

PHASE-LABELED DOUBLES AND DISJOINT HANDLESIDE CUBES WITH COMPLETE MARKINGS

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ABSTRACT. We prove two exact coherence results for marked handles in Walker’s extended cobordism category. At fixed disjoint sites, a matching pair of forward and reverse phases contributes one local $S^1 \times S^2$ summand, whereas a mismatch contributes an identity cylinder. Thus the resulting double depends only on the total Walker weight and the number of phase matches, not on the orders of attachment. For positive handleslides supported in pairwise-disjoint protected sectors, we construct the complete marked Boolean cube and prove that every two-face commutes with weight zero; consequently every finite route has the same composite and the endpoint identities telescope. An optional odd-prime $\mathrm{SO}(3)$ -TQFT corollary evaluates the fixed-site double as $\kappa_p^N (D_p^{\mathrm{GM}})^b$ times the identity. Overlapping slides and their pentagon are treated by a separate companion theorem.

1. MAIN RESULTS

This paper studies two operations on marked handles. First, what is the composite of forward and reverse attachments when several marked sites are present? Second, what extra work is needed when the marking itself changes along a disjoint handleslide cube with complete markings? We call the composite of forward and reverse attachments a *mixed double*, allowing their phase labels to be chosen independently. The first question is local and fixed-site; the second concerns the vertices of the cube and requires supplied protected slide sectors.

Classical cut-system and Heegaard-diagram complexes already contain square and pentagon relation cells for handleslides [HT80, JTZ21]. The claims here are therefore limited to the printed refinement: supplied pairwise-disjoint protected sectors, complete endpoint markings, actual supported maps, a common Walker Lagrangian, and literal equalities with the declared weights. No priority claim is made for a square or order relation after those qualifiers are dropped. In the case without distinguished disks, Qin’s sutured compression-body complex likewise uses isotopy classes of cut-systems and triangle, square, and pentagon relations to prove simple connectivity [Qin25, Theorem 1.1, Definition 2.5, and Section 4, especially Definition 4.2]. Our disjoint cube instead keeps the specified complete marking and selected smooth representatives; it does not supply the full move-complex conclusion.

At fixed sites, the Boolean coordinate records which handles have been attached, and different orders are histories with common endpoints in one pulled-back chart. In the slide layer, every Boolean coordinate records a completed change of marking; the source and target rows of later edges therefore depend on the current vertex. Commutation on homology alone does not identify the corresponding Walker morphisms, because endpoint Lagrangians and ordered Maslov corrections must also agree.

Our fixed-site calculation begins locally. We retain the actual supported half-turn and separating-disk twist, calculate the four weighted same-site composites, and prove every heterogeneous interchange before passing to order-independent histories. An independently constructed simultaneous trace then identifies the topology of the mixed double without retaining an artificial decomposition in the output. This yields [Theorem 1.1](#).

Theorem A (Fixed-site phase double). *A matching forward/reverse phase leaves one local $S^1 \times S^2$ factor, while a mismatching phase leaves an identity cylinder; the global mixed double*

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is the product of these local contributions, with the Walker weight carried separately by the identity-cylinder part.

For the moving layer, one fully specified two-site positive-slide representative is imported from the companion pentagon theorem [Sun26, Theorem 2.2 and Appendix F]. Here a protected sector is a specified genus-two neighborhood containing the complete labelled local data for one slide direction and meeting its complement along a collared interface disk. The complete marking includes the oriented curves, framings, collars, and cut-face data needed by the local representative. We prove the additional statements needed to place disjoint copies into those sectors: compatibility of all completed vertex rows, actual supported edge maps and inverses, signed integral homology, a common rational Lagrangian, phase-zero and all-phase Walker covariance, literal two-faces, and finite-route closure. These arguments produce Theorem 1.2; none is attributed to the imported two-site construction.

Theorem B (Disjoint handleslide cube). *A cube edge is not merely a commuting homology matrix: it is a completed marked Walker cobordism whose endpoint Lagrangians, phase data, and weights have been matched. Once those data are aligned, every two-face is literally equal, and every finite route has the same composite.*

1.1. Fixed marked sites. For a positive integer q , write $[q] = \{1, \dots, q\}$. Fix $g, k \geq 1$, a connected marked outgoing surface $(\Sigma_g^{\text{out}}, \lambda_g)$, and k pairwise-disjoint, distinctly labelled disk sites with disjoint closures and complete local markings. Fix phase vectors $\epsilon, \eta \in \{0, 1\}^k$, label-attached weight vectors

$$\mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^- \in \mathbb{Z}^k,$$

and independent attachment orders $\alpha, \beta \in S_k$. A coordinate u_t^+ with $\epsilon_t = 0$, or u_t^- with $\eta_t = 0$, is inactive and ignored. Write

$$n_t = c_t + r_t + \epsilon_t u_t^+ + \eta_t u_t^-, \quad N = \sum_{t=1}^k n_t,$$

and define the Hamming distance by

$$d_H(\epsilon, \eta) := |\{t \in [k] : \epsilon_t \neq \eta_t\}|.$$

Then

$$\mathcal{M}(\epsilon, \eta) = \{t \in [k] : \epsilon_t = \eta_t\}, \quad b = |\mathcal{M}(\epsilon, \eta)| = k - d_H(\epsilon, \eta).$$

For $I \subseteq [k]$, let Σ_I^{out} be the marked outgoing surface obtained by attaching exactly the sites in I . For $t \notin I$, the constructed phase arrows have types

$$J_{t|I}^e : \Sigma_I^{\text{out}} \longrightarrow \Sigma_{I \cup \{t\}}^{\text{out}}, \quad R_{t|I}^v : \Sigma_{I \cup \{t\}}^{\text{out}} \longrightarrow \Sigma_I^{\text{out}}, \quad e, v \in \{0, 1\}.$$

For $n \in \mathbb{Z}$, write $C(\text{id}_\Sigma, n)$ for the identity mapping cylinder of Walker weight n , and $\mathcal{D}_{t|I}(n)$ for the preferred local double of weight n in $\text{End}(\Sigma_I^{\text{out}})$. At preferred weight zero, write $1 = C(\text{id}_{\Sigma_I^{\text{out}}}, 0)$ and $\mathcal{D} = \mathcal{D}_{t|I}(0)$ in that typed endomorphism set. The symbols $J_\alpha^\epsilon, R_\beta^\eta$ denote the corresponding ordered traces, $J_{\text{sim}}^\epsilon, R_{\text{sim}}^\eta$ their independently constructed simultaneous counterparts, and $M_{\alpha, \beta}^{\epsilon, \eta} := R_\beta^\eta J_\alpha^\epsilon$ their mixed double. Fixed weight arguments are suppressed when no ambiguity can arise, and $(-)^{\dagger}$ denotes source orientation reversal. The detailed constructions appear in Sections 3 to 5.

The weighted same-site table identifies the local effect of each phase pair in one pulled-back marked chart. The heterogeneous interchanges then let arbitrary histories be reordered and compared with an independently constructed simultaneous trace. In that geometric trace, the matching pieces determine the connected-sum topology, while the accumulated weight belongs to the identity cylinder. Source reversal is applied after this marked source topology has been identified.

Smallest fixed-site picture. When there is one active site, the four phase choices already display the local dichotomy: matching forward and reverse phases produce the preferred local double, while either mismatch produces an identity cylinder. This is the smallest setting in which the phase labels have geometric meaning. It also shows why the later multi-site result is not just a permutation of attachments: the local type is determined before histories are reordered, and the Walker weight is carried by the resulting cylinder.

	$\eta_t = 0$	$\eta_t = 1$
$\epsilon_t = 0$	$\mathcal{D}_{t I}(n_t)$	$C(\text{id}_{\Sigma_I^{\text{out}}, n_t})$
$\epsilon_t = 1$	$C(\text{id}_{\Sigma_I^{\text{out}}, n_t})$	$\mathcal{D}_{t I}(n_t)$

FIGURE 1. The one-site phase diagram. Diagonal phase choices match and leave a local $S^1 \times S^2$ factor; off-diagonal choices leave the identity cylinder. The Walker weight n_t is shown separately from this topological dichotomy.

A worked $k = 2$ phase choice. Take $\epsilon = (0, 1)$ and $\eta = (1, 1)$. Site 1 is a mismatch and therefore contributes an identity cylinder; site 2 is a match and contributes one local double. Thus

$$d_H(\epsilon, \eta) = 1, \quad \mathcal{M}(\epsilon, \eta) = \{2\}, \quad b = 1,$$

and for arbitrary allowed weights the theorem gives

$$M^{\epsilon, \eta} \cong C(\text{id}_{\Sigma_g^{\text{out}}, N}) \# (S^1 \times S^2, 0), \quad \beta_1(M^{\epsilon, \eta}) = 2g + 1.$$

This specialization exposes both local outcomes while leaving the order of the two attachments irrelevant.

Theorem 1.1 (Fixed-site phase choices and completed traces). *Uniformly over the preceding data, the following statements hold.*

- (1) *The Boolean family of attachment states, its Walker Lagrangians, and every phase-zero or phase-one forward and reverse handle arrow are fully typed in one pulled-back Walker chart. The local weights are $c_t + \epsilon_t u_t^+$ and $r_t + \eta_t u_t^-$, respectively, and every spectator marking is fixed.*
- (2) *For every $t \in [k]$ and $I \subseteq [k] \setminus \{t\}$, the localized marked half-turn, fourth-power separating-disk twist, capping-kernel identity, the two reverse-absorption identities, and the elementary source-reversal identities hold with their actual Walker seams. In particular, source reversal inverts the supported slope and negates its weight; it is not a Hilbert-space adjoint and is not a slide inverse.*
- (3) *The weighted same-site table is*

$$R_{t|I}^{\eta_t} J_{t|I}^{\epsilon_t} = \begin{cases} \mathcal{D}_{t|I}(n_t), & \epsilon_t = \eta_t, \\ C(\text{id}_{\Sigma_I^{\text{out}}, n_t}), & \epsilon_t \neq \eta_t, \end{cases}$$

for every $t \in [k]$ and $I \subseteq [k] \setminus \{t\}$. Moreover, for every $n \in \mathbb{Z}$,

$$\mathcal{D}_{t|I}(n) \cong C(\text{id}_{\Sigma_I^{\text{out}}, n}) \# (S^1 \times S^2, 0).$$

Hence the preferred-weight table is $(\mathcal{D}, 1, 1, \mathcal{D})$, and a matching-phase entry is not an identity cylinder of another weight.

- (4) *All four forward and all four reverse phase interchanges hold, including the heterogeneous cases, with subset tags and label-attached weights unchanged. Each fixed phase vector therefore has literal order independence. The associated Coxeter-shaped diagrams are diagrams of equality only; they construct no comparison cylinder and no permutation action.*
- (5) *For every $\alpha, \beta \in S_k$, the ordered histories agree literally with independently constructed simultaneous traces:*

$$J_{\alpha}^{\epsilon}(\mathbf{c}, \mathbf{u}^+) = J_{\text{sim}}^{\epsilon}(\mathbf{c}, \mathbf{u}^+), \quad R_{\beta}^{\eta}(\mathbf{r}, \mathbf{u}^-) = R_{\text{sim}}^{\eta}(\mathbf{r}, \mathbf{u}^-).$$

Their homology, transported Lagrangians, complete endpoint markings, and Walker weights all agree.

- (6) *The mixed double $M_{\alpha,\beta}^{\epsilon,\eta}$ is independent of both attachment orders; denoting its common value by $M^{\epsilon,\eta}$, it satisfies*

$$M^{\epsilon,\eta} \cong C(\text{id}_{\Sigma_g^{\text{out}}}, N) \# \left(\#_{t \in \mathcal{M}(\epsilon,\eta)} (S^1 \times S^2, 0) \right), \quad \beta_1(M^{\epsilon,\eta}) = 2g + b.$$

It is the identity cylinder of weight N exactly when $b = 0$; for $b > 0$ it is not an identity cylinder of any weight. No separating spheres or embedded labelled match set are retained as output data.

- (7) *Source reversal is contravariant, reverses products, and negates every Walker weight. In the order notation above,*

$$(J_\alpha^\epsilon)^\dagger = R_\alpha^\epsilon(-\mathbf{c}, -\mathbf{u}^+), \quad (R_\beta^\eta)^\dagger = J_\beta^\eta(-\mathbf{r}, -\mathbf{u}^-),$$

and

$$(M_{\alpha,\beta}^{\epsilon,\eta})^\dagger = M_{\beta,\alpha}^{\eta,\epsilon}(-\mathbf{r}, -\mathbf{c}, -\mathbf{u}^-, -\mathbf{u}^+).$$

Here equality means literal equality in the indicated marked Walker Hom set, including its integer weight. The symbol $\cong$ denotes a boundary-marking-preserving diffeomorphism retaining the displayed weight data. The theorem includes $g = 1$, $k = 1$, empty phase words, complementary phase vectors, and arbitrary positive, zero, or negative active weights.

1.2. Disjoint positive-slide sectors. Now fix $s \geq 1$ and $k \geq 2s$, a connected compression body with complete marking

$$W: X = \Sigma_g^{\text{out}} \longrightarrow Y = \Sigma_{g+k}^{\text{out}},$$

and a rational disk-kernel Lagrangian $\lambda_g \subset H_1(X; \mathbb{Q})$. The old marking fixes an inclusion $H_1(X; \mathbb{Q}) \hookrightarrow H_1(Y; \mathbb{Q})$, through which we regard $H_1(X; \mathbb{Q})$ and λ_g as subspaces of $H_1(Y; \mathbb{Q})$. Supply an injective ordered-pair assignment

$$a \longmapsto (i_a, j_a), \quad a \in [s],$$

with $2s$ distinct active labels, and s pairwise-disjoint protected genus-two sectors carrying the complete local data and positive product-zero-framed slide specified in [Section 6](#). At the reference chart, assume that the one-holed-torus phase-support collars of all k labels are pairwise disjoint.

For $A \subseteq [s]$, let $\mathfrak{C}_A$ be the directly completed vertex. For $a \notin A$, let $F_{a|A}$ be the supported compression-body edge, $h_{a|A}$ its outgoing transition, and

$$H_{a|A} = C(h_{a|A}, 0)$$

its marked weight-zero Walker cylinder. In the complete old marking, declare

$$H_1(Y; \mathbb{Q}) = H_1(X; \mathbb{Q}) \oplus \bigoplus_{t=1}^k \text{span}_{\mathbb{Q}}\{[m_t], [\ell_t]\},$$

and

$$\Lambda = \lambda_g \oplus \bigoplus_{t=1}^k \langle [m_t] \rangle_{\mathbb{Q}}.$$

For later weight formulas, put

$$c_\Sigma = \sum_{t=1}^k c_t, \quad r_\Sigma = \sum_{t=1}^k r_t, \quad U^+(\epsilon) = \sum_{\epsilon_t=1} u_t^+, \quad U^-(\eta) = \sum_{\eta_t=1} u_t^-.$$

The slide proof first fixes the complete protected sectors and directly completed vertices. It then constructs the actual edge maps with common collars, computes their signed homology actions, and proves the common rational Lagrangian. Next it checks Walker covariance for every phase and weight. The final step compares disjoint edges pointwise, which yields literal squares, finite route independence, and telescoping.

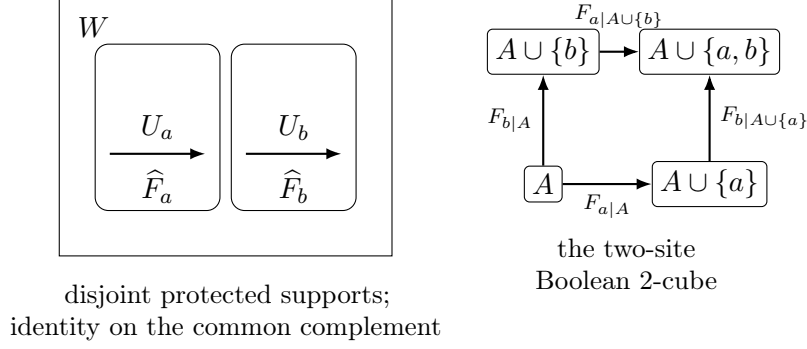


FIGURE 2. The local-support picture and the smallest disjoint handleslide cube. The left panel records only disjoint supports and identity on their common complement; it encodes no unstated meridian, longitude, framing, or orientation data. In the square, the intermediate-set tags are part of every edge, and the two composites are equal as complete-marked smooth maps before passing to homology or Walker cylinders.

Theorem 1.2 (Disjoint positive handleslide cube with complete markings). *Uniformly over all of the preceding slide data, the following six conclusions hold.*

- (1) Every $\mathfrak{C}_A$, $F_{a|A}$, $h_{a|A}$, and $H_{a|A}$ is well defined with complete marking and cube-compatible endpoint rows. The edge $F_{a|A}$ has a smooth inverse, has the identity germ on the full interface collar, and fixes pointwise the incoming boundary, the old region, every other protected sector, and every unused handle.
- (2) Every edge fixes the old and spectator homology summands and has active integral action

$$\begin{aligned} (h_{a|A})_*[m_{i_a}^A] &= [m_{i_a}^{A \cup \{a\}}] = [m_{i_a}^A] + [m_{j_a}^A], & (h_{a|A})_*[\ell_{i_a}^A] &= [\ell_{i_a}^{A \cup \{a\}}] = [\ell_{i_a}^A], \\ (h_{a|A})_*[m_{j_a}^A] &= [m_{j_a}^{A \cup \{a\}}] = [m_{j_a}^A], & (h_{a|A})_*[\ell_{j_a}^A] &= [\ell_{j_a}^{A \cup \{a\}}] = [\ell_{j_a}^A] - [\ell_{i_a}^A]. \end{aligned}$$

This action is integral symplectic. Every vertex Lagrangian satisfies $\Lambda_A = \Lambda$, and every edge preserves Λ in both directions. The actual inverse carries the target marked row back to the source marked row; it is not the separate operation of expressing source classes in target coordinates.

- (3) For arbitrary $\mathbf{c}, \mathbf{r}$, the forward and reverse ordinary phase-zero identities are separately typed, and their Walker lifts are

$$H_{a|A} J_A^0(\mathbf{c}) = J_{A \cup \{a\}}^0(\mathbf{c}), \quad R_{A \cup \{a\}}^0(\mathbf{r}) H_{a|A} = R_A^0(\mathbf{r}).$$

The two ordered Maslov corrections vanish independently. The common weights are c_Σ and r_Σ , respectively, and the completed morphisms retain no intrinsic decomposition of these totals.

- (4) At each completed vertex, every phase word is defined from its actual oriented and framed meridian-longitude row. The supported labelled words are Walker-exact and order independent, agree with the fixed-site words at $A = \emptyset$, and satisfy complete-marking conjugacy across an edge. All four ordered phase-cylinder corrections vanish in their actual multiplication orders. Consequently

$$H_{a|A} J_A^e(\mathbf{c}, \mathbf{u}^+) = J_{A \cup \{a\}}^e(\mathbf{c}, \mathbf{u}^+),$$

$$R_{A \cup \{a\}}^\eta(\mathbf{r}, \mathbf{u}^-) H_{a|A} = R_A^\eta(\mathbf{r}, \mathbf{u}^-),$$

with common weights $c_\Sigma + U^+(\epsilon)$ and $r_\Sigma + U^-(\eta)$. Weight coordinates remain attached to their labels during every replacement; only the displayed totals are intrinsic to the completed morphisms.

(5) For $A \subseteq [s]$ and distinct $a, b \notin A$, the two routes around the face first agree as literal smooth maps on W :

$$(1.1) \quad F_{b|A \cup \{a\}} F_{a|A} = F_{a|A \cup \{b\}} F_{b|A}.$$

They therefore induce the same marked outgoing map. Their two Walker corrections vanish separately and give the literal weight-zero square

$$(1.2) \quad H_{b|A \cup \{a\}} H_{a|A} = H_{a|A \cup \{b\}} H_{b|A}.$$

(6) For a slide order $\omega = (\omega_1, \dots, \omega_s)$, put $A_q^\omega = \{\omega_1, \dots, \omega_q\}$, $A_0^\omega = \emptyset$, and

$$H_\omega = H_{\omega_s|A_{s-1}^\omega} \cdots H_{\omega_1|\emptyset}.$$

All H_ω have one common value $H_{[s]}$ of Walker weight zero. The involutive, distant, and braid-shaped adjacent-replacement diagrams close as equalities, but yield no retained swap morphism or permutation action. The endpoint covariances telescope to

$$\begin{aligned} H_{[s]} J_\emptyset^0(\mathbf{c}) &= J_{[s]}^0(\mathbf{c}), & R_{[s]}^0(\mathbf{r}) H_{[s]} &= R_\emptyset^0(\mathbf{r}), \\ H_{[s]} J_\emptyset^\epsilon(\mathbf{c}, \mathbf{u}^+) &= J_{[s]}^\epsilon(\mathbf{c}, \mathbf{u}^+), & R_{[s]}^\eta(\mathbf{r}, \mathbf{u}^-) H_{[s]} &= R_\emptyset^\eta(\mathbf{r}, \mathbf{u}^-). \end{aligned}$$

The terminal subscript $[s]$ in the last clause is the vertex at which all s slide directions have been applied; it is not the handle-label set $[k]$. Theorem 1.2 includes $g = 1$, $s = 1$, $k = 2s$, empty phase words, arbitrary zero or negative weight coordinates, the reference and terminal vertices, and every unused spectator when $k > 2s$.

1.3. Scope. The two theorems prove source geometry in Walker's extended cobordism category. They do not prove an overlapping-slide relation, an arbitrary handleslide groupoid, relations among pentagons, stabilization, Cerf completeness, Heegaard invariance, a TQFT construction, WRT invariance, a Habiro lift, or a $\hat{Z}$ comparison. The overlapping three-handle pentagon remains an independent companion theorem. A normalized ordinary $\mathrm{SO}(3)$ -TQFT scalar for the fixed-site double is recorded only after all source-topological arguments, in Section 10.

Organization. Section 2 fixes the common Walker conventions and separates the two Boolean state spaces. Sections 3 to 5 prove the fixed-site theorem. Sections 6 to 9 construct and prove the slide cube. The optional ordinary scalar appears in Section 10; boundary cases, reusable interfaces, and exact nonclaims are collected in Section 11.

2. WALKER CONVENTIONS AND THE TWO BOOLEAN STATE SPACES

An attachment state records which handles are present; a slide state records which complete markings have changed after all handles are present. Both use the same labelled handles, terminal surface, endpoint marking, and Walker multiplication law. Their Boolean indices nevertheless describe different geometric operations.

We first fix the marked surface, its rational Lagrangian, and the Walker multiplication convention. The common relative extension then lets later order and coherence arguments compare routes within each chart while keeping the two kinds of state distinct.

2.1. Marked Walker cylinders. All manifolds and surfaces are oriented. Composition and mapping-class products are written right-factor first: in a product MN , the factor N is traversed first. Let (Σ, λ) be a marked oriented surface with declared rational Lagrangian $\lambda \subset H_1(\Sigma; \mathbb{Q})$. For an orientation-preserving marked diffeomorphism ϕ of Σ , write $C(\phi, m)$ for its Walker mapping cylinder of integer weight m . Our convention is

$$(2.1) \quad C(\psi, n)C(\phi, m) = C(\psi\phi, n + m - \mu_\Sigma(\phi_*\lambda, \lambda, \psi_*^{-1}\lambda)).$$

We use the extended-surface convention of [Wal91] and [GM13, arXiv v3, §3, eq. (2)]. We use throughout the direct-sum and repeated-pair identities

$$(2.2) \quad \mu(L_1 \oplus L'_1, L_2 \oplus L'_2, L_3 \oplus L'_3) = \mu(L_1, L_2, L_3) + \mu(L'_1, L'_2, L'_3),$$

$$(2.3) \quad \mu(L_1, L_2, L_3) = 0 \quad \text{if two of the three entries coincide.}$$

Every later calculation first displays the three Lagrangians in the order forced by (2.1) and only then applies these two identities.

Equality of Walker morphisms means literal equality in one marked Hom set, with both endpoint markings and the integer weight retained. The notation $M \cong N$ instead records a boundary-marking-preserving diffeomorphism of the underlying cobordisms retaining the displayed weights. If an underline is used, $\underline{M}$ forgets only the Walker integer and retains all marking data. The symbol $M^\dagger$ is reserved for source orientation reversal; it is neither a Hilbert-space adjoint nor the inverse of a slide map. For an oriented product-framed simple closed curve γ , write T_γ for the positive right Dehn twist in its specified collar, extended by the identity outside that collar.

2.2. The attachment-state Boolean chart. For a positive integer n , put $[n] = \{1, \dots, n\}$. Fix $g, k \geq 1$, a connected marked outgoing surface $(\Sigma_g^{\text{out}}, \lambda_g)$, and k pairwise-disjoint, distinctly labelled disk sites $d_1, \dots, d_k$ with disjoint closures and complete local markings. A fixed-site *attachment state* is a subset

$$(2.4) \quad I \subseteq [k].$$

Attaching the labelled one-holed torus at precisely the sites in I gives the outgoing surface Σ_I^{out} . Its marked homology and declared Lagrangian split as

$$(2.5) \quad H_1(\Sigma_I^{\text{out}}; \mathbb{Q}) = H_1(\Sigma_g^{\text{out}}; \mathbb{Q}) \oplus \bigoplus_{t \in I} \text{span}_{\mathbb{Q}}\{[m_t], [\ell_t]\},$$

$$(2.6) \quad \lambda_I = \lambda_g \oplus \bigoplus_{t \in I} \langle [m_t] \rangle_{\mathbb{Q}}.$$

The marking at site t includes its ordered attaching feet and joining band, oriented co-core meridian m_t , oriented product-framed longitude ℓ_t , and separating collar, with $[m_t] \cdot [\ell_t] = 1$. The old surface, every handle indexed by I , and every unused disk are labelled spectators for an attachment at a new site.

Let

$$(2.7) \quad Q_{g,k}^{\text{out}} := \text{Mod} \left(\Sigma_g^{\text{out}} \setminus \bigcup_{t=1}^k \text{int}(d_t), \partial \right),$$

and let

$$(2.8) \quad e_I : Q_{g,k}^{\text{out}} \longrightarrow \Gamma(\Sigma_I^{\text{out}})$$

be capping followed by the identity extension over the attached labelled tori.

Proposition 2.1 (Common Boolean Walker extension). *The Walker two-cocycles pulled back along the maps e_I , $I \subseteq [k]$, are equal. Consequently there is one central extension $\tilde{Q}_{g,k}^{\text{out}}$ and, for every attachment state I , a homomorphism*

$$(2.9) \quad \tilde{e}_I : \tilde{Q}_{g,k}^{\text{out}} \longrightarrow \tilde{\Gamma}(\Sigma_I^{\text{out}}, \lambda_I), \quad \tilde{e}_I(q, n) = C(e_I(q), n).$$

Thus one relative mapping class and its integer weight have compatible images at every vertex of the attachment-state Boolean chart.

Proof. Each e_I fixes the attached torus summands, so its pulled-back cocycle splits into the old relative term and repeated meridian pairs. The pairs contribute zero. The cocycle is therefore independent of the attachment state, giving one central extension of the shared relative source.

Let $f, h \in Q_{g,k}^{\text{out}}$. At the state I , the correction for the product of their extensions is

$$-\mu((e_I(f))_* \lambda_I, \lambda_I, (e_I(h))_*^{-1} \lambda_I).$$

Both mapping classes are the identity on every attached labelled torus. Using (2.5) and (2.6), the displayed correction is

$$- \mu((e_\emptyset(f))_* \lambda_g, \lambda_g, (e_\emptyset(h))_*^{-1} \lambda_g) \\ - \sum_{t \in I} \mu(\langle [m_t] \rangle, \langle [m_t] \rangle, \langle [m_t] \rangle).$$

Every term in the sum is zero by (2.3). The pullback cocycle is therefore the one at $I = \emptyset$, independently of I , and (2.9) respects (2.1). The common source is the relative group $Q_{g,k}^{\text{out}}$; this construction introduces no comparison homomorphism between unrelated closed Walker groups. $\square$

The terminal attachment state fixes the notation used by the slide layer:

$$(2.10) \quad X := \Sigma_\emptyset^{\text{out}} = \Sigma_g^{\text{out}}, \quad Y := \Sigma_{[k]}^{\text{out}} = \Sigma_{g+k}^{\text{out}}, \quad \Lambda := \lambda_{[k]}.$$

The terminal attachment trace is a compression body with complete marking $W : X \rightarrow Y$. Its complete marking contains more than the symplectic splitting: it retains the ordered feet and bands, oriented co-core disks, embedded product-framed meridian-longitude rows, endpoint parameterizations, signed cut faces and pairings, full product collars, dual ribbons, and the interface and detector germs used in the slide construction. The old marking also fixes the inclusion $H_1(X; \mathbb{Q}) \hookrightarrow H_1(Y; \mathbb{Q})$, through which we regard $H_1(X; \mathbb{Q})$ and λ_g as subspaces of $H_1(Y; \mathbb{Q})$. In particular,

$$(2.11) \quad H_1(Y; \mathbb{Q}) = H_1(X; \mathbb{Q}) \oplus \bigoplus_{t=1}^k \text{span}_{\mathbb{Q}}\{[m_t], [\ell_t]\},$$

$$(2.12) \quad \Lambda = \lambda_g \oplus \bigoplus_{t=1}^k \langle [m_t] \rangle_{\mathbb{Q}}.$$

For an attachment order $\alpha = (\alpha_1, \dots, \alpha_k) \in S_k$, define

$$(2.13) \quad I_q^\alpha := \{\alpha_1, \dots, \alpha_q\}, \quad I_0^\alpha := \emptyset.$$

Independent forward and reverse attachment orders will be denoted $\alpha, \beta \in S_k$. These symbols are never used for slide order.

2.3. The completed slide-state Boolean cube. The slide enhancement is available only after all k handles have been attached. It requires integers $s \geq 1$ and $k \geq 2s$, together with the protected data specified in Section 6. A *slide state* is a subset

$$(2.14) \quad A \subseteq [s].$$

At every such state the underlying outgoing surface is the same surface Y ; the state records which of the s prescribed positive slides have been completed in the marking. We write $\mathfrak{C}_A$ for that completed marking, p_A for its outgoing parameterization, and $m_t^A, \ell_t^A \subset Y$ for its actual oriented product-framed row. The associated Lagrangian will be

$$(2.15) \quad \Lambda_A := \lambda_g \oplus \bigoplus_{t=1}^k \langle [m_t^A] \rangle_{\mathbb{Q}}.$$

Only the reference identity $\Lambda_\emptyset = \Lambda$ is immediate from the old marking. The equality $\Lambda_A = \Lambda$ for arbitrary A is a later theorem, proved from the signed integral edge formulas in both directions; it is not part of the definition above.

For a slide order $\omega = (\omega_1, \dots, \omega_s) \in S_s$, put

$$(2.16) \quad A_q^\omega := \{\omega_1, \dots, \omega_q\}, \quad A_0^\omega := \emptyset.$$

Thus $I = \emptyset$ is the genus- g attachment state, while $A = \emptyset$ is the reference marking on the already completed genus- $g + k$ boundary. In particular,

$$(2.17) \quad \Sigma_\emptyset^{\text{out}} = X \quad \text{but} \quad (W, \mathfrak{C}_\emptyset) \text{ has outgoing boundary } Y.$$

No argument may identify these two empty states.

2.4. Phases, labelled weights, and two reusable lemmas. Fix phase assignments and independent label-attached weight vectors

$$(2.18) \quad \epsilon, \eta \in \{0, 1\}^k, \quad \mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^- \in \mathbb{Z}^k.$$

A coordinate u_t^+ with $\epsilon_t = 0$, or u_t^- with $\eta_t = 0$, is inactive and ignored. Write

$$(2.19) \quad \begin{aligned} c_\Sigma &:= \sum_{t=1}^k c_t, & r_\Sigma &:= \sum_{t=1}^k r_t, \\ U^+(\epsilon) &:= \sum_{\epsilon_t=1} u_t^+, & U^-(\eta) &:= \sum_{\eta_t=1} u_t^-. \end{aligned}$$

The coordinates remain attached to their labels during every equality replacement. Only the displayed totals are intrinsic to a completed morphism after the labelled construction has been performed.

The following lemma isolates the one phase-cylinder calculation common to the two layers. It does not concern handle arrows or slide edges.

Lemma 2.2 (Disjoint phase cylinders). *Let (Z, Λ_Z) be a marked surface with a symplectic splitting*

$$H_1(Z; \mathbb{Q}) = H_{\text{old}} \oplus \bigoplus_{t \in K} P_t, \quad \Lambda_Z = \Lambda_{\text{old}} \oplus \bigoplus_{t \in K} \langle [m_t] \rangle_{\mathbb{Q}},$$

where $P_t = \text{span}_{\mathbb{Q}}\{[m_t], [\ell_t]\}$, and suppose the corresponding one-holed-torus collars are pairwise disjoint. Put

$$\sigma_t := T_{m_t} T_{\ell_t} T_{m_t}, \quad \Theta_t(u) := C(\sigma_t, u).$$

For distinct $i, j \in K$ and $u, v \in \mathbb{Z}$,

$$(2.20) \quad \Theta_j(v) \Theta_i(u) = C(\sigma_j \sigma_i, u + v) = C(\sigma_i \sigma_j, u + v) = \Theta_i(u) \Theta_j(v).$$

More generally, every labelled product of distinct $\Theta_t(u_t)$ is independent of its order as a Walker morphism and has weight $\sum_t u_t$.

Proof. The calculation separates over the disjoint symplectic planes. The Walker correction decomposes into the corresponding Maslov contributions, and each active plane contributes a triple of meridian/longitude lines whose index is zero. The mapping classes commute because their collars are disjoint; the longer-product statement then follows by adding one untouched plane at a time.

The action of σ_t sends the meridian line in P_t to the longitude line and fixes every other summand. In the first product in (2.20), the correction dictated by (2.1) is

$$\mu_Z((\sigma_i)_* \Lambda_Z, \Lambda_Z, (\sigma_j)^{-1} \Lambda_Z).$$

On P_i the three lines are $(\langle \ell_i \rangle, \langle m_i \rangle, \langle m_i \rangle)$; on P_j they are $(\langle m_j \rangle, \langle m_j \rangle, \langle \ell_j \rangle)$; and on every other summand all three entries agree. The correction is zero. Reversing i and j gives the independently ordered triples (m_i, m_i, ℓ_i) and (ℓ_j, m_j, m_j) , so that correction also vanishes. The mapping classes commute pointwise because their collars are disjoint, proving (2.20).

For a longer prefix, the new factor changes one previously untouched symplectic plane. Each old active plane contributes a triple of the form (ℓ, m, m) , the new plane contributes (m, m, ℓ) , and every other plane is constant. Induction and (2.2)–(2.3) prove the last assertion. $\square$

The second reusable lemma contains only the combinatorics of literal equalities. All geometric and Walker hypotheses must be verified before it is invoked.

Lemma 2.3 (Adjacent-equality confluence). *Let P be a finite set, let Z_B be an object for every $B \subseteq P$, and for $p \notin B$ let $\Phi_{p|B} : Z_B \rightarrow Z_{B \cup \{p\}}$ be a typed arrow. Suppose that for every distinct $p, q \notin B$ one has the literal equality*

$$(2.21) \quad \Phi_{q|B \cup \{p\}} \Phi_{p|B} = \Phi_{p|B \cup \{q\}} \Phi_{q|B}.$$

Then the total composite is independent of the order of P . Adjacent equality replacements close around the involutive, distant, and three-term Coxeter-shaped diagrams. These diagrams retain no comparison arrow and define no permutation action. The same assertion holds in the opposite category.

Proof. Every two orders of P are related by adjacent transpositions. At an adjacent transposition, the preceding entries determine one subset B , and the affected two-factor block is exactly one instance of (2.21); all exterior factors have the same typed endpoints. Repeated substitution proves order independence.

Applying the same equality twice gives the involutive loop. Replacements in disjoint pairs of positions commute because they produce the same labelled factors with the same preceding subsets. For three consecutive labels, the two sequences of three adjacent replacements begin with the same expression and end with the same reversed expression; every intermediate edge is an instance of (2.21). Transitivity of literal equality closes the two routes. No new arrow has been constructed. Passing to the opposite category proves the final statement. $\square$

3. FIXED-SITE LOCAL PHASE CALCULUS AND WEIGHTED TABLE

The local calculation isolates the only place where the two phase choices can interact. Once the old surface and the unused sites are treated as spectators, the problem becomes a calculation in one marked one-holed torus, with the Walker weight kept alongside the supported mapping class.

Work in the attachment-state Boolean chart of Section 2. Fix $I \subseteq [k]$ and a label $i \notin I$, and abbreviate

$$X_I := \Sigma_I^{\text{out}}, \quad Y_{i,I} := \Sigma_{I \cup \{i\}}^{\text{out}}.$$

The old surface, the handles indexed by I , and every unused disk are spectators. We use the homology and Lagrangian splittings (2.5)–(2.6) and the Walker convention (2.1). Every local map below is supported in the i -th handle neighborhood with its complete marking and is the identity on all spectators. Products act from the right, and every displayed integer is a Walker weight rather than a power of a mapping class.

The elementary compression body and its literal reverse are

$$(3.1) \quad \mathcal{C}_{i|I}(c) : X_I \longrightarrow Y_{i,I},$$

$$(3.2) \quad \mathcal{R}_{i|I}^{\text{rev}}(r) : Y_{i,I} \longrightarrow X_I.$$

Here $c, r \in \mathbb{Z}$ are independently assigned weights. The superscript on $\mathcal{R}^{\text{rev}}$ records the literal reverse family; it does not say that $\mathcal{R}^{\text{rev}}(r)$ is the actual dagger of $\mathcal{C}(r)$.

Recall that T_γ denotes the right Dehn twist about γ . On the new one-holed torus fix the exact right-twist word

$$(3.3) \quad \varsigma_i = T_{m_i} T_{\ell_i} T_{m_i}, \quad (\varsigma_i)_* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\sigma_{i|I}$ be its extension by the identity over the old surface and all spectators, and put

$$(3.4) \quad \Theta_{i|I}(u) := C(\sigma_{i|I}, u) : Y_{i,I} \longrightarrow Y_{i,I}, \quad \mathcal{S}_{i|I} := \Theta_{i|I}(0).$$

Thus

$$(3.5) \quad (\sigma_{i|I})_* \langle m_i \rangle = \langle \ell_i \rangle, \quad (\sigma_{i|I})_*^{-1} \langle m_i \rangle = \langle \ell_i \rangle,$$

where signs of individual generators disappear only because these are Lagrangian lines.

Fix the phase and weight vectors

$$\begin{aligned} \epsilon, \eta &\in \{0, 1\}^k, \\ \mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^- &\in \mathbb{Z}^k. \end{aligned}$$

The weight vectors are independent. At the chosen label abbreviate

$$\begin{aligned} e &:= \epsilon_i, & v &:= \eta_i, & c &:= c_i, & r &:= r_i, \\ u^+ &:= u_i^+, & u^- &:= u_i^-. \end{aligned}$$

For these local data define

$$(3.6) \quad J_{i|I}^0(c) := \mathcal{C}_{i|I}(c),$$

$$(3.7) \quad J_{i|I}^1(c, u^+) := \Theta_{i|I}(u^+) \mathcal{C}_{i|I}(c),$$

$$(3.8) \quad R_{i|I}^0(r) := \mathcal{R}_{i|I}^{\text{rev}}(r),$$

$$(3.9) \quad R_{i|I}^1(r, u^-) := \mathcal{R}_{i|I}^{\text{rev}}(r) \Theta_{i|I}(u^-).$$

Every $J_{i|I}^e$ has type $X_I \rightarrow Y_{i,I}$, and every $R_{i|I}^v$ has type $Y_{i,I} \rightarrow X_I$. When $e = 0$, the coordinate u^+ is inactive and is ignored; when $v = 0$, the coordinate u^- is inactive and is ignored. The superscripts e, v are the only powers in this notation. Put

$$(3.10) \quad \begin{aligned} a_i &:= c + eu^+, & b'_i &:= r + vu^-, \\ \text{that is} \quad a_i &= c_i + \epsilon_i u_i^+, & b'_i &= r_i + \eta_i u_i^-. \end{aligned}$$

Lemma 3.1 (Exact elementary phase weights). *The actual Walker weights of $J_{i|I}^e(c, u^+)$ and $R_{i|I}^v(r, u^-)$ are a_i and b'_i , respectively.*

Proof. The weight is computed on the one active symplectic plane; all old and unused summands are spectators. Phase zero has no internal seam, while phase one transports the new meridian line through the marked half-turn. Direct-sum additivity then reduces the Maslov calculation to the displayed local correction.

There is no internal seam in a phase-zero arrow. For a phase-one forward arrow, direct-sum additivity gives

$$(3.11) \quad \begin{aligned} &\mu(\lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, (\sigma_{i|I})_*^{-1} \lambda_{I \cup \{i\}}) \\ &= \mu(\lambda_g, \lambda_g, \lambda_g) + \sum_{s \in I} \mu(\langle m_s \rangle, \langle m_s \rangle, \langle m_s \rangle) + \mu(\langle m_i \rangle, \langle m_i \rangle, \langle \ell_i \rangle) = 0. \end{aligned}$$

For a phase-one reverse arrow the seam is

$$(3.12) \quad \begin{aligned} &\mu((\sigma_{i|I})_* \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}) \\ &= \mu(\lambda_g, \lambda_g, \lambda_g) + \sum_{s \in I} \mu(\langle m_s \rangle, \langle m_s \rangle, \langle m_s \rangle) + \mu(\langle \ell_i \rangle, \langle m_i \rangle, \langle m_i \rangle) = 0. \end{aligned}$$

Every term vanishes by a repeated pair. Hence the constituent weights add with no correction, proving the two formulas. $\square$

3.1. The localized marked half-turn. Let $\delta_{i|I}$ be the chosen separating curve in the collar of the i -th attached one-holed torus. Let $d_{i|I}$ be the positive twist about this curve, extended by the identity on the old surface and all spectators, and write

$$(3.13) \quad V_{d_{i|I}} := C(d_{i|I}, 0).$$

Proposition 3.2 (Relative half-turn localization). *For every $I \subseteq [k]$ and $i \notin I$, the exact relative half-turn extends across the compression body, fixes the incoming boundary and every spectator region, and satisfies*

$$(H1) \quad \boxed{\mathcal{S}_{i|I}^2 \mathcal{C}_{i|I}(0) = \mathcal{C}_{i|I}(0)}.$$

The equality is literal in the marked Walker Hom set and has weight zero.

The local compression model and its reflected reverse. Build the compression body in a product neighborhood of the active site, attach the handle at the two marked feet, and use the explicit reflection to identify the reverse side. The co-core supplies the meridian, while the joining band supplies the framed longitude, so the boundary map and its marked germ are determined together.

Proof. Build the local compression body from $X_I \times [0, 1]$ by attaching

$$H_i = [-1, 1] \times D^2$$

at the two ordered feet belonging to d_i . Its co-core is $\{0\} \times D^2$, with boundary m_i ; the chosen joining band and the product push-off of the core determine the framed longitude ℓ_i . On H_i use

$$(t, z) \mapsto (-t, \bar{z}).$$

It is orientation preserving, exchanges the ordered feet, preserves the co-core setwise, and reverses both the core and its product-framed longitude. Indeed, reflection of the interval coordinate and complex conjugation of the disk coordinate each reverse orientation, so their product preserves the orientation of H_i . On the two attaching rectangles these coordinates agree with the prescribed exchange of the feet. Extend the map over the joining band by the corresponding half-turn. Inside the chosen separating collar, interpolate from this rotation to the identity while keeping the outer edge of the collar fixed, and take the identity on $X_I \times \{0\}$. The construction is therefore relative to the incoming boundary and to a full outer collar, rather than merely up to an unmarked isotopy. Because the active handle neighborhood is disjoint from every T_s , $s \in I$, and from every unused disk, the extension is pointwise the identity on all spectator regions.

Cutting the outgoing one-holed torus along $m_i \cup \ell_i$ gives the marked square determined by the feet and band. Reading its paired edges in the fixed positive-twist convention gives the exact relative word

$$(3.14) \quad (T_{m_i} T_{\ell_i})^3 = (T_{m_i} T_{\ell_i} T_{m_i})^2 = \sigma_{i|I}^2.$$

This is stronger than its action on homology. The homology matrix alone cannot detect a power of the separating twist. With the cut square and its boundary marking fixed, any alternative relative lift with the same homological action would have the form $T_{\delta_{i|I}}^q \sigma_{i|I}^2$. Track one oriented, product-framed arc crossing the separating collar once. Squaring the explicit half-turn rotates the normal framing of this arc through $+2\pi$, and hence produces exactly the positive twist $T_{\delta_{i|I}}$. The square of the alternative lift would instead be $T_{\delta_{i|I}}^{2q+1}$. The separating twist has infinite order in the relative marked mapping-class group [FM12], so comparison with the constructed square forces $q = 0$. Thus no boundary-twist power is hidden in the extension, and the outgoing restriction is the written relative word, not merely a representative with the same homology action.

It remains to check the Walker lift. First, the only correction in the product defining $\mathcal{S}_{i|I}^2$ is

$$(3.15) \quad \begin{aligned} & \mu((\sigma_{i|I})_* \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, (\sigma_{i|I})_*^{-1} \lambda_{I \cup \{i\}}) \\ &= \mu(\lambda_g, \lambda_g, \lambda_g) + \sum_{s \in I} \mu(\langle m_s \rangle, \langle m_s \rangle, \langle m_s \rangle) + \mu(\langle \ell_i \rangle, \langle m_i \rangle, \langle \ell_i \rangle) = 0. \end{aligned}$$

Thus $\mathcal{S}_{i|I}^2$ has weight zero. Its composition with the compression body has correction

$$(3.16) \quad \begin{aligned} & \mu(\lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, (\sigma_{i|I})_*^{-2} \lambda_{I \cup \{i\}}) \\ &= \mu(\lambda_g, \lambda_g, \lambda_g) + \sum_{s \in I} \mu(\langle m_s \rangle, \langle m_s \rangle, \langle m_s \rangle) + \mu(\langle m_i \rangle, \langle m_i \rangle, \langle m_i \rangle) = 0, \end{aligned}$$

because $\sigma_{i|I}^2$ is -1 on the active plane and the identity on every other summand. The ordinary relative equality therefore lifts with no central shift, proving (H1). $\square$

Proposition 3.3 (Fourth power and capping-kernel absorption). *The exact localized identities*

$$(H2) \quad \boxed{\mathcal{S}_{i|I}^4 = V_{d_{i|I}}}$$

and

$$(H3) \quad \boxed{\mathcal{R}_{i|I}^{\text{rev}}(0) V_{d_{i|I}} = \mathcal{R}_{i|I}^{\text{rev}}(0)}$$

hold as weight-zero marked Walker morphisms.

The two-chain relation and capping. The two-chain relation converts the chosen half-turn word into a boundary twist after taking its fourth power. The twist bounds the separating disk that caps the active one-holed torus, so the resulting mapping cylinder is absorbed by the declared capping kernel.

Proof. The braid relation for the two curves meeting once and the two-chain relation in the fixed one-holed-torus marking [FM12] give

$$(3.17) \quad \sigma_{i|I}^2 = (T_{m_i} T_{\ell_i})^3, \quad \sigma_{i|I}^4 = (T_{m_i} T_{\ell_i})^6 = T_{\delta_{i|I}}.$$

The first equality is the braid calculation for the three-twist word; the second is the relative two-chain relation. The final twist is positive: the separating curve is unoriented, so the opposite boundary orientations on its two sides do not invert the ambient right Dehn twist. Equation (3.15) already shows that the first square has weight zero. Squaring it has correction

$$(3.18) \quad \begin{aligned} & \mu((\sigma_{i|I})_*^2 \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, (\sigma_{i|I})_*^{-2} \lambda_{I \cup \{i\}}) \\ &= \mu(\lambda_g, \lambda_g, \lambda_g) + \sum_{s \in I} \mu(\langle m_s \rangle, \langle m_s \rangle, \langle m_s \rangle) + \mu(\langle m_i \rangle, \langle m_i \rangle, \langle m_i \rangle) = 0. \end{aligned}$$

Hence (3.17) lifts exactly to (H2).

For (H3), take the positive boundary-parallel twist τ_i in the relative k -disk surface, with weight zero in the common pulled-back Walker extension of Proposition 2.1. Its support is a small annulus around the i -th boundary circle. Capping at the i -th site in the high vertex turns this annular twist into the separating twist, whereas capping by the original disk in the low vertex makes it isotopic to the identity relative to the remaining marking. Thus

$$e_{I \cup \{i\}}(\tau_i) = d_{i|I}, \quad e_I(\tau_i) = \text{id}_{X_I}.$$

The twist extends through the literal reverse compression body: the separating curve bounds the retained compression disk there, and twisting a collar of that disk realizes the boundary twist in the three-manifold. After compression the same twist is supported in the capped disk and is trivial on the lower boundary. The extension is supported in the active handle region and fixes both endpoint markings elsewhere. The equality of the underlying relative cobordisms is therefore

$$\mathcal{R}_{i|I}^{\text{rev}}(0)C(d_{i|I}, 0) = C(\text{id}_{X_I}, 0)\mathcal{R}_{i|I}^{\text{rev}}(0).$$

The two Walker interfaces are, respectively,

$$\begin{aligned} & \mu((d_{i|I})_* \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}) = 0, \\ & \mu(\lambda_I, \lambda_I, \lambda_I) = 0. \end{aligned}$$

Here $d_{i|I}$ is separating and acts trivially on first homology. Thus the relative covariance equality has no central correction and is exactly (H3). This argument localizes the capping-kernel mechanism at the typed vertex; it does not import an arbitrary-genus half-turn identity. $\square$

Proposition 3.4 (Inverse and positive reverse absorption). *One has*

$$(H4) \quad \boxed{\mathcal{R}_{i|I}^{\text{rev}}(0)\mathcal{S}_{i|I}^{-2} = \mathcal{R}_{i|I}^{\text{rev}}(0)}$$

and

$$(H5) \quad \boxed{\mathcal{R}_{i|I}^{\text{rev}}(0)\mathcal{S}_{i|I}^2 = \mathcal{R}_{i|I}^{\text{rev}}(0)}.$$

The same identities hold after assigning an arbitrary weight r to the reverse handle.

Proof. Reverse the actual marked cobordism, rather than rewriting its homology row. Contravariance reverses the boundary order, and the interface Maslov term is zero because the transported active line agrees with the declared line at the seam. The same argument handles both phase choices.

Actual orientation reversal of (H1) reverses the order, inverts the outgoing cylinder, and negates the zero weight. It gives (H4). Its interface correction is

$$\mu((\sigma_{i|I})_*^{-2} \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}) = 0$$

by the same old/spectator/active direct-sum splitting as (3.16).

Now use the fourth power rather than identifying the half-turn square with the identity:

$$\begin{aligned} \mathcal{R}_{i|I}^{\text{rev}}(0) \mathcal{S}_{i|I}^2 &= \mathcal{R}_{i|I}^{\text{rev}}(0) \mathcal{S}_{i|I}^{-2} \mathcal{S}_{i|I}^4 \\ &= \mathcal{R}_{i|I}^{\text{rev}}(0) V_{d_{i|I}} \\ &= \mathcal{R}_{i|I}^{\text{rev}}(0). \end{aligned} \tag{3.19}$$

All even powers preserve the declared Lagrangian, so the displayed reassociation introduces no Maslov correction. The last equality is (H3). A central weight shift on the reverse handle passes unchanged through the same zero-seam calculation, proving the assertion for arbitrary r . Neither equation asserts $\mathcal{S}_{i|I}^2 = 1$; indeed $\mathcal{S}_{i|I}^4 = V_{d_{i|I}}$ retains the nontrivial separating twist. $\square$

3.2. Actual elementary source dagger. Actual source orientation reversal keeps the fixed endpoint Lagrangians fixed. It reverses the arrow and negates its Walker weight. Thus, for the exact slope cylinder,

$$\Theta_{i|I}(u)^\dagger = C(\sigma_{i|I}^{-1}, -u). \tag{H6}$$

In particular, the right side is not $\Theta_{i|I}(-u)$. The two mapping classes have the same image of the meridian as an unoriented line, but they are distinct relative marked mapping classes. The preferred inverse cylinder is nevertheless

$$\mathcal{S}_{i|I}^{-1} = C(\sigma_{i|I}^{-1}, 0),$$

because its product with $\mathcal{S}_{i|I}$ has active correction $\mu(\langle \ell_i \rangle, \langle m_i \rangle, \langle \ell_i \rangle) = 0$, while every spectator contribution has a repeated pair.

Proposition 3.5 (Elementary phase-arrow dagger). *For $e, v \in \{0, 1\}$, with inactive slope coordinates ignored,*

$$(J_{i|I}^e(c, u^+))^\dagger = R_{i|I}^e(-c, -u^+) \tag{H7}$$

$$(R_{i|I}^v(r, u^-))^\dagger = J_{i|I}^v(-r, -u^-). \tag{H8}$$

The weights on the two sides are, respectively, $-a_i$ and $-b'_i$.

Source reversal and the dagger identities. Source reversal is applied only after the elementary compression and reverse families have been constructed. The local dagger identities follow from orientation reversal of the marked product model, with weight negation and the written factor order retained.

Proof. For phase zero, actual orientation reversal of the weighted compression body gives

$$\mathcal{C}_{i|I}(c)^\dagger = \mathcal{R}_{i|I}^{\text{rev}}(-c), \quad (\mathcal{R}_{i|I}^{\text{rev}}(r))^\dagger = \mathcal{C}_{i|I}(-r).$$

For phase one, contravariance and (H6) first give the literal formulas

$$(\Theta_{i|I}(u^+) \mathcal{C}_{i|I}(c))^\dagger = \mathcal{R}_{i|I}^{\text{rev}}(-c) C(\sigma_{i|I}^{-1}, -u^+), \tag{3.20}$$

$$(\mathcal{R}_{i|I}^{\text{rev}}(r) \Theta_{i|I}(u^-))^\dagger = C(\sigma_{i|I}^{-1}, -u^-) \mathcal{C}_{i|I}(-r). \tag{3.21}$$

The proved absorptions, not an equality of slope generators, now give

$$\mathcal{R}_{i|I}^{\text{rev}}(0) \mathcal{S}_{i|I}^{-1} = \mathcal{R}_{i|I}^{\text{rev}}(0) \mathcal{S}_{i|I}, \tag{3.22}$$

$$\mathcal{S}_{i|I}^{-1} \mathcal{C}_{i|I}(0) = \mathcal{S}_{i|I} \mathcal{C}_{i|I}(0). \tag{3.23}$$

Indeed, the first follows by multiplying $\mathcal{R}^{\text{rev}} \mathcal{S}^2 = \mathcal{R}^{\text{rev}}$ on the right by $\mathcal{S}^{-1}$, and the second follows by applying $\mathcal{S}^{-1}$ on the left to $\mathcal{S}^2 \mathcal{C} = \mathcal{C}$.

Both signs of the slope have zero handle seam. Explicitly, the active summand in the first equality is

$$\mu(\langle \ell_i \rangle, \langle m_i \rangle, \langle m_i \rangle) = 0,$$

because both $\sigma_{i|I}$ and $\sigma_{i|I}^{-1}$ carry the meridian line to the longitude line. For the second equality it is

$$\mu(\langle m_i \rangle, \langle m_i \rangle, \langle \ell_i \rangle) = 0.$$

The old and spectator summands contain three equal entries. Consequently central shifts may be restored in (3.22) and (3.23) without changing either weight. Equations (3.20) and (3.21) therefore become the phase-one instances of (H7) and (H8). Lemma 3.1 shows that the common weights are $-c - u^+ = -a_i$ and $-r - u^- = -b'_i$. $\square$

3.3. The complete weighted same-site table. Define the preferred local double in its actual endpoint Hom set by

$$(L0) \quad \mathcal{D}_{i|I}(0) := R_{i|I}^0(0) J_{i|I}^0(0) = \mathcal{R}_{i|I}^{\text{rev}}(0) \mathcal{C}_{i|I}(0) \in \text{End}(X_I).$$

For $n \in \mathbb{Z}$, $\mathcal{D}_{i|I}(n)$ denotes the same marked cobordism with total Walker weight n . This is a central weight shift inside $\text{End}(X_I)$; it is not an operator that has been identified across different labels or subset vertices.

For the fixed local phases e, v , put

$$(L1) \quad n_i := a_i + b'_i = c + r + eu^+ + vu^-.$$

Theorem 3.6 (Weighted local phase table). *For every $I \subseteq [k]$, $i \notin I$, every $e, v \in \{0, 1\}$, and every active constituent integer weight,*

$$(L2) \quad R_{i|I}^v(r, u^-) J_{i|I}^e(c, u^+) = \begin{cases} \mathcal{D}_{i|I}(n_i), & e = v, \\ C(\text{id}_{X_I}, n_i), & e \neq v. \end{cases}$$

At preferred weight zero this is

$$(L3) \quad \begin{array}{c|cc} & e = 0 & e = 1 \\ \hline v = 0 & \mathcal{D} & 1 \\ v = 1 & 1 & \mathcal{D} \end{array}.$$

Here the table entries abbreviate $\mathcal{D} = \mathcal{D}_{i|I}(0)$ and $1 = C(\text{id}_{X_I}, 0)$.

Matching phases in the common seam. Transport the low-end and high-end Lagrangian lines into one common seam calculation. Matching phases cancel the active slope change and leave the local double; mismatching phases leave the identity on the spectator surface. The topology of the matching entry is then determined by the same marked one-holed-torus model.

Proof. Seam and weight. We first calculate the gluing seam independently of the topology. The line transported to the high endpoint by the forward phase arrow is

$$L_J^e = \begin{cases} \langle m_i \rangle, & e = 0, \\ \langle \ell_i \rangle, & e = 1, \end{cases}$$

and the line inverse-transported from the low endpoint by the reverse phase arrow is

$$L_R^v = \begin{cases} \langle m_i \rangle, & v = 0, \\ \langle \ell_i \rangle, & v = 1. \end{cases}$$

These statements use the exact cylinder $\sigma_{i|I}$; they are not definitions made from the desired table. On the old surface and on each $s \in I$, the corresponding three entries are $\lambda_g, \lambda_g, \lambda_g$ and

$\langle m_s \rangle, \langle m_s \rangle, \langle m_s \rangle$. The four active-plane seams are therefore

$$(L4) \quad \begin{array}{c|cc} & e=0 & e=1 \\ \hline v=0 & \mu(\langle m_i \rangle, \langle m_i \rangle, \langle m_i \rangle) & \mu(\langle \ell_i \rangle, \langle m_i \rangle, \langle m_i \rangle) \\ v=1 & \mu(\langle m_i \rangle, \langle m_i \rangle, \langle \ell_i \rangle) & \mu(\langle \ell_i \rangle, \langle m_i \rangle, \langle \ell_i \rangle) \end{array} = 0.$$

In particular, the $(e, v) = (1, 1)$ seam is the two-slope seam

$$\mu(\langle \ell_i \rangle, \langle m_i \rangle, \langle \ell_i \rangle) = 0;$$

it is not omitted or replaced by a parity slogan. Together with (3.11) and (3.12), this proves that every constituent weight contributes exactly once and that the total is n_i .

Matching phases. It remains to identify the four underlying marked cobordisms. At preferred weight zero their right-factor-first words are

$$(3.24) \quad \begin{array}{c|cc} & e=0 & e=1 \\ \hline v=0 & \mathcal{R}^{\text{rev}}\mathcal{C} & \mathcal{R}^{\text{rev}}\mathcal{SC} \\ v=1 & \mathcal{R}^{\text{rev}}\mathcal{SC} & \mathcal{R}^{\text{rev}}\mathcal{S}^2\mathcal{C}, \end{array}$$

where every symbol in this display carries the fixed tag $i \mid I$. The first word is $\mathcal{D}_{i \mid I}(0)$. The last word is the same double, because either $\mathcal{S}^2\mathcal{C} = \mathcal{C}$ by (H1), or $\mathcal{R}^{\text{rev}}\mathcal{S}^2 = \mathcal{R}^{\text{rev}}$ by (H5). Thus the $(1, 1)$ entry is $\mathcal{D}$, not the identity.

Mismatching phases. For either mismatch, the common middle word is $\mathcal{R}^{\text{rev}}\mathcal{SC}$. We now identify it with the actual product relative to both endpoint markings. Before the slope is inserted, the return compression disk has boundary m_i . Pulling that boundary through the exact word $T_{m_i}T_{\ell_i}T_{m_i}$ gives the actual product-framed longitude, not only its homology class, and this curve meets the belt meridian once. Take a small neighborhood of the ordered feet, joining band, one-handle, and return compression disk. In these coordinates the return attaching curve crosses the belt circle exactly once and its normal framing is the product framing. The standard local cancellation deletes the one-handle and the return two-handle inside this neighborhood while leaving the complement and both endpoint collars fixed.

Excluding a boundary twist. This is an exact marked cancellation, not a conclusion from the intersection number alone: the attaching feet, band, co-core, actual curves, and product normal have all been retained. To see that no relative boundary twist remains, cut the local one-holed torus along $m_i \cup \ell_i$. The written three-twist word carries the marked meridian edge to the marked longitude edge with the collar framing fixed by the same crossing arc used in the proof of Proposition 3.2. A word differing by $T_{\delta_{i \mid I}}^q$ would turn that framed arc through an additional $2\pi q$, whereas the explicit word has $q = 0$. The cancellation is therefore the product $X_I \times [0, 1]$ with its fixed endpoint parameterizations. It follows that

$$\mathcal{R}_{i \mid I}^{\text{rev}}(0)\mathcal{S}_{i \mid I}\mathcal{C}_{i \mid I}(0) = C(\text{id}_{X_I}, 0).$$

Restoring the weight. Both mismatch entries use this same actual word; only the label-attached phase bookkeeping says whether the slope belongs to the forward or reverse constituent. Restoring the already calculated weight n_i proves all four cases of (L2), and (L3) is its zero-weight specialization. $\square$

Proposition 3.7 (Bare local topology and first-homology separation). *For every $n \in \mathbb{Z}$,*

$$(L5) \quad \boxed{\mathcal{D}_{i \mid I}(n) \cong C(\text{id}_{X_I}, n) \# (S^1 \times S^2, 0).}$$

Consequently,

$$(L6) \quad \boxed{\mathcal{D}_{i \mid I}(n) \neq C(\text{id}_{X_I}, m) \quad \text{for all } m, n \in \mathbb{Z}.}$$

The separating-sphere identification of the double. Close a regular neighborhood of the active forward–reverse pair by its separating collar. The exterior remains a product, while the interior is the genus-one meridian-to-meridian double, which is $S^1 \times S^2$. This separates the local topological contribution from the boundary marking and Walker-weight bookkeeping.

Proof. The enclosed local piece. Use the chosen separating collar to close the boundary of a regular neighborhood of the active forward–reverse handle pair. The resulting two-sphere encloses only the i -th handle region because the exterior product structure is fixed throughout the composite. Outside that sphere the bare double $\mathcal{R}_{i|I}^{\text{rev}}\mathcal{C}_{i|I}$ is the product on X_I . Inside it, the forward and return compression disks have the same marked meridian m_i . Equivalently, the two punctured solid tori are glued meridian to meridian. This is the genus-one Heegaard description of $S^1 \times S^2$, rather than the meridian-to-longitude gluing that would give the cancelling local product. The incoming and outgoing local interfaces account for the two removed balls, so the enclosed marked piece is $S^1 \times S^2$ with two open balls removed. Gluing it back to the exterior product is precisely an interior connected sum and gives the ordinary relative cobordism decomposition in (L5).

Walker weight. Its Walker seam is

$$\mu(\langle m_i \rangle, \langle m_i \rangle, \langle m_i \rangle) = 0,$$

and the old and spectator terms also have three equal entries. The separating sphere has zero first homology and creates no further Maslov term. Hence the complete weight n may be placed on the product cylinder and the closed summand may be recorded at weight zero, proving the extended statement (L5).

Homological separation. Let $g_I = g + |I|$ be the genus of X_I . The underlying identity cylinder has

$$\beta_1(X_I \times [0, 1]) = 2g_I,$$

whereas connected sum in the interior with $S^1 \times S^2$ gives

$$\beta_1((X_I \times [0, 1]) \# (S^1 \times S^2)) = 2g_I + 1.$$

Walker weight does not change the underlying three-manifold. Therefore no choice of m, n can make the two marked extended cobordisms equal, which proves (L6). $\square$

Remark 3.8 (Scope of the local calculation). Every equality in this section was proved in a fixed marked Walker Hom set from the handle model, the exact slope word, relative covariance, and explicit Maslov calculations. No target matrix, target faithfulness, finite-dimensional value, or physical TQFT was used. The argument proves no interchange between different sites, no order independence, and no global connected-sum theorem; those questions require the disjoint-support calculus of the next section. It makes no statement for repeated, overlapping, nested, moved, or relabelled sites, and uses no refined genus-at-least-three carrier. It also never identifies $\mathcal{S}_{i|I}^2$ with the identity or $\sigma_{i|I}^{-1}$ with $\sigma_{i|I}$.

4. FIXED-SITE INTERCHANGE, ATTACHMENT ORDERS, AND COMPLETED TRACES

The preceding local table answers one site at a time; this section explains why those local answers can be assembled in any order. The key issue is not the commutativity of disjoint supports alone, but the preservation of subset tags, endpoint markings, and Walker weights while the history is reordered.

First establish the local interchange identities for bare, phased, forward, and reverse arrows while retaining every site tag and weight. Next use these adjacent swaps to compare tagged order words inside their fixed Hom sets; the result is an equality-only closure, not a new global groupoid assertion. Then compute the homological relations and Walker seams for the ordered histories, including both internal and inter-factor phase corrections. Finally replace those ordered histories by the independently constructed simultaneous trace, which is the geometric input used by the later match-count argument.

Throughout this section composition is right-factor first. We use the localized arrows proved in the preceding section, with the types

$$J_{i|I}^e : \Sigma_I^{\text{out}} \longrightarrow \Sigma_{I \cup \{i\}}^{\text{out}}, \quad R_{i|I}^v : \Sigma_{I \cup \{i\}}^{\text{out}} \longrightarrow \Sigma_I^{\text{out}}.$$

Their actual Walker weights are

$$a_i = c_i + eu_i^+, \quad b'_i = r_i + vu_i^-,$$

where an inactive slope coordinate is absent, not a zero-weight additional arrow. We also use only the already proved elementary handle-arrow dagger identities

$$(J_{i|I}^e(c_i, u_i^+))^\dagger = R_{i|I}^e(-c_i, -u_i^+), \quad (R_{i|I}^v(r_i, u_i^-))^\dagger = J_{i|I}^v(-r_i, -u_i^-),$$

and the already proved weighted same-site table. In particular, no global dagger formula and no order-independent trace is an input below.

For a bit $q \in \{0, 1\}$, put

$$(4.1) \quad L_s^q := \begin{cases} \langle m_s \rangle, & q = 0, \\ \langle \ell_s \rangle, & q = 1. \end{cases}$$

The sign with which the half-turn sends m_s to ℓ_s is irrelevant for this one-dimensional subspace. This notation does not identify the marked mapping classes σ_s and σ_s^{-1} .

Proposition 4.1 (Disjoint-support interchange). *Let $I \subseteq [k]$ and let $i, j \notin I$ be distinct. For arbitrary independent integers r_i, r_j, u_i, u_j , the following are literal equalities of marked Walker extended cobordisms. The bare reverse handles satisfy*

$$(4.2) \quad \begin{aligned} & \mathcal{R}_{i|I}^{\text{rev}}(r_i) \mathcal{R}_{j|I \cup \{i\}}^{\text{rev}}(r_j) \\ &= \mathcal{R}_{j|I}^{\text{rev}}(r_j) \mathcal{R}_{i|I \cup \{j\}}^{\text{rev}}(r_i) : \Sigma_{I \cup \{i, j\}}^{\text{out}} \longrightarrow \Sigma_I^{\text{out}}, \end{aligned}$$

with common weight $r_i + r_j$. On the common high surface, the slope cylinders satisfy

$$(4.3) \quad \begin{aligned} & \Theta_{i|I \cup \{j\}}(u_i) \Theta_{j|I \cup \{i\}}(u_j) \\ &= \Theta_{j|I \cup \{i\}}(u_j) \Theta_{i|I \cup \{j\}}(u_i) \in \text{End}(\Sigma_{I \cup \{i, j\}}^{\text{out}}), \end{aligned}$$

with common weight $u_i + u_j$. A slope cylinder also passes through the reverse compression at the other site:

$$(4.4) \quad \begin{aligned} & \Theta_{i|I}(u_i) \mathcal{R}_{j|I \cup \{i\}}^{\text{rev}}(r_j) \\ &= \mathcal{R}_{j|I \cup \{i\}}^{\text{rev}}(r_j) \Theta_{i|I \cup \{j\}}(u_i) : \Sigma_{I \cup \{i, j\}}^{\text{out}} \longrightarrow \Sigma_{I \cup \{i\}}^{\text{out}}, \end{aligned}$$

with common weight $u_i + r_j$.

For a generic attachment state $I' \subseteq [k] \setminus \{s\}$, define the slope-corrected reverse at a site by

$$\mathcal{R}_{s|I'}^\sigma(r_s, u_s) := \mathcal{R}_{s|I'}^{\text{rev}}(r_s) \Theta_{s|I'}(u_s).$$

Then

$$(4.5) \quad \begin{aligned} & \mathcal{R}_{i|I}^\sigma(r_i, u_i) \mathcal{R}_{j|I \cup \{i\}}^\sigma(r_j, u_j) \\ &= \mathcal{R}_{j|I}^\sigma(r_j, u_j) \mathcal{R}_{i|I \cup \{j\}}^\sigma(r_i, u_i) : \Sigma_{I \cup \{i, j\}}^{\text{out}} \longrightarrow \Sigma_I^{\text{out}}, \end{aligned}$$

with common weight $r_i + r_j + u_i + u_j$. Every equality fixes the old surface, every attached spectator, and every unused disk, and it preserves the label attached to each weight.

Disjoint supports and the Walker seam. Because the two active neighborhoods are disjoint, the geometric maps commute on the local pieces. The remaining issue is the Walker seam: old, attached, and unused spectators contribute repeated Lagrangian lines, so only the two active planes require a direct check.

Proof. All active maps are supported in disjoint neighborhoods of the i - and j -sites. On an already attached spectator plane, the incoming, declared, transported, and outgoing lines are all $\langle m_s \rangle$, so direct-sum additivity contributes only

$$\mu(\langle m_s \rangle, \langle m_s \rangle, \langle m_s \rangle) = 0.$$

An unused disk contributes no symplectic plane, and each active map extends by the identity near its boundary circle. It is therefore enough to check the two active sites, while retaining the fixed endpoint parameterizations.

The two bare reverse traces remove handles in disjoint labelled regions. Exchanging their critical heights gives a diffeomorphism relative to both boundary parameterizations and preserves each handle label. In the two orders the Walker interface triples are, respectively,

$$\mu(\lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}) = 0, \quad \mu(\lambda_{I \cup \{j\}}, \lambda_{I \cup \{j\}}, \lambda_{I \cup \{j\}}) = 0.$$

Thus the underlying marked equality lifts with weight $r_i + r_j$, proving (4.2).

The two slope mapping classes have disjoint supports and commute. For the order on the left of (4.3), the Walker correction splits as

$$\begin{aligned} \mu(\lambda_g, \lambda_g, \lambda_g) + \sum_{s \in I} \mu(\langle m_s \rangle, \langle m_s \rangle, \langle m_s \rangle) \\ + \mu(\langle m_i \rangle, \langle m_i \rangle, \langle \ell_i \rangle) + \mu(\langle \ell_j \rangle, \langle m_j \rangle, \langle m_j \rangle) = 0. \end{aligned}$$

For the opposite order, the last two terms are

$$\mu(\langle \ell_i \rangle, \langle m_i \rangle, \langle m_i \rangle) + \mu(\langle m_j \rangle, \langle m_j \rangle, \langle \ell_j \rangle) = 0.$$

Hence both products retain weight $u_i + u_j$, proving (4.3).

Because $i \neq j$, the mapping class supported at i extends by the identity across the reverse compression that removes the j -handle. This gives the underlying relative-boundary equality in (4.4). Its two Walker corrections are

$$\begin{aligned} \mu((\sigma_{i|I \cup \{j\}})_* \lambda_{I \cup \{i,j\}}, \lambda_{I \cup \{i,j\}}, \lambda_{I \cup \{i,j\}}) = 0, \\ \mu(\lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, (\sigma_{i|I})_*^{-1} \lambda_{I \cup \{i\}}) = 0. \end{aligned}$$

Each has a repeated pair, so both sides have weight $u_i + r_j$.

Finally, expand both corrected reverses. The preceding three identities give the typed calculation

$$\begin{aligned} \mathcal{R}_{i|I}^{\text{rev}}(r_i) \Theta_{i|I}(u_i) \mathcal{R}_{j|I \cup \{i\}}^{\text{rev}}(r_j) \Theta_{j|I \cup \{i\}}(u_j) \\ = \mathcal{R}_{i|I}^{\text{rev}}(r_i) \mathcal{R}_{j|I \cup \{i\}}^{\text{rev}}(r_j) \Theta_{i|I \cup \{j\}}(u_i) \Theta_{j|I \cup \{i\}}(u_j) \\ = \mathcal{R}_{j|I}^{\text{rev}}(r_j) \mathcal{R}_{i|I \cup \{j\}}^{\text{rev}}(r_i) \Theta_{j|I \cup \{i\}}(u_j) \Theta_{i|I \cup \{j\}}(u_i) \\ = \mathcal{R}_{j|I}^{\text{rev}}(r_j) \Theta_{j|I}(u_j) \mathcal{R}_{i|I \cup \{j\}}^{\text{rev}}(r_i) \Theta_{i|I \cup \{j\}}(u_i). \end{aligned}$$

The first and last steps use (4.4); the middle step uses (4.2) and (4.3). Every interface correction has already been computed, so the common weight is $r_i + r_j + u_i + u_j$. This proves (4.5). $\square$

4.1. Heterogeneous adjacent swaps.

Theorem 4.2 (Fixed-site heterogeneous phase interchange). (R) *Let $I \subseteq [k]$ and let $i, j \notin I$ be distinct. For every $(\eta_i, \eta_j) \in \{0, 1\}^2$, one has the typed equality*

$$(4.6) \quad \begin{aligned} R_{i|I}^{\eta_i} R_{j|I \cup \{i\}}^{\eta_j} \\ = R_{j|I}^{\eta_j} R_{i|I \cup \{j\}}^{\eta_i} : \Sigma_{I \cup \{i,j\}}^{\text{out}} \longrightarrow \Sigma_I^{\text{out}}. \end{aligned}$$

The common Walker weight is

$$(4.7) \quad b'_i + b'_j = r_i + r_j + \eta_i u_i^- + \eta_j u_j^-.$$

Here and below a phase-zero factor has only its handle-weight argument, whereas a phase-one factor has its handle and active slope-weight arguments.

(F) Under the same hypotheses, for every $(\epsilon_i, \epsilon_j) \in \{0, 1\}^2$,

$$(4.8) \quad \begin{aligned} J_{j|I \cup \{i\}}^{\epsilon_j} J_{i|I}^{\epsilon_i} \\ = J_{i|I \cup \{j\}}^{\epsilon_i} J_{j|I}^{\epsilon_j} : \Sigma_I^{\text{out}} \longrightarrow \Sigma_{I \cup \{i,j\}}^{\text{out}}. \end{aligned}$$

The common Walker weight is

$$(4.9) \quad a_i + a_j = c_i + c_j + \epsilon_i u_i^+ + \epsilon_j u_j^+.$$

The phase-dependent argument convention is the one stated after (4.6); in particular no inactive u_s^+ is passed to a phase-zero arrow.

Four local phase cases and source reversal. Prove the reverse interchange in the four phase cases, keeping the subset tags visible. The mixed cases move a slope cylinder only across the different-site reverse handle; source reversal then supplies the forward interchange with the correct factor order and weight.

Proof of the reverse relations. We prove the four cases before using any order theorem.

If $(\eta_i, \eta_j) = (0, 0)$, equation (4.6) is (4.2), with its full subset tags and common weight $r_i + r_j$.

If $(\eta_i, \eta_j) = (1, 1)$, expand both phase-one arrows as $R^1 = \mathcal{R}^{\text{rev}}\Theta$. Equation (4.5), with $u_s = u_s^-$, gives the required equality and common weight $r_i + r_j + u_i^- + u_j^-$. Thus this case uses the exact slope words and not an assertion that a half-turn squares to the identity.

For $(\eta_i, \eta_j) = (0, 1)$, all subset tags are visible in the following calculation:

$$\begin{aligned} & \mathcal{R}_{i|I}^{\text{rev}}(r_i) \mathcal{R}_{j|I \cup \{i\}}^{\text{rev}}(r_j) \Theta_{j|I \cup \{i\}}(u_j^-) \\ &= \mathcal{R}_{j|I}^{\text{rev}}(r_j) \mathcal{R}_{i|I \cup \{j\}}^{\text{rev}}(r_i) \Theta_{j|I \cup \{i\}}(u_j^-) && \text{by (4.2)} \\ &= \mathcal{R}_{j|I}^{\text{rev}}(r_j) \Theta_{j|I}(u_j^-) \mathcal{R}_{i|I \cup \{j\}}^{\text{rev}}(r_i) && \text{by (4.4), read in reverse} \\ &= R_{j|I}^1(r_j, u_j^-) R_{i|I \cup \{j\}}^0(r_i). \end{aligned}$$

The second equality slides the slope supported at j only past the different-site reverse supported at i . Every line has type $\Sigma_{I \cup \{i, j\}}^{\text{out}} \rightarrow \Sigma_I^{\text{out}}$ and exact weight $r_i + r_j + u_j^-$.

For $(\eta_i, \eta_j) = (1, 0)$, the other mixed calculation is

$$\begin{aligned} & \mathcal{R}_{i|I}^{\text{rev}}(r_i) \Theta_{i|I}(u_i^-) \mathcal{R}_{j|I \cup \{i\}}^{\text{rev}}(r_j) \\ &= \mathcal{R}_{i|I}^{\text{rev}}(r_i) \mathcal{R}_{j|I \cup \{i\}}^{\text{rev}}(r_j) \Theta_{i|I \cup \{j\}}(u_i^-) && \text{by (4.4)} \\ &= \mathcal{R}_{j|I}^{\text{rev}}(r_j) \mathcal{R}_{i|I \cup \{j\}}^{\text{rev}}(r_i) \Theta_{i|I \cup \{j\}}(u_i^-) && \text{by (4.2)} \\ &= R_{j|I}^0(r_j) R_{i|I \cup \{j\}}^1(r_i, u_i^-). \end{aligned}$$

Here the first equality slides the i -slope only past the different-site j -reverse. Every line again has the required endpoints, now with exact weight $r_i + r_j + u_i^-$. These four calculations prove both (4.6) and (4.7) without a same-site cancellation, a left-inverse relation, or an order-independence premise. Indeed, before any surrounding history is added, the outer reverse seam on the two sides is respectively

$$\mu(\lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, \lambda_I \oplus L_i^{\eta_i}) = 0, \quad \mu(\lambda_{I \cup \{j\}}, \lambda_{I \cup \{j\}}, \lambda_I \oplus L_j^{\eta_j}) = 0.$$

The phase-one internal seams were proved in the localized phase calculation. Thus the displayed sums are already the actual weights; the proposition below computes the additional mixed spectator lines created by a surrounding history. $\square$

Proof of the forward relations. The forward relation is not proved by introducing a new local calculation. Apply source reversal to the already established reverse relation, negate the constituent weights, and use the elementary dagger identity at each tagged site. This preserves the distinction between a slope reversal and a coordinate rewrite.

Apply the already proved reverse relation (4.6) with $\eta_s = \epsilon_s$, $r_s = -c_s$, and, when the phase is active, $u_s^- = -u_s^+$, for $s = i, j$. This is legitimate because the reverse relations have already been proved in all four cases. Actual source orientation reversal reverses the product order and negates its Walker weight. The already proved elementary identity for either $s \in \{i, j\}$ and its relevant subset tag I' is

$$(R_{s|I'}^{\epsilon_s})^\dagger = J_{s|I'}^{\epsilon_s}$$

Assign the handle weight $-c_s$ and the active slope weight $-u_s^+$; after dagger these weights are c_s, u_s^+ . With these weights, the elementary identity turns the dagger of the left side of (4.6) into the left side of (4.8), and similarly on the right. The common reverse weight $-a_i - a_j$ becomes $a_i + a_j$. No generator-level replacement of $\Theta(u)^\dagger$ by $\Theta(-u)$, and no global dagger theorem, occurs in this argument. $\square$

Proposition 4.3 (Mixed spectator Maslov terms). *Every Walker interface used in the reverse and forward parts of Theorem 4.2, including an occurrence inside a longer phase history, has total Maslov correction zero. This remains true with the entire old surface, every attached site in I , and every unused disk retained.*

Spectator splitting and Maslov cancellation. Decompose every intermediate interface into old, active, spectator, and unused summands. Repeated pairs vanish in the old and spectator summands; the active triples are exactly the local phase triples already calculated. This reduces every surrounding-history correction to the local zero calculation.

Proof. All maps in an active swap are supported in the i - and j -regions. By the chosen splitting of first homology, an interface decomposes into the old-surface summand, the active i, j planes that are present at that interface, and one plane for every attached spectator. The old-surface term is

$$\mu(\lambda_g, \lambda_g, \lambda_g) = 0.$$

An unused disk has not acquired a torus plane and contributes no symplectic summand.

We now calculate rather than infer the attached-spectator terms. A phase history supported at a spectator $s \in I$ can put either L_s^0 or L_s^1 on each transported side, while the declared middle line is always $\langle m_s \rangle$: a phase-zero local word fixes the meridian line, a phase-one local word sends it to the longitude line, and every different-site factor is the identity on this plane. Thus the complete list is

(q, q')	spectator triple
$(0, 0)$	$(\langle m_s \rangle, \langle m_s \rangle, \langle m_s \rangle)$
$(0, 1)$	$(\langle m_s \rangle, \langle m_s \rangle, \langle \ell_s \rangle)$
$(1, 0)$	$(\langle \ell_s \rangle, \langle m_s \rangle, \langle m_s \rangle)$
$(1, 1)$	$(\langle \ell_s \rangle, \langle m_s \rangle, \langle \ell_s \rangle)$

Every row has a repeated pair, so every spectator term vanishes. In particular, this explicitly includes

$$\mu(\langle m_s \rangle, \langle m_s \rangle, \langle \ell_s \rangle), \quad \mu(\langle \ell_s \rangle, \langle m_s \rangle, \langle m_s \rangle), \quad \mu(\langle \ell_s \rangle, \langle m_s \rangle, \langle \ell_s \rangle).$$

For completeness, the active-plane seams in the displayed reverse words are also explicit. A bare reverse gives (m, m, m) . At a slope-different-site-reverse slide the two sides give (ℓ, m, m) and (m, m, ℓ) , respectively. Expanding the corrected interchange introduces only these triples and the two-slope possibility (ℓ, m, ℓ) . The forward seams obtained by orientation reversal have the same list with first and third entries exchanged. Hence each active term also has a repeated pair. In particular, the mixed 01 and 10 proofs above use only the zero-correction interfaces (4.4) and (4.2), while the 11 proof uses the zero-correction expansion of (4.5).

Direct-sum additivity now gives, at every interface,

$$(4.11) \quad \mu_{\text{total}} = \mu_{\text{old}} + \mu_i + \mu_j + \sum_{s \in I} \mu_s = 0,$$

with a missing active plane simply omitted. Thus the weights used in (4.7) and (4.9) are the actual Walker weights on the complete chart, not merely the empty-spectator weights. $\square$

4.2. Order words and equality-only Coxeter closure. To pass from one adjacent interchange to arbitrary attachment histories, we need order words that keep each phase bit and constituent weight attached to its disk label. The following notation also makes the fixed source and target of the two directions explicit.

Definition 4.4 (Tagged phased order words). For $\alpha = (\alpha_1, \dots, \alpha_k) \in S_k$, set

$$(4.12) \quad I_a^\alpha := \{\alpha_1, \dots, \alpha_a\}, \quad I_0^\alpha := \emptyset.$$

Define

$$(4.13) \quad J_\alpha^\epsilon(\mathbf{c}, \mathbf{u}^+) := J_{\alpha_k|I_{k-1}^\alpha}^{\epsilon_{\alpha_k}} \cdots J_{\alpha_1|\emptyset}^{\epsilon_{\alpha_1}} : \Sigma_{\emptyset}^{\text{out}} \longrightarrow \Sigma_{[k]}^{\text{out}},$$

$$(4.14) \quad R_\alpha^\eta(\mathbf{r}, \mathbf{u}^-) := R_{\alpha_1|\emptyset}^{\eta_{\alpha_1}} \cdots R_{\alpha_k|I_{k-1}^\alpha}^{\eta_{\alpha_k}} : \Sigma_{[k]}^{\text{out}} \longrightarrow \Sigma_{\emptyset}^{\text{out}}.$$

Every phase bit and every constituent weight in these expressions remains attached to its disk label. The tag α on the reverse word is its matching attachment word; chronologically, the right-factor-first reverse word removes $\alpha_k, \dots, \alpha_1$.

Corollary 4.5 (Phased order independence). *For each fixed phase vector and all label-attached weights, the expressions in Definition 4.4 have literal common values*

$$(4.15) \quad J_\alpha^\epsilon(\mathbf{c}, \mathbf{u}^+) =: J_{[k]}^\epsilon(\mathbf{c}, \mathbf{u}^+),$$

$$(4.16) \quad R_\alpha^\eta(\mathbf{r}, \mathbf{u}^-) =: R_{[k]}^\eta(\mathbf{r}, \mathbf{u}^-),$$

in their two fixed-endpoint Hom sets, for every $\alpha \in S_k$.

Proof. Any two order words are joined by finitely many adjacent transpositions. Suppose positions $a, a+1$ contain i, j , and let I be the set of the labels in the preceding positions. The affected block of the forward word is exactly

$$J_{j|I \cup \{i\}}^{\epsilon_j} J_{i|I}^{\epsilon_i},$$

so Theorem 4.2 replaces it by the block with i, j interchanged. The affected algebraic block of the reverse word is exactly

$$R_{i|I}^{\eta_i} R_{j|I \cup \{i\}}^{\eta_j},$$

so Theorem 4.2 performs the matching replacement. All outside factors have literally matching tagged endpoints. Proposition 4.3 supplies the full Walker weight calculation when the replacement sits inside the longer history. A swap therefore moves labels, phases, and their constituent weights together and does not change the sum. Finite induction proves (4.15)–(4.16). $\square$

Proposition 4.6 (Equality-only Coxeter closure). *For the forward words and reverse words separately, and for each fixed phase vector, let τ_a denote only the equality replacement interchanging the entries in positions $a, a+1$. These replacements close around*

$$(4.17) \quad \tau_a^2 = 1, \quad \tau_a \tau_b = \tau_b \tau_a \quad (|a - b| > 1), \quad \tau_a \tau_{a+1} \tau_a = \tau_{a+1} \tau_a \tau_{a+1}.$$

These are loops of literal equalities of expressions. They retain no swap cylinder, comparison isotopy, or S_k -action.

Proof. Check the three Coxeter-shaped relations at the level of labelled expressions. Involutivity reverses one equality, distant swaps involve disjoint positions, and the braid uses the same three labels with different subset tags. Since these are equality paths, no swap cobordism or permutation action is created.

For τ_a^2 , the second step traverses the same instance of (4.6) or (4.8) backward. The preceding set, both phases, and all label-attached weights are unchanged.

If $|a - b| > 1$, the two swaps involve disjoint pairs of positions. Either order produces the same four labeled factors. Every subset tag is the set of labels preceding a position, not an ordering of that set; consequently the two routes have identical tagged endpoints and weights.

For the braid, let the three consecutive labels be i, j, t , and let I be the set of earlier labels. The first route has label orders

$$(i, j, t) \longrightarrow (j, i, t) \longrightarrow (j, t, i) \longrightarrow (t, j, i),$$

whereas the second has

$$(i, j, t) \longrightarrow (i, t, j) \longrightarrow (t, i, j) \longrightarrow (t, j, i).$$

The active pair and exact base tag on the two routes are

$$\begin{array}{l|l} \text{first route} & (i, j | I) \quad (i, t | I \cup \{j\}) \quad (j, t | I) \\ \text{second route} & (j, t | I \cup \{i\}) \quad (i, t | I) \quad (i, j | I \cup \{t\}). \end{array}$$

Thus every edge is a correctly typed instance of (4.6) or (4.8); phases and weights follow i, j, t , not the three positions. Both routes end at the same expression, so transitivity of literal equality closes the diagram. For $k = 1$ there is no generator; for $k = 2$ only the involutive loop is present. $\square$

4.3. Ordered histories before a common trace. For a cobordism $W : \Sigma_- \rightarrow \Sigma_+$, write

$$(4.18) \quad \mathcal{N}(W) := \ker(H_1(\overline{\Sigma_-} \sqcup \Sigma_+; \mathbb{Q}) \longrightarrow H_1(W; \mathbb{Q}))$$

for its homological Lagrangian relation, and use $\mathcal{N}(W)_*$ and $\mathcal{N}(W)^*$ for relational image and inverse image. For a phase assignment q on I , put

$$(4.19) \quad \lambda_I^q := \lambda_q \oplus \bigoplus_{s \in I} L_s^{q_s}.$$

Lemma 4.7 (One-edge transport and internal phase seams). *For $i \notin I$ and $e, v \in \{0, 1\}$, the underlying three-manifold of each phase arrow is the corresponding compression body. There is an identification*

$$(4.20) \quad H_1(J_{i|I}^e; \mathbb{Q}) \cong H_1(\Sigma_I^{\text{out}}; \mathbb{Q}) \oplus \mathbb{Q}x_i,$$

and the same statement holds for $R_{i|I}^v$. Put

$$P_i := \text{Span}(m_i, \ell_i),$$

and let $j_0 : P_i \rightarrow \mathbb{Q}x_i$ be the active-plane part of the bare high-boundary inclusion. Under the homology identifications above, the forward-target and reverse-source inclusions are distinguished by

$$(4.21) \quad j_{J,0} = j_0, \quad j_{J,1} = j_0 \circ (\sigma_{i|I})_*^{-1}, \quad j_{R,0} = j_0, \quad j_{R,1} = j_0 \circ (\sigma_{i|I})_*.$$

Explicitly,

$$(4.22) \quad \begin{array}{c|cc} & m_i & \ell_i \\ \hline j_{J,0} & 0 & x_i \\ j_{J,1} & x_i & 0 \\ j_{R,0} & 0 & x_i \\ j_{R,1} & -x_i & 0 \end{array}.$$

Invariantly,

$$(4.23) \quad \begin{aligned} \ker(j_{J,e}) &= L_i^e, & \ker(j_{R,v}) &= L_i^v, \\ P_i/L_i^e &\xrightarrow[j_{J,e}]{\cong} \mathbb{Q}x_i, \\ P_i/L_i^v &\xrightarrow[j_{R,v}]{\cong} \mathbb{Q}x_i. \end{aligned}$$

The exact one-edge transports are

$$(4.24) \quad \mathcal{N}(J_{i|I}^e)_* \lambda_I = \lambda_I \oplus L_i^e, \quad \mathcal{N}(J_{i|I}^e)^* \lambda_{I \cup \{i\}} = \lambda_I,$$

$$(4.25) \quad \mathcal{N}(R_{i|I}^v)_* \lambda_{I \cup \{i\}} = \lambda_I, \quad \mathcal{N}(R_{i|I}^v)^* \lambda_I = \lambda_I \oplus L_i^v.$$

For a phase-one forward or reverse arrow, its sole internal slope/handle seam has Maslov index zero; hence its actual weights are $c_i + u_i^+$ and $r_i + u_i^-$, respectively.

Core classes, phase lines, and seam cancellation. The one-edge calculation separates the new handle plane from the old and spectator summands. The bare compression relation supplies the core class, the phase cylinder precomposes the distinguished boundary inclusion by the slope action, and the resulting kernel and quotient are the displayed phased lines. Composing that relation with the compression relation gives the transport formulas, while the internal Walker seam is reduced separately to a repeated Lagrangian pair.

Proof. A slope cylinder changes neither the underlying three-manifold nor its homology. For the bare compression body, the handle core contributes x_i , the co-core bounds m_i , and the framed longitude maps to x_i . Thus $j_0(m_i) = 0$ and $j_0(\ell_i) = x_i$. The chosen exact slope action is

$$(\sigma_{i|I})_* m_i = -\ell_i, \quad (\sigma_{i|I})_* \ell_i = m_i,$$

and hence

$$(\sigma_{i|I})_*^{-1} m_i = \ell_i, \quad (\sigma_{i|I})_*^{-1} \ell_i = -m_i.$$

The forward phase cylinder changes the target inclusion by precomposition with $(\sigma_{i|I})_*^{-1}$, whereas the reverse phase cylinder changes the source inclusion by precomposition with $(\sigma_{i|I})_*$. This proves (4.20), (4.21), and the complete table (4.22). Its kernels are exactly the phased lines L_i^ϵ and L_i^ν , and each remaining one-dimensional quotient maps isomorphically to $\mathbb{Q}x_i$, proving (4.23).

Composing these distinguished inclusions, equivalently the graph relation of the appropriate slope cylinder, with the compression relation gives (4.24) and (4.25); every old and spectator summand is fixed. Notice that this is a statement about Lagrangian lines, not an equality $\sigma_i = \sigma_i^{-1}$.

For $J^1 = \Theta\mathcal{C}$, the internal seam is

$$\mu(\lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}, \lambda_I \oplus \langle \ell_i \rangle) = 0.$$

For $R^1 = \mathcal{R}^{\text{rev}}\Theta$, it is

$$\mu(\lambda_I \oplus \langle \ell_i \rangle, \lambda_{I \cup \{i\}}, \lambda_{I \cup \{i\}}) = 0.$$

The entries agree on the old and spectator summands, and on the active plane the displayed triples are respectively (m, m, ℓ) and (ℓ, m, m) . Repeated-pair vanishing and direct-sum additivity prove the claim and recover the active-weight formulas rather than assuming them. $\square$

Proposition 4.8 (Every mixed ordered history). *For every phase vector and every order α , calculated separately before introducing a simultaneous trace, define the partial words*

$$(4.26) \quad J_{\alpha, \leq a}^\epsilon := J_{\alpha_a | I_{a-1}^\alpha}^{\epsilon_{\alpha_a}} \cdots J_{\alpha_1 | \emptyset}^{\epsilon_{\alpha_1}},$$

$$(4.27) \quad R_{\alpha, \geq a}^\eta := R_{\alpha_a | I_{a-1}^\alpha}^{\eta_{\alpha_a}} \cdots R_{\alpha_k | I_{k-1}^\alpha}^{\eta_{\alpha_k}}.$$

At all intermediate interfaces their exact transports are

$$(4.28) \quad \mathcal{N}(J_{\alpha, \leq a}^\epsilon)_* \lambda_g = \lambda_g \oplus \bigoplus_{t=1}^a L_{\alpha_t}^{\epsilon_{\alpha_t}}, \quad \mathcal{N}(J_{\alpha, \leq a}^\epsilon)^* \lambda_{I_a^\alpha} = \lambda_g,$$

$$(4.29) \quad \mathcal{N}(R_{\alpha, \geq a}^\eta)_* \lambda_{[k]} = \lambda_{I_{a-1}^\alpha}, \quad \mathcal{N}(R_{\alpha, \geq a}^\eta)^* \lambda_{I_{a-1}^\alpha} = \lambda_{I_{a-1}^\alpha} \oplus \bigoplus_{t=a}^k L_{\alpha_t}^{\eta_{\alpha_t}}.$$

Moreover,

$$(4.30) \quad H_1(J_\alpha^\epsilon; \mathbb{Q}) \cong H_1(\Sigma_g^{\text{out}}; \mathbb{Q}) \oplus \bigoplus_{a=1}^k \mathbb{Q}x_{\alpha_a}, \quad \beta_1(J_\alpha^\epsilon) = 2g + k,$$

$$(4.31) \quad H_1(R_\alpha^\eta; \mathbb{Q}) \cong H_1(\Sigma_g^{\text{out}}; \mathbb{Q}) \oplus \bigoplus_{a=1}^k \mathbb{Q}x_{\alpha_a}, \quad \beta_1(R_\alpha^\eta) = 2g + k.$$

Writing wt for the integer Walker weight, the total weights are

$$(4.32) \quad \text{wt}(J_\alpha^\epsilon) = \sum_{i=1}^k a_i, \quad \text{wt}(R_\alpha^\eta) = \sum_{i=1}^k b'_i.$$

All intermediate and endpoint markings are the fixed label-preserving markings.

Induction over the labelled sites. Compute each ordered history before invoking order independence. Induction adds one independent core class per labelled site and transports the phase line through the next interface; repeated-pair Maslov triples make every inter-factor correction vanish. The total weight is therefore the sum of the independently attached local weights.

Proof. Iterating (4.20) in the literal order $\alpha_1, \dots, \alpha_k$ adds one independent core class at each distinctly labeled site and no relation between disjoint sites. Reversing the compression bodies leaves the same underlying homology calculation. This proves (4.30) and (4.31) separately for each α . Induction using (4.24) and (4.25) gives (4.28) and (4.29); different-site factors fix every previously created phase line.

We next compute every inter-factor seam. For $a = 2, \dots, k$, immediately before the a -th forward attachment, the relational image of the initial Lagrangian is

$$\lambda_g \oplus \bigoplus_{t < a} L_{\alpha_t}^{\epsilon_{\alpha_t}}.$$

The declared middle term is $\lambda_{I_{a-1}^\alpha}$. By (4.24), the next inverse image is the same term. Thus an earlier phase-one spectator contributes (ℓ, m, m) , while a phase-zero spectator contributes (m, m, m) . The old-surface term also has a repeated pair, so the full seam is zero.

Chronologically the reverse word removes $\alpha_k, \dots, \alpha_1$. For $a = 2, \dots, k$, at the seam before removing α_{a-1} , the relational image of the already applied higher reverse factor is the declared $\lambda_{I_{a-1}^\alpha}$, while the inverse image through the next reverse factor differs from it only by replacing the line at α_{a-1} by $L_{\alpha_{a-1}}^{\eta_{\alpha_{a-1}}}$. Its local triple is therefore (m, m, ℓ) in phase one and (m, m, m) in phase zero. All remaining spectator and old-surface terms again have a repeated pair. This calculates every reverse inter-factor correction as zero.

Lemma 4.7 already calculated every internal phase-one seam. Adding its elementary weights across the now-verified zero inter-factor seams gives (4.32). Finally, every local arrow fixes all other labels and every composition identifies the exact tagged intermediate surface. The slope word is supported inside its labeled torus and does not relabel an endpoint. Hence the external markings are exactly the prescribed markings, with no comparison isotopy retained. None of these calculations uses a common trace or order independence. $\square$

4.4. The independently constructed simultaneous trace. Order independence gives a common value for every ordered history, but the next topological argument needs one actual cobordism in which all labelled local pieces coexist. The simultaneous trace is therefore constructed directly in the disjoint product neighborhoods, before any order is chosen, and is only afterward compared with the ordered traces. This turns the order calculation into a geometric object for the match-count argument without defining that object by a preferred word or retaining a level-moving isotopy.

Proposition 4.9 (Geometric simultaneous phase traces). *There are marked Walker extended cobordisms*

$$J_{\text{sim}}^\epsilon(\mathbf{c}, \mathbf{u}^+) : \Sigma_\emptyset^{\text{out}} \longrightarrow \Sigma_{[k]}^{\text{out}}, \quad R_{\text{sim}}^\eta(\mathbf{r}, \mathbf{u}^-) : \Sigma_{[k]}^{\text{out}} \longrightarrow \Sigma_\emptyset^{\text{out}},$$

constructed directly from the k disjoint labeled neighborhoods, such that

$$(4.33) \quad J_\alpha^\epsilon(\mathbf{c}, \mathbf{u}^+) = J_{\text{sim}}^\epsilon(\mathbf{c}, \mathbf{u}^+), \quad R_\beta^\eta(\mathbf{r}, \mathbf{u}^-) = R_{\text{sim}}^\eta(\mathbf{r}, \mathbf{u}^-)$$

for every two, independently chosen, order tags α, β . Their weights are the two sums in (4.32). When no ambiguity can arise, we suppress these fixed label-attached weight arguments below.

Proof. Construct the simultaneous trace directly in pairwise-disjoint product neighborhoods, synchronizing the local critical levels rather than defining it from an order word. Separating those levels one at a time invokes the already proved interchange and spectator identities, while the independent history calculation verifies the markings, transported lines, homology, and weight.

Choose, before choosing an order word, pairwise-disjoint product neighborhoods N_i of the chosen disks, attaching feet, bands, co-cores, longitudes, and separating collars. Outside $\bigcup_i N_i$, take the product cobordism. Inside every N_i , place the local forward handle trace and, when $\epsilon_i = 1$, its supported slope cylinder in a common high-boundary collar. Synchronizing the k disjoint critical levels defines J_{sim}^ϵ geometrically. Similarly, put the phase- η_i slope collar before the local reverse trace in N_i and synchronize the reverse levels to define R_{sim}^η . This construction uses the neighborhoods and local arrows only; neither object is defined as the value of an order word. Equip the two directly constructed extended cobordisms with the integers $\sum_i a_i$ and $\sum_i b'_i$, respectively. These are intrinsic parts of the simultaneous constructions, not values transported from a chosen order.

Now separate the disjoint critical levels one at a time. Interchanging two neighboring levels is the relative-boundary equality already proved in the forward or reverse part of Theorem 4.2,

with the full correction supplied by Proposition 4.3. Any separation order therefore yields the corresponding ordered trace. Proposition 4.8 independently shows that its core classes, transported lines, all internal and inter-factor corrections, endpoint marking, and total weight agree with the simultaneous construction. This proves (4.33) without retaining a chosen level-moving isotopy. $\square$

5. FIXED-SITE MATCH-COUNT TOPOLOGY AND SOURCE DAGGER

Once order independence is known, the composite can be understood by asking which local forward and reverse phases match. This section turns that count into the topology of the simultaneous double and separately records the contravariant source-reversal operation.

First use the two order-independence results to form the forward–reverse double with its two independent phase records. Then isolate each labelled neighborhood by a separating sphere and classify its local double: matching phases retain a punctured $S^1 \times S^2$ piece, while mismatching phases retain the product piece. The disjoint connected-sum decomposition turns that local table into the global match count and Betti-number formula. The source dagger is a separate contravariant operation on the marked cobordism, so it is recorded after the topology of the double has been identified.

5.1. The global match count.

Definition 5.1 (Independent double and match data). For independent order tags $\alpha, \beta \in S_k$, define

$$(5.1) \quad M_{\alpha, \beta}^{\epsilon, \eta}(\mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^-) := R_{\beta}^{\eta}(\mathbf{r}, \mathbf{u}^-) J_{\alpha}^{\epsilon}(\mathbf{c}, \mathbf{u}^+) \in \text{End}(\Sigma_g^{\text{out}}).$$

Set

$$(5.2) \quad \mathcal{M} := \mathcal{M}(\epsilon, \eta) := \{i \in [k] : \epsilon_i = \eta_i\}, \quad b := b(\epsilon, \eta) := |\mathcal{M}|,$$

$$(5.3) \quad N := \sum_{i=1}^k (c_i + r_i + \epsilon_i u_i^+ + \eta_i u_i^-) = \sum_{i=1}^k (a_i + b'_i).$$

Also write

$$(5.4) \quad d_H(\epsilon, \eta) := |\{i \in [k] : \epsilon_i \neq \eta_i\}|$$

for the Hamming distance. Inactive slope coordinates do not occur in N .

Lemma 5.2 (Order independence of the double). *The morphism in (5.1) is independent of α and β separately. Denote its common value by*

$$M^{\epsilon, \eta}(\mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^-).$$

We suppress the fixed label-attached weight arguments below.

Proof. Equations (4.15) and (4.16) are literal equalities in their fixed Hom sets. Substitute both into (5.1) to prove the claim. No local double is moved between different Hom sets. $\square$

Theorem 5.3 (Simultaneous local pieces and match-count topology). *For every allowed g, k , phase pair, and active constituent weights,*

$$(5.5) \quad M^{\epsilon, \eta} \cong C(\text{id}_{\Sigma_g^{\text{out}}}, N) \# \left(\#_{i \in \mathcal{M}} (S^1 \times S^2, 0) \right).$$

Writing $\beta_1(W) := \dim_{\mathbb{Q}} H_1(W; \mathbb{Q})$, the number of displayed closed summands is

$$(5.6) \quad b = k - d_H(\epsilon, \eta),$$

and

$$(5.7) \quad \beta_1(M^{\epsilon, \eta}) = 2g + b.$$

Consequently,

$$(5.8) \quad b = 0 \implies M^{\epsilon, \eta} = C(\text{id}_{\Sigma_g^{\text{out}}}, N),$$

whereas $b > 0$ implies that $M^{\epsilon, \eta}$ is not an identity cylinder of any Walker weight.

Local doubles and the global match count. Use the independently constructed simultaneous traces and surround each labelled neighborhood by a disjoint separating sphere. A matching phase pair gives the local meridian-to-meridian double, hence one punctured $S^1 \times S^2$; a mismatching pair gives the product piece. Disjoint connected-sum decomposition then counts the matching sites, while the history calculation supplies the global Walker weight.

Proof. Compose the independently constructed traces $R_{\text{sim}}^{\eta} J_{\text{sim}}^{\epsilon}$. In the geometric construction, choose one auxiliary separating sphere around each N_i . These spheres are pairwise disjoint and lie in one actual global cobordism. We inspect the pieces they cut off; we do not form, multiply, or commute the typed symbols $\mathcal{D}_{i|I}$ from different endomorphism sets.

Inside the i -th sphere the local word is the already proved same-site word $R_i^{\eta_i} J_i^{\epsilon_i}$, with its actual markings. If $\epsilon_i = \eta_i$, the local table and its topology identify the piece as punctured $S^1 \times S^2$. This includes both the 00 and the 11 match; in particular an all-one phase pair contributes at every site. If $\epsilon_i \neq \eta_i$, the exact marked slope-word calculation identifies the local piece as the punctured S^3 product piece. It therefore disappears from connected sum with the outside product.

The Walker calculation is equally local but is made in this single global trace. Proposition 4.8 gives the forward and reverse internal and inter-factor corrections and their weights. At the middle forward/reverse interface, the chosen direct-sum splitting and the endpoint cases of (4.28) and (4.29) give the exact correction

$$(5.9) \quad \mu(\lambda_{[k]}^{\epsilon}, \lambda_{[k]}, \lambda_{[k]}^{\eta}) = \mu(\lambda_g, \lambda_g, \lambda_g) + \sum_{i=1}^k \mu(L_i^{\epsilon_i}, \langle m_i \rangle, L_i^{\eta_i}) = 0.$$

The four terms in the sum are exactly the four already proved local-table seams and the four rows of (4.10); every one is zero. In particular, the matching 11 term is

$$\mu(\langle \ell_i \rangle, \langle m_i \rangle, \langle \ell_i \rangle) = 0.$$

A separating sphere has zero first homology and hence contributes no Maslov term. The total Walker weight is therefore exactly

$$\sum_i a_i + \sum_i b'_i = N.$$

The central shift is recorded on the outside cylinder in (5.5); every displayed closed summand has weight zero. The complement of the local spheres is the product on the old surface. There is one $S^1 \times S^2$ summand for precisely each $i \in \mathcal{M}$, which proves (5.5).

A coordinate is outside $\mathcal{M}$ exactly when it contributes one to the Hamming distance, proving (5.6). The product cylinder has first Betti number $2g$, and connected sum with each $S^1 \times S^2$ adds one independent first-homology generator. Hence (5.7). If $b = 0$, every local piece is the exact punctured S^3 product with the prescribed markings, so removing all auxiliary spheres leaves the literal identity cylinder of weight N , proving (5.8). If $b > 0$, then $\beta_1(M) = 2g + b > 2g$, whereas every weighted identity cylinder has first Betti number $2g$; no choice of Walker weight can make them equal.

The spheres and the subset $\mathcal{M}$ index this proof only. The construction neither chooses nor retains a sphere system or an embedded copy of the match subset as part of its output; the globally detected data here are the count b and the independently retained total weight N . In particular, equal values of b do not erase different values of N . For $k = 1$, this is exactly the four-entry local table. For $k = 2$, the choice $\epsilon = (0, 1)$, $\eta = (0, 0)$ has $b = 1$. Equal phase vectors, including the all-one pair, have $b = k$; complementary phase vectors have $b = 0$. When $g = 1$, formula (5.7) reads $\beta_1 = 2 + b$. $\square$

Remark 5.4 (Limitations). Every argument above uses one fixed chart of pairwise-disjoint, distinctly labeled sites. It does not cover repeated, overlapping, nested, intersecting, slid, born, canceled, unlabelled, or relabeled sites. The elementary dagger used only to derive (4.8) is not a

global dagger theorem. No scalar target value, refined finite carrier, physical handle map, Cerf relation, Heegaard invariant, or $\widehat{Z}$ comparison is asserted here.

5.2. Typed global words and orientation reversal. Fix integers $g, k \geq 1$, the tagged outgoing objects Σ_I^{out} for $I \subseteq [k]$, the two phase vectors $\epsilon, \eta \in \{0, 1\}^k$, and the four label-attached weight vectors

$$\mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^- \in \mathbb{Z}^k.$$

All products are right-factor first. An inactive coordinate u_i^+ when $\epsilon_i = 0$, or u_i^- when $\eta_i = 0$, is ignored in every arrow and every formula below.

For an order word $\alpha = (\alpha_1, \dots, \alpha_k) \in S_k$, set

$$(5.10) \quad I_a^\alpha = \{\alpha_1, \dots, \alpha_a\}, \quad I_0^\alpha = \emptyset.$$

The two already constructed blocks are

$$(5.11) \quad \begin{aligned} J_\alpha^\epsilon(\mathbf{c}, \mathbf{u}^+) &:= J_{\alpha_k|I_{k-1}^\alpha}^{\epsilon_{\alpha_k}}(c_{\alpha_k}, u_{\alpha_k}^+) \cdots J_{\alpha_1|\emptyset}^{\epsilon_{\alpha_1}}(c_{\alpha_1}, u_{\alpha_1}^+) \\ &: \Sigma_\emptyset^{\text{out}} \longrightarrow \Sigma_{[k]}^{\text{out}}, \end{aligned}$$

$$(5.12) \quad \begin{aligned} R_\alpha^\eta(\mathbf{r}, \mathbf{u}^-) &:= R_{\alpha_1|\emptyset}^{\eta_{\alpha_1}}(r_{\alpha_1}, u_{\alpha_1}^-) \cdots R_{\alpha_k|I_{k-1}^\alpha}^{\eta_{\alpha_k}}(r_{\alpha_k}, u_{\alpha_k}^-) \\ &: \Sigma_{[k]}^{\text{out}} \longrightarrow \Sigma_\emptyset^{\text{out}}. \end{aligned}$$

The arguments displayed in these formulas are understood by the inactive coordinate convention. Because products act from the right, the reverse word chronologically removes $\alpha_k, \dots, \alpha_1$. Thus α is the matching attachment tag, not an inverse-permutation tag.

The elementary source-dagger identities proved before any global argument are, for $\xi \in \{0, 1\}$,

$$(5.13) \quad (J_{i|I}^\xi(c_i, u_i))^\dagger = R_{i|I}^\xi(-c_i, -u_i),$$

$$(5.14) \quad (R_{i|I}^\xi(r_i, u_i))^\dagger = J_{i|I}^\xi(-r_i, -u_i).$$

They are handle-arrow identities. They were obtained from $\Theta_{i|I}(u)^\dagger = C(\sigma_{i|I}^{-1}, -u)$ only after the proved half-turn absorption relations; no equality $C(\sigma^{-1}, -u) = C(\sigma, -u)$ is being asserted.

Proposition 5.5 (Exact dagger of the two global blocks). *For every order tag $\alpha \in S_k$,*

$$(DG0) \quad \boxed{(J_\alpha^\epsilon(\mathbf{c}, \mathbf{u}^+))^\dagger = R_\alpha^\epsilon(-\mathbf{c}, -\mathbf{u}^+).}$$

For every order tag $\beta \in S_k$,

$$(DG1) \quad \boxed{(R_\beta^\eta(\mathbf{r}, \mathbf{u}^-))^\dagger = J_\beta^\eta(-\mathbf{r}, -\mathbf{u}^-).}$$

Both are literal equalities of marked Walker extended cobordisms, including the negated total integer weights.

Proof. Contravariance reverses the written factors of (5.11). Applying (5.13) factor by factor gives

$$\begin{aligned} (J_\alpha^\epsilon(\mathbf{c}, \mathbf{u}^+))^\dagger &= R_{\alpha_1|\emptyset}^{\epsilon_{\alpha_1}}(-c_{\alpha_1}, -u_{\alpha_1}^+) \cdots R_{\alpha_k|I_{k-1}^\alpha}^{\epsilon_{\alpha_k}}(-c_{\alpha_k}, -u_{\alpha_k}^+) \\ &= R_\alpha^\epsilon(-\mathbf{c}, -\mathbf{u}^+). \end{aligned}$$

This is exactly the reverse word with attachment tag α ; no inverse permutation is introduced. The forward block has weight $\sum_i (c_i + \epsilon_i u_i^+)$. Orientation reversal negates it, and the block on the right has weight $\sum_i (-c_i - \epsilon_i u_i^+)$, so the Walker weights agree exactly.

The same calculation applied to (5.12), now using (5.14), yields

$$\begin{aligned} (R_\beta^\eta(\mathbf{r}, \mathbf{u}^-))^\dagger &= J_{\beta_k|I_{k-1}^\beta}^{\eta_{\beta_k}}(-r_{\beta_k}, -u_{\beta_k}^-) \cdots J_{\beta_1|\emptyset}^{\eta_{\beta_1}}(-r_{\beta_1}, -u_{\beta_1}^-) \\ &= J_\beta^\eta(-\mathbf{r}, -\mathbf{u}^-). \end{aligned}$$

Its weight is the negative of $\sum_i (r_i + \eta_i u_i^-)$. This proves both formulas with the endpoint markings and integer weights retained. $\square$

For independent tags $\alpha, \beta \in S_k$, retain the typed double

$$(5.15) \quad M_{\alpha, \beta}^{\epsilon, \eta}(\mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^-) := R_{\beta}^{\eta}(\mathbf{r}, \mathbf{u}^-) J_{\alpha}^{\epsilon}(\mathbf{c}, \mathbf{u}^+) \in \text{End}(\Sigma_g^{\text{out}}).$$

Its already proved total weight and match data are

$$(5.16) \quad \begin{aligned} \mathcal{M} &:= \mathcal{M}(\epsilon, \eta) = \{i \in [k] : \epsilon_i = \eta_i\}, & b &:= b(\epsilon, \eta) = |\mathcal{M}|, \\ N &= \sum_{i=1}^k (c_i + r_i + \epsilon_i u_i^+ + \eta_i u_i^-). \end{aligned}$$

Corollary 5.6 (Exact dagger of the mixed double). *One has*

$$(DG2) \quad \boxed{\begin{aligned} (M_{\alpha, \beta}^{\epsilon, \eta}(\mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^-))^{\dagger} \\ = M_{\beta, \alpha}^{\eta, \epsilon}(-\mathbf{r}, -\mathbf{c}, -\mathbf{u}^-, -\mathbf{u}^+). \end{aligned}}$$

The transformation sends N to $-N$ and leaves b unchanged.

Proof. Using contravariance and then (DG0)–(DG1),

$$\begin{aligned} (R_{\beta}^{\eta} J_{\alpha}^{\epsilon})^{\dagger} &= (J_{\alpha}^{\epsilon})^{\dagger} (R_{\beta}^{\eta})^{\dagger} \\ &= R_{\alpha}^{\epsilon}(-\mathbf{c}, -\mathbf{u}^+) J_{\beta}^{\eta}(-\mathbf{r}, -\mathbf{u}^-). \end{aligned}$$

By the definition (5.15), this last word is the right side of (DG2): its forward parameter block is $(-\mathbf{r}, -\mathbf{u}^-)$, and its reverse parameter block is $(-\mathbf{c}, -\mathbf{u}^+)$. The phase pair and the order tags are therefore exchanged exactly as displayed. Substitution in (5.16) gives $-N$, while equality of the two phase bits is symmetric, so the match count remains b . $\square$

The symbol $\dagger$ in this subsection denotes source orientation reversal only. None of the three displayed dagger identities is a Hermitian-adjoint identity in a target theory.

Proof of Theorem 1.1. The local phase table and typed interchanges reduce the ordered histories to the simultaneous trace. Its geometric decomposition gives the match-count topology and total weight, after which block reversal gives the dagger identities. These steps retain the distinct conclusions of literal equality, marked diffeomorphism, topological connected sum, and source orientation reversal.

The common pulled-back chart and its Walker extension are Proposition 2.1. The localized half-turn, capping, absorption, and elementary source-reversal statements are Propositions 3.2 to 3.5. The four-entry weighted table and the topology distinguishing its matching entries are Theorem 3.6 and Proposition 3.7.

The separately typed forward and reverse interchanges are Theorem 4.2; their spectator corrections are checked in Proposition 4.3. Literal order independence and equality-only confluence then follow from Corollary 4.5 and Proposition 4.6. The ordered history calculation and its comparison with the independently constructed simultaneous trace are Propositions 4.8 and 4.9. Finally, Lemma 5.2 and Theorem 5.3 give the order-independent mixed double, its match-count topology, and its first Betti number, while Proposition 5.5 and Corollary 5.6 give the two block reversals and the mixed-double reversal. These results prove the seven clauses in their stated order. $\square$

6. SLIDE DATA AND LOCAL POSITIVE-SLIDE INSERTION

After all k labelled handles have been attached, a slide changes the marking of the completed compression body. Protected sectors specify where each local move is defined; their common collars allow the moves to be inserted simultaneously. The reference chart is identified with the terminal fixed-site chart before any slide is made. That identification imports no assertion about a moved marking or a face of the slide cube.

6.1. Protected sectors and completed charts. Fix

$$g \geq 1, \quad s \geq 1, \quad k \geq 2s,$$

and an injective assignment of ordered active pairs

$$(6.1) \quad a \mapsto (i_a, j_a), \quad a \in [s],$$

such that the $2s$ labels i_a, j_a are all distinct. On the outgoing boundary of the reference chart $\mathfrak{C}_\emptyset$, write $\bar{m}_t^\emptyset, \bar{\ell}_t^\emptyset$ for the geometric marked curves and $m_t, \ell_t \subset Y$ for their images in the fixed abstract surface. These are the rows of the terminal attachment marking from Section 2; in particular, they realize (2.11) and (2.12). Their one-holed-torus support collars are pairwise disjoint and are disjoint from the old-region marking.

For each $a \in [s]$, choose a protected genus-two sector $U_a \subset W$ containing exactly the complete marking data at the labels i_a, j_a . At a vertex $A \subseteq [s]$, the actual co-core at label t is denoted D_t^A . The reference sector contains $D_{i_a}^\emptyset, D_{j_a}^\emptyset$, the prescribed positive push-off $D_{j_a}^{\text{po}}$ of the second co-core, the positive product-zero-framed slide band, and the dual-longitude band. It also contains the complete old and slid cut-cellulations, all site collars, the outer detector collar with its framed crossing arc, the dual ribbons, and all signed faces and face pairings. The sector meets the closure of its complement in a single interface disk E_a , supplied with a full product collar.

The closures of the U_a , including every named collar and protected product neighborhood, are pairwise disjoint. Moreover, U_a is disjoint from the feet, bands, co-cores, marked curves, framing collars, and product neighborhoods of every label outside $\{i_a, j_a\}$. Labels in no active pair are called *unused*. Thus, when direction a is inserted, the incoming boundary and its collar, the old genus- g region, every U_b with $b \neq a$, and every unused handle together with a neighborhood of its complete marking lie in the identity exterior of U_a . This is a pointwise separation hypothesis, not merely a disjointness assertion on homology classes.

For $A \subseteq [s]$, define $\mathfrak{C}_A$ directly: in U_b , use the complete old marked row if $b \notin A$ and the complete positively slid row if $b \in A$; on the common complement, use the old row. The protected closures are disjoint, so these insertions are simultaneous. This definition never passes through a history of partially attached or partially slid handles.

Let $\bar{m}_t^A, \bar{\ell}_t^A$ be the actual geometric rows on $\partial_+(W, \mathfrak{C}_A)$, with their orientations, product framings, and support collars, and let D_t^A be the actual co-core disk in that completed marking. The abstract rows $m_t^A, \ell_t^A \subset Y$ are obtained by inserting the corresponding complete coordinate row in every protected sector. The outgoing parameterization is the simultaneous gluing

$$(6.2) \quad p_A: \partial_+(W, \mathfrak{C}_A) \longrightarrow Y, \quad p_A(\bar{m}_t^A) = m_t^A, \quad p_A(\bar{\ell}_t^A) = \ell_t^A.$$

The equalities retain labels, orientations, product framings, and support collars. Old and slid local rows agree literally on every full interface collar and on the common complement, so there is no independent postcomposition of one local piece of p_A . The transported support collars remain pairwise disjoint: the collars within one active sector belong to its complete local model, distinct protected sectors have disjoint closures, and every unused collar lies in the common identity exterior. The declared vertex Lagrangian is the subspace Λ_A defined in (2.15); its equality with Λ away from the reference vertex is not assumed here.

6.2. The terminal/reference-vertex interface. The same terminal compression body carries two bookkeeping systems. Fixed-site notation remembers the attachment history and ends at the state $I = [k]$; slide notation starts with all handles present and calls its old completed marking $A = \emptyset$. The following proposition is the entire interface between those systems.

Proposition 6.1 (Terminal/reference-vertex interface). *At the reference slide vertex, the completely marked compression body $(W, \mathfrak{C}_\emptyset): X \rightarrow Y$ is the terminal simultaneous fixed-site trace. In particular,*

$$(6.3) \quad m_t^\emptyset = m_t, \quad \ell_t^\emptyset = \ell_t, \quad \Lambda_\emptyset = \Lambda = \lambda_{[k]}.$$

The equality of compression bodies retains the endpoint parameterizations, oriented co-cores, framed rows, signed cut faces and pairings, collars, ribbons, detector germs, and label data. The

first two displayed equalities are equalities of embedded oriented product-framed curves in the fixed surface Y .

Let

$$(6.4) \quad \sigma_t^{\text{ref}} := \sigma_{t[k] \setminus \{t\}} = T_{m_t} T_{\ell_t} T_{m_t}, \quad \sigma_{t;\emptyset} := T_{m_t^\emptyset} T_{\ell_t^\emptyset} T_{m_t^\emptyset},$$

where each word is extended by the identity outside its specified one-holed-torus collar. Then $\sigma_{t;\emptyset} = \sigma_t^{\text{ref}}$ as a supported marked mapping class. If $t_1, \dots, t_r$ are the labels for which $\epsilon_t = 1$, in any order, set

$$(6.5) \quad \sigma_{\text{ref}}^\epsilon := \sigma_{t_r}^{\text{ref}} \cdots \sigma_{t_1}^{\text{ref}}.$$

This mapping class and the associated Walker cylinder are independent of the displayed order. Define σ_{ref}^η by the same rule using the labels for which $\eta_t = 1$.

Define the directly completed phase-zero reference traces by giving $(W, \mathfrak{C}_\emptyset)$ and its literal reverse the weights c_Σ and r_Σ , respectively. With the slide-state subscript $\emptyset$, they satisfy the literal equalities

$$(6.6) \quad J_\emptyset^0(\mathbf{c}) = J_{[k]}^0(\mathbf{c}) = J_{\text{sim}}^0(\mathbf{c}): X \longrightarrow Y,$$

$$(6.7) \quad R_\emptyset^0(\mathbf{r}) = R_{[k]}^0(\mathbf{r}) = R_{\text{sim}}^0(\mathbf{r}): Y \longrightarrow X.$$

For arbitrary phase vectors, put

$$(6.8) \quad J_\emptyset^\epsilon(\mathbf{c}, \mathbf{u}^+) := C(\sigma_{\text{ref}}^\epsilon, U^+(\epsilon)) J_\emptyset^0(\mathbf{c}),$$

$$(6.9) \quad R_\emptyset^\eta(\mathbf{r}, \mathbf{u}^-) := R_\emptyset^0(\mathbf{r}) C(\sigma_{\text{ref}}^\eta, U^-(\eta)).$$

Then

$$(6.10) \quad J_\emptyset^\epsilon(\mathbf{c}, \mathbf{u}^+) = J_{[k]}^\epsilon(\mathbf{c}, \mathbf{u}^+) = J_{\text{sim}}^\epsilon(\mathbf{c}, \mathbf{u}^+),$$

$$(6.11) \quad R_\emptyset^\eta(\mathbf{r}, \mathbf{u}^-) = R_{[k]}^\eta(\mathbf{r}, \mathbf{u}^-) = R_{\text{sim}}^\eta(\mathbf{r}, \mathbf{u}^-).$$

The respective weights are $c_\Sigma + U^+(\epsilon)$ and $r_\Sigma + U^-(\eta)$.

Proof. The interface is proved by comparing the two complete gluings piece by piece: the reference chart uses exactly the terminal old rows on every protected sector and on the common complement. Once those rows, collars, and endpoint parameterizations are identified, the phase words have the same supported curves as the terminal fixed-site words. Disjoint supports then make the phase factors commute and remove the intermediate Walker seams. The proposition thus transfers the terminal trace at the old vertex only; slide edges require the local insertion constructed below.

The reference chart uses the complete old row in every protected sector and the old row on the common complement. Those are precisely the pieces of the terminal attachment marking, with the same product coordinates on every interface collar. Their simultaneous gluing is therefore the terminal parameterization, not a parameterization related to it by an unrecorded postcomposition. This proves the equality of compression bodies with complete marking and all the identities in (6.3). The last identity also follows directly by substituting $m_t^\emptyset = m_t$ into (2.15) and comparing with (2.12).

The same equality of framed rows and support collars gives $\sigma_{t;\emptyset} = \sigma_t^{\text{ref}}$ in (6.4). Distinct phase words have disjoint supports. The calculation in Lemma 2.2 therefore shows both that their mapping classes commute and that every Walker correction between them vanishes. Hence (6.5) is independent of the chosen order and its labelled cylinders combine to weight $U^+(\epsilon)$, or to $U^-(\eta)$ in the reverse phase product.

At phase zero, the directly completed reference cobordism and the fixed-site simultaneous trace have the same underlying oriented compression body (respectively its literal reverse), the same complete endpoint marking, and the same respective total weights c_Σ and r_Σ . Thus they are literally the same Walker morphisms, proving (6.6) and (6.7). The simultaneous-trace clause of Theorem 1.1 identifies these morphisms with every fixed-site ordered history. Adding the identical supported phase words, with the zero corrections just calculated, proves (6.10) and (6.11), including the stated weights. $\square$

The proposition transfers only the reference marking, the terminal fixed-site phase words, the label-attached weights, and the completed terminal traces. It transfers no word at $A \neq \emptyset$, no slide edge or inverse, no edge conjugacy, no Walker seam across an edge, no two-face, and no finite slide-order conclusion.

6.3. Local-model identification and supported insertion. We now specify the geometric input for one positive edge. For every $a \in [s]$, fix orientation-preserving product coordinate identifications that preserve the complete marking,

$$(6.12) \quad \chi_a^- : (U_a, E_a, \text{old local data}) \longrightarrow (U_\gamma, E_\gamma, \text{old local data}),$$

$$(6.13) \quad \chi_a^+ : (U_a, E_a, \text{slid local data}) \longrightarrow (U_\gamma^\gamma, E_\gamma, \text{slid local data}).$$

They use the same product coordinates on E_a and its full collar. Under this dictionary, the local labels 1, 2 correspond respectively to i_a, j_a , and $D_{j_a}^{\text{po}}$ corresponds to the local D_2^{po} . The positive band, dual band, cut balls, complete cellulation, signed faces and face pairings, co-core orientations, dual ribbons, detector, product framings, and every collar germ also agree with their local-model counterparts.

Lemma 6.2 (Local-model identification and identity extension). *Let $F_\gamma : U_\gamma \rightarrow U_\gamma^\gamma$ be the complete-marked positive-slide representative constructed in [Sun26, Theorem 2.2 and Appendix F]. Then*

$$(6.14) \quad f_a := (\chi_a^+)^{-1} F_\gamma \chi_a^- : U_a \longrightarrow U_a$$

has the coordinate-identity product germ on the full collar of E_a . The piecewise formula

$$(6.15) \quad \widehat{F}_a(x) = \begin{cases} f_a(x), & x \in \text{int}(U_a), \\ x, & x \in W \setminus \text{int}(U_a) \end{cases}$$

defines a smooth orientation-preserving diffeomorphism of W , with a smooth inverse, supported in U_a . It is the identity on the incoming boundary and its collar, the old genus- g region, every U_b with $b \neq a$, every unused handle, and a neighborhood of all corresponding complete marking data.

Proof. The local model supplies a map preserving the complete marking, not merely an underlying diffeomorphism. Transport through the two collar-preserving charts gives a map with the identity germ needed for the piecewise formula to glue smoothly. The support-separation hypotheses then extend that local map to all of W while fixing every other marked sector, so the same supported map can be reused at each slide state; only the typed boundary transition changes with A .

The cited Appendix F constructs both F_γ and its inverse with the coordinate-identity germ on a full interface collar, while preserving every item in the preceding dictionary. Transport through (6.12)–(6.13) preserves that germ. Consequently the two branches in (6.15), and separately the two inverse branches, agree on an open product neighborhood with all derivatives. They glue to mutually inverse smooth maps, and both branches have positive orientation sign. The pointwise fixation statement follows from the separation hypotheses: the incoming boundary and collar, the old region, every other protected sector, and every unused site lie, together with neighborhoods of their complete markings, in the identity exterior. $\square$

For $A \subseteq [s]$ and $a \notin A$, use the same ambient map at every source vertex and define

$$(6.16) \quad F_{a|A} := \widehat{F}_a : (W, \mathfrak{C}_A) \longrightarrow (W, \mathfrak{C}_{A \cup \{a\}}).$$

Inside U_a , complete local transport carries the old row to the slid row; outside U_a , every row is fixed pointwise. Hence (6.16) has the displayed source and target types. Its formula is independent of A , since all already-slid sectors are disjoint from its support. Define the outgoing transition and its weight-zero Walker lift by

$$(6.17) \quad h_{a|A} := p_{A \cup \{a\}} \circ (F_{a|A}|_{\partial_+ W}) \circ p_A^{-1}, \quad H_{a|A} := C(h_{a|A}, 0).$$

Both endpoint parameterizations are part of this type. The zero is part of the definition; a cylinder of another weight is not a slide edge here.

The transport also retains the residual-factor tests attached to the chosen local representative. A meridian-twist postfactor would change the winding of a transported dual ribbon, while the prescribed ribbon row has winding zero. A separating twist about the outer collar would change the full turn number of the framed detector crossing, while the transported detector has turn zero. After cutting along the two active co-cores, any remaining cell-fixing postfactor fixes every vertex and edge of the complete cellulation and the boundary of each disk or rectangular two-cell. Cellwise Alexander isotopies, paired across the signed faces, make that postfactor topologically isotopic to the identity relative to the complete cellular marking. This topological statement does not assert smooth-relative uniqueness. The smooth edge used here is the particular representative transported in (6.14). Its calibration uses the ribbons, detector, collars, and complete cellulation; it is not inferred from the four homology formulas proved later.

The citation in Lemma 6.2 supplies one local geometric representative in a complete genus-two model, with its residual-factor calibration. Transport through the two product-coordinate identifications retains its identity on the full interface collar; the separation and identity-exterior hypotheses then give the spectator fixation proved above. These are the inputs to the completed edge. Walker equality, phase and weight identities, insertion for arbitrary spectators, and compatibility between slides are not supplied by that citation. The homology action, common Lagrangian, edge covariance, squares, and finite closure are established in the subsequent sections from the chosen edge.

7. COMPLETED VERTICES, HOMOLOGY, AND PHASE-ZERO COVARIANCE

The supported representatives of Section 6 must fit together as maps of one global marked boundary. The completed charts $\mathfrak{C}_A$, their outgoing parameterizations p_A , and the edge maps $F_{a|A}$ have already been defined. We now check that their meridian and longitude rows glue with the complete marking intact. This geometric check makes the subsequent homology action meaningful and supplies the common rational Lagrangian that controls the Walker seams.

First glue the selected local rows to the common complement, so that each vertex is represented by one boundary with complete marking rather than by independent sector choices. Next compare the meridian spans of adjacent vertices and compute the integral homology action of an edge, including its dual-longitude correction. The actual inverse is then distinguished from a mere coordinate rewrite, after which the common Lagrangian and the phase-zero Walker identities follow. The order is essential: the homology formulas record the already constructed marked maps; they do not replace the gluing and marking argument.

7.1. Compatible vertex rows and the common Lagrangian. Recall that $\bar{m}_t^A, \bar{\ell}_t^A$ are the geometric curves on the outgoing boundary of $(W, \mathfrak{C}_A)$, whereas $m_t^A, \ell_t^A \subset Y$ are their counterparts in the fixed abstract surface.

Proposition 7.1 (Compatible completed-vertex rows). *For every subset $A \subseteq [s]$, the sector and complement rows defining p_A glue smoothly and orientation-preservingly, and*

$$(7.1) \quad p_A(\bar{m}_t^A) = m_t^A, \quad p_A(\bar{\ell}_t^A) = \ell_t^A \quad (1 \leq t \leq k),$$

with labels, orientations, product framings, and support collars retained. There is no independent postcomposition of an individual glued row. Moreover, the vertex Lagrangian from (2.15) satisfies

$$(7.2) \quad \boxed{\Lambda_A = \Lambda} \quad (A \subseteq [s])$$

as a subspace of the one fixed group $H_1(Y; \mathbb{Q})$.

Proof. Choose the old or slid complete row in each protected sector and glue it to the unchanged complement through the common product collars. Disjoint collars make this a simultaneous global marking rather than a collection of independent choices. The homology calculation then compares the old and slid meridian spans in each active handle, which proves equality of the vertex Lagrangians in the one fixed homology group.

In each protected sector, choose the complete old row or the complete slid row according as the sector label lies outside or inside A . The two coordinate identifications in (6.12)–(6.13) use the

same product coordinates on the entire interface collar. Both local rows agree there, as smooth product germs, with the common complement row. Distinct closed sector collars are disjoint, so no triple-overlap compatibility condition occurs. The selected local rows therefore glue to one smooth global parameterization. Every piece preserves orientation, and the pieces agree on their overlaps; hence the glued map does as well. The construction retains every oriented framed support collar and gives (7.1). Since this simultaneous gluing is the definition of p_A , it leaves no further postcomposition choice.

It remains to compare meridian spans. If $b \notin A$, the sector meridians are the old classes $[m_{i_b}]$ and $[m_{j_b}]$. If $b \in A$, the transported positive-slide row gives

$$(7.3) \quad [m_{i_b}^A] = [m_{i_b}] + [m_{j_b}], \quad [m_{j_b}^A] = [m_{j_b}].$$

Thus the slid meridian plane is contained in the old plane. The converse inclusion follows from

$$(7.4) \quad [m_{i_b}] = [m_{i_b}^A] - [m_{j_b}^A], \quad [m_{j_b}] = [m_{j_b}^A].$$

The active pairs use $2s$ distinct labels, so these calculations lie in disjoint handle summands. The old summand λ_g and every unused meridian line are unchanged. Taking their direct sum proves both inclusions in (7.2). This also covers $k = 2s$, when the sum over unused meridians is empty. $\square$

7.2. Complete edge marking and integral homology. The edge transition $h_{a|A}$ and its weight-zero cylinder $H_{a|A}$ were defined in (6.17). Their types include both endpoint parameterizations.

Proposition 7.2 (Edge marking, homology, and Lagrangian transport). *For every $A \subseteq [s]$ and $a \notin A$, the boundary transition transports the actual embedded active rows by*

$$(7.5) \quad \begin{aligned} h_{a|A}(m_{i_a}^A) &= m_{i_a}^{A \cup \{a\}}, & h_{a|A}(\ell_{i_a}^A) &= \ell_{i_a}^{A \cup \{a\}}, \\ h_{a|A}(m_{j_a}^A) &= m_{j_a}^{A \cup \{a\}}, & h_{a|A}(\ell_{j_a}^A) &= \ell_{j_a}^{A \cup \{a\}}, \end{aligned}$$

with orientations, product framings, ribbons, detector, and support collars retained. It fixes the complete marked row of every other label in its product coordinates. On integral homology,

$$(7.6) \quad \begin{aligned} (h_{a|A})_*[m_{i_a}^A] &= [m_{i_a}^{A \cup \{a\}}] = [m_{i_a}^A] + [m_{j_a}^A], & (h_{a|A})_*[\ell_{i_a}^A] &= [\ell_{i_a}^{A \cup \{a\}}] = [\ell_{i_a}^A], \\ (h_{a|A})_*[m_{j_a}^A] &= [m_{j_a}^{A \cup \{a\}}] = [m_{j_a}^A], & (h_{a|A})_*[\ell_{j_a}^A] &= [\ell_{j_a}^{A \cup \{a\}}] = [\ell_{j_a}^A] - [\ell_{i_a}^A]. \end{aligned}$$

For every $t \notin \{i_a, j_a\}$, including active labels in other sectors and unused labels,

$$(7.7) \quad \begin{aligned} (h_{a|A})_*[m_t^A] &= [m_t^{A \cup \{a\}}] = [m_t^A], \\ (h_{a|A})_*[\ell_t^A] &= [\ell_t^{A \cup \{a\}}] = [\ell_t^A]. \end{aligned}$$

The map also fixes the old summand $H_1(X; \mathbb{Z})$ pointwise.

As an actual map from the target marked basis to the source marked basis, its inverse satisfies

$$(7.8) \quad \begin{aligned} (h_{a|A}^{-1})_*[m_{i_a}^{A \cup \{a\}}] &= [m_{i_a}^A], & (h_{a|A}^{-1})_*[\ell_{i_a}^{A \cup \{a\}}] &= [\ell_{i_a}^A], \\ (h_{a|A}^{-1})_*[m_{j_a}^{A \cup \{a\}}] &= [m_{j_a}^A], & (h_{a|A}^{-1})_*[\ell_{j_a}^{A \cup \{a\}}] &= [\ell_{j_a}^A]. \end{aligned}$$

By contrast, merely rewriting the source classes in target-vertex coordinates inside the fixed abstract group gives

$$(7.9) \quad \begin{aligned} [m_{i_a}^A] &= [m_{i_a}^{A \cup \{a\}}] - [m_{j_a}^{A \cup \{a\}}], & [\ell_{i_a}^A] &= [\ell_{i_a}^{A \cup \{a\}}], \\ [m_{j_a}^A] &= [m_{j_a}^{A \cup \{a\}}], & [\ell_{j_a}^A] &= [\ell_{j_a}^{A \cup \{a\}}] + [\ell_{i_a}^{A \cup \{a\}}]. \end{aligned}$$

Equation (7.9) is a coordinate rewrite, not the action of $(h_{a|A}^{-1})_*$.

Finally, $(h_{a|A})_*$ is an integral symplectic automorphism, and

$$(7.10) \quad \boxed{(h_{a|A})_*\Lambda_A = \Lambda_{A \cup \{a\}} = \Lambda, \quad (h_{a|A}^{-1})_*\Lambda = \Lambda.}$$

Supported transport on homology. The supported map first identifies the active complete rows and fixes the spectators. Its local positive slide gives the meridian shear and the dual longitude correction; the inverse relations distinguish actual transport from a mere coordinate rewrite. The resulting integral symplectic map carries the common meridian span to the next vertex Lagrangian.

Proof. By Lemma 6.2, the supported representative transports the complete local marking in the a -sector and is the identity on every other row. Conjugation by p_A and $p_{A \cup \{a\}}$ proves (7.5). The compatibility of the complete markings from Section 6, including the ribbon and detector tests, ensures that these identities specify the selected relative class and are not inferred from homology alone.

The positive local slide has the meridian plus sign and the compensating dual-longitude minus sign. Transport through the orientation-preserving coordinates, with the ordered relabelling $1 \mapsto i_a$ and $2 \mapsto j_a$, gives (7.6). No permutation or orientation reversal of the ordered active pair occurs. The identity exterior gives (7.7) and the pointwise identity on the old summand. Applying the actual inverse to the four target marked classes gives (7.8); solving the last equalities in (7.6) inside the common abstract group instead gives (7.9). The resulting change of basis is unitriangular on the active four-dimensional summand and the identity elsewhere, hence is an integral automorphism.

To check symplecticity without suppressing signs, abbreviate $m_i = m_{i_a}^A$, $m_j = m_{j_a}^A$, $\ell_i = \ell_{i_a}^A$, and $\ell_j = \ell_{j_a}^A$. Since $a \notin A$, this is the old row in the active sector. Using $m_r \cdot \ell_t = \delta_{rt}$, the matching pairings are

$$(7.11) \quad (m_i + m_j) \cdot \ell_i = 1, \quad m_j \cdot (\ell_j - \ell_i) = 1,$$

and the only nontrivial cross calculation is

$$(7.12) \quad (m_i + m_j) \cdot (\ell_j - \ell_i) = 0 - 1 + 1 - 0 = 0.$$

The other cross pairing is $m_j \cdot \ell_i = 0$; transformed meridians pair mutually to zero, as do transformed longitudes. Skew symmetry and identity on the orthogonal complementary summands complete the integral symplecticity check. In particular, the minus sign in the last formula of (7.6) is forced: a plus sign would make (7.12) equal to 2.

For the forward Lagrangian direction, retaining every summand gives

$$\begin{aligned} (h_{a|A})_* \Lambda_A &= \lambda_g \oplus \langle [m_i] + [m_j], [m_j] \rangle_{\mathbb{Q}} \oplus \bigoplus_{t \notin \{i_a, j_a\}} \langle [m_t^A] \rangle_{\mathbb{Q}} \\ &= \lambda_g \oplus \langle [m_i], [m_j] \rangle_{\mathbb{Q}} \oplus \bigoplus_{t \notin \{i_a, j_a\}} \langle [m_t^A] \rangle_{\mathbb{Q}} = \Lambda. \end{aligned}$$

The transformed active span lies in the old active span, and the reverse inclusion uses $[m_i] = ([m_i] + [m_j]) - [m_j]$. Thus neither inclusion is implicit. Proposition 7.1 identifies both endpoint Lagrangians with Λ . For the reverse direction, (7.8) maps the two target active meridians to the two source active meridians and fixes every other meridian line. This proves the second equality in (7.10) directly. $\square$

7.3. Direct phase-zero traces. For each $A \subseteq [s]$, let $J_A^0(\mathbf{0}): X \rightarrow Y$ be the completed forward compression body in the chart $\mathfrak{C}_A$, with outgoing parameterization p_A and Walker weight zero. Let $R_A^0(\mathbf{0}): Y \rightarrow X$ be its reverse, again with the completed endpoint marking and weight zero. These are direct constructions at the completed vertex. They do not factor through partially slid charts or elementary attachment arrows.

For arbitrary external vectors $\mathbf{c}, \mathbf{r} \in \mathbb{Z}^k$, set

$$(7.13) \quad c_{\Sigma} = \sum_{t=1}^k c_t, \quad r_{\Sigma} = \sum_{t=1}^k r_t,$$

and define

$$(7.14) \quad \begin{aligned} J_A^0(\mathbf{c}) &:= C(\text{id}_Y, c_\Sigma) J_A^0(\mathbf{0}), \\ R_A^0(\mathbf{r}) &:= R_A^0(\mathbf{0}) C(\text{id}_Y, r_\Sigma). \end{aligned}$$

The coordinates of $\mathbf{c}$ and $\mathbf{r}$ remain label-attached external data. The complete morphisms in (7.14) depend on them only through the totals c_Σ and r_Σ ; an individual coordinate cannot be recovered from the resulting morphism.

At the old vertex $A = \emptyset$, the ordinary cobordisms with complete marking underlying (7.14) agree with the old-vertex traces established in Theorem 1.1; the endpoint rows and the assigned totals c_Σ and r_Σ agree as well. This comparison uses only the old completed chart and imports no slid-site word or edge covariance. We do not use it here to identify a Walker composite before its ordered seam has been computed.

7.4. Ordinary forward and reverse covariance. For a Walker morphism M , write $\underline{M}$ for the same completely marked cobordism with only its integer weight forgotten.

Proposition 7.3 (Ordinary phase-zero edge covariance). *For every $A \subseteq [s]$, every $a \notin A$, and all $\mathbf{c}, \mathbf{r} \in \mathbb{Z}^k$, one has*

$$(7.15) \quad \boxed{\underline{H}_{a|A} \underline{J}_A^0(\mathbf{c}) = \underline{J}_{A \cup \{a\}}^0(\mathbf{c})}$$

and

$$(7.16) \quad \boxed{\underline{R}_{A \cup \{a\}}^0(\mathbf{r}) \underline{H}_{a|A} = \underline{R}_A^0(\mathbf{r})}.$$

Both formulas use the same source-to-target cylinder $H_{a|A} = C(h_{a|A}, 0)$, not a cylinder of its inverse. These are ordinary completely marked identities; the ordered Maslov corrections and the corresponding Walker-weighted identities will be computed separately.

Collar collapse and completed vertices. Collapse the source-to-target cylinder onto the outgoing collar of the forward trace and use the parameterization square to identify the resulting marked cobordism with the directly completed next vertex. The reverse identity is checked independently with the inverse ambient map, while the same source cylinder and its direction are retained. Forgetting the central integer isolates this geometric equality; the ordered Walker corrections are computed only afterward.

Proof. Write $f_{a|A,+} = F_{a|A}|_{\partial_+ W}$. The definition of the boundary transition is equivalent to

$$(7.17) \quad p_{A \cup \{a\}} \circ f_{a|A,+} = h_{a|A} \circ p_A.$$

Collapse $\underline{H}_{a|A}$ onto the outgoing collar of the compression body representing $\underline{J}_A^0(\mathbf{c})$. The exposed outgoing row is $h_{a|A} \circ p_A$. By (7.17), and because $F_{a|A}$ is the identity on X and its collar, that ambient map is an isomorphism from the collapsed representative to the directly completed compression body at $A \cup \{a\}$. It matches both endpoint parameterizations and every component of the complete marking. Forgetting the central integer makes the same argument valid for every $\mathbf{c}$, which proves (7.15). The occurrence of $h_{a|A}$, rather than its inverse, is forced by the source-to-target direction in (7.17).

The reverse equality is typed independently. Reverse the orientations and the endpoint roles, and use the orientation-preserving map

$$(7.18) \quad \overline{F_{a|A}^{-1}}: \overline{(W, \mathfrak{C}_{A \cup \{a\}})} \longrightarrow \overline{(W, \mathfrak{C}_A)}.$$

It is the identity on the common boundary X and its collar. Collapsing $\underline{H}_{a|A}$ against the incoming collar of the reverse compression body at $A \cup \{a\}$ exposes $h_{a|A}^{-1} \circ p_{A \cup \{a\}}$. Rearranging (7.17) gives

$$(7.19) \quad p_A \circ f_{a|A,+}^{-1} = h_{a|A}^{-1} \circ p_{A \cup \{a\}}.$$

Consequently $\overline{F_{a|A}^{-1}}$ identifies the collapsed composite with the reverse completed compression body at A , retaining both endpoint rows and the complete marking. Again the underline removes

the central integer and nothing else, so the argument holds for every $\mathbf{r}$. This proves (7.16). The inverse in (7.19) comes from the incoming collapse of the same $h_{a|A}$ -cylinder; it does not change that cylinder to $C(h_{a|A}^{-1}, 0)$. $\square$

The arguments include the single-direction case $s = 1$, retaining all $k - 2$ spectators; when also $g = 1$ and $k = 2$, they reduce to the local marking, homology, Lagrangian, and ordinary covariance identities. For $s = 2$ and $s = 3$, the same gluing and direct-sum proofs apply sector by sector, with no extra compatibility choice. The unused-label sum is empty when $k = 2s$, while for $k > 2s$ every unused row is fixed pointwise. Arbitrary positive, zero, and negative coordinates in $\mathbf{c}$ and $\mathbf{r}$ are allowed. Finally, no argument depends on whether active labels are numerically adjacent: the pairwise-disjoint protected sectors and the injectivity of $a \mapsto (i_a, j_a)$ control all supports and direct summands.

8. WALKER-EXACT COVARIANCE AND ACTUAL-VERTEX PHASES

The phase-zero calculation gives the geometric skeleton. Here we restore the integer weights and check that the same transported Lagrangian makes the Walker corrections agree at every actual vertex and in both directions.

The notation in this half of the paper records a completed slide vertex. Thus S_A^ϵ is the complete phase word at the vertex A ; it is not the fixed-site local cylinder $S_{i|I}$ used in the first half. The former varies over a Boolean family of completed slides, whereas the latter records one local phase choice at a fixed handle site.

We now restore the Walker integers that were suppressed in the ordinary covariance identities of Proposition 7.3. The calculation uses the common Lagrangian proved in the preceding section: for every $A \subseteq [s]$ and every $a \notin A$,

$$(8.1) \quad \Lambda_A = \Lambda, \quad (h_{a|A})_* \Lambda = \Lambda, \quad (h_{a|A}^{-1})_* \Lambda = \Lambda.$$

The edge cylinder remains $H_{a|A} = C(h_{a|A}, 0)$; its weight is part of its definition. All products below are read right-factor first, and each Maslov triple is written in its composition order before it is simplified.

The common Lagrangian first determines the two phase-zero seam weights. For nonzero phases, marked conjugacy transports the words built from the actual curves and support collars at each vertex. Checking the forward and reverse Maslov triples separately then gives phase-cylinder covariance, which combines with the phase-zero identities to give completed-endpoint covariance. Thus the weights are computed from the transported marking data, rather than from an old phase word assigned to a moved vertex.

8.1. The two phase-zero seams.

Lemma 8.1 (Seam Lagrangians and central weights). *At a completed vertex A , the forward compression body transports its incoming declared Lagrangian to $\Lambda_A = \Lambda$ on the high boundary, whereas the reverse compression body inverse-transport its outgoing declared Lagrangian to the same high-boundary subspace. For a completely marked cobordism M and a displayed boundary copy S , write $\lambda_M(S)$ for the induced boundary Lagrangian transported into the fixed group $H_1(Y; \mathbb{Q})$ through the retained parameterization; on an oppositely oriented target copy, this uses inverse transport. On the two sides of the edge cylinder one has*

$$(8.2) \quad \lambda_{H_{a|A}}(Y) = (h_{a|A})_* \Lambda, \quad \lambda_{H_{a|A}}(\bar{Y}) = (h_{a|A}^{-1})_* \Lambda.$$

Consequently, for the totals in (7.13),

$$(8.3) \quad \text{wt}(J_A^0(\mathbf{c})) = c_\Sigma, \quad \text{wt}(R_A^0(\mathbf{r})) = r_\Sigma.$$

Proof. First identify both high-boundary Lagrangians with the common meridian span. Then read the two edge-cylinder seam subspaces from the chosen source and target parameterizations. In each composition order the seam has three equal entries, so the correction vanishes and only the assigned integer total remains.

The high-boundary kernel of the forward compression body is the old disk-kernel together with the lines of all actual co-core meridians. Its transported Lagrangian is therefore

$$\lambda_g \oplus \bigoplus_{t=1}^k \langle [m_t^A] \rangle_{\mathbb{Q}} = \Lambda_A = \Lambda.$$

Orientation reversal gives the same complete meridian span as the inverse-transported high-boundary Lagrangian of the reverse compression body. These are statements about the directly completed compression bodies. The two identities in (8.2) follow from the source parameterization id_Y and target parameterization $h_{a|A}$ of the mapping cylinder.

For the forward central factor in (7.14), the seam triple in actual product order is

$$(\Lambda, \Lambda, \text{id}_{Y*}^{-1} \Lambda) = (\Lambda, \Lambda, \Lambda).$$

Thus its weight is $0 + c_{\Sigma} - 0 = c_{\Sigma}$. For the independently typed reverse central factor, the seam triple is

$$(\text{id}_{Y*} \Lambda, \Lambda, \Lambda) = (\Lambda, \Lambda, \Lambda),$$

and its weight is $r_{\Sigma} + 0 - 0 = r_{\Sigma}$. These computations see $\mathbf{c}$ and $\mathbf{r}$ only through their totals; their coordinates remain label-attached external data. $\square$

Proposition 8.2 (Weighted phase-zero edge covariance). *For every $A \subseteq [s]$, every $a \notin A$, and all $\mathbf{c}, \mathbf{r} \in \mathbb{Z}^k$, the two ordered corrections are*

$$(8.4) \quad \boxed{\mu_Y(\Lambda, \Lambda, (h_{a|A}^{-1})_* \Lambda) = 0}$$

and

$$(8.5) \quad \boxed{\mu_Y((h_{a|A})_* \Lambda, \Lambda, \Lambda) = 0}.$$

They give equalities of completely marked Walker morphisms

$$(8.6) \quad \boxed{H_{a|A} J_A^0(\mathbf{c}) = J_{A \cup \{a\}}^0(\mathbf{c})}$$

and

$$(8.7) \quad \boxed{R_{A \cup \{a\}}^0(\mathbf{r}) H_{a|A} = R_A^0(\mathbf{r})}.$$

The common weights in the two displays are c_{Σ} and r_{Σ} , respectively. At $A = \emptyset$, these phase-zero traces agree with the fixed-site traces established in Theorem 1.1, with the same endpoint markings and totals. This comparison is made only at the old completed vertex.

The phase-zero collar comparison. The phase-zero edge identity is obtained by comparing the exposed outgoing row after composing with the edge cylinder. The endpoint parameterization square makes the ambient maps agree on the complete marking, while the common Lagrangian data make both ordered Maslov triples vanish. Forgetting the central integer is therefore a geometric equality before the Walker weights are restored.

Proof. In the forward composite $H_{a|A} J_A^0(\mathbf{c})$, the right factor transports the incoming Lagrangian to Λ , the declared seam Lagrangian is Λ , and the left cylinder inverse-transport its target Lagrangian to $(h_{a|A}^{-1})_* \Lambda$. This is exactly the ordered triple in (8.4). After its order has been fixed, (8.1) identifies the third entry with Λ , so repeated-pair vanishing gives zero. The composite weight is therefore

$$c_{\Sigma} + 0 - \mu_Y(\Lambda, \Lambda, (h_{a|A}^{-1})_* \Lambda) = c_{\Sigma}.$$

The forward ordinary equality (7.15) retains both endpoint markings, and the right side has weight c_{Σ} by Lemma 8.1; hence it upgrades to (8.6).

The reverse product is calculated separately. In $R_{A \cup \{a\}}^0(\mathbf{r}) H_{a|A}$, the right cylinder transports the source Lagrangian to $(h_{a|A})_* \Lambda$, the seam Lagrangian is Λ , and the reverse compression body inverse-transport its low-boundary Lagrangian to Λ . Thus the ordered triple is the one in (8.5);

it is not a permutation of the forward triple. Once again (8.1) and repeated-pair vanishing give zero, and the weight is

$$0 + r_\Sigma - \mu_Y((h_{a|A})_*\Lambda, \Lambda, \Lambda) = r_\Sigma.$$

The independently typed ordinary equality (7.16), together with the weight of its right side, proves (8.7). In both directions the slide cylinder contributes exactly zero.

At $A = \emptyset$, the direct completed chart, complete compression body, endpoint rows, and endpoint parameterizations are those used in the fixed-site construction. Its phase-zero theorem identifies every labelled order with that completed trace and gives weights c_Σ and r_Σ . Equality of the underlying cobordisms with complete marking and of these integers yields the stated comparison. No phase word at a slid vertex is involved. $\square$

8.2. Phase words from completed markings. For every completed vertex A and every label $t \in [k]$, define the supported three-twist word from the actual oriented product-framed row at that vertex:

$$(8.8) \quad \sigma_{t;A} := T_{m_t^A} T_{\ell_t^A} T_{m_t^A}, \quad \Theta_{t;A}(u) := C(\sigma_{t;A}, u).$$

It is extended by the identity outside the actual one-holed-torus support collar of label t . Thus $\sigma_{t;A}$ is defined afresh from $\mathfrak{C}_A$, rather than transported from a phase word at a different vertex.

For $\epsilon \in \{0, 1\}^k$, put $I_\epsilon = \{t : \epsilon_t = 1\}$. If $\rho = (\rho_1, \dots, \rho_q)$ is an ordering of I_ϵ , set

$$(8.9) \quad S_{A,\rho}^\epsilon(\mathbf{u}^+) := \Theta_{\rho_q;A}(u_{\rho_q}^+) \cdots \Theta_{\rho_1;A}(u_{\rho_1}^+).$$

The labelled pair (t, u_t^+) moves as one unit under an order replacement. When I_ϵ is empty, the right side means $C(\text{id}_Y, 0)$.

Proposition 8.3 (Supported phase words and completed traces). *For every $A \subseteq [s]$, every $\epsilon, \eta \in \{0, 1\}^k$, and every $\mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^- \in \mathbb{Z}^k$, the word in (8.9) is independent of ρ as a Walker morphism. More precisely,*

$$(8.10) \quad S_{A,\rho}^\epsilon(\mathbf{u}^+) = S_A^\epsilon(\mathbf{u}^+) := C(\sigma_A^\epsilon, U^+(\epsilon)),$$

where

$$(8.11) \quad \sigma_A^\epsilon := \prod_{t \in I_\epsilon} \sigma_{t;A}, \quad U^+(\epsilon) := \sum_{t \in I_\epsilon} u_t^+,$$

and the mapping-class product denotes the common product. The same construction, with $I_\eta := \{t : \eta_t = 1\}$, defines $S_A^\eta(\mathbf{u}^-) = C(\sigma_A^\eta, U^-(\eta))$, including the empty word, where $U^-(\eta) = \sum_{t \in I_\eta} u_t^-$.

Define the completed-endpoint traces by

$$(8.12) \quad \begin{aligned} J_A^\epsilon(\mathbf{c}, \mathbf{u}^+) &:= S_A^\epsilon(\mathbf{u}^+) J_A^0(\mathbf{c}), \\ R_A^\eta(\mathbf{r}, \mathbf{u}^-) &:= R_A^0(\mathbf{r}) S_A^\eta(\mathbf{u}^-). \end{aligned}$$

Their weights are

$$(8.13) \quad \text{wt}(J_A^\epsilon) = c_\Sigma + U^+(\epsilon), \quad \text{wt}(R_A^\eta) = r_\Sigma + U^-(\eta).$$

Only at $A = \emptyset$ do we invoke the assertion that these morphisms agree with the order-independent full phase composites established in Theorem 1.1, with the same labelled parameters.

Vertexwise phase words and seam cancellation. Build the phase word at the completed vertex from the actual marked three-twist neighborhoods, not from a word transported from another vertex. Disjoint supports make adjacent phase factors commute, while the common meridian Lagrangian makes each ordered Walker seam vanish. The direct compression trace can then be joined to the phase word with the same label-attached weights; only at the old vertex is this construction compared with the previously established fixed-site word.

Proof. The support collars of distinct labels are disjoint at every completed vertex: this holds within a transported two-label sector, between distinct protected sectors, and between either kind of sector and every unused site. On the symplectic plane

$$P_t^A := \text{span}_{\mathbb{Q}}\{[m_t^A], [\ell_t^A]\} \subset H_1(Y; \mathbb{Q}),$$

the positive three-twist convention gives

$$(8.14) \quad (\sigma_{t;A})_*[m_t^A] = -[\ell_t^A], \quad (\sigma_{t;A})_*[\ell_t^A] = [m_t^A],$$

and both $\sigma_{t;A}$ and its inverse are the identity on every other direct summand.

Let $i \neq j$. Applying the Walker product law first to the ordered product $\Theta_{j;A}(v)\Theta_{i;A}(u)$ gives the correction

$$(8.15) \quad \mu_Y((\sigma_{i;A})_*\Lambda, \Lambda, (\sigma_{j;A})_*^{-1}\Lambda).$$

On the i - and j -planes its triples of lines are, respectively,

$$(8.16) \quad (\langle \ell_i^A \rangle, \langle m_i^A \rangle, \langle m_i^A \rangle), \quad (\langle m_j^A \rangle, \langle m_j^A \rangle, \langle \ell_j^A \rangle).$$

Every remaining summand has three equal entries. Direct-sum additivity and repeated-pair vanishing make (8.15) zero. The opposite ordered product $\Theta_{i;A}(u)\Theta_{j;A}(v)$ has the independently ordered correction

$$(8.17) \quad \mu_Y((\sigma_{j;A})_*\Lambda, \Lambda, (\sigma_{i;A})_*^{-1}\Lambda),$$

whose i - and j -plane triples are

$$(\langle m_i^A \rangle, \langle m_i^A \rangle, \langle \ell_i^A \rangle), \quad (\langle \ell_j^A \rangle, \langle m_j^A \rangle, \langle m_j^A \rangle).$$

It vanishes separately. Since the two marked mapping classes have disjoint supports, they commute pointwise, and hence

$$(8.18) \quad \Theta_{j;A}(v)\Theta_{i;A}(u) = C(\sigma_{j;A}\sigma_{i;A}, u+v) = \Theta_{i;A}(u)\Theta_{j;A}(v).$$

There is no additional correction in a longer word. Let $\kappa = (\kappa_1, \dots, \kappa_q)$ be an ordered prefix of distinct labels, put

$$K_\kappa := \{\kappa_1, \dots, \kappa_q\}, \quad \sigma_{\kappa;A} := \sigma_{\kappa_q;A} \cdots \sigma_{\kappa_1;A},$$

and adjoin a new label $j \notin K_\kappa$. Its correction is

$$\mu_Y((\sigma_{\kappa;A})_*\Lambda, \Lambda, (\sigma_{j;A})_*^{-1}\Lambda).$$

On a plane indexed by $t \in K_\kappa$ the three lines are $(\langle \ell_t^A \rangle, \langle m_t^A \rangle, \langle m_t^A \rangle)$; on the j -plane they are $(\langle m_j^A \rangle, \langle m_j^A \rangle, \langle \ell_j^A \rangle)$; elsewhere they coincide. It is therefore zero. Induction followed by finite adjacent replacements proves (8.10), with each weight kept attached to its label and summed only after the order is fixed. The same proof applies to the minus-labelled parameters.

For the forward composite in (8.12), the ordered internal seam is

$$(8.19) \quad \mu_Y(\Lambda, \Lambda, (\sigma_A^\epsilon)_*^{-1}\Lambda) = 0,$$

while the independently typed reverse seam is

$$(8.20) \quad \mu_Y((\sigma_A^\eta)_*\Lambda, \Lambda, \Lambda) = 0.$$

Their repeated pairs, together with Lemma 8.1, prove (8.13).

At the old vertex, each actual curve, framing, and support collar is the corresponding fixed-site datum, so each $\sigma_{t;\emptyset}$ is literally the supported three-twist word used in Theorem 1.1. The direct completed compression body and its endpoint rows are also the same. The fixed-site theorem assigns weights $c_\Sigma + U^+(\epsilon)$ and $r_\Sigma + U^-(\eta)$, exactly as above. Equality of the marked cobordisms and Walker integers proves the old-vertex comparison. This comparison follows the construction of the actual-vertex words and supplies no word, edge, or covariance statement at a slid vertex. $\square$

8.3. Complete conjugacy and all-phase covariance.

Proposition 8.4 (Phase-cylinder commutation across an edge). *Let $A \subseteq [s]$, $a \notin A$, and $B = A \cup \{a\}$. Complete marking transport gives, for every $t \in [k]$,*

$$(8.21) \quad h_{a|A} \sigma_{t;A} h_{a|A}^{-1} = \sigma_{t;B}$$

as marked mapping classes with their support collars and product framings. Consequently,

$$(8.22) \quad \boxed{H_{a|A} S_A^\epsilon(\mathbf{u}^+) = S_B^\epsilon(\mathbf{u}^+) H_{a|A}}$$

and, with the reverse-trace typing,

$$(8.23) \quad \boxed{S_B^\eta(\mathbf{u}^-) H_{a|A} = H_{a|A} S_A^\eta(\mathbf{u}^-)}.$$

Every side of the first equality has weight $U^+(\epsilon)$, and every side of the second has weight $U^-(\eta)$.

Naturality of the complete markings. Transport the complete meridian, longitude, framing, and support-collar data through the edge map. Naturality of the three twists gives the conjugacy of the marked mapping classes; the two directions are then checked separately by writing their ordered Maslov triples. The common Lagrangian forces the relevant repeated pair, so the edge transports the phase word without adding a new central weight.

Proof. An orientation-preserving marked diffeomorphism conjugates a positive Dehn twist about an oriented product-framed curve to the positive twist about the transported curve. For $t = i_a, j_a$, Proposition 7.2 transports the actual embedded meridian and longitude, their framings, and their complete support collars from the row at A to the row at B . For every other label, the supported edge fixes a neighborhood of the complete marking pointwise. Applying twist naturality to all three factors proves (8.21); this is framed marking conjugacy, not an inference from homology. Multiplication in any of the verified supported orders gives

$$(8.24) \quad h_{a|A} \sigma_A^\epsilon = \sigma_B^\epsilon h_{a|A}, \quad h_{a|A} \sigma_A^\eta = \sigma_B^\eta h_{a|A}.$$

It remains to compare the Walker integers. The two corrections in the forward-typed equality, in the orders in which the factors occur, are

$$(8.25) \quad \mu_Y((\sigma_A^\epsilon)_* \Lambda, \Lambda, (h_{a|A}^{-1})_* \Lambda) = 0,$$

$$(8.26) \quad \mu_Y((h_{a|A})_* \Lambda, \Lambda, (\sigma_B^\epsilon)^{-1} \Lambda) = 0.$$

The first vanishes because its last two entries agree by (8.1); the second vanishes because its first two entries agree. Thus the two sides of (8.22) are

$$\begin{aligned} C(h_{a|A}, 0) C(\sigma_A^\epsilon, U^+(\epsilon)) &= C(h_{a|A} \sigma_A^\epsilon, U^+(\epsilon)), \\ C(\sigma_B^\epsilon, U^+(\epsilon)) C(h_{a|A}, 0) &= C(\sigma_B^\epsilon h_{a|A}, U^+(\epsilon)), \end{aligned}$$

and (8.24) identifies their marked mapping classes.

For the reverse-trace typing, the two corrections are separately

$$(8.27) \quad \mu_Y((h_{a|A})_* \Lambda, \Lambda, (\sigma_B^\eta)^{-1} \Lambda) = 0,$$

$$(8.28) \quad \mu_Y((\sigma_A^\eta)_* \Lambda, \Lambda, (h_{a|A}^{-1})_* \Lambda) = 0.$$

The first has equal first and second entries, while the second has equal second and third entries. These are the corrections for $C(\sigma_B^\eta, U^-(\eta)) C(h_{a|A}, 0)$ and $C(h_{a|A}, 0) C(\sigma_A^\eta, U^-(\eta))$, respectively, in their actual order. They vanish independently, and the second equality in (8.24) identifies the marked maps. This proves (8.23) with unchanged phase weight and zero slide weight. $\square$

Proposition 8.5 (All-phase completed-endpoint edge covariance). *For every $A \subseteq [s]$, every $a \notin A$, all $\epsilon, \eta \in \{0, 1\}^k$, and all $\mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^- \in \mathbb{Z}^k$,*

$$(8.29) \quad \boxed{H_{a|A} J_A^\epsilon(\mathbf{c}, \mathbf{u}^+) = J_{A \cup \{a\}}^\epsilon(\mathbf{c}, \mathbf{u}^+)}$$

and

$$(8.30) \quad \boxed{R_{A \cup \{a\}}^\eta(\mathbf{r}, \mathbf{u}^-) H_{a|A} = R_A^\eta(\mathbf{r}, \mathbf{u}^-)}.$$

Their common weights are, respectively, $c_\Sigma + U^+(\epsilon)$ and $r_\Sigma + U^-(\eta)$.

Proof. At each vertex, disjoint support collars make the elementary phase maps commute on the marked surface, so their labelled products depend only on the active set and their total weight. The edge carries those phase maps to the next vertex, and the phase-zero covariance then supplies the completed trace identity. The argument keeps the forward and reverse composites separately typed.

In the displayed calculation below, the fixed arguments $(\mathbf{c}, \mathbf{r}, \mathbf{u}^+, \mathbf{u}^-)$ are suppressed from the symbols J , R , and S , but are unchanged at every equality. Using the completed-endpoint definition, the forward phase-cylinder commutation, and then phase-zero covariance gives the typed chain

$$\begin{aligned} H_{a|A} J_A^\epsilon &= H_{a|A} S_A^\epsilon J_A^0 \\ &= S_{A \cup \{a\}}^\epsilon H_{a|A} J_A^0 \\ &= S_{A \cup \{a\}}^\epsilon J_{A \cup \{a\}}^0 = J_{A \cup \{a\}}^\epsilon. \end{aligned}$$

Every step is an equality of Walker morphisms by [Propositions 8.2 and 8.4](#), so no correction is suppressed. Its common weight is $c_\Sigma + U^+(\epsilon)$.

The reverse calculation is independently typed:

$$\begin{aligned} R_{A \cup \{a\}}^\eta H_{a|A} &= R_{A \cup \{a\}}^0 S_{A \cup \{a\}}^\eta H_{a|A} \\ &= R_{A \cup \{a\}}^0 H_{a|A} S_A^\eta \\ &= R_A^0 S_A^\eta = R_A^\eta, \end{aligned}$$

with common weight $r_\Sigma + U^-(\eta)$. Throughout both calculations the parameters remain label-indexed. The resulting complete morphisms retain only the total base weight and the active phase-weight sum; they neither transport nor recover an individual summand. $\square$

9. LITERAL SQUARES, FINITE ORDER, AND TELESOPING

The cube is the point at which local compatibility becomes a global relation. We first compare the supported maps as smooth maps of the common compression body, then transport that equality to the outgoing boundary and finally to Walker cylinders. The order and telescoping statements are consequences of that pointwise geometry.

For reference, the edge notation records successive views of one supported move. The imported local model F_γ is transported to f_a , extended by the identity to the ambient map $\widehat{F}_a$, and then typed at its source vertex as $F_{a|A}$. Its outgoing boundary map is $h_{a|A}$, and $H_{a|A} = C(h_{a|A}, 0)$ is the corresponding weight-zero Walker cylinder. The set A records which other disjoint slides have already been completed; it does not change the supported ambient map in direction a .

We now assemble the pairwise-disjoint edge directions into the finite cube. The geometric statement comes first. By [\(6.16\)](#), every edge in direction a is represented by the same ambient map $\widehat{F}_a$, independently of the source vertex, and the closed protected sectors—including their full product collars—are pairwise disjoint. We will use these facts to prove equality of the two smooth composites pointwise on the common compression body. The outgoing and Walker squares will then follow from that equality; they are not inferred from commuting homology matrices.

Throughout the section, composition is right-factor first. We also retain the common-Lagrangian identities

$$(9.1) \quad (h_{a|A})_* \Lambda = \Lambda, \quad (h_{a|A}^{-1})_* \Lambda = \Lambda$$

from [Proposition 7.2](#).

9.1. The pointwise smooth square.

Proposition 9.1 (Literal disjoint-support square). *Let $A \subseteq [s]$, and let $a, b \notin A$ be distinct. Then, as literal smooth maps on the common underlying compression body W ,*

$$(9.2) \quad \boxed{F_{b|A \cup \{a\}} \circ F_{a|A} = F_{a|A \cup \{b\}} \circ F_{b|A}}.$$

Both sides are the map $\Phi_{a,b}: W \rightarrow W$ given pointwise by

$$(9.3) \quad \Phi_{a,b}(x) = \begin{cases} \widehat{F}_a(x), & x \in \text{int}(U_a), \\ \widehat{F}_b(x), & x \in \text{int}(U_b), \\ x, & x \in W \setminus (\text{int } U_a \cup \text{int } U_b). \end{cases}$$

The branches in (9.3) agree on full product neighborhoods of the interfaces. In particular, $\Phi_{a,b}$ is smooth, and (9.2) is an equality of complete-marked relative maps from $\mathfrak{C}_A$ to $\mathfrak{C}_{A \cup \{a,b\}}$, with every spectator marking fixed.

Proof. Disjoint supports allow the two fixed ambient maps to be evaluated pointwise: on the first support only the a -map acts, on the second only the b -map acts, and outside both supports the identity remains. The full product collars make these descriptions agree smoothly at the interfaces, yielding a literal equality of marked maps before any homology or Walker calculation.

The four active labels i_a, j_a, i_b, j_b are distinct, but the support hypothesis is stronger than this label condition: the closures of U_a and U_b , with every protected neighborhood and full product collar, are disjoint. By Lemma 6.2, the map $\widehat{F}_a$ carries U_a to itself and is the identity on a full neighborhood of the complement of its support. The same holds with a and b interchanged.

If $x \in U_a$, then $\widehat{F}_b$ fixes both x and $\widehat{F}_a(x) \in U_a$. Hence

$$(\widehat{F}_b \widehat{F}_a)(x) = \widehat{F}_a(x) = (\widehat{F}_a \widehat{F}_b)(x).$$

The same argument on U_b gives the common value $\widehat{F}_b(x)$. Outside the two interiors both maps are the identity. On an interface collar these descriptions overlap, but the supported map has the identity product germ on an open neighborhood of the collar. Thus the three branches in (9.3) agree smoothly, not merely along the interface disk.

Finally,

$$F_{a|A} = F_{a|A \cup \{b\}} = \widehat{F}_a, \quad F_{b|A} = F_{b|A \cup \{a\}} = \widehat{F}_b$$

as underlying maps. The two composites therefore equal $\Phi_{a,b}$ pointwise. Each route begins with the same completed source chart and ends with the same completed target chart. The literal equality also retains every label, framing, collar, ribbon, detector, cellulation, and spectator row carried by the two constituent edges. No comparison isotopy or homological uniqueness assertion is used. $\square$

Proposition 9.2 (Outgoing square). *With the cube-compatible endpoint parameterizations,*

$$(9.4) \quad \boxed{h_{b|A \cup \{a\}} h_{a|A} = h_{a|A \cup \{b\}} h_{b|A}}.$$

This is equality of the actual marked outgoing maps on Y , with the source row p_A and target row $p_{A \cup \{a,b\}}$ retained.

Proof. Expanding the left side from (6.17) gives

$$\begin{aligned} h_{b|A \cup \{a\}} h_{a|A} &= p_{A \cup \{a,b\}} (F_{b|A \cup \{a\}}|_{\partial_+ W}) p_{A \cup \{a\}}^{-1} p_{A \cup \{a\}} (F_{a|A}|_{\partial_+ W}) p_A^{-1} \\ &= p_{A \cup \{a,b\}} ((F_{b|A \cup \{a\}} F_{a|A})|_{\partial_+ W}) p_A^{-1}. \end{aligned}$$

The middle parameterizations cancel literally. Expanding the other route gives

$$p_{A \cup \{a,b\}} ((F_{a|A \cup \{b\}} F_{b|A})|_{\partial_+ W}) p_A^{-1}.$$

The middle maps agree by Proposition 9.1, which proves (9.4). This argument is exactly where the simultaneous, cube-compatible construction of the rows is needed: an independent postcomposition of an intermediate p_A would destroy the literal cancellation. $\square$

9.2. The weight-zero Walker square. The equality of outgoing maps does not by itself determine the Walker integers. We therefore calculate the two products separately, in the order prescribed by (2.1).

Lemma 9.3 (The two square corrections). *For the left route through the square, the actual ordered correction is*

$$(9.5) \quad \mu_Y\left((h_{a|A})_*\Lambda, \Lambda, (h_{b|A \cup \{a\}}^{-1})_*\Lambda\right) = \mu_Y(\Lambda, \Lambda, \Lambda) = 0.$$

For the right route, the separately derived correction is

$$(9.6) \quad \mu_Y\left((h_{b|A})_*\Lambda, \Lambda, (h_{a|A \cup \{b\}}^{-1})_*\Lambda\right) = \mu_Y(\Lambda, \Lambda, \Lambda) = 0.$$

Proof. Compute the two routes separately in right-factor-first order: each route places its image, seam, and inverse-image Lagrangians in its own Maslov triple, and only then does the common Lagrangian reduce that triple to a repeated pair. The right route is not obtained by permuting or reversing the left calculation; these two independent zero corrections are the input to the literal Walker square.

On the left route, the first applied cylinder is $H_{a|A}$ and the second is $H_{b|A \cup \{a\}}$. Walker multiplication therefore puts the image through the first map in the first Maslov slot, the declared seam Lagrangian in the second slot, and the inverse image through the second map in the third slot. This gives the ordered triple in (9.5) before simplification. Applying (9.1) to the two source-state-tagged edges turns it into $(\Lambda, \Lambda, \Lambda)$, whose Maslov index vanishes.

On the right route, the first applied cylinder is $H_{b|A}$ and the second is $H_{a|A \cup \{b\}}$. A new application of the multiplication formula gives the triple in (9.6). It is not obtained by permuting or formally reversing the first calculation. Equation (9.1) again makes all three entries equal, so this correction also vanishes. $\square$

Proposition 9.4 (Literal Walker square). *In the Walker endomorphism set of (Y, Λ) ,*

$$(9.7) \quad \boxed{H_{b|A \cup \{a\}} H_{a|A} = H_{a|A \cup \{b\}} H_{b|A}}.$$

Both composites have actual integer weight zero.

Proof. Every edge cylinder has weight zero by definition. Hence the two composite weights are, respectively,

$$\begin{aligned} 0 + 0 - \mu_Y\left((h_{a|A})_*\Lambda, \Lambda, (h_{b|A \cup \{a\}}^{-1})_*\Lambda\right) &= 0, \\ 0 + 0 - \mu_Y\left((h_{b|A})_*\Lambda, \Lambda, (h_{a|A \cup \{b\}}^{-1})_*\Lambda\right) &= 0 \end{aligned}$$

by Lemma 9.3. Their underlying marked outgoing maps agree by Proposition 9.2. Equality of that map together with equality of the retained integer is precisely the equality of Walker cylinders in (9.7); no projectivization or target evaluation is involved. $\square$

9.3. Finite order independence. Let $\mathfrak{S}_s$ denote the set of orders of the slide directions. For $\omega = (\omega_1, \dots, \omega_s) \in \mathfrak{S}_s$, set

$$A_q^\omega := \{\omega_1, \dots, \omega_q\}, \quad A_0^\omega := \emptyset.$$

Define the ordered total cylinder and its prefixes by

$$(9.8) \quad H_\omega := H_{\omega_s|A_{s-1}^\omega} \cdots H_{\omega_1|\emptyset}, \quad H_{\omega, \leq q} := H_{\omega_q|A_{q-1}^\omega} \cdots H_{\omega_1|\emptyset},$$

where $H_{\omega, \leq 0} := C(\text{id}_Y, 0)$. Let $\varphi_{\omega, \leq q}$ denote the underlying marked map of $H_{\omega, \leq q}$, with $\varphi_{\omega, \leq 0} = \text{id}_Y$.

Lemma 9.5 (Zero weight of every prefix). *For every $\omega \in \mathfrak{S}_s$ and $0 \leq q \leq s$,*

$$(9.9) \quad (\varphi_{\omega, \leq q})_*\Lambda = \Lambda, \quad H_{\omega, \leq q} = C(\varphi_{\omega, \leq q}, 0).$$

In particular, every prefix has Walker weight zero.

Proof. The prefix induction is typed at every stage: assume the current marked map preserves the common Lagrangian and has weight zero, then append the next edge on the left. The edge's common-Lagrangian property forces the ordered Walker triple to be a repeated pair, so the correction stays zero and the induction retains both assertions.

The assertion is immediate for the empty prefix. Suppose it holds through $q - 1$. The next edge map $h_{\omega_q|A_{q-1}^\omega}$ preserves Λ by (9.1), so its composite with $\varphi_{\omega, \leq q-1}$ does as well. Before simplification, the Walker correction for adjoining this edge on the left is

$$\mu_Y\left((\varphi_{\omega, \leq q-1})_*\Lambda, \Lambda, (h_{\omega_q|A_{q-1}^\omega}^{-1})_*\Lambda\right).$$

The induction hypothesis and (9.1) reduce this ordered triple to $(\Lambda, \Lambda, \Lambda)$. Its index is zero; the old prefix and the new edge both have weight zero. This proves (9.9) at q . $\square$

Proposition 9.6 (Order-independent total slide cylinder). *All expressions in (9.8) have one common value:*

$$(9.10) \quad \boxed{H_\omega =: H_{[s]} \quad (\omega \in \mathfrak{S}_s).}$$

This is literal equality in the Walker category, and the common cylinder has weight zero. The subscript $[s]$ records the terminal set of slide directions; it is not the handle-label set $[k]$.

Proof. An adjacent interchange is replaced by the already proved literal weight-zero Walker square. The reached vertex is the same after the two factors, so every later tagged edge is unchanged; reducing arbitrary order words by adjacent inversions then identifies all total cylinders with the reference order.

First suppose two order words differ by interchanging adjacent entries at positions $q, q + 1$. Let P be the set of the preceding $q - 1$ slide directions, and call the adjacent entries a, b . The two affected right-factor-first fragments are

$$H_{b|P \cup \{a\}} H_{a|P} \quad \text{and} \quad H_{a|P \cup \{b\}} H_{b|P}.$$

They are literally equal by Proposition 9.4. After either fragment has been applied, the reached vertex is $P \cup \{a, b\}$. Every later left factor therefore has the same source-state tag on both sides, and every earlier right factor is unchanged. Substitution gives equality of the two full Walker composites.

Any order can be reduced to $(1, \dots, s)$ by finitely many adjacent interchanges: move 1 left to the first position, then move 2 left within the remaining suffix, and continue. The inversion number decreases at every step, so the procedure terminates. Repeated substitution of the literal square equality identifies every H_ω with the reference-order composite, and hence with every other $H_{\omega'}$. Taking $q = s$ in Lemma 9.5 proves that this common value has weight zero. $\square$

The preceding proof uses adjacent replacement only as a method for proving equality. The next statement records the resulting Coxeter-shaped closure without adding swap morphisms to the construction.

Proposition 9.7 (Equality-only Coxeter-shaped closure). *For an order expression, let τ_q mean only the replacement of the entries in positions $q, q + 1$ by the opposite order, using the literal Walker equality (9.7). With 1 denoting the unchanged equality path, the following precisely ranged replacement diagrams close:*

$$(9.11) \quad \tau_q^2 = 1 \quad (1 \leq q \leq s - 1),$$

$$(9.12) \quad \tau_q \tau_r = \tau_r \tau_q \quad (1 \leq q, r \leq s - 1, |q - r| > 1),$$

$$(9.13) \quad \tau_q \tau_{q+1} \tau_q = \tau_{q+1} \tau_q \tau_{q+1} \quad (1 \leq q \leq s - 2).$$

These formulas concern paths of equality substitutions. There is no mapping cylinder τ_q , no retained swap map or comparison isotopy, and no $\mathfrak{S}_s$ -action.

Proof. Each τ_q is only a local equality substitution at the subset tag determined by the preceding entries. Involution, distant commutation, and the three-term relation follow by comparing the

resulting order words; their stated ranges handle the low-dimensional cases without introducing a swap cylinder or a symmetric-group action.

Fix an order expression. A single τ_q -step substitutes (9.7) at the prefix set determined by the entries before positions $q, q+1$.

For (9.11), the first substitution changes the adjacent substring (a, b) to (b, a) ; applying the same literal equality in the opposite direction returns to (a, b) . This uses neither an inverse slide nor a cylinder representing τ_q .

For (9.12), the two affected pairs occupy disjoint positions. Replacing one pair does not change the set of indices preceding the other pair; it changes only their order. Both routes therefore use the same two square equalities in opposite succession and end at the identical order expression.

For (9.13), it is enough to display the affected three consecutive entries. The two routes are

$$(a, b, c) \mapsto (b, a, c) \mapsto (b, c, a) \mapsto (c, b, a)$$

and

$$(a, b, c) \mapsto (a, c, b) \mapsto (c, a, b) \mapsto (c, b, a).$$

At every arrow, the entries before the affected pair determine the exact subset tag, and the two entries are distinct because the word is an order of $[s]$. Hence each arrow is an instance of (9.7), and both replacement routes have the same initial and terminal Walker expressions.

For $s = 1$, all three families are vacuous. For $s = 2$, only the involution family occurs, with $q = 1$; the distant and three-term ranges are empty. Thus the displayed ranges introduce no exceptional low-dimensional case and no action. $\square$

9.4. Telescoping to the terminal vertex.

Proposition 9.8 (Phase-zero and all-phase telescoping). *For arbitrary base-weight vectors $\mathbf{c}, \mathbf{r} \in \mathbb{Z}^k$,*

$$(9.14) \quad \boxed{H_{[s]} J_{\emptyset}^0(\mathbf{c}) = J_{[s]}^0(\mathbf{c}), \quad R_{[s]}^0(\mathbf{r}) H_{[s]} = R_{\emptyset}^0(\mathbf{r}).}$$

For every $\epsilon, \eta \in \{0, 1\}^k$ and arbitrary $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{Z}^k$,

$$(9.15) \quad \boxed{H_{[s]} J_{\emptyset}^{\epsilon}(\mathbf{c}, \mathbf{u}^+) = J_{[s]}^{\epsilon}(\mathbf{c}, \mathbf{u}^+)}$$

and

$$(9.16) \quad \boxed{R_{[s]}^{\eta}(\mathbf{r}, \mathbf{u}^-) H_{[s]} = R_{\emptyset}^{\eta}(\mathbf{r}, \mathbf{u}^-).}$$

The common weights in (9.14) are c_{Σ} and r_{Σ} . Those in (9.15) and (9.16) are $c_{\Sigma} + U^+(\epsilon)$ and $r_{\Sigma} + U^-(\eta)$, respectively. The vectors remain label-attached throughout the substitutions; the completed Walker morphisms retain only the base totals and active phase-weight sums described in Proposition 8.3.

Route replacement and telescoping. Choose one order only as a temporary route through the already constructed edge family. The phase-zero and all-phase covariance identities replace each successive edge-plus-trace composite by the trace at the next vertex, while finite order independence replaces the chosen total edge word by the terminal one. Because every replacement is already a Walker equality with its weight retained, the telescoping changes the endpoint but introduces no new seam correction.

Proof. Fix $\omega \in \mathfrak{S}_s$, abbreviate $A_q = A_q^{\omega}$, and use the prefixes from (9.8). For $1 \leq q \leq s$, the two phase-zero edge equalities of Proposition 8.2 are

$$H_{\omega_q | A_{q-1}} J_{A_{q-1}}^0(\mathbf{c}) = J_{A_q}^0(\mathbf{c}), \quad R_{A_q}^0(\mathbf{r}) H_{\omega_q | A_{q-1}} = R_{A_{q-1}}^0(\mathbf{r}).$$

Successive substitution gives

$$(9.17) \quad H_{\omega, \leq q} J_{\emptyset}^0(\mathbf{c}) = J_{A_q}^0(\mathbf{c}), \quad R_{A_q}^0(\mathbf{r}) H_{\omega, \leq q} = R_{\emptyset}^0(\mathbf{r}).$$

At $q = s$, the reached set is $[s]$, and Proposition 9.6 replaces $H_{\omega, \leq s}$ by $H_{[s]}$. This proves (9.14).

The all-phase edge identities of [Proposition 8.5](#) give, with the same source-state tags,

$$H_{\omega_q|A_{q-1}} J_{A_{q-1}}^\epsilon(\mathbf{c}, \mathbf{u}^+) = J_{A_q}^\epsilon(\mathbf{c}, \mathbf{u}^+)$$

and

$$R_{A_q}^\eta(\mathbf{r}, \mathbf{u}^-) H_{\omega_q|A_{q-1}} = R_{A_{q-1}}^\eta(\mathbf{r}, \mathbf{u}^-).$$

The identical finite substitution yields

$$H_{\omega, \leq q} J_\emptyset^\epsilon(\mathbf{c}, \mathbf{u}^+) = J_{A_q}^\epsilon(\mathbf{c}, \mathbf{u}^+)$$

and

$$R_{A_q}^\eta(\mathbf{r}, \mathbf{u}^-) H_{\omega, \leq q} = R_\emptyset^\eta(\mathbf{r}, \mathbf{u}^-).$$

Taking $q = s$ proves (9.15)–(9.16). Every step is already an equality of Walker morphisms with its integer weight retained, so telescoping creates no additional seam correction. The zero weight of $H_{[s]}$, together with the endpoint-weight calculation in [Proposition 8.3](#), gives the four weights stated above. $\square$

When $s = 1$, there is no two-face and the order statement is tautological; the telescoped identities reduce to the single-edge covariances. For $s \geq 2$, every replacement used above involves distinct directions with disjoint protected sectors. Nothing in the argument supplies a relation for shared-label or overlapping slides, a retained permutation action, or a general slide sequence outside this fixed finite family.

Assembly of the disjoint cube. The final theorem packages the construction in the order forced by its geometry. First, the protected local model gives compatible vertices and supported edge maps with a common Lagrangian. Second, the phase-zero and all-phase calculations supply the edge covariances and their Walker weights. Third, disjoint supports give literal two-faces, while prefix substitutions and finite route replacement remove order dependence. Finally, the endpoint telescoping identities assemble these local equalities into the stated cube and preserve the theorem’s equality-only scope.

Proof of Theorem 1.2. The protected-sector construction, reference interface, local-model identification, and identity extension are established in [Proposition 6.1](#) and [Lemma 6.2](#). [Propositions 7.1](#) and [7.2](#) then prove complete compatibility of the vertices and edges, the actual inverse, the signed integral symplectic formulas, and both directions of the common-Lagrangian assertion.

Ordinary phase-zero covariance is [Proposition 7.3](#); its two independently ordered Walker lifts are [Lemma 8.1](#) and [Proposition 8.2](#). The actual-vertex phase words, their fixed-site reference comparison, conjugacy preserving the complete markings, four phase-cylinder corrections, and all-phase covariance are proved in [Propositions 8.3](#) to [8.5](#).

For two distinct directions, [Propositions 9.1](#) and [9.2](#) establish the pointwise and outgoing squares before [Lemma 9.3](#) and [Proposition 9.4](#) establish the literal weight-zero Walker square. The prefix calculation, finite route replacement, equality-only closure, and endpoint telescoping are [Lemma 9.5](#) and [Propositions 9.6](#) to [9.8](#). These results prove the six clauses of the theorem without importing any several-slide assertion from the two-site companion construction. $\square$

10. THE OPTIONAL ORDINARY ROOTWISE SCALAR

Only after the source cobordism has been identified do we ask what an ordinary root-of-unity theory sees. The source topology determines the scalar calculation; the target theory is used here as a consequence, not as a substitute for the geometric coherence proof.

All results through [Section 9](#) are statements in the source category. Only in this section do we apply a target theory. By [Theorem 1.1](#), the order-independent fixed-site double has the marked source topology

$$(10.1) \quad M^{\epsilon, \eta} \cong C(\text{id}_{\Sigma_g^{\text{out}}}, N) \# \left(\#_{t \in \mathcal{M}(\epsilon, \eta)} (S^1 \times S^2, 0) \right), \quad b = |\mathcal{M}(\epsilon, \eta)|.$$

The separating spheres and the indexed match set in this formula are proof witnesses. Neither is retained as part of the resulting marked cobordism.

Fix an odd prime $p \geq 5$ and a primitive p -th root ζ_p . Set

$$(10.2) \quad A_p = -\zeta_p^{(p+1)/2}, \quad A_p^2 = \zeta_p, \quad K_p = \mathbb{Q}(\zeta_p).$$

Choose once and globally a correlated pair $D_p^{\text{GM}}, \kappa_p \neq 0$ in a finite extension $\tilde{K}_p/K_p$ such that

$$(10.3) \quad (D_p^{\text{GM}})^2 = -\frac{p}{(A_p^2 - A_p^{-2})^2}, \quad \kappa_p^2 = A_p^{-6-p(p+1)/2}.$$

This is the ordinary BHMV–Gilmer–Masbaum normalization [BHMV95, GM07, GM13]. Let Z_p denote the normalized ordinary BHMV theory and put

$$V_{p,g}^{\text{out}} = Z_p(\Sigma_g^{\text{out}}), \quad \tilde{V}_{p,g}^{\text{out}} = \tilde{K}_p \otimes_{K_p} V_{p,g}^{\text{out}}.$$

In this scalar-extended theory, increasing a Walker weight by one multiplies the TQFT map by κ_p [GM13, arXiv v3, Section 11, p. 24]. The BHMV surgery normalization gives a factor D_p^{GM} for an internal connected sum with one weight-zero $S^1 \times S^2$ [BHMV95, GM07]. The two signed choices belong to one global normalization and are never selected independently at different sites.

Corollary 10.1 (Optional ordinary scalar). *On $\tilde{V}_{p,g}^{\text{out}}$,*

$$(10.4) \quad \boxed{Z_p(M^{\epsilon,\eta}) = \kappa_p^N (D_p^{\text{GM}})^b \text{id}_{\tilde{V}_{p,g}^{\text{out}}}.$$

This is an ordinary, unrefined, rootwise, scalar-extended conclusion.

Proof. The geometric theorem first identifies the mixed double as an identity cylinder together with the actual matching $S^1 \times S^2$ summands. The ordinary functor then evaluates these two topological pieces separately: Walker weight contributes κ_p , while each genuine connected-sum factor contributes the same globally chosen D_p^{GM} . This is why the exponent b counts the geometric match set and why the conclusion remains an optional rootwise scalar rather than a refined state-level result.

Apply the normalized ordinary functor to (10.1). The identity cylinder of weight N acts by κ_p^N times the identity for every $N \in \mathbb{Z}$. The connected-sum rule contributes one copy of the same globally chosen D_p^{GM} for each of the b actual $S^1 \times S^2$ factors. Successive application therefore gives $(D_p^{\text{GM}})^b$; no sign is selected independently at a site. Multiplying the two contributions proves the formula. $\square$

Remark 10.2 (No refined exponentiation). The refined genus-one-to-two scalar factor belongs to a different, retained-label finite-coefficient theorem. No input used here constructs refined carriers in genus at least three, simultaneous refined handle maps, cross-handle finite covariance, a refined connected-sum rule, or sewing. Consequently neither (10.1) nor (10.4) implies, defines, or even types a b -fold power of such a refined factor.

11. BOUNDARY CASES, REUSABLE INTERFACES, AND EXACT NONCLAIMS

The final section records how the theorems behave at their parameter boundaries and which interfaces can safely be reused. It is a mathematical scope statement: the nonclaims below prevent the local source relations from being read as stronger global or operator-level coherence.

11.1. Quantified boundary cases.

Proposition 11.1 (Fixed-site boundary cases). *Theorem 1.1 includes $g = 1$, $k = 1$, the all-zero and all-one phase vectors, complementary phase vectors, empty phase words, and arbitrary positive, zero, or negative active weights. If $\epsilon = \eta$, then $b = k$; if $\eta = \mathbf{1} - \epsilon$, where $\mathbf{1} = (1, \dots, 1)$, then $b = 0$. For $k = 1$, the four preferred-weight entries of the local table are $(\mathcal{D}, 1, 1, \mathcal{D})$.*

Proof. These cases are direct substitutions in

$$b = \#\{t : \epsilon_t = \eta_t\}, \quad \beta_1(M^{\epsilon, \eta}) = 2g + b,$$

and in the four-entry local table. The proofs of the Walker equalities are integer identities involving only weight sums and vanishing Maslov corrections, so zero and negative active coordinates require no separate convention. $\square$

Proposition 11.2 (Slide-cube boundary cases). *Theorem 1.2 includes $g = 1$, $s = 1$, $k = 2s$, every unused spectator when $k > 2s$, the empty phase word, zero and negative base or phase weights, the reference vertex $A = \emptyset$, and the terminal vertex $A = [s]$, without an additional hypothesis.*

Proof. The slide boundary cases reduce to the already proved one-edge identities when $s = 1$. Extra spectators lie in the identity exterior, while zero or negative weights remain covered by the same integer identities. The reference and terminal vertices are constructed directly, and the edge quantifier itself explains why the terminal vertex has no outgoing edge.

The old summand is used only through $g \geq 1$. If $s = 1$, there is one slide order; the assertions involving two distinct directions are vacuous, whereas order independence and telescoping reduce to the proved one-edge identities. If $k = 2s$, the unused-spectator sum is empty. If $k > 2s$, the identity exterior fixes every extra site, its complete marking, and its homology plane pointwise.

For the zero phase assignment, the supported phase product is the weight-zero identity cylinder, so the all-phase identities reduce literally to their phase-zero forms. Every weight calculation is an identity in $\mathbb{Z}$, hence permits zero and negative coordinates. The completed construction defines the reference and terminal vertices directly. Edges are asserted only for $a \notin A$, so the terminal vertex has no outgoing edge, exactly as the quantifiers require. $\square$

11.2. Input and output interfaces. The fixed-site part of this paper ends with the completed forward and reverse traces, their literal order independence, the source-reversal formulas, and the match-count topology of their double. These outputs are used at the reference slide vertex only. A fixed-site attachment arrow is never renamed as a slide edge.

The local positive-slide input from [Sun26, Theorem 2.2 and Appendix F] ends with one constructed two-site map and its complete geometric calibration. The identification in Section 6.3 transports that map into a supplied protected sector and verifies that the incoming boundary, old region, other sectors, and unused handles lie in its identity exterior. Spectator insertion is a conclusion of that identification and of the separation hypotheses; it is not a conclusion of the imported two-site construction. That construction contributes no phase, Walker-weight, several-slide, square, or finite-order assertion.

The reusable slide output is precisely the directly completed Boolean family $\{\mathfrak{C}_A\}_{A \subseteq [s]}$, its supported positive edges and signed integral homology actions, the common Lagrangian, phase-zero and all-phase Walker covariance, literal disjoint two-faces, and the order-independent terminal cylinder with its telescoped endpoint identities. A later application may cite this result only after supplying the same complete marking and pairwise-disjoint protected-sector hypotheses.

11.3. Exact nonclaims. The two main theorems do not assert any of the following:

- (1) a negative slide or the opposite band orientation;
- (2) isotopy independence for arbitrary choices of slide band;
- (3) shared-label, overlapping, intersecting, or nested slide sectors;
- (4) a braid relation for overlapping slides;
- (5) an arbitrary slide sequence, slide groupoid, or higher coherence;
- (6) a vertex or arrow with only part of a local slide completed;
- (7) a slide cylinder of nonzero Walker weight;
- (8) birth, death, cancellation, stabilization, or Cerf relations;
- (9) a cap-adjointed functor, a cap-absorption statement for a slide edge, or any same-fixed-cap absorption beyond the fixed-site identities in Theorem 1.1;

- (10) a realization-kernel, faithfulness, fullness, or completeness theorem;
- (11) a finite refined carrier or a physical slide operator;
- (12) unrestricted Heegaard or Reidemeister–Singer descent;
- (13) a gluing, sewing, monoidal, or TQFT theorem;
- (14) a closed three-manifold invariant;
- (15) a WRT comparison; or
- (16) a comparison with $\widehat{Z}$ or with Habiro completion.

The inverse $F_{a|A}^{-1}$ is available only as the literal inverse of the constructed positive edge, with its stated source and target complete markings. It is not an arbitrary negative-band transition. Likewise, the adjacent-replacement symbols used in the slide-cube proof name substitutions of literal Walker equalities between order expressions. They are not mapping cylinders, comparison isotopies, retained swap maps, or generators of a permutation-group action.

The disjoint cube and the overlapping-slide pentagon answer different coherence questions. The present square comes from supported maps whose protected closures are disjoint. The companion theorem treats a prescribed overlapping-support configuration. Neither result, alone or together, extends the present theorem to shared labels, intersecting bands, or an unrestricted handleslide calculus.

11.4. Geometric conclusion. Within those boundaries, the paper supplies two complete source-level coherence mechanisms. At one fixed handle site, doubling reduces to a match-count calculation and a corresponding Walker-weight identity. For pairwise-disjoint slide sectors, the completed vertices form a Boolean family whose supported maps commute pointwise, whose Walker faces commute literally, and whose terminal cylinder is independent of the order of the slides. These two mechanisms are the reusable geometric output: one controls local phase choices, and the other controls disjoint global assembly without replacing the separate overlapping-slide pentagon.

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