

IMPROVED KISSING NUMBERS IN NINETEEN AND TWENTY-ONE DIMENSIONS

ABSTRACT. We prove that the kissing numbers in 19 and 21 dimensions are at least 12460 and 31560, improving the best published bounds 11948 (Ho) and 29768 (Cohn and Li). Both constructions start from the odd-sign configurations of Cohn and Li, which leave room around the sign vectors they add. We use this room to reshape the added vectors so that codewords closer than $n/4$ can be used together. In 21 dimensions each added vector has entries ± 2 on one line of the projective plane of order 4 formed by the weight-5 words of the punctured Golay code, and ± 3 elsewhere; a linear labelling of the codewords by lines gives 3840 new points, and the proof is by hand. In 19 dimensions we cut the 20-dimensional configuration by a hyperplane and project the sign vectors of the 20-dimensional code into it. The projections have a 16-coordinate head and a three-dimensional tail, the tails separate words whose heads are close, and one point for each of the 1792 heads with a nonzero tail can be added, where Ho adds 1280.

1. INTRODUCTION

The kissing problem in $\mathbb{R}^n$ asks how many unit spheres can touch a central unit sphere without overlapping [3, Ch. 1, §2.2]. In terms of the points of tangency, it asks for the largest set X of unit vectors in $\mathbb{R}^n$ with $\langle x, y \rangle \leq 1/2$ for all distinct $x, y \in X$. We write τ_n for this maximum. It is known only for $n \in \{1, 2, 3, 4, 8, 24\}$; see [2, Table 1.1] for references.

For $17 \leq n \leq 23$ the best constructions were Leech's cross sections of the Leech lattice minimal vectors [5] until Cohn and Li [2] improved dimensions 17 through 21. In dimensions 19 to 21 their construction is as follows. Take the classical configuration built from a constant-weight code [6], and put an odd number of minus signs on each codeword instead of an even number. Every sign vector of a code orthogonal to the constant-weight code then has inner product at most 6 with the old points, which have squared norm 8. The kissing condition allows $\frac{1}{2}\sqrt{8n}$, since the sign vector has squared norm n , and this is larger than 6 for $n \geq 19$. The room is small: $\frac{1}{2}\sqrt{8n}$ is about 6.16 for $n = 19$ and 6.48 for $n = 21$. So these sign vectors can be added, one per word of a code with minimum distance at least $n/4$. Cohn and Li obtained $\tau_{19} \geq 11692$ and $\tau_{21} \geq 29768$ in this way. Ho [4] found a larger code of 1280 words for dimension 19, reaching $\tau_{19} \geq 11948$.

Theorem 1.1. $\tau_{19} \geq 12460$ and $\tau_{21} \geq 31560$.

Table 1 compares these values with the best published lower bounds and the known upper bounds. The gains over the published bounds are 512 in dimension 19 and 1792 in dimension 21.

The idea. Cohn and Li add one sign vector for each word of a binary code, and two of these vectors fit together only when their words differ in at least $n/4$ places. Against the old points, however, an added vector has inner product 6 where $\frac{1}{2}\sqrt{8n}$ is allowed. We use this room to change the shape of the added vectors, so that vectors whose words are closer than $n/4$ can still fit.

2020 *Mathematics Subject Classification.* 52C17; 05B40; 94B05.

Key words and phrases. kissing number, spherical codes, Golay code.

TABLE 1. Kissing number bounds in dimensions 19 and 21. The previous lower bounds are the best published ones, as listed in Cohn’s table [1]. The upper bounds are the floors of numerical semidefinite programming values [7] and are not interval-certified.

dimension	previous lower bound	new lower bound	upper bound
19	11948 [4]	12460	24417
21	29768 [2]	31560	53524

In 21 dimensions the only close pairs are words of the punctured Golay code that differ by one of its 21 words of weight 5, which are the lines of a projective plane of order 4. We give each added vector entries ± 2 on one line and ± 3 elsewhere. Two such vectors whose words differ by a line P fit exactly when their two lines and P are three different lines, and a linear choice of lines achieves this for all 3840 words outside a subcode of size 256 (Section 3).

In 19 dimensions we cut Cohn and Li’s 20-dimensional configuration by the hyperplane orthogonal to a word T of weight 4 of the 20-dimensional code, and project the sign vectors of that code into the hyperplane. A projection has a 16-coordinate head and a three-dimensional tail. Two heads differ in at least four places, and when they differ in exactly four, their inner product is 8, while a new point has squared norm about 18 or 19 and the limit for two new points is between 9 and 9.8. So the tails may contribute at most about 1 to 1.8. This fails only for equal tails, which the choice of words avoids at head distance four, and in the tightest remaining case one head entry is lowered. This gives 1792 added vectors where Ho’s code has 1280 (Section 4).

The 21-dimensional proof is by hand. The 19-dimensional proof is by hand except for one check of the printed choice in Table 5; Section 5 says what the computer does.

2. THE ODD-SIGN CONFIGURATIONS

We identify a binary word with its support. For $c \in \mathbb{F}_2^n$ we write $\text{wt}(c)$ for its weight, $(-1)^c \in \{\pm 1\}^n$ for its sign vector, and $\lambda \circ q$ for the coordinatewise product of two vectors. We write $\mathbf{1}$ for the all-ones vector and $\mathbf{1}_T$ for the indicator vector of a set T of coordinates.

Let $\mathcal{O}$ be a set of words of length n with weight 8 and pairwise intersections of size at most 4. The *odd configuration* $B(\mathcal{O})$ consists of the $4\binom{n}{2}$ roots $\pm 2e_i \pm 2e_j$ and, for each $o \in \mathcal{O}$, the 128 vectors with entries ± 1 on o , zeros elsewhere and an odd number of minus signs [2, §2]. All have squared norm 8. We call these 128 vectors the *octad vectors* on o .

Lemma 2.1 (Cohn and Li). *Distinct $b, b' \in B(\mathcal{O})$ satisfy $\langle b, b' \rangle \leq 4$.*

Proof. Two distinct roots have inner product in $\{0, \pm 4, -8\}$. A root and an octad vector share at most two coordinates, each contributing at most 2, so their inner product is at most 4. Two vectors on the same octad have odd sign patterns, so they differ in an even positive number d of places and have inner product $8 - 2d \leq 4$. Two vectors on different octads share at most four coordinates. $\square$

The next lemma is the reason for odd signs. It is the weighted form of the argument in [2, §2].

Lemma 2.2. *Let $c \in \mathbb{F}_2^n$ meet every word of $\mathcal{O}$ in an even number of coordinates, let $q = (-1)^c$, and let $\lambda \in \mathbb{R}_{>0}^n$. Let b be an octad vector on $o \in \mathcal{O}$. Then q and b disagree in an odd number*

of coordinates of o , and

$$\langle \lambda \circ q, b \rangle \leq \sum_{j \in o} \lambda_j - 2 \min_{j \in o} \lambda_j,$$

with equality for some b on o . In particular $\langle q, b \rangle \leq 6$. For a root $b = \pm 2e_i \pm 2e_j$ the inner product is at most $2(\lambda_i + \lambda_j)$.

Proof. The sign vector q has an even number of minus signs on o , and b has an odd number. So they disagree in an odd number of coordinates of o , in particular in at least one, and the bound follows. Equality holds for the odd pattern that agrees with q except at one coordinate where λ is smallest. $\square$

For nonzero x, y the angle between them is at least 60° if and only if

$$(2.1) \quad \langle x, y \rangle \leq 0 \quad \text{or} \quad 4\langle x, y \rangle^2 \leq |x|^2 |y|^2.$$

We call such a pair *compatible*. If every pair from a finite set of nonzero vectors is compatible, then the vectors normalized to unit length form a kissing configuration, and distinct vectors give distinct points because (2.1) excludes cosine 1. We often check compatibility in the form $\langle x, y \rangle \leq \frac{1}{2}|x||y|$.

3. DIMENSION 21

The codes. Coordinates are $1, \dots, 24$, and the construction lives on the first 21. Let $\mathcal{G}$ be the extended binary Golay code: the cyclic code of length 23 generated by $1+t+t^5+t^6+t^7+t^9+t^{11}$, with the coefficient of t^k at coordinate $k+1$, extended by an overall parity bit at coordinate 24 [3, Ch. 3, §2.8]. Its nonzero weights are 8, 12, 16 and 24, two octads meet in 0, 2 or 4 coordinates, and every three coordinates lie in exactly 21 octads [3, Ch. 10, §2]. Let $Q = \{22, 23, 24\}$ and let $\mathcal{O}$ be the 210 octads of $\mathcal{G}$ disjoint from Q . Then $B = B(\mathcal{O})$ is Cohn and Li's 27720-point configuration in $\mathbb{R}^{21}$.

Let D be $\mathcal{G}$ punctured to the first 21 coordinates, a $[21, 12, 5]$ code. Puncturing is injective because $\mathcal{G}$ has minimum distance $8 > 3$, so D has 4096 words. Because $\mathcal{G}$ is self-dual and the octads in $\mathcal{O}$ vanish on Q , every word of D meets every octad of $\mathcal{O}$ evenly. A word $d \in D$ comes from a unique $g \in \mathcal{G}$, and $\text{wt}(d) = \text{wt}(g) - |g \cap Q|$ with $|g \cap Q| \leq 3$. So the words of weight 5, 6 and 7 come from octads g with $|g \cap Q| = 3, 2$ and 1 , and all other nonzero words have weight at least 8.

We call the 21 words of weight 5 in D *lines*. They are the octads through Q with Q removed. Two of these octads share the three points of Q , so they meet in exactly 4 coordinates, and two lines meet in exactly one coordinate. Through each coordinate j pass five lines, from the five octads containing the four points of $Q \cup \{j\}$, so the lines form a projective plane of order 4 on the 21 coordinates. For a line L we write $\hat{L} = L \cup Q$ for its octad.

Two sign vectors of distinct words of D at distance d have cosine $1 - 2d/21$, which exceeds $1/2$ only for $d = 5$, where it is $11/21$. Cohn and Li add 2048 sign vectors, from a subcode of index 2 in D with minimum distance 6 [2, §2]. To use more words of D we must allow pairs of words that differ by a line.

Lemma 3.1. *Let L be a line and $d \in D$ a nonzero word, and put $a = |d \cap L|$.*

- (1) *Every octad of $\mathcal{O}$ meets L in 0, 2 or 4 coordinates.*
- (2) *If $\text{wt}(d) = 5$, then d is a line, and $a = 5$ if $d = L$ and $a = 1$ otherwise.*
- (3) *If $\text{wt}(d) = 6$, then $a \leq 2$. If $\text{wt}(d) = 7$, then $a \leq 3$.*

Proof. (1) An octad $o \in \mathcal{O}$ is disjoint from Q , so $o \cap L = o \cap \hat{L}$, and two distinct octads meet in 0, 2 or 4 coordinates. (2) is the definition of lines and the fact that two lines meet in one coordinate. (3) Let d come from g . If $\text{wt}(d) = 6$, then g is an octad with $|g \cap Q| = 2$, so $g \neq \hat{L}$ and $|g \cap \hat{L}| \in \{2, 4\}$. Since $g \cap \hat{L}$ contains the two points of $g \cap Q$, $a = |g \cap \hat{L}| - 2 \leq 2$. If $\text{wt}(d) = 7$, the same argument with $|g \cap Q| = 1$ gives $a \leq 4 - 1 = 3$. $\square$

Line-weighted sign vectors. For a line L let $\lambda_L \in \{2, 3\}^{21}$ have entries 2 on L and 3 elsewhere, so $|\lambda_L|^2 = 5 \cdot 4 + 16 \cdot 9 = 164$. For $w \in D$ and a line L put

$$x(w, L) = \lambda_L \circ (-1)^w.$$

Here is the reason for the weights. If two words differ by a line P , their sign vectors disagree on the 5 coordinates of P and agree on the other 16. With all entries equal to 3 the inner product would be $9(16 - 5) = 99$, above the limit $\frac{1}{2} \cdot 9 \cdot 21 = 94.5$. Lowering an entry from 3 to 2 at a coordinate where the signs agree removes 3 from the inner product and only 5 from each squared norm. So each vector is lightened on a line, and two vectors whose words differ by P fit when their lines avoid P and differ from each other; Lemma 3.3 makes this exact. With entries a on the line and b elsewhere, the base requires $a/b \geq \sqrt{2/5}$ (the case of Lemma 3.2) and the pairs with three distinct lines not through one point require $a/b \leq 2/3$, so the weights 2 and 3 sit at the end of the window, where these pairs touch.

Lemma 3.2. *For every $w \in D$ and every line L , the point $x(w, L)$ is compatible with every point of B . The largest inner product is 18, against an octad vector, and the largest cosine is $18/\sqrt{1312} \approx 0.49694$.*

Proof. By Lemma 2.2, the inner product with an octad vector on o is at most $\sum_{j \in o} \lambda_j - 2 \min_{j \in o} \lambda_j$. By Lemma 3.1(1), o meets L in 0, 2 or 4 coordinates, and the bound is $24 - 6 = 18$, $(4 + 18) - 4 = 18$ or $(8 + 12) - 4 = 16$. Against a root the inner product is at most $2(3 + 3) = 12$. Both are compatible, since $4 \cdot 18^2 = 1296 \leq 1312 = 164 \cdot 8$. $\square$

Lemma 3.3. *Let $w \neq w'$ be words of D , let L, L' be lines, and put $d = w + w'$. If d is a line and L, L', d are not three distinct lines, then $\langle x(w, L), x(w', L') \rangle > 82 = \frac{1}{2} |\lambda_L|^2$. Otherwise the inner product is at most 82. It equals 80 when d is a line and L, L', d are three distinct lines through one coordinate, and 82 when they are three distinct lines not through one coordinate.*

Proof. Write $\sigma = (-1)^d$, $\delta = \text{wt}(d)$, $a = |d \cap L|$ and $a' = |d \cap L'|$, and let e_L be the indicator vector of L . Since $\lambda_L = 3 \cdot \mathbf{1} - e_L$, the inner product is

$$\sum_j (9 - 3e_L - 3e_{L'} + e_L e_{L'})_j \sigma_j = 9 \sum_j \sigma_j - 3 \sum_{j \in L} \sigma_j - 3 \sum_{j \in L'} \sigma_j + \sum_{j \in L \cap L'} \sigma_j.$$

The first three sums are $21 - 2\delta$, $5 - 2a$ and $5 - 2a'$. So

$$(3.1) \quad \langle x(w, L), x(w', L') \rangle = 159 - 18\delta + 6(a + a') + \sum_{j \in L \cap L'} \sigma_j.$$

The last sum is $5 - 2a$ if $L = L'$. If $L \neq L'$, the lines meet in one coordinate p , and the sum is -1 if $p \in d$ and 1 otherwise.

Case $\delta \geq 6$. Then d is not a line, and we show the inner product is at most 82. By Lemma 3.1, $a, a' \leq a_{\max}$, where $a_{\max} = 2, 3, 5$ for $\delta = 6, \delta = 7$ and $\delta \geq 8$. If $L = L'$, (3.1) reads $164 - 18\delta + 10a$. If $L \neq L'$, it is at most $160 - 18\delta + 6(a + a')$. In both cases the inner product is at most

$$164 - 18\delta + 12a_{\max} = 80, 74, 80 \quad \text{for } \delta = 6, 7, 8,$$

and smaller for $\delta > 8$. All are at most 82.

Case $\delta = 5$. Then d is a line P , and by Lemma 3.1(2), $a = 5$ if $P = L$ and $a = 1$ otherwise, and likewise for a' . We evaluate (3.1).

- $L = L' = P$: $a = 5$, and the inner product is $164 - 90 + 50 = 124$.
- $L = L' \neq P$: $a = 1$, and the inner product is $164 - 90 + 10 = 84$.
- $L \neq L'$ and $P = L$: $a = 5$, $a' = 1$, and $p = L \cap L'$ lies on P , so the last sum is -1 . The inner product is $159 - 90 + 36 - 1 = 104$. The case $P = L'$ is the same.
- L, L', P distinct: $a = a' = 1$, and the inner product is $81 + \sigma_p$, which is 80 if $p \in P$, that is, if the three lines pass through one coordinate, and 82 otherwise.

The first three cases are exactly those where L, L', P are not three distinct lines, and there the inner product exceeds 82. In the last case it is at most 82. $\square$

So a set of points $x(w, L(w))$, one for each w in a set $W \subseteq D$, is pairwise compatible if and only if

$$(3.2) \quad w, w + P \in W \text{ for a line } P \implies L(w), L(w + P), P \text{ are three distinct lines.}$$

In particular $w + L(w) \notin W$ for every $w \in W$. Two points on the same word are never compatible: by (3.1) with $\delta = 0$ their inner product is 164 if $L = L'$ and 160 otherwise, both above 82.

The configuration. The line rule is satisfied by a coset construction. Suppose $K \subseteq D$ is a linear subcode such that no line lies in K and every nonzero coset of K in D contains a line. Choose one line $f(\kappa)$ in each nonzero coset κ , and put

$$W = D \setminus K, \quad L(w) = f(w + K) \quad (w \in W).$$

Then $w + L(w) \in K$ for every $w \in W$. Since there are only 21 lines, K has at most 21 nonzero cosets, so its codimension k satisfies $2^k - 1 \leq 21$, that is, $k \leq 4$; and $|W| = 4096(1 - 2^{-k})$ is largest for $k = 4$, where $|W| = 3840$.

Lemma 3.4. *There is a subcode $K \subseteq D$ of dimension 8 that contains no line and meets every nonzero coset of K in a line. One such K is the kernel of the linear map $\eta: D \rightarrow \mathbb{F}_2^4$,*

$$\eta(w) = (w_3 + w_4, \quad w_2 + w_{11}, \quad w_2 + w_5 + w_6 + w_{10}, \quad w_1 + w_3 + w_8 + w_{14}),$$

where $w_j \in \mathbb{F}_2$ is the j th coordinate of w .

Proof. Table 2 lists the 21 lines and their values $\eta(P)$. Every line has $\eta(P) \neq 0$, so no line lies in $\ker \eta$, and every nonzero value of $\mathbb{F}_2^4$ occurs, so η is onto, $\ker \eta$ has dimension 8, and each of its 15 nonzero cosets contains a line. $\square$

TABLE 2. The 21 lines P of D and the values $\eta(P)$, written as four bits. The 15 lines marked with an asterisk are the chosen lines $f(\kappa)$.

line	η	line	η	line	η
$\{4, 5, 6, 8, 10\}$	1011*	$\{2, 3, 6, 7, 15\}$	1101*	$\{1, 4, 9, 14, 15\}$	1000*
$\{3, 9, 10, 11, 16\}$	1111*	$\{1, 6, 12, 13, 16\}$	0011*	$\{7, 8, 9, 12, 17\}$	0001*
$\{4, 7, 11, 13, 18\}$	1100*	$\{2, 10, 12, 14, 18\}$	0101*	$\{1, 3, 5, 17, 18\}$	1010*
$\{2, 5, 9, 13, 19\}$	0100*	$\{6, 11, 14, 17, 19\}$	0111*	$\{8, 15, 16, 18, 19\}$	0001
$\{1, 2, 8, 11, 20\}$	0010*	$\{5, 7, 14, 16, 20\}$	0011	$\{10, 13, 15, 17, 20\}$	0010
$\{3, 4, 12, 19, 20\}$	0001	$\{3, 8, 13, 14, 21\}$	1001*	$\{5, 11, 12, 15, 21\}$	0110*
$\{2, 4, 16, 17, 21\}$	1110*	$\{1, 7, 10, 19, 21\}$	0011	$\{6, 9, 18, 20, 21\}$	0010

Let $K = \ker \eta$ and let $f(\kappa)$ be the line marked with an asterisk in Table 2 whose value η is κ . The configuration is all of B , the 27720 points of Cohn and Li, together with the 3840 vectors $x(w, L(w))$, $w \in W = D \setminus K$: integer vectors with entries ± 2 on the line $L(w)$, ± 3 elsewhere, and signs given by w . The total is $27720 + 3840 = 31560$.

Proposition 3.5. *These 31560 vectors are pairwise compatible, nonzero and span $\mathbb{R}^{21}$. Hence $\tau_{21} \geq 31560$.*

Proof. Base pairs are compatible by Lemma 2.1, and base-new pairs by Lemma 3.2. For new pairs we check the line rule (3.2). Let w and $w' = w + P$ be in W for a line P . Then $L(w) \neq P$, since $L(w) = P$ would give $w' = w + L(w) \in K$; likewise $L(w') \neq P$. And $L(w) \neq L(w')$: since $P \notin K$, the words w and w' lie in different cosets of K , and a line lies in only one coset. So the three lines are distinct, and Lemma 3.3 gives compatibility. The base contains all 840 roots, which span $\mathbb{R}^{21}$. $\square$

This proves Theorem 1.1 for $n = 21$. The proof uses no computer calculation beyond listing the 21 lines of Table 2 and their 84 parities.

Remark 3.6. The line rule bounds this method. Each $w \in W$ has the neighbour $w + L(w)$ outside W , and each word outside W is $w + P$ for at most 21 pairs (w, P) . The relation $w \sim w + P$ preserves the cosets of the span of the lines, which is a $[21, 10]$ code (a finite computation), so D splits into four classes of 1024 words, and in each class at most $\lfloor 1024 \cdot 21/22 \rfloor = 977$ words lie in W . So line-weighted sign vectors with any labelling give at most $4 \cdot 977 = 3908$ new points.

4. DIMENSION 19

Here is the idea. Cohn and Li's 20-dimensional configuration B_{20} is the odd configuration on the 130 octads of a shortened Golay code; the orthogonal code E , the Golay code punctured at four coordinates, has 4096 words, among them five disjoint words of weight 4, the *tetrads*. We fix a tetrad T and cut B_{20} by the hyperplane $\mathbf{1}_T^\perp$; the 10668 points of B_{20} in it are Cohn and Li's 19-dimensional base. A sign vector $(-1)^c$, $c \in E$, projected into $\mathbf{1}_T^\perp$ has a 16-coordinate *head* on the coordinates outside T and a three-dimensional *tail*. The words c and $c + T$ have the same head and opposite tails, and apart from this two heads differ in at least four coordinates; keeping one word of each pair $\{c, c + T\}$ with nonzero tail gives 1792 heads, one point each. Two heads at distance 4 have inner product 8, and the limit for two new points is between 9 and 9.8, so the tails may add at most about 1 to 1.8. The rest of the section arranges this: the tails come in two shapes, each needs one adjustment, and the word kept from each pair is chosen so that no two kept heads at distance 4 have equal tails.

The codes. Coordinates are $1, \dots, 20$. Let $E \subseteq \mathbb{F}_2^{20}$ be the span of the twelve words

$$\begin{aligned} &\{1, 8, 9, 12, 16, 17, 18, 19\}, \{2, 10, 11, 14, 15, 17, 18, 20\}, \{3, 7, 9, 13, 15, 16, 17, 19\}, \\ &\{4, 7, 8, 10, 12, 15, 16, 19\}, \{5, 10, 12, 13, 15, 16, 17, 18\}, \{6, 7, 8, 9, 10, 13, 16, 18\}, \\ &\{1, 4, 7, 9\}, \{1, 5, 6, 18, 20\}, \{1, 3, 12, 15, 20\}, \{1, 10, 13, 19, 20\}, \\ &\{1, 3, 5, 6, 7, 13, 14, 15, 18, 20\}, \{2, 4, 6, 7, 8, 13, 14, 16, 17, 18\}. \end{aligned}$$

On the first 19 coordinates these are Ho's twelve generators [4, §2], in his order. Our coordinate 20 is Ho's coordinate 24, the last column of his extension matrix, so we have added it to the words where that column has a 1. Ho shows that the full extension to length 24 is the extended binary Golay code [4, Prop. 2]. So E is the Golay code punctured at Ho's coordinates 20 to 23, and its dual $E^\perp$ is the Golay code shortened at the same four coordinates [8]. The 130

words of weight 8 in $E^\perp$ are the *octads*. They form the shortened Steiner system $S(5, 8, 24)$ with $n = 20$ of [2, Table 2.1], so $B_{20} = B(\text{octads})$ is Cohn and Li's 17400-point configuration in $\mathbb{R}^{20}$.

The five octads of the Golay code that contain the four punctured coordinates partition the other twenty coordinates into five tetrads [3, Ch. 10], and puncturing turns them into five words of weight 4 in E , the *tetrads*. Let $T = \{1, 4, 7, 9\}$, one of them, and let H be the other 16 coordinates, the *head*. For $c \in E$ we write c_T and c_H for its parts on T and H , and likewise v_T, v_H for a vector $v \in \mathbb{R}^{20}$.

Lemma 4.1. *The following hold.*

- (1) *The five tetrads are pairwise disjoint and cover all 20 coordinates.*
- (2) *Two distinct octads meet in at most 4 coordinates. The octads meet T in 0, 2 or 4 coordinates, for 30, 96 and 4 octads respectively.*
- (3) *For every $d \in E$ other than 0 and T , $|d_H|$ is even and at least 4. The words with $|d_H| = 4$ have $|d_T| = 0, 1, 2, 3, 4$ for 4, 64, 144, 64 and 4 words respectively, and those with $d_T = \emptyset$ are the four tetrads other than T .*

Proof. Let S be the set of Ho's four punctured coordinates, so that $\Omega = S \cup T$ is an octad of the Golay code, and let $d \in E$ come from the Golay word g . We use the standard facts [3, Ch. 10, §2]: every 4-set lies in exactly 5 octads; two octads meet in 0, 2 or 4 points; a dodecad meets every octad in 2, 4 or 6 points; and for a fixed octad, exactly 30 octads are disjoint from it and exactly 16 meet it in a given pair of its points.

(1) The five octads through S meet pairwise in exactly S , so the tetrads are disjoint, and they cover the 20 coordinates because every point outside S lies in an octad through S .

(2) The octads of $E^\perp$ are the octads of the Golay code disjoint from S , so two of them meet in at most 4 points, and such an octad o meets T in $o \cap \Omega$, which has 0, 2 or 4 points. There are 30 octads disjoint from Ω ; the 4 octads through T other than Ω meet Ω exactly in T ; and for each of the 6 pairs of T there are 16 octads meeting Ω exactly in that pair, which gives 96.

(3) $|d_H| = \text{wt}(g) - |g \cap \Omega|$. An octad g meets Ω in 0, 2, 4 or 8 points, giving $|d_H| = 8, 6, 4, 0$, and 0 only for $g = \Omega$, that is, $d = T$. A dodecad gives $|d_H| \in \{10, 8, 6\}$. A word of weight 16 is the complement of an octad o and meets Ω in $8 - |o \cap \Omega| \in \{8, 6, 4, 0\}$ points, giving $|d_H| \in \{8, 10, 12, 16\}$, and the all-ones word gives 16. So $|d_H|$ is even and at least 4 for $d \neq 0, T$, with $|d_H| = 4$ exactly when g is an octad meeting Ω in a 4-set A . Each of the 70 subsets A lies in 4 octads other than Ω , and for them $d_T = A \cap T$. The number of A with $|A \cap T| = j$ is $\binom{4}{j} \binom{4}{4-j}$, which gives $4 \cdot 1, 4 \cdot 16, 4 \cdot 36, 4 \cdot 16$ and $4 \cdot 1$ words. For $j = 0$, $A = S$, and the four octads through S other than Ω give the other four tetrads. $\square$

By (3), two words of E have the same head exactly when they are equal or differ by T . Let U be the span of the five tetrads, a code of 32 words, and let $U_0 \subseteq U$ be the 16 sums of an even number of tetrads, so that $U = U_0 \cup (T + U_0)$ and U_0 is spanned by the four words $u + T$, u a tetrad other than T .

The body. *The isometry.* Let

$$R = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

acting on the coordinates $(1, 4, 7, 9)$ of T . The rows of $R/2$ are an orthonormal basis of $\mathbf{1}_T^\perp \cap \mathbb{R}^T$, so $(R/2)^\top (R/2) = I - J/4$ with J the all-ones matrix, and the map $\varphi(v) = (v_H, Rv_T/2) \in$

$\mathbb{R}^{16} \times \mathbb{R}^3$ satisfies

$$(4.1) \quad \langle \varphi(a), \varphi(v) \rangle = \langle a, v \rangle - \frac{1}{4} \left(\sum_{j \in T} a_j \right) \left(\sum_{j \in T} v_j \right).$$

It is an isometry from the hyperplane $\mathbf{1}_T^\perp$ onto $\mathbb{R}^{19}$. Any orthonormal basis of $\mathbf{1}_T^\perp \cap \mathbb{R}^T$ would do; this one keeps the entries simple. We call the first 16 coordinates of $\mathbb{R}^{19}$ the *head* and the last three the *tail*, index the head coordinates by H with unit vectors e_j , and write f_1, f_2, f_3 for the standard basis of the tail. For distinct $i, l \in T$ the vector $R(e_i - e_l)/2$ has two entries ± 1 and one entry 0.

The section. There are 10668 points of B_{20} orthogonal to $\mathbf{1}_T$, which is the size of Cohn and Li's 19-dimensional base [2, Table 2.1]. They are of five kinds, listed in Table 3: the 480 roots inside H ; the 12 roots $\pm 2(e_i - e_l)$ with $i, l \in T$; the 3840 octad vectors on the 30 octads disjoint from T ; the 6144 octad vectors on the 96 octads meeting T twice whose two T -entries have opposite signs; and the 192 octad vectors on the four octads containing T with two plus and two minus signs on T . In $\mathbb{R}^{19}$ the first four kinds are head roots; tail roots $\varphi(\pm 2(e_i - e_l))$, with zero head and tail entries $0, \pm 2, \pm 2$; octad vectors with eight head entries ± 1 and zero tail; and vectors with six head entries ± 1 and tail $Rb_T/2$ with entries $0, \pm 1, \pm 1$. The fifth kind has four head entries ± 1 , on a tetrad other than T , and tail $\pm 2f_k$.

The body. The body is the image $\varphi(B_{20} \cap \mathbf{1}_T^\perp)$ of the whole section: 10668 points of squared norm 8. Distinct body points have inner product at most 4, by Lemma 2.1 and (4.1).

The new points. Let C be the set of the 3584 words $c \in E$ with $|c_T| \in \{1, 2, 3\}$. The other 512 words, with $c_T = \emptyset$ or $c_T = T$, are left out: their projections have zero tail and squared norm 16, and inner product 6 with an octad vector on an octad disjoint from T , above $\frac{1}{2}\sqrt{8 \cdot 16} \approx 5.66$. For $c \in C$ let $q = (-1)^c$ and let $t_c = Rq_T/2 \in \mathbb{R}^3$ be the tail of the projected sign vector, so that φ maps the projection of q into $\mathbf{1}_T^\perp$ to (q_H, t_c) . We call c an *axis word* if $|c_T| = 2$; then $\sum_T q_j = 0$ and $t_c = \pm 2f_k$ for some k . We call c a *diagonal word* if $|c_T|$ is odd; then $\sum_T q_j = \pm 2$ and $t_c \in \{\pm 1\}^3$. There are 1536 axis and 2048 diagonal words. Since $T \in E$, the words c and $c + T$ are both in C , of the same kind, with the same head and opposite tails, $t_{c+T} = -t_c$. By Lemma 4.1(3) the 3584 words form 1792 such pairs, one for each head. Two words of the same kind have $t_c = t_{c'}$ exactly when $c + c'$ vanishes on T : the vector $q_T - q'_T$ is zero off d_T and ± 2 on d_T , and it is killed by R only if it is a multiple of $\mathbf{1}_T$, which for words of C means $d_T = \emptyset$.

Let $W \subseteq C$ contain exactly one word of each pair $\{c, c + T\}$, and give each axis word $c \in W$ a head coordinate $m(c) \in H$, its *mark*. Fix $\mu > 0$. For an axis word $c \in W$ let

$$x_c = (\lambda_c \circ q_H, t_c), \quad \lambda_c \in \mathbb{R}^H \text{ with entries } 1, \text{ except } \mu \text{ at } m(c), \quad |x_c|^2 = 19 + \mu^2,$$

the projected sign vector with one head entry lowered to μ . For a diagonal word $c \in W$ let

$$y_c = (q_H, \sqrt{2/3} t_c), \quad |y_c|^2 = 16 + 2 = 18,$$

the projected sign vector with its tail shortened from squared length 3 to 2, the length at which the octad vectors touch it. The new points are the 768 axis points x_c and the 1024 diagonal points y_c , $c \in W$. Their heads are pairwise different. The total is $10668 + 1792 = 12460$. The choice of W and the marks comes at the end of this section. We take

$$\mu = \frac{373}{500};$$

the proofs below work for every μ with $\sqrt{60} - 7 \leq \mu \leq \mu_{\max}$, where $\mu_{\max} \approx 0.7492$ is the smaller root of $(7 + \mu + 2\sqrt{2/3})^2 = \frac{9}{2}(19 + \mu^2)$. The lower end comes from the body (Lemma 4.2)

and the upper end from the diagonal points (Lemma 4.3). The interval is narrow, about $[0.7460, 0.7492]$, and $373/500$ is a rational point inside it.

TABLE 3. The blocks of the 19-dimensional configuration. The body blocks are described by their preimages under φ .

block	squared norm	size
roots in H	8	480
roots in T	8	12
octad vectors, octad disjoint from T	8	3840
octad vectors, octad meeting T twice	8	6144
octad vectors, octad containing T	8	192
axis points x_c	$19 + \mu^2$	768
diagonal points y_c	18	1024
total		12460

Inner products. For an axis word c let $v_c \in \mathbb{R}^{20}$ have $(v_c)_H = \lambda_c \circ q_H$ and $(v_c)_T = q_T$. Then $v_c \perp \mathbf{1}_T$ and $x_c = \varphi(v_c)$, so for $b \in \mathbf{1}_T^\perp$, by (4.1),

$$(4.2) \quad \langle x_c, \varphi(b) \rangle = \langle v_c, b \rangle.$$

For a diagonal word c and $b \in \mathbf{1}_T^\perp$, the tails satisfy $\langle Rq_T/2, Rb_T/2 \rangle = \langle q_T, b_T \rangle$, again by $(R/2)^\top(R/2) = I - J/4$ and $\sum_T b_j = 0$. So

$$(4.3) \quad \langle y_c, \varphi(b) \rangle = \langle q_H, b_H \rangle + \sqrt{2/3} \langle q_T, b_T \rangle.$$

Lemma 4.2. *Every body point is compatible with every new point. Against an axis point the inner product is at most $7 - \mu$, and $(7 - \mu)^2 \leq 2(19 + \mu^2)$ because $\mu \geq \sqrt{60} - 7$. Against a diagonal point it is at most $6 = \frac{1}{2}\sqrt{8 \cdot 18}$.*

Proof. Let $b \in B_{20}$ with $b \perp \mathbf{1}_T$. *Axis points.* By (4.2) and Lemma 2.2, since $c \in E$ meets every octad evenly, $\langle x_c, \varphi(b) \rangle = \langle v_c, b \rangle$ is at most $\sum_{j \in o} \lambda_j - 2 \min_{j \in o} \lambda_j$ for an octad vector on o , which is $7 + \mu - 2\mu = 7 - \mu$ if $m(c) \in o$ and $8 - 2 = 6$ otherwise, and at most 4 for a root. All are at most $7 - \mu$, and $(7 - \mu)^2 \leq 2(19 + \mu^2)$ is $\mu^2 + 14\mu - 11 \geq 0$.

Diagonal points. Put $h = \langle q_H, b_H \rangle$ and $s = \langle q_T, b_T \rangle$, so that (4.3) reads $\langle y_c, \varphi(b) \rangle = h + \sqrt{2/3}s$, and $h + s = \langle q, b \rangle \leq 6$ by Lemma 2.2. If $s \geq 0$ the inner product is at most $h + s \leq 6$. If $s < 0$ it is less than h , and $h \leq 6$ because b then meets T and has at most six nonzero head coordinates. $\square$

Now two new points. Heads at distance 4 have inner product 8, and the thresholds in the last column of Table 4 lie between 9 and 9.8, so everything depends on the tails and the light entries. Let $c \neq c'$ be words of W and $d = c + c'$. Since W contains one word of each pair $\{c, c + T\}$, $d \neq T$, so $|d_H|$ is even and at least 4 by Lemma 4.1(3), and $\langle q_H, q'_H \rangle = 16 - 2|d_H|$. The light entries change the head product by at most $2(1 - \mu)$: by exactly $\pm(1 - \mu)$ for an axis-diagonal pair, with $+$ when $m(c) \in d$, and by at most $1 - \mu^2$ or $2(1 - \mu)$ for two axis points with equal or different marks. For the tails, $\langle t_c, t_{c'} \rangle$ is 4, 0 or -4 for two axis words (4 exactly when $d_T = \emptyset$), 3, 1, -1 , -3 for two diagonal words (3 exactly when $d_T = \emptyset$), and $4 - 2|d_T| = \pm 2$ for an axis and a diagonal word, since then $|d_T|$ is odd and the computation behind (4.3) gives $\langle t_c, t_{c'} \rangle = \langle q_T, q'_T \rangle$ when $\sum_T q_j = 0$. Table 4 collects the resulting bounds.

Lemma 4.3. *Let $c \neq c'$ be words of W and $d = c + c'$.*

TABLE 4. Inner products of two new points, by the kind of pair and the difference $d = c + c'$, with $\nu = 1 - \mu \approx 0.254$ and $\rho = 2\sqrt{2/3} \approx 1.633$. The last column is $\frac{1}{2}$ times the product of the norms. A case marked “fails” exceeds the threshold. For an axis and a diagonal word with $|d_H| = 4$ the value is $8 + \nu - \rho$ if $|d_T| = 3$, and $8 - \nu + \rho$ or $8 + \nu + \rho$ if $|d_T| = 1$, according as the mark lies outside or inside d .

pair	$ d_H \geq 6$	$ d_H = 4, d_T = \emptyset$	$ d_H = 4, d_T \neq \emptyset$	$\frac{1}{2} x y $
diagonal, diagonal	$\leq 4 + 2 = 6$	$8 + 2 = 10$, fails	$\leq 8 + \frac{2}{3}$	9
axis, axis	$\leq 4 + 2\nu + 4$	$\geq 8 - 2\nu + 4$, fails	$\leq 8 + 2\nu$	$\frac{1}{2}(19 + \mu^2) \approx 9.78$
axis, diagonal	$\leq 4 + \nu + \rho$	none	$\leq 8 + \nu - \rho$ or $8 \mp \nu + \rho$	$\frac{1}{2}\sqrt{18(19 + \mu^2)} \approx 9.38$

- (1) If c, c' are both diagonal or both axis words, their points are compatible unless d is a tetrad other than T , and then they are not.
- (2) If c is an axis word and c' a diagonal word, their points are compatible unless $|d_H| = 4$, $|d_T| = 1$ and $m(c) \in d$, and then they are not.

Proof. (1) is the first two rows of Table 4, where $\nu = 1 - \mu$: the only case above the threshold is $|d_H| = 4, d_T = \emptyset$, and by Lemma 4.1(3) this is the case where d is a tetrad other than T . (2) For $|d_H| \geq 6$ the table applies. For $|d_H| = 4$ and $|d_T| = 3$ the tail product is -2 , and the inner product is at most $8 + (1 - \mu) - 2\sqrt{2/3} < 8$. For $|d_H| = 4$ and $|d_T| = 1$ the tail product is $+2$, and the inner product is $8 - (1 - \mu) + 2\sqrt{2/3} = 7 + \mu + 2\sqrt{2/3}$ if $m(c) \notin d$ and $9 - \mu + 2\sqrt{2/3}$ if $m(c) \in d$. The first is at most $\frac{1}{2}\sqrt{18(19 + \mu^2)}$ exactly when $\mu \leq \mu_{\max}$; for $\mu = 373/500$ this is the exact inequality $4(11619 + 1000\sqrt{6})^2 < 162 \cdot 4889129$, where both sides are scaled by 1500^2 , which holds because $-228034254 + 92952000\sqrt{6} < 0$. The second exceeds 9.8, while the threshold is below 9.4. $\square$

Proposition 4.4. *Let $W \subseteq C$ contain one word of each pair $\{c, c+T\}$, and let each axis word of W have a mark. The body and the new points form a kissing configuration if and only if*

- (W1) *no two words of W differ by a tetrad other than T ; equivalently, W is a union of cosets of U_0 , and so contains exactly one of the two cosets of U_0 in each coset of U in C ; and*
- (W2) *for every axis word $c \in W$, the mark $m(c)$ lies outside d for every $d \in E$ with $|d_H| = 4$, $|d_T| = 1$ and $c + d \in W$.*

Proof. Body pairs are compatible by Lemma 2.1 and (4.1), body-new pairs by Lemma 4.2, and new pairs exactly under (W1) and (W2) by Lemma 4.3. For the equivalence in (W1): if $c \in W$ and u is a tetrad other than T , then $c + u \notin W$ is the same as $c + u + T \in W$, because W meets the pair $\{c + u, c + u + T\}$ once. So (W1) says that W is closed under adding the words $u + T$, which span U_0 . The cosets of U inside C consist of words of one kind, and each splits into the two cosets $c + U_0$ and $c + T + U_0$ of U_0 , exchanged by adding T . $\square$

The choice of W . There are $3584/32 = 112$ cosets of U in C : 48 of axis words and 64 of diagonal words. By (W1), W is fixed by choosing one coset of U_0 in each of them, 112 binary choices. Condition (W2) is about the marks. For an axis word $c \in W$ let $\Delta(c)$ be the set of words $d \in E$ with $|d_H| = 4$, $|d_T| = 1$ and $c + d \in W$, and let

$$F(c) = H \setminus \bigcup_{d \in \Delta(c)} d$$

be the set of *free* head coordinates of c . By Lemma 4.1(3) there are 64 words d with $|d_H| = 4$ and $|d_T| = 1$; for each of them $c + d$ is a diagonal word, and exactly one of $c + d$ and $c + d + T$ lies in W . Condition (W2) says $m(c) \in F(c)$. So a choice of the 112 cosets works if and only if every axis word of W has a free coordinate.

Let W be the union of the 112 cosets $c + U_0$, for the 112 words c of Table 5. There a word is written as the integer $\sum_{j \in c} 2^{j-1}$, and each listed word is the smallest word of W in its coset of U ; recall that U_0 consists of the 16 sums of an even number of the tetrads $\{1, 4, 7, 9\}$, $\{2, 11, 14, 20\}$, $\{3, 6, 8, 19\}$, $\{5, 12, 13, 16\}$ and $\{10, 15, 17, 18\}$.

TABLE 5. The 112 coset representatives of W : 48 axis words, then 64 diagonal words.

axis words							
1134	2453	4337	5590	6500	7434	8363	9413
10359	11600	12378	15592	17037	18346	19224	20342
21301	24455	26184	27571	29399	30393	31243	32556
66280	69210	70169	71287	72389	73698	74307	75620
76758	77752	79580	80679	82021	82955	84153	85406
87290	88321	90503	93493	94271	95512	96682	97732
diagonal words							
986	1972	2822	3617	4907	5644	6846	7888
9073	10015	10980	12227	13184	13991	15196	16178
16414	17520	18626	19941	20902	21633	22579	23645
25084	26002	26729	27982	28740	30051	30872	31990
65842	66581	67751	68809	70083	71085	71967	72760
73936	75255	75788	76898	77857	78927	80308	81043
82678	83921	84835	85773	86862	87840	88978	89781
90717	92026	92801	93935	95205	96139	96880	98135

Lemma 4.5. *The 112 cosets $c + U_0$ lie in distinct cosets of U , so W satisfies (W1) and has 1792 words, 768 axis and 1024 diagonal. For every axis word $c \in W$, $|\Delta(c)| = 24$ and $F(c)$ consists of exactly one coordinate.*

Proof. A finite computation over the 768 axis words of W and the 64 words d . $\square$

We let $m(c)$ be the free coordinate of c . Then (W2) holds.

Proposition 4.6. *The 10668 body points and the 1792 new points $x_c, y_c, c \in W$, are 12460 pairwise compatible nonzero vectors spanning $\mathbb{R}^{19}$. Hence $\tau_{19} \geq 12460$.*

Proof. Compatibility follows from Proposition 4.4 and Lemma 4.5. The head roots span the head and the tail roots span the tail. $\square$

This proves Theorem 1.1 for $n = 19$. The proof uses the computer only for Lemma 4.5.

Remark 4.7. The 112 choices of halves in Table 5 were found by computer search. The proof does not use the search: Lemma 4.5 checks the table directly, and since every axis word has exactly one free coordinate, the marks are forced by W .

5. VERIFICATION

The computer is used for three finite facts, all in exact arithmetic: Lemma 4.5, the list of lines in Table 2 with their 84 parities, and the count in Remark 3.6. Everything else is proved by hand.

All pairs. The accompanying data list the two configurations. In dimension 21 the coordinates are integers. In dimension 19 the axis points are scaled by 500 and the diagonal points by 3, so that all coordinates lie in $\mathbb{Z}[\sqrt{6}]$: axis points have head entries ± 500 except one entry ± 373 and tail $\pm 1000f_k$, and diagonal points have head entries ± 3 and tail entries $\pm \sqrt{6}$. A program written independently of the programs that built the configurations checks each configuration on all pairs, 77 619 570 in dimension 19 and 498 001 020 in dimension 21, with no symmetry reduction and no floating point in any decision. It confirms the number of points and that every point is nonzero, decides (2.1) for every pair of distinct points, exactly in $\mathbb{Z}[\sqrt{6}]$ in dimension 19, and confirms that no two points have the same direction. Every pair passes, and in each configuration the largest cosine is exactly $\frac{1}{2}$. In dimension 19 there are 2 814 168 contacts, of which 344 064 involve a new point, all between a diagonal point and a body point; no axis point touches anything, and the largest cosine of an axis point with a body point is about 0.4999967 and with a diagonal point about 0.4998897. In dimension 21 the contacts that involve a new point number 34048, all between two new points; no base point touches a new point. This check uses none of the lemmas, so it confirms Propositions 4.6 and 3.5 independently.

AI USAGE STATEMENT

GPT-6 Astra Pro, Claude Opus 5.5 and Claude Fable 5.1 were used to find these constructions, prepare supplementary data and draft this paper. The manuscript was proofread and lightly revised by the author. The author welcomes detailed questions regarding the use of AI in this work.

ACKNOWLEDGEMENTS

The author thanks Evolving Programs for providing tokens at no cost. The author also thanks Henry Cohn for correspondence.

DATA AVAILABILITY

The coordinates and an exact checking program accompany this paper. They can be found at <https://github.com/Felpix-Studios/kissing-number>.

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