

ALGEBRAICITY OF WEIL CLASSES ON ABELIAN SIXFOLDS OF WEIL TYPE FOR $\mathbb{Q}(\sqrt{-3})$ WITH DISCRIMINANT -2

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ABSTRACT. Let $K = \mathbb{Q}(\sqrt{-3})$. We prove that the Weil classes of every polarized abelian sixfold of Weil type for K with Hermitian discriminant -2 are algebraic. This discriminant is not split, so the theorems of Schoen and of Markman on split Weil classes do not apply. The statement also follows from a companion paper, which treats every discriminant through split eightfolds and a degeneration; the proof here is independent of it, uses no eightfolds and no degeneration, takes every cycle from Schoen's theorem on cyclic covers, and rests on exact certificates only. The sixfolds come from Galois covers of the projective line with group the complex reflection group G_{32} , branched over thirteen points: the multiplicity space of the reflection representation in the first cohomology of such a cover is the first cohomology of an abelian sixfold of Weil type. On every such sixfold the Weil classes are algebraic: they are obtained from Schoen's algebraic Weil classes on fifty-one cyclic quotients of the cover, through a correspondence that Milne's theorem on Lefschetz classes makes algebraic, and the choice of quotients is governed by an integral identity among characters, checked by an exact computation. The family of these sixfolds fills an open set of the nine-dimensional moduli space: along an explicit isomonodromic deformation of one explicit cover, a 9×9 minor of the codifferential of the period map has a nonzero Taylor coefficient of order seven, an exact rational number. A Hilbert-scheme and Baire-category argument then reaches the whole moduli space. The discriminant of the family is computed exactly. For the members whose Hodge group is the full special unitary group, every Hodge class on every power is algebraic.

1. INTRODUCTION

1.1. Weil classes. Let A be a complex abelian variety of dimension $2n$ with an action $\iota: K \rightarrow \text{End}(A) \otimes \mathbb{Q}$ of an imaginary quadratic field K , such that the two embeddings of K in $\mathbb{C}$ each occur with multiplicity n on the tangent space. Then the K -line

$$W_K(A) = \bigwedge_K^{2n} H^1(A, \mathbb{Q}) \subset H^{2n}(A, \mathbb{Q})$$

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is a two-dimensional space of Hodge classes of type (n, n) , the *Weil classes* of A [23]. For a general such A they are not in the ring generated by divisor classes [20, §5]. A polarization compatible with ι determines a K -Hermitian form H of signature (n, n) on $H_1(A, \mathbb{Q})$, and its discriminant $\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$ is an invariant of the polarized rational Hodge structure. The polarized abelian $2n$ -folds of Weil type for K with a fixed discriminant form one family, parametrized by a quotient of the Hermitian symmetric domain of $SU(n, n)$, of dimension n^2 . The class $(-1)^n$ is called *split*.

Let $K = \mathbb{Q}(\sqrt{-3})$. The rational primes $p \equiv 2 \pmod{3}$ are inert in K , so a norm from K has even valuation at each of them. For sixfolds the split class is -1 , and the class -2 is not split: -2 and $-2/(-1) = 2$ have odd valuation at the inert prime 2, so neither is a norm.

1.2. What was known. For abelian fourfolds of Weil type the Weil classes are algebraic for every K and every discriminant (Markman [11]). For sixfolds the split case is known: Schoen for $\mathbb{Q}(\sqrt{-3})$ [16], and Markman for every K [11]. The companion papers treat further non-split cases by other routes: [19] proves algebraicity for sixfolds of every discriminant for $K = \mathbb{Q}(\sqrt{-3})$, through split eightfolds and Schoen's product argument, and [18] treats the class -3 for $\mathbb{Q}(i)$. In particular Theorem A below is also a consequence of [19]. The present paper gives an independent proof. It uses no eightfolds, no product argument and no boundary degeneration; every cycle it uses comes from Schoen's theorem on cyclic covers, transported by a correspondence; and its three certificates (the character identity, the dominance and the discriminant) are exact. The method needs only a finite group with a suitable representation and an integral character identity, so it applies to class data where the degeneration of [19] is not available; Remark 5.13 explains why no degeneration of the G_{32} family reaches the class -2 .

Sections 2, 3 and 7, the set-up of Section 4, Lemma 4.3, Lemma 6.2 and Lemma 9.1 adapt the corresponding parts of [19] and [18]. They are stated here with proofs, so that this paper can be read alone.

1.3. Results.

Theorem A. *Let $K = \mathbb{Q}(\sqrt{-3})$. The Weil classes of every polarized abelian sixfold of Weil type for K with Hermitian discriminant -2 in $\mathbb{Q}^\times / \text{Nm } K^\times$ are algebraic.*

Theorem B. *Let A be an abelian sixfold as in Theorem A whose Hodge group is SU_H , the special unitary group of its Hermitian form. Then every Hodge class on every power A^m , $m \geq 1$, is algebraic.*

The hypothesis of Theorem B holds outside a countable union of proper closed subvarieties of the moduli space [20, Theorem 6.11]. For $m = 1$ it is the Hodge conjecture for A itself: by [20, Theorem 6.12] the Hodge classes of A are then generated by divisor classes and the Weil classes.

Corollary C (Corollary 8.2). *Let A be a simple abelian sixfold of type III whose endomorphism algebra D contains K , and whose induced Hermitian discriminant for K is -2 . Then $W_K(A)$ is algebraic. If moreover the PEL family of A is effective in Abdulali's sense (its arithmetic group is torsion free and its generic member has endomorphism algebra D ; Section 8), then the Hodge conjecture holds for every power of every general member of that family, that is, every member whose Hodge group is that of the generic member. This includes families with $D = (-1, -1)_{\mathbb{Q}}$ for which no imaginary quadratic subfield of D gives a split discriminant (Remark 8.3).*

1.4. The construction in one page. *The family.* Let $G \subset GL_4(K)$ be the Shephard–Todd group G_{32} , of order 155 520, generated by the four matrices of Appendix A, and ρ its natural representation on K^4 , with character field K . Its reflections have order three, with nontrivial eigenvalue ω (class $\mathcal{R}_1$) or ω^2 (class $\mathcal{R}_2$), $\omega = e^{2\pi i/3}$. Let $\mathcal{I}$ be the class of elements with eigenvalues $(1, 1, -1, -1)$. We consider Galois covers $C \rightarrow \mathbb{P}^1$ with group G , branched over thirteen points, with local monodromy in $\mathcal{R}_1$ at six of them, in $\mathcal{R}_2$ at six, and in $\mathcal{I}$ at one. Such a cover has genus 505 441, by Riemann–Hurwitz. By the Chevalley–Weil formula, ρ occurs in $H^1(C)$ with multiplicity six and Hodge numbers $(3, 3)$ (Lemma 3.6); the multiplicity space is the first cohomology of an abelian sixfold $A_\rho(C) \subset J(C)$ of Weil type for K , the image of an endomorphism of $J(C)$ given by an element of $\mathbb{Q}[G]$. Without the point in $\mathcal{I}$ the multiplicity would be four, giving fourfolds. Thirteen points give a family of dimension $13 - 3 = 10$, one more than the dimension 9 of the moduli space.

Cycles (Section 4). For a subgroup $H' \leq G$ and a character χ of H' of order $m \geq 3$, the curve $C/\ker \chi$ is a cyclic cover of C/H' . When its branch exponents pair off as $\alpha, -\alpha$, Schoen proved that the Weil classes of the primitive part of its Jacobian are algebraic [15]. That primitive part is a factor of $J(C)$, and the centralizer U of G in the symplectic group of $H^1(C)$ acts on its Weil classes through characters of the abelianization of U ; it acts on the Weil line $\det H^1(A_\rho)_\rho$ through a character e_ρ . If e_ρ is an integer combination of the characters of such pairs, a product of the corresponding Schoen classes, contracted against a U -invariant class, maps onto the Weil classes of $A_\rho(C)$, and Milne's theorem on Lefschetz classes [12] makes the contracting class algebraic. For G_{32} an identity with fifty-one pairs holds (Proposition 4.5); it is printed in Appendix B.

Dominance (Section 5). An explicit cover C_0 is the pullback of a rigid rank-four Fuchsian system on $\mathbb{P}^1$ minus four points along a rational function of degree six. At C_0 the codifferential of the period map, a 9×10 matrix over $\mathbb{Q}$, has rank only 5. Moving the twelve reflection points of C_0 along a Galois-equivariant rational direction and solving the Schlesinger equations as power series in the deformation parameter θ , all Taylor coefficients lie in $\mathbb{Q}$; a 9×9 minor of the codifferential has its first nonzero coefficient at θ^7 ,

an exact rational number (Proposition 5.9). So the period map has rank 9 at nearby covers.

Discriminant and spreading (Sections 6 and 7). The Hermitian form of $A_\rho(C)$ is computed from the monodromy as a parabolic cup product, exactly over K , with determinant $-2/81$ (Proposition 6.3). The set of points of the moduli space where the Weil classes are algebraic is a countable union of closed algebraic subsets and contains an open set, so it is everything.

1.5. What is new, and the inputs. The mechanisms are known: Schoen’s cycles on cyclic covers [15], their transfer to a non-abelian factor (van Geemen and Verra [21, Corollaries 4.9 and 4.10], who use one unramified cyclic subcover of order four and its quaternionic translates), Lefschetz classes [12], and dominance of a family of Prym varieties followed by spreading [9]. New here are the G_{32} family, the integral character identity with fifty-one cyclic quotients that combines Schoen’s classes into the Weil classes of its factor, an exact certificate of its dominance, and the exact computation of its discriminant.

Published results used: Schoen’s theorem [15, Theorem 2.0 and Corollary 3.1], with a replacement for one step (Lemma 4.3); Milne’s theorem on Lefschetz classes [12, Theorem 3.2]; Katz’s rigidity theorem [8, Theorem 1.1.2]; Landherr’s theorem and van Geemen’s description of the moduli [20]; the Chevalley–Weil formula and the codifferential of the period map of a family of Galois covers [4]; the indices of the maximal subgroups of $PSp_4(3)$ [2]; Abdulali’s theorem on type III [1, Theorem 4.1]; the first fundamental theorem for SL_n [10, §8.4]; and standard sheaf theory (proper base change, relative Hilbert schemes [22]).

Finite computations, each exact and described in the appendices: the order and generation of G , and the group-theoretic inputs of the monodromy argument (Appendix A); the character identity of Proposition 4.5 (Appendix B); the rank certificate of Proposition 5.9 (Appendix C); and the Hermitian matrix of Proposition 6.3. All of them run in characteristic zero, without floating point.

1.6. Organization. Section 2 recalls abelian varieties of Weil type and their moduli, and Section 3 constructs the factor $A_\rho(C)$ and the G_{32} family. Section 4 proves that the Weil classes of every member are algebraic. Section 5 proves dominance, Section 6 computes the discriminant, and Section 7 spreads algebraicity to the moduli space and proves Theorem A. Section 8 treats type III and Section 9 proves Theorem B. The appendices give the data.

Main notation

K, ω, δ	$\mathbb{Q}(\sqrt{-3}), e^{2\pi i/3}, \delta = \sqrt{-3} = 1 + 2\omega$
G, ρ	the group $G_{32} \subset GL_4(K)$ and its natural representation on $V_\rho = K^4$
$\mathcal{R}_1, \mathcal{R}_2, \mathcal{I}$	reflections with eigenvalue ω , with eigenvalue ω^2 ; elements with eigenvalues $(1, 1, -1, -1)$
$\text{Hur}, \text{Hur}^\circ$	covers with the class datum of Definition 3.5; the component through C_0
$A_\rho(C), M_\tau, m_\tau$	the Weil-type factor; multiplicity space and multiplicity of τ in $H^1(C)$
$U, T, X^*(T)$	centralizer of G in $Sp(H^1(C))$, its maximal abelian quotient, its characters
$(H', \chi), a(\chi), e_\rho$	a supplier, its character in $X^*(T)$, the character of $\det M_\rho$
$\mathcal{D}_\rho(C), \mathcal{D}_{\bar{\rho}}(C), \mu_C$	$\text{Hom}_G(\rho, H^0(K_C)), \text{Hom}_G(\bar{\rho}, H^0(K_C))$, the codifferential
$\mathbb{L}, \varphi, C_0$	the rigid local system on $\mathbb{P}_z^1$, the map of degree six, the base cover
$\mathcal{M}, \mathcal{H}_n$	the moduli space and the period domain

2. ABELIAN VARIETIES OF WEIL TYPE

Throughout, K is an imaginary quadratic field and $\vartheta \in K$ a nonzero element with $\bar{\vartheta} = -\vartheta$; for $K = \mathbb{Q}(\sqrt{-3})$ we take $\vartheta = \delta = \sqrt{-3}$.

Definition 2.1. A *polarized abelian $2n$ -fold of Weil type for K* is a triple (A, ι, E) : a complex abelian variety A of dimension $2n$, an embedding $\iota: K \rightarrow \text{End}(A) \otimes \mathbb{Q}$ such that both embeddings of K occur with multiplicity n on $T_0 A$, and a polarization E (an alternating form on $H_1(A, \mathbb{Q})$) with $E(\iota(a)x, y) = E(x, \iota(\bar{a})y)$.

Following van Geemen [20, Lemma 5.2], the form $H(x, y) = E(x, \iota(\vartheta)y) + \vartheta E(x, y)$ is K -Hermitian of signature (n, n) on the $2n$ -dimensional K -space $H_1(A, \mathbb{Q})$. Its discriminant $\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$ does not depend on ϑ , and it is unchanged by a K -linear isogeny that multiplies the polarization by a positive rational number, since such a change multiplies $\det H$ by a $2n$ -th power of a rational number, which is a norm. By Landherr's theorem [20, §5.4], a nondegenerate K -Hermitian space is determined up to isometry by its dimension, signature and discriminant; those of signature (n, n) with an n -dimensional totally isotropic subspace are those of discriminant $(-1)^n$. We write SU_H for the special unitary group of $(H_1(A, \mathbb{Q}), H)$, an algebraic group over $\mathbb{Q}$.

2.1. Moduli. Fix a K -Hermitian space (V, H) of dimension $2n$, signature (n, n) and discriminant $\mathfrak{d}$, and an $\mathcal{O}_K$ -stable lattice $\Lambda \subset V$ on which the alternating form $\text{Tr}_{K/\mathbb{Q}}(H/2\vartheta)$, scaled by a positive rational, is integral. The space $\mathcal{H}_n$ of n -dimensional complex subspaces of $V_{\mathbb{R}}$ (for the complex structure $\iota(\vartheta)/|\vartheta|$) on which H is positive is the Hermitian symmetric domain of $SU(n, n)$, connected of dimension n^2 . A point of $\mathcal{H}_n$ determines a Hodge

structure of weight one on V of Weil type for K , polarized by $\mathrm{Tr}_{K/\mathbb{Q}}(H/2\vartheta)$, and every such polarized Hodge structure arises in this way [20, 5.10 and the proof of 6.11]. For $\ell \geq 3$ the congruence subgroup $\Gamma(\ell)$ of the unitary group of (Λ, H) is torsion free, and $\mathcal{M} = \Gamma(\ell) \backslash \mathcal{H}_n$ is a smooth connected quasi-projective variety with a universal abelian scheme $f: \mathcal{A} \rightarrow \mathcal{M}$ with K -action [20, §§5.5–5.11].

Lemma 2.2 (Isogeny invariance). *If $\psi: A' \rightarrow A$ is a K -linear isogeny, then $W_K(A)$ is spanned by algebraic classes if and only if $W_K(A')$ is. Every polarized abelian $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}$ is K -isogenous to a fibre of $\mathcal{A} \rightarrow \mathcal{M}$.*

Proof. Pullback and pushforward by ψ are algebraic correspondences, and ψ^* maps $W_K(A)$ isomorphically onto $W_K(A')$, since it is a K -linear isomorphism on $H^1(-, \mathbb{Q})$. For the second statement, Landherr's theorem gives a K -isometry from $(H_1(A, \mathbb{Q}), H)$ to (V, H) ; the image of $H_1(A, \mathbb{Z})$ is commensurable with Λ , so an integer multiple of it lies in Λ , and multiplication by that integer is a K -linear isogeny from A to the fibre at the point given by the complex structure of A . $\square$

3. THE WEIL-TYPE FACTOR AND THE G_{32} FAMILY

3.1. Hurwitz spaces. Fix a finite group G and conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_r$ of G , $r \geq 4$. The *Nielsen class* is the set of tuples $(g_1, \dots, g_r)$ with $g_j \in \mathcal{C}_j$, $g_1 \cdots g_r = 1$ and $\langle g_1, \dots, g_r \rangle = G$. By the Riemann existence theorem, a connected Galois cover $C \rightarrow \mathbb{P}^1$ with group G , branched over r ordered points $p_1, \dots, p_r$, is the same as such a tuple up to simultaneous conjugation, once standard generators of $\pi_1(\mathbb{P}^1 - \{p_j\})$ are fixed: g_j is the monodromy along a loop around p_j . More generally, a homomorphism $\pi_1(\mathbb{P}^1 - \{p_j\}) \rightarrow G$ that is onto determines a connected Galois G -cover. The Hurwitz space Hur of such covers is finite and étale over the configuration space of r distinct ordered points; its connected components are smooth, hence irreducible, of dimension $r - 3$ modulo PGL_2 [5].

For an irreducible complex representation τ of G write $H^1(C, \mathbb{C}) = \bigoplus_{\tau} \tau \otimes M_{\tau}$ and $m_{\tau} = \dim M_{\tau}$, with $M_{\tau} = \mathrm{Hom}_G(\tau, H^1(C, \mathbb{C}))$. Since G acts by morphisms of Hodge structures, M_{τ} is a complex vector space with a decomposition $M_{\tau} = \mathrm{Hom}_G(\tau, H^{1,0}) \oplus \mathrm{Hom}_G(\tau, H^{0,1})$; it is not in general defined over $\mathbb{Q}$. For a representation realizable over K , as in Subsection 3.2, M_{τ} is the complexification of a K -space M (proof of Lemma 3.3) along one embedding of K . For $g \in G$ write $c_{\tau}(g) = \mathrm{rk}(\tau(g) - 1)$, and $\mathrm{age} \tau(g) = \sum_k \alpha_k$ when the eigenvalues of $\tau(g)$ are $e^{2\pi i \alpha_k}$ with $0 \leq \alpha_k < 1$.

Proposition 3.1. *If τ is irreducible and nontrivial of dimension d , then*

$$m_{\tau} = -2d + \sum_{j=1}^r c_{\tau}(g_j), \quad \dim \mathrm{Hom}_G(\tau, H^0(C, K_C)) = \sum_{j=1}^r \mathrm{age} \tau(g_j^{-1}) - d.$$

Proof. Let $U_0 = \mathbb{P}^1 - \{p_j\}$ and $\mathbb{V}$ the local system on U_0 attached to τ . Then m_τ is the dimension of the parabolic cohomology $H^1(\mathbb{P}^1, j_*\mathbb{V})$, the image of $H_c^1(U_0, \mathbb{V}) \rightarrow H^1(U_0, \mathbb{V})$. Since τ is irreducible and nontrivial, $H^0(U_0, \mathbb{V}) = 0$, and $H^2(U_0, \mathbb{V}) = 0$ because U_0 is an affine curve; so $\dim H^1(U_0, \mathbb{V}) = (r-2)d$. Likewise $H_c^0 = H_c^2 = 0$, the latter by duality with $H^0(U_0, \mathbb{V}^\vee) = 0$, so $\dim H_c^1(U_0, \mathbb{V}) = (r-2)d$. The long exact sequence for U_0 and small punctured discs Δ_j^* around the p_j begins

$$0 = H^0(U_0, \mathbb{V}) \rightarrow \bigoplus_j H^0(\Delta_j^*, \mathbb{V}) \rightarrow H_c^1(U_0, \mathbb{V}) \rightarrow H^1(U_0, \mathbb{V}) \rightarrow \cdots,$$

so the image of $H_c^1 \rightarrow H^1$ has dimension $(r-2)d - \sum_j (d - c_\tau(g_j))$. This gives the first formula. The second is the Chevalley–Weil formula [4, Theorem 2.10]. $\square$

3.2. The Weil-type factor. Let σ be an irreducible representation of G whose character field is K and which is realizable over K , on $V_\sigma = K^d$. The simple factor of $\mathbb{Q}[G]$ attached to σ is $\mathbb{Q}[G]\varepsilon \cong \text{End}_K(V_\sigma) \cong M_d(K)$, with centre K . Averaging gives a G -invariant positive definite K -Hermitian form h_V on V_σ ; the Rosati involution $g \mapsto g^{-1}$ acts on $\mathbb{Q}[G]\varepsilon$ as the h_V -adjoint and on its centre as complex conjugation.

Definition 3.2. Choose $v \in V_\sigma$ with $h_V(v, v) \neq 0$, and let $e \in \mathbb{Q}[G]\varepsilon$ be the h_V -orthogonal projection onto the line Kv , a primitive idempotent fixed by the Rosati involution. Choose an integer $N_0 \geq 1$ with $N_0 e \in \mathbb{Z}[G]$; then $N_0 e$ is an endomorphism of $J(C)$, not itself idempotent. Put

$$A_\sigma(C) = \text{im}(N_0 e: J(C) \rightarrow J(C)),$$

with the restricted polarization E and the action ι of the centre K . It does not depend on N_0 , and other choices of v give K -isogenous abelian varieties.

The σ -isotypic part of $J(C)$ is isogenous to $A_\sigma(C)^d$. Factors of this kind are studied as Prym–Tyurin varieties in [3, Ch. 12].

Lemma 3.3. *($A_\sigma(C), \iota, E$) is a polarized abelian variety of dimension m_σ , and the two embeddings of K occur on its tangent space with multiplicities the Hodge numbers (p_σ, q_σ) of M_σ . If $m_\sigma = 2n$ and $(p_\sigma, q_\sigma) = (n, n)$, it is a polarized abelian $2n$ -fold of Weil type for K .*

Proof. The σ -isotypic part of $H_1(J(C), \mathbb{Q})$ is $V_\sigma \otimes_K M$ with $M = \text{Hom}_{K[G]}(V_\sigma, H_1(J(C), \mathbb{Q})_\varepsilon)$, and $H_1(A_\sigma, \mathbb{Q}) = Kv \otimes_K M \cong M$. After extending scalars along the two embeddings of K , M becomes M_σ and $M_{\bar{\sigma}}$. The Rosati involution of E restricts on $\iota(K)$ to that of $\mathbb{Q}[G]$ on the centre, which is complex conjugation. $\square$

Lemma 3.4. *On a connected component S of Hur, the discriminant of the Hermitian form of $A_\sigma(C)$ does not depend on $C \in S$.*

Proof. The spaces $H_1(A_\sigma(C), \mathbb{Q})$ with ι and E form a local system over S , since e and ι act by flat endomorphisms and E is flat; parallel transport is a K -isometry. $\square$

3.3. The group G_{32} and the class datum. Let $K = \mathbb{Q}(\sqrt{-3})$, and let $G \subset GL_4(K)$ be the group generated by the matrices r_1, r_2, x, y of Appendix A. It has order 155 520, and it is the Shephard–Todd group G_{32} ; we use only the following properties, which are checked in Appendix A. Its natural representation ρ on $V_\rho = K^4$ is absolutely irreducible, so its character field is K and it is realizable over K . The matrices preserve the standard Hermitian form, which is therefore h_V . The centre $Z(G)$ has order six, $G/Z(G) \cong PSp_4(3)$ is simple of order 25 920, the derived group of G is the kernel of $\det \rho$, of index three, and every irreducible character of G takes values in K . Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the classes of the reflections with nontrivial eigenvalue ω and ω^2 respectively; they are the classes of r_1 and r_2 , and they consist of mutually inverse elements. Let $\mathcal{I}$ be the class of x^2 , whose eigenvalues are $(1, 1, -1, -1)$.

Definition 3.5. Let Hur be the Hurwitz space of connected Galois covers $C \rightarrow \mathbb{P}^1$ with group G , branched over thirteen points, with local monodromy in $\mathcal{R}_1$ at six of them, in $\mathcal{R}_2$ at six, and in $\mathcal{I}$ at one. We call this list of classes the *class datum*, and write Hur° for the connected component of Hur that contains the cover C_0 constructed in Lemma 5.2.

Lemma 3.6. *For every $C \in \text{Hur}$, ρ occurs in $H^1(C)$ with multiplicity 6 and Hodge numbers $(3, 3)$, so $A_\rho(C)$ is a polarized abelian sixfold of Weil type for K . The genus of C is 505 441.*

Proof. A reflection has $c_\rho = 1$, and an element of $\mathcal{I}$ has $c_\rho = 2$, so Proposition 3.1 gives $m_\rho = -8 + 12 + 2 = 6$. The inverse of an element of $\mathcal{R}_1$ has eigenvalues $(\omega^2, 1, 1, 1)$, of age $\frac{2}{3}$; that of an element of $\mathcal{R}_2$ has age $\frac{1}{3}$; an element of $\mathcal{I}$ has age 1. So $\dim \text{Hom}_G(\rho, H^0(K_C)) = 6 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} + 1 - 4 = 3$, and the Hodge numbers are $(3, 3)$. Conclude with Lemma 3.3. Riemann–Hurwitz gives $2g - 2 = 155\,520 \left(-2 + 12 \cdot \frac{2}{3} + \frac{1}{2}\right) = 1\,010\,880$. $\square$

Remark 3.7 ($A_\rho(C)$ is not of CM type). The family $A_\rho(C)$, $C \in \text{Hur}^\circ$, fills an open set of its moduli space (Corollary 5.11), and the Hodge group of a very general point of that space is SU_H [20, Theorem 6.11]. So for C outside a countable union of proper analytic subsets, $A_\rho(C)$ is not of CM type and not isogenous to a power of an elliptic curve.

4. THE WEIL CLASSES OF EVERY MEMBER ARE ALGEBRAIC

We first outline the argument. The centralizer U of G in the symplectic group of $H^1(C)$ acts on the Weil line $\det M_\rho \subset H^6(A_\rho(C), \mathbb{C})$ through a character e_ρ of its maximal abelian quotient T . A Schoen class on a factor of $J(C)$ spans, together with its Galois conjugates, eigenlines of U with characters $\pm a(\chi)$. If $e_\rho = \sum_i n_i a(\chi_i)$ with integers n_i , then the class Γ obtained by tensoring the generator of $\det M_\rho$ with the complementary eigenlines of the Schoen factors is invariant under U , hence under Milne’s group of the

power of $J(C)$ that contains all the factors, hence algebraic by Milne's theorem; and Γ , used as a correspondence, maps a product of Schoen classes to a nonzero multiple of the Weil class.

4.1. Schoen suppliers.

Definition 4.1. A *supplier* is a pair (H', χ) of a subgroup $H' \leq G$ and a character $\chi: H' \rightarrow \mathbb{C}^\times$ of order $m \geq 3$.

For $C \in \text{Hur}$, the curve $C' = C/\ker \chi$ is a connected cyclic cover of C/H' with group $H'/\ker \chi \cong \mathbb{Z}/m$. Its *primitive part* $P_\chi \subset J(C)$ is the abelian subvariety on which $\mathbb{Z}/m$ acts through primitive characters; it is the image of $N\pi_\chi$ for a rational idempotent π_χ of $\mathbb{Q}[H']$ and an integer N , and $F_\chi = \mathbb{Q}(\mu_m)$ acts on it. For $t \in (\mathbb{Z}/m)^\times$ let $V_t \subset H^1(P_\chi, \mathbb{C})$ be the eigenspace where H' acts by χ^t ; then

$$V_t = \bigoplus_\tau \text{Hom}_{H'}(\chi^t, \tau|_{H'}) \otimes M_\tau.$$

The points of C/H' over p_j correspond to the double cosets $H' \backslash G / \langle g_j \rangle$; over the double coset of x the local monodromy is $xg_j^\ell x^{-1} \in H'$, with ℓ the least positive integer for which this lies in H' . Let q be the genus of C/H' , s the number of branch points of $C' \rightarrow C/H'$, and $\alpha \in (\mathbb{Z}/m - \{0\})^s$ their exponents with respect to a generator, defined up to $(\mathbb{Z}/m)^\times$ and order. All V_t have the same dimension $h = 2q - 2 + s$ [15, Lemma 1.5], and the Weil classes $W_{F_\chi}(P_\chi) = \bigwedge_{F_\chi}^h H^1(P_\chi, \mathbb{Q})$ satisfy $W_{F_\chi}(P_\chi) \otimes \mathbb{C} = \bigoplus_t \det V_t$. Following Schoen [15, §1], α is *simple* if, after reordering, $\alpha_{2i} = -\alpha_{2i-1}$ for all i ; then s and h are even. A supplier is *admissible* if $h > 0$ and α is simple; for $s = 0$ simplicity is automatic. These data depend only on G and the classes $\mathcal{C}_j$, since replacing g_j by a conjugate yg_jy^{-1} replaces the double coset of x by that of xy^{-1} , with the same local monodromy.

Lemma 4.2 (Schoen). *If (H', χ) is admissible, then $W_{F_\chi}(P_\chi)$ is spanned by classes of algebraic cycles, for every $C \in \text{Hur}$.*

Proof. We quote Schoen [15, §2]. Let C' be a smooth projective irreducible curve over $\mathbb{C}$ with a faithful action $\mathbb{Z}/m \hookrightarrow \text{Aut}(C')$, $m > 1$, whose invariant h is even, so that the quotient map has an even number $2r$ of branch points, with exponents $1 \leq \alpha_j \leq m-1$. *Theorem 2.0* (p. 11): if the $(\mathbb{Z}/m)^\times$ -orbit of $(\alpha_1, \dots, \alpha_{2r})$ is simple, then the relevant subspace U of $H^h(C'^h)$ is generated by fundamental classes of algebraic cycles. *Corollary 3.1* (p. 24), for $m > 2$: under the same hypotheses, the corresponding subspace of $H^h(B, \mathbb{Q})$ is generated by classes of algebraic cycles of codimension $h/2$, where $B \subseteq \text{Alb}(C')$ is the primitive part, the image of $\prod_{1 \neq Z' \leq \mathbb{Z}/m} \sum_{t \in Z'} (\text{Id} - t)$. The statements have no coprimality hypothesis: Schoen's §1 treats a branch point with $\gcd(\alpha_j, m) = d_j > 1$ through the local model $y^{m/d_j} = w^{\alpha_j/d_j}$. The unramified case $r = 0$ is part of Theorem 2.0, where simplicity is automatic; Schoen proves it separately in §2, where β_0 is étale and P_0 splits into m connected components, using the first case of his Lemma 2.6. His Lemma 2.8

is not used when $r = 0$, since there are no blow-ups. Six rows of Table 1 are unramified.

Apply this to $C' = C/\ker \chi \rightarrow C/H'$, which is smooth, projective and irreducible as a quotient of C , with $H'/\ker \chi \cong \mathbb{Z}/m$ acting faithfully. The subspace of Corollary 3.1 is W_{F_χ} of the primitive part of $J(C')$, which is isogenous to P_χ compatibly with F_χ . When every exponent is $m/2$ and $4 \mid m$, one step of Schoen's proof is replaced by Lemma 4.3. $\square$

Three suppliers of the identity of Proposition 4.5 (rows 13, 14 and 15 of Table 1) have $m = 12$ and every exponent equal to $6 = m/2$. For such covers one step of Schoen's proof needs a replacement. Recall Schoen's construction [15, §2, pp. 11–19], for $C' \rightarrow X$ as in Lemma 4.2 with branch points ordered so that $\alpha_{2i} = -\alpha_{2i-1}$, $1 \leq i \leq r$. Let $W_0 = C'^h/G_0$, where $G_0 \subset (\mathbb{Z}/m)^h \rtimes S_h$ is the kernel of the sum map extended by S_h ; the residual group $\mathbb{Z}/m$ acts on W_0 , and $\beta_0: W_0 \rightarrow S^h X$ is a $\mathbb{Z}/m$ -cover branched along the divisors $S_j = b_j + S^{h-1}X$. Schoen chooses a linear subspace $P_0 \subset S^h X$ of dimension $h/2$ (his Lemma 2.3) and, for $i = 1, \dots, r$, blows up $(S^h X)_{i-1}$ along $K_{i-1} = S_{2i-1,i-1} \cap S_{2i,i-1}$, the intersection of the strict transforms of S_{2i-1} and S_{2i} ; the exceptional divisor E_i is a $\mathbb{P}^1$ -bundle over K_{i-1} . Let W_i be the normalization of the pulled-back cover, $\beta_i: W_i \rightarrow (S^h X)_i$, and $F_i = \beta_i^{-1}(E_i)$. To see that his cycles have nonzero classes, Schoen's Lemma 2.8 reduces to the statement that no primitive character of $\mathbb{Z}/m$ occurs in $H^*(F_i, \mathbb{C})$ [15, p. 18], which he proves (Sublemma 2.9, p. 19) by choosing a prime $p \mid m$ such that the $\mathbb{Z}/p$ -quotient of $C' \rightarrow X$ is ramified over b_{2i-1} . When every exponent is $m/2$ and $4 \mid m$, no such prime exists: for $m = 12$ and exponent 6, the images of 6 in $\mathbb{Z}/2$ and in $\mathbb{Z}/3$ are zero.

Lemma 4.3 (Schoen's Theorem 2.0 without the choice of a prime). *In the situation of Schoen's Theorem 2.0, for every $i \leq r$ and every primitive character χ of $\mathbb{Z}/m$, the χ -part of $H^*(F_i, \mathbb{C})$ vanishes. Hence Theorem 2.0 and Corollary 3.1 of [15] hold under their stated hypotheses, without the choice of a prime in Sublemma 2.9.*

Proof. Since β_i is finite, $H^*(F_i, \mathbb{C})_\chi = H^*(E_i, (\beta_{i*}\mathbb{C})_\chi|_{E_i})$. The stalk of $\beta_{i*}\mathbb{C}$ at a point is the permutation module on the points above it, and its χ -part vanishes at every point whose stabilizer contains nontrivial inertia, because a primitive character is faithful. By Schoen's Lemma 2.5, E_i is not a branch divisor of β_i : its multiplicity in the branch locus is $\alpha_{2i-1} + \alpha_{2i} \equiv 0 \pmod{m}$. So on E_i the sheaf $(\beta_{i*}\mathbb{C})_\chi$ is $j_!\mathcal{L}$, where $\mathcal{L}$ is a rank-one local system on the complement in E_i of the strict transforms $S_{j,i}$, with monodromy $\chi(\alpha_j) \neq 1$ around $S_{j,i}$. Here $j: U_i \rightarrow E_i$ is the inclusion of the complement U_i of all the $S_{j,i}$. The strict transforms $S_{2i-1,i}$ and $S_{2i,i}$ meet E_i in two disjoint sections of the $\mathbb{P}^1$ -bundle $p: E_i \rightarrow K_{i-1}$, and every other $S_{j,i}$ meets E_i in a union of whole fibres [15, §2]. Fix $k \in K_{i-1}$, let ι_k be the inclusion of the fibre $\Pi_k = p^{-1}(k)$, and j_k that of $\Pi_k \cap U_i$ in Π_k . Extension by zero commutes with restriction to a closed subvariety (both sides have the

same stalks), so $\iota_k^* j_! \mathcal{L} = (j_k)_! (\mathcal{L}|_{\Pi_k \cap U_i})$. There are two cases. If Π_k lies in some $S_{j,i}$ with $j \neq 2i-1, 2i$, then $\Pi_k \cap U_i$ is empty and $\iota_k^* j_! \mathcal{L} = 0$. Otherwise Π_k meets no $S_{j,i}$ with $j \neq 2i-1, 2i$, and it meets $S_{2i-1,i}$ and $S_{2i,i}$ transversally in one point each, so $\Pi_k \cap U_i \cong \mathbb{C}^\times$, and a small loop in Π_k around the first point is a meridian of $S_{2i-1,i}$. So $\mathcal{L}|_{\Pi_k \cap U_i}$ is a rank-one local system on $\mathbb{C}^\times$ with monodromy $\chi(\alpha_{2i-1}) \neq 1$, and $H_c^*(\mathbb{C}^\times, \mathcal{L}|_{\Pi_k \cap U_i}) = 0$: H_c^0 vanishes since $\mathbb{C}^\times$ is not compact, H_c^2 is dual to the invariants of the dual system, which vanish, and the Euler characteristic of $\mathbb{C}^\times$ is 0. In both cases $H^*(\Pi_k, \iota_k^* j_! \mathcal{L}) = 0$. Since p is proper, proper base change [14, Theorem 11.3.6] identifies the stalk of $R^b p_*(j_! \mathcal{L})$ at k with $H^b(\Pi_k, \iota_k^* j_! \mathcal{L}) = 0$. So $Rp_* j_! \mathcal{L} = 0$ and $H^*(E_i, j_! \mathcal{L}) = H^*(K_{i-1}, Rp_* j_! \mathcal{L}) = 0$. This uses only $\alpha_{2i-1} \not\equiv 0 \pmod{m}$. The rest of Schoen's proof is unchanged. $\square$

4.2. The character group. Let U be the centralizer of G in the symplectic group of $(H^1(C, \mathbb{Q}), E)$; we only need its complex points. Recall that an irreducible complex representation τ of a finite group is *orthogonal* or *quaternionic* if it is self-dual and carries a nonzero invariant symmetric, respectively alternating, bilinear form, and that otherwise $\tau \not\cong \bar{\tau}$. In $H^1(C, \mathbb{C}) = \bigoplus_\tau \tau \otimes M_\tau$ the G -invariant cup-product form pairs $\tau \otimes M_\tau$ with $\bar{\tau} \otimes M_{\bar{\tau}}$ and vanishes between other summands. By Schur's lemma its restriction is the tensor product of the invariant form on τ , unique up to scalar, with a nondegenerate form on the multiplicity spaces: a duality between M_τ and $M_{\bar{\tau}}$ if $\tau \not\cong \bar{\tau}$, a symmetric form on M_τ if τ is quaternionic, an alternating one if τ is orthogonal. Hence $U(\mathbb{C})$ is the product of the groups $GL(M_\tau)$ over the pairs $\{\tau, \bar{\tau}\}$ of non-self-dual representations with $m_\tau > 0$, of $O(M_\tau)$ over the quaternionic τ with $m_\tau > 0$, and of $Sp(M_\tau)$ over the orthogonal τ with $m_\tau > 0$. Let $T = U(\mathbb{C})/[U(\mathbb{C}), U(\mathbb{C})]$. Its character group is

$$X^*(T) = \mathbb{Z}^{\text{pairs}} \oplus (\mathbb{Z}/2)^{\text{quat}},$$

with one free coordinate for each pair (the determinant on M_τ , for a chosen member τ of the pair; for the pair of ρ we choose ρ), and one coordinate of order two for each quaternionic τ with $m_\tau > 0$ (the determinant of $O(M_\tau)$). For a supplier (H', χ) the line $\det V_\chi$, $V_\chi = \bigoplus_\tau \text{Hom}_{H'}(\chi, \tau|_{H'}) \otimes M_\tau$, is a U -eigenspace, with character

$$a(\chi) = (\langle \tau|_{H'}, \chi \rangle - \langle \bar{\tau}|_{H'}, \chi \rangle)_{\text{pairs}} \oplus (\langle \tau|_{H'}, \chi \rangle \bmod 2)_{\text{quat}}.$$

The Weil line $\det M_\rho$ has character e_ρ , the coordinate vector of the pair of ρ .

Lemma 4.4. *For $G = G_{32}$ and every supplier (H', χ) and $t \in (\mathbb{Z}/m)^\times$, the character of $\det V_{\chi^t}$ is $a(\chi)$ on the free coordinates if the automorphism $\zeta_m \mapsto \zeta_m^t$ fixes ω (extended to fix ω when $3 \nmid m$), and $-a(\chi)$ otherwise; on the coordinates of order two it is $a(\chi)$.*

Proof. Let γ be an automorphism of $\mathbb{Q}(\zeta_m, \omega)$ with $\gamma(\zeta_m) = \zeta_m^t$. Then $\langle \tau|_{H'}, \chi^t \rangle = \langle \tau^\gamma|_{H'}, \chi \rangle$, where τ^γ has character $\gamma \circ \text{tr } \tau$. Since every character

of G is K -valued, $\tau^\gamma = \tau$ if γ fixes ω and $\tau^\gamma = \bar{\tau}$ otherwise; the second case exchanges the two terms of each free coordinate, and a quaternionic τ is self-conjugate. $\square$

If $3 \nmid m$, then $\mathbb{Q}(\zeta_m)$ and K are linearly disjoint, so every t extends both to an automorphism fixing ω and to one moving it; the lemma then gives $a(\chi) = -a(\chi)$, so $a(\chi)$ vanishes on the free coordinates. Every supplier of Table 1 has $3 \mid m$ ($m \in \{3, 6, 12, 15\}$). Then $\omega = \zeta_m^{m/3}$, and $t = -1$ moves ω , as used in the proof of Proposition 4.6.

Proposition 4.5 (The character identity). *In $X^*(T) = \mathbb{Z}^{44} \oplus (\mathbb{Z}/2)^4$,*

$$e_\rho = \sum_{i=1}^{51} n_i a(\chi_i)$$

for the fifty-one admissible suppliers (H'_i, χ_i) and the integers n_i of Appendix B, with $\sum_i |n_i| = 56$.

Proof. Appendix B prints, for each supplier, its branch data (which show that it is admissible) and the nonzero coordinates of $a(\chi_i)$, so that the identity can be checked by adding 51 sparse integer vectors. The appendix also describes how these data are computed from the group. $\square$

4.3. The contraction.

Proposition 4.6. *For every $C \in \text{Hur}$, $W_K(A_\rho(C))$ is spanned by classes of algebraic cycles.*

Proof. Write $e_\rho = \sum_i n_i a(\chi_i)$ as in Proposition 4.5, put $t_i = \text{sign } n_i \in \{\pm 1\} \subset (\mathbb{Z}/m_i)^\times$, $P_i = P_{\chi_i}$, $F_i = F_{\chi_i}$, and let $V_{i,t}$ be the eigenspaces of P_i . By Lemma 4.2 choose a nonzero rational algebraic class $y_i \in W_{F_i}(P_i)$. Put $B = \prod_i P_i^{|n_i|}$ and $Y = \boxtimes_i y_i^{\boxtimes |n_i|} \in H^*(B, \mathbb{Q})$, an algebraic class.

The correspondence. Choose generators $z_{i,t}$ of the lines $\det V_{i,t}$ and put $Z_i = \prod_{t \neq t_i} z_{i,t}$. The product $\prod_t z_{i,t}$ spans the top cohomology of P_i , on which U acts trivially, so Z_i has the character opposite to that of $\det V_{i,t_i}$, which is $-t_i a(\chi_i)$ on the free coordinates (Lemma 4.4, $t = -1$ moving ω) and $a(\chi_i)$ on the others. Let w generate $\det M_\rho \subset H^6(A_\rho, \mathbb{C})$ and put

$$\Gamma = \left(\boxtimes_i Z_i^{\boxtimes |n_i|} \right) \boxtimes w \in H^*(B \times A_\rho, \mathbb{C}).$$

Degrees. Write ϕ for Euler's function. P_i has dimension $\phi(m_i)h_i/2$, each $z_{i,t}$ and y_i has degree h_i , Z_i has degree $(\phi(m_i) - 1)h_i$, and w has degree 6. So Γ has degree $\sum_i |n_i|(\phi(m_i) - 1)h_i + 6$, and for the algebraic class Y of degree $\sum_i |n_i|h_i$ the class $\text{pr}_{2*}(\text{pr}_1^* Y \cup \Gamma)$ below has degree $\sum_i |n_i|h_i + \deg \Gamma - 2 \dim B = 6$.

Its character is $e_\rho - \sum_i n_i a(\chi_i) = 0$ on the free coordinates and $\sum_i |n_i| a(\chi_i)$ on the coordinates of order two, which vanishes because $n_i \equiv |n_i|$ and the identity holds there too. Characters of $U(\mathbb{C})$ factor through T , so Γ is fixed by all of $U(\mathbb{C})$.

Γ is algebraic. Each factor X' of $B \times A_\rho$ (a copy of some P_i , or A_ρ) is the image of an endomorphism $N_0 e_{X'}$ of $J(C)$ with $e_{X'}$ an idempotent of

$\mathbb{Q}[G]$; choose one integer N_0 for all of them. The inclusions $X' \hookrightarrow J(C)$ and the homomorphisms $N_0 e_{X'}: J(C) \rightarrow X'$ (the endomorphism $N_0 e_{X'}$ with its target restricted to its image) give

$$B \times A_\rho \xrightarrow{\iota_0} J(C)^R \xrightarrow{\pi} B \times A_\rho, \quad \pi \circ \iota_0 = [N_0], \quad R = 1 + \sum_i |n_i|.$$

They are built from elements of $\mathbb{Q}[G]$, so they commute with U , and π^* is U -equivariant for the diagonal action of U on $H^1(J(C)^R) = H^1(C)^{\oplus R}$; hence $\pi^*\Gamma$ is fixed by $U(\mathbb{C})$. Milne defines $S(A)$, for an abelian variety A , as the algebraic group whose points are the elements γ of the centralizer of $\text{End}^0(A)$ with $\gamma^\dagger \gamma = 1$ for the Rosati involution, “the largest algebraic subgroup of Sp whose elements commute with the endomorphisms of A ” [12, §1]; it acts on $H^*(A^r) \cong \bigwedge (H^1(A)^{\oplus r})$ diagonally. Since $\mathbb{Q}[G] \subseteq \text{End}^0(J(C))$, the elements of $S(J(C))$ commute with G and preserve E , so $S(J(C)) \subseteq U$, and $\pi^*\Gamma$ is $S(J(C))$ -invariant. Milne’s Theorem 3.2 [12] states: *For any abelian variety A over Ω and integer $r \geq 0$, the k -algebra $H^*(A^r)^{S(A)}$ is generated by divisor classes*; here $\Omega = \mathbb{C}$ and $k = \mathbb{Q}$, and by his Lemma 3.1 the invariants of the complexification are the complexification of the rational invariants. So $\pi^*\Gamma$ is a complex linear combination of products of divisor classes on $J(C)^R$. Pullback by ι_0 preserves algebraic classes, and $\iota_0^* \pi^* \Gamma = [N_0]^* \Gamma = N_0^{\deg \Gamma} \Gamma$. So Γ is a linear combination, with complex coefficients, of classes of algebraic cycles.

The contraction is nonzero. Write $y_i = \sum_t c_{i,t} v_{i,t}$ with $v_{i,t} \in \det V_{i,t}$ normalized by $\int_{P_i} v_{i,t_i} \cup Z_i = 1$. For $t \neq t_i$ the product $v_{i,t} \cup Z_i$ contains $\det V_{i,t}$ twice and vanishes. Each $c_{i,t}$ is nonzero, because $W_{F_i}(P_i)$ is a one-dimensional F_i -vector space and a nonzero vector has a nonzero component under every embedding of F_i . Hence

$$\Gamma_*(Y) = \text{pr}_{2*}(\text{pr}_1^* Y \cup \Gamma) = \pm \prod_i c_{i,t_i}^{|n_i|} w \neq 0,$$

and w lies in the complex span of algebraic classes. The same argument with every t_i replaced by $-t_i$ gives $\bar{w}$.

Rational descent. Let $\mathcal{Z} \subseteq H^6(A_\rho, \mathbb{Q})$ be the span of algebraic classes. Then $W_K(A_\rho) \otimes \mathbb{C} = \mathbb{C}w \oplus \mathbb{C}\bar{w} \subseteq \mathcal{Z} \otimes \mathbb{C}$, so $(W_K(A_\rho) + \mathcal{Z}) \otimes \mathbb{C} = \mathcal{Z} \otimes \mathbb{C}$, and $W_K(A_\rho) \subseteq \mathcal{Z}$ by comparing dimensions. $\square$

5. DOMINANCE

5.1. The codifferential. Let $C \in \text{Hur}$ with branch set B_C , $|B_C| = 13$, $\infty \notin B_C$, and put $\mathcal{D}_\rho(C) = \text{Hom}_G(\rho, H^0(C, K_C))$ and $\mathcal{D}_{\bar{\rho}}(C) = \text{Hom}_G(\bar{\rho}, H^0(C, K_C))$, each of dimension three. The tangent space to Hur at C modulo PGL_2 is Serre dual to $H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$, of dimension 10. Let

$$\mu_C: \mathcal{D}_\rho(C) \otimes \mathcal{D}_{\bar{\rho}}(C) \rightarrow H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$$

be multiplication of differentials followed by contraction of $\rho \otimes \bar{\rho}$ against its invariant line, with values in $H^0(C, K_C^{\otimes 2})^G = H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$.

Lemma 5.1 (Codifferential). *The cotangent space of $\mathcal{H}_3$ at the period point of $A_\rho(C)$ is $\mathcal{D}_\rho(C) \otimes \mathcal{D}_{\bar{\rho}}(C)$, of dimension 9, and μ_C is, up to a nonzero normalization, the codifferential at C of the period map of the family A_ρ . If $\text{rk } \mu_{C_1} = 9$ for one C_1 in a connected component S of Hur , the period map is a submersion at C_1 and has rank 9 on a nonempty Zariski open subset of S .*

Proof. The codifferential of the period map of $J(C)$ along the Hurwitz family is multiplication $H^0(K_C) \otimes H^0(K_C) \rightarrow H^0(K_C^{\otimes 2})$ followed by restriction to G -invariants [4, §§2–3]. Under $H^0(C, K_C) = \bigoplus_\tau \tau \otimes \text{Hom}_G(\tau, H^0(K_C))$, the part that pairs the ρ - and $\bar{\rho}$ -components and lands in the invariants is μ_C . The domain $\mathcal{H}_3$ parametrizes the ρ -eigenspace of $H^{1,0}(A_\rho)$, which is $\mathcal{D}_\rho(C)$, and its tangent space is $\text{Hom}(H_\rho^{1,0}, H_\rho^{0,1})$; the polarization identifies $H_\rho^{0,1}$ with the dual of $H_{\bar{\rho}}^{1,0} = \mathcal{D}_{\bar{\rho}}(C)$, so the cotangent space is $\mathcal{D}_\rho(C) \otimes \mathcal{D}_{\bar{\rho}}(C)$. The rank is lower semicontinuous and 9 is its maximum, and S is irreducible. $\square$

5.2. An explicit cover. Let $\mathbb{L}$ be the rank-four local system on $\mathbb{P}_z^1 - \{0, 5, -7, \infty\}$ of the Fuchsian system $dY = \left(\frac{B_0}{z} + \frac{B_5}{z-5} + \frac{B_{-7}}{z+7}\right)Y dz$ with

$$B_0 = \text{diag}\left(-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{2}, \frac{3}{4}\right), \quad B_5 = u_1 v_1^T, \quad B_{-7} = u_2 v_2^T,$$

with $B_\infty = -(B_0 + B_5 + B_{-7})$ and the rational vectors u_i, v_i of Appendix A. One checks there that $\text{tr } B_5 = \frac{1}{3}$, $\text{tr } B_{-7} = \frac{2}{3}$, and that B_∞ is diagonalizable with eigenvalues $\pm \frac{2}{3}$, each twice. No two eigenvalues of a residue differ by a nonzero integer, so each local monodromy is conjugate to $\exp(2\pi i B)$ for the residue B : it lies in $\mathcal{R}_1$ at 5, in $\mathcal{R}_2$ at -7 , in the class of x (eigenvalues $(i, -1, -1, -i)$) at 0, and in the class of y (eigenvalues $(\omega, \omega, \omega^2, \omega^2)$, order three) at ∞ .

The monodromy of $\mathbb{L}$. The tuple (r_1, r_2, x, y) of Appendix A has product one, lies in the same four conjugacy classes of $GL_4(\mathbb{C})$, generates G , and is absolutely irreducible (the K -algebra generated by its matrices is $M_4(K)$). The dimensions of the four classes are $6 + 6 + 10 + 8 = 30 = 2(4^2 - 1)$. Let $\mathbb{F}$ be the local system of the tuple (r_1, r_2, x, y) on $\mathbb{P}^1$ minus $\{5, -7, 0, \infty\}$, and recall that an irreducible local system is *physically rigid* if every local system with the same local monodromy classes at the same points is isomorphic to it. By Katz’s theorem [8, Theorem 1.1.2], an irreducible local system whose local classes have total dimension $2(n^2 - 1)$, n the rank, is physically rigid. This applies to $\mathbb{F}$, and only $\mathbb{F}$ needs to be irreducible; we recall the argument, since $\mathbb{L}$ is not yet known to be irreducible. Let $\mathcal{H} = \text{Hom}(\mathbb{F}, \mathbb{L})$ and j the inclusion of the complement of the four points. Since the local monodromies of $\mathbb{F}$ and $\mathbb{L}$ are conjugate, the invariants of $\mathcal{H}$ at each point have the dimension of the centralizer of the class, $16 - \dim(\text{class})$, so

$$\chi(\mathbb{P}^1, j_* \mathcal{H}) = (2 - 4) \cdot 16 + \sum (16 - \dim) = -32 + 64 - 30 = 2.$$

Hence $h^0(j_* \mathcal{H}) + h^2(j_* \mathcal{H}) \geq 2$, and Poincaré–Verdier duality gives $H^2(\mathbb{P}^1, j_* \mathcal{H})^\vee \cong H^0(\mathbb{P}^1, j_* \mathcal{H}^\vee)$ with $\mathcal{H}^\vee = \text{Hom}(\mathbb{L}, \mathbb{F})$. So there is a nonzero

homomorphism $\mathbb{F} \rightarrow \mathbb{L}$ or $\mathbb{L} \rightarrow \mathbb{F}$. Since $\mathbb{F}$ is irreducible, the first is injective and the second surjective, and both are isomorphisms because the ranks are equal. So $\mathbb{L} \cong \mathbb{F}$: its monodromy tuple with respect to standard loops around $5, -7, 0, \infty$ is simultaneously conjugate to (r_1, r_2, x, y) , it is irreducible, and its monodromy group is $\rho(G)$.

The pullback. Let

$$\varphi(s) = \frac{s^2(s-3)^4}{(s-1)^3(s+2)^3},$$

of degree six. Its fibre over 0 has type $[2, 4]$, over ∞ type $[3, 3]$, its fibres over 5 and -7 are reduced, and its two other critical values are the roots of $729z^2 - 3674z + 5120$, which are not real. At a point of ramification index e over a branch value with local monodromy g , the local monodromy of $\varphi^*\mathbb{L}$ is conjugate to g^e . So $\varphi^*\mathbb{L}$ has local monodromy in $\mathcal{R}_1$ at the six roots of $P_5 = s^2(s-3)^4 - 5(s-1)^3(s+2)^3$, in $\mathcal{R}_2$ at the six roots of $P_{-7} = s^2(s-3)^4 + 7(s-1)^3(s+2)^3$, in $\mathcal{I}$ at $s = 0$ (class of x^2), and trivial local monodromy at $s = 3$ (since $x^4 = 1$) and at $s = 1, -2$ (since $y^3 = 1$).

Lemma 5.2. *The monodromy group of $\varphi^*\mathbb{L}$ is $\rho(G)$, so $\varphi^*\mathbb{L}$ is the ρ -local system of a cover $C_0 \in \text{Hur}$, branched over the thirteen points 0 and the roots of $P_5 P_{-7}$.*

Proof. Over the complement of the branch values, φ is an unramified cover of degree six, so the monodromy group Π of $\varphi^*\mathbb{L}$ is the image of a subgroup of index six of π_1 of the complement downstairs, and $[\rho(G) : \Pi] \leq 6$. The image of Π in $G/Z(G) \cong PSp_4(3)$ has index at most 6, and every proper subgroup of $PSp_4(3)$ has index at least 27 [2]; so $G = \Pi Z(G)$ (identifying G with $\rho(G)$), and since $Z(G)$ is central, $[G, G] = [\Pi, \Pi] \subseteq \Pi$. The derived group has index three and is the kernel of $\det \rho$, and Π contains a reflection in $\mathcal{R}_1$, of determinant $\omega \neq 1$; so $\Pi = G$. The local system $\varphi^*\mathbb{L}$ thus defines a surjection from π_1 of the complement of the thirteen points onto G , hence a connected Galois G -cover, with the local classes listed above. $\square$

5.3. Differentials from a Fuchsian system. Let $C \in \text{Hur}$, and suppose that the ρ -local system of C is the local system of a Fuchsian system $dY = \Omega Y$, $\Omega = \sum_{p \in P} A_p ds/(s-p)$, on the trivial bundle of rank four over $\mathbb{P}_s^1$, whose set of poles $P \subset \mathbb{C}$ contains the branch set B_C ; ∞ is not a pole, so $\sum_p A_p = 0$. At the poles not in B_C (*apparent singularities*) the local monodromy is trivial. Let Y be a fundamental solution near a base point.

Local form. At $p \in P$ let $\Lambda_p = \text{diag}(\lambda_{p,1}, \dots, \lambda_{p,4})$ be the eigenvalues of A_p . We say that Y has *normal form* at p if near p there is a fundamental solution

$$Y_p = P_p H_p(s) (s-p)^{\Lambda_p}, \quad H_p \text{ holomorphic}, \quad H_p(p) = 1,$$

with P_p an eigenbasis of A_p . Substituting gives the recursion $kH_k + H_k \Lambda_p - \Lambda_p H_k = R_k$ for the Taylor coefficients H_k of $P_p^{-1} \cdot (P_p H_p)$, with R_k determined by $H_0, \dots, H_{k-1}$ and the Taylor coefficients of $\sum_{q \neq p} A_q/(s-q)$ at

p ; entrywise $(k + \lambda_{p,j} - \lambda_{p,i})(H_k)_{ij} = (R_k)_{ij}$. If no two eigenvalues of A_p differ by a positive integer, the recursion has a unique solution and normal form holds. At a *resonance* $k = \lambda_{p,i} - \lambda_{p,j}$ the entry $(H_k)_{ij}$ is free, and normal form holds if and only if the corresponding entries $(R_k)_{ij}$, the *logarithmic obstructions*, vanish; then, if Y_p is one solution in normal form, so is $Y_p(1 + cE_{ij}) = P_p H_p(1 + cE_{ij}(s-p)^k)(s-p)^{\Lambda_p}$, because $(s-p)^{\Lambda_p}$ conjugates $1 + cE_{ij}$ to $1 + cE_{ij}(s-p)^k$; this changes $(H_k)_{ij}$ by c and no coefficient of lower order. So every choice of the free entries gives a solution in normal form.

Lemma 5.3 (The dictionary). *Suppose that Y has normal form at every $p \in P$. Up to exchanging ρ and $\bar{\rho}$ (which does not affect μ_C), $\mathcal{D}_\rho(C)$ is identified with the space of $r \in \mathbb{C}(s)^4$ such that the vector of forms $Y^{-1}r ds$ is holomorphic on C , and $\mathcal{D}_{\bar{\rho}}(C)$ with the space of $q \in \mathbb{C}(s)^4$ such that $Y^T q ds$ is holomorphic on C . Concretely, r belongs to the first space if and only if*

$$\text{ord}_p[(P_p H_p)^{-1}r]_j \geq \lfloor \lambda_{p,j} \rfloor \quad (p \in P, j = 1, \dots, 4),$$

r is holomorphic on $\mathbb{C} - P$, and $\text{ord}_\infty r \geq 2$; for q the same holds with $(P_p H_p)^T$ and $\lfloor -\lambda_{p,j} \rfloor$. Under these identifications, $\mu_C(r \otimes q) = q^T r ds^2$.

Proof. Pulled back to the complement of the ramification points in C , Y is single-valued, because its monodromy factors through ρ and the loops that lift to closed loops on C map to 1 in G ; and $g^*Y = Y\rho'(g)$ for $g \in G$, with ρ' equal to ρ or $\bar{\rho}$ depending on conventions. For a rational vector r , the vector $\omega = Y^{-1}r ds$ of meromorphic forms then transforms by $g^*\omega = \rho'(g)^{-1}\omega$, so its components span the image of a G -map from $\rho^\vee$ to the meromorphic forms on C . Conversely, a G -map from $\rho^\vee$ to $H^0(K_C)$ is a vector ω of holomorphic forms with this transformation law, and $r = Y\omega/ds$ is G -invariant, hence descends to $\mathbb{P}_s^1 - P$; it is meromorphic at P because Y and Y^{-1} have moderate growth at the poles of a Fuchsian system, and at ∞ , which is neither a pole nor a branch point, Y is holomorphic and invertible, so r vanishes to order at least two there, ds having a double pole. So r is rational. Since ρ is unitary, $\rho^\vee \cong \bar{\rho}'$; the same argument with Y^T gives the second space, and $q^T Y \cdot Y^{-1}r = q^T r$ is the invariant contraction.

The order condition: at $p \in P$ the local monodromy has finite order e , so $e\lambda_{p,j} \in \mathbb{Z}$; in a local coordinate y on C with $s-p = y^e$ the j -th component of $Y_p^{-1}r ds$ is $[(P_p H_p)^{-1}r]_j (s-p)^{-\lambda_{p,j}} ds$, of order $e \text{ord}_p[(P_p H_p)^{-1}r]_j - e\lambda_{p,j} + e - 1$ in y , and this is ≥ 0 exactly when $\text{ord}_p[(P_p H_p)^{-1}r]_j \geq \lfloor \lambda_{p,j} \rfloor$. Holomorphy of the vector $Y^{-1}r ds$ does not depend on the choice of fundamental solution, since two of them differ by a constant matrix. $\square$

Lemma 5.4. *In the situation of Lemma 5.3, put $M_p = \max(0, -\min_j \lfloor \lambda_{p,j} \rfloor)$.*

- (1) *Every $r \in \mathcal{D}_\rho(C)$ is uniquely of the form $r = \sum_{p \in P} \sum_{k=1}^{M_p} c_{p,k}(s-p)^{-k}$ with $c_{p,k} \in \mathbb{C}^4$; likewise for $\mathcal{D}_{\bar{\rho}}(C)$ with $-\lambda_{p,j}$ in place of $\lambda_{p,j}$.*

- (2) $\mathcal{D}_\rho(C)$ is the kernel of the linear map Φ from the space of these coefficients to the Laurent coefficients that the order conditions require to vanish, together with the coefficient of s^{-1} at ∞ ; it has dimension 3.
- (3) If a square submatrix $\Phi_{R,F'}$ of size $n-3$ (n the number of unknowns), on a row set R and the complement F' of three columns F , is invertible, then the vectors x_f , $f \in F$, with coordinate 1 at f , 0 at the other columns of F , and $-\Phi_{R,F'}^{-1}\Phi_{R,f}$ on F' , form a basis of $\ker \Phi$.

Proof. (1) Away from P , Y is holomorphic and invertible, so r is holomorphic there; at $p \in P$, $P_p H_p$ is holomorphic and invertible, so $\text{ord}_p r \geq -M_p$; and r vanishes at ∞ , so it is the sum of its principal parts, which are independent. (2) Each condition is the vanishing of finitely many Laurent coefficients, linear in the unknowns; the dimension is 3 by Lemma 3.6. (3) The x_f lie in the kernel of the rows R , which has dimension 3 since those rows have rank $n-3$, and they are independent; this kernel contains $\ker \Phi$, of dimension 3. $\square$

Lemma 5.5. *Let $B \subset \mathbb{C}$ be a set of 13 points and D_B the monic polynomial with roots B . Then $H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B)) = \{N(s) ds^2 / D_B(s) : \deg N \leq 9\}$.*

Proof. A section is $g(s) ds^2$ with g holomorphic on $\mathbb{C} - B$, with at most simple poles on B , and, since ds^2 has a pole of order four at ∞ , $g = O(s^{-4})$ there. So gD_B is an entire function of growth $O(s^9)$, a polynomial of degree at most 9; conversely every such quotient qualifies. $\square$

We call the coefficients of N the *numerator coordinates*. In these coordinates and in bases of $\mathcal{D}_\rho(C)$ and $\mathcal{D}_{\bar{\rho}}(C)$, μ_C is a 9×10 matrix.

5.4. The isomonodromic deformation. Let $P = \{t_q\}$ be the poles of a Fuchsian system with residues A_q , $\sum_q A_q = 0$. Let $t_q(\theta)$ be analytic functions of a complex parameter θ near 0, with $t_q(0) = t_q$, pairwise distinct, and consider the *Schlesinger equations*

$$(1) \quad \frac{dA_p}{d\theta} = \sum_{q \neq p} [A_q, A_p] \frac{d}{d\theta} \log(t_p(\theta) - t_q(\theta)), \quad A_p(0) = A_p.$$

Lemma 5.6. *The solution of (1) is analytic near $\theta = 0$, and its Taylor series is the unique formal power series solution. For θ near 0:*

- (1) $\sum_q A_q(\theta) = 0$, and each $A_q(\theta)$ is conjugate to A_q ;
- (2) the connection $\Omega = \sum_q A_q(\theta) d \log(s - t_q(\theta))$ on $\{(s, \theta) : s \neq t_q(\theta)\}$ is flat, so the monodromy representation of the system at θ , transported along the fibration, does not depend on θ ; in particular local monodromies stay trivial at the apparent singularities;
- (3) if a pole t_p does not move ($t_p(\theta) = t_p$) and the system at $\theta = 0$ has normal form at t_p , then the system at θ has normal form at t_p , with $P_{t_p}(\theta)$ and $H_{t_p}(s, \theta)$ analytic in θ .

Proof. Equation (1) is an analytic ordinary differential equation, so the initial value problem has a unique analytic solution, whose Taylor coefficients satisfy the recursion obtained by expanding (1); that recursion determines them uniquely.

(1) The derivative of $\sum_p A_p$ is $\sum_p \sum_{q \neq p} [A_q, A_p] m_{pq}$ with $m_{pq} = \frac{d}{d\theta} \log(t_p - t_q)$ symmetric in p, q , which vanishes by antisymmetry. Each equation has the form $A'_p = [S_p, A_p]$, so $A_p(\theta) = G_p(\theta) A_p G_p(\theta)^{-1}$ with $G'_p = S_p G_p$, $G_p(0) = 1$.

(2) Put $\ell_q = d \log(s - t_q(\theta))$. Then $d\Omega = \sum_q dA_q \wedge \ell_q$ and $\Omega \wedge \Omega = \sum_{p < q} [A_p, A_q] \ell_p \wedge \ell_q$. With $a = s - t_p$ and $b = s - t_q$ one has $\ell_p \wedge \ell_q = d \log(a - b) \wedge (\ell_q - \ell_p)$, and $d \log(a - b) = d \log(t_q - t_p) =: m_{pq} d\theta$. Hence $\Omega \wedge \Omega = \sum_q (\sum_{p \neq q} [A_p, A_q] m_{pq} d\theta) \wedge \ell_q$, and the flatness condition $d\Omega = \Omega \wedge \Omega$ is (1). For θ in a small disc the space $\{(s, \theta) : s \neq t_q(\theta)\}$ is a locally trivial fibration over the disc, and parallel transport of a flat connection along a path in the base identifies the monodromy representations of the fibres.

(3) Near $s = t_p$, the θ -component of Ω is $\Omega_\theta = -\sum_{q \neq p} A_q(\theta) t'_q(\theta)/(s - t_q(\theta))$ (the term $q = p$ vanishes because $t'_p = 0$), which is holomorphic in (s, θ) near $(t_p, 0)$. Let $G(s, \theta)$ solve $\partial_\theta G = \Omega_\theta G$, $G(s, 0) = 1$; it is holomorphic near $(t_p, 0)$. By flatness, $G(s, \theta) P_{t_p} H_{t_p}(s) (s - t_p)^\Lambda$ is a fundamental solution of the system at θ , and it equals $P(\theta) H(s, \theta) (s - t_p)^\Lambda$ with $P(\theta) = G(t_p, \theta) P_{t_p}$ and $H(s, \theta) = P(\theta)^{-1} G(s, \theta) P_{t_p} H_{t_p}(s)$, holomorphic with $H(t_p, \theta) = 1$. Since $A_p(\theta)$ is the residue, $P(\theta)$ is an eigenbasis of it. $\square$

5.5. The base cover and the certificate. At C_0 the Fuchsian system is $\varphi^* \mathbb{L}$, written on $\mathbb{P}_s^1$: its poles are $s = 0$ (residue $2B_0$, eigenvalues $-\frac{3}{2}, -1, -1, \frac{3}{2}$), the twelve roots of $P_5 P_{-7}$, the *reflection poles* (residues equal to B_5 at the roots of P_5 and to B_{-7} at the roots of P_{-7} , since φ is unramified there and $\varphi^* \frac{dz}{z-5}$ has residue 1 at a simple root of $\varphi - 5$; eigenvalues $(0, 0, 0, \frac{1}{3})$ and $(0, 0, 0, \frac{2}{3})$), and three apparent singularities, $s = 3$ (residue $4B_0$, eigenvalues $-3, -2, -2, 3$) and $s = 1, -2$ (residue $3B_\infty$, eigenvalues $-2, -2, 2, 2$). The residue at ∞ is 0, since $\varphi(\infty) = 1$ is not a branch value.

Lemma 5.7. *The system at C_0 has normal form at every pole.*

Proof. At the twelve reflection poles no two eigenvalues differ by a positive integer. At $s = 0$, the system $\mathbb{L}$ has a solution $Y_{\mathbb{L}} = F(z) z^{B_0}$ near $z = 0$ with F holomorphic and $F(0) = 1$, since B_0 is diagonal and nonresonant; with $\varphi(s) = s^2 u(s)$, $u(0) \neq 0$, the pullback $F(\varphi(s)) u(s)^{B_0} s^{2B_0}$ is in normal form with $\Lambda = 2B_0$. The same argument applies at $s = 3$, where $\varphi = (s - 3)^4 u$, and at $s = 1, -2$, where $1/\varphi$ vanishes to order three and $\mathbb{L}$ has a solution $F_\infty(1/z) z^{-B_\infty}$ in a basis diagonalizing B_∞ . $\square$

The direction. Let $E_5 = \mathbb{Q}[s]/(P_5)$ and $E_{-7} = \mathbb{Q}[s]/(P_{-7})$; both are fields of degree six. For polynomials $V_5, V_{-7} \in \mathbb{Q}[s]$ of degree at most five, move

each root t of P_5 to $t + \theta V_5(t)$ and each root t of P_{-7} to $t + \theta V_{-7}(t)$, keeping $s = 0$ and the three apparent singularities fixed; the reflection poles are then the *moving poles*. Let $A_p(\theta)$ be the solution of (1). By Lemma 5.6(2), for small θ the system at θ is the ρ -local system of the cover $C_\theta \in \text{Hur}$ obtained from C_0 by moving its branch points along this path, so C_θ lies on the component Hur° of C_0 . By Lemma 5.6(3) and Lemma 5.7 it has normal form at $0, 3, 1, -2$, and at the moving poles the eigenvalues stay nonresonant; so Lemmas 5.3 and 5.4 apply at every small θ .

What is computed. We fix the choices that enter Lemma 5.4, and show that every Taylor coefficient is rational.

- (1) *The residues over $\mathbb{Q}$.* For $\epsilon \in \{5, -7\}$ the six embeddings $E_\epsilon \rightarrow \mathbb{C}$ send the class of s to the six roots of P_ϵ ; for $X \in M_4(E_\epsilon)[[\theta]]$ and a root t write $X(t)$ for the image. Consider the system of formal differential equations in $\mathcal{A}_5 \in M_4(E_5)[[\theta]]$, $\mathcal{A}_{-7} \in M_4(E_{-7})[[\theta]]$ and $A_p \in M_4(\mathbb{Q})[[\theta]]$, $p = 0, 3, 1, -2$, with $\mathcal{A}_\epsilon(0) = B_\epsilon$ and $A_p(0)$ the residues at $\theta = 0$, whose right-hand sides are the following elements, written for $\mathcal{A}_5$.

- The other roots of P_5 : for roots $a \neq b$ one has $t_a(\theta) - t_b(\theta) = (a - b)(1 + \theta \Delta(a, b))$ with the divided difference $\Delta(a, b) = \frac{V_5(a) - V_5(b)}{a - b}$, a polynomial in a, b with rational coefficients. So the term of b in (1) at a is $[\mathcal{A}_5(b), \mathcal{A}_5(a)] \Delta(a, b) / (1 + \theta \Delta(a, b))$, with no division by $a - b$. Summing over all b , the diagonal included, is the trace from $E_5 \otimes_{\mathbb{Q}} E_5$ to the first factor; the diagonal term, where $\Delta(a, a) = V_5'(a)$, is a commutator $[\mathcal{A}_5(a), \mathcal{A}_5(a)] = 0$. So the sum is the image of an element of $M_4(E_5)[[\theta]]$.
- The roots of P_{-7} : the term of a root b of P_{-7} involves $\frac{V_5(a) - V_{-7}(b)}{a - b + \theta(V_5(a) - V_{-7}(b))}$. In the algebra $E_5 \otimes_{\mathbb{Q}} E_{-7}$, which is finite étale over $\mathbb{Q}$, hence a finite product of fields, $a \otimes 1 - 1 \otimes b$ is a unit, since P_5 and P_{-7} have no common root (they are fibres of φ over different points). So the term is the image of an element of $M_4(E_5 \otimes E_{-7})[[\theta]]$, and the sum over b is its trace to E_5 .
- The fixed poles q : the term is $[A_q, \mathcal{A}_5(a)] V_5(a) / (a - q + \theta V_5(a))$, and $a - q$ is a unit of E_5 .

The equations for $\mathcal{A}_{-7}$ are symmetric, and those for A_p involve sums over all roots of P_5 and of P_{-7} , which are traces to $\mathbb{Q}$. This system has a unique formal solution, with coefficients in $M_4(E_5)$, $M_4(E_{-7})$ and $M_4(\mathbb{Q})$, computed order by order. Its images under the embeddings, $A_t(\theta) = \mathcal{A}_\epsilon(\theta)(t)$ at the roots t and $A_p(\theta)$ at the fixed poles, satisfy (1), so by uniqueness they are its solution. Every later sum over the moving poles is likewise a trace to $\mathbb{Q}$.

- (2) *Fixed poles.* At $p = 0, 3, 1, -2$ the eigenvalues of $A_p(\theta)$ are constant (Lemma 5.6(1)). Take $P_p(0)$ the basis of eigenvectors of $A_p(0)$ given by the reduced echelon bases of the eigenspaces, ordered by increasing eigenvalue, and $P_p(\theta) = P_p(0)(1 + Y(\theta))$ with $Y(0) = 0$ and

every entry of Y between two equal eigenvalues equal to 0; and take the solution H_p of the recursion of Subsection 5.3 with every free resonant entry equal to 0.

- (3) *Moving poles.* At a root t of P_5 let $w_t(\theta)$ be the left eigenvector of $A_t(\theta)$ for the eigenvalue $\frac{1}{3}$ with $w_t(0) = v_1$ and $w_t(\theta)^T u_1 = \frac{1}{3}$; at a root of P_{-7} the left eigenvector for $\frac{2}{3}$ with $w_t(0) = v_2$ and $w_t(\theta)^T u_2 = \frac{2}{3}$. At $\theta = 0$ these conditions hold because $v_i^T u_i = \text{tr } B$.
- (4) *Unknowns.* For $\mathcal{D}_\rho$: the coefficients $c_{p,k} \in \mathbb{C}^4$ of Lemma 5.4(1) at $p = 0, 3, 1, -2$, with $k \leq 2, 3, 2, 2$; 36 unknowns, ordered by p in this order, then by k , then by coordinate. There are none at the moving poles, where $\lfloor \lambda \rfloor \geq 0$. For $\mathcal{D}_{\bar{\rho}}$: the same 36, and, since $\lfloor -\frac{1}{3} \rfloor = \lfloor -\frac{2}{3} \rfloor = -1$, a simple pole at each moving pole, whose residue must be a multiple of $w_t(\theta)$ by the order condition of Lemma 5.3. By Lagrange interpolation that multiple is $\beta_5(t) = \sum_{v=0}^5 \beta_{5,v} t^v$ at the roots of P_5 and $\beta_{-7}(t) = \sum_v \beta_{-7,v} t^v$ at the roots of P_{-7} ; the twelve scalars $\beta_{5,0}, \dots, \beta_{5,5}, \beta_{-7,0}, \dots, \beta_{-7,5}$ come last, 48 unknowns in all.
- (5) *Equations.* The rows of Φ are the Laurent coefficients required to vanish by Lemma 5.3 at $p = 0, 3, 1, -2$, ordered by p , then by the component j , then by the order, from -3 to $\lfloor \pm \lambda_{p,j} \rfloor - 1$; then the four coordinates of the coefficient of s^{-1} at ∞ : 45 rows for $\mathcal{D}_\rho$ and 57 for $\mathcal{D}_{\bar{\rho}}$. Their coefficients are rational, by (1).
- (6) *Kernel bases.* $\Phi(0)$ has rank 33, resp. 45. The free columns F (numbered from 1 in the order of (4)) are 11, 23, 31 for $\mathcal{D}_\rho$, the third coordinate of $c_{p,1}$ at $p = 3, 1, -2$, and 43, 47, 48 for $\mathcal{D}_{\bar{\rho}}$, the scalars $\beta_{-7,0}, \beta_{-7,4}, \beta_{-7,5}$. The row set R consists of the first rows, in the order of (5), that are independent on the columns F' . Since the unknowns are rational coordinates, the vectors $x_f(\theta)$ of Lemma 5.4(3) have rational Taylor coefficients, and so do the bases r_1, r_2, r_3 of $\mathcal{D}_\rho$ and q_1, q_2, q_3 of $\mathcal{D}_{\bar{\rho}}$ that they define.
- (7) *The matrix.* Let $D_{5,\theta}(s) = \prod_t (s - t - \theta V_5(t))$ over the roots t of P_5 , likewise $D_{-7,\theta}$, and $D_\theta = s D_{5,\theta} D_{-7,\theta}$, the monic polynomial whose roots are the thirteen branch points of C_θ ; its coefficients are rational in each order, as symmetric functions. By Lemma 5.5, $q_b^T r_a ds^2 = N_{ab}(s) ds^2 / D_\theta(s)$ with $\deg N_{ab} \leq 9$. Let $\mathbf{M}(\theta)$ be the 9×10 matrix whose row $3(a-1) + b$ lists the coefficients of $s^0, \dots, s^9$ of N_{ab} .

Lemma 5.8 (Jets). *The quantities of (2)–(7) are analytic in θ near 0, $\mathbf{M}(\theta)$ is the matrix of μ_{C_θ} in the bases (r_a) , (q_b) and in numerator coordinates, and the Taylor series of all of them are the unique formal solutions of their defining equations, computed order by order. Their Taylor coefficients are rational.*

Proof. The residues are analytic by Lemma 5.6. The spectral projectors of $A_p(\theta)$ for its constant eigenvalues are analytic (Riesz integrals), so an

analytic eigenbasis exists, and right multiplication by an analytic block-diagonal matrix gives the normalization of (2); it is unique, so it coincides with the formal solution. The same holds for the eigenvector of the simple eigenvalue in (3), since $w_t(0)^T u_i \neq 0$. The recursion for H_k has nonzero denominators off the resonant entries, which are set to 0, so each $H_k(\theta)$ is a polynomial in analytic quantities; by Lemma 5.6(3) and the remark before Lemma 5.3 the logarithmic obstructions vanish and this choice is a solution in normal form. The conditions involve only finitely many H_k . The block $\Phi_{R,F'}(0)$ is invertible, so its inverse is analytic near 0; since $\dim \ker \Phi(\theta) = 3$ for every small θ (Lemma 5.4(2)), the $x_f(\theta)$ span it. Finally D_θ has coefficients analytic in θ , and the numerator of $q^T r$ is a polynomial in the previous data. Rationality was shown in (1)–(7). $\square$

Proposition 5.9. *Let*

$$V_5(s) = -\frac{3}{7}s^5 + \frac{3}{7}s^4 + \frac{3}{7}s^3 + \frac{3}{7}s^2 + \frac{1}{7}s - \frac{2}{7}, \quad V_{-7}(s) = \frac{3}{7}s^3 - \frac{3}{7}s - \frac{1}{7}.$$

(1) $\text{rk } \mathbf{M}(0) = \text{rk } \mu_{C_0} = 5$.

(2) *Let $m(\theta)$ be the 9×9 minor of $\mathbf{M}(\theta)$ that omits the column of the coefficient of s^0 . Then*

$$m(\theta) = c\theta^7 + O(\theta^8), \quad c \approx -1.031 \cdot 10^{42},$$

where $2^{72}c$ is the integer

$$-4869337150302392927709794715527302500429442156335511356109310625.$$

(3) *Hence $\text{rk } \mu_{C_\theta} = 9$ for every sufficiently small $\theta \neq 0$.*

Proof. Parts (1) and (2) are an exact computation over $\mathbb{Q}$ of the Taylor coefficients of $\mathbf{M}(\theta)$ to order 12, described in Appendix C; by Lemma 5.8 they are the Taylor coefficients of the analytic matrix $\mathbf{M}(\theta)$. Part (3): m is analytic and not identically zero, so it has no zeros in a punctured neighbourhood of 0. $\square$

The value of c depends on the normalizations of (2)–(7); its nonvanishing does not. All ten 9×9 minors of $\mathbf{M}(\theta)$ have their first nonzero coefficient at θ^7 (Appendix C). An invariant form of this statement is the following.

Remark 5.10 (The Smith form). Over $\mathbb{C}[[\theta]]$ the Smith normal form of $\mathbf{M}(\theta)$ is $\text{diag}(1, 1, 1, 1, 1, \theta, \theta^2, \theta^2, \theta^2)$. In particular the ideal generated by its 9×9 minors is (θ^7) . This does not depend on the bases: analytic changes of the bases of $\mathcal{D}_\rho(C_\theta)$, $\mathcal{D}_{\bar{\rho}}(C_\theta)$ and of the numerator coordinates multiply $\mathbf{M}(\theta)$ on both sides by invertible matrices over $\mathbb{C}[[\theta]]$. To see the Smith form, write $\mathbf{M}(\theta)$ in constant bases adapted to the kernel and image of $\mathbf{M}(0)$, so that the upper left 5×5 block $\mathbf{M}_{11}(\theta)$ is invertible. The Schur complement $\mathbf{S}(\theta) = \mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{12}$, a 4×5 matrix, has the same invariant factors as $\mathbf{M}$ apart from five units, and $\mathbf{S}(\theta) = \theta\mathbf{S}_1 + \theta^2\mathbf{S}_2 + O(\theta^3)$. The computation gives $\text{rk } \mathbf{S}_1 = 1$, and the 3×4 block of $\mathbf{S}_2$ between the cokernel and the kernel of $\mathbf{S}_1$ has rank 3; so the invariant factors of $\mathbf{S}$ are $\theta, \theta^2, \theta^2, \theta^2$. Since $\mathbf{S}_1$ has rank one, no argument at first order in θ gives rank 9.

Corollary 5.11. *On the component Hur° through C_0 the period map to $\mathcal{H}_3$ has rank 9 on a nonempty Zariski open set, and it is a submersion at C_θ for small $\theta \neq 0$.*

Proof. Lemma 5.1 and Proposition 5.9. $\square$

Remark 5.12 (The rank at C_0). Let $C_{\mathbb{L}} \rightarrow \mathbb{P}_z^1$ be the G -cover with ρ -local system $\mathbb{L}$, branched over $0, 5, -7, \infty$. By Proposition 3.1, ρ occurs in $H^1(C_{\mathbb{L}})$ with multiplicity $-8 + 1 + 1 + 4 + 4 = 2$ and Hodge numbers $(1, 1)$ (the ages of the inverses of r_1, r_2, x, y are $\frac{2}{3}, \frac{1}{3}, 2, 2$). The cover C_0 is the normalization of the fibre product $C_{\mathbb{L}} \times_{\mathbb{P}_z^1} \mathbb{P}_s^1$, connected by Lemma 5.2, so it maps G -equivariantly onto $C_{\mathbb{L}}$ with degree six, and pullback embeds $H^1(C_{\mathbb{L}})$ in $H^1(C_0)$ as a sub-Hodge structure. Its ρ -part, of dimension 2 and type $(1, 1)$, and its orthogonal complement under the polarization, of dimension 4 and type $(2, 2)$, split M_ρ , and $A_\rho(C_0)$ is isogenous to a product of abelian varieties of Weil type of dimensions 2 and 4, with Hermitian forms of signatures $(1, 1)$ and $(2, 2)$. The same holds for every cover obtained in this way from the system $\mathbb{L}$ with its four singular points moved and from a rational function with the same ramification as φ . So the period points of these covers lie on a special subvariety of $\mathcal{H}_3$ of type $U(1, 1) \times U(2, 2)$, of dimension $1 + 4 = 5$. The proof does not use this remark.

Remark 5.13 (No degeneration argument). The boundary criterion of [19] is not available here. The Hermitian space of discriminant -2 has Witt index 2: an indefinite Hermitian space of dimension at least 3 over K is isotropic, since its trace form is an indefinite rational quadratic form of dimension at least 6, so the space splits off two hyperbolic planes, leaving a plane of discriminant -2 , which is isotropic only when its discriminant is -1 . At a cusp stabilizing a two-dimensional isotropic subspace, the logarithms of a degeneration span at most a four-dimensional space of directions, less than nine.

6. THE DISCRIMINANT

6.1. The Hermitian form as a parabolic cup product. Let $\pi: C \rightarrow \mathbb{P}^1$ be a cover in Hur , $U_C = \mathbb{P}^1 - B_C$, and $\mathcal{R} = \pi_* \mathbb{Q}|_{U_C}$, a local system with fibre $\mathbb{Q}[G]$. Put $\mathbb{L}' = \text{Hom}_G(V_\rho, \mathcal{R} \otimes K)$, a local system of K -vector spaces of rank four. The fibre pairing of $\mathcal{R}$ (multiplication followed by the sum over the fibre), made conjugate-linear in the first variable, restricts on $V_\rho \otimes \mathbb{L}'$ to $h_V \otimes h_{\mathbb{L}'}$ for a flat positive definite Hermitian form $h_{\mathbb{L}'}$ on $\mathbb{L}'$, unique up to a positive rational factor. Let

$$V_{\text{par}} = H_{\text{par}}^1(U_C, \mathbb{L}'), \quad \mathcal{S}(a, b) = \int_{U_C} h_{\mathbb{L}'}(a \cup b), \quad h_{\text{par}} = \delta \mathcal{S},$$

the parabolic cohomology (the image of compactly supported cohomology), a K -space of dimension six, with its skew-Hermitian cup product $\mathcal{S}$ and the Hermitian form h_{par} .

Proposition 6.1. *There is a K -linear or K -antilinear isomorphism $H_1(A_\rho(C), \mathbb{Q}) \cong V_{\text{par}}$ under which the Hermitian form H of $A_\rho(C)$ is a nonzero rational multiple of h_{par} , or of its conjugate. In particular H and h_{par} have the same discriminant and, up to order, the same signature.*

Proof. Step 1. On $H_1(A, \mathbb{Q})$ the form $\mathcal{T} = H/2\delta$ is the unique K -sesquilinear skew-Hermitian form with $\text{Tr}_{K/\mathbb{Q}} \mathcal{T} = E$.

Step 2. Let $N' = H_1(J(C), \mathbb{Q})_\varepsilon$ and $M = \text{Hom}_{K[G]}(V_\rho, N')$. Evaluation $V_\rho \otimes_K M \rightarrow N'$ is a G -equivariant isomorphism, because $\text{End}_{K[G]}(V_\rho) = K$. A G -invariant sesquilinear form on N' is $h_V \otimes \psi$ for a unique sesquilinear form ψ on M , since the G -invariant sesquilinear forms on V_ρ are the multiples of h_V . The form $\mathcal{T}$ on N' is G -invariant by the uniqueness in Step 1, so $\mathcal{T} = h_V \otimes \psi$, and $\mathcal{T}|_{A_\rho} = h_V(v, v) \psi$ under $m \mapsto v \otimes m$, because e is fixed by the Rosati involution.

Step 3. Since π is finite and C is normal, $H^1(C, K) = H_{\text{par}}^1(U_C, \mathcal{R} \otimes K)$, with the cup product given by the fibre pairing. The functor $\text{Hom}_{K[G]}(V_\rho, -)$ is exact and commutes with these operations, so it identifies the ρ -part of the skew-Hermitian lift of the cup product with $h_V \otimes \mathcal{S}$.

Step 4. Poincaré duality on C and the identification of H_1 with H^1 by E are G -equivariant and carry $\mathcal{T}$ to $\pm$ the lift of Step 3. They carry the ρ -component to the ρ - or the $\bar{\rho}$ -component, according to conventions; accordingly the induced isomorphism $M \cong V_{\text{par}}$ is K -linear or K -antilinear, and $\psi = c\mathcal{S}$ or $\psi = c\bar{\mathcal{S}}$ with $c \in K^\times$. Both forms are skew-Hermitian, so $c = \bar{c}$ is rational. So $H|_{A_\rho} = 2\delta h_V(v, v) c\mathcal{S} = 2h_V(v, v) c h_{\text{par}}$, or the conjugate. Scaling a Hermitian form of dimension six by a rational number r multiplies its determinant by $r^6 = \text{Nm}(r^3)$, and $\det \bar{h} = \det h$. $\square$

The form $\mathcal{S}$ is computable from a monodromy tuple. Let $\mathcal{W}$ be a local system of K -vector spaces on $\mathbb{P}^1$ minus r points with a flat Hermitian form $h_{\mathcal{W}}$ and unitary monodromy $T_1, \dots, T_r$ along standard loops with $\ell_1 \cdots \ell_r = 1$, generating a finite group. Put $Q_i = T_1 \cdots T_{i-1}$. A parabolic cocycle is given by vectors a_i with $\sum_i Q_i(1 - T_i)a_i = 0$; its value on ℓ_i is $c_i = (1 - T_i)a_i$. Put $F_i = \sum_{j < i} Q_j c_j$.

Lemma 6.2. *For parabolic cocycles a, b , the cup product of their classes with the fibre pairing $h_{\mathcal{W}}$, on the fundamental class oriented so that the loops run clockwise, is*

$$\mathcal{S}(a, b) = \sum_{i=1}^r h_{\mathcal{W}}(Q_i^{-1} F_i^a + a_i, c_i^b).$$

It does not depend on the choice of the vectors a_i , which are determined by the cocycle modulo $\ker(1 - T_i)$; it vanishes on coboundaries and is skew-Hermitian.

Proof. We use the following Δ -complex structure on $\mathbb{P}^1$ (Figure 1). Its vertices are a base point b_0 , a second point b_∞ , and the punctures $p_1, \dots, p_r$. Its edges are $e_0, \dots, e_{r-1}$ from b_0 to b_∞ , cutting the sphere into r sectors,

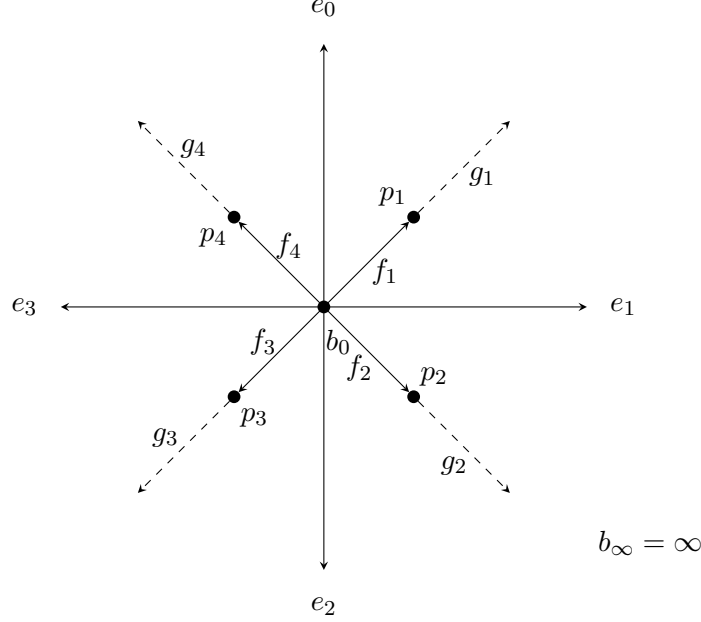


FIGURE 1. The Δ -complex of Lemma 6.2 for $r = 4$, drawn in $\mathbb{P}^1 - \{\infty\}$. The edges e_i and g_i end at $b_\infty = \infty$. The sector between e_{i-1} and e_i contains p_i and is cut by the path $f_i \cup g_i$ into the two triangles at p_i . The loop ℓ_i , out along e_{i-1} and back along e_i , runs clockwise around p_i .

with the sector between e_{i-1} and e_i containing p_i (indices of e modulo r); and, for each i , an edge f_i from b_0 to p_i and an edge g_i from p_i to b_∞ inside that sector. Its faces are the $2r$ triangles with vertices (b_0, p_i, b_∞) : for each i one with edges f_i, g_i, e_{i-1} and one with edges f_i, g_i, e_i . The loop ℓ_i goes out along e_{i-1} and back along e_i , and we orient $\mathbb{P}^1$ so that it runs clockwise around p_i .

The monodromy group is finite, so there is a finite Galois cover branched over the punctures on which W becomes trivial; lift the complex to it. There the cup product of 1-cochains followed by h_W , evaluated on the fundamental class, is the simplicial cup product $(f \cup f')[v_0, v_1, v_2] = h_W(f[v_0, v_1], f'[v_1, v_2])$ [7, §3.2]. Equivariant cochains contribute the same term on every lift of a triangle, so dividing by the degree gives a sum over the triangles of $\mathbb{P}^1$. Choose the lifts of the two triangles at p_i so that they share their edge g_i ; their edges f_i are then two lifts that differ by the local monodromy at p_i , and the two triangles enter the fundamental class with opposite signs. The terms of the two triangles combine, by the cocycle condition on each triangle, into

$$\mathcal{S}_{\text{simp}}(f, f') = \sum_i h_W(v_{e_i}^f - v_{e_{i-1}}^f, v_{g_i}^{f'}),$$

with v_ϵ^f the value of f on the chosen lift of ϵ . Write a, c, F for the data of f and a', c', F' for those of f' . The cocycle conditions on the two triangles at the i -th puncture give $a_i = Q_i^{-1}v_{f_i}$ and $v_{e_{i-1}} = v_{e_0} - F_i$. Substituting these into $\mathcal{S}_{\text{simp}}$ and summing gives

$$\mathcal{S}_{\text{simp}}(f, f') = - \sum_i [h_W(c_i, a'_i) + h_W(Q_i c_i, F'_i)] + h_W\left(\sum_i Q_i c_i, v_{e_0}^{f'}\right),$$

and the last term vanishes by the cocycle relation $\sum_i Q_i c_i = 0$. By unitarity of Q_i , $h_W(c_i, Q_i^{-1}F'_i) = h_W(Q_i c_i, F'_i)$, so

$$\overline{\mathcal{S}(b, a)} = \sum_i h_W(c_i, Q_i^{-1}F'_i + a'_i) = \sum_i [h_W(c_i, a'_i) + h_W(Q_i c_i, F'_i)],$$

and $\mathcal{S}_{\text{simp}}(f, f') = -\overline{\mathcal{S}(b, a)}$. It remains to show $\mathcal{S}(a, b) + \overline{\mathcal{S}(b, a)} = 0$. Since T_i is unitary and $c_i = (1 - T_i)a_i$, one has $h_W(c_i, c'_i) = h_W(a_i, c'_i) + h_W(c_i, a'_i)$, and since $F_{i+1} = F_i + Q_i c_i$,

$$h_W(F_{i+1}, F'_{i+1}) - h_W(F_i, F'_i) = h_W(F_i, Q_i c'_i) + h_W(Q_i c_i, F'_i) + h_W(c_i, c'_i).$$

Hence

$$\begin{aligned} \mathcal{S}(a, b) + \overline{\mathcal{S}(b, a)} &= \sum_i [h_W(F_{i+1}, F'_{i+1}) - h_W(F_i, F'_i)] \\ &= h_W(F_{r+1}, F'_{r+1}) - h_W(F_1, F'_1) = 0, \end{aligned}$$

because $F_1 = 0$ and $F_{r+1} = \sum_i Q_i c_i = 0$. So $\mathcal{S}_{\text{simp}} = \mathcal{S}$, and $\mathcal{S}$ is skew-Hermitian. A coboundary pairs to zero by the same telescoping. Finally, replacing a_i by $a_i + k$ with $T_i k = k$ changes neither c_i nor any F_j , and changes $\mathcal{S}(a, b)$ by $h_W(k, c'_i) = h_W(k, a'_i) - h_W(T_i^{-1}k, a'_i) = 0$, by unitarity; the second variable enters only through the c'_i . $\square$

Proposition 6.3. *For every $C \in \text{Hur}^\circ$, $A_\rho(C)$ has Hermitian signature $(3, 3)$ and discriminant -2 in $\mathbb{Q}^\times / \text{Nm } K^\times$.*

Proof. By Lemma 3.4 it suffices to treat C_0 . By Proposition 6.1 we need $\det h_{\text{par}}$ for $\mathbb{L}'$, which is $\varphi^*\mathbb{L}$ or its conjugate; the conjugate has the conjugate form, with the same determinant. Since φ is finite, $H_{\text{par}}^1(\mathbb{P}_s^1, \varphi^*\mathbb{L}) = H_{\text{par}}^1(\mathbb{P}_z^1, \mathbb{L} \otimes \varphi_* K)$ with the same cup product, where $\varphi_* K$ is the local system of the permutation representation of φ , with its standard form. This is a parabolic cohomology on $\mathbb{P}_z^1$ minus six points: $5, -7, 0$, the two other critical values c_1, c_2 of φ , and ∞ . Choose standard loops around them in this order. With respect to them the monodromy of $\mathbb{L}$ is $(M_5, M_{-7}, M_0, 1, 1, M_\infty)$, and $(M_5, M_{-7}, M_0, M_\infty)$ is a product-one tuple in the classes of r_1, r_2, x, y with respect to standard loops of $\mathbb{P}_z^1 - \{5, -7, 0, \infty\}$; by rigidity (Section 5.2) it is conjugate to (r_1, r_2, x, y) , by an element of $GL_4(K)$ since both tuples are defined over K and absolutely irreducible, and the conjugation carries $h_{\mathbb{L}}$ to a positive rational multiple of the standard form, which does not change the discriminant. The monodromy of $\varphi_* K$ is a product-one tuple $(1, 1, \pi_0, \pi_1, \pi_2, \pi_\infty)$ of permutations of cycle types $[2, 4], [2, 1^4], [2, 1^4], [3, 3]$ generating a transitive group. The transitive product-one tuples of these

cycle types form exactly four classes under simultaneous conjugation by S_6 , with representatives $(\pi_0, \pi_1, \pi_2, \pi_\infty)$:

$$\begin{aligned} &((1, 5, 2, 4)(3, 6), (1, 6), (2, 3), (1, 4, 3)(2, 5, 6)), \\ &((1, 5, 2, 4)(3, 6), (1, 6), (1, 5), (1, 4, 2)(3, 5, 6)), \\ &((1, 5, 2, 4)(3, 6), (1, 6), (4, 6), (1, 6, 3)(2, 5, 4)), \\ &((1, 5, 2, 4)(3, 6), (1, 4), (1, 6), (1, 6, 3)(2, 5, 4)). \end{aligned}$$

The computation below gives the same matrix for each, so it is not necessary to determine which class φ realizes. For each of the four, Lemma 6.2 applied to the tuple $(T_1, \dots, T_6) = (r_1 \otimes 1, r_2 \otimes 1, x \otimes \pi_0, 1 \otimes \pi_1, 1 \otimes \pi_2, y \otimes \pi_\infty)$ of rank 24, with h_W the tensor product of the standard forms, computes $\mathcal{S}$ exactly over K . The parameters are the tuples $(a_1, \dots, a_6) \in (K^{24})^6$ with $\sum_i Q_i(1 - T_i)a_i = 0$; they form a space Z of dimension 120. On Z the form has rank 6; its radical, of dimension 114, is spanned by the tuples coming from coboundaries and the tuples with every $a_i \in \ker(1 - T_i)$. Let $z_1, \dots, z_{120}$ be the reduced row echelon basis of Z , in the coordinates $(a_1, \dots, a_6)$, and $z_{j_1}, \dots, z_{j_6}$ the first six of them that are independent modulo the radical. On them $h_{\text{par}} = \delta\mathcal{S}$ has the Gram matrix

$$h_0 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & -\omega - \frac{1}{3} & -\frac{2}{3}\omega & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & \omega + \frac{2}{3} & \frac{1}{3}\omega & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3}\omega \\ \omega + \frac{2}{3} & -\omega - \frac{1}{3} & 0 & \frac{2}{3} & -\frac{2}{3}\omega & 0 \\ \frac{2}{3}\omega + \frac{2}{3} & -\frac{1}{3}\omega - \frac{1}{3} & 0 & \frac{2}{3}\omega + \frac{2}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3}\omega + \frac{1}{3} & 0 & 0 & -\frac{1}{3} \end{pmatrix},$$

the same matrix for all four classes. Its characteristic polynomial is

$$x^6 - \frac{4}{3}x^5 - \frac{26}{9}x^4 + \frac{41}{27}x^3 + \frac{19}{27}x^2 - \frac{22}{81}x - \frac{2}{81} = \left(x^2 - \frac{2}{9}\right)\left(x^4 - \frac{4}{3}x^3 - \frac{8}{3}x^2 + \frac{11}{9}x + \frac{1}{9}\right),$$

with six real roots, approximately -1.326 , -0.471 , -0.078 , 0.471 , 0.474 and 2.263 . So the signature is $(3, 3)$ and $\det h_{\text{par}} = -\frac{2}{81} = -2 \cdot \text{Nm}(\frac{1}{9})$, whose class is -2 . $\square$

7. FROM AN OPEN SET TO THE WHOLE MODULI SPACE

Let $\mathcal{M} = \mathcal{M}(K, 3, -2)$ be the moduli space of Subsection 2.1.

Lemma 7.1. *Let $\mathcal{W} = \bigwedge_K^6 R^1 f_* \mathbb{Q} \subset R^6 f_* \mathbb{Q}$ on $\mathcal{M}$. The set $\mathcal{M}^{\text{alg}}$ of points t at which $\mathcal{W}_t$ is spanned by classes of algebraic cycles is a countable union of closed algebraic subsets of $\mathcal{M}$.*

Proof. This is the standard argument with relative Hilbert schemes [22, §3.3.2, pp. 90–94, and the proofs of Theorems 7.19 and 8.26]. Fix a relatively ample line bundle on $f: \mathcal{A} \rightarrow \mathcal{M}$. For each Hilbert polynomial, the relative Hilbert scheme of three-dimensional closed subschemes of the fibres is projective over $\mathcal{M}$; there are countably many Hilbert polynomials, and each relative Hilbert scheme has finitely many irreducible components. A

rational algebraic class in $H^6(\mathcal{A}_t, \mathbb{Q})$ is a rational combination of classes of such subschemes.

Fix a finite tuple $\mathcal{H}_1, \dots, \mathcal{H}_k$ of components, let T be an irreducible component of $\mathcal{H}_1 \times_{\mathcal{M}} \dots \times_{\mathcal{M}} \mathcal{H}_k$, $\tilde{T} \rightarrow T$ a resolution, and $g: \tilde{T} \rightarrow \mathcal{M}$ the composite; g is proper, so $g(\tilde{T})$ is closed. The universal subschemes pull back to subschemes $\mathcal{Z}_l$ of $\mathcal{A} \times_{\mathcal{M}} \tilde{T}$, flat over $\tilde{T}$, and their fibre classes are global flat sections of $g^* R^6 f_* \mathbb{Q}$ [22, proof of Theorem 3.33, p. 94]. Let $\mathcal{F}_u$ be the span of $[\mathcal{Z}_{1,u}], \dots, [\mathcal{Z}_{k,u}]$. Parallel transport carries each of these global sections to itself, so it carries $\mathcal{F}_u$ onto $\mathcal{F}_{u'}$ and $g^* \mathcal{W}_u$ onto $g^* \mathcal{W}_{u'}$; so the condition $g^* \mathcal{W}_u \subseteq \mathcal{F}_u$ holds on all of the connected $\tilde{T}$ or nowhere. Call T *good* in the first case; then $\mathcal{W}_t$ is algebraic for every $t \in g(\tilde{T})$. Conversely, if $\mathcal{W}_t$ is spanned by algebraic classes, choose subschemes $Z_1, \dots, Z_k$ of $\mathcal{A}_t$ whose classes span a space containing $\mathcal{W}_t$; the point $(Z_1, \dots, Z_k)$ lies on a component T , which is good. So $\mathcal{M}^{\text{alg}}$ is the union of the $g(\tilde{T})$ over the countably many good T . $\square$

Theorem 7.2. *Let S be a connected component of Hur on which $A_\rho(C)$ has discriminant -2 , such that $W_K(A_\rho(C))$ is algebraic for every $C \in S$ and $\text{rk } \mu_{C_1} = 9$ at some $C_1 \in S$. Then the Weil classes of every polarized abelian sixfold of Weil type for K with discriminant -2 are algebraic.*

Proof. Choose a small polydisc $\Delta \ni C_1$ in the configuration space over which the covers form an analytic family. Over the simply connected Δ the spaces $H_1(A_\rho(C), \mathbb{Q})$ with ι and H form a trivial flat bundle of Hermitian spaces of discriminant -2 , so Landherr's theorem gives a K -isometry $m: H_1(A_\rho(C_1), \mathbb{Q}) \rightarrow (V, H)$, which we transport over Δ . Sending C to the complex structure of $H_1(A_\rho(C), \mathbb{R})$, transported by m , gives the period map $\Phi: \Delta \rightarrow \mathcal{H}_3$, a submersion at C_1 by Lemma 5.1. Shrinking Δ , $\Phi(\Delta)$ contains a ball $\mathcal{B}$, whose image in $\mathcal{M}$ is open, since $\Gamma(\ell)$ acts properly and freely.

Let $C \in \Delta$ with $z = \Phi(C) \in \mathcal{B}$, and let A_z be the fibre of $\mathcal{A}$ over the image of z . The lattice $m(H_1(A_\rho(C), \mathbb{Z}))$ is commensurable with Λ ; if $N_1 m(H_1(A_\rho(C), \mathbb{Z})) \subseteq \Lambda$, then $N_1 m$ is a K -linear map of lattices compatible with the complex structures, hence a K -linear isogeny $A_\rho(C) \rightarrow A_z$. By hypothesis and Lemma 2.2, the Weil classes of A_z are algebraic. So $\mathcal{M}^{\text{alg}}$ contains a nonempty open set. A nonempty open subset of $\mathcal{M}$ is a Baire space, so by Lemma 7.1 one of the closed sets has nonempty interior; since $\mathcal{M}$ is irreducible, it is all of $\mathcal{M}$. Every polarized sixfold of Weil type for K with discriminant -2 is K -isogenous to a fibre of $\mathcal{A}$ (Lemma 2.2). $\square$

Proof of Theorem A. By Lemma 3.6, $A_\rho(C)$ is a polarized sixfold of Weil type for K for every $C \in \text{Hur}$; by Proposition 4.6 its Weil classes are algebraic; by Proposition 6.3 its discriminant is -2 on Hur° ; and by Corollary 5.11 $\text{rk } \mu_{C_\theta} = 9$ at points of Hur° . Theorem 7.2 with $S = \text{Hur}^\circ$ concludes. $\square$

8. TYPE III SIXFOLDS

Let A be a simple complex abelian sixfold with $\text{End}^0(A) = D$, a totally definite quaternion algebra over $\mathbb{Q}$, with canonical involution $b \mapsto \bar{b}$. Then $H_1(A, \mathbb{Q}) \cong D^3$ as a left D -module, and the polarization is $E(x, y) = \text{Trd}_{D/\mathbb{Q}} T(x, y)$ for a nondegenerate form T on D^3 with $T(ax, by) = aT(x, y)\bar{b}$ and $\overline{T(y, x)} = -T(x, y)$ [3, §9.6]. Such a T is diagonal in a suitable D -basis, with entries pure quaternions $\lambda_1, \lambda_2, \lambda_3$ (since $T(x, x) \neq 0$ for some x , a Gram–Schmidt argument applies), and $\text{disc}(T) = \lambda_1^2 \lambda_2^2 \lambda_3^2 \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2}$ does not depend on the basis, since a change of basis by P multiplies it by $\text{Nrd}(P)^2$. Suppose $K = \mathbb{Q}(u) \subset D$ with $u^2 = -3$; choose a pure quaternion j with $ju = -uj$, and put $b = j^2 < 0$, so that $D = (-3, b)_\mathbb{Q}$ and $D = K \oplus Kj$.

Lemma 8.1. *With K acting by left multiplication, (A, K, E) is a polarized sixfold of Weil type for K , and its Hermitian discriminant is $-b \text{disc}(T)$ in $\mathbb{Q}^\times / \text{Nm } K^\times$.*

Proof. The Rosati involution of E is the canonical involution of D , which restricts to complex conjugation on K . Since j anticommutes with u and commutes with the complex structure, it exchanges the two eigenspaces of u on $H^{1,0}(A)$, so both embeddings of K occur three times.

For $q \in D$ write $q = q_1 + q_2 j$ with $q_1, q_2 \in K$ and put $\text{pr}(q) = q_1$. Then $T_K = \text{pr} \circ T$ is K -sesquilinear (linear in the first variable) and skew-Hermitian, and $\text{Tr}_{K/\mathbb{Q}} T_K = \text{Trd} \circ T = E$, because $q_2 j$ is a pure quaternion. Van Geemen’s form $H(x, y) = E(x, \iota(u)y) + uE(x, y)$ is linear in the second variable; with $t = T_K(x, y)$ one has $T_K(x, uy) = -ut$, so $H(x, y) = \text{Tr}_{K/\mathbb{Q}}(-ut) + u \text{Tr}_{K/\mathbb{Q}}(t) = 2u\bar{t}$, that is $H = 2u\overline{T_K}$. Since ${}^t\overline{T_K} = -T_K$ and the size is six, $\det T_K$ is rational, and $\det H = (2u)^6 \det T_K = -27 \cdot 64 \det T_K$, which is $-\det T_K$ in $\mathbb{Q}^\times / \text{Nm } K^\times$ because $27 = \text{Nm}(3u)$ and $64 = \text{Nm}(8)$. On a block De with $T(e, e) = \lambda = \ell_1 u + \ell_2 j + \ell_3 u j$, take the K -basis (e, je) ; using $jz = \bar{z}j$ for $z \in K$ one finds

$$T_K = \begin{pmatrix} \ell_1 u & -b(\ell_2 + \ell_3 u) \\ b(\ell_2 - \ell_3 u) & b\ell_1 u \end{pmatrix},$$

with $\det T_K = -3b\ell_1^2 + b^2\ell_2^2 + 3b^2\ell_3^2 = b\lambda^2$ on this block. Taking the three blocks, $\det T_K = b^3 \text{disc}(T)$ for the whole form, and $\text{disc } H = -b^3 \text{disc}(T) = -b \text{disc}(T)$, since $b^2 = \text{Nm}(b)$. $\square$

Corollary 8.2. *If $-b \text{disc}(T) = -2$ in $\mathbb{Q}^\times / \text{Nm } K^\times$, then $W_K(A)$ is spanned by algebraic classes. If moreover the PEL family of type III defined by (D, T) is effective, then the Hodge conjecture holds for every power of every general member of that family.*

Proof. The first statement is Theorem A with Lemma 8.1. For the second, Abdulali calls a PEL type *effective* if its arithmetic group is torsion free and the generic member has endomorphism algebra D [1, §2.5, p. 670], and

a member *general* if its Hodge group is that of the generic member [1, §2, p. 670]. His Theorem 4.1 [1, p. 673] states: *Let A be a general member of an effective PEL-family of type III. The Hodge ring of A is generated by divisors, together with the Weil cycles associated to a maximal subfield of $\text{End}_{\mathbb{Q}}(A)$. The algebraicity of the Weil cycles on A implies the usual Hodge conjecture for all powers of A .* The field K is a maximal subfield of D . $\square$

Remark 8.3. Take $D = (-1, -1)_{\mathbb{Q}}$, ramified exactly at 2 and ∞ . It contains K , since 2 is inert in K , and $D \cong (-3, -2)_{\mathbb{Q}}$, so we may take $b = -2$. Let i, j be the standard generators of $(-3, -2)_{\mathbb{Q}}$, $k = ij$, and $T = \text{diag}(i, j, 3j+2k)$. The squares of the entries are $-3, -2$ and $-2 \cdot 9 - 6 \cdot 4 = -42$, so $\text{disc}(T) = -252 = -7 \cdot 6^2$ and $-b \text{disc}(T) = -504 = -2 \cdot 252$, with $252 = \text{Nm}(3 + 9u)$; the class is -2 . No imaginary quadratic subfield $K' = \mathbb{Q}(u') \subset D$, $u'^2 = -d'$, gives a split class: writing $D = (-d', b')_{\mathbb{Q}}$, the class for K' is $-b' \text{disc}(T)$, which is split exactly when $b' \text{disc}(T) \in \text{Nm } K'^{\times}$, that is, when the quaternion algebras $(-d', \text{disc}(T))_{\mathbb{Q}}$ and $(-d', b')_{\mathbb{Q}} = D$ coincide. The first contains $\mathbb{Q}(\sqrt{\text{disc } T}) = \mathbb{Q}(\sqrt{-7})$, in which 2 splits; a quaternion algebra containing a quadratic field is split at every prime that splits in that field, since a division algebra over $\mathbb{Q}_2$ contains no copy of $\mathbb{Q}_2 \times \mathbb{Q}_2$. So the first algebra is split at 2, while D is ramified at 2, and they differ. So neither Markman's theorem nor Schoen's split case applies to this family.

9. POWERS: PROOF OF THEOREM B

Let W be a complex vector space of dimension N , and U_1, U_2 complex vector spaces of dimension m . Put

$$\Lambda = \bigwedge (W \otimes U_1) \otimes \bigwedge (W^* \otimes U_2) = \bigwedge ((W \otimes U_1) \oplus (W^* \otimes U_2)),$$

with the commuting actions of $GL(W)$ and $GL(U_1) \times GL(U_2)$. For $u \in U_1$, $u' \in U_2$ let $\Theta(u, u') = \sum_i (w_i \otimes u) \wedge (w_i^* \otimes u')$ for dual bases, and for $x \in U_1$ let $\det(W \otimes x) \in \bigwedge^N (W \otimes U_1)$ be the image of a generator of $\bigwedge^N W$ under $w \mapsto w \otimes x$; define $\det(W^* \otimes x')$ similarly.

Lemma 9.1. $\Lambda^{SL(W)}$ is generated by the $\Theta(u, u')$, the $\det(W \otimes x)$ and the $\det(W^* \otimes x')$.

Proof. We deduce it from the first fundamental theorem for $SL(W)$, in the form of Kraft and Procesi [10, §8.4, Theorem (FFT for SL_n), p. 90]: *the ring of invariants $K[V^p \oplus V^{*q}]^{SL(V)}$ is generated by the scalar products $\langle j | i \rangle$ and the determinants $[i_1, \dots, i_n]$ and $[j_1, \dots, j_n]^*$, for a vector space V of dimension n over a field of characteristic zero.* An $SL(W)$ -invariant element of $W^{\otimes p} \otimes W^{*\otimes q}$ is an invariant polynomial on $W^{*p} \oplus W^q$ that is homogeneous of degree one in each of the $p + q$ vector variables; apply the theorem with $V = W^*$. A polynomial in the scalar products and determinants that is of degree one in each variable is a linear combination of products in which every variable occurs exactly once. So the invariants in $W^{\otimes p} \otimes W^{*\otimes q}$ are spanned by tensor products of contractions $\sum_i w_i \otimes w_i^*$ (one factor W and

one factor W^*), determinant tensors $\sum_{\pi \in S_N} \text{sgn}(\pi) w_{\pi(1)} \otimes \cdots \otimes w_{\pi(N)}$ (in N factors W), and dual determinant tensors.

The homogeneous piece $\bigwedge^p(W \otimes U_1) \otimes \bigwedge^q(W^* \otimes U_2)$ is the image of $W^{\otimes p} \otimes W^{*\otimes q} \otimes U_1^{\otimes p} \otimes U_2^{\otimes q}$ under antisymmetrization, which is equivariant and surjective; since $SL(W)$ is reductive, the invariants of the image are the image of the invariants. Antisymmetrization turns tensor products into wedge products, up to sign. The image of a contraction in the factors $(w \otimes u_k)$ and $(w^* \otimes u'_l)$ is $\Theta(u_k, u'_l)$. The image of a determinant tensor in the factors $u_{k_1}, \dots, u_{k_N}$ is the full polarization of the homogeneous map $x \mapsto N! \det(W \otimes x)$ of degree N , evaluated at $(u_{k_1}, \dots, u_{k_N})$, which by the polarization identity is a linear combination of values $\det(W \otimes x)$; likewise for the dual determinant tensors. $\square$

Theorem 9.2. *Let (A, ι, E) be a polarized abelian $2n$ -fold of Weil type for K , $n \geq 2$, whose Weil classes are algebraic and whose Hodge group is SU_H . Then every Hodge class on every power A^m is algebraic.*

Proof. Let $H^1(A, \mathbb{C}) = W \oplus W'$ be the decomposition into the two eigenspaces of K . The polarization identifies W' with W^* as representations of $SU_H(\mathbb{C}) = SL(W)$. The Hodge group of A^m is SU_H acting diagonally, so the Hodge classes on A^m , tensored with $\mathbb{C}$, are $\Lambda^{SL(W)}$ with $N = 2n$ and $U_1 = U_2 = \mathbb{C}^m$, the coordinates of A^m kept separate on the two eigenspaces. The $\Theta(u, u')$ are Hodge classes of degree two, hence combinations of divisor classes. Let $\mathcal{O} = \{y \in K : \iota(y) \in \text{End}(A)\}$. For $y \in \mathcal{O}^m$ the homomorphism $f_y: A^m \rightarrow A$, $(a_k) \mapsto \sum_k \iota(y_k) a_k$, pulls the two Weil lines of A back to $\det(W \otimes \tilde{y})$ and $\det(W^* \otimes \tilde{y})$ separately, with $\tilde{y} \in \mathbb{C}^m$ the image of y under the given embedding; these are algebraic. Since $\mathcal{O}$ is a lattice in K , the set of $\tilde{y}$ is Zariski dense in U_1 , and so is the set of $\tilde{y}$ in U_2 ; so these classes span all $\det(W \otimes x)$ and all $\det(W^* \otimes x')$. By Lemma 9.1, products of algebraic classes span all Hodge classes over $\mathbb{C}$, and rational descent as at the end of the proof of Proposition 4.6 finishes the proof. $\square$

Proof of Theorem B. Theorem A and Theorem 9.2 with $n = 3$. $\square$

Members of $\mathcal{M}$ whose Hodge group is smaller than SU_H carry further Hodge classes on their powers; we make no claim about them.

APPENDIX A. THE GROUP, THE RIGID SYSTEM AND THE BASE COVER

With $\omega = e^{2\pi i/3}$, let

$$r_1 = \begin{pmatrix} \frac{\omega+2}{3} & \frac{2\omega+1}{3} & 0 & -\frac{2\omega+1}{3} \\ -\frac{\omega+2}{3} & \frac{\omega+2}{3} & 0 & \frac{1-\omega}{3} \\ 0 & 0 & 1 & 0 \\ \frac{\omega+2}{3} & \frac{1-\omega}{3} & 0 & \frac{\omega+2}{3} \end{pmatrix}, \quad r_2 = \begin{pmatrix} \frac{1-\omega}{3} & \frac{2\omega+1}{3} & \frac{2\omega+1}{3} & 0 \\ \frac{1-\omega}{3} & \frac{1-\omega}{3} & -\frac{\omega+2}{3} & 0 \\ \frac{1-\omega}{3} & -\frac{\omega+2}{3} & \frac{1-\omega}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{\omega+2}{3} & -\frac{2\omega+1}{3} & \frac{1-\omega}{3} \\ 0 & \frac{\omega+2}{3} & -\frac{\omega+2}{3} & \frac{\omega+2}{3} \\ 0 & -\frac{\omega+2}{3} & \frac{\omega-1}{3} & \frac{2\omega+1}{3} \end{pmatrix},$$

$$y = \begin{pmatrix} \frac{\omega-1}{3} & 0 & -\frac{\omega+2}{3} & -\frac{\omega+2}{3} \\ 0 & \frac{\omega-1}{3} & \frac{1-\omega}{3} & \frac{\omega-1}{3} \\ \frac{1-\omega}{3} & -\frac{\omega+2}{3} & -\frac{\omega+2}{3} & 0 \\ \frac{1-\omega}{3} & \frac{\omega+2}{3} & 0 & -\frac{\omega+2}{3} \end{pmatrix},$$

matrices acting on column vectors. They are unitary for the standard Hermitian form, $r_1 r_2 x y = 1$, r_1 has eigenvalues $(\omega, 1, 1, 1)$, r_2 has $(\omega^2, 1, 1, 1)$, x has $(i, -1, -1, -i)$ with x^2 of eigenvalues $(1, 1, -1, -1)$ and $x^4 = 1$, and y has order three with eigenvalues $(\omega, \omega, \omega^2, \omega^2)$. The group $G = \langle r_1, r_2, x, y \rangle$ has order 155 520; the K -algebra generated by r_1, r_2, x is $M_4(K)$; the derived group of G has index three and is $\ker \det$; $G/Z(G)$ has order 25 920; every irreducible character of G is K -valued; and $\det r_1 = \omega$. These facts are checked by the certificate scripts in GAP 4.14 [6] and SageMath 10.7 [13] described in Appendix D.

The residues of the rigid system of Subsection 5.2 are $B_0 = \text{diag}(-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{2}, \frac{3}{4})$, $B_5 = u_1 v_1^T$ and $B_{-7} = u_2 v_2^T$ with

$$\begin{aligned} u_1 &= (1, 1, 0, 1), \\ v_1 &= \left(\frac{119}{648}, \frac{28}{135}, -\frac{41472}{289}, -\frac{187}{3240} \right), \\ u_2 &= \left(-\frac{5598720}{14161}, 0, 1, \frac{5598720}{22253} \right), \\ v_2 &= \left(-\frac{2648107}{3627970560}, -\frac{1090397}{755827200}, \frac{56}{135}, -\frac{2648107}{18139852800} \right). \end{aligned}$$

One checks directly that $\text{tr } B_5 = \frac{1}{3}$, $\text{tr } B_{-7} = \frac{2}{3}$, and that $B_\infty = -(B_0 + B_5 + B_{-7})$ is diagonalizable with eigenvalues $\pm \frac{2}{3}$, each twice. The derivative of φ is

$$\varphi'(s) = \frac{3s(s-3)^3(5s^2-3s+4)}{(s-1)^4(s+2)^4},$$

so the critical points are 0, 3, 1, -2 and the roots of $5s^2 - 3s + 4$, whose critical values are the roots of $729z^2 - 3674z + 5120$; $\varphi(\infty) = 1$; and the fibres over 5 and -7 consist of the six distinct roots of P_5 and of P_{-7} . The permutation monodromy of φ generates S_6 .

APPENDIX B. THE CHARACTER IDENTITY

Table 1 lists the fifty-one suppliers (H'_i, χ_i) of Proposition 4.5. The columns are: the coefficient n_i ; the order of H'_i ; the order m of χ_i ; the genus q of C/H'_i ; the number s of branch points of $C/\ker \chi_i \rightarrow C/H'_i$; $h = 2q - 2 + s$; $d = \gcd(m, \alpha)$ for ramified rows; the exponents, written a^k for the exponent a occurring k times; and the type: U for unramified, S for ramified with simple exponents, and S* for the three rows that use Lemma 4.3. Every row is admissible: in each ramified row the exponents a

and $m - a$ occur equally often (for $a = m/2$ an even number of times), and h is even and positive.

Table 1: The suppliers.

i	n_i	$ H'_i $	m	q	s	h	d	exponents	type
1	-1	155520	3	0	12	10	1	$1^6, 2^6$	S
2	-1	3456	6	108	12	226	3	3^{12}	S
3	1	4320	6	115	12	240	3	3^{12}	S
4	1	1728	3	221	108	548	1	$1^{54}, 2^{54}$	S
5	-1	486	3	905	288	2096	1	$1^{144}, 2^{144}$	S
6	1	144	6	3505	12	7020	3	3^{12}	S
7	1	1944	3	229	96	552	1	$1^{48}, 2^{48}$	S
8	-1	648	6	685	288	1656	2	$2^{144}, 4^{144}$	S
9	-2	486	6	913	96	1920	2	$2^{48}, 4^{48}$	S
10	-3	486	6	913	384	2208	2	$2^{192}, 4^{192}$	S
11	-1	486	6	913	96	1920	2	$2^{48}, 4^{48}$	S
12	-2	486	6	913	384	2208	2	$2^{192}, 4^{192}$	S
13	-1	24	12	21025	72	42120	6	6^{72}	S*
14	-1	24	12	21025	72	42120	6	6^{72}	S*
15	-1	96	12	5257	18	10530	6	6^{18}	S*
16	1	15	15	33697	0	67392	-	-	U
17	-1	1296	6	339	144	820	2	$2^{72}, 4^{72}$	S
18	-1	36	6	14017	96	28128	3	3^{96}	S
19	-1	1152	6	436	12	882	3	3^{12}	S
20	1	144	6	3397	324	7116	2	$2^{162}, 4^{162}$	S
21	1	972	3	449	192	1088	1	$1^{96}, 2^{96}$	S
22	-1	576	6	871	0	1740	-	-	U
23	1	108	3	4385	432	9200	1	$1^{216}, 2^{216}$	S
24	-1	1152	6	430	36	894	3	3^{36}	S
25	-1	576	6	871	12	1752	3	3^{12}	S
26	-1	576	6	769	324	1860	2	$2^{162}, 4^{162}$	S
27	-1	486	3	913	96	1920	1	$1^{48}, 2^{48}$	S
28	-1	486	6	913	288	2112	2	$2^{144}, 4^{144}$	S
29	-1	486	3	913	384	2208	1	$1^{192}, 2^{192}$	S
30	-1	324	6	1537	48	3120	3	3^{48}	S
31	1	18	6	28057	96	56208	3	3^{96}	S
32	-1	648	3	685	288	1656	1	$1^{144}, 2^{144}$	S
33	-1	432	3	989	108	2084	1	$1^{54}, 2^{54}$	S
34	-1	480	15	1045	0	2088	-	-	U
35	-1	432	3	989	540	2516	1	$1^{270}, 2^{270}$	S
36	-1	384	6	1306	36	2646	3	3^{36}	S
37	-1	144	6	3073	12	6156	3	3^{12}	S
38	-1	144	3	3505	0	7008	-	-	U
39	1	144	6	3505	12	7020	3	3^{12}	S
40	1	144	3	3397	324	7116	1	$1^{162}, 2^{162}$	S
41	1	54	6	9361	0	18720	-	-	U
42	1	576	3	769	324	1860	1	$1^{162}, 2^{162}$	S
43	-1	432	6	989	4	1980	3	3^4	S

i	n_i	$ H'_i $	m	q	s	h	d	exponents	type
44	-1	432	3	989	108	2084	1	$1^{54}, 2^{54}$	S
45	-1	486	3	905	384	2192	1	$1^{192}, 2^{192}$	S
46	-2	432	6	1159	48	2364	3	3^{48}	S
47	-1	432	3	989	432	2408	1	$1^{216}, 2^{216}$	S
48	-1	384	6	1306	0	2610	-	-	U
49	-1	324	12	1369	288	3024	4	$4^{144}, 8^{144}$	S
50	1	243	3	1825	576	4224	1	$1^{288}, 2^{288}$	S
51	-1	216	3	1979	1080	5036	1	$1^{540}, 2^{540}$	S

The vectors. The coordinates of $X^*(T) = \mathbb{Z}^{44} \oplus (\mathbb{Z}/2)^4$ are numbered $1, \dots, 44$ for the pairs of non-real irreducible characters with $m_\tau > 0$ and $45, \dots, 48$ for the quaternionic ones, in the order in which the certificate enumerates the irreducible characters of G ; the pair of ρ is coordinate 3, so e_ρ is the third coordinate vector. Table 2 lists the nonzero coordinates of each $a(\chi_i)$ as pairs (coordinate:value). Summing $n_i a(\chi_i)$ gives 1 in coordinate 3, 0 in the other free coordinates, and even values in coordinates 45 to 48.

Table 2: The nonzero coordinates of $a(\chi_i)$.

i	n_i	$a(\chi_i)$
1	-1	1:1
2	-1	6:1, 31:1
3	1	8:1, 26:1
4	1	25:1, 40:1
5	-1	2:1, 9:1, 23:-1, 27:1
6	1	36:1, 37:1, 39:-1, 46:1
7	1	1:1, 6:1, 7:1, 12:1, 22:1, 26:1
8	-1	13:1, 22:1, 26:1, 34:1, 35:1, 44:1
9	-2	19:1, 28:1, 37:1, 38:1, 42:1, 43:1
10	-3	17:1, 29:1, 38:1, 39:1, 42:1, 43:1
11	-1	3:1, 16:1, 18:1, 28:1, 37:2, 38:1, 39:1
12	-2	4:1, 16:1, 20:1, 29:1, 37:1, 38:1, 39:2
13	-1	2:-1, 3:1, 4:-1, 15:-1, 17:-1, 19:1, 27:-3, 28:3, 29:-3, 36:2, 37:2, 39:-2, 46:1
14	-1	2:-1, 3:-1, 15:-1, 16:-1, 17:1, 18:2, 20:2, 27:-3, 28:-3, 36:2, 37:-1, 39:1, 42:-2, 43:2, 46:1
15	-1	16:1, 17:1, 18:1, 19:1, 20:1, 28:2, 29:2, 37:4, 38:4, 39:4, 42:4, 43:5
16	1	3:1, 4:1, 6:1, 7:1, 8:1, 10:2, 11:2, 12:3, 13:3, 16:4, 17:4, 18:4, 19:4, 20:4, 21:4, 22:5, 24:6, 25:6, 26:6, 28:7, 29:7, 31:8, 32:8, 34:9, 35:9, 37:12, 38:12, 39:12, 40:12, 41:13, 42:13, 43:16, 44:16
17	-1	13:1, 22:1, 44:1
18	-1	5:-1, 23:1, 30:-1
19	-1	7:1, 25:1, 32:1, 40:1
20	1	9:1, 23:-1, 30:-1, 33:-1
21	1	10:1, 12:1, 22:1, 24:1, 44:1
22	-1	21:1, 22:1, 41:1, 44:2
23	1	9:1, 23:-2, 30:-1, 33:-1

i	n_i	$a(\chi_i)$
24	-1	6:1, 7:1, 13:1, 26:1, 31:1, 32:1
25	-1	12:1, 21:1, 26:1, 40:1, 41:1, 44:1
26	-1	10:1, 24:1, 32:1, 34:1, 41:1, 44:1
27	-1	25:1, 31:1, 35:1, 40:1, 41:1, 44:1
28	-1	15:1, 27:1, 36:-1, 46:1, 47:1, 48:1
29	-1	24:1, 32:1, 34:1, 40:1, 41:1, 44:1
30	-1	2:1, 3:-1, 4:1, 27:1, 28:-1, 29:1
31	1	5:-1, 23:1, 30:-1, 45:1, 46:1, 48:1
32	-1	1:1, 12:1, 21:1, 22:1, 26:1, 34:1, 35:1, 40:1
33	-1	11:1, 25:2, 31:1, 35:1, 40:1, 41:1, 44:1
34	-1	12:1, 21:1, 22:1, 24:1, 25:1, 40:1, 41:1, 44:1
35	-1	13:1, 22:1, 24:1, 34:1, 35:1, 40:2, 44:1
36	-1	26:1, 31:1, 32:1, 34:1, 35:1, 40:1, 41:1, 44:1
37	-1	3:-1, 19:-1, 36:-1, 37:-1, 38:-1, 42:1, 43:1, 46:1
38	-1	7:1, 8:-1, 12:1, 21:-1, 23:-1, 24:1, 32:1, 41:-1
39	1	4:1, 17:1, 36:-1, 38:1, 39:1, 42:-1, 43:-1, 46:1
40	1	1:1, 13:1, 21:1, 22:1, 23:-1, 24:-1, 26:1, 40:-1
41	1	4:1, 16:1, 17:-1, 20:1, 37:1, 39:1, 42:-1, 43:-1
42	1	7:1, 8:1, 12:1, 21:1, 24:2, 32:1, 40:1, 41:1
43	-1	3:1, 16:1, 19:1, 28:2, 37:2, 38:1, 42:1
44	-1	8:1, 11:1, 21:1, 25:1, 31:1, 35:1, 41:2, 44:1
45	-1	4:1, 10:1, 12:1, 22:1, 24:1, 29:1, 37:1, 39:1, 44:1
46	-2	8:1, 26:2, 31:1, 32:1, 34:1, 35:1, 40:1, 41:1
47	-1	8:1, 10:1, 21:1, 24:1, 32:1, 34:1, 41:2, 44:1
48	-1	12:1, 21:1, 22:1, 24:1, 25:1, 40:1, 41:1, 44:2
49	-1	16:1, 28:1, 29:1, 37:1, 38:1, 39:1, 42:2, 43:1
50	1	15:-1, 23:1, 27:-1, 30:1, 33:1, 36:1, 46:1, 47:1, 48:1
51	-1	9:1, 23:-1, 27:2, 30:-1, 33:-1, 36:-1, 45:1, 48:1

How the data are computed. The group is built in GAP from the four matrices of Appendix A, with its character table. A product-one generating tuple in the class datum is fixed. For each supplier, H'_i is given by words in the generators and χ_i by its values on them; the branch data come from the double cosets $H'_i \backslash G / \langle g_j \rangle$, and $a(\chi_i)$ from the restrictions $\tau|_{H'_i}$ by Frobenius reciprocity. The checking script reads these fixed data, recomputes every row, and checks the identity without reference to the order of the characters: for every non-real τ with $m_\tau > 0$ it computes $\sum_i n_i (\langle \tau|_{H'_i}, \chi_i \rangle - \langle \bar{\tau}|_{H'_i}, \chi_i \rangle)$ and requires +1 at $\tau = \rho$, -1 at $\tau = \bar{\rho}$ and 0 otherwise, and for every quaternionic τ with $m_\tau > 0$ it requires $\sum_i n_i \langle \tau|_{H'_i}, \chi_i \rangle$ to be even. A second script, not using GAP, rechecks the identity and the simplicity of the exponents from the exported data. The suppliers were found among the linear characters, of orders 3 to 18, of representatives of the conjugacy classes of subgroups of G .

APPENDIX C. THE RANK CERTIFICATE

The computation. The program computes, exactly over $\mathbb{Q}$ and modulo θ^{13} , the quantities (1)–(7) of Subsection 5.5 along the direction (V_5, V_{-7}) of Proposition 5.9: the residues, with each residue at a root of P_ϵ represented by one 4×4 matrix over $E_\epsilon[\theta]/(\theta^{13})$ and the sums of item (1) computed as traces; the local data at $s = 0, 3, 1, -2$ and the eigenvectors at the moving poles; the matrices Φ for $\mathcal{D}_\rho$ (36 unknowns, 45 rows) and for $\mathcal{D}_{\bar{\rho}}$ (48 unknowns, 57 rows); the kernel bases for the free columns of item (6); and the matrix $\mathbf{M}(\theta)$ of item (7).

The result. $\text{rk } \mathbf{M}(0) = 5$, and each of the ten 9×9 minors $m_j(\theta)$ of $\mathbf{M}(\theta)$, omitting the column of s^j , has its first nonzero Taylor coefficient at θ^7 . The coefficients c_j are rational numbers with denominator 2^{72} and numerators of 64 to 66 digits; to four digits they are

j	0	1	2	3	4
$c_j/10^{42}$	-1.031	1.609	-1.701	1.946	-2.572
j	5	6	7	8	9
$c_j/10^{42}$	4.150	-7.929	16.51	-33.78	60.57

The exact value of c_0 is printed in Proposition 5.9. In the Schur form of Remark 5.10, $\mathbf{S}_1$ has rank one, with nonzero entry $16770025629/32000$ in adapted bases, and the 3×4 block of $\mathbf{S}_2$ has rank three.

Identities verified along the way. The following consequences of the lemmas of Section 5 hold exactly modulo θ^{13} : the sum of all residues is 0; the characteristic polynomial of every residue is constant; the logarithmic obstructions at $s = 0, 3, 1, -2$ vanish; the vectors w_t are left eigenvectors; $\Phi x_f = 0$ on all rows, not only on the rows R ; the polar coefficients of $q^T r$ of order at most -2 at $s = 0$ and at most -1 at the apparent singularities vanish; and each numerator has degree at most 9.

Independent computations. Two computations in interval arithmetic, written separately from the exact program, integrate the Schlesinger equations numerically with rigorous error bounds. Along the direction of Proposition 5.9, at $\theta = 10^{-13}$ and $2 \cdot 10^{-13}$, the second reproduces ratios of the minors m_j predicted by the exact jets, and its minors grow by the factor 2^7 when θ is doubled. The first, along another direction, also finds a 9×9 minor vanishing to order 7.

APPENDIX D. SUPPLEMENTARY FILES

The programs and their outputs are in the certificate archive [17], doi:10.5281/zenodo.23549518, whose README lists every file named below with its path in the archive. The rank certificate of Appendix C is in the directory

dominance-exact :

the program `g32exact.sage` (SageMath 10.7 [13]; run as `sage g32exact.sage 12`, one process, under 4 GB), its output, the exact coefficients c_j in `minor_coeffs.json`, and the two interval-arithmetic computations. The other computations are in the directory `certificate`, in four subdirectories: `A`, the character identity (GAP); `B`, the facts about G and the tuple of Appendix A (Sage, exact); `D`, the discriminant computation of Proposition 6.3 (`disc.cert.sage`); `E`, a separately written computation of the pairing of Lemma 6.2 on the same local system and on small covers. (Subdirectory `C` holds an earlier numerical rank certificate, superseded by Appendix C.) A README in each directory lists the commands and the SHA-256 hashes of the outputs.

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