–Perron dimension vector. We obtain the following result.

Theorem 3.5. *For every grading-restricted generalized $\mathcal{W}_{\sqrt{p}Q}$-module X, we have*

$$\mathrm{FPdim}_{\mathcal{D}}(X) = \lim_{y\downarrow 0} \frac{\mathrm{ch}_X(\mathrm{i}y)}{\mathrm{ch}_{\mathcal{W}_{\sqrt{p}Q}}(\mathrm{i}y)}.$$

In particular, for every $\gamma \in Q^*/pQ$,

$$\mathrm{FPdim}_{\mathcal{D}}(V_{\sqrt{p}Q+\gamma/\sqrt{p}}) = p^{N_l}(p/t)^{N_s} = D_p.$$

In Section 4 we recall the charges of the screening operators in $V_{\sqrt{p}Q}$, which are given by rescaled coroots, such that their braiding matrix q_{ij} and Nichols algebra are those of the quantum group of the Langlands dual Lie algebra $u_q(\mathfrak{g}^\vee)$ with $q^{2p} = 1$.

In Section 5 we repeat a construction in [CLR23, Len25] with improved methods:

In Section 5.1 we state a quasi-version of [LM25], which in itself is a relative version of Tannaka–Krein reconstruction and of the Radford projection theorem.

Lemma 5.1. *Let $\mathcal{C}$ be a braided fusion category, $\mathcal{B}$ a finite tensor category with a specified central $\mathcal{C}$-action, and $\mathfrak{l} : \mathcal{B} \to \mathcal{C}$ a faithful exact $\mathcal{C}$-module functor. Assume that $\mathfrak{l}$ has the structure of a quasi-monoidal functor and satisfies the compatibilities specified in (5.1), (5.2), (5.3) and (5.4). Then there is a coquasi-Hopf algebra $\mathfrak{N}$ in $\mathcal{C}$ such that*

$$\mathcal{B} \simeq \mathrm{Comod}_{\mathcal{C}}(\mathfrak{N}),$$

and $\mathfrak{l}$ is the functor forgetting the left coaction.

So the point of the new version is that $\mathfrak{l}$ needs to preserve tensor products, but not in a coherent way in terms of pentagon identities, and we reconstruct merely a coquasi-Hopf algebra. Note that the finiteness assumptions in this assertion are not essential and can be replaced by explicit weaker knowledge, see [CL26, Section 7.2].

In Section 5.2 we give a general situation in which the previous reconstruction Lemma applies:

Lemma 5.3. *Let $\mathcal{B}$ be a finite monoidal category and $\mathcal{C}$ a finite braided monoidal central subcategory, such that $\mathcal{C}$ is semisimple and contains all simple objects in $\mathcal{B}$. Then there is an exact faithful quasi-tensor functor*

$$\mathfrak{l} : \mathcal{B}^{\mathrm{split}} \longrightarrow \mathcal{C}, \qquad \mathfrak{l} \circ \iota \simeq \mathrm{id}_{\mathcal{C}}.$$

that sends an object to the direct sum of its composition factors.

In Section 6 we first follow [AAM17, AKM15] and discuss Nichols algebra cohomology and the following favourable properties which are essential for the Andruskiewitsch–Schneider program and play a prominent role in our context in [Len25, CL26]: We call the Nichols algebra $\mathfrak{B}(M)$ *robust* if any connected coquasi-Hopf algebra $\mathfrak{N}$ with $\mathrm{gr}\,\mathfrak{N} \cong \mathfrak{B}(M)$ is isomorphic to $\mathfrak{B}(M)$ up to a twist. This means that two obstructions vanish: One in $\mathrm{HH}^2_{\mathcal{C}}(\mathfrak{N}, \mathfrak{N}_1)$, which measures possible

deformations or liftings of the graded Nichols algebra to a merely filtered Hopf algebra, the other one is $\mathrm{HH}^3_{\mathcal{C}}(\mathfrak{N}, 1)$, which measures possible nontrivial quasi-structure.

In Corollary 6.6 and Lemma 6.7, which are new in the present article, we establish that this is the case for the Nichols algebra of screenings, which is the quantum Borel part of a small quantum group. Most of this is an extension of an argument in [AKM15] to the even-order case. The two particularly small cases $A_2, 2$ and $B_2, 4$ have to be settled by explicit computation.

We remark that it would also be possible to circumvent the cohomology calculation by going, as in [CLR23, Len25], to a slightly larger lattice, so that the degrees relevant to the relations and to the third cohomology remain separated. We have called such a situation *sufficiently unrolled*. Since the previous argument is more complicated but exposes more of the actual issue, we have decided to include it.

In Section 7 we conclude the proof of Theorem 7.1 along the lines of [Len25].

a) Rigid braided monoidal category: The C_2-cofiniteness and rigidity results give the finite ribbon category $\mathcal{D} = \mathrm{Rep}_{\mathrm{gr}}(\mathcal{W}_{\sqrt{p}Q})$.

b) Commutative algebra: We regard $A = V_{\sqrt{p}Q}$ as an exact commutative algebra in $\mathcal{D}$ and identify its local modules with the lattice category $\mathcal{C} = \mathrm{Rep}(V_{\sqrt{p}Q})$, indexed by $\Gamma = Q^*/pQ$.

c) Schauenburg functor: We have an equivalence $\mathcal{D} \simeq \mathcal{Z}_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{B})$ for $\mathcal{B} = \mathcal{D}_A$, with the specified central copy of $\mathcal{C}^{\mathrm{rev}}$.

d) Reconstruction and splitting: We identify $\mathcal{B}^{\mathrm{split}}$ with the category of left comodules over a connected coquasi-Hopf algebra $\mathfrak{N}$ in $\mathcal{C}^{\mathrm{rev}}$.

e) Primitives: The nonsplit screening extensions supply the primitive elements and the corresponding Nichols algebra realizing the split part of the category.

f) Frobenius–Perron bound: The character formula gives $\mathrm{FPdim}(A) = \dim \mathfrak{B}(q_{ij})$. Equality in the dimension bounds then gives $\mathcal{B}^{\mathrm{split}} = \mathcal{B}$ and $\mathrm{gr}\,\mathfrak{N} \cong \mathfrak{B}(q)$.

g) Removing quasi and lifting: The two algebraic assertions of Section 6 remove the extra reassociator and the filtered lifting while retaining the central structure and the ambient Cartan associator.

h) Twist: The lattice conformal twists determine a unique ribbon structure on the target. With this normalization the braided equivalence is a ribbon equivalence, including on extensions.

Use of AI

The entire article, with exceptions mentioned below, is understood and written by us. The main new arguments in Sections 2 and 3 were originally obtained by AI, subsequently we have understood, rearranged and considerably rewritten them, in particular with respect to a broader applicable principle and to alternative proof strategies we had in mind. The improvement of the arguments in Sections 5 and 6 was suggested by AI, and the technical argument in Section 6.2 extending [AAM17] to even orders and the calculation in Section 6.3 dealing with two exceptional cases were left as written by AI.

Part I
The Feigin–Tipunin vertex algebra

1 Preliminaries

We first recall the conventions used throughout the rest of the paper.

1.1 Vertex algebra conventions and notation

We work over $\mathbb{C}$. If $A(z) \in W[[z, z^{-1}]]$ is a formal power series, by $[z^j]A(z)$ we mean the coefficient of the z^j-term. A VOA of CFT type has $V = \bigoplus_{n\geq 0} V_n$ with $V_0 = \mathbb{C}\mathbf{1}$. We use the standard conventions $Y(v,z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$, $T = L_{-1}$. We write $\mathrm{Rep}_{\mathrm{gr}}(V)$ for the category of grading-restricted generalized modules. Our convention for the invariant bilinear form is

$$\langle Y(u,z)v, w\rangle = \langle v, Y(e^{zL_1}(-z^{-2})^{L_0}u, z^{-1})w\rangle.$$

When possible, we will normalize the bilinear form by $\langle \mathbf{1}, \mathbf{1}\rangle = 1$. For a conformally graded vertex algebra V, put

$$C_2(V) = \mathrm{span}\{u_{(-2)}v : u, v \in V\}, \qquad R_V = V/C_2(V). \tag{1.1}$$

We say that V is *C_2-cofinite* if $\dim_{\mathbb{C}} R_V < \infty$. The normally ordered product descends to multiplication on R_V, and the zeroth product gives its Poisson bracket. For a V-module M, the same construction gives the R_V-module

$$R_V(M) = M/\,\mathrm{span}\{u_{(-2)}v : u \in V,\ v \in M\}.$$

Following [Ara15], we say that M is strongly finitely generated if $R_V(M)$ is finitely generated over R_V in the usual commutative algebra sense. We will also need the $\hbar$-deformed Zhu algebra $A_\hbar(V)$ of a VOA, but delay its discussion to subsection 1.4 for expository reasons.

1.2 The Feigin–Tipunin vertex algebra

Let G be a simply connected simple complex group with simple finite-dimensional Lie algebra $\mathfrak{g}$, Cartan subalgebra $\mathfrak{h}$, root system Φ and Weyl group W. Put $r = \dim \mathfrak{h}$. Write $Q \subset P$ for the root and weight lattices, and P_+ for the dominant weights. We use the invariant form to identify the Cartan space with its dual. Long roots have squared length 2 and short roots have squared length $2/t$, where t is the lacing number. For a choice of positive roots, ρ is their half-sum and $\rho^\vee$ is the half-sum of their coroots, with $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. If M is a finite-dimensional G-module, we denote by M_λ its λ-weight space, for $\lambda \in \mathfrak{h}^*$.

We take $p = tm$, where m is an integer at least two. We assume the vacuum condition $(\rho^\vee, \theta_s) \leq m$ of Sugimoto and Li [SL26, Lemma 2.9(1)]. Here θ_s is the highest short root, or the highest root in simply laced type. With this normalization $\sqrt{p}\,Q$ is even. The corresponding lattice vertex algebra decomposes into Heisenberg Fock spaces $M(1)e^\beta$, indexed by $\beta \in \sqrt{p}Q$, where $M(1)$ is the free boson VOA based on the dual Cartan $\mathfrak{h}^*$ with its invariant bilinear form:

$$V_{\sqrt{p}Q} = \bigoplus_{\beta\in\sqrt{p}\,Q} M(1)e^\beta.$$

We write $b(n) = (b(-1)\mathbf{1})_{(n)}$ for a Heisenberg mode.

In the Feigin–Tipunin construction, we modify the usual lattice conformal vector of $V_{\sqrt{p}Q}$ so that the screening operators below have degree zero. Let $a = \sqrt{p}\rho - \frac{\rho^\vee}{\sqrt{p}} \in \mathfrak{h}^*$. Adding $a(-2)\mathbf{1}$ to the usual lattice conformal vector gives [SL26, (2.6) and (2.11)]

$$L_0 = L_0^{\mathrm{std}} - a(0), \qquad \Delta(\beta) = \tfrac{1}{2}(\beta,\beta) - (a,\beta), \tag{1.2}$$

where $\Delta(\beta)$ is the conformal weight of e^β, the lowest vector of $M(1)e^\beta$.

The two screening operators for each simple root α_i are given in the vacuum lattice module by

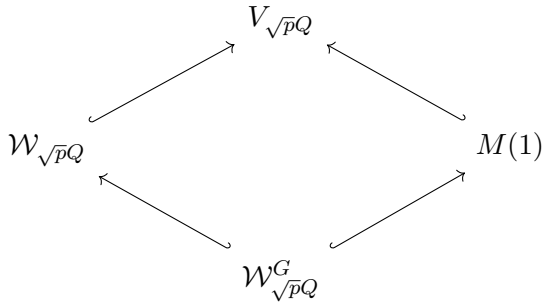
$$\begin{aligned} S_i^- &= \mathrm{Res}_z Y(e^{-\alpha_i^\vee/\sqrt{p}}, z) \quad &&: V_{\sqrt{p}Q} \longrightarrow V_{\sqrt{p}Q - \alpha_i^\vee/\sqrt{p}} \quad \text{(short)}, \\ S_i^+ &= \mathrm{Res}_z Y(e^{\sqrt{p}\,\alpha_i}, z) \quad &&: V_{\sqrt{p}Q} \longrightarrow V_{\sqrt{p}Q} \qquad\qquad \text{(long)}. \end{aligned} \tag{1.3}$$

Both exponential vectors have conformal weight one by (1.2). The short operators define the Feigin–Tipunin vertex algebra $\mathcal{W}_{\sqrt{p}Q} = \bigcap_{i=1}^r \ker S_i^-$ inside $V_{\sqrt{p}Q}$. The long operators preserve this common kernel. Together with the Cartan operators $-\alpha_i^\vee(0)/\sqrt{p}$, they give the action of a Borel subgroup B on it [SL26, (2.8), (2.12), Lemmas 2.10–2.11 and Definition 3.1]. In particular $M(1)e^{-\sqrt{p}\nu}$ has Cartan weight ν.

Feigin and Tipunin proposed that this action of B extends to G through a bundle on G/B [FT10]. This was proved in the simply laced case by Sugimoto [Sug21, Main Theorem (1)]. In the present setting, Sugimoto and Li show that evaluation of global sections of the bundle $G \times_B V_{\sqrt{p}Q}$ at the base point $eB \in G/B$ induces a B-equivariant isomorphism [SL26, Theorem 3.2(1)]

$$H^0(G/B, G \times_B V_{\sqrt{p}Q}) \xrightarrow{\sim} \mathcal{W}_{\sqrt{p}Q}.$$

The action of G on the sections supplies the desired action on $\mathcal{W}_{\sqrt{p}Q}$. A G-invariant has Cartan weight zero and so belongs to the summand $M(1)e^0 = M(1)$ of the lattice VOA $V_{\sqrt{p}Q}$. We use the following inclusions throughout Part I:



1.3 The Sugimoto–Li decomposition

In [DLM96] Dong-Li-Mason show that any simple VOA V with a compact Lie-group G acting on it has a $G \times V^G$ Schur Weyl decomposition:

$$V = \bigoplus_{\lambda \in \widehat{G}} L_\lambda \otimes M_\lambda$$

Sugimoto and Li established an analogous decomposition for the Feigin–Tipunin algebra, where G is the simply connected Lie group of $\mathfrak{g}$. In particular, there is a $G \times \mathcal{W}^G_{\sqrt{p}Q}$-module decomposition [SL26, (3.1) and Theorem 3.2(1)]

$$\mathcal{W}_{\sqrt{p}Q} = \bigoplus_{\lambda \in P_+ \cap Q} L(\lambda) \otimes M_\lambda. \tag{1.4}$$

Here $L(\lambda)$ is the irreducible G-representation of highest weight λ, and M_λ is its multiplicity space. In the authors' notation $M_\lambda = \mathcal{W}_{-\lambda}$. In particular $M_0 = \mathcal{W}^G_{\sqrt{p}Q}$. The algebra $\mathcal{W}_{\sqrt{p}Q}$ is simple and each M_λ is a simple $\mathcal{W}^G_{\sqrt{p}Q}$-module [SL26, Corollary 3.4]. Under the vacuum condition of Section 1.2, their duality theorem also gives self-contragredience of $\mathcal{W}_{\sqrt{p}Q}$ [SL26, Lemma 2.9(1) and Theorem 3.2(2)].

There are two identifications in their Theorem 4.3 that we use. First, the inclusion of the invariants into $M(1)$ is the principal Miura realization at level k [SL26, (4.3)–(4.6), Theorem 4.3],

$$\mathcal{W}^G_{\sqrt{p}Q} = \bigcap_i \ker(S_i^+|_{M(1)}) \simeq \mathrm{W}^k(\mathfrak{g}), \qquad k + h^\vee = 1/p. \tag{1.5}$$

Here $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. We use this identification throughout the rest of the paper. Second, each multiplicity space is the principal Drinfeld–Sokolov reduction of the affine Weyl module induced from $L(\lambda)$,

$$M_\lambda \simeq T^{1/p}_{\lambda,0} = H^0_{\mathrm{DS},+}(\widehat{V}^k(L(\lambda))). \tag{1.6}$$

In the highest-weight realization its lowest space is the line spanned by $e^{-\sqrt{p}\lambda}$ [SL26, Lemma 4.1 and Theorem 4.3]. Thus the least conformal weight in the whole $L(\lambda)$-summand of (1.4) is

$$h_\lambda = \Delta(-\sqrt{p}\lambda) = \tfrac{p}{2}(\lambda,\lambda) + (p\rho - \rho^\vee, \lambda). \tag{1.7}$$

In particular, $\mathcal{W}_{\sqrt{p}Q}$ is of CFT type. Indeed, a nonzero dominant $\lambda \in Q$ is a nonnegative integral combination of simple roots, and $(p\rho - \rho^\vee, \alpha_i) = p(\alpha_i,\alpha_i)/2 - 1 > 0$. Hence $h_\lambda > 0$ for $\lambda \neq 0$. The remaining summand $\mathrm{W}^k(\mathfrak{g}) \subset M(1)$ has only the vacuum in degree zero.

1.4 The deformed Zhu algebra

Two key results for the Feigin–Tipunin algebra depend on certain characters of the C_2-algebra $R_{\mathcal{W}_{\sqrt{p}Q}}$ and the Zhu algebra $A(\mathcal{W}_{\sqrt{p}Q})$. Both of these associative algebras can be given in a single deformed $\hbar$-family $A_\hbar(\mathcal{W}_{\sqrt{p}Q})$; see [DSK06] and [Cas26], from which we recall the results stated in this subsection. This allows us to provide a uniform proof for the key technical mechanisms needed in the proof of C_2-cofiniteness and in establishing the Frobenius–Perron dimensions. For this reason, we recall the definition and some of its properties.

Definition 1.1. [DSK06] Let V be a $\mathbb{Z}$-graded vertex algebra and let $u \in V$ be homogeneous of conformal weight Δ. Set $Y_\hbar(u,z) := (1+\hbar z)^\Delta Y(u,z)$, where $\hbar$ is an indeterminate. The $*_\hbar$ and $\circ_\hbar$-products are defined as

$$u *_\hbar v := \mathrm{Res}_z z^{-1} Y_\hbar(u,z)v, \qquad u \circ_\hbar v = \mathrm{Res}_z z^{-2} Y_\hbar(u,z)v.$$

Let $O_\hbar(V)$ be the $\mathbb{C}[\hbar]$-span of the circle products. The *deformed Zhu algebra* is $A_\hbar(V) := V[\hbar]/O_\hbar(V)$. The product $*_\hbar$ descends to an associative product with unit $[\mathbf{1}]$ [DSK06, Section 2.1, Theorem 2.7]. For the deformed fields see also [Cas26, Remark 3.4].

The parameter $\hbar$ keeps track of the correction terms to the ordinary product. Expanding the factor $(1+\hbar z)^\Delta$ in the two residues gives

$$u *_\hbar v = u_{(-1)}v + \hbar\Delta\, u_{(0)}v + \cdots, \qquad u \circ_\hbar v = u_{(-2)}v + \hbar\Delta\, u_{(-1)}v + \cdots.$$

At $\hbar = 0$, the relations become precisely those defining $C_2(V)$, and the product on the quotient is the normally ordered product. At $\hbar = 1$, the full binomial sums give Zhu's product and circle relations [Zhu96]. In this paper, we take the $\hbar = 1$ specialization as the definition of $A(V)$; see [DSK06, Section 2.1] for details. Thus the family has fibers

$$A_\hbar(V)/\hbar A_\hbar(V) \simeq R_V, \qquad A_\hbar(V)/(\hbar - 1)A_\hbar(V) \simeq A(V).$$

The commutator also records the Poisson bracket of the first fiber. In $A_\hbar(V)$ it satisfies [DSK06, Theorem 2.7]

$$[u *_\hbar v - v *_\hbar u] = \hbar[\mathrm{Res}_z (1+\hbar z)^{\Delta - 1} Y(u,z)v]. \tag{1.8}$$

The residue specializes to $u_{(0)}v$, giving $\{\bar{u}, \bar{v}\} = \overline{u_{(0)}v}$ on R_V. We will apply the same family of circle relations in both fibers.

We will also use the higher circle products

$$I_{n,\hbar}(u,v) = \mathrm{Res}_z z^{-n-2}(1+\hbar z)^{\Delta} Y(u,z)v \ \in O_\hbar(V) \qquad (n \geq 0). \tag{1.9}$$

The claim that $I_{n,\hbar}(u,v) \in O_\hbar(V)$ follows from a basic induction and integration by parts. We refer the reader to [DSK06],[Cas26] for this argument and all other details we have omitted in this subsection.

1.5 Evaluation on the Heisenberg algebra

A character of a polynomial algebra is specified by its values on the linear functions. In this paper we will be interested in characters of $A_\hbar(M(1))$ which as the following proposition shows are determined by evaluation at points.

Proposition 1.2. *The correspondence $b \mapsto [b(-1)\mathbf{1}]$ gives an isomorphism of associative unital algebras*

$$S(\mathfrak{h})[\hbar] = \mathbb{C}[\mathfrak{h}^*][\hbar] \xrightarrow{\sim} A_\hbar(M(1)).$$

Write $\mathrm{ev}_{\hbar,x} : A_\hbar(M(1)) \to \mathbb{C}[\hbar]$ for evaluation at $x \in \mathfrak{h}^*$. On oscillator monomials its composite with the quotient map is

$$\mathrm{ev}_{\hbar,x}(b_1(-n_1)\cdots b_k(-n_k)\mathbf{1}) = \prod_{i=1}^{k} (-\hbar)^{n_i - 1}(b_i, x), \qquad n_i \geq 1. \tag{1.10}$$

In each complex fiber $\hbar = t_0$ these are all its complex characters, with a unique x for each. Equivalently, over the base ring a polynomial family of points gives a $\mathbb{C}[\hbar]$-valued character.

Proof. On $M(1)$, $b(0)$ acts as the zero operator, so left star multiplication by $b(-1)\mathbf{1}$ acts by ordinary creation $b(-1)$. Applying (1.9) to this current gives $[b(-n)v] = -\hbar[b(-n+1)v]$ for $n \geq 2$. Consequently the currents commute and generate therefore there is a surjective commutative unital algebra map $S(\mathfrak{h})[\hbar] \to A_\hbar(M(1))$, and the displayed evaluation formula is forced.

To show that the surjection is also injective take $t_0 \neq 0$. The map $u \mapsto t_0^{\mathrm{wt}\, u} u$ identifies the fiber at t_0 with the usual Zhu algebra: on homogeneous inputs their weights in the star product give this identity term by term, and a circle product acquires one extra common factor t_0. On the highest-weight line of the Heisenberg module indexed by x/t_0, the image of $[b(-1)\mathbf{1}]$ acts by (b,x). A polynomial relation among the currents would thus vanish at every (t_0, x) with $t_0 \neq 0$. These points are Zariski dense in $\mathbb{A}^1 \times \mathfrak{h}^*$, so the relation is zero. $\square$

There is a second way to evaluate a Heisenberg state which is crucial to the strategy in this part of the paper. Let $x \in \mathfrak{h}^*$ and normalize the standard Heisenberg form by $\langle \mathbf{1}, \mathbf{1} \rangle_{M(1)} = 1$.

Lemma 1.3 (Vacuum formula for Heisenberg evaluation)**.** *Let $w_\hbar(z) = \hbar^{-1} \log(1 + \hbar z)$ be the formal power series in $\hbar$ and z, with $w_0(z) = z$, and set*

$$D_{\hbar,x} = \mathrm{Res}_z w_\hbar(z) Y(x(-1)\mathbf{1}, z) = \sum_{n \geq 1} \frac{(-\hbar)^{n-1}}{n} x(n). \tag{1.11}$$

For $v \in M(1)$, evaluation of its class in $A_\hbar(M(1))$ is given by

$$\mathrm{ev}_{\hbar,x}([v]) = \langle e^{D_{\hbar,x}} v, \mathbf{1} \rangle_{M(1)}.$$

In the two fibers $D_{0,x} = x(1)$ and $D_{1,x} = x[1]$, the first square-bracket mode.

Proof. Every mode $x(n)$ occurring in $D_{\hbar,x}$ has $n \geq 1$ and lowers conformal weight by n. Thus $D_{\hbar,x}$ is locally nilpotent, and its exponential acts by a finite sum on every state. The usual free boson OPEs imply that $[D_{\hbar,x}, b(-n)] = (-\hbar)^{n-1}(x, b)$, and so conjugating by $e^{D_{\hbar,x}}$ translates each creation operator by this scalar. Taking the vacuum coefficient after this translation gives precisely (1.10). For the last assertion, substitute $w = \log(1 + z)$ in $\mathrm{Res}_w w Y[x, w]$, with $Y[x, w] = e^w Y(x(-1)\mathbf{1}, e^w - 1)$. □

In the lattice construction, $Y(u, z)v$ has coefficients in $M(1)$ whenever $u \in M(1)e^\beta$ and $v \in M(1)e^{-\beta}$. For instance, take $u = e^\beta$ and normalize $v \in \mathbb{C}e^{-\beta}$ so that the leading coefficient of $Y(u, z)v$ is the vacuum. The lattice vertex operator then gives

$$Y(u, z)v = z^{-(\beta,\beta)} \exp\left(\sum_{n \geq 1} \frac{\beta(-n)}{n} z^n \right) \mathbf{1}.$$

Applying Lemma 1.3 and the Baker–Campbell–Hausdorff formula to the coefficients gives

$$\mathrm{ev}_{\hbar,x}(Y(u, z)v) = z^{-(\beta,\beta)} \exp\big((\beta, x) w_\hbar(z)\big). \tag{1.12}$$

At $\hbar = 0$ the last factor is $e^{(\beta,x)z}$, and at $\hbar = 1$ it is $(1 + z)^{(\beta,x)}$. With oscillator factors present these exponentials and powers acquire polynomial coefficients. We will use the first expansion to locate common zeros of invariant polynomials which will allow us to prove $\mathcal{W}_{\sqrt{p}Q}$ is C_2-cofinite and the second to restrict the possible Zhu characters.

In Section 2.2 the vectors u, v will lie in a copy of the adjoint representation and will be dual with respect to the invariant form of $\mathcal{W}_{\sqrt{p}Q}$. On the intersection the two vacuum pairings satisfy

$$\langle b, \mathbf{1} \rangle_{M(1)} = \langle b, \mathbf{1} \rangle_{\mathcal{W}_{\sqrt{p}Q}}, \qquad b \in M(1) \cap \mathcal{W}_{\sqrt{p}Q}. \tag{1.13}$$

Indeed the conformal shift changes L_0 by a Heisenberg zero mode, which vanishes on $M(1)$. Both pairings annihilate the positive-degree subspaces there and take value 1 on the vacuum. Thus the scalar in the leading coefficient of a product of opposite adjoint vectors can be computed with the form in which they are dual.

2 C_2-cofiniteness

We prove C_2-cofiniteness of $\mathcal{W}_{\sqrt{p}Q}$ using the G-action and the identification of its invariant subalgebra with the principal W-algebra. Recall that $\mathcal{W}^G_{\sqrt{p}Q} = \mathrm{W}^{1/p-h^\vee}(\mathfrak{g})$. For brevity, set $U := \mathrm{W}^{1/p-h^\vee}(\mathfrak{g})$. Its Zhu algebra is isomorphic to the algebra of Weyl-invariant polynomial functions on $\mathfrak{h}^*$. Since $C_2(U) \subseteq C_2(\mathcal{W}_{\sqrt{p}Q})$, the inclusion induces a homomorphism of graded Poisson algebras

$$\rho : R_U \longrightarrow R_{\mathcal{W}_{\sqrt{p}Q}}, \qquad [u]_U = u + C_2(U) \longmapsto [u]_{\mathcal{W}_{\sqrt{p}Q}} = u + C_2(\mathcal{W}_{\sqrt{p}Q}) \quad (u \in U).$$

We abbreviate either class to $[u]$ when its quotient is clear. Even though the vertex algebra inclusion is injective, new C_2-relations among its invariants can appear in the larger algebra. Our argument uses precisely these relations. We prove two assertions.

(i) Finite generation: $R_{\mathcal{W}_{\sqrt{p}Q}}$ is finitely generated as an R_U-module through ρ.

(ii) Finite-dimensional image: The image $\rho(R_U)$ is finite-dimensional over $\mathbb{C}$.

These two statements imply the desired C_2-cofiniteness. The two assertions use different features of the G-action on $\mathcal{W}_{\sqrt{p}Q}$. The finite generation assertion uses the Schur–Weyl decomposition of Sugimoto and Li recalled in (1.4): the irreducible U-modules occurring in this decomposition are indexed by weights $\lambda \in P_+ \cap Q$, and we show that only finitely many corresponding G-types survive in $R_{\mathcal{W}_{\sqrt{p}Q}}$. The finite-dimensional image assertion follows by adapting to the Feigin–Tipunin setting a lattice construction given previously in [Fre85, (3.24)–(3.26) and Proposition 3.5] and [BM13], which allows one to construct G-invariant fields. The classes of these fields lie in the kernel and generate a cofinite ideal; more precisely, their zero set is the nullcone of R_U. From these two steps we conclude C_2-cofiniteness in Theorem 2.5.

2.1 Finite generation over R_U

Recall the decomposition of Sugimoto and Li [SL26, (3.1) and Theorem 3.2(1)],

$$\mathcal{W}_{\sqrt{p}Q} = \bigoplus_{\lambda \in P_+ \cap Q} L(\lambda) \otimes M_\lambda.$$

Here G acts on $L(\lambda)$ and U acts on its multiplicity space M_λ. Recall that Arakawa essentially showed [Ara15, Cor. 4.5.10(ii)] that for such M_λ:

$$R_{M_\lambda} = R_U \otimes L(\lambda)$$

so in particular M_λ is strongly finitely generated by at most $\dim L(\lambda) < \infty$ elements. Therefore, to show the finite generation result it suffices to show that only finitely many weights have non-zero C_2-image.

We use a conformal-weight bound. Suppose we have a homogeneous vector $u \in M(1)e^{-\sqrt{p}\mu}$ of weight b, with $\mu \neq 0$ dominant. The normally ordered product of n copies has weight nb and lies in $M(1)e^{-n\sqrt{p}\mu}$, whose lowest conformal weight is

$$h_{n\mu} = \tfrac{p}{2} n^2 (\mu, \mu) + n(p\rho - \rho^\vee, \mu).$$

The product therefore vanishes for large n. The same observation allows us to show the result in general.

Proposition 2.1. *Only finitely many highest weights $\lambda \in P_+ \cap Q$ occur in $R_{\mathcal{W}_{\sqrt{p}Q}}$.*

Proof. Write $\lambda = \beta_1 + \cdots + \beta_n$ using a fixed finite set of nonzero generators of $P_+ \cap Q$. Let each u_i range over the homogeneous vectors in the highest-weight copy $L(\beta_i)_{\beta_i} \otimes M_{\beta_i}$ in (1.4), and consider the space

$$N = \mathrm{span}_{\mathbb{C}}\{(u_1)_{(-m_1-1)} \cdots (u_{n-1})_{(-m_{n-1}-1)} u_n : m_i \in \mathbb{Z}\}.$$

By using the commutator formula, it is not difficult to see that N is a non-zero U-submodule. Furthermore, $N \subset L(\lambda)_\lambda \otimes M_\lambda \cong M_\lambda$ and as M_λ is a simple U-module, $N = M_\lambda$. Since G acts by automorphisms of $R_{\mathcal{W}_{\sqrt{p}Q}}$, it suffices to show that $[N] = 0$ for sufficiently large n.

Suppose $[N] \neq 0$. Among the mode products spanning N with nonzero class, choose w with minimal total input weight $S = \sum_i \mathrm{wt}\, u_i$. For each i, fix a homogeneous strong generating set for the U-module M_{β_i} and expand u_i as a linear combination of strong generators and descendants $a_{(-k)}v$, all of weight $\mathrm{wt}\, u_i$. Here $a \in U_{>0}$ is homogeneous and $k \geq 1$.

Consider a product in this expansion with a descendant input $a_{(-k)}v$. The iterate and commutator formulas replace that input by v and express the product as a sum of terms in which a mode of a acts either on the output or on another input. In the first case the term is $a_{(-\ell)}w'$, with $\ell \geq 1$, and w' has total input weight $S - (\mathrm{wt}\, a + k - 1) < S$. By minimality $w' \in C_2$, and the outer negative mode preserves C_2. In the second case a mode $a_{(j)}$, with $j \geq 0$, acts on another input, giving a product of total input weight $S - k - j < S$. Its class also vanishes by minimality. Thus we may choose the minimizing product w with every input among the chosen strong generators. Now $\mathrm{wt}\, w = \sum_i \mathrm{wt}\, u_i + \sum_{i<n} m_i$, and one can show that $w \in C_2$ if $\sum_{i<n} m_i > 0$ [Li05, Proposition 2.11 and Corollary 3.6]. Therefore $\sum_{i<n} m_i \leq 0$. Let b be a common upper bound on the conformal weights of the chosen strong generators of the finitely many U-modules M_β . For this choice of w,

$$h_\lambda = \tfrac{p}{2}(\lambda, \lambda) + (p\rho - \rho^\vee, \lambda) \leq \mathrm{wt}\, w \leq \sum_i \mathrm{wt}\, u_i \leq nb.$$

Now notice that $c := \min\{(\beta_i, \beta_j)\} > 0$, so $(\lambda, \lambda) = \sum_{i,j}(\beta_i, \beta_j) \geq cn^2$. Since $(p\rho - \rho^\vee, \lambda) = \sum_i (p\rho - \rho^\vee, \beta_i)$ is bounded below by $-Cn$ for a fixed $C \geq 0$, we obtain $\frac{p}{2}cn^2 \leq (b + C)n$. This bounds n uniformly, leaving only finitely many possible λ. □

Combining the above result with strong finite U-generation, we obtain the desired finiteness result.

Corollary 2.2. *$R_{\mathcal{W}_{\sqrt{p}Q}}$ is a finite R_U-module.*

2.2 The image of R_U

It remains to show that the image of ρ is finite-dimensional. To find enough relations in its kernel, we use the Miura map $\iota : U \to M(1)$. At the level of C_2-algebras, it induces the embedding [ACL19, Section 6, Lemma 6.1]

$$R_U = \mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow R_{M(1)} = \mathbb{C}[\mathfrak{h}^*]. \tag{2.1}$$

Thus a G-invariant field of $\mathcal{W}^G_{\sqrt{p}Q}$ gives a polynomial whose values we can compute by Heisenberg evaluation. The idea for showing $\dim \rho(R_U) < \infty$ and completing C_2-cofiniteness is to construct a family of G-invariant fields $\mathcal{Q}_j$ for $j \geq 2$ whose C_2-classes $f_j := [\mathcal{Q}_j]$ vanish under ρ and have the

origin as their only common zero on $\mathfrak{h}^*/W$. Namely, the ideal $I = \langle f_2, \cdots \rangle$ has radical $(R_U)_{>0}$, so R_U/I and hence $\rho(R_U)$ are finite-dimensional. Together with Corollary 2.2, this implies that $\mathcal{W}_{\sqrt{p}Q}$ is C_2-cofinite.

The proof of this fact can actually be seen as adapting a classical invariant theory result called Hilbert's nullcone theorem to the setting of VOAs. To explain this connection and hopefully elucidate the proof we show how the proof strategy works in a toy case.

Warm-up with simply laced root lattices. Let $p = 1$ and assume in this case that Q is simply laced and give V_Q its ordinary lattice conformal vector. In the Frenkel–Kac realization $V_Q \simeq L_1(\mathfrak{g})$, the Cartan currents and the vectors e^α for $\alpha \in \Phi$ form the adjoint representation in weight one. In particular, the zero modes of e^{α_i} and $e^{-\alpha_i}$, with the usual normalizations, give the Chevalley generators. The resulting G-action and its fixed points give the level-one model for our inclusion [FK80, Fre85, BM13],

$$\mathrm{W}^{1-h^\vee}(\mathfrak{g}) \simeq V_Q^G \ \subset \ V_Q.$$

Suppose we wanted to see directly, using only this inclusion, why the image of $R_{V_Q^G} \to R_{V_Q}$ is finite-dimensional. We outline the construction of I. Frenkel from [Fre85, (3.24)–(3.26) and Proposition 3.5] and [BM13] which allows us to do this.

As $V_Q \cong L_1(\mathfrak{g})$ the most natural G-invariant is of course the quadratic Casimir. Choose dual bases u_a, u^a with respect to the Killing form of the weight 1-space $(V_Q)_1 \cong \mathfrak{g}t^{-1}$. Constructing the usual Segal-Sugawara vector (up to a scalar) gives the invariant $\sum_a (u_a)_{(-1)}u^a \in V_Q^G$. Replacing the mode (-1) by $(-j)$ gives further invariant coefficients

$$\mathcal{Q}_j^{(1)} = \sum_a (u_a)_{(-j)}u^a \qquad \text{of} \qquad \mathcal{Q}^{(1)}(z) = \sum_a Y(u_a, z)u^a.$$

For $j \geq 2$ every term belongs to $C_2(V_Q)$. The individual currents need not be G-invariant, however, so their sum can give a nonzero class in $R_{V_Q^G}$.

We can identify the classes $f_j^{(1)} := [\mathcal{Q}_j^{(1)}]$ by evaluating $\mathcal{Q}^{(1)}(z)$ at a point $x \in \mathfrak{h}^*$. For each root the contribution is given by (1.12) at $\hbar = 0$, with the indicated dual normalization. The Cartan currents contribute their double poles and their normal products. Hence

$$z^2 \operatorname{ev}_{0,x} \mathcal{Q}^{(1)}(z) = r + (x,x)z^2 + \sum_{\alpha \in \Phi} e^{(\alpha,x)z}. \tag{2.2}$$

For $j \geq 2$ the coefficient of z^{j+1} is $\operatorname{tr}((\operatorname{ad} x)^{j+1})/(j+1)!$. Thus,

$$\operatorname{ev}_{0,x}(f_j^{(1)}) = \frac{\operatorname{tr}((\operatorname{ad} x)^{j+1})}{(j+1)!}.$$

In particular, the set of common zeros $V(f_j^{(1)} : j \geq 2) = \mathcal{N}$ is the nilpotent cone, [Kos63, Prop. 16] (see also [Mor19, Example 17]). Since $R_{V_Q^G} = \mathbb{C}[\mathfrak{h}^*]^W$, this implies that $R_{V_Q^G}$ is a finitely generated $\mathbb{C}[f_j^{(1)} : j \geq 2]$-module by standard invariant theory; see, for example, [DK02, Lem. 2.4.5]. The classes of $\mathcal{Q}_j^{(1)}$ vanish in R_{V_Q} for $j \geq 2$, so the image of $R_{V_Q^G} \to R_{V_Q}$ is finite-dimensional. We now carry the same construction over to $\mathcal{W}_{\sqrt{p}Q}$. We need a copy of the adjoint representation and an invariant form with which to pair its vectors. Evaluating elements of the Heisenberg algebra then allows us to determine the $\mathcal{W}_{\sqrt{p}Q}$ analogue of (2.2).

The Feigin–Tipunin algebra. Let θ be the highest root of $\mathfrak{g}$ and recall that there is a copy of the adjoint representation inside of $\mathcal{W}_{\sqrt{p}Q}$ generated by

$$\mathcal{E} := U(\mathfrak{g})e^{-\sqrt{p}\theta}, \qquad d = h_\theta = ph^\vee - h + 1, \tag{2.3}$$

where h is the Coxeter number. The formula for d follows from (1.7) and the heights $h-1$ and $h^\vee - 1$ of θ and $\theta^\vee$.

Lemma 2.3. *$\mathcal{E}$ is the unique copy of the adjoint representation in this degree of $\mathcal{W}_{\sqrt{p}Q}$. Its vectors are Virasoro-primary, and the invariant form restricts on it to a nonzero multiple of the Killing form. Its Cartan decomposition is*

$$\mathcal{E} = \mathcal{E}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathcal{E}_\alpha, \qquad \mathcal{E}_0 \subset M(1), \qquad \mathcal{E}_\alpha \subset M(1)e^{-\sqrt{p}\alpha}.$$

Consequently, the zero-weight space has dimension r, and $\mathcal{E}_\alpha, \mathcal{E}_{-\alpha}$ pair nondegenerately.

Proof. In the Sugimoto–Li decomposition (1.4), the adjoint occurs with multiplicity space M_θ. Its lowest space is the one-dimensional line of $e^{-\sqrt{p}\theta}$, of weight d, by (1.6)–(1.7). The map $U(\mathfrak{g}) \to \mathcal{E}$ taking u to $ue^{-\sqrt{p}\theta}$ therefore factors through a nonzero quotient of $L(\theta)$. This identifies $\mathcal{E}$ with $L(\theta) \simeq \mathfrak{g}$ and proves uniqueness in degree d. The positive Virasoro modes commute with G and there is no adjoint copy in a lower degree, so they annihilate $\mathcal{E}$.

The normalized invariant form of $\mathcal{W}_{\sqrt{p}Q}$ is nondegenerate and unique, hence is preserved by G. It pairs vectors of equal conformal degree and pairs a G-type only with its dual. Since the adjoint is self-dual and occurs just once in this degree, its restriction to $\mathcal{E}$ is nondegenerate, hence a nonzero multiple of the Killing form. The Cartan action recalled in Section 1.2 places its weight-α line in $M(1)e^{-\sqrt{p}\alpha}$. The dimensions and pairings are then those of the adjoint representation. □

With Lemma 2.3 established, choose a weight basis E_a of $\mathcal{E}$ and its dual E^a for this form. That is, $E_a \in \mathcal{E}_\alpha, E^a \in \mathcal{E}_{-\alpha}$ when $\alpha \neq 0$. As in the level-one example, the invariant tensor $\sum_a E_a \otimes E^a$ gives a series with coefficients in U,

$$\mathcal{Q}(z) = \sum_a Y(E_a, z)E^a, \qquad \mathcal{Q}_j = \sum_a (E_a)_{(-j)}E^a, \qquad f_j = [\mathcal{Q}_j] \in R_U. \tag{2.4}$$

As in the level-one calculation, $f_j \in \ker\rho$ for $j \geq 2$, since each term of $\mathcal{Q}_j$ belongs to $C_2(\mathcal{W}_{\sqrt{p}Q})$. We now follow the same strategy as in the $p = 1$ case. The Miura map allows us to consider f_j as functions on $\mathfrak{h}^*$, and Lemma 1.3 will show that their only common zero is the origin.

To avoid repeating calculations that will be needed later in Section 3.1 we will calculate the following for any $\hbar$

$$[z^{j-1}](1+\hbar z)^d \mathcal{Q}(z) \in U[\hbar] \cap O_\hbar(\mathcal{W}_{\sqrt{p}Q}), \qquad j \geq 2. \tag{2.5}$$

Lemma 2.4. *Fix $x \in \mathfrak{h}^*$, and group the sum $\mathcal{Q}(z) = \sum_a Y(E_a, z)E^a$ according to $\beta \in \{0\} \cup \{-\sqrt{p}\,\alpha : \alpha \in \Phi\}$, collecting the terms with $E_a \in M(1)e^\beta$. Then*

$$\mathrm{ev}_{\hbar,x}\, \mathcal{Q}(z) = \frac{1}{z^{2d}(1+\hbar z)^d} \sum_\beta e^{(\beta,x)w_\hbar(z)}(1+\hbar z)^{\Delta(\beta)} P_\beta(z), \tag{2.6}$$

where each $P_\beta(z)$ is a polynomial in z satisfying

$$P_0(0) = r(-1)^d, \qquad P_{-\sqrt{p}\,\alpha}(0) = (-1)^d \quad (\alpha \in \Phi).$$

Proof. First, the mode commutator gives

$$[D_{\hbar,x}, Y(u,z)] = \sum_{n\geq 0} \frac{\partial_z^n w_\hbar(z)}{n!}\, Y(x(n)u, z).$$

On $M(1)e^\beta$, the zero mode $x(0)$ acts as the scalar (β, x), whereas for $n \geq 1$

$$\frac{\partial_z^n w_\hbar(z)}{n!} = (-\hbar)^{n-1}/(n(1+\hbar z)^n)$$

and $x(n)$ lowers conformal weight by n. Since the Heisenberg modes $x(n)$ $(n \geq 0)$ mutually commute, exponentiating $\mathrm{ad}_{D_{\hbar,x}}$ yields

$$e^{D_{\hbar,x}} Y(u,z) e^{-D_{\hbar,x}} = e^{(\beta,x)w_\hbar(z)} \sum_{i\geq 0} (1+\hbar z)^{-i} Y(u_i, z),$$

where u_i denotes the component of $e^{D_{\hbar,x}}u$ of conformal weight $d-i$.

Similarly, write $e^{D_{\hbar,x}} v = \sum_{j\geq 0} v_j$ with v_j of weight $d-j$. By Lemma 1.3 and the conjugation formula above,

$$\begin{aligned} \mathrm{ev}_{\hbar,x}(Y(u,z)v) &= \left\langle (e^{D_{\hbar,x}} Y(u,z) e^{-D_{\hbar,x}}) e^{D_{\hbar,x}} v, \mathbf{1} \right\rangle_{M(1)} \\ &= e^{(\beta,x)w_\hbar(z)} \sum_{i,j\geq 0} (1+\hbar z)^{-i} \left\langle Y(u_i,z)v_j, \mathbf{1}\right\rangle_{M(1)}. \end{aligned}$$

Since $(u_i)_{(n)}v_j$ has conformal weight $2d-i-j-n-1$, only the mode $n = 2d-i-j-1$ pairs nontrivially with the vacuum $\mathbf{1}$, so

$$\left\langle Y(u_i,z)v_j, \mathbf{1}\right\rangle_{M(1)} = z^{-2d+i+j} \left\langle (u_i)_{(2d-i-j-1)} v_j, \mathbf{1}\right\rangle_{M(1)}.$$

Summing over $E_a \in M(1)e^\beta$ and factoring out $z^{-2d} e^{(\beta,x)w_\hbar(z)} (1+\hbar z)^{\Delta(\beta)-d}$, we obtain

$$P_\beta(z) = \sum_{E_a \in M(1)e^\beta} \sum_{i,j\geq 0} z^{i+j} (1+\hbar z)^{d-\Delta(\beta)-i} \left\langle (u_i)_{(2d-i-j-1)} v_j, \mathbf{1}\right\rangle_{M(1)},$$

where u_i and v_j are associated to $u = E_a$ and $v = E^a$ in each summand. The lowest conformal weights of $M(1)e^\beta$ and $M(1)e^{-\beta}$ impose the bounds $0 \leq i \leq d - \Delta(\beta)$ and $0 \leq j \leq d - \Delta(-\beta)$. Thus the sum is finite and every power of $1+\hbar z$ is nonnegative, so $P_\beta(z)$ is a polynomial in z.

Setting $z = 0$ leaves only the $i = j = 0$ terms, for which $u_0 = u$ and $v_0 = v$. By (1.13) and recalling from Lemma 2.3 that u, v are highest weight vectors for the Virasoro, each dual pair contributes

$$\langle u_{(2d-1)} v, \mathbf{1}\rangle_{M(1)} = \langle u_{(2d-1)} v, \mathbf{1}\rangle_{\mathcal{W}_{\sqrt{p}Q}} = (-1)^d \langle u, v\rangle_{\mathcal{W}_{\sqrt{p}Q}} = (-1)^d.$$

Because there are r dual pairs for the Cartan subspace ($\beta = 0$) and one dual pair for each root line ($\beta = -\sqrt{p}\,\alpha$, $\alpha \in \Phi$), evaluating at $z = 0$ yields the stated constant terms. □

With everything in place we are now ready to conclude C_2-cofiniteness.

Theorem 2.5. *The Feigin–Tipunin vertex algebra $\mathcal{W}_{\sqrt{p}Q}$ is C_2-cofinite for the parameters of Section 1.2.*

Proof. By Corollary 2.2, it suffices to show that $\rho(R_U)$ is finite-dimensional. Setting $\hbar = 0$ in Lemma 2.4 gives

$$z^{2d}\,\mathrm{ev}_{0,x}\,\mathcal{Q}(z) = P_0(z) + \sum_{\alpha\in\Phi} e^{-\sqrt{p}(\alpha,x)z} P_{-\sqrt{p}\alpha}(z). \tag{2.7}$$

The coefficient of z^{2d+j-1} on the left is $f_j(x)$. Thus if all the f_j, $j \geq 2$, vanish at x, the displayed series is a polynomial of degree at most $2d$.

A nonzero (α, x) in the sum on the right prevents this. Indeed, collect the roots with the same value of $-\sqrt{p}(\alpha, x)$. Their coefficient polynomial has constant term $(-1)^d$ times the number of roots in that group. On the other hand, exponentials with distinct frequencies are linearly independent over $\mathbb{C}[z]$. Consequently all roots vanish on x, so $x = 0$. Conversely every f_j vanishes at zero because it is homogeneous of positive degree. By the Nullstellensatz the ideal $I = (f_j : j \geq 2) \subset R_U$ has radical the maximal ideal of the origin. It follows that R_U/I is finite-dimensional. Since $I \subset \ker\rho$, its quotient $\rho(R_U)$ is finite-dimensional as well.

The finite set of R_U-module generators from Corollary 2.2 also generates over $\rho(R_U)$. Hence $\dim_{\mathbb{C}} R_{\mathcal{W}_{\sqrt{p}Q}} < \infty$. □

With C_2-cofiniteness established, we can now apply a rigidity theorem of McRae [McR26b, Theorem 4.5], building on work of Etingof and Penneys [EP26], to conclude that the category $\mathcal{D} = \mathrm{Rep}_{\mathrm{gr}}(\mathcal{W}_{\sqrt{p}Q})$ of grading-restricted generalized modules is a factorizable finite ribbon category.

Corollary 2.6. *The category $\mathcal{D} = \mathrm{Rep}_{\mathrm{gr}}(\mathcal{W}_{\sqrt{p}Q})$ is a factorizable finite ribbon category, with duals given by contragredient modules.*

Proof. The algebra $\mathcal{W}_{\sqrt{p}Q}$ is simple, self-contragredient and nonnegatively graded by Section 1.3, and C_2-cofinite by Theorem 2.5. The assertion follows from McRae's rigidity theorem [McR26b, Theorem 4.5]. □

3 Zhu characters and Frobenius–Perron dimensions

Recall that $Q^* = \{y \in \mathfrak{h}^* : (y, Q) \subset \mathbb{Z}\}$ is the dual root lattice with respect to the chosen invariant form. Having established C_2-cofiniteness and that the category $\mathrm{Rep}_{\mathrm{gr}}(\mathcal{W}_{\sqrt{p}Q})$ is a factorizable finite ribbon category, we now turn to the next vertex-algebraic ingredient needed to complete the logarithmic Kazhdan–Lusztig equivalence: determining the Frobenius–Perron dimensions of the simple lattice modules $A_\gamma = V_{\sqrt{p}Q+\gamma/\sqrt{p}}$, regarded as objects of $\mathcal{D}$. More precisely, we will establish that

$$\mathrm{FPdim}_{\mathcal{D}}(A_\gamma) = p^{N_l}(p/t)^{N_s} = D_p, \qquad \gamma \in Q^*/pQ,$$

where N_l and N_s denote the numbers of long and short positive roots, respectively, and t is the lacing number.

Let us first recall the underlying strategy. The Perron–Frobenius theorem states that a square matrix B with strictly positive entries has a largest real eigenvalue $\rho(B)$, equal to its spectral radius, which is algebraically simple. Moreover, its corresponding eigenspace is spanned by a strictly positive eigenvector, unique up to scaling. In particular, let $X_0 = \mathbf{1}, X_1, \ldots, X_n$ denote the simple objects of a finite tensor category, and consider $T = \bigoplus_{i=0}^n X_i$. By rigidity, its fusion matrix $M = (N_{T,X_i}^{X_j})_{i,j}$ has strictly positive entries. Its unique positive eigenvector, normalized to have vacuum component equal to 1, is precisely the Frobenius–Perron dimension vector $(\mathrm{FPdim}(X_i))_i$.

In the unitary rational setting, this vector is given by the vacuum column of the modular S-matrix, so that

$$\mathrm{FPdim}(X_i) = \frac{S_{i0}}{S_{00}}.$$

In the nonsemisimple setting, we follow the familiar strategy of studying quantum dimensions through modular transformations and character asymptotics; see, for example, [Zhu96, Hua08, DJX13, Miy04, AN13, Fio16, BM15, BM17, CM14, CMW17, CG17, GR19, Cre24] and references therein. We first establish, without explicitly determining the full Zhu algebra of $\mathcal{W}_{\sqrt{p}Q}$, that all simple modules have real lowest conformal weights and that there is a unique simple module of globally minimal conformal weight. The character formulas of Sugimoto–Li [SL26], together with Poisson summation, then allow us to calculate the quantum dimensions of the lattice modules explicitly and show that they are positive.

To identify these quantum dimensions with Frobenius–Perron dimensions, we use Miyamoto's modular invariance theorem for pseudotraces [Miy04] to isolate the polynomial coefficients corresponding to the minimal conformal weight in the modular transformations of the simple characters. McRae's open Hopf link theorem [McR26a, Theorem 5.2] then shows that the resulting vector is a generalized eigenvector of the fusion matrix. The vacuum-character asymptotic established in the analytic-index calculation supplies the required nonvanishing, and Perron–Frobenius theory identifies this vector with the Frobenius–Perron dimension vector. Consequently, for every grading-restricted generalized $\mathcal{W}_{\sqrt{p}Q}$-module X,

$$\mathrm{FPdim}_{\mathcal{D}}(X) = \lim_{y \downarrow 0} \frac{\mathrm{ch}_X(iy)}{\mathrm{ch}_{\mathcal{W}_{\sqrt{p}Q}}(iy)}.$$

Combining this identification with the explicit character asymptotics obtained from Sugimoto–Li gives the desired Frobenius–Perron dimensions of the lattice modules.

3.1 Principal characters and the least conformal weight

As in the C_2-cofiniteness argument, we use the action of G to study the Feigin–Tipunin algebra $\mathcal{W}_{\sqrt{p}Q}$ through the principal W-algebra $U = \mathcal{W}^G_{\sqrt{p}Q} \simeq \mathrm{W}^{\frac{1}{p}-h^\vee}(\mathfrak{g})$. Recall that (1.4) gives a $G \times U$-module decomposition of $\mathcal{W}_{\sqrt{p}Q}$. Here we use a similar decomposition on the lowest-weight spaces of simple $\mathcal{W}_{\sqrt{p}Q}$-modules.

Recall that under the Miura map, the Zhu algebra $A(U)$ is identified with the algebra of polynomial functions on $\mathfrak{h}^*$ invariant under a shifted Weyl group action [ACL19, Proposition 6.2 and (29)]. More precisely, two Heisenberg highest-weight parameters $\lambda, \mu \in \mathfrak{h}^*$ determine the same character precisely when

$$\lambda + \rho - (k + h^\vee)\rho^\vee \in W(\mu + \rho - (k + h^\vee)\rho^\vee).$$

Additionally, recall that in the Miura realization of Sugimoto–Li, the simple-root Heisenberg currents are normalized as $-\alpha_i/\sqrt{p}$, so that the corresponding screening operators agree with the long screenings $S_i^+ = e^{\sqrt{p}\alpha_i}_{(0)}$ [SL26, (4.3)–(4.5)]. Consequently, the Heisenberg evaluation $\mathrm{ev}_{1,x}$ corresponds to the highest-weight parameter $\lambda = -x/\sqrt{p}$. Since $k + h^\vee = 1/p$, the affine change of coordinates

$$\mathfrak{h}^* \longrightarrow \mathfrak{h}^*, \qquad x \mapsto \zeta = p\rho - \rho^\vee - \sqrt{p}\, x$$

identifies the shifted Weyl action with the ordinary Weyl action. Thus two Heisenberg evaluations $\mathrm{ev}_{1,x}$ and $\mathrm{ev}_{1,x'}$ determine the same character of $A(U)$ precisely when

$$p\rho - \rho^\vee - \sqrt{p}\,x' \in W(p\rho - \rho^\vee - \sqrt{p}\,x).$$

Equivalently, the characters of $A(U)$ are parametrized by ordinary Weyl orbits $W\zeta$. The conformal weight associated with this character is

$$\mathrm{ev}_{1,x}([\omega]) = \frac{1}{2}(x,x) - \frac{1}{\sqrt{p}}(p\rho - \rho^\vee, x) = h_* + \frac{(\zeta,\zeta)}{2p}, \tag{3.1}$$

where $h_* = (c-r)/24$ and $r = \operatorname{rank}\mathfrak{g}$.

To bound the lowest conformal weight of a simple $\mathcal{W}_{\sqrt{p}Q}$-module, we first show that the lowest-weight space of every simple $\mathcal{W}_{\sqrt{p}Q}$-module has a G-isotypic decomposition whose multiplicity spaces define characters of $A(U)$. Since $U \subset \mathcal{W}_{\sqrt{p}Q}$ is conformal, this expresses the lowest conformal weight of every simple module as

$$h_S = h_* + \frac{(\zeta,\zeta)}{2p}$$

for some Harish–Chandra parameter ζ. We next show that the parameters arising in this way belong to the real lattice Q^*, which gives the lower bound $h_S \geq h_*$. Finally, we construct a simple module attaining the bound and prove that it is unique. We begin with the G-action on simple Zhu modules and the resulting Schur–Weyl-type decomposition.

Lemma 3.1. *Every simple $A(\mathcal{W}_{\sqrt{p}Q})$-module T admits a compatible rational G-action, and hence a decomposition*

$$T = \bigoplus_{\lambda \in P_+} L(\lambda) \otimes E_{T,\lambda}, \qquad E_{T,\lambda} = \mathrm{Hom}_G(L(\lambda), T).$$

Each multiplicity space $E_{T,\lambda}$ is naturally an $A(U)$-module. Moreover, every nonzero $E_{T,\lambda}$ is one-dimensional, and the resulting characters of $A(U)$ are distinct as the pair (T,λ) varies over all simple $A(\mathcal{W}_{\sqrt{p}Q})$-modules and their G-types.

Proof. By Theorem 2.5, the Zhu algebra $A(\mathcal{W}_{\sqrt{p}Q})$ is finite-dimensional. Its Jacobson radical is G-stable, and its semisimple quotient has the form

$$\frac{A(\mathcal{W}_{\sqrt{p}Q})}{\mathrm{Jac}(A(\mathcal{W}_{\sqrt{p}Q}))} \cong \prod_{T \in \mathrm{Irr}(A(\mathcal{W}_{\sqrt{p}Q}))} \mathrm{End}(T). \tag{3.2}$$

Since G is connected, it preserves every matrix-algebra factor. By Skolem–Noether [The, Lemma 11.6.2], its action on each factor gives a homomorphism $G \to \mathrm{PGL}(T)$. Since G is simply connected and semisimple, this projective action lifts to a rational action on T: its differential lifts through $\mathrm{Der}(\mathrm{End}\,T) = \mathfrak{sl}(T)$ and then integrates to G. Consequently, complete reducibility of finite-dimensional G-representations gives the asserted decomposition.

Now notice, the G-action is compatible with the Zhu-algebra module structure:

$$g \cdot (u \cdot v) = (g \cdot u) \cdot (g \cdot v), \qquad u \in A(\mathcal{W}_{\sqrt{p}Q}), \quad v \in T. \tag{3.3}$$

and elements of $A(U)$ have G-invariant images in $A(\mathcal{W}_{\sqrt{p}Q})$, so their action on T commutes with G. Hence they act on the multiplicity spaces through

$$\mathrm{End}_G(T) \cong \prod_{\lambda : E_{T,\lambda} \neq 0} \mathrm{End}(E_{T,\lambda}).$$

Taking G-invariants of the quotient in (3.2) is exact and gives the surjection

$$A(\mathcal{W}_{\sqrt{p}Q})^G \twoheadrightarrow \left(\frac{A(\mathcal{W}_{\sqrt{p}Q})}{\mathrm{Jac}(A(\mathcal{W}_{\sqrt{p}Q}))}\right)^G \cong \prod_{T,\lambda:E_{T,\lambda}\neq 0} \mathrm{End}(E_{T,\lambda}). \tag{3.4}$$

On the other hand, by exactness of invariants $A(U)$ surjects onto $A(\mathcal{W}_{\sqrt{p}Q})^G$ and $A(U)$ is commutative. Thus each nonzero $E_{T,\lambda}$ must be one-dimensional. The distinct coordinate characters of the product remain distinct after pulling back through the surjections from $A(\mathcal{W}_{\sqrt{p}Q})^G$ and $A(U)$. □

If S is a simple $\mathcal{W}_{\sqrt{p}Q}$-module with lowest-weight space T, then each nonzero multiplicity space $E_{T,\lambda}$, $\lambda \in P_+$, determines a Harish–Chandra parameter $\zeta_{T,\lambda}$ up to the Weyl group, and

$$h_S = h_* + \frac{(\zeta_{T,\lambda}, \zeta_{T,\lambda})}{2p} \qquad \text{for every } \lambda \text{ with } E_{T,\lambda} \neq 0.$$

To show that h_* is a lower bound, we now show that $\zeta_{T,\lambda}$ lies in the dual root lattice Q^*, so that $h_S \geq h_*$.

Theorem 3.2. *If a character of $A(U)$ factors through $A(\mathcal{W}_{\sqrt{p}Q})^G$, its Harish–Chandra parameter ζ belongs to the dual root lattice Q^*.*

Proof. Choose a Heisenberg evaluation $\mathrm{ev}_{1,x}$ representing the character, so that

$$\zeta = p\rho - \rho^\vee - \sqrt{p}\, x.$$

Recall the invariant field $\mathcal{Q}(z)$ from (2.4). In the proof of C_2-cofiniteness, we evaluated this field at $\hbar = 0$; here we instead take $\hbar = 1$ to obtain its Zhu specialization. In particular, since the coefficients in (2.5) belong to $O(\mathcal{W}_{\sqrt{p}Q})$ at $\hbar = 1$, we see that

$$z^{2d}(1+z)^d \,\mathrm{ev}_{1,x}\, \mathcal{Q}(z) \tag{3.5}$$

is a polynomial in z of degree at most $2d$. By Lemma 2.4, this polynomial takes the form

$$z^{2d}(1+z)^d \,\mathrm{ev}_{1,x}\, \mathcal{Q}(z) = P_0(z) + \sum_{\alpha\in\Phi} (1+z)^{\frac{p}{2}(\alpha,\alpha)+(\alpha,\zeta)} P_\alpha(z), \tag{3.6}$$

where $P_0, P_\alpha \in \mathbb{C}[z]$ and $P_\alpha(0) = (-1)^d$ for every root α. Since $\sqrt{p}\,Q$ is even, we have $\frac{p}{2}(\alpha,\alpha) \in \mathbb{Z}$. For each class $\lambda \in \mathbb{C}/\mathbb{Z}$, define

$$F_\lambda(z) = \sum_{\substack{\alpha\in\Phi \\ (\alpha,\zeta)\equiv\lambda \pmod{\mathbb{Z}}}} (1+z)^{\frac{p}{2}(\alpha,\alpha)+(\alpha,\zeta)} P_\alpha(z).$$

Analytic continuation around $z = -1$ multiplies F_λ by $e^{2\pi i\lambda}$. Since the left-hand side of (3.6) is a polynomial, applying successive powers of monodromy yields a Vandermonde system with distinct eigenvalues, forcing

$$F_\lambda(z) = 0 \qquad \text{whenever } \lambda \neq 0.$$

However, for every class containing at least one root,

$$F_\lambda(0) = (-1)^d \#\{\alpha \in \Phi : (\alpha,\zeta) \equiv \lambda \pmod{\mathbb{Z}}\} \neq 0.$$

Thus every root must satisfy $(\alpha,\zeta) \in \mathbb{Z}$, and hence $\zeta \in Q^*$. □

Now that we have established h_* is a lower bound on conformal weights of simple modules, we can show that this bound is achieved by a unique simple module.

Theorem 3.3. *Every simple grading-restricted $\mathcal{W}_{\sqrt{p}Q}$-module has real lowest conformal weight at least $h_* = (c-r)/24$. Exactly one simple module S_* has lowest weight h_*, and its lowest space is one-dimensional.*

Proof. To see that the bound is attained, observe that

$$(p\rho - \rho^\vee, \alpha_i) = \frac{p}{2}(\alpha_i, \alpha_i) - 1 \in \mathbb{Z}$$

for every simple root. Thus

$$\frac{p\rho - \rho^\vee}{\sqrt{p}} \in (\sqrt{p}\, Q)^*,$$

and the shifted lattice module $V_{\sqrt{p}Q + (p\rho - \rho^\vee)/\sqrt{p}}$ is defined. We will use this lattice module to construct a simple $\mathcal{W}_{\sqrt{p}Q}$-module attaining the bound. To that end, consider lattice elements of the form

$$e^{(p\rho - \rho^\vee)/\sqrt{p} + \sqrt{p}\,\gamma}, \qquad \gamma \in Q.$$

By (1.2), their conformal weights are

$$\begin{aligned}\Delta\left(\frac{p\rho - \rho^\vee}{\sqrt{p}} + \sqrt{p}\,\gamma\right) &= \frac{p}{2}(\gamma,\gamma) - \frac{(p\rho - \rho^\vee, p\rho - \rho^\vee)}{2p} \\ &= h_* + \frac{p}{2}(\gamma,\gamma).\end{aligned}$$

Thus the lowest conformal weight is attained at $\gamma = 0$, and the lowest-weight space is

$$\mathbb{C} e^{(p\rho - \rho^\vee)/\sqrt{p}}.$$

On restriction to $\mathcal{W}_{\sqrt{p}Q}$, this one-dimensional lowest-weight space is a simple $A(\mathcal{W}_{\sqrt{p}Q})$-module. Zhu's correspondence therefore gives a simple module S_* of lowest conformal weight h_*.

Finally, any simple module attaining h_* has the principal character corresponding to $\zeta = 0$. By Lemma 3.1, this character occurs in at most one simple Zhu module, so Zhu's correspondence gives uniqueness of S_*. □

3.2 Frobenius–Perron dimensions

Having established the existence of a unique simple module of least conformal weight, we can now study quantum dimensions using modular transformations of characters in the non-semisimple setting. In particular, our goal is to show that the quantum dimension

$$\operatorname{qdim}(X) = \lim_{y \downarrow 0} \frac{\operatorname{ch}_X(iy)}{\operatorname{ch}_{\mathcal{W}_{\sqrt{p}Q}}(iy)}$$

exists and agrees with the Frobenius–Perron dimension of X. Here, the character of a grading-restricted generalized $\mathcal{W}_{\sqrt{p}Q}$-module X is defined by

$$\operatorname{ch}_X(\tau) = \operatorname{tr}_X e^{2\pi i \tau (L_0 - c/24)}.$$

In the non-semisimple setting, ordinary characters are generally no longer closed under modular transformations. Instead, one needs generalized characters (defined using pseudotraces; see [Miy04, McR26a]), which we use in the proof of our dimension formula.

As explained in the introduction, the argument uses modular transformations and the existence of a unique simple module of least conformal weight, together with McRae's open Hopf link formula [McR26a], to establish this equality. We then use the character formula of Sugimoto and Li [SL26] to compute the Frobenius–Perron dimension of the lattice extension $V_{\sqrt{p}Q}$.

We begin with the lattice character calculation. Recall that N_l and N_s denote the numbers of long and short positive roots of $\mathfrak{g}$, respectively, and set $D_p = p^{N_l}(\frac{p}{t})^{N_s}$. For the sake of brevity, denote for $\gamma \in Q^*/pQ$,

$$A_\gamma = V_{\sqrt{p}Q+\gamma/\sqrt{p}}$$

for the corresponding lattice module, regarded as a $\mathcal{W}_{\sqrt{p}Q}$-module by restriction.

Theorem 3.4 (Analytic lattice index)**.** *For every $\gamma \in Q^*/pQ$,*

$$\lim_{y\downarrow 0} \frac{\mathrm{ch}_{A_\gamma}(iy)}{\mathrm{ch}_{\mathcal{W}_{\sqrt{p}Q}}(iy)} = D_p.$$

Proof. Write $N_+ = |\Phi_+|$ and $\mathscr{D}(x) = \prod_{\alpha>0}(x, \alpha^\vee)$. Taking characters in the decomposition (1.4) and applying the Weyl dimension formula gives

$$\begin{aligned}\mathrm{ch}_{\mathcal{W}_{\sqrt{p}Q}}(\tau) &= \sum_{\lambda\in P_+\cap Q} \dim L(\lambda)\,\mathrm{ch}_{M_\lambda}(\tau)\\ &= \frac{1}{\mathscr{D}(\rho)} \sum_{\lambda\in P_+\cap Q} \mathscr{D}(\lambda+\rho)\,\mathrm{ch}_{M_\lambda}(\tau).\end{aligned}$$

By the character formula of Sugimoto and Li [SL26, Corollary 3.3, (3.2)]

$$\eta(iy)^r\,\mathrm{ch}_{M_\lambda}(iy) = \sum_{w\in W} \det(w) e^{-\pi p y|w(\lambda+\rho)-\rho^\vee/p|^2}.$$

Here $\eta(\tau)$ denotes Dedekind's eta function. Substituting this formula gives

$$\begin{aligned}\eta(iy)^r\,\mathrm{ch}_{\mathcal{W}_{\sqrt{p}Q}}(iy) &= \frac{1}{\mathscr{D}(\rho)} \sum_{\lambda\in P_+\cap Q} \mathscr{D}(\lambda+\rho) \sum_{w\in W} \det(w) e^{-\pi p y|w(\lambda+\rho)-\rho^\vee/p|^2}\\ &= \frac{1}{\mathscr{D}(\rho)} \sum_{x\in Q+\rho} \mathscr{D}(x) e^{-\pi p y|x-\rho^\vee/p|^2}.\end{aligned} \tag{3.7}$$

Indeed, the regular elements of $Q+\rho$ form the disjoint union of the Weyl orbits $W(\lambda+\rho)$, for $\lambda \in P_+ \cap Q$, while the discriminant vanishes on singular elements.

Applying the Poisson summation formula to (3.7), we obtain

$$\eta(\mathrm{i}y)^r\,\mathrm{ch}_{\mathcal{W}_{\sqrt{p}Q}}(\mathrm{i}y) = \frac{1}{\mathrm{vol}(Q)\mathscr{D}(\rho)} \sum_{\xi\in Q^*} e^{2\pi\mathrm{i}(\rho,\xi)}\,\widehat{f}_y(\xi),$$

where

$$\widehat{f}_y(\xi) = \int_{\mathbb{R}^r} \mathscr{D}(x) e^{-\pi p y|x-\rho^\vee/p|^2 - 2\pi\mathrm{i}(x,\xi)}\,dx.$$

Recall that the Weyl discriminant is harmonic [Kos63], so $\Delta\mathscr{D} = 0$. Completing the square and translating the Gaussian integral gives

$$\begin{aligned}\widehat{f}_y(\xi) &= e^{-2\pi\mathrm{i}(\rho^\vee/p,\xi)-\pi|\xi|^2/(py)} \int_{\mathbb{R}^r} e^{-\pi py|x|^2}\,\mathscr{D}\left(x + \frac{\rho^\vee}{p} - \frac{\mathrm{i}\xi}{py}\right) dx \\ &= (py)^{-r/2} e^{-2\pi\mathrm{i}(\rho^\vee/p,\xi)-\pi|\xi|^2/(py)} \left(e^{\Delta/(4\pi py)}\mathscr{D}\right)\left(\frac{\rho^\vee}{p} - \frac{\mathrm{i}\xi}{py}\right) \\ &= (py)^{-r/2} e^{-2\pi\mathrm{i}(\rho^\vee/p,\xi)-\pi|\xi|^2/(py)}\,\mathscr{D}\left(\frac{\rho^\vee}{p} - \frac{\mathrm{i}\xi}{py}\right).\end{aligned}$$

Here the second equality follows from Gaussian averaging, which acts on polynomials by $e^{\Delta/(4\pi py)}$, see [Eva10, Section 2.3.1] . The final equality follows from $\Delta\mathscr{D} = 0$, since all higher terms in the exponential vanish.

Substituting this into the Poisson summation formula yields

$$\eta(\mathrm{i}y)^r \operatorname{ch}_{\mathcal{W}_{\sqrt{p}Q}}(\mathrm{i}y) = \frac{(py)^{-r/2}}{\operatorname{vol}(Q)\mathscr{D}(\rho)} \sum_{\xi\in Q^*} e^{2\pi\mathrm{i}(\rho-\rho^\vee/p,\xi)-\pi|\xi|^2/(py)} \cdot \mathscr{D}\left(\frac{\rho^\vee}{p} - \frac{\mathrm{i}\xi}{py}\right).$$

As $y \to 0^+$, the terms with $\xi \neq 0$ are exponentially suppressed, since the discriminant grows at most polynomially in y^{-1}. Thus, the leading contribution comes from $\xi = 0$, giving

$$\eta(\mathrm{i}y)^r \operatorname{ch}_{\mathcal{W}_{\sqrt{p}Q}}(\mathrm{i}y) \sim \frac{(py)^{-r/2}}{\operatorname{vol}(Q)} \frac{\mathscr{D}(\rho^\vee/p)}{\mathscr{D}(\rho)}.$$

On the other hand, by the conformal weight formula (1.2), the lattice character is

$$\eta(\mathrm{i}y)^r \operatorname{ch}_{A_\gamma}(\mathrm{i}y) = \sum_{\delta\in Q} e^{-\pi py|\delta+(\gamma-p\rho+\rho^\vee)/p|^2} \sim \frac{(py)^{-r/2}}{\operatorname{vol}(Q)},$$

where the asymptotic again follows from the zero-frequency term in Poisson summation. Consequently,

$$\lim_{y\downarrow 0} \frac{\operatorname{ch}_{A_\gamma}(\mathrm{i}y)}{\operatorname{ch}_{\mathcal{W}_{\sqrt{p}Q}}(\mathrm{i}y)} = p^{N_+} \frac{\mathscr{D}(\rho)}{\mathscr{D}(\rho^\vee)}.$$

Finally, the standard relation between root heights and Weyl-group exponents [Hum90, Section 3.20] shows that the products of the heights of the positive roots and positive coroots agree, giving $\mathscr{D}(\rho^\vee) = t^{N_s}\mathscr{D}(\rho)$. As $N_+ = N_s + N_l$ the limit is $p^{N_l}(p/t)^{N_s} = D_p$.

□

With everything in place we can now prove the desired formula for Frobenius–Perron dimensions.

Theorem 3.5. *For every $X \in \mathcal{D}$,*

$$\operatorname{FPdim}_{\mathcal{D}}(X) = \lim_{y\downarrow 0} \frac{\operatorname{ch}_X(iy)}{\operatorname{ch}_{\mathcal{W}_{\sqrt{p}Q}}(iy)}.$$

In particular, for every $\gamma \in Q^*/pQ$, including $\gamma = 0$,

$$\operatorname{FPdim}_{\mathcal{D}}(V_{\sqrt{p}Q+\gamma/\sqrt{p}}) = p^{N_l}(p/t)^{N_s} = D_p. \tag{3.8}$$

Proof. Let $X_0 = \mathcal{W}_{\sqrt{p}Q}, X_1, \dots, X_b$ be the simple $\mathcal{W}_{\sqrt{p}Q}$-modules. By Miyamoto's theorem [Miy04, Theorem 5.5], the space of genus-one one-point functions is spanned by pseudotrace functions associated with n-th generalized Verma modules, for n sufficiently large. Choose such modules P_a with corresponding symmetric linear functionals φ_a on the higher Zhu algebra $A_n(\mathcal{W}_{\sqrt{p}Q})$. As $\mathcal{D}$ is rigid, we can apply McRae's open Hopf link formula [McR26a, Theorem 5.2, (5.5)] to obtain

$$\mathrm{ch}_{X_i}(-1/\tau) = \sum_a \mathrm{ptr}_{P_a}^{\varphi_a}\left(h_{X_i,P_a} q^{L_0 - c/24}\right), \qquad q = e^{2\pi \mathrm{i}\tau}.$$

To assist with some bookkeeping, for each P_a, denote its set of conformal weights by

$$\mathrm{wt}(P_a) = \{h \in \mathbb{C} : (P_a)_{[h]} \neq 0\},$$

so that $P_a = \bigoplus_{h \in \mathrm{wt}(P_a)} (P_a)_{[h]}$ is its decomposition into generalized L_0-eigenspaces. On this space, write $L_0 = h\,\mathrm{id} + N_{a,h}$, where $N_{a,h}$ is nilpotent of order at most $d_{a,h}$. We see that

$$\begin{aligned} q^{L_0 - c/24}\big|_{(P_a)_{[h]}} &= q^{h-c/24} e^{2\pi\mathrm{i}\tau N_{a,h}} \\ &= q^{h-c/24} \sum_{k=0}^{d_{a,h}-1} \frac{(2\pi\mathrm{i}\tau)^k}{k!} N_{a,h}^k. \end{aligned}$$

Substituting into McRae's formula gives

$$\mathrm{ch}_{X_i}(-1/\tau) = \sum_a \sum_{h \in \mathrm{wt}(P_a)} q^{h-c/24} \sum_{k=0}^{d_{a,h}-1} \frac{(2\pi\mathrm{i}\tau)^k}{k!} \cdot \mathrm{ptr}_{(P_a)_{[h]}}^{\varphi_a}\left(h_{X_i,P_a} N_{a,h}^k\right).$$

Thus each conformal weight contributes a power of q multiplied by a polynomial in τ.

By Theorem 3.3, there is a unique simple module S_* of minimal conformal weight h_*. Since each P_a has finite length, every generalized conformal weight satisfies $h \geq h_*$, and the weight h_* can occur only through composition factors isomorphic to S_*. Setting $\tau = \mathrm{i}Y$, we collect the contributions of weight h_* into the polynomial

$$b_i(Y) = \sum_a \mathrm{ptr}_{(P_a)_{[h_*]}}^{\varphi_a}\left(h_{X_i,P_a} e^{-2\pi Y N_{a,h_*}}\right).$$

Here the exponential is a polynomial in Y, since N_{a,h_*} is nilpotent. Thus, for some $\delta > 0$ and $D \geq 0$,

$$\mathrm{ch}_{X_i}(\mathrm{i}/Y) = e^{2\pi(c/24 - h_*)Y}\left(b_i(Y) + O(Y^D e^{-2\pi\delta Y})\right).$$

With this in mind, consider the vector

$$b(Y) = (b_0(Y), \dots, b_b(Y))^{\mathsf{t}}.$$

We claim that $b(Y)$ is nonzero and componentwise nonnegative for all sufficiently large Y. To see this, note the ordinary characters satisfy $\mathrm{ch}_{X_i}(\mathrm{i}/Y) > 0$ for every $Y > 0$. Suppose that $b_i(Y)$ is not identically zero, and write

$$b_i(Y) = a_i Y^{d_i^\vee} + O(Y^{d_i^\vee - 1}), \qquad a_i \neq 0.$$

Since the remaining terms in the character expansion are exponentially suppressed, we have

$$\lim_{Y \to \infty} \frac{\mathrm{ch}_{X_i}(\mathrm{i}/Y)}{e^{2\pi(c/24 - h_*)Y} Y^{d_i^\vee}} = a_i.$$

The left-hand side is a limit of positive real numbers, so $a_i > 0$. Hence every nonzero $b_i(Y)$ is positive for sufficiently large Y. It remains to show that $b(Y)$ is not identically zero. By the vacuum-character asymptotic established in the proof of Theorem 3.4,

$$\mathrm{ch}_{X_0}(\mathrm{i}/Y) \sim Ce^{\pi r Y/12}, \qquad C = \frac{p^{-r/2}}{\mathrm{vol}(Q)} \frac{\mathscr{D}(\rho^\vee/p)}{\mathscr{D}(\rho)} > 0.$$

Since $h_* = (c-r)/24$, comparison with the pseudotrace expansion gives

$$\lim_{Y\to\infty} b_0(Y) = C.$$

As $b_0(Y)$ is a polynomial, it follows that $b_0(Y) = C > 0$. In particular, $b(Y)$ is nonzero.

We next show that $b(Y)$ is an eigenvector of a positive fusion matrix, which will allow us to identify its components with the Frobenius–Perron dimensions of the simple modules. To that end, set

$$\mathsf{T} = \bigoplus_{i=0}^{b} X_i, \qquad M_{ij} = [\mathsf{T} \otimes X_i : X_j],$$

and notice that $M = (M_{ij})$ is a strictly positive matrix as rigidity gives an epimorphism

$$(X_j \otimes X_i^*) \otimes X_i \twoheadrightarrow X_j.$$

Since tensor products are exact, some simple composition factor F of $X_j \otimes X_i^*$ must satisfy

$$[F \otimes X_i : X_j] > 0.$$

As F occurs as a summand of T, we conclude that $M_{ij} > 0$ for every i, j. Now, as S_* is the unique simple module of least conformal weight h_*, by Schur's lemma, the open Hopf link acts on S_* by a scalar:

$$h_{\mathsf{T},S_*} = \lambda \,\mathrm{id}_{S_*}.$$

With this in mind, for each P_a, choose a composition series

$$0 = P_a^0 \subset P_a^1 \subset \cdots \subset P_a^{\ell_a^\vee} = P_a.$$

Taking generalized weight-h_* spaces gives a filtration of $(P_a)_{[h_*]}$. By Theorem 3.3, its nonzero successive quotients can only come from S_*. Since $h_{\mathsf{T},S_*} = \lambda \,\mathrm{id}_{S_*}$, naturality of open Hopf links implies that $h_{\mathsf{T},P_a} - \lambda$ lowers the filtration on $(P_a)_{[h_*]}$. Consequently, for some common positive integer d,

$$(h_{\mathsf{T},P_a} - \lambda)^d\big|_{(P_a)_{[h_*]}} = 0$$

for every a. This implies that $b(Y)$ is a generalized eigenvector of eigenvalue λ

$$(M - \lambda I)^d b(Y) = 0.$$

To see this, let $b_Z(Y)$ denote the coefficient of the lowest exponential in the character of any object $Z \in \mathcal{D}$. By McRae's formula and the multiplicativity of open Hopf links,

$$b_{\mathsf{T}^{\otimes j}\otimes X_i}(Y) = \sum_a \mathrm{ptr}^{\varphi_a}_{(P_a)_{[h_*]}} \left(h_{X_i,P_a} h^j_{\mathsf{T},P_a} e^{-2\pi Y N_{a,h_*}} \right).$$

Taking the binomial combination and using linearity of pseudotraces gives

$$\sum_{j=0}^{d} \binom{d}{j} (-\lambda)^{d-j} b_{\mathsf{T}^{\otimes j} \otimes X_i}(Y) = \sum_a \operatorname{ptr}^{\varphi_a}_{(P_a)_{[h_*]}} \left(h_{X_i,P_a} (h_{\mathsf{T},P_a} - \lambda)^d e^{-2\pi Y N_{a,h_*}} \right) = 0.$$

Here the last equality follows from the preceding nilpotence relation on $(P_a)_{[h_*]}$. On the other hand, by the definition of M and additivity of ordinary characters,

$$b_{\mathsf{T}^{\otimes j} \otimes X_i}(Y) = \sum_k (M^j)_{ik} b_k(Y) = (M^j b(Y))_i.$$

Substituting into the preceding identity gives

$$\big((M - \lambda I)^d b(Y)\big)_i = 0.$$

Since i was arbitrary, the claim follows.

By the Perron–Frobenius theorem, the strictly positive matrix M has a strictly positive left eigenvector u with eigenvalue equal to its spectral radius $\rho(M)$. For sufficiently large Y, we have shown that $b(Y)$ is nonzero and componentwise nonnegative. Hence

$$u^{\mathsf{t}} b(Y) > 0.$$

Pairing the preceding relation with u^{t} gives

$$0 = (\rho(M) - \lambda)^d u^{\mathsf{t}} b(Y).$$

It follows that $\lambda = \rho(M)$. Since the Perron eigenvalue of a strictly positive matrix is algebraically simple, its generalized eigenspace coincides with its ordinary one-dimensional eigenspace. Consequently,

$$M b(Y) = \rho(M) b(Y).$$

By uniqueness, we obtain

$$b_i(Y) = b_0(Y) \operatorname{FPdim}(X_i).$$

Finally, comparing the leading terms in the character asymptotics gives

$$\lim_{Y \to \infty} \frac{\operatorname{ch}_{X_i}(\mathrm{i}/Y)}{\operatorname{ch}_{X_0}(\mathrm{i}/Y)} = \operatorname{FPdim}(X_i).$$

This proves the claim.

□

Part II
Reconstruction and the logarithmic correspondence

4 Dictionary

4.1 Lattices

Let Q be the root lattice with simple roots $\alpha_1, \dots, \alpha_n$, normalized so that long roots have squared length 2 and short roots have squared length $2/t$, and let P be the weight lattice and $\lambda_1, \dots, \lambda_n$ the fundamental weights. If p is divisible by the lacing number t, then $\sqrt{p}Q$ is an even integral lattice.

Let $Q^\vee$ be the coroot lattice with simple coroots $\alpha_1^\vee, \cdots, \alpha_n^\vee$, normalized so that long roots have squared length $2t$ and short roots have squared length 2. Define $d_i^\vee = (\alpha_i^\vee, \alpha_i^\vee)/2$ and $\ell_i^\vee = p/d_i^\vee$. In particular, $Q^\vee$ (an even integral lattice) is dual to P and $\frac{1}{\sqrt{p}}Q^\vee$ is dual to $\sqrt{p}P$ which contains $\sqrt{p}Q$.

4.2 Lattice vertex algebra

The category of representations of the lattice vertex algebra is

$$\mathcal{C} = \mathrm{Rep}(V_{\sqrt{p}Q}) = \mathrm{Vect}_\Gamma^{\sigma,\omega}$$

where $\Gamma = Q^*/pQ$ and ω is the usual associator and σ the usual braiding associated to the discriminant form, see e.g. [Len25, Section 6.1]. For $\gamma \in \Gamma$, write $C_\gamma = V_{\sqrt{p}Q+\gamma/\sqrt{p}}$ for the corresponding simple object of $\mathcal{C} = \mathrm{Rep}(V_{\sqrt{p}Q})$. The conformal shift (1.2) multiplies the usual balancing by the character induced by $e^{-2\pi \mathrm{i} a(0)}$.

We have an exact sequence $0 \to Q^\vee/pQ \to \Gamma \to Q^*/Q^\vee \to 0$ with $Q^\vee/pQ = \bigoplus_i \mathbb{Z}/\ell_i^\vee\mathbb{Z}$ from $p\alpha_i = \ell_i^\vee \alpha_i^\vee$. The exact sequence gives $|\Gamma| = |P/Q| \prod_i \ell_i^\vee$.

4.3 Nichols algebra of screenings

The short screening operators are defined by lattice elements $-\alpha_i^\vee/\sqrt{p}$. Consequently the braiding scalars of pairs of screenings in lattice coordinates are represented by

$$q_{ij} = \exp(\pi \mathrm{i}(\alpha_i^\vee, \alpha_j^\vee)/p).$$

The Nichols algebra $\mathfrak{B}$ associated to this diagonal braiding matrix is the nilpotent part of the small quantum group $u_q(\mathfrak{g}^\vee)$ for $q = e^{\frac{\pi \mathrm{i}}{p}}$. In our reconstruction we use the inverse braiding matrix since our Nichols algebra lives in $\mathcal{C}^{rev}$. Lusztig's PBW basis $\prod_\alpha E_{\alpha^\vee}^{N_\alpha}$ with $N_\alpha < \ell_\alpha^\vee$ readily gives the following dimension

$$\dim \mathfrak{B}(q) = p^{N_l} m^{N_s} \tag{4.1}$$

with $m = p/t$.

5 Relative reconstruction from a splitting functor

5.1 A reconstruction theorem

Let $\mathcal{C}$ be a finite rigid braided monoidal category and let $\mathcal{B}$ be a finite rigid monoidal category. Assume we have a fully faithful monoidal functor

$$\iota : \mathcal{B} \longleftarrow \mathcal{C}$$

which is central, meaning it lifts to a braided monoidal functor $(\iota, c) : \mathcal{C} \to \mathcal{Z}(\mathcal{B})$ with half-braidings $c_{\iota(V),X} : \iota(V) \otimes X \xrightarrow{\sim} X \otimes \iota(V)$. Suppose we have on the other hand an exact faithful monoidal functor

$$\mathfrak{l} : \mathcal{B} \longrightarrow \mathcal{C}$$

such that $\mathfrak{l} \circ \iota \cong \mathrm{id}$. Then the main result in [LM25] is that $\mathcal{B}$ can be reconstructed as a category of comodules (or modules) over a Hopf algebra $\mathfrak{N}$ in $\mathcal{C}$. This extends Tannaka–Krein reconstruction for $\mathcal{C} = \mathrm{Vect}$, for which the existence of ι is trivial, and is a categorical version of the Radford–Majid projection theorem, for which $\iota, \mathfrak{l}$ translate to a pair of quotient and embedding between Hopf algebras realizing $\mathcal{C}, \mathcal{B}$. Note that without the existence of ι we can write $\mathcal{B}$ as modules over a Hopf monad in $\mathcal{C}$.

In the following we require a slightly more general version of [LM25] where $\mathfrak{l}$ is merely a quasi-monoidal functor, that is, there are choices of isomorphisms $J_{X,Y} : \mathfrak{l}(X) \otimes \mathfrak{l}(Y) \xrightarrow{\sim} \mathfrak{l}(X \otimes Y)$, which are normalized, but which are not assumed to fulfill the pentagon coherence condition. Then we claim that we can reconstruct $\mathcal{B}$ as comodules over a coquasi-Hopf algebra $\mathfrak{N}$ with reassociator $\Phi : \mathfrak{N}^{\otimes 3} \to 1$. For the classical quasi-Hopf reconstruction, see [Maj92].

Lemma 5.1 (Relative quasi-reconstruction)**.** *Let $\mathcal{C}$ be a finite rigid braided monoidal category, $\mathcal{B}$ a finite rigid monoidal category, $\iota : \mathcal{B} \hookleftarrow \mathcal{C}$ a fully faithful central monoidal functor and $\mathfrak{l} : \mathcal{B} \to \mathcal{C}$ a faithful exact quasi-monoidal functor, such that $\mathfrak{l} \circ \iota \cong \mathrm{id}$. Assume in addition that $\mathfrak{l}$ is compatible with the $\mathcal{C}$-bimodule structure via ι, meaning concretely that there are normalized natural isomorphisms*

$$L_{V,X} : V \otimes \mathfrak{l}(X) \longrightarrow \mathfrak{l}(\iota(V) \otimes X), \qquad R_{X,V} : \mathfrak{l}(X) \otimes V \longrightarrow \mathfrak{l}(X \otimes \iota(V)),$$

with bimodule coherences

$$\begin{aligned} L_{V\otimes W,X} &= L_{V,\iota(W)\otimes X} \circ (\mathrm{id} \otimes L_{W,X}), \\ R_{X,V\otimes W} &= R_{X\otimes\iota(V),W}(R_{X,V} \otimes \mathrm{id}), \\ R_{\iota(V)\otimes X,W}(L_{V,X} \otimes \mathrm{id}) &= L_{V,X\otimes\iota(W)}(\mathrm{id} \otimes R_{X,W}), \end{aligned} \tag{5.1}$$

assume it is compatible with the central structure

$$\mathfrak{l}(c_{V,X})\, L_{V,X} = R_{X,V}\, c_{V,\mathfrak{l}(X)}, \tag{5.2}$$

and assume it is compatible with the chosen quasi-tensor structure in restricting to it

$$J_{\iota(V),X} = L_{V,X}, \qquad J_{X,\iota(V)} = R_{X,V} \tag{5.3}$$

and satisfying the pentagon identities involving two copies of $\mathcal{B}$ (associators in $\mathcal{B}, \mathcal{C}$ are not written)

$$\begin{aligned} J_{\iota(V)\otimes X,Y}(L_{V,X} \otimes 1) &= L_{V,X\otimes Y}(\mathrm{id} \otimes J_{X,Y}), \\ J_{X,Y\otimes\iota(V)}(\mathrm{id} \otimes R_{Y,V}) &= R_{X\otimes Y,V}(J_{X,Y} \otimes \mathrm{id}), \\ J_{X\otimes\iota(V),Y}(R_{X,V} \otimes \mathrm{id}) &= J_{X,\iota(V)\otimes Y}(\mathrm{id} \otimes L_{V,Y}). \end{aligned} \tag{5.4}$$

Then there exists a coquasi-Hopf algebra $(\mathfrak{N}, \Phi)$ *in* $\mathcal{C}$ *and an equivalence of monoidal categories*

$$\mathcal{B} \xrightarrow{\sim} \mathrm{Comod}_{\mathcal{C}}(\mathfrak{N})$$

such that the functor to $\mathcal{C}$ *forgetting the coaction corresponds to* $\mathfrak{l}$ *and the functor to* $\mathcal{B}$ *endowing an object with the trivial coaction corresponds to* ι *and* L, R *are the identity.*

Proof. [LM25] applied to the right module functor $\mathfrak{l}, R$ produces a relative coend $\mathfrak{N}$ and Proposition 6.4 and Theorem 6.7 establish that it can be endowed with a coalgebra structure giving an equivalence of abelian categories $\mathcal{B} \xrightarrow{\sim} \mathrm{Comod}_{\mathcal{C}}(\mathfrak{N})$. Section 6.4 describes how to endow $\mathfrak{N}$ with an algebra structure such that this becomes an equivalence of monoidal categories if $\mathfrak{l}$ is a monoidal functor. If $\mathfrak{l}$ is merely a quasi-monoidal functor with quasi-monoidal structure $J_{X,Y}$, we proceed similarly:

The left action comes from the assumed central structure and (5.2). For left comodules $M_1, \dots, M_n$, let

$$\rho^{(n)} : \bigotimes_j \mathfrak{l}(M_j) \longrightarrow \mathfrak{N}^{\otimes n} \otimes \bigotimes_j \mathfrak{l}(M_j)$$

be the tensor product of their coactions, which uses the braiding. A natural transformation

$$\bigotimes_j \mathfrak{l}(M_j) \longrightarrow W \otimes \bigotimes_j \mathfrak{l}(M_j)$$

is called balanced if it respects the left action in the first factor, the right action in the last factor, and the identification of a right action on the j-th factor with the left action on the $j+1$-th factor. In the left module constraint, a new object passes W by the braiding, while the right constraint keeps W on the left.

Such natural transformations precisely correspond to maps:

$$\mathrm{Hom}_{\mathcal{C}}(\mathfrak{N}^{\otimes n}, W) \cong \mathrm{Nat}_{\mathcal{C}}^{\mathrm{balanced}}(\bigotimes_j \mathfrak{l}(M_j), W \otimes \bigotimes_j \mathfrak{l}(M_j)). \tag{5.5}$$

Indeed, the coaction $M \to \mathfrak{N} \otimes \mathfrak{l}(M)$ is a comodule morphism into a cofree left comodule. Naturality and balancing can be used to move out $\mathfrak{l}(M_j)$. Applying the counits gives

$$\Theta_{M_1,\dots,M_n} = (f \otimes 1)\rho^{(n)}_{M_1,\dots,M_n}, \qquad f = (1_W \otimes \varepsilon_N^{\otimes n})\Theta_{\mathfrak{N},\dots,\mathfrak{N}}. \tag{5.6}$$

Conversely, that formula defines a balanced family for every f, by naturality of the coactions and of the braiding. This proves both existence and uniqueness in (5.5). We next discuss multiplication and reassociator: Transport the coaction on $\mathfrak{l}(X \otimes Y)$ through J:

$$\rho_{X,Y} = (1 \otimes J^{-1}_{X,Y})\rho_{X\otimes Y} J_{X,Y}. \tag{5.7}$$

The three identities (5.4) make this a balanced family. Equations (5.5)–(5.6) give a unique morphism $\mu : \mathfrak{N} \otimes \mathfrak{N} \to \mathfrak{N}$ with $\rho_{X,Y} = (m \otimes 1)\rho^{(2)}_{X,Y}$. The coaction identities say that m is a morphism of coalgebras for the braided tensor-coalgebra structure. The coaction on the tensor unit supplies a group-like unit; the unit identities for m follow from (5.3). Write $a^{\mathcal{B}}$ and $a^{\mathcal{C}}$ for the ambient associators. Define the invertible natural transformation $\Phi_{X,Y,Z}$ of $(\mathfrak{l}(X) \otimes \mathfrak{l}(Y)) \otimes \mathfrak{l}(Z)$ by

$$\begin{aligned} J_{X,Y\otimes Z}(1 \otimes J_{Y,Z}) a^{\mathcal{C}}_{\mathfrak{l}(X),\mathfrak{l}(Y),\mathfrak{l}(Z)} \Phi_{X,Y,Z} \\ = \mathfrak{l}(a^{\mathcal{B}}_{X,Y,Z}) J_{X\otimes Y,Z}(J_{X,Y} \otimes 1). \end{aligned} \tag{5.8}$$

The mixed identities make Φ balanced. Under the correspondence it becomes a (convolution-invertible) morphism $\Phi : \mathfrak{N}^{\otimes 3} \to 1$. The condition that the transported associator is a comodule morphism is the coquasi-associativity identity. Applying the coefficient correspondence for $n = 4$ to the pentagon in $\mathcal{B}$ gives the coquasi pentagon. The unit and boundary identities give the counital normalizations of Φ. We have hence constructed a coquasi-bialgebra.

We now consider $\mathfrak{l}(\iota(V))$: By (5.2) and (5.3) this is the trivial comodule structure on N and its assigned half-braiding is the braiding with $\mathfrak{l}(-)$, so the central lift is preserved. Because all simple objects come from $\mathcal{C}$, the subcoalgebras are all $\eta(1)$ and hence $\mathfrak{N}$ is connected. Rigidity supplies the antipode in the Hopf reconstruction. □

5.2 The split scenario

We now assume $\iota : \mathcal{C} \hookrightarrow \mathcal{B}$ is a full central semisimple subcategory closed under tensor product and duals, and assume that every simple object in $\mathcal{B}$ is contained in the image of $\mathcal{C}$. Then there is an obvious choice of a splitting functor

Definition 5.2. Let $\mathfrak{l} : \mathcal{B} \to \mathcal{C}$ be the functor sending any object X in $\mathcal{B}$ to the direct sum of its composition factors, which lie in $\mathcal{C}$, and restricting any morphism to the respective composition factors as made precise below.

We give a more formal construction of this functor, which exposes additional interesting structure: Let $\varepsilon : P_1 \to 1$ be the projective cover of the tensor unit in $\mathcal{B}$. We can now consider the exact functor $\mathcal{C} \to \mathcal{B}$ given by $F : V \mapsto V \otimes P_1$. Let $\mathfrak{l}$ be the right adjoint functor, which is the internal $\underline{Hom}(P_1, -)$ as a $\mathcal{C}$ module category. It exists by right exactness and finiteness, and it has the defining property

$$\mathrm{Hom}_{\mathcal{C}}(V, \mathfrak{l}(X)) = \mathrm{Hom}_{\mathcal{B}}(\iota(V) \otimes P_1, X),$$

From this definition, it is clear that $\mathfrak{l}(X)$ is the direct sum of its composition factors, since $\iota(V) \otimes P_1$ is a projective cover of $\iota(V)$. The counit of the adjunction is $e : \iota(\mathfrak{l}(X)) \otimes P_1 \to X$, which can be interpreted as embedding any composition factor of X into X, which is possible after tensoring with P_1.

We now discuss how this adjoint functor obtains a quasi-monoidal structure: Consider the fibre product $P_1 \times_\varepsilon P_1$ and the map $P_1 \otimes P_1 \longrightarrow P_1 \times_\varepsilon P_1$. Projectivity of P_1 lifts the identity map to

$$\Delta_{P_1} : \; P_1 \longrightarrow P_1 \otimes P_1$$

which is by construction counital, but not a priori coassociative. We remark that this is the shadow of a homotopy coherent coalgebra object $P_\bullet$ in the derived category.

Lemma 5.3. *Assume that $\mathcal{C}$ is a finite rigid semisimple braided monoidal category, $\mathcal{B}$ a finite rigid monoidal category. Assume $\iota : \mathcal{C} \hookrightarrow \mathcal{B}$ is a full central subcategory closed under tensor product and duals, and assume that every simple object in $\mathcal{B}$ is contained in the image of $\mathcal{C}$. Then $\mathfrak{l}$ is a faithful exact functor that fulfills $\mathfrak{l} \circ \iota = \mathrm{id}$ and carries a quasi-monoidal structure.*

Proof. All properties follow from standard adjoint functor calculus:

- $\mathrm{Hom}_{\mathcal{C}}(V, \mathfrak{l}(\iota(W))) \cong \mathrm{Hom}_{\mathcal{B}}(\iota(V) \otimes P_1, \iota(W))$ and the right-hand side contains for simple V, W a unique morphism given by $\varepsilon : P_1 \to 1$ iff $V \cong W$. Hence $\mathfrak{l} \circ \iota = \mathrm{id}$.

- As a right adjoint, $\mathfrak{l}$ is automatically left exact, and it is right exact because $F(V)$ is projective in a rigid category.
- An exact functor $\mathfrak{l}$ is faithful if it does not kill any nonzero object X. Since any nonzero X has a simple quotient $\iota(V)$ and ε lifts to a nonzero map $\iota(V) \otimes P_1 \to \iota(V)$, the object X is not sent to zero.
- A right adjoint to an oplax monoidal functor is lax monoidal, and the same holds without assuming pentagon identities. The functor F has an oplax quasi-monoidal structure $F(V \otimes W) \to F(V) \otimes F(W)$ using Δ_{P_1} and the central structure, so the right adjoint $\mathfrak{l}, J$ is lax quasi-monoidal. To prove that it is strong quasi-monoidal, we use that a natural transformation between biexact functors is a natural isomorphism iff it is an isomorphism for all pairs of simple objects. But for simple objects it is clear that the quasi-monoidal structure J is an isomorphism.

□

We now show that the Reconstruction Lemma 5.1 can be applied in this situation, and to do so we have to provide the additional invertible bimodule constraints $L_{V,X} : V \otimes \mathfrak{l}(X) \to \mathfrak{l}(\iota(V) \otimes X)$ and $R_{X,V} : \mathfrak{l}(X) \otimes V \to \mathfrak{l}(X \otimes \iota(V))$. Under the adjunction, these morphisms in $\mathcal{C}$ can be defined from the corresponding morphisms in $\mathcal{B}$

$$\widehat{L_{V,X}} : \iota(V \otimes \mathfrak{l}(X)) \otimes P_1 \longrightarrow \iota(V) \otimes X$$
$$\widehat{R_{X,V}} : \iota(\mathfrak{l}(X) \otimes V) \otimes P_1 \longrightarrow X \otimes \iota(V)$$

The morphism $\widehat{L_{V,X}}$ can be defined from the monoidal structure of ι and e, while the morphism $\widehat{R_{X,V}}$ can be defined from the monoidal and the central structure of ι and e; note that this asymmetry comes from the choice $(-) \otimes P_1$ versus $P_1 \otimes (-)$. Now coherence conditions in Lemma 5.1 can be obtained from all available compatibilities. We summarize this in the following table: For $g : V \to \mathfrak{l}(X)$ and $h : W \to \mathfrak{l}(Y)$ we can compute the mates $\widehat{g}, \widehat{h}$ under the adjunction

$$\widehat{L_{V,X}(1 \otimes g)} = 1 \otimes \hat{g},$$
$$\widehat{R_{X,V}(g \otimes 1)} = (\hat{g} \otimes 1)(1 \otimes c_{V,P_1}),$$
$$\widehat{J_{X,Y}(g \otimes h)} = (\hat{g} \otimes \hat{h})(1 \otimes \tau_{W,P_1} \otimes 1)(1 \otimes d).$$

Identity	**Identity in $\mathcal{A}$**	**Structure used**
(i) $L_{V \otimes d} = L_V(1 \otimes L_d)$	Both sides are $1 \otimes e_X$	Monoidal structure of ι
(i) $R_{X,V \otimes W} = R(R \otimes 1)$	$c_{V \otimes W,P_1}$ $= (c_{V,P_1} \otimes 1)(1 \otimes c_{W,P_1})$	ι is monoidal into $Z(\mathcal{B})$ (half-braiding of a tensor product)
(i) left and right commute	Left action is on the far left; associativity	Associator of $\mathcal{B}$
(i) units; constraints on $\iota(\mathcal{C})$	$e_{\iota(W)} = 1 \otimes \varepsilon$; $(\varepsilon \otimes 1)c_{W,P_1} = 1 \otimes \varepsilon$	Naturality of c in $\varepsilon : P_1 \to 1$

Identity	Reduced identity in $\mathcal{A}$	What it uses
(ii) $\mathsf{L}(c_{V,X})\ell = r\,\tau_{V,\mathfrak{l}(X)}$	$c_{V,X}(1\otimes e_X)$ $= (e_X\otimes 1)(1\otimes c_{V,P_1})(c_{V,\iota(\mathfrak{l}(X))}\otimes 1)$	Naturality of c in e_X; hexagon in the second variable; braided lift $c_{V,\iota(U)} = \iota(\sigma_{V,U})$
(iii) $J_{\iota(V),Y} = L_{V,Y}$	$(\varepsilon\otimes 1)W = 1$	Left counitality, c natural in ε
(iii) $J_{X,\iota(V)} = R_{X,V}$	$(1\otimes\varepsilon)W = 1$	Right counitality
(iii) $J_{\mathbf{1},X} = 1 = J_{X,\mathbf{1}}$	The cases $V = 1$ above	Both counitalities
(iv) $J(L\otimes 1) = L(1\otimes J)$	The two mates are literally equal	Nothing
(iv) $J(1\otimes R) = R(J\otimes 1)$	$(W\otimes 1)\tau_{V,P_1} = c_{V,P_1\otimes P_1}(1\otimes W)$, $c_{V,P_1\otimes P_1} = (1\otimes\tau_{V,P_1})(\tau_{V,P_1}\otimes 1)$	Naturality of c_V in W; Hexagon
(iv) $J(R\otimes 1) = J(1\otimes L)$	$\tau_{V\otimes U,P_1} = (c_{V,P_1}\otimes 1)(1\otimes\tau_{U,P_1})$ with $U = \mathfrak{l}(Y)$	Hexagon

As a consequence we can apply Lemma 5.1 and obtain:

Lemma 5.4. *Assume that $\mathcal{C}$ is a finite rigid semisimple braided monoidal category, $\mathcal{B}$ a finite rigid monoidal category. Assume $\iota : \mathcal{C} \hookrightarrow \mathcal{B}$ is a full central subcategory closed under tensor product and duals, and assume that every simple object in $\mathcal{B}$ is contained in the image of $\mathcal{C}$. Then there exists a coquasi-Hopf algebra $(\mathfrak{N}, \Phi)$ in $\mathcal{C}$ and an equivalence of monoidal categories*

$$\mathcal{B} \xrightarrow{\sim} \mathrm{Comod}_{\mathcal{C}}(\mathfrak{N})$$

Since all simple objects are by construction trivial comodules, $\mathfrak{N}$ is connected.

6 Cohomology and liftings of Nichols algebras

6.1 Cohomology and lifting

Let $\mathcal{C}$ be a semisimple $\Bbbk$-linear abelian braided monoidal category with bilinear tensor product. Let $\mathfrak{N}$ be an algebra with augmentation $\epsilon : \mathfrak{N} \to 1$. Define $\mathfrak{N}^+ = \ker(\epsilon)$ and consider 1 a $\mathfrak{N}$-module via ϵ. Tensoring the normalized bar resolution of the trivial module 1_ϵ with 1_ϵ gives the reduced bar complex

$$1 \xleftarrow{\mathrm{d}_1=0} \mathfrak{N}^+ \xleftarrow{\mathrm{d}_2} (\mathfrak{N}^+)^{\otimes 2} \xleftarrow{d_3} (\mathfrak{N}^+)^{\otimes 3} \cdots$$

$$d(b_1\otimes\cdots\otimes b_n) = \sum_{i=1}^{n-1}(-1)^i\,(b_1\otimes\cdots b_ib_{i+1}\cdots\otimes b_n)$$

The homology of this complex is $\mathrm{Tor}_n^{\mathfrak{N}}(1,1)$. In small degrees

$$\mathrm{Tor}_1^{\mathfrak{N}}(1,1) = \mathfrak{N}^+/(\mathfrak{N}^+)^2$$
$$\mathrm{Tor}_2^{\mathfrak{N}}(1,1) = \ker(\mathfrak{N}^+\otimes_{\mathfrak{N}}\mathfrak{N}^+ \to \mathfrak{N}^+)$$

The interpretation for a graded algebra $\mathfrak{N}$ is that Tor_1 are the generators, Tor_2 are the relations, see [AKM15] Corollary 4.4, then further Tor_3 are relations in the ideal of relations and so on. For any trivial $\mathfrak{N}$-bimodule X we have for semisimple $\mathcal{C}$ a comparison to the Hochschild cohomology computed inside $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{Tor}_n^{\mathfrak{N}}(1,1), X) = \mathrm{HH}^n_{\mathcal{C}}(\mathfrak{N}, X).$$

Remark 6.1. Semisimplicity of $\mathcal{C}$ can be replaced for $n = 2$ by the splitting of the multiplication map as a morphism in $\mathcal{C}$, see [MW09] Section 4.1 and [AKM15] Lemma 4.1

Remark 6.2. For an augmented algebra in vector spaces and X a trivial bimodule, a classical result is

$$\mathrm{HH}^n(\mathfrak{N}, X) \cong \mathrm{Ext}^n_{\mathfrak{N}}(\mathbf{1}, X) \cong \mathrm{Hom}_{\Bbbk}\big(\mathrm{Tor}^{\mathfrak{N}}_n(\mathbf{1}, \mathbf{1}), X\big).$$

If $F : \mathcal{C} \to \mathrm{Vect}$ is a fibre functor to vector spaces, then the Tor in both categories coincide, and the identification of Hochschild cohomology with respective homomorphisms gives an embedding

$$\mathrm{HH}^n_{\mathcal{C}}(\mathfrak{N}, X) \subset \mathrm{HH}^n(F(\mathfrak{N}), F(X))$$

The relation between these cohomologies is discussed in [AAM17] Remark 3.1 and [AAM17] Corollary 4.3 and Proposition 4.7, following [AMŞ07].

Theorem 6.3. *Let $\mathcal{C}$ be a rigid finite semisimple braided monoidal category and H, Φ a connected coquasi-bialgebra in $\mathcal{C}$, where $\Phi : H^{\otimes 3} \to 1$ is the associator, and let $\mathfrak{N} := \mathrm{gr}(H)$, which is a bialgebra* [AAM17] *Theorem 1.5.*

a) Suppose $\mathrm{HH}^3_{\mathcal{C}}(\mathfrak{N}, 1) = 0$, then the quasi-bialgebra H is equivalent to a bialgebra H' (with the same graded algebra) up to gauge transformation in $\mathcal{C}$.

b) Suppose $\mathrm{HH}^2_{\mathcal{C}}(\mathfrak{N}, \mathfrak{N}_1) = 0$ and $\mathfrak{N}$ generated in degree 1, then $\mathfrak{N} \cong H'$.

Proof. Assertion a) is proven in [AAM17] Theorem 3.2 if $\mathcal{C}$ is a category of Yetter–Drinfeld modules, but the proof carries over to an arbitrary rigid finite braided monoidal category, which in particular has an exact tensor product.

For convenience we summarize the proof: Let $\Phi : H^{\otimes 3} \to 1$ be the associator of the coquasi-bialgebra. The degree 0 term is $\Phi_0 = \epsilon^{\otimes 3}$. Let Φ_d be the first nontrivial degree, then the pentagon implies $\mathrm{d}\Phi_d = 0$. The assumed vanishing of the third cohomology group implies the existence of J, normalized and of degree d, such that $\mathrm{d}J = \Phi_d$. Clearly $\epsilon^{\otimes 2} + J$ is invertible by connectedness, and after gauging by J the first nontrivial degree is higher. By [AAM17] Proposition 2.6 gauging does not change the $\mathrm{gr}(H)$, so the assumption still holds and by induction we can continue and remove every degree. Note that we used that as object in $\mathcal{C}$ the filtration of H can be split. The induction terminates as for an object in a locally finite category (such as finite-dimensional vector spaces) the filtration terminates.

Assertion b) is the first main result in [AKM15] Theorem 5.3 for a category with a fibre functor to vector spaces. A relation of minimal degree maps to a primitive element, giving a morphism $\mathrm{Tor}_2 \to \mathfrak{N}_1$, which vanishes. Then we obtain an isomorphism, since it is an isomorphism on the graded algebra.

□

We remark that an alternative way is to pull back along $\mathbb{Z}^n \to \Gamma$, since on the free group $\mathbb{Z}^n$ every quadratic form comes from a bicharacter χ. Hence Tor in $\mathcal{C}$ is the pullback of the ordinary $\mathbb{Z}^n$-graded Tor.

6.2 Robustness

We call a Nichols algebra $\mathfrak{N}$ in a locally finite semisimple braided monoidal category $\mathcal{C}$ *robust* if it fulfills the cohomological conditions in Theorem 6.3, and thus any connected coquasi-Hopf algebra H with $\mathrm{gr}(H) = \mathfrak{N}$ is gauge equivalent to $\mathfrak{N}$. We now give conditions for robustness:

Lemma 6.4. *Let* $\mathcal{C} = \mathrm{Vect}_\Gamma^{\sigma,\omega}$ *where* Γ *is* sufficiently unrolled *in the sense that if* $D \subset \mathbb{Z}^n$ *is the set of all degrees appearing in* $\mathfrak{N}$ *then the map* $\mathbb{Z}^n \to \Gamma$ *is injective on* $D + D + D$. *Then* $\mathfrak{N}$ *is robust.*

Lemma 6.5. *For the Nichols algebra* $\mathfrak{N} = \mathfrak{B}(q)$ *in* $\mathrm{Vect}_\Gamma^{\mathfrak{q}}$, *the* $\mathbb{Z}^n$*-degrees of* $\mathrm{Tor}_2(1,1)$ *are contained in*

$$\ell_\beta^\vee \beta, \qquad \beta + \gamma \quad (\beta, \gamma \text{ distinct positive roots}). \tag{6.1}$$

and the $\mathbb{Z}^n$*-degrees of* $\mathrm{Tor}_3(1,1)$ *are contained in*

$$\ell_\beta^\vee \beta + \gamma, \qquad (\beta, \gamma \text{ may coincide}), \qquad \beta_1 + \beta_2 + \beta_3 \quad (\beta_1, \beta_2, \beta_3 \text{ distinct positive roots}).$$

Proof. Choose a convex Lyndon PBW order on the positive coroots. The straightening formulas have skew-commuting leading terms and corrections supported strictly between the endpoint roots [Ang15, Theorems 3.11, 3.16 and Lemmas 4.5, 4.7–4.8]. If the ordered roots are $\beta^{(1)}, \dots, \beta^{(s)}$, give the jth root vector weight M^{s-j}, with M greater than all total correction exponents. Every correction then has smaller weight. The root powers vanish, and the bounded PBW basis [Ang15, Theorem 4.9] identifies the associated graded with the quantum complete intersection

$$S = \mathbb{C}\langle z_\beta : \beta \in \Phi_+^\vee \rangle / (z_\beta^{\ell_\beta^\vee},\ z_\beta z_\gamma - q_{\beta\gamma} z_\gamma z_\beta \ (\beta \prec \gamma)).$$

This filtration preserves fine degree.

The twisted tensor product of the periodic one-variable resolutions [MPSW10, Section 4] has one-variable generator degrees $\beta, \ell_\beta^\vee \beta, (\ell_\beta^\vee + 1)\beta$ in homological degrees one, two, and three. Splitting homological degree two or three among distinct variables gives exactly the lists above. The same periodic resolution applies when $\ell_\beta^\vee = 2$. The filtered bar complex gives a spectral sequence from the bar homology of S to that of B. Its differentials preserve fine degree, so the abutment has no additional supporting degree. □

Corollary 6.6. *The Nichols algebra* $\mathfrak{N} = \mathfrak{B}(q)$ *Cartan type with* $\mathrm{ord}(q_{ii}) > -a_{ij}$, $\ell_\beta^\vee = \mathrm{ord}(q_\beta)$ *in* $\mathrm{Vect}_\Gamma^{\sigma,\omega}$ *given by the Feigin-Tipunin conjecture is robust, except possibly in the cases* $(A_2, p = 2)$ *and* $(B_2, p = 4)$

Proof. Cartan integrality gives $t(\alpha,\alpha)/2 \in \mathbb{Z}$ for $\alpha \in Q$. Positive definiteness and the coroot lengths imply

$$\|p\alpha\| \geq m\sqrt{2t} \quad (\alpha \in Q \setminus \{0\}), \qquad \|\beta\| \leq \sqrt{2t} \quad (\beta \text{ a coroot}). \tag{6.2}$$

The parameter range gives $m \geq 2$. Since $\ell_\beta^\vee \beta = p\alpha \in pQ$, a mixed degree could vanish only if $\gamma \in pQ$, contradicting the bound. For three distinct positive coroots,

$$0 < \|\beta_1 + \beta_2 + \beta_3\| < 3\sqrt{2t}.$$

Strictness follows because distinct roots in a reduced root system are not positively collinear. This excludes a period when $m \geq 3$.

When $m = 2$, the inequality $(\rho^\vee, \theta_s) \leq 2$ leaves $A_1, A_2, B_2 = C_2$. There is no triple in A_1. In A_2 the three positive roots sum to $2\alpha_1 + 2\alpha_2$. In B_2, take $Q = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, with positive coroots $e_1 - e_2, e_1 + e_2, 2e_1, 2e_2$. The four triple sums are

$$4e_1, \quad 2e_1 + 2e_2, \quad 3e_1 + e_2, \quad 3e_1 + 3e_2.$$

Only $4e_1$ belongs to $4Q$; its simple-coroot degree is $(4,2)$.

Now we turn to HH^2: By (6.1), a relation degree is either $\ell_\beta^\vee \beta$, which is zero in H, or $\beta+\gamma$ for distinct positive coroots. A primitive degree β_i is nonzero by (6.2). For $m \geq 3$, equality of the second type with β_i modulo pQ would give a nonzero period with

$$\|\beta+\gamma-\beta_i\| < 3\sqrt{2t} \leq m\sqrt{2t}.$$

The vector is nonzero because a simple coroot is not a sum of two positive coroots; the inequality is strict by reducedness. This is impossible. At $m=2$, the presentations in Lemma 6.7, together with $B(A_1, p=2) = \mathbb{C}[x]/(x^2)$, have all relation degrees zero in H except the B_2 degree $(3,1)$. It differs from the primitive degrees $(1,0),(0,1)$ in $\mathbb{Z}/4 \oplus \mathbb{Z}/2$. □

6.3 Robustness for $(A_2,2)$ and $(B_2,4)$

Lemma 6.7 (The two exceptional minimal-syzygy spaces vanish)**.** *For the finite Cartan-type Nichols algebras in the indicated bicharacter normalization,*

$$\mathrm{Tor}_3^{B(A_2,2)}(\mathbb{C},\mathbb{C})_{(2,2)} = 0, \qquad \mathrm{Tor}_3^{B(B_2,4)}(\mathbb{C},\mathbb{C})_{(4,2)} = 0.$$

Proof. For an algebra $A = \mathbb{C}\langle x,y\rangle/(f_1,\dots,f_s)$, set $P_1 = Ae_x \oplus Ae_y$ and $P_2 = \bigoplus_j Ae_{f_j}$. Use the last-letter decomposition $f = f_x x + f_y y$ to define

$$P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} A \xrightarrow{\varepsilon} \mathbb{C} \longrightarrow 0, \qquad d_2(e_f) = [f_x]e_x + [f_y]e_y. \tag{6.3}$$

This sequence is exact at P_1: differentiation of a two-sided relation multiple gives a left multiple of its derivative, while a positive right factor contributes coefficients already zero in A. With $K = \ker d_2$, dimension shifting gives

$$\mathrm{Tor}_3^A(\mathbb{C},\mathbb{C}) \cong \ker(\mathbb{C}\otimes_A K \to \mathbb{C}\otimes_A P_2) \subseteq K/A_+K. \tag{6.4}$$

Thus $K_\nu \subseteq (A_+K)_\nu$ suffices; the presentation need not be minimal.

For $A_2, p=2$, put $z = xy+yx$. The presentation and PBW basis are

$$A = \mathbb{C}\langle x,y\rangle/(x^2,y^2,z^2), \qquad \{x^a z^b y^c : 0 \leq a,b,c < 2\},$$

by [Ang09, Theorem 5.14]. In degree $\nu=(2,2)$, $(P_2)_\nu = \mathbb{C}e_{z^2}$ because $A_{(0,2)} = A_{(2,0)} = 0$. Moreover

$$d_2(e_{z^2}) = zy\,e_x + xz\,e_y \neq 0.$$

The coefficients are nonzero PBW monomials, so $K_\nu = 0$.

For $B_2, p=4$, take $q_{xx} = -\mathrm{i}$, $q_{yy} = q_{xy} = -1$, $q_{yx} = 1$, and put $z = xy+yx$, $w = xz - \mathrm{i}zx$, $s = xw - wx$. The specialized rank-two presentation [Ang09, Theorem 5.19] is

$$A = \mathbb{C}\langle x,y\rangle/(x^4,y^2,s,w^2,z^4), \qquad \{x^a w^b z^c y^d : 0 \leq a,c < 4,\ 0 \leq b,d < 2\}. \tag{6.5}$$

The root-vector degrees are $(1,0),(2,1),(1,1),(0,1)$. The identities needed below are

$$yx = -xy + z, \qquad zx = -\mathrm{i}xz + \mathrm{i}w, \qquad wx = xw. \tag{6.6}$$

The remaining rank-two commutations follow from the relations: $yz - zy = y^2x - xy^2$; expansion modulo (y^2) gives $yw + wy = (1-\mathrm{i})z^2$ and $zw - wz = \mathrm{i}(ys - sy)$.

In degree $\nu = (4,2)$,

$$(P_2)_\nu = \mathbb{C}xy\, e_s \oplus \mathbb{C}z\, e_s \oplus \mathbb{C}e_{w^2}. \tag{6.7}$$

The x^4, y^2 summands have zero coefficients, and the z^4 relation has larger degree. Set $a = x^2y - \mathrm{i}xz - w$. Then

$$d_2(e_s) = ae_x + x^3e_y, \qquad d_2(e_{w^2}) = w(xy - \mathrm{i}z)e_x + x^2we_y.$$

Using (6.6) gives $xyx^3 = zx^3 = \mathrm{i}x^3z + x^2w$. The e_y coefficients of the three columns in (6.7) are therefore

$$\mathrm{i}x^3z + x^2w, \qquad \mathrm{i}x^3z + x^2w, \qquad x^2w.$$

Independence of the PBW monomials x^3z, x^2w shows that the rank is at least two. On the other hand,

$$\kappa = xe_s + ye_{x^4} \in (P_2)_{(4,1)}, \qquad d_2\kappa = 0,$$

because $xa + yx^3 = -\mathrm{i}s$ in the free algebra and $x^4 = 0$ in A. Moreover $y\kappa = (z - xy)e_s \neq 0$. Thus K_ν is the one-dimensional span of $y\kappa$ and is contained in A_+K. Equation (6.4) proves the second vanishing. □

7 Concluding the proof

We apply the reconstruction of [CLR23, Len25] to the lattice extension $\mathcal{W}_{\sqrt{p}Q} \subset V_{\sqrt{p}Q}$. The character calculation in Part I will tell us when the screening extensions account for the entire category of modules over the lattice algebra.

Theorem 7.1 (Logarithmic Kazhdan–Lusztig correspondence)**.** *For the parameters of Section 1.2, there is a braided equivalence*

$$\mathrm{Rep}_{\mathrm{gr}}(\mathcal{W}_{\sqrt{p}Q}) \simeq_{\mathrm{br}} \mathcal{Z}_{\mathcal{C}^{\mathrm{rev}}}\big(\mathrm{Comod}_{\mathcal{C}^{\mathrm{rev}}}(\mathfrak{B}(q))\big), \qquad \mathcal{C} = \mathrm{Rep}(V_{\sqrt{p}Q}).$$

Here $\mathfrak{B}(q) = \mathfrak{B}_{\mathcal{C}^{\mathrm{rev}}}(X)$ is the screening Nichols algebra of Section 4.3, and all comodules are left comodules. The relative center is taken with respect to the specified central copy of $\mathcal{C}^{\mathrm{rev}}$, with its pointed associator. With the lattice normalization (7.5), the equivalence is ribbon.

Proof. We follow the order of the introduction.

a) Put $\mathcal{D} = \mathrm{Rep}_{\mathrm{gr}}(\mathcal{W}_{\sqrt{p}Q})$. By Theorem 2.5 and McRae's rigidity theorem, $\mathcal{D}$ is a factorizable finite ribbon category (Corollary 2.6).

b) The lattice extension $A = V_{\sqrt{p}Q}$ is a commutative algebra in $\mathcal{D}$. We first check the hypotheses for passing to its category of modules. An ideal subobject is stable under every coefficient of the extension multiplication and hence is a vertex-algebra ideal. Simplicity of the lattice vertex algebra makes A simple as an algebra object. It is also haploid. A map from the unit sends the vacuum into $\ker L(-1)$, and translation is unchanged by the conformal shift. In the unshifted positive Hermitian lattice grading a vector v of degree $n > 0$ satisfies

$$\|L(-1)v\|^2 = 2n\|v\|^2 + \|L(1)v\|^2.$$

Only the vacuum line lies in the kernel. Finally the twist on A is trivial, since $\Delta(\sqrt{p}\, q) = \frac{p}{2}(q,q) - (s,q) \in \mathbb{Z}$ for $q \in Q$. Here $\sqrt{p}\, Q$ is even and $(s, \alpha_i) = p/d_i^\vee - 1 \in \mathbb{Z}$. Exactness of this simple algebra follows from [CSZ25, Theorem 7.1].

Thus the right A-modules form a finite tensor category $\mathcal{B} = \mathcal{D}_A$. To form the relative tensor product, the right action ρ_M induces the left action $\rho_M c^{-1}_{M,A}$, where c is the braiding of $\mathcal{D}$. This is the convention of [SY24] applied to $\mathcal{D}^{\mathrm{rev}}$. On local modules the two induced actions agree. The extension theorem [CKM24] therefore identifies the local subcategory, with its ordinary tensor product, as

$$\mathcal{C} = \mathcal{D}_A^{\mathrm{loc}} = \mathrm{Rep}(A), \qquad \Gamma = Q^*/pQ \simeq (\sqrt{p}\,Q)^*/(\sqrt{p}\,Q).$$

Write C_γ for its simple object of degree $\gamma \in \Gamma$.

c) There are two central functors into $Z(\mathcal{B})$. For a local L, the inverse braiding $c^{-1}_{M,L}$ descends to $L \otimes_A M \to M \otimes_A L$. On two local modules this is the reversed lattice braiding, giving the specified inclusion $i : \mathcal{C}^{\mathrm{rev}} \to Z(\mathcal{B})$. For the free functor $F_A(X) = X \otimes A$, the central lift from $\mathcal{D}$ uses $c_{X,M}$, under $F_A(X) \otimes_A M \simeq X \otimes M$ and $M \otimes_A F_A(X) \simeq M \otimes X$.

The Schauenburg comparison identifies $Z(\mathcal{B}) \simeq_{\mathrm{br}} \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{D}$ with these central functors [SY24, Corollary 5.14], [CLR23, Corollary 3.8]. In particular, on writing $\delta_A = \mathrm{FPdim}_{\mathcal{D}}(A)$, we have [SY24, Lemmas 5.15 and 5.17]

$$\mathcal{D} \simeq_{\mathrm{br}} Z_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{B}), \qquad \mathrm{FPdim}(\mathcal{B}) = |\Gamma|\delta_A, \qquad \mathrm{FPdim}(\mathcal{D}) = |\Gamma|\delta_A^2. \tag{7.1}$$

It remains to identify $\mathcal{B}$ together with this central lattice category.

d) Begin with the full Serre subcategory $\mathcal{B}^{\mathrm{split}}$ of modules whose composition factors are the C_γ. The C_γ are invertible, so exactness of tensor products and duality make $\mathcal{B}^{\mathrm{split}}$ a finite tensor subcategory. It inherits the specified central lift. Let $P \twoheadrightarrow \mathbf{1}$ be the projective cover in $\mathcal{B}^{\mathrm{split}}$. Lemma 5.3 gives the exact faithful functor

$$\mathsf{L}(M) = \bigoplus_{\gamma \in \Gamma} \mathrm{Hom}_{\mathcal{B}^{\mathrm{split}}}(C_\gamma \otimes P, M) \otimes_{\mathbb{C}} C_\gamma.$$

Its tensor comparison satisfies the central compatibilities of Lemma 5.1. Lemma 5.4 therefore identifies $\mathcal{B}^{\mathrm{split}}$ with $\mathrm{Comod}_{\mathcal{C}^{\mathrm{rev}}}(N)$, centrally over $\mathcal{C}^{\mathrm{rev}}$, for a connected coquasi-Hopf algebra N. We must determine N and decide whether $\mathcal{B}^{\mathrm{split}}$ exhausts $\mathcal{B}$.

e) The screenings give a lower bound for the size of N. Set $\beta_i = \alpha_i^\vee$. The rank-one construction of [Len25, Lemmas 6.2–6.3] produces nonsplit extensions in $\mathcal{B}^{\mathrm{split}}$,

$$0 \longrightarrow \mathbf{1} \longrightarrow M_i \longrightarrow C_{-\beta_i} \longrightarrow 0.$$

Indeed, for $\xi_i = -\beta_i/\sqrt{p}$, the rank-one lattice has the required parameter $\ell_i^\vee$, since

$$\mathbb{Z}\xi_i \cap (\sqrt{p}\,Q) = \ell_i^\vee \mathbb{Z}\xi_i = \mathbb{Z}\sqrt{p}\alpha_i, \qquad |\sqrt{p}\alpha_i|^2 = 2\ell_i^\vee.$$

Its orthogonal sum with $(\sqrt{p}\,Q) \cap \alpha_i^\perp$ has finite index in $\sqrt{p}\,Q$. The extension thus passes through the finite-index simple-current extension of [Len25, Lemma 6.3]. The full embedding of [Len25, Lemma 6.2] preserves its nonsplitting.

Choose a splitting of the underlying object in the pointed category. On generators x of the subobject and y over the quotient, the left coaction is

$$x \longmapsto 1 \otimes x, \qquad y \longmapsto 1 \otimes y + n_i \otimes x.$$

Nonsplitting gives $n_i \neq 0$. Coassociativity gives $\Delta(n_i) = 1 \otimes n_i + n_i \otimes 1$. Thus the screening object $X = \bigoplus_i C_{-\beta_i}$ embeds in the first graded piece of the coradical filtration.

For clarity, this is the internal filtration

$$F_a N = \ker\bigl(q^{\otimes(a+1)}\Delta^{(a)}\bigr), \qquad q : N \to N/\mathbf{1}, \quad a \geq 0,$$

where $\Delta^{(a)}$ has $a+1$ factors. The socle of the regular comodule N is $\mathbf{1}$. Indeed a trivial subcomodule satisfies $\Delta = \eta \otimes \mathrm{id}$ on its image, so the counit puts it in $\eta(\mathbf{1})$. All simple comodules have trivial coaction, hence this is the finite Loewy filtration, starting with $F_0 N = \mathbf{1}$. Multiplication preserves it because it is a coaugmentation-preserving coalgebra map from the tensor coalgebra with filtration $\sum_{i+j\leq a} F_i N \otimes F_j N$. In the highest filtration degree only the counital term of the normalized reassociator contributes. Thus $\operatorname{gr} N$ is an associative graded braided bialgebra in $\mathcal{C}^{\mathrm{rev}}$, with its given associator [AAM17, Theorem 1.5].

Let $N' \subseteq \operatorname{gr} N$ be the graded sub-bialgebra generated by X. Its defining ideal in $T(X)$ is a graded braided Hopf ideal with zero degree-one part, hence is contained in the Nichols ideal. We have obtained a surjection $N' \twoheadrightarrow \mathfrak{B}(q)$ and the bound

$$D_p = \dim \mathfrak{B}(q) \leq \dim N' \leq \dim N.$$

f) Part I supplies the matching upper bound. By Theorem 3.5, $\delta_A = D_p$. On the reconstructed side the C_γ with trivial coaction are the simple objects. The cofree comodule N is injective, with socle $\mathbf{1}$, so it is the injective hull of $\mathbf{1}$ and is projective. Tensoring it with the C_γ gives all the indecomposable projectives. Each has Frobenius–Perron dimension $\dim N$, since the underlying dimension in the pointed category is a positive character of the Grothendieck ring by the tensor comparisons of the forgetful functor. Consequently $\mathrm{FPdim}(\mathcal{B}^{\mathrm{split}}) = |\Gamma| \dim N$.

Comparing the full Serre inclusion with (7.1) now gives just one chain of inequalities,

$$|\Gamma| D_p \leq |\Gamma| \dim N' \leq |\Gamma| \dim N = \mathrm{FPdim}(\mathcal{B}^{\mathrm{split}}) \leq \mathrm{FPdim}(\mathcal{B}) = |\Gamma| D_p.$$

Equality gives $\mathcal{B}^{\mathrm{split}} = \mathcal{B}$ and makes both $N' \hookrightarrow \operatorname{gr} N$ and $N' \twoheadrightarrow \mathfrak{B}(q)$ isomorphisms [Len25, Section 6, Lemma 6.5]. Thus every simple A-module is local and $\operatorname{gr} N \simeq \mathfrak{B}(q)$.

g) We can now apply robustness to N itself. It is supported on $\mathsf{H} = Q^\vee/pQ$. Corollary 6.6 and Lemma 6.7 give $H^3_{\mathcal{C}^{\mathrm{rev}}}(\mathfrak{B}(q), \mathbf{1}) = 0$, so the normalized internal gauge of Theorem 6.3(a) removes its extra reassociator. The proof of Corollary 6.6 and Theorem 6.3(b) then identify the filtered bialgebra with $\mathfrak{B}(q)$. These are precisely the two steps in [AAM17, Theorem 3.2] and [Len25, Corollary 5.8] used in Section 6. We therefore obtain a tensor equivalence central over the full lattice category,

$$K : \mathcal{B} \xrightarrow{\sim} \mathrm{Comod}_{\mathcal{C}^{\mathrm{rev}}}(\mathfrak{B}(q)). \tag{7.2}$$

The forgetful functor on the right, composed with K, gives the strong monoidal retraction

$$\mathrm{For}_{\mathfrak{B}(q)} \circ K : \mathcal{B} \longrightarrow \mathcal{C}^{\mathrm{rev}}, \qquad (\mathrm{For}_{\mathfrak{B}(q)} \circ K)\Big|_{\mathcal{C}^{\mathrm{rev}}} \simeq \mathrm{id}_{\mathcal{C}^{\mathrm{rev}}} \tag{7.3}$$

on the underlying lattice subcategory. In particular the central comparison uses the original pointed associator throughout.

h) Because K preserves the specified central lift, it induces an equivalence on the relative centers. Substituting (7.2) into (7.1) gives the braided equivalence

$$\Xi : \mathcal{D} \simeq_{\mathrm{br}} Z_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{B}) \simeq_{\mathrm{br}} Z_{\mathcal{C}^{\mathrm{rev}}}\big(\mathrm{Comod}_{\mathcal{C}^{\mathrm{rev}}}(\mathfrak{B}(q))\big) =: \mathcal{Z}. \tag{7.4}$$

i) It remains to fix the ribbon structure. Let $\mathsf{U} : \mathcal{Z} \to \mathrm{Comod}_{\mathcal{C}^{\mathrm{rev}}}(\mathfrak{B}(q))$ forget the half-braiding and put $Q_\gamma = \mathsf{U}^{\mathrm{ra}}(C_\gamma)$. Compatibility of Ξ with the free functor and its right adjoint identifies Q_γ with $\Xi(\mathrm{Res}\, C_\gamma)$. The lattice conformal weights therefore give

$$\theta_{Q_\gamma} = e^{\pi \mathrm{i}(|\gamma|^2 - 2(s,\gamma))/p} \mathrm{id}_{Q_\gamma}, \qquad s = p\rho - \rho^\vee. \tag{7.5}$$

Changing γ by pQ changes its conformal weight by an integer. To see uniqueness, every simple T of the center embeds in some Q_γ by adjunction from a simple quotient of $\mathsf{U}(T)$. Two ribbon twists with the prescribed values thus agree on simples. Their ratio a is a monoidal automorphism of the identity. Monoidality gives $a_{X^*} = (a_X^*)^{-1}$, while ribbon duality gives $a_{X^*} = a_X^*$, so $a_X^2 = 1$. A composition series makes $a_X - 1$ nilpotent, and in characteristic zero an involution with this property is the identity. This proves uniqueness on extensions as well and completes the ribbon equivalence.

□

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