

Cubic augmentation of a two-moment model: an exact finite absorption theorem and its limits

Kunal Tyagi*

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Abstract

We study adjoining a half-bandwidth signed cubic trace and a spectral count ladder to a bandwidth-one trace/Frobenius model motivated by zeta-zero certificates. In a finite circle model with mass $N = 64$, flat windows, and explicit budget centers, we prove exact absorption over the full atom-only class: the augmented and baseline optima agree for both the distinct-count and simple-fraction objectives. Three explicitly specified integer-marked grid columns attain the lower bounds. Their mixing weights are defined algebraically, and an Arb certificate proves all strict feasibility and inertia inequalities. The witness satisfies the ladder at every positive threshold, not just on a finite threshold grid. We also give the exact conjugate-pair spectrum, the isolated-block cubic identity, grid Parseval, and the sharp conditional spectral-capacity constant $2\sqrt{2}$. The result is a finite model theorem, not an unconditional statement about zeta zeros. Version 2 corrects an infeasible witness, a cubic-tolerance mismatch, and an incorrect rationality claim in version 1. Global absorption in the pair-containing or pinned classes remains unproved; equality on a searched column dictionary and heuristic pricing do not establish it.

1 Problem, contribution and revision

The two-moment certificate in the work of Alpöge and Furman [1] motivates adding a signed cubic trace at a second bandwidth. Classical higher-level correlations [2] suggest suitable model budgets, but passing from those correlation statements to every row of a particular finite model is an additional task. We make the model and its proved scope explicit.

The main result is Theorem 3.2: one precisely specified $N = 64$ flat-window atom-only model has an exactly base-optimal law satisfying its half-bandwidth Frobenius row, cubic row and all-threshold count ladder. The proof combines an analytic atom-only lower bound with a finite algebraic construction and certified interval inequalities. It does not require optimizing over a numerically generated dictionary.

Correction of version 1. Version 1, titled *The two-moment certificate is robust under Rudnick–Sarnak-range cubic augmentation with capacity control*, stated broader exact absorption claims than its proofs and computations supported. Those full-class conclusions are withdrawn. Its printed clean-grid witness missed the implemented cubic band, its displayed cubic tolerance differed from the code’s, and selected row values are irrational algebraic numbers, not rational numbers. This

*hello.jay.tyagi@gmail.com. *Use of AI.* The original work used Claude (Anthropic). This revision, proof review, and companion certificates used OpenAI Codex. The author directed the work and is responsible for the content. Code and publication artifacts: <https://github.com/x67ai/riemann-rh-program>.

revision replaces the witness and states its budgets exactly. Section 6 separates the theorem proved here, the earlier numerical observations, and the open analytic bridges.

2 The finite model

Fix a circle of circumference N and a finite symmetric integer band B , with uniform harmonic weights $1/M$, $M = |B|$. A configuration consists of real positions θ_a with positive integer marks m_a , and optionally conjugate pairs $\theta_p \pm id_p$ of common positive integer mark m_p . Positions are taken modulo N . Its total mass is $\sum_a m_a + 2 \sum_p m_p = N$. Put

$$c_s = \sum_a m_a e^{-2\pi i s \theta_a / N} + 2 \sum_p m_p \cosh(2\pi s d_p / N) e^{-2\pi i s \theta_p / N}, \quad H_B[j, k] = \frac{c_{j-k}}{M} \quad (j, k \in B).$$

The identity $c_{-s} = \overline{c_s}$ makes H_B Hermitian and $\text{tr } H_B = N$. At the two bands used here,

$$B_1 = \{-32, \dots, 32\}, \quad M_1 = 65, \quad B' = \{-16, \dots, 16\}, \quad M' = 33,$$

write $F_1 = \text{tr } H_{B_1}^2$, $F' = \text{tr } H_{B'}^2$, $C' = \text{tr } H_{B'}^3$, and $n(V) = \#\{\lambda \in \text{Spec}(H_{B'}) : |\lambda| \geq V\}$. An *atom-only* configuration has no conjugate pairs; its matrices are positive semidefinite. Marks at coincident real positions are combined. Its objectives are $N_d = \#\{a\}$ and $p_1 = \#\{a : m_a = 1\}/N$. A law is a probability distribution on configurations, with finite expected F_1 ; each constraint below is an expectation constraint.

2.1 Exact budgets and the meaning of the cubic slack

All decimal constants in the following display mean exact rational numbers:

$$\begin{aligned} N &= 64, & \varepsilon &= 0.05 = 1/20, & C_{\text{led}} &= 10, \\ B_1^{\text{cen}} &= 84.71214548308222, & B_2^{\text{cen}} &= 135.09544203420174, & B_3^{\text{cen}} &= 304.5549106659001. \end{aligned}$$

The superscript distinguishes a budget center from a harmonic band. Below, write $b_j = B_j^{\text{cen}}$ and $U = b_1(1 + \varepsilon)$. These are fixed model parameters inherited from the primary finite cell of version 1. No equality to a zeta statistic, or to an unrounded random-matrix expectation, is assumed.

The baseline atom problem imposes mass 64 and

$$b_1(1 - \varepsilon) \leq E[F_1] \leq b_1(1 + \varepsilon). \quad (1)$$

The augmented problem adds

$$b_2(1 - \varepsilon) \leq E[F'] \leq b_2(1 + \varepsilon), \quad (2)$$

$$|E[C'] - b_3| \leq \varepsilon N, \quad (3)$$

$$E[n(V)] \leq C_{\text{led}} N V^{-4} \quad \text{for every } V > 0. \quad (4)$$

Thus the cubic half-width is the *absolute* amount $\varepsilon N = 3.2$. Version 1's code used the different relative amount εb_3 . The witness below has $E[C'] = b_3$ exactly and satisfies both conventions. No auxiliary garnish variable is needed for this theorem.

3 Exact atom-only absorption

3.1 The analytic lower bound

For any atom-only configuration and a symmetric band, the position kernel $D_B(x) = M^{-1} \sum_{j \in B} e^{-2\pi i j x / N}$ is real. Therefore

$$\mathrm{tr} H_B^2 = \sum_{a,a'} m_a m_{a'} D_B(\theta_a - \theta_{a'})^2 \geq \sum_a m_a^2.$$

The terms with distinct positions are nonnegative; equality requires their kernels to vanish when both marks are positive. For integer $m \geq 1$,

$$1 \geq (3m - m^2)/2, \quad \mathbf{1}_{m=1} \geq 2m - m^2,$$

with equality in both precisely for $m \in \{1, 2\}$. Summing gives

$$E[N_d]/N \geq \frac{3N - U}{2N}, \quad E[p_1] \geq 2 - \frac{U}{N} \quad (5)$$

under the upper baseline constraint. This reasoning covers arbitrary real positions and unbounded integer mark alphabets, not merely a grid dictionary. The integrality device is classical; see [3].

Lemma 3.1 (grid Parseval). *Let B consist of M consecutive integers and put $\theta_k = kN/M$, $0 \leq k < M$. For any nonnegative marks on these grid sites, $\text{tr } H_B^2 = \sum_k m_k^2$.*

Proof. Let $A[j, k] = M^{-1/2} e^{-2\pi i j k / M}$, $j \in B$, $0 \leq k < M$. Orthogonality of the M characters makes A unitary. The frequency matrix is $A \text{diag}(m_k) A^*$, so its eigenvalues are exactly the marks, including zero marks at unused sites. Taking the squared trace proves the statement. This proof also covers a translated consecutive band. $\square$

3.2 Three exact columns

For $N = 64$, use the 65 grid sites $\theta_k = 64k/65$. Table 1 specifies each column completely: each digit is the mark at one site, and zero denotes an unused site. The two strings in a row are concatenated in the displayed order, beginning at $k = 0$. Their $(N_d, \#\{m = 1\}, F_1)$ values are respectively

[illegible]

Table 1: Exact integer data. Each column has mass 64.

$(64, 64, 64)$, $(48, 32, 96)$ and $(32, 0, 128)$. Let $f_i = F'(A_i)$ and $q_i = C'(A_i)$, defined by the exact matrices of Section 2; they are real algebraic numbers in the real subfield of the 65th cyclotomic field. In particular they are not defined by rounding their displayed decimal approximations.

Put

$$r = \frac{U - 64}{32}, \quad w_2 = \frac{b_3 - q_0(1 - r) - q_1 r}{q_0 - 2q_1 + q_2}, \quad w_1 = r - 2w_2, \quad w_0 = 1 - r + w_2. \quad (6)$$

These equations impose

$$\sum_i w_i = 1, \quad \sum_i w_i F_1(A_i) = U, \quad \sum_i w_i q_i = b_3 \quad (7)$$

identically, provided the denominator is nonzero.

Theorem 3.2 (finite atom-only absorption). *For the exact model parameters in Section 2, the law on A_0, A_1, A_2 with weights (6) is feasible for (1)–(4). For both objectives, the baseline and augmented infima over all atom-only configurations at arbitrary positions are equal, with values*

$$\begin{aligned} \inf E[N_d]/N &= \frac{3N - U}{2N} = \frac{103052247242763669}{128000000000000000}, \\ \inf E[p_1] &= 2 - U/N = \frac{39052247242763669}{64000000000000000}. \end{aligned}$$

The same law works for every $C_{\text{led}} \geq 10$ and for the broader relative cubic band $|E[C'] - b_3| \leq \varepsilon b_3$.

Proof. The following finite inequalities are certified by the companion Arb computation, at 256 bits. The displayed decimal intervals are deliberately wider than its output enclosures:

$$\begin{aligned} q_0 &\in (245.45087235, 245.45087237), \\ q_1 &\in (411.08651303, 411.08651305), \\ q_2 &\in (301.35412844, 301.35412847), \\ w_0 &\in (0.47469100426, 0.47469100428), \\ w_1 &\in (0.27100071778, 0.27100071782), \\ w_2 &\in (0.25430827791, 0.25430827796), \\ \sum_i w_i f_i &\in (135.43015807, 135.43015809). \end{aligned}$$

The denominator in (6) is strictly negative. In particular all weights are positive. The last interval lies strictly inside $[b_2(1 - \varepsilon), b_2(1 + \varepsilon)]$. The exact identities (7) prove the baseline and cubic constraints without a numerical feasibility tolerance.

For each of the three 33×33 half-band matrices, certified Hermitian LDL^* pivots for $vI - H$, $v = 2, 3, 4$, exclude zero. Their signs give the counts above these thresholds:

	2	3	4
A_0	0	0	0
A_1	16	2	0
A_2	3	2	0

No eigenvalue equals a tested threshold, and all three matrices are positive semidefinite. Write $\bar{n}_v = \sum_i w_i n_{A_i}(v)$. For $0 < V \leq 2$, $E[n(V)] \leq 33 < 640/16$. For $2 < V \leq 3$, $E[n(V)] \leq \bar{n}_2 < 640/81$, the strict margin exceeding 2.8022. For $3 < V < 4$, $E[n(V)] \leq \bar{n}_3 < 640/256$, the margin exceeding 1.4493. At $V \geq 4$ every count is zero. Monotonicity therefore proves (4) at every threshold.

Each column has only marks 1, 2, and Lemma 3.1 gives equality in the analytic lower-bound chain. Hence its mixture attains both bounds (5). The augmented infimum cannot be below the baseline infimum; the same law attains both, proving the theorem. Increasing the ladder constant or widening the cubic band preserves feasibility. $\square$

Certificate and trust base. The companion `certificate.py` takes the integer mark vectors in `witness-data.json`, forms the Fourier matrices directly with Arb balls, computes their traces, solves the algebraic weight formulas, and checks every strict margin. It never rationalizes a floating-point row or adds feasibility grace. A Hermitian LDL^* decomposition without pivoting is accepted only when every real pivot excludes zero; congruence then certifies the inertia. The exact equalities are algebraic consequences of (6), rather than interval equality tests. The trust base is Python’s integer/rational arithmetic, the displayed finite algorithms, and Arb’s inclusion contract through python-flint 0.6.0 [5]. This is a computer-assisted finite proof, not a new Lean formalization.

4 Structural explanation

4.1 The true conjugate-pair spectrum

For a pair of depth d and mark m , set

$$p_j = M^{-1/2} e^{-2\pi i j \theta / N} e^{2\pi j d / N}, \quad q_j = M^{-1/2} e^{-2\pi i j \theta / N} e^{-2\pi j d / N}.$$

Its contribution is $m(pq^* + qp^*)$. Symmetry of the band gives $\langle p, q \rangle = 1$ and $\|p\|^2 = \|q\|^2 = A$, where

$$A = M^{-1} \sum_{j \in B} \cosh(4\pi j d / N) \geq 1.$$

The vectors $p + q$ and $p - q$ are orthogonal and give eigenvalues $m(1 + A)$ and $m(1 - A)$. At $d = 0$ the latter vanishes. Thus the normalized pair charges are

$$(\text{mass}, F, C) = (2m, 2m^2(1 + A^2), 2m^3(1 + 3A^2)).$$

In the shallow limit they become $(2m, 4m^2, 8m^3)$, with spectrum $(2m, 0)$, not a symmetric pair $(+2m, -2m)$. For a flat continuum band, $A = \sinh(2\pi \lambda d) / (2\pi \lambda d)$.

Proposition 4.1 (isolated-block cubic identity). *For a direct sum of isolated atom and pair blocks with total mass L , squared trace F and signed cubic trace C ,*

$$C - 3F + 2L = \sum_{\text{zeros}} m(m-1)(m-2) + 6 \sum_{\text{pairs}} m_p^2(m_p-1)A_p^2.$$

The first sum counts each atom once and each pair twice, using the mark on each zero. In particular it vanishes on mark- $\{1, 2\}$ atoms and multiplicity-one pairs, where $C = 3F - 2L$.

Proof. An atom contributes $m^3 - 3m^2 + 2m = m(m-1)(m-2)$. A pair contributes $2m^3(1 + 3A^2) - 6m^2(1 + A^2) + 4m = 2m(m-1)(m-2) + 6m^2(m-1)A^2$. Sum the identities. $\square$

The cubic row is redundant with mass and Frobenius *on this isolated block class*. A deep pair’s raw cubic value is larger than a double’s; there is no assertion that the two raw values agree at all depths. Interactions between different blocks are outside the identity.

4.2 Why a second bandwidth can read arrangement

At the 65-site grid let $c_s = \sum_k m_k e^{-2\pi i s k / 65}$. The half-band row is

$$F' = \frac{1}{33^2} \sum_{s=-32}^{32} (33 - |s|) |c_s|^2.$$

Unlike the full-band matrix, it is not a complete unitary conjugation of the mark diagonal. Its rows can therefore change with relative positions. For example two simple atoms at separations 64/65 and 128/65 have $F' = 2 + 2D'(64/65)^2$ and $2 + 2D'(128/65)^2$, respectively. Here

$$D'(64/65) = \frac{\sin(33\pi/65)}{33\sin(\pi/65)}, \quad |D'(128/65)| = \frac{1}{66\cos(\pi/65)},$$

and the first magnitude exceeds 0.6 while the second is below 0.02. Both full-band Frobenius values are 2.

A common translation, however, changes neither trace. In particular the vacancy column A_0 has $F' = 4128/33$ for *every* vacancy location: $c_0 = 64$ and $|c_s| = 1$ for $s \not\equiv 0 \pmod{65}$. Version 1's contrary sentence about detecting the vacancy location was incorrect.

4.3 Sine-process budgets and their limited role

For the intensity-one determinantal sine process and a flat continuum window $\psi_\lambda(x) = \sin(\pi\lambda x)/(\pi\lambda x)$, direct integration of the two- and three-point correlation determinants gives

$$m_2(\lambda) = \lambda^{-1} + \lambda/3, \quad m_3(\lambda) = 1 + \lambda^{-2} \quad (0 < \lambda \leq 1).$$

For clarity, put $u = \lambda^{-1}\mathbf{1}_{[-\lambda/2, \lambda/2]}$, $b = \mathbf{1}_{[-1/2, 1/2]}$ and $t(\alpha) = (1 - |\alpha|)_+$. Expanding the determinants and applying Plancherel yields

$$\begin{aligned} m_2 &= 1 + \int u^2 - \int t(u * u), \\ m_3 &= 1 + 3 \left(\int u^2 - \int t(u * u) \right) + \int u^3 - 3 \int (t * u)u^2 + 2 \int (b * u)^3. \end{aligned}$$

For this flat u , the five integrals are, in order, λ^{-1} , $1 - \lambda/3$, λ^{-2} , $\lambda^{-1} - 1/3$ and $1 - \lambda/2$. Substitution proves the formulas. In particular the centers 4/3, 13/6, 5 arise naturally as continuum model values. These formulas do not identify a finite model's cubic row with a zeta statistic. Theorem 3.2 uses its displayed exact parameters and does not depend on such an identification.

5 Two spectral-capacity facts

5.1 Sharp capacity conditional on a genuine ladder

Let μ be a positive measure on $[1, \infty)$ representing absolute escaped eigenvalue heights per unit mass. Suppose

$$\int h^2 d\mu \leq \eta, \quad \mu([h, \infty)) \leq Ch^{-4} \quad (h \geq 1).$$

Theorem 5.1 (sharp continuum capacity). *For $C > 0$ and $0 < \eta \leq 2C$,*

$$\sup_{\mu} \int h^3 d\mu = 2\sqrt{2C\eta}.$$

Equality is attained by density $4Ch^{-5}$ on (a, ∞) and no mass below a , where $a = \sqrt{2C/\eta}$.

Proof. For $a \geq 1$, $h^3 \leq ah^2 + (h^3 - ah^2)_+$. Tonelli and the tail bound give

$$\int h^3 d\mu \leq a\eta + C \int_a^\infty (3h^2 - 2ah)h^{-4} dh = a\eta + 2C/a.$$

The minimizing $a = \sqrt{2C/\eta}$ is at least 1 and gives the asserted bound. The displayed density has second moment $2C/a^2 = \eta$, third moment $4C/a = 2\sqrt{2C\eta}$, and tail $C \min(a^{-4}, h^{-4})$, proving attainment. $\square$

At fixed C the allowed cubic shift is $o(N)$ if $\eta = o(1)$. The theorem assumes a bound at every threshold; seven threshold inequalities alone do not imply its tail estimate. Theorem 3.2 verifies its own finite witness ladder directly, not a ladder for actual zeta matrices.

5.2 A divergent-cutoff observation in the isolated-spectrum model

Suppose $N \rightarrow \infty$, $V_0 \rightarrow \infty$, and an isolated-spectrum family has mass N , $F = \kappa N + o(N)$, bounded cubic deficit $\Delta_N N$ with $0 \leq \Delta_N \leq D$, and a fixed positive fraction of simple positive atoms. Choose

$$h = \min(\lfloor V_0/2 \rfloor, \lfloor N^{1/4} \rfloor), \quad k = \lceil \Delta_N N / h^3 \rceil.$$

Add k isolated positive atoms of integer mark h and delete kh simple atoms. There are enough simples for large N , since $kh = o(N)$. The mass is unchanged, the Frobenius change is $o(N)$, and the cubic change is $\Delta_N N + o(N)$. Each added eigenvalue is below V_0 , so rows seeing only $|\lambda| \geq V_0$ do not see the additions. The distinct and simple objectives do not increase.

This elementary construction explains why a single divergent-cutoff cubic tail constraint can be ineffective in that isolated-spectrum setting. It is not a realization theorem for arbitrary interacting configurations on a fixed circle; the direct-sum and deficit assumptions are part of the claim.

6 Pair-containing classes and unresolved scope

The analytic lower bound in Section 3 uses atom-only positivity. For pairs the mixed position couplings can be negative, so that proof cannot be applied unchanged. A weaker, uniform spectral bound remains available. For a configuration containing atoms and pairs define $N_d = \# \text{atoms} + 2\# \text{pairs}$. Each atom block has positive inertia at most one, and each pair block $m(pq^* + qp^*)$ also has positive inertia at most one. Subadditivity gives $n_+(H) \leq \# \text{atoms} + \# \text{pairs} \leq N_d$. Since $3x - x^2 \leq (9/4)\mathbf{1}_{x>0}$ for real x ,

$$N_d \geq \frac{4}{9}(3N - F_1).$$

At the ideal center $F_1 \leq (4/3)N$ this gives $N_d/N \geq 20/27$, which is weaker than the atom-only corner $5/6$. It does not establish full pair-class absorption.

What numerical dictionary equality means. Version 1 reported 940 small marginal differences on accumulated column dictionaries under its implemented relative cubic slack, together with multistart pricing and finite pair-interference tests. Those are numerical observations about the searched dictionaries. Adding a new column can lower the base optimum without lowering the augmented optimum: the new column may violate an added row. An already feasible atom witness does not exclude this possibility when the true baseline optimum over a larger class is unknown. Likewise, a finite collection of near-CUE witnesses and heuristic pricing does not prove global optimality in a pinned class. Neither full-class nor pinned-class exact absorption is asserted in this revision.

Analytic licensing. The signed cubic budget’s identification with a sine-process value and the upgrade from one spectral cutoff to an all-threshold ladder were left open in version 1. They remain open here. The classical Rudnick–Sarnak support range does not on its own identify every finite row or justify the required uniformity. The unconditional pair-correlation theorem of Baluyot–Goldston–Suriajaya–Turnage-Butterbaugh [4] is a second-order result and supplies no third-order identification. These are separate from the exact finite atom-only theorem proved above.

Formalization. The earlier Lean files at repository tag `paper-2026-08-28` certify selected grid Parseval and integer-corner statements. They do not certify the full LP, the revised algebraic witness, or any unresolved analytic bridge. The new witness is certified by the accompanying Arb computation; no fresh Lean build or additional kernel-checked theorem is claimed.

Algebraic versus rational rows. The vacancy row $F' = 4128/33$ is rational, but the selected columns A_1, A_2 have irrational half-band Frobenius values. The companion exact polynomial check reduces their Fourier numerators modulo Φ_{65} and obtains nonconstant remainders. Thus rounding rows to decimal rationals and checking Fraction arithmetic is not an exact check of the corresponding geometric configurations. Version 1’s checker also allowed positive feasibility grace; neither device is used for Theorem 3.2.

Data, code and revision record

The accompanying `cubic-augmentation-v2-source-and-checks.zip` contains this source, exact input data, the portable Arb certificate and output, an exact cyclotomic row check, requirements, build instructions and a change record. Run `python3 certificate.py` in the archive’s `anc/` folder. The input vectors in Table 1 match that file exactly. Historical version-1 numerical data remain available at the original publication record and the public repository tag `paper-2026-08-28`. Those historical computations are not invoked as exact proof certificates in this revision. The concept DOI is 10.5281/zenodo.22171688.

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