

C_2 -cofiniteness of logarithmic W -algebras

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Abstract

Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra with root lattice Q and lacing number t , equipped with an invariant bilinear form normalized so that long roots have squared length two. We prove C_2 -cofiniteness of the Feigin–Tipunin logarithmic W -algebra $\mathcal{W}_p(Q)$ for $p = tm \geq 2$ and $m \geq h_{L^\vee}^\vee - 1$, where $h_{L^\vee}^\vee$ is the dual Coxeter number of the Langlands dual. Drinfeld–Sokolov reduction of induced affine Weyl modules gives graded generators for the multiplicity spaces of the conformal group action. Their degree bounds show that the intrinsic C_2 -algebra is a finite module over the polynomial C_2 -algebra of its principal invariant subalgebra. Adjoint contractions produce a finite relation ideal in this polynomial ring. Heisenberg normal ordering and a constant-coefficient differential equation show that its common zero is the origin. The argument gives explicit sufficient bounds and strong finite generation in the stated parameter range.

1 Introduction and statement of the result

The triplet vertex algebra $\mathcal{W}(p)$ is a fundamental example of a nonrational, C_2 -cofinite vertex operator algebra [5]. Its higher-rank generalizations were proposed by Feigin and Tipunin through lattice-screening constructions [6], and subsequently studied by Sugimoto, who established their geometric construction and simplicity properties [7, 8]. More recently, the shift-system framework of Sugimoto and Li has provided a uniform lattice construction for all finite simple Lie types, together with an identification of the principal invariant algebra and its multiplicity modules [1].

These structural results describe the algebra as a sum of group representations tensored with principal W -algebra modules. To prove C_2 -cofiniteness, one must control the intrinsic quotient of the whole screening kernel. The invariant subalgebra has a polynomial C_2 -algebra, so both the number of surviving group types and the polynomial action must be made finite. We prove these two finiteness statements and combine them.

Let Φ be the root system of $\mathfrak{g}$, with positive roots Φ_+ , simple roots $\alpha_1, \dots, \alpha_r$, root lattice Q , weight lattice P , dominant cone P_+ and Weyl group W . Let G be the adjoint algebraic group and $\mathfrak{h}$ the chosen Cartan subalgebra. Write $Q_+ = \sum_i \mathbb{Z}_{\geq 0} \alpha_i$. The invariant form identifies the Cartan space with its dual. Use the positive-definite real root form, extended complex bilinearly, normalized by

$$(\alpha, \alpha) = 2 \quad (\alpha \text{ long}), \quad (\alpha, \alpha) = 2/t \quad (\alpha \text{ short}),$$

where $t \in \{1, 2, 3\}$ is the lacing number. Define

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}, \quad \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha, \quad \rho^\vee = \frac{1}{2} \sum_{\alpha > 0} \alpha^\vee.$$

Thus $(\rho^\vee, \alpha_i) = 1$ and $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$. Write $h, h^\vee$ for the Coxeter and dual Coxeter numbers of $\mathfrak{g}$, and $h_L^\vee = h_{L^\vee}^\vee$ for the dual Coxeter number of its Langlands dual.

Let $p = tm$, $m \in \mathbb{Z}_{>0}$. The lattice $\sqrt{p}Q$ is even and integral: its simple-root Gram matrix has integral entries and even diagonal entries. The divisibility is necessary for this ordinary lattice construction: a scaled root of minimal length has squared length $2p/t$, which must be an even integer. Give its lattice vertex algebra the conformal vector

$$a_p = \sqrt{p}\rho - \frac{\rho^\vee}{\sqrt{p}}, \quad \omega = \frac{1}{2} \sum_j b^j(-1)b_j(-1)\mathbf{1} + a_p(-2)\mathbf{1}, \quad (1.1)$$

where $(b^j), (b_j)$ are dual Cartan bases. The central charge and the weight of a pure exponential are

$$c = r - 12(a_p, a_p), \quad \Delta(\beta) = \frac{1}{2}(\beta, \beta) - (a_p, \beta). \quad (1.2)$$

With $Y(u, z) = \sum_{j \in \mathbb{Z}} u_{(j)} z^{-j-1}$, define

$$\mathcal{V} = \mathcal{W}_p(Q) = \bigcap_{i=1}^r \ker \text{Res}_z Y(e^{-\alpha_i^\vee/\sqrt{p}}, z) \subset V_{\sqrt{p}Q}. \quad (1.3)$$

These short screenings map to charged lattice modules. Their kernels on the vacuum lattice space form the vertex algebra considered here. This is $W_{\sqrt{p}Q}$ in [1, Section 2.4.1]; in simply-laced type it is also denoted $W(p)_Q$ in [7]. Both $e^{\sqrt{p}\alpha_i}$ and $e^{-\alpha_i^\vee/\sqrt{p}}$ have conformal weight one under (1.1). Write $T = L_{-1} = \omega_{(0)}$ for the translation operator.

For a vertex algebra B and a B -module M , set

$$\begin{aligned} C_2(B) &= \text{span}\{u_{(-2)}v : u, v \in B\}, & R_B &= B/C_2(B), \\ C_2^B(M) &= \text{span}\{u_{(-2)}v : u \in B, v \in M\}. \end{aligned} \quad (1.4)$$

The algebra R_B is the commutative C_2 Poisson algebra of B . Write $R_B(M) = M/C_2^B(M)$. Its R_B -action, like the product in R_B , is induced by the (-1) product.

Theorem 1.1. *For every finite-dimensional simple complex Lie algebra and every parameter*

$$p = tm \geq 2, \quad m \in \mathbb{Z}_{>0}, \quad m \geq h_L^\vee - 1, \quad (1.5)$$

the logarithmic W -algebra $\mathcal{W}_p(Q)$ is C_2 -cofinite. More precisely, for $A = \mathcal{V}^G$, $R_{\mathcal{V}}$ is a finite R_A -module. The contraction ideal $I_{\Phi, p} \subset R_A$ defined in Section 6 has finite codimension and annihilates $R_{\mathcal{V}}$. Strong generators can be chosen of weights at most $\max\{h, pR_\Phi^2 + (p+1)R_\Phi\}$, where the sufficient cutoff R_Φ is defined in (4.2).

The parameter range is

Type	Lacing number t	Allowed p
A_1	1	$p \geq 2$
A_n ($n \geq 2$)	1	$p \geq n$
D_n ($n \geq 4$)	1	$p \geq 2n - 3$
E_6	1	$p \geq 11$
E_7	1	$p \geq 17$
E_8	1	$p \geq 29$
B_n ($n \geq 2$)	2	even $p \geq 2n$
C_n ($n \geq 2$)	2	even $p \geq 4n - 4$
F_4	2	even $p \geq 16$
G_2	3	$3 \mid p, p \geq 9$

The lower bound in (1.5) is the vacuum strong condition in the structural theorem of [1]. The theorem concerns the ordinary lattice family (1.3). Smaller admissible parameters, and the odd-parameter type- B construction with a fermion factor, are outside its statement.

The published module identification reduces each multiplicity space to the principal reduction of an induced affine Weyl module. Classical reduction with its finite-dimensional coefficient representation determines the number and degrees of generators in the intrinsic module C_2 quotient. The linear spread of these degrees is smaller than the quadratic growth of lattice-exponential products in sufficiently large group types, which then vanish in R_V . Invariant contractions of the adjoint component give relations in R_A . Their Cartan symbols form exponential polynomials with nonzero constant terms at each root exponent. A finite set of coefficients and a differential equation force their common zero to be the origin, making the polynomial action finite dimensional.

AI contribution

OpenAI's ChatGPT was used extensively to develop proof strategies and mathematical arguments, draft and revise the text, assist with literature searches, and write code for exact symbolic computations. The author directed this work and revised the manuscript.

2 The group decomposition and the principal invariant algebra

Use the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ associated with Φ_+ . Let $N \subset G$ be the positive maximal unipotent subgroup and $\mathfrak{n} = \text{Lie } N = \mathfrak{n}_+$. Fix a principal $\mathfrak{sl}_2$ -triple $(e, 2\rho^\vee, f)$ with $f \in \mathfrak{n}_-$. The principal Kostant slice is $\mathcal{S} = f + \mathfrak{g}^e$, where $\mathfrak{g}^e = \{y \in \mathfrak{g} : [e, y] = 0\}$. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by the invariant form; $\mathfrak{n}^\perp$ is the annihilator of $\mathfrak{n}$.

Distinguish the shifted level κ from the affine level k . The notation $\mathcal{W}^\ell(\mathfrak{g})$ denotes the universal principal algebra at affine level ℓ , and $\mathcal{W}_\ell(\mathfrak{g})$ its simple graded quotient. For the algebra used in the reduction argument,

$$k = \frac{1}{p} - h^\vee, \quad \kappa = k + h^\vee = \frac{1}{p}, \quad \check{k} = m - h_L^\vee, \quad \check{\kappa} = \check{k} + h_L^\vee = m. \quad (2.1)$$

Thus $t\kappa\check{\kappa} = 1$, the Feigin–Frenkel duality relation. We perform untwisted principal reduction on $\hat{\mathfrak{g}}$ at k .

Lemma 2.1. *Under (1.5), $\mathcal{V}$ is simple, self-contragredient and of CFT type, with a conformal algebraic G -action. It has a decomposition*

$$\mathcal{V} = \bigoplus_{\lambda \in P_+ \cap Q} L(\lambda) \otimes M_\lambda, \quad A = \mathcal{V}^G = \bigcap_i \ker(e^{\sqrt{p}\alpha_i})_{(0)}|_{\mathcal{H}}, \quad (2.2)$$

Here $L(\lambda)$ is the irreducible finite-dimensional $\mathfrak{g}$ -module of highest weight λ , and $\mathcal{H}$ is the neutral Heisenberg algebra. The multiplicity space M_λ is a simple A -module, with a one-dimensional top generated by $e^{-\sqrt{p}\lambda}$ in its highest-weight realization. Its lowest conformal degree is

$$H(\lambda) = (\rho^\vee, \lambda), \quad J(\lambda) = (\rho, \lambda), \quad h_\lambda = \frac{p}{2}(\lambda, \lambda) + pJ(\lambda) - H(\lambda). \quad (2.3)$$

Through its specified Miura inclusion $A \subset \mathcal{H}$, one has

$$A \simeq \mathcal{W}^k(\mathfrak{g}) \simeq \mathcal{W}_k(\mathfrak{g}). \quad (2.4)$$

Under this identification,

$$M_\lambda \simeq T_{\lambda,0}^{1/p} = H_{\text{DS},+}^0(\widehat{V}^k(L(\lambda))) \quad (2.5)$$

as A -modules with their conformal grading.

Proof. The vacuum specialization of [1, Theorem 3.2, Corollary 3.4, equations (4.3)–(4.6), Theorem 4.3] gives the group action, simplicity, the Miura identification (2.4), and (2.5). Self-contragredience is the zero-coset case of its Theorem 3.2(2). Let θ_s be the highest short root, taking $\theta_s = \theta$ in simply-laced type. In the coroot-space convention of that source, ${}^L\theta = t\theta_s$ and $(\rho^\vee, \theta_s) = h_L^\vee - 1$. Hence its vacuum strong condition is

$$(\rho^\vee, {}^L\theta) \leq p \iff t(h_L^\vee - 1) \leq tm.$$

Thus the endpoint in (1.5) is included. Its Lemma 4.1 identifies the one-dimensional top.

The algebra and module identifications at the rational shifted level $\kappa = 1/p$ are those of [1, Theorem 4.3]. In its notation, the coset parameter is zero and the dominant index is our λ , so the module is $T_{\lambda,0}^{1/p}$. The algebra identification is through its Miura inclusion in equation (4.3). Thus (2.5) is a graded isomorphism for the specified $A \subset \mathcal{H}$ action and transports the intrinsic subspace C_2^A itself. The convention for $T_{\lambda,0}^\kappa$ is that of [3, Sections 2.1–2.2]; the rational-level identification used here is supplied by [1].

The centre of the simply connected group acts on $L(\lambda)$ by the class of λ in P/Q ; all indices here belong to Q . Hence the action factors through the adjoint group. Formula (1.2) at $\beta = -\sqrt{p}\lambda$ gives (2.3). Dominant weights have nonnegative simple-root coordinates, because the inverse finite Cartan matrix has positive entries. On a simple root,

$$(p\rho - \rho^\vee, \alpha_i) = \frac{p}{2}(\alpha_i, \alpha_i) - 1 > 0$$

in our parameter range. Consequently $h_\lambda > 0$ for nonzero λ . Relative Fock degrees are nonnegative integers, and the vacuum multiplicity space has a one-dimensional degree-zero part. This proves CFT type. All homogeneous pieces are finite dimensional. $\square$

For later use, if $\mu - \lambda = \delta \in Q_+ \setminus \{0\}$ and both weights are dominant, then

$$h_\mu - h_\lambda = \frac{p}{2}(\delta, \mu + \lambda) + (p\rho - \rho^\vee, \delta) > 0. \quad (2.6)$$

The first term is nonnegative by dominance; the second is strictly positive. The short screening on $e^{-\sqrt{p}\lambda}$ starts with $z^{(\alpha_i^\vee, \lambda)}$, so has zero residue. Its group Cartan weight is λ , since the group Cartan operators on a lattice charge β are $-\alpha_i^\vee(0)/\sqrt{p}$. If $L(\mu)$ contains that weight, then $\mu - \lambda \in Q_+$. Formula (2.6) therefore shows that this exponential is in precisely the λ -isotypic component at its lowest degree. This component contains a single copy of $L(\lambda)$ in degree h_λ .

Lemma 2.2. *Heisenberg symbol projection induces an injection*

$$R_A \hookrightarrow R_{\mathcal{H}} = \mathbb{C}[\mathfrak{h}^*], \quad \text{im } R_A = \mathbb{C}[\mathfrak{h}^*]^W = \mathbb{C}[u_1, \dots, u_r]. \quad (2.7)$$

The degrees of u_i are the basic invariant degrees. Every point of $\text{Spec } R_A$ lifts to a Cartan point.

Proof. The principal PBW and Miura symbol statements [4, Section 5, equations (23)–(25); proof of Lemma 6.1] apply to the universal algebra $\mathcal{W}^k(\mathfrak{g})$ at any noncritical level. Here A is this algebra by (2.4), and $k + h^\vee = 1/p \neq 0$. At Li degree zero the cited symbol theorem gives the Chevalley restriction map $\mathbb{C}[\mathcal{S}] \simeq \mathbb{C}[\mathfrak{h}^*]^W \subset \mathbb{C}[\mathfrak{h}^*]$, for the principal Kostant slice $\mathcal{S}$. Under the

physical lattice currents, the level-1/ p Miura current is $b'_i = -\beta_i/p$, $\beta_i = \sqrt{p}\alpha_i$, with Gram matrix $(b'_i, b'_j) = (\alpha_i, \alpha_j)/p$. Its screening $\exp(-\kappa^{-1} \int b'_i)$ is exactly $e^{\sqrt{p}\alpha_i}$. This is the embedding in [1, equation (4.3)]. Scaling the Cartan variables by a common nonzero scalar preserves their invariant ring. This proves (2.7) for the specified inclusion $A \subset \mathcal{H}$. In particular the symbol map has zero kernel: if an element of A has zero Heisenberg symbol, its class in R_A is zero. This is the degree-zero assertion of the cited Miura symbol theorem.

The Cartan coordinate ring is integral over its invariants: each polynomial a satisfies the monic invariant-coefficient equation $\prod_{w \in W} (X - wa) = 0$. Lying over, over $\mathbb{C}$, gives the final assertion. $\square$

3 Reduced Weyl modules and their generator degrees

For $E = L(\lambda)$ let

$$\widehat{V}^k(E) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} E, \quad t\mathfrak{g}[t]E = 0, \quad K|_E = k, \quad N_\lambda = H_{\text{DS},+}^0(\widehat{V}^k(E)).$$

Our convention is Arakawa's $H_f^{\infty/2+i}$ for a principal nilpotent $f \in \mathfrak{n}_-$ and character $\Psi(xt^j) = \delta_{j,-1}(f, x)$. It is the zero-twist instance of [3, Section 2.1]. The principal good-grading element is $\rho^\vee$.

Proposition 3.1. *The intrinsic module quotient $R_A(N_\lambda)$ is a free R_A -module of rank $\dim L(\lambda)$. A vector of weight ν in $L(\lambda)$ contributes a generator in degree*

$$\Delta_\lambda^{\text{aff}} - (\rho^\vee, \nu), \quad \Delta_\lambda^{\text{aff}} = \frac{(\lambda, \lambda + 2\rho)}{2\kappa} = \frac{p}{2}(\lambda, \lambda + 2\rho). \quad (3.1)$$

Thus $R_A(M_\lambda)$ has at most $\dim L(\lambda)$ homogeneous generators, all in $[h_\lambda, h_\lambda + 2H(\lambda)]$. Their homogeneous lifts strongly span M_λ using negative A -modes.

Proof. Let $F^\bullet$ denote the intrinsic module Li filtration. For $E = L(\lambda)$, the affine PBW description is

$$\widehat{V}^k(E) = U(t^{-1}\mathfrak{g}[t^{-1}]) \otimes E.$$

Giving xt^{-j} degree j makes its graded pieces finite-dimensional algebraic $\mathfrak{g}$ -modules. The affine Sugawara energies are $\Delta_\lambda^{\text{aff}} + \mathbb{Z}_{\geq 0}$, with finite-dimensional eigenspaces, and the $\mathfrak{g}$ -action is locally finite. The module is generated by E and has a Weyl flag of length one. Thus it is a finitely generated object of KL_k in the sense of [2, Section 2.6], at the level in (2.1).

The intrinsic affine Li-degree-zero quotient is

$$\widehat{V}^k(E)/C_2^{V^k(\mathfrak{g})}(\widehat{V}^k(E)) \simeq \mathbb{C}[\mathfrak{g}^*] \otimes E, \quad (3.2)$$

with polynomial multiplication on the first factor and diagonal $\mathfrak{g}$ -action. Theorem 4.5.9(ii) and Corollary 4.5.10(ii) of [2] identify the filtration induced by reduction with the intrinsic module filtration and give

$$R_A(N_\lambda) \simeq (\mathbb{C}[f + \mathfrak{n}^\perp] \otimes E)^N. \quad (3.3)$$

Here the principal grading is integral, so $\mathfrak{n} = \mathfrak{n}_+$ and there is no neutral-ghost factor. The moment constraints act on the coordinate factor in (3.2). This identifies the intrinsic module quotient as an R_A -module.

For clarity, $F^1 N_\lambda = C_2^A(N_\lambda)$. The intrinsic module Li filtration has this property by its definition; equivalently, modes of index at most -2 are translated (-2) -products and their span is stable under the remaining negative modes. In the present convention,

$$a_{(-j)}v = \frac{(T^{j-2}a)_{(-2)}v}{(j-1)!} \quad (j \geq 2).$$

The principal Kostant isomorphism is

$$N \times \mathcal{S} \longrightarrow f + \mathfrak{n}^\perp, \quad (n, s) \longmapsto \text{Ad}(n)s, \quad \mathcal{S} = f + \mathfrak{g}^e.$$

View $\mathbb{C}[f + \mathfrak{n}^\perp] \otimes E$ as polynomial E -valued functions. An invariant for the diagonal action satisfies

$$F(\text{Ad}(n)x) = nF(x).$$

Consequently restriction to $\mathcal{S}$ is an isomorphism

$$(\mathbb{C}[f + \mathfrak{n}^\perp] \otimes E)^N \longrightarrow \mathbb{C}[\mathcal{S}] \otimes E. \quad (3.4)$$

Its inverse sends $q \otimes v$ to

$$F_{q,v}(\text{Ad}(n)s) = q(s)nv.$$

This is polynomial because the Kostant map is an isomorphism and the N -representation on E is algebraic. It is $\mathbb{C}[\mathcal{S}]$ -linear. Since $R_A = \mathbb{C}[\mathcal{S}]$, equations (3.3)–(3.4) prove freeness of rank $\dim E$.

We next compute the generator degrees for use in Section 4. Let

$$\Delta_\lambda^{\text{aff}} = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)} = \frac{p}{2}(\lambda, \lambda + 2\rho).$$

The reduced Hamiltonian is the affine Hamiltonian minus $\rho^\vee(0)$, with the charged-ghost contribution. On the function realization its action is

$$(\mathcal{G}_z F)(x) = z^{\Delta_\lambda^{\text{aff}}} z^{-\rho^\vee} F(z \text{Ad}(z^{\rho^\vee})x). \quad (3.5)$$

The coordinate action fixes the principal f and preserves $\mathcal{S}$. Under the Kostant map it sends

$$(n, s) \longmapsto (z^{\rho^\vee} n z^{-\rho^\vee}, z \text{Ad}(z^{\rho^\vee})s).$$

For $v \in E_\nu$, put $F_v = F_{1,v}$. Substitution into (3.5) gives

$$\begin{aligned} (\mathcal{G}_z F_v)(\text{Ad}(n)s) &= z^{\Delta_\lambda^{\text{aff}}} z^{-\rho^\vee} (z^{\rho^\vee} n z^{-\rho^\vee})v \\ &= z^{\Delta_\lambda^{\text{aff}} - (\rho^\vee, \nu)} n v. \end{aligned}$$

Thus the generator associated to v has degree

$$\Delta_\lambda^{\text{aff}} - (\rho^\vee, \nu). \quad (3.6)$$

For weights of E , the extreme values of $(\rho^\vee, \nu)$ are $H(\lambda)$ and $-H(\lambda)$, because $\rho^\vee$ is dominant and $w_0 \rho^\vee = -\rho^\vee$. The lower endpoint is

$$\Delta_\lambda^{\text{aff}} - H(\lambda) = \frac{p}{2}(\lambda, \lambda) + p(\rho, \lambda) - (\rho^\vee, \lambda) = h_\lambda.$$

All generators therefore have degrees in $[h_\lambda, h_\lambda + 2H(\lambda)]$. Transport them by the graded isomorphism (2.5).

Finally choose homogeneous lifts of these module generators and of the generators of R_A . Modulo $C_2^A(M_\lambda)$ every vector is a polynomial in the latter acting on the former. Lift this expression using (-1) -products. The remainder is a sum of $a_{(-2)}v$. Terms with $a = \mathbf{1}$ vanish, and every other term has inner vector of strictly smaller conformal degree. Induction on the nonnegative integral relative degree strongly spans M_λ by the chosen lifts. $\square$

4 Finitely many surviving group types

We write $V_\lambda = L(\lambda) \otimes M_\lambda$ and P_λ for its isotypic projector. Each P_λ commutes with all A -modes and preserves $C_2(\mathcal{V})$: the group action commutes with invariant fields, and $C_2(\mathcal{V})$ is group-stable on each finite semisimple homogeneous piece.

Lemma 4.1. *Let $U, W \subset \mathcal{V}$ be lower-bounded graded A -modules strongly spanned by homogeneous finite sets S_U, S_W using negative A -modes. Let $\Pi \in \text{End}(\mathcal{V})$ commute with A -modes. Every coefficient $\Pi(u_{(j)}v)$ is a linear combination of negative A -modes applied to coefficients $\Pi(s_{(l)}t)$, $s \in S_U, t \in S_W$. If Π preserves $C_2(\mathcal{V})$ and these generator coefficients vanish for $l \geq -1$, then all the projected coefficients belong to $C_2(\mathcal{V})$.*

Proof. Set $I(u)_{(j)}v = \Pi(u_{(j)}v)$. For $a \in A_{>0}$ and $k \geq 1$, the iterate and commutator formulas give

$$\begin{aligned} I(a_{(-k)}u)_{(j)}v &= \sum_{i \geq 0} (-1)^i \binom{-k}{i} (a_{(-k-i)}I(u)_{(j+i)}v - (-1)^k I(u)_{(j-k-i)}a_{(i)}v), \\ I(u)_{(j)}a_{(-k)}v &= a_{(-k)}I(u)_{(j)}v - \sum_{i \geq 0} \binom{-k}{i} I(a_{(i)}u)_{(j-k-i)}v. \end{aligned}$$

The sums truncate on fixed inputs. Induct, for all j together, on the sum of the two relative input weights. Removing the outer negative mode reduces this sum by $\text{wt}(a) + k - 1 > 0$. In an iterate correction the pair $(a_{(-k)}u, v)$ is replaced by $(u, a_{(i)}v)$; the total input weight decreases by $k + i > 0$. The commutator correction has the same decrease with $(a_{(i)}u, v)$. All external output modes are negative. This proves the first assertion. Modes of index at most -2 are translated (-2) products, hence lie in $C_2(\mathcal{V})$ after projection. For $j \geq 2$, $b_{(-j)}v = (T^{j-2}b)_{(-2)}v/(j-1)!$. Stability under (-1) products follows from

$$a_{(-1)}b_{(-2)}v = b_{(-2)}a_{(-1)}v + \sum_{i \geq 0} (-1)^i (a_{(i)}b)_{(-3-i)}v.$$

All correction terms are again translated (-2) products. Thus that space is stable under every negative A -mode. Together with the assumed vanishing at indices at least -1 , this proves the second assertion. $\square$

Proposition 4.2. *If $\lambda = \mu + \nu$ for dominant root-lattice weights and*

$$p(\mu, \nu) > 2H(\mu) + 2H(\nu), \tag{4.1}$$

then $V_\lambda \subset C_2(\mathcal{V})$.

Proof. By Proposition 3.1, generator-weight sums for V_μ, V_ν are at most $h_\mu + h_\nu + 2H(\mu) + 2H(\nu) < h_\lambda$, since $h_{\mu+\nu} - h_\mu - h_\nu = p(\mu, \nu)$. Their projected products of index at least -1 vanish by this degree bound. Lemma 4.1 puts every projected coefficient in $C_2(\mathcal{V})$. Let $\mathcal{C}_\lambda$ be the span of all these projected coefficients. It is G -stable by equivariance. For $a \in A$ and any $q \in \mathbb{Z}$, the commutator identity gives

$$a_{(q)}P_\lambda(u_{(j)}v) = P_\lambda(u_{(j)}a_{(q)}v) + \sum_{i \geq 0} \binom{q}{i} P_\lambda((a_{(i)}u)_{(q+j-i)}v).$$

Since V_μ and V_ν are A -modules, every term belongs to $\mathcal{C}_\lambda$. Hence this coefficient span is stable under all A -modes. It contains

$$(e^{-\sqrt{p}\mu})_{(-1-p(\mu,\nu))}e^{-\sqrt{p}\nu} = \epsilon(-\sqrt{p}\mu, -\sqrt{p}\nu)e^{-\sqrt{p}\lambda}.$$

The cocycle value is nonzero. The exponential is its own isotypic projection by (2.6); its G -orbit spans $L(\lambda) \otimes \mathbb{C}v_\lambda$ in the top degree, where v_λ is the multiplicity highest vector. The A -submodule generated by v_λ is all of the simple module M_λ . Thus $\mathcal{C}_\lambda = V_\lambda$, proving the assertion. $\square$

Here is an explicit uniform cutoff for each type. Use the Cartan matrix convention $C_{ij} = (\alpha_i^\vee, \alpha_j)$, and fundamental weights ϖ_i . Let

$$\begin{aligned} \tau &= \det C, \quad G_{ij} = (\varpi_i, \varpi_j), \quad H_i = (\rho^\vee, \varpi_i), \quad c_0 = \min_{i,j} \frac{\tau G_{ij}}{H_j}, \\ p_* &= \max\{2, t(h_L^\vee - 1)\}, \quad n_0 = \left\lceil \frac{4}{p_* c_0} \right\rceil, \quad L = n_0 \tau, \quad B = L \max_i H_i, \\ R_\Phi &= \lceil \max\{2B, (L-1) \sum_i H_i\} \rceil. \end{aligned} \tag{4.2}$$

Write $D_{ii} = (\alpha_i, \alpha_i)/2$. The simple-root Gram matrix is DC , and the fundamental weights are the columns of C^{-1} in the simple-root basis. Hence

$$G = (C^{-1})^\top D, \quad H_i = \sum_j (C^{-1})_{ji}.$$

These entries are positive. Write $C = 2I - \mathcal{A}$. The matrix $\mathcal{A}$ has nonnegative entries and its connected graph is the Dynkin graph. Since C is symmetrizable positive definite, its eigenvalues are positive. Perron–Frobenius therefore gives $\text{spr}(\mathcal{A}) < 2$, and $C^{-1} = \frac{1}{2} \sum_{j \geq 0} (\mathcal{A}/2)^j$ has positive entries. The rank-one case is immediate. The adjugate formula gives $\tau \varpi_i \in \bar{Q}$.

Lemma 4.3. *For $H(\lambda) > R_\Phi$, $\lambda \in P_+ \cap Q$, the decomposition in Proposition 4.2 exists for every p in (1.5).*

Proof. Write $\lambda = \sum_i a_i \varpi_i$, $a_i \in \mathbb{Z}_{\geq 0}$. If all $a_i \leq L-1$, then $H(\lambda) \leq (L-1) \sum_i H_i$, a contradiction. Choose $a_i \geq L$ and put $\mu = L\varpi_i$, $\nu = \lambda - \mu$. Both are dominant and belong to Q . We have $H(\mu) \leq B$ and $H(\nu) > H(\mu)$. For every j ,

$$p(\mu, \varpi_j) = p n_0 \tau G_{ij} \geq p_* n_0 c_0 H_j \geq 4H_j.$$

Thus $p(\mu, \nu) \geq 4H(\nu) > 2H(\mu) + 2H(\nu)$. $\square$

It follows that R_ν is a finite R_A -module, generated by at most

$$K_\Phi = \sum_{\substack{\lambda \in P_+ \cap Q \\ H(\lambda) \leq R_\Phi}} (\dim L(\lambda))^2 \tag{4.3}$$

elements. There are finitely many indices because all $H_i > 0$. Explicitly, the inclusions $C_2^A(M_\lambda) \subset C_2(\mathcal{V})$ and the eliminated components give the surjection

$$\bigoplus_{H(\lambda) \leq R_\Phi} L(\lambda) \otimes R_A(M_\lambda) \twoheadrightarrow R_\nu. \tag{4.4}$$

5 Invariant adjoint contractions

Let θ be the highest long root. Since $(\rho, \theta) = h^\vee - 1$ and $(\rho^\vee, \theta) = h - 1$, its extremal exponential has weight

$$E = e^{-\sqrt{p}\theta}, \quad d = h_\theta = ph^\vee - h + 1. \tag{5.1}$$

Its group orbit spans $\mathcal{E} \simeq \mathfrak{g}$ in $\mathcal{V}_d$.

Lemma 5.1. *The subspace $\mathcal{E}$ is the unique adjoint copy in $\mathcal{V}_d$ and consists of Virasoro primary vectors. The normalized invariant bilinear form of $\mathcal{V}$ restricts nondegenerately to $\mathcal{E}$ and pairs opposite lattice charges. The nonzero charge spaces of $\mathcal{E}$ are one-dimensional and indexed by $\sqrt{p}\Phi$; its neutral space has dimension r .*

Proof. The decomposition in Lemma 2.1 and the one-dimensional top of M_θ give the multiplicity assertion. For $n > 0$,

$$L_n = \frac{1}{2} \sum_{j,k} : b^j(k) b_j(n-k) : - (n+1) a_p(n).$$

Every summand annihilates the Fock highest vector E . Thus E is primary. The conformal G -action makes every vector of $\mathcal{E}$ primary as well.

Let $\langle, \rangle$ be the invariant bilinear form normalized by $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. Simplicity and self-contragredience make it nondegenerate. Pullback by a conformal automorphism gives another normalized invariant form, which agrees with it by uniqueness. The form is therefore G -invariant and pairs only equal conformal degrees. In the finite semisimple G -module $\mathcal{V}_d$, an adjoint copy can pair only with an adjoint copy. Since there is exactly one such copy, the form restricts nondegenerately to $\mathcal{E}$.

Denote the group Cartan operators by h_i . They act on a vector of charge β by $-(\alpha_i^\vee, \beta)/\sqrt{p}$. Infinitesimal G -invariance gives

$$\langle h_i u, v \rangle + \langle u, h_i v \rangle = 0.$$

If vectors of charges β, γ pair nontrivially, this identity implies $(\alpha_i^\vee, \beta + \gamma) = 0$ for every i , hence $\gamma = -\beta$. The weights of the adjoint representation are the roots with multiplicity one and zero with multiplicity r . $\square$

Choose a basis (E_a) of $\mathcal{E}$ with each E_a in a single lattice charge space, and put $\mathcal{B}_{ab} = \langle E_a, E_b \rangle$ and let $\mathcal{B}^{ab}$ be the entries of its inverse. Set

$$\mathcal{Q}(z) = \sum_{a,b} \mathcal{B}^{ab} Y(E_a, z) E_b, \quad \mathcal{Q}_s = \sum_{a,b} \mathcal{B}^{ab} (E_a)_{(-s)} E_b, \quad f_s = [\mathcal{Q}_s]_{R_A}. \quad (5.2)$$

Invariance of the tensor implies $\mathcal{Q}_s \in A$. Its degree is $2d + s - 1$. For $s \geq 2$, it is in $C_2(\mathcal{V})$. Let $\varphi : R_A \rightarrow R_{\mathcal{V}}$ be the algebra map induced by inclusion. Then $\varphi(f_s) = 0$, so $f_s \cdot [v] = \varphi(f_s)[v] = 0$ for every $[v] \in R_{\mathcal{V}}$.

Let π_x denote Heisenberg symbol projection followed by evaluation at the Cartan point x : $a(-1)$ evaluates to $a(x)$, and $a(-n)$ evaluates to zero for $n \geq 2$. We identify the root space with its dual by the normalized form and write $\alpha(x)$ for this evaluation.

Lemma 5.2. *For $\xi_\alpha = \sqrt{p} \alpha(x)$, the normalized contraction $H_x(z) \in \mathbb{C}[[z]]$ has the form*

$$H_x(z) = z^{2d} \pi_x \mathcal{Q}(z) = P_0(z) + \sum_{\alpha \in \Phi} e^{\xi_\alpha z} P_\alpha(z), \quad (5.3)$$

where

$$\deg P_0 \leq 2d, \quad \deg P_\alpha \leq 2d - p(\alpha, \alpha), \quad P_\alpha(0) = (-1)^d. \quad (5.4)$$

Moreover $P_0(0) = r(-1)^d$. The degree bound retains the squared length of each root.

Proof. By Lemma 5.1, every nonzero term has opposite input charges. Consider monomials

$$u = \prod_{i=1}^a a_i(-m_i) e^\beta, \quad v = \prod_{j=1}^b b_j(-n_j) e^{-\beta}, \quad m_i, n_j \geq 1.$$

Introduce independent auxiliary variables s_i, t_j and define the formal states

$$U(\mathbf{s}) = \exp\left(\sum_i s_i a_i(-m_i)\right) e^\beta, \quad V(\mathbf{t}) = \exp\left(\sum_j t_j b_j(-n_j)\right) e^{-\beta}.$$

Their coefficients are ordinary Fock states. Set

$$\begin{aligned} L_i &= \delta_{m_i,1} a_i(x) + (-1)^{m_i} (a_i, \beta) z^{-m_i}, \\ R_j &= \delta_{n_j,1} b_j(x) - (\beta, b_j) z^{-n_j}, \\ K_{ij} &= (-1)^{m_i-1} n_j \binom{m_i + n_j - 1}{m_i - 1} (a_i, b_j) z^{-m_i - n_j}. \end{aligned} \tag{5.5}$$

The coefficientwise generating identity is

$$\begin{aligned} \pi_x Y(U(\mathbf{s}), z) V(\mathbf{t}) &= \epsilon(\beta, -\beta) z^{-(\beta, \beta)} e^{\beta(x)z} \\ &\cdot \exp\left(\sum_i s_i L_i + \sum_j t_j R_j + \sum_{i,j} s_i t_j K_{ij}\right). \end{aligned} \tag{5.6}$$

Here the left and right auxiliary degrees, rather than powers of z , are used to extract individual states.

To verify (5.6), use

$$Y(a(-m)w, z) =: \frac{\partial^{m-1} a(z)}{(m-1)!} Y(w, z) :.$$

The left creation parts stand before the exponential field; the left annihilation parts stand after it. Among creation modes, and among nonnegative modes, Heisenberg commutators vanish. Thus no pairing of two left oscillators occurs. The surviving creation term is $a(-1)$ only when $m = 1$. The left zero mode on charge $-\beta$ gives $(-1)^m (a, \beta) z^{-m}$. These are the two terms of L_i .

The annihilation exponential in $Y(e^\beta, z)$ shifts a right oscillator by

$$b(-n) \mapsto b(-n) - (\beta, b) z^{-n},$$

which gives R_j after projection. Finally the scalar commutator of a positive left mode with a right oscillator is

$$\frac{\partial_z^{m-1}}{(m-1)!} (n(a, b) z^{-n-1}) = (-1)^{m-1} n \binom{m+n-1}{m-1} (a, b) z^{-m-n}.$$

It gives K_{ij} . All these commutators are scalar, so their exponential is exactly the last exponential in (5.6). The creation exponential projects to $e^{\beta(x)z}$; the remaining lattice factor is $\epsilon(\beta, -\beta) z^{-(\beta, \beta)}$. This proves the identity.

Taking the coefficient of $s_1 \cdots s_a t_1 \cdots t_b$ gives

$$\pi_x Y(u, z) v = \epsilon(\beta, -\beta) z^{-(\beta, \beta)} e^{\beta(x)z} \sum_M \prod_{(i,j) \in M} K_{ij} \prod_{i \text{ unpaired}} L_i \prod_{j \text{ unpaired}} R_j, \tag{5.7}$$

where M runs through partial matchings of *labelled* left and right oscillators. Labelling preserves multiplicities for repeated oscillators.

Suppose u, v both have conformal weight d . Their total oscillator energy is

$$N_\beta = 2d - \Delta(\beta) - \Delta(-\beta) = 2d - (\beta, \beta).$$

Every monomial in the sum in (5.7) has z -degree between $-N_\beta$ and zero: each oscillator is either unpaired or used once in a cross-contraction. Consequently multiplication by z^{N_β} turns that sum into a polynomial of degree at most N_β . Linearity gives the same bound for arbitrary homogeneous states of these charges. This proves the polynomial bounds in (5.4), both for $\beta = \sqrt{p}\alpha$ and for $\beta = 0$.

For primary vectors u, v of weight d , invariance of the bilinear form gives

$$u_{(2d-1)}v = (-1)^d \langle u, v \rangle \mathbf{1}. \quad (5.8)$$

Indeed pair $Y(u, z)v$ with $\mathbf{1}$ and use

$$\langle Y(u, z)v, \mathbf{1} \rangle = (-1)^d z^{-2d} \langle v, Y(u, z^{-1})\mathbf{1} \rangle, \quad Y(u, z^{-1})\mathbf{1} = e^{z^{-1}T}u.$$

Only the term u survives the pairing by conformal degree.

Fix a root charge β . Its line pairs nontrivially with the opposite line; write their pairing as $b_\beta \neq 0$. Their form matrix and inverse are

$$\begin{pmatrix} 0 & b_\beta \\ b_\beta & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b_\beta^{-1} \\ b_\beta^{-1} & 0 \end{pmatrix}.$$

There is exactly one ordered summand with left charge β . By (5.8) its normalized constant term is $(-1)^d$. The opposite ordered summand has the opposite exponential. The same inverse-form contraction on the neutral block has constant term $r(-1)^d$. This establishes the claimed constants for every compatible choice of lattice cocycle. $\square$

Lemma 5.3. *If $H_x(z)$ is a polynomial, then $x = 0$.*

Proof. Group the roots by equal scalar exponents $\xi = \sqrt{p}\alpha(x)$. For each nonzero ξ , the grouped coefficient polynomial has constant term

$$\#\{\alpha \in \Phi : \sqrt{p}\alpha(x) = \xi\} (-1)^d \neq 0.$$

This includes collisions between roots of different lengths. Distinct scalar exponentials are linearly independent over $\mathbb{C}[z]$: apply sufficiently large powers of $\partial_z - \eta$ to kill all other exponent groups, including the polynomial group at exponent zero. On the remaining coefficient at ξ , each resulting factor $\partial_z + \xi - \eta$ is invertible on the finite-dimensional space of polynomials of bounded degree, since its constant coefficient is nonzero. Hence no nonzero exponent group can occur in a polynomial H_x . Every root must evaluate to zero at x , and the roots span the Cartan space. $\square$

Skew symmetry gives $\mathcal{Q}(z) = e^{zT}\mathcal{Q}(-z)$. All its coefficients are neutral and symbol projection kills translation. Hence $f_s = 0$ for even s ; the relation ideal may equivalently be generated by the odd indices in its defining range.

6 A finite relation ideal and proof of the theorem

Define the length-dependent ODE multiplicities and relation cutoff

$$m_\alpha = 2d - p(\alpha, \alpha) + 1, \quad S = 1 + \sum_{\alpha \in \Phi} m_\alpha, \quad I_{\Phi, p} = (f_s : 2 \leq s \leq S) \subset R_A. \quad (6.1)$$

They are positive integers. At both opposite root charges there are weight- d vectors, so their total oscillator energy is $2d - p(\alpha, \alpha) \geq 0$. Equivalently, putting $n_l = |\Phi_l|$, $n_s = |\Phi_s|$ in non-simply-laced type,

$$S = 1 + n_l(2d - 2p + 1) + n_s(2d - 2p/t + 1). \quad (6.2)$$

In all types one may instead write $S = 1 + |\Phi|(2d + 1) - 2prh^\vee$, using $\sum_{\alpha \in \Phi} (\alpha, \alpha) = 2rh^\vee$.

Proposition 6.1. *The radical of $I_{\Phi,p}$ is the homogeneous maximal ideal of $R_A = \mathbb{C}[u_1, \dots, u_r]$. In particular, $D_{\Phi,p} := \dim_{\mathbb{C}} R_A/I_{\Phi,p} < \infty$.*

Proof. Let $c \in \text{Spec } R_A$ be a common zero of the displayed generators and lift it to a Cartan point x using (2.7). Put $\xi_\alpha = \sqrt{p}\alpha(x)$. Lemma 5.2 implies that the formal series $H_x \in \mathbb{C}[[z]]$ is annihilated by the monic constant-coefficient differential operator

$$\mathcal{D}_x = \partial_z^{2d+1} \prod_{\alpha \in \Phi} (\partial_z - \xi_\alpha)^{m_\alpha}, \quad q = \text{ord } \mathcal{D}_x = 2d + S. \quad (6.3)$$

Coincident exponents merely repeat factors; zero exponents add factors of ∂_z . Thus the same operator annihilates the series at these points.

Let J_x be the Taylor polynomial of H_x through degree $2d$. Since the factors in $\mathcal{D}_x$ commute and $\partial_z^{2d+1} J_x = 0$, both H_x and J_x are annihilated by $\mathcal{D}_x$.

The coefficient of z^{2d+s-1} in H_x is $f_s(c)$. The series $H_x - J_x$ therefore has zero coefficients in exactly the following consecutive initial range:

Degrees	Reason for vanishing
$0, \dots, 2d$	Definition of J_x
$2d + 1, \dots, 2d + S - 1 = q - 1$	Relations $f_2(c) = \dots = f_S(c) = 0$

There are q initial coefficients, as required for an operator of order q .

The uniqueness argument is formal. Write

$$\mathcal{D}_x = \partial_z^q + \sum_{j=0}^{q-1} c_j \partial_z^j, \quad H_x - J_x = \sum_{n \geq 0} a_n z^n.$$

The coefficient of z^n in $\mathcal{D}_x(H_x - J_x) = 0$ is

$$\frac{(n+q)!}{n!} a_{n+q} + \sum_{j=0}^{q-1} c_j \frac{(n+j)!}{n!} a_{n+j} = 0. \quad (6.4)$$

The leading factor is nonzero over $\mathbb{C}$. Starting from $a_0 = \dots = a_{q-1} = 0$, induction gives $a_n = 0$ for every n . Thus $H_x = J_x$ is polynomial. Lemma 5.3 gives $x = 0$, whence $c = 0$.

Conversely each f_s is homogeneous of positive conformal weight $2d + s - 1$, and its Miura symbol vanishes at the origin. The common zero set is therefore exactly the origin. The Nullstellensatz gives the radical assertion. For each i , some power $u_i^{N_i}$ belongs to $I_{\Phi,p}$; the monomials with exponents below these powers span the quotient. This proves finite codimension. $\square$

Proof of Theorem 1.1. By (5.2), the algebra map $\varphi : R_A \rightarrow R_{\mathcal{V}}$ satisfies $\varphi(I_{\Phi,p}) = 0$. Hence $I_{\Phi,p}$ annihilates the entire R_A -module $R_{\mathcal{V}}$. The finite-module surjection (4.4) therefore factors through

$$\bigoplus_{H(\lambda) \leq R_{\Phi}} L(\lambda) \otimes \frac{R_A(M_{\lambda})}{I_{\Phi,p} R_A(M_{\lambda})} \twoheadrightarrow R_{\mathcal{V}}.$$

Each summand is a quotient of at most $(\dim L(\lambda))^2$ copies of $R_A/I_{\Phi,p}$, by Proposition 3.1. Thus

$$\dim_{\mathbb{C}} R_{\mathcal{V}} \leq K_{\Phi} D_{\Phi,p} < \infty. \quad (6.5)$$

Homogeneous lifts of algebra generators of the finite-dimensional intrinsic quotient strongly generate the CFT-type vertex algebra by weight induction on the remaining (-2) products. $\square$

Corollary 6.2. *Strong generators can be chosen of weights at most $\max\{h, pR_{\Phi}^2 + (p+1)R_{\Phi}\}$.*

Proof. For a dominant root-lattice weight $\lambda = \sum b_i \alpha_i$, $b_i \geq 0$ and $H(\lambda) = \sum b_i$. Long roots have the maximum length $\sqrt{2}$, so $\|\lambda\| \leq \sqrt{2}H(\lambda)$. Also $J(\lambda) = \sum b_i(\alpha_i, \alpha_i)/2 \leq H(\lambda)$. Equation (2.3) gives $h_{\lambda} + 2H(\lambda) \leq pH(\lambda)^2 + (p+1)H(\lambda)$. On the surviving components this is at most the claimed bound; the invariant algebra has basic generator degrees at most h . Choose lifts of these invariant generators and of the homogeneous module generators in (4.4), including bases of their finite $L(\lambda)$ factors. Their classes algebra-generate $R_{\mathcal{V}}$: polynomial combinations of the invariant classes multiply the module generators to span every element of $R_{\mathcal{V}}$. For a homogeneous vector of $\mathcal{V}$, lift such an expression by (-1) products. The remainder lies in $C_2(\mathcal{V})$ and is a sum of $a_{(-2)}b$ with positive-weight a and strictly smaller-weight a, b . Induction on conformal weight gives strong generation by the chosen lifts, with the stated weight bound. $\square$

Corollary 6.3. *Tensor products of these screening-kernel algebras are C_2 -cofinite. In particular the corresponding construction for a semisimple root system, with allowed parameters on its simple factors, has the same property.*

Proof. For vertex algebras U, V , $R_{U \otimes V} \simeq R_U \otimes R_V$. This follows by expanding the tensor-product fields and using translation to put modes of index at most -2 in the tensor factors' C_2 spaces. Alternatively use the tensor product of their Li filtrations at degree zero. The factorwise screening kernels also tensor, by elementary linear algebra over $\mathbb{C}$. Finiteness is preserved. $\square$

A Source conventions

All internal numbers in citations refer to the arXiv versions specified in the bibliography. We use [1, Theorem 2.13(1), Theorem 3.2, Corollary 3.4, Theorem 4.3] in the ordinary even-lattice case, with vacuum coset parameter zero. Its strong condition is $m \geq h_L^{\vee} - 1$; together with $p = tm \geq 2$ this is (1.5). The module $T_{\lambda,0}^{1/p}$ is the untwisted principal reduction of the induced affine Weyl module at $k = 1/p - h^{\vee}$. The Miura embedding is the one in [1, equation (4.3)], transported by $b'_i = -\beta_i/p$.

For intrinsic quotients, [2, Theorem 4.5.9(ii), Corollary 4.5.10(ii)] are applied to a finitely generated KL_k module with a Weyl flag of length one. Its PBW description and finite coefficient representation are given in Section 3. For the invariant algebra, the noncritical PBW and Miura symbol statements are [4, Section 5, equations (23)–(25); proof of Lemma 6.1]. The shifted level is $\kappa = 1/p \neq 0$ throughout. The specified Miura realization identifies A with the universal principal algebra, and the graded module isomorphism (2.5) identifies the intrinsic module quotients used in Proposition 3.1.

References

- [1] S. Sugimoto and H. Li, *A Lie algebraic pattern behind logarithmic CFTs*, Comm. Math. Phys. **407** (2026), article 202. Numbered source: arXiv:2409.07381v4.
- [2] T. Arakawa, *Associated varieties of modules over Kac–Moody algebras and C_2 -cofiniteness of W -algebras*, Int. Math. Res. Not. IMRN **2015** (2015), no. 22, 11605–11666. Numbered source: arXiv:1004.1554v5.
- [3] T. Arakawa and E. Frenkel, *Quantum Langlands duality of representations of $\mathcal{W}$ -algebras*, Compos. Math. **155** (2019), no. 12, 2235–2262. Numbered source: arXiv:1807.01536v3.
- [4] T. Arakawa, T. Creutzig and A. R. Linshaw, *W -algebras as coset vertex algebras*, Invent. Math. **218** (2019), no. 1, 145–195. Numbered source: arXiv:1801.03822v4.
- [5] D. Adamović and A. Milas, *On the triplet vertex algebra $\mathcal{W}(p)$* , Adv. Math. **217** (2008), 2664–2699. arXiv:0707.1857v2.
- [6] B. L. Feigin and I. Yu. Tipunin, *Logarithmic CFTs connected with simple Lie algebras*, preprint (2010). arXiv:1002.5047.
- [7] S. Sugimoto, *On the Feigin–Tipunin conjecture*, Selecta Math. (N.S.) **27** (2021), article 86. Numbered source: arXiv:2004.05769v3.
- [8] S. Sugimoto, *Simplicity of higher rank triplet W -algebras*, Int. Math. Res. Not. IMRN **2023** (2023), no. 8, 7169–7199. Numbered source: arXiv:2105.00638v3.