

ALGEBRAICITY OF WEIL CLASSES ON ABELIAN SIXFOLDS OF WEIL TYPE FOR $\mathbb{Q}(\sqrt{-3})$ WITH DISCRIMINANT -2

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ABSTRACT. Let $K = \mathbb{Q}(\sqrt{-3})$. We prove that the Weil classes of every polarized abelian sixfold of Weil type for K with Hermitian discriminant -2 are algebraic. This is a non-split discriminant, to which the theorems of Schoen and of Markman on split Weil classes do not apply. The proof uses Prym–Tyurin varieties of covers of $\mathbb{P}^1$ with monodromy the complex reflection group G_{32} , and combines known mechanisms in a new configuration. First, on every such variety the Weil classes of the six-dimensional isotypic factor are algebraic. They are obtained, by a Lefschetz-class correspondence supplied by Milne’s theorem, from Schoen’s algebraic Weil classes on fifty-one cyclic intermediate quotients; the choice of quotients is governed by an integral identity among characters of a torus, verified by an exact computation. Second, the ten-dimensional family of these covers dominates the nine-dimensional moduli space: we exhibit an explicit cover at which the period map has rank nine, by a certified computation. Third, a Hilbert-scheme and Baire-category argument spreads algebraicity from an open set to the whole moduli space. The discriminant of the family is identified exactly. All computations are supplied as reproducible certificates. For the members whose special Mumford–Tate group is the full special unitary group, an invariant-theoretic lemma then gives the Hodge conjecture for every power.

1. INTRODUCTION

Let A be a complex abelian variety of dimension $2n$ on which an imaginary quadratic field K acts, with both embeddings of K occurring n times on the tangent space. Weil observed that the K -line

$$W_K(A) = \bigwedge_K^{2n} H^1(A, \mathbb{Q}) \subset H^{2n}(A, \mathbb{Q})$$

is a two-dimensional space of Hodge classes, which for general such A are not generated by divisor classes [26], [23, §5]. A polarization compatible with K defines a K -Hermitian form H of signature (n, n) on $H_1(A, \mathbb{Q})$, and its discriminant $\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$ is an isogeny invariant [23, Lemma 5.2].

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Weil-type abelian varieties of dimension $2n$ with fixed (K, disc) form a family of dimension n^2 .

The algebraicity of the Weil classes is known when the discriminant is split, that is $\text{disc } H = (-1)^n$: for $K = \mathbb{Q}(\sqrt{-3})$ and $n = 3$ by Schoen [20], and for every K and $n = 3$ by Markman [10], with consequences for the Hodge conjecture for abelian fourfolds. For a non-split discriminant in dimension six, the only case we know is a seven-dimensional subfamily treated in the author's companion note [22], discussed below; no complete moduli space of non-split sixfolds was previously known. Our main result is the following.

Theorem 1.1. *Let $K = \mathbb{Q}(\sqrt{-3})$ and let (A, ι, E) be a polarized abelian sixfold of Weil type for K whose Hermitian discriminant is -2 in $\mathbb{Q}^\times / \text{Nm } K^\times$. Then $W_K(A)$ is spanned by classes of algebraic cycles.*

The class -2 is not a norm and differs from the split class -1 by 2 , which is inert in K ; so these sixfolds are not of split type.

Corollary 1.2. *Let (A, ι, E) be as in Theorem 1.1, and suppose that the special Mumford–Tate group of A is SU_H , the special unitary group of the Hermitian form H (the hypothesis of [23, Theorem 6.12]). This holds for the period point of A outside a countable union of proper closed real-analytic subsets of the period domain $\mathcal{H}_3$ [23, Theorem 6.11 and its proof], and hence off the image of such a union in the moduli space. Then every Hodge class on A is algebraic.*

Proof. By [23, Theorem 6.12], $\dim B^p(A) = 1$ for $p \neq 3$. The p -th power of the polarization class is a nonzero element of $D^p(A) \subseteq B^p(A)$, so $B^p(A) = D^p(A)$, and $B^3(A) = D^3(A) \oplus W_K(A)$, where B^p denotes the Hodge classes of codimension p and D^p the span of products of divisor classes. Divisor classes are algebraic, and $W_K(A)$ is spanned by algebraic classes by Theorem 1.1. $\square$

Section 8 applies Theorem 1.1 to abelian sixfolds of type III whose quaternion algebra contains K : when the induced class is -2 , Abdulali's theorem gives the Hodge conjecture for every power of the general member (Corollary 8.2), including a family with $D = (-1, -1)_{\mathbb{Q}}$ that no split theorem reaches (Remark 8.3). Corollary 1.2 concerns A itself. Section 9 extends it to every power: for A as in Corollary 1.2 and every $m \geq 1$, the Hodge conjecture holds for A^m (Theorem 9.1), by a lemma on the invariants of SL_6 in exterior algebras.

Strategy. The proof has three ingredients, which we describe now.

Supply (Section 4). Let $G = G_{32}$ be the Shephard–Todd complex reflection group of order 155520, whose reflection representation ρ is four-dimensional and defined over K . For a G -cover $C \rightarrow \mathbb{P}^1$ with the branch datum of Section 3, the ρ -isotypic part of $J(C)$ is isogenous to a power of a sixfold A_ρ of Weil type for K . Each pair (H, χ) of a subgroup $H \leq G$ and

a character χ of H of order $m \geq 3$ gives a cyclic cover $C/\ker \chi \rightarrow C/H$, whose primitive Prym part carries cyclotomic Weil classes; Schoen proved these are algebraic whenever the branch exponents of the cyclic cover are simple, that is, pair off as opposites, and in particular for unramified covers [19]. We show that suitable products of these classes, contracted by an algebraic correspondence supplied by Milne's theorem on Lefschetz classes [14], give $W_K(A_\rho)$, provided an explicit character of a torus lies in the lattice spanned by the characters of the chosen suppliers. For G_{32} we verify this lattice condition by an exact computation. Transferring Schoen's classes from a cyclic subcover to a non-abelian factor is due to van Geemen and Verra [24, Corollaries 4.9 and 4.10], who use one unramified cyclic subcover of order four and its quaternionic translates; here fifty-one subgroups contribute, and an integral lattice of characters governs the contraction.

Dominance (Section 5). The G_{32} covers with this branch datum form a ten-dimensional family, and the moduli space of the sixfolds in Theorem 1.1 is nine-dimensional. We construct an explicit cover, as the pullback of a rigid rank-four local system along a rational function of degree six, and certify that the codifferential of the period map has rank nine at a nearby point reached by an isomonodromic deformation. The certificate is a nonzero 9×9 minor computed in ball arithmetic.

Spreading (Section 7). Algebraicity on an open subset of the moduli space spreads to all of it: the locus where the whole Weil plane is algebraic is a countable union of closed algebraic subsets, by relative Hilbert schemes, and Baire's theorem does the rest. This step uses neither the variational Hodge conjecture nor Deligne's Principle B.

Finally, Section 6 shows that the factor A_ρ has discriminant -2 , so that the family lands in the moduli space of Theorem 1.1.

What is new. None of the mechanisms is new: Schoen's cycles on cyclic covers [19], their transfer to a non-abelian Prym factor [24], Lefschetz classes [14], and dominance of a Prym family followed by spreading [12]. What is new is a realization of them that reaches a non-split discriminant: the G_{32} family, the integral character identity that combines fifty-one cyclic suppliers into the Weil classes of its isotypic factor, the certified dominance of the family, and the exact identification of its discriminant.

What is computer-assisted. Four statements rest on computations, all in characteristic zero. Each is supplied as a certificate: a script, its exact inputs, a single command, and a transcript, described in Appendix C. They are

- the integral lattice identity of Proposition 4.3 (exact, in GAP);
- the identification of the monodromy of the explicit cover of Section 5 (exact, in GAP);

- the rank statement of Proposition 5.1 (exact linear algebra over $\mathbb{Q}$ at the base point, outward-rounded ball arithmetic at the nearby point);
- the discriminant of Proposition 6.1 (an exhaustive enumeration and exact Hermitian linear algebra over K).

Related work. Schoen’s construction [19, 20] produces Weil classes on Pryms of cyclic covers; Patel and Zhang extend it to finite abelian unramified covers [18]. Van Geemen and Verra transfer Schoen’s classes from a cyclic subcover to quaternionic Pryms [24]. Koike combines Schoen’s cycles with the dominance of a Prym map onto a Shimura family of Weil sixfolds [12]; Sections 5 and 7 follow the same pattern for a non-abelian family. Markman proves the algebraicity of the Weil classes of every sixfold of Weil type with split discriminant, for every K [10]; see also [13]. In his account Schoen’s sixfold case is a split case, and no non-split discriminant is treated. Prym–Tyurin factors of covers with monodromy a reflection group, with a dominance argument for the Prym–Tyurin map, were used by Alexeev, Donagi, Farkas, Izadi and Ortega to uniformize the whole moduli space of principally polarized sixfolds by $W(E_6)$ -covers [3, 4]; there the group is a real reflection group, and no Hermitian discriminant arises. The supply step is close in spirit to the determinant criterion of [16, Theorem 2.5] and to the cohomology supply by abelian covers of [17], two manuscripts of 30 September 2026 in the `openai/math` repository, which predate this work. Those manuscripts work with abelian covers and with Markman’s split theorem as input; combined with a third manuscript of the same repository on split abelian eightfolds [15], whose analytic core we have not been able to verify, their criterion would give the Hodge conjecture for every power of the sixfolds of Theorem 1.1. The proof below does not use [15], and it proves less: the algebraicity of $W_K(A)$ for A itself. In a companion note [22] the author shows that the Weil classes of Pryms of balanced cyclic covers are algebraic under a Fermat-type hypothesis; for $m = 6$ and branch datum $(1, 3, 4, 4)$ over genus-two curves this gives a seven-dimensional family of sixfolds of Weil type for K with discriminant -2 . Theorem 1.1 reaches the whole nine-dimensional moduli space, and its proof does not use [22]: every supplier it needs is covered by Schoen’s theorem.

2. WEIL-TYPE SIXFOLDS AND THEIR MODULI

Throughout, $K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\omega)$, $\delta = \sqrt{-3}$.

Definition 2.1. A *polarized abelian sixfold of Weil type for K* is a triple (A, ι, E) where A is a complex abelian sixfold, $\iota: K \rightarrow \text{End}^0(A)$ acts on the tangent space with both embeddings of multiplicity three, and E is a polarization with $E(\iota(a)x, y) = E(x, \iota(\bar{a})y)$.

Following van Geemen [23, Lemma 5.2], put

$$H(x, y) = E(x, \iota(\delta)y) + \delta E(x, y)$$

on the six-dimensional K -space $V = H_1(A, \mathbb{Q})$. Then H is a nondegenerate K -Hermitian form of signature $(3, 3)$, its discriminant $\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$ does not depend on the choice of basis or lattice, and it is an isogeny invariant [23, Lemma 5.2(2)–(4)]. The Weil plane $W_K(A) = \bigwedge_K^6 H^1(A, \mathbb{Q})$ is two-dimensional over $\mathbb{Q}$ and consists of Hodge classes [23, Lemma 5.2(6)].

By Landherr's theorem, a K -Hermitian form is determined up to isometry by its dimension, signature and discriminant [23, §5.4]. Fix (V, H) of dimension six, signature $(3, 3)$ and discriminant -2 , and a lattice $\Lambda \subset V$ stable under the maximal order of K . Let $\mathcal{H}_3$ be the space of three-dimensional subspaces of $V_{\mathbb{R}}$ (with its complex structure from $\iota(\delta)/\sqrt{3}$) on which H is positive; it is the Hermitian symmetric domain $SU(3, 3)/S(U(3) \times U(3))$, connected of complex dimension nine [23, Lemma 5.10]. For $n \geq 3$ let $\Gamma(n)$ be the subgroup of $U(H)(\mathbb{Q})$ preserving Λ and acting trivially on $\Lambda/n\Lambda$. It is torsion free, $M_n = \Gamma(n) \backslash \mathcal{H}_3$ is a smooth connected quasi-projective variety [23, §5.11], hence irreducible, and it carries a universal abelian scheme $f: \mathcal{A} \rightarrow M_n$ with K -action [23, §5.5–5.8].

Lemma 2.2 (Isogeny invariance). *Let $\psi: A' \rightarrow A$ be a K -linear isogeny of abelian varieties of Weil type. Then $W_K(A)$ is algebraic if and only if $W_K(A')$ is. Every polarized sixfold of Weil type for K with discriminant -2 is K -isogenous to a fibre of $\mathcal{A} \rightarrow M_n$.*

Proof. ψ^* is a ring isomorphism $H^*(A, \mathbb{Q}) \rightarrow H^*(A', \mathbb{Q})$, K -linear on H^1 , so $\psi^*W_K(A) = W_K(A')$. Pullback and pushforward preserve algebraic classes and $\psi_*\psi^* = \deg \psi$. For the second statement, Landherr's theorem gives a K -isometry $(H_1(A', \mathbb{Q}), H') \cong (V, H)$; transporting the complex structure gives a point of $\mathcal{H}_3$ [23, §5.7–5.8], and if N clears denominators so that $N \cdot H_1(A', \mathbb{Z}) \subset \Lambda$, multiplication by N is a K -linear isogeny onto the corresponding fibre. (It scales the polarization by N^2 , which is harmless.) $\square$

3. THE G_{32} FAMILY

Let $G = G_{32}$, of order 155520, with centre of order six and $G/Z(G)$ simple of order 25920. Its reflection representation $\rho: G \rightarrow GL_4(K)$ is absolutely irreducible and has Schur index one over K . Let $\mathcal{R}_1$ (respectively $\mathcal{R}_2$) be the conjugacy class of reflections with nontrivial eigenvalue ω (respectively ω^2), each of order three, and let $\mathcal{J}$ be the class of the element J with $\rho(J)$ of eigenvalues $(1, 1, -1, -1)$.

Definition 3.1. A *balanced G_{32} cover* is a connected Galois cover $C \rightarrow \mathbb{P}^1$ with group G , branched over thirteen points, with local monodromy in $\mathcal{R}_1$ at six of them, in $\mathcal{R}_2$ at six, and in $\mathcal{J}$ at one.

Such covers exist (Section 5 constructs one), and they vary in a family of dimension $13 - 3 = 10$, a union of components of a Hurwitz space S . By the Chevalley–Weil formula, $H^1(C, \mathbb{C}) = \bigoplus_{\sigma} \sigma \otimes M_{\sigma}$ over the irreducible representations σ of G , and for $\sigma = \rho$ the multiplicity space M_{ρ} has dimension six with Hodge numbers $(3, 3)$. Let $\varepsilon \in \mathbb{Q}[G]$ be the central idempotent of the

simple factor $\mathbb{Q}[G]\varepsilon \cong M_4(K)$ belonging to ρ , and e a primitive idempotent in it, which we may take fixed by the Rosati involution $g \mapsto g^{-1}$. Put

$$A_\rho = A_\rho(C) = \text{im}(Ne: J(C) \rightarrow J(C))$$

for N with $Ne \in \mathbb{Z}[G]$, with the restricted polarization and the action of the centre K of $M_4(K)$ [6, §13.6]. Then $H_1(A_\rho, \mathbb{Q})$ is a six-dimensional K -space, the centre acts with multiplicities $(3, 3)$, and the Rosati involution acts on K as complex conjugation, so (A_ρ, ι, E) is a polarized sixfold of Weil type for K .

4. SUPPLY: THE WEIL CLASSES ARE ALGEBRAIC ON EVERY MEMBER

4.1. Suppliers. Fix a balanced G_{32} cover C . For a subgroup $H \leq G$ and a linear character χ of H of order $m \geq 3$, the quotient $C' = C/\ker \chi$ is a connected cyclic $\mathbb{Z}/m$ -cover of the curve C/H , of genus q say. Let $P_\chi \subset J(C)$ be its primitive part, cut out by a rational idempotent of $\mathbb{Q}[H] \subset \mathbb{Q}[G]$, on which $E_\chi = \mathbb{Q}(\mu_m)$ acts. Then

$$H^1(P_\chi, \mathbb{C}) = \bigoplus_{t \in (\mathbb{Z}/m)^\times} V_t, \quad V_t = \bigoplus_{\sigma} \text{Hom}_H(\chi^t, \sigma|_H) \otimes M_\sigma,$$

with $\dim V_t = h$ independent of t , and $W_{E_\chi}(P_\chi) = \bigwedge_{E_\chi}^h H^1(P_\chi, \mathbb{Q})$ satisfies $W_{E_\chi}(P_\chi) \otimes \mathbb{C} = \bigoplus_t \det V_t$.

The branch data of $C' \rightarrow C/H$ are computed from double cosets: the points of C/H over a branch point b_j of $C \rightarrow \mathbb{P}^1$ with monodromy g_j correspond to $H \backslash G / \langle g_j \rangle$, with local monodromy $xg_j^\ell x^{-1} \in H$. They depend only on the conjugacy classes of the g_j , so they are the same for every balanced cover. Let s be the number of branch points of $C' \rightarrow C/H$; then $h = 2q - 2 + s$.

Write $\alpha = (\alpha_1, \dots, \alpha_s) \in (\mathbb{Z}/m - \{0\})^s$ for the local exponents of $C' \rightarrow C/H$ with respect to a generator of $\mathbb{Z}/m$; they are defined up to the diagonal action of $(\mathbb{Z}/m)^\times$, and no α_j need be prime to m . Following Schoen [19, §1, p. 11], α is *simple* if s is even and, after a permutation, $\alpha_{2i} = -\alpha_{2i-1}$ for every i . Equivalently, for every $a \in \mathbb{Z}/m - \{0\}$ the number of j with $\alpha_j = a$ equals the number with $\alpha_j = -a$, and it is even when $2a = 0$. A simple tuple satisfies Schoen's balance condition [19, Corollary 1.9], so $W_{E_\chi}(P_\chi)$ then consists of Hodge classes.

Definition 4.1. A supplier (H, χ) is *admissible* if h is even and α is simple; for $s = 0$ (the cover $C' \rightarrow C/H$ is unramified) this is automatic.

Lemma 4.2. *If (H, χ) is admissible, then $W_{E_\chi}(P_\chi)$ is spanned by classes of algebraic cycles, at every balanced cover.*

Proof. The curve $C' = C/\ker \chi$ is smooth, projective and irreducible, being a quotient of the irreducible curve C , and $H/\ker \chi \cong \mathbb{Z}/m$ acts on it faithfully. Schoen's Theorem 2.0 [19] applies to such a curve whenever the invariant h is even and the $(\mathbb{Z}/m)^\times$ -orbit of its exponent tuple is simple;

his §1 allows exponents with $\gcd(\alpha_j, m) > 1$. His Corollary 3.1 then states that the Weil classes $U' \subset H^h(B, \mathbb{Q})$ of the abelian subvariety $B \subset \text{Alb}(C')$ on which $\mathbb{Z}/m$ acts through primitive characters are generated by algebraic cycles. This B is the primitive part of $J(C')$, which is isogenous to P_χ compatibly with the action of E_χ , and $U' = W_{E_\chi}(B)$. For $s = 0$ the simplicity hypothesis is empty [19, p. 12]; see also [18, Theorems 1.1, 1.2 and 5.3]. The exponents depend only on the conjugacy classes of the local monodromies of $C \rightarrow \mathbb{P}^1$, so the hypotheses hold at every balanced cover once they hold at one. $\square$

4.2. Characters. Let U be the centralizer of G in $\text{Sp}(H^1(C, \mathbb{Q}), E)$. Over $\mathbb{C}$ it is a product of $GL(M_\sigma)$ over pairs $\{\sigma, \bar{\sigma}\}$ of complex representations, $O(M_\sigma)$ over quaternionic σ and $\text{Sp}(M_\sigma)$ over orthogonal σ . Let T be its maximal abelian quotient, with character group $X^*(T) = \mathbb{Z}^{\text{pairs}} \oplus (\mathbb{Z}/2)^{\text{quat}}$. Each $\det V_t$ is a U -eigenline, with character

$$a_t(\chi) = \left(\langle \sigma|_H, \chi^t \rangle - \langle \bar{\sigma}|_H, \chi^t \rangle \right)_{\text{pairs}} \oplus \left(\langle \sigma|_H, \chi^t \rangle \bmod 2 \right)_{\text{quat}}.$$

Every irreducible character of G_{32} has values in K (certificate A checks this from the character table). Hence, for the automorphism $\gamma_t: \zeta_m \mapsto \zeta_m^t$, $\langle \sigma|_H, \chi^t \rangle = \langle \sigma^{\gamma_t}|_H, \chi \rangle$ with $\sigma^{\gamma_t} = \sigma$ if γ_t fixes K and $\sigma^{\gamma_t} = \bar{\sigma}$ otherwise; so $a_t(\chi) = a(\chi)$ in the first case and $a_t(\chi) = -a(\chi)$ in the second, on the free coordinates, while the $\mathbb{Z}/2$ coordinates are unchanged because a quaternionic σ is self-conjugate. Certificate A also verifies $a_t(\chi_i) = \pm a(\chi_i)$ directly for every supplier of the witness and every t . Let $e_\rho \in X^*(T)$ be the character by which U acts on $\det M_\rho$, and let L be the subgroup of $X^*(T)$ generated by the $a(\chi)$ of admissible suppliers.

Proposition 4.3 (Lattice condition). *For $G = G_{32}$ with the balanced branch datum, $e_\rho \in L$. An explicit expression $e_\rho = \sum_i n_i a(\chi_i)$ is given in Appendix A.*

This is an exact finite computation in GAP over all conjugacy classes of subgroups of G and all their linear characters of order 3 to 18; Appendix A lists the suppliers, their branch data and their admissibility, and certificate A of Appendix C gives every subgroup by generators, every character by its values, every vector $a(\chi_i)$, and the integral check of the identity in $\mathbb{Z}^{\text{pairs}} \oplus (\mathbb{Z}/2)^{\text{quat}}$. The witness of Appendix A has 51 suppliers: 6 unramified and 45 ramified, every one admissible. (The greedy selection depends on the order in which GAP enumerates subgroup classes; an earlier session found a different witness with 53 suppliers, equally valid. The paper uses the one certificate A produces and checks.) In 27 of the ramified rows the exponents share a factor with m ; they are simple all the same, which is all Schoen's theorem needs. Unramified suppliers are indispensable: e_ρ does not lie even in the rational span of the characters of all ramified suppliers with $h \geq 2$, since an explicit rational linear form on the free part of $X^*(T)$ vanishes on every one of them and takes the value 1 on e_ρ .

4.3. The contraction.

Proposition 4.4. *Let C be any balanced G_{32} cover. Then $W_K(A_\rho(C))$ is spanned by classes of algebraic cycles.*

Proof. Write $e_\rho = \sum_i n_i a(\chi_i)$ with admissible (H_i, χ_i) (Proposition 4.3) and choose for each i a nonzero algebraic class $y_i \in W_{E_i}(P_i)$, $E_i = E_{\chi_i}$, $P_i = P_{\chi_i}$. Put $t_i = 1$ if $n_i > 0$ and $t_i = -1$ if $n_i < 0$. Let $B = \prod_i P_i^{|n_i|}$ and $Y = \boxtimes_i y_i^{\boxtimes |n_i|} \in H^*(B, \mathbb{Q})$.

For each i choose generators $z_{i,t}$ of the lines $\det V_{i,t}$ and put $Z_i = \prod_{t \neq t_i} z_{i,t}$. Let w be a generator of $\det M_\rho \subset H^6(A_\rho, \mathbb{C})$ and

$$\Gamma = \left(\boxtimes_i Z_i^{\boxtimes |n_i|} \right) \boxtimes w \in H^*(B \times A_\rho, \mathbb{C}).$$

Since U acts symplectically on each $H^1(P_i)$, the top class $\prod_t \det V_{i,t}$ is U -invariant, so Z_i has character $-a_{t_i}(\chi_i)$; by the choice of t_i and the lattice identity, Γ is U -invariant, including on the $\mathbb{Z}/2$ coordinates.

Γ is algebraic. Let $r = 1 + \sum_i |n_i|$. Each factor X of $B \times A_\rho$ (a copy of some P_i , or A_ρ) is the image of an endomorphism Ne_X of $J(C)$ with e_X an idempotent of $\mathbb{Q}[G]$ and $Ne_X \in \mathbb{Z}[G]$. Write $\iota_X: X \hookrightarrow J(C)$ for the inclusion and $\pi_X: J(C) \rightarrow X$ for the corestriction of Ne_X , so that $\pi_X \circ \iota_X = [N]_X$. Taking products gives

$$B \times A_\rho \xrightarrow{\iota = \prod \iota_X} J(C)^r \xrightarrow{\pi = \prod \pi_X} B \times A_\rho, \quad \pi \circ \iota = [N],$$

and both maps commute with U , which commutes with $\mathbb{Q}[G]$. The class Γ is homogeneous of degree $q_\Gamma = \sum_i |n_i|(\varphi(m_i) - 1)h_i + 6$, where $m_i = \text{ord } \chi_i$ and $h_i = h(\chi_i)$, so $[N]^* \Gamma = N^{q_\Gamma} \Gamma$. Milne's group $S(J(C))$, the largest algebraic subgroup of the symplectic group commuting with $\text{End}^0(J(C))$ [14, §3], commutes in particular with $\mathbb{Q}[G]$, so $S(J(C)) \subseteq U$, and $\pi^* \Gamma$ is $S(J(C))$ -invariant. By Milne's theorem [14, Theorem 3.2], which states that $H^*(A^r)^{S(A)}$ is generated by divisor classes for every abelian variety A and $r \geq 0$, together with [14, Lemma 3.1] for the extension of scalars to $\mathbb{C}$, $\pi^* \Gamma$ is a $\mathbb{C}$ -linear combination of products of divisor classes. Pullback along the homomorphism ι preserves algebraic classes, so $\iota^* \pi^* \Gamma = (\pi \circ \iota)^* \Gamma = N^{q_\Gamma} \Gamma$ is a $\mathbb{C}$ -linear combination of algebraic classes on $B \times A_\rho$, and so is Γ . We use Γ as a correspondence from B to A_ρ : $\Gamma_*(Y) = \text{pr}_{2*}(\text{pr}_1^* Y \cup \Gamma)$, which sends algebraic classes with complex coefficients to algebraic classes with complex coefficients.

The contraction is nonzero. Write $y_i = \sum_t c_{i,t} v_{i,t}$ with $v_{i,t} \in \det V_{i,t}$, normalized by $\int_{P_i} v_{i,t_i} \cup Z_i = 1$. If $t \neq t_i$ then $v_{i,t} \cup Z_i$ contains $\det V_{i,t}$ twice and vanishes. Hence

$$\Gamma_*(Y) = \pm \prod_i c_{i,t_i}^{|n_i|} w.$$

Every $c_{i,t}$ is nonzero: $W_{E_i}(P_i)$ is a one-dimensional E_i -vector space and $W_{E_i}(P_i) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_\tau W_{E_i}(P_i) \otimes_{E_i, \tau} \mathbb{C}$, so a nonzero vector has nonzero image

under every embedding. Thus w lies in the $\mathbb{C}$ -span of algebraic classes, and so does $\bar{w}$ by the same argument with t_i replaced by $-t_i$.

Rational descent. Let $\mathcal{C} \subset H^6(A_\rho, \mathbb{Q})$ be the $\mathbb{Q}$ -span of the classes of algebraic cycles. We have shown $W_K(A_\rho) \otimes \mathbb{C} = \mathbb{C}w \oplus \mathbb{C}\bar{w} \subseteq \mathcal{C} \otimes \mathbb{C}$. For $\mathbb{Q}$ -subspaces $W, \mathcal{C}$ of a $\mathbb{Q}$ -vector space, $W \otimes \mathbb{C} \subseteq \mathcal{C} \otimes \mathbb{C}$ implies $W \subseteq \mathcal{C}$, since $(W + \mathcal{C}) \otimes \mathbb{C} = \mathcal{C} \otimes \mathbb{C}$ forces $\dim_{\mathbb{Q}}(W + \mathcal{C}) = \dim_{\mathbb{Q}} \mathcal{C}$. So $W_K(A_\rho) \subseteq \mathcal{C}$. $\square$

Remark 4.5. No relative cycle over the family is constructed or needed: the statement is pointwise, and Section 7 uses it only pointwise on an open set.

5. DOMINANCE

5.1. The codifferential. Let $C \rightarrow \mathbb{P}^1$ be a balanced G_{32} cover with branch locus S , $|S| = 13$, and put $A = \text{Hom}_G(\rho, H^0(C, K_C))$, $B = \text{Hom}_G(\bar{\rho}, H^0(C, K_C))$, each of dimension three by Chevalley–Weil. The Hurwitz space of balanced covers is, locally at C , the space of deformations of $(\mathbb{P}^1, S)$, with tangent space $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}(-S))$, dual by Serre duality to $H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(S))$, of dimension 10. The codifferential of the period map of $J(C)$ along this family is the multiplication $H^0(K_C) \otimes H^0(K_C) \rightarrow H^0(K_C^{\otimes 2})$ followed by restriction to G -invariants, and $H^0(C, K_C^{\otimes 2})^G = H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(S))$ [9, §2–3]. Under $H^0(C, K_C) = \bigoplus_{\sigma} \sigma \otimes \text{Hom}_G(\sigma, H^0(K_C))$, the part of the multiplication that lands in the invariants and pairs the ρ - and $\bar{\rho}$ -components is

$$\mu: A \otimes B \longrightarrow H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(S)),$$

obtained by contracting $\rho \otimes \bar{\rho}$ against its unique invariant line. Since $H^{1,0}(A_\rho)$ and $H^{0,1}(A_\rho)$ are identified with A and $\bar{B}$, the cotangent space of $\mathcal{H}_3$ at the period point is $A \otimes B$, of dimension nine, and μ is the codifferential of the period map of the family of A_ρ [9]. If μ has rank nine at one cover, that period map is a submersion there.

5.2. An explicit cover. Let L be the rank-four local system on $\mathbb{P}^1 \setminus \{0, 5, -7, \infty\}$ given by the Fuchsian system with residues

$$B_0 = \text{diag}(-\tfrac{3}{4}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{3}{4}), \quad B_5 = u_1 v_1^T, \quad B_{-7} = u_2 v_2^T, \\ B_\infty = -(B_0 + B_5 + B_{-7}),$$

with $\text{tr } B_5 = \frac{1}{3}$, $\text{tr } B_{-7} = \frac{2}{3}$ and B_∞ semisimple with eigenvalues $(-\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$; explicit rational u_i, v_i are given in Appendix B. The residue algebra is $M_4(\mathbb{Q})$, so L is irreducible. Its local monodromies lie in $\mathcal{R}_1$ at 5, in $\mathcal{R}_2$ at -7 , in the class of an element x with eigenvalues $(i, -1, -1, -i)$ and $x^2 \in \mathcal{J}$ at 0, and in the class of an element y of order three with eigenvalues $(\omega, \omega, \omega^2, \omega^2)$ at ∞ . The dimensions of the four local conjugacy classes are $6 + 6 + 10 + 8 = 30 = 2(4^2 - 1)$, so L is rigid.

Identification of the monodromy. Certificate B exhibits explicit matrices $r_1, r_2, x, y \in \rho(G_{32})$ with $r_1 r_2 x y = 1$ (in the order matching the loops around $5, -7, 0, \infty$), $r_1 \in \mathcal{R}_1$, $r_2 \in \mathcal{R}_2$, x with eigenvalues $(i, -1, -1, -i)$

and $x^2 \in \mathcal{J}$, y with eigenvalues $(\omega, \omega, \omega^2, \omega^2)$, generating $\rho(G_{32})$, and absolutely irreducible. The local monodromies of L lie in the same four $GL_4(\mathbb{C})$ -conjugacy classes, because no two eigenvalues of a residue differ by a nonzero integer, so each local monodromy is conjugate to $\exp(2\pi i B_p)$. By Katz's theorem [11, §1.1] an irreducible local system on $\mathbb{P}^1$ minus four points with rigidity index 2 is determined up to isomorphism by its local monodromy classes. Both L and the local system of the tuple (r_1, r_2, x, y) are irreducible with these classes, so they are isomorphic, and the monodromy group of L is $\rho(G_{32})$.

Let

$$\varphi(s) = \frac{s^2(s-3)^4}{(s-1)^3(s+2)^3},$$

of degree six. Its fibre over 0 has type $[2, 4]$, its fibre over ∞ has type $[3, 3]$, its fibres over 5 and -7 are reduced, and its two remaining critical values are the nonreal roots of $729z^2 - 3674z + 5120$. The pullback φ^*L has local monodromy in $\mathcal{R}_1$ at six points, in $\mathcal{R}_2$ at six points, in $\mathcal{J}$ at the point $s = 0$, and trivial monodromy at the points over ∞ and at $s = 3$. At a point of ramification index e over a branch value with local monodromy g , the local monodromy of φ^*L is conjugate to g^e . Certificate B checks the passport of φ and, from the ramification indices and $x^4 = y^3 = 1$, that the thirteen nontrivial local classes are six in $\mathcal{R}_1$, six in $\mathcal{R}_2$ and one in $\mathcal{J}$. It does not compute the local monodromies as explicit words, and it does not compute the monodromy group of φ^*L ; that group is determined by the following argument.

The monodromy group of φ^*L is all of $\rho(G_{32})$. Over the complement of the branch values, φ is an unramified cover of degree six, so π_1 of the complement upstairs is a subgroup of index six in π_1 downstairs, and the monodromy group Γ of φ^*L is its image. Filling in the punctures with trivial monodromy does not change the image. Hence $[\rho(G_{32}) : \Gamma] \leq 6$. The image of Γ in $G_{32}/Z(G_{32}) \cong PSp_4(3)$ then has index at most 6, and every proper subgroup of $PSp_4(3)$ has index at least 27 [5]; so Γ maps onto $PSp_4(3)$. The group G_{32} is generated by its reflections, which form the two classes $\mathcal{R}_1$ and $\mathcal{R}_2$ of mutually inverse elements, so its abelianization is cyclic of order 3, given by $\det \rho$. Since Γ maps onto $G_{32}/Z(G_{32})$, we have $G_{32} = \Gamma Z(G_{32})$, and as $Z(G_{32})$ is central,

$$[G_{32}, G_{32}] = [\Gamma Z(G_{32}), \Gamma Z(G_{32})] = [\Gamma, \Gamma] \subseteq \Gamma.$$

The derived group is $\ker \det \rho$, of index 3 (indeed $G_{32} \cong \mu_3 \times Sp_4(3)$, checked in GAP). Since Γ contains a reflection in $\mathcal{R}_1$, of determinant $\omega \neq 1$, it follows that $\Gamma = \rho(G_{32})$. So φ^*L is the ρ -local system of a balanced G_{32} cover C_0 .

5.3. The rank. At C_0 the spaces A and B are computed exactly from the local Frobenius data of φ^*L , and μ is a 9×10 matrix over $\mathbb{Q}$.

Proposition 5.1. (1) *At C_0 , $\dim A = \dim B = 3$ and $\text{rk } \mu = 5$.*

- (2) Let C_1 be the cover obtained from C_0 by the Schlesinger isomonodromic deformation, moving the thirteen branch points to $t_i + \tau v_i$ with explicit rational velocities v_i and $\tau = 10^{-8}$. Then $\text{rk } \mu(C_1) = 9$.

Part (1) is exact. For part (2), the Schlesinger equations are solved by a Taylor series in τ computed in ball arithmetic at 800 bits to order 70. The tail is bounded by a coarse rational Cauchy majorant, giving a radius of at most $2 \cdot 10^{-139}$ on every residue entry at $\tau = 10^{-8}$. The spaces A and B at C_1 are three-dimensional by Chevalley–Weil, and Lemma B.1 shows that inverting square pivot blocks in ball arithmetic encloses bases of them. The certificate records $\mu(C_1)$ in residue coordinates, as a 9×13 matrix; by Lemma B.2 its rank is the rank of $\mu(C_1)$. A 9×9 minor of that matrix is enclosed in a ball of radius about $4 \cdot 10^{-177}$ around a point of absolute value $2.60 \cdot 10^{-80}$, so it is nonzero. Details are in Appendix B and certificate C.

The deformed system. The Schlesinger flow is isomonodromic: along it the monodromy representation of $\pi_1(\mathbb{P}^1 \setminus \{\text{poles}\})$ is constant, exactly, and the residues stay in their adjoint orbits. At the three apparent singularities ($s = 3, 1, -2$ at $\tau = 0$) the local monodromy is trivial and the residue eigenvalues are integers; both properties are preserved exactly, so these points remain apparent singularities with trivial monodromy for every small τ . Hence, for small τ , the system at C_1 has the same monodromy as φ^*L on the complement of the thirteen moved genuine poles, with trivial monodromy at the apparent points. It is therefore the ρ -local system of the balanced G_{32} cover C_1 branched at the moved points, obtained from C_0 by moving the branch points along a path in the configuration space. The ball enclosures of the logarithmic obstructions at the apparent points, which contain 0, serve only as a consistency check of the numerical solution.

The component. The Hurwitz space of balanced covers is finite and étale over the configuration space of thirteen distinct points of $\mathbb{P}^1$, so each of its connected components is smooth, hence irreducible; C_0 and C_1 lie on the same component S° .

Corollary 5.2. *On the connected component S° of the Hurwitz space through C_0 , the period map to $\mathcal{H}_3$ has generic rank nine, and is a submersion at C_1 .*

Proof. The locus where $\text{rk } \mu \geq 9$ is Zariski open, and nonempty by Proposition 5.1(2); since $\dim A \otimes B = 9$, the generic rank is nine. $\square$

6. THE DISCRIMINANT

Proposition 6.1. *For every C in S° , $(A_\rho(C), \iota, E)$ has Hermitian signature $(3, 3)$ and discriminant -2 in $\mathbb{Q}^\times / \text{Nm } K^\times$.*

Proof. The Hermitian form is a flat invariant along S° , so it suffices to treat C_0 . Throughout, $U = \mathbb{P}^1 \setminus S$ and $\pi: C_0 \rightarrow \mathbb{P}^1$.

Step 1: the skew-Hermitian lift. On $V_A = H_1(A, \mathbb{Q})$ for any (A, ι, E) of Weil type for K , put $T = H/(2\delta)$, with H as in Section 2. A direct computation from $H(x, y) = E(x, \iota(\delta)y) + \delta E(x, y)$ gives that T is K -sesquilinear and skew-Hermitian, with $T + \bar{T} = E$, that is, $\text{Tr}_{K/\mathbb{Q}} T = E$. Since δ is a fixed scalar, $\det H = (2\delta)^6 \det T = -27 \cdot 64 \cdot \det T$ for the K -matrix of T . We never use $\det T$: the determinant computed below is that of a Hermitian matrix.

Step 2: Schur. Let $V_\rho = K^4$ carry ρ , and h_V a G -invariant positive definite Hermitian form on V_ρ , which we may take with values in K (for the matrices of certificate B, h_V is the standard form). The isotypic part of $H_1(J(C_0), \mathbb{Q})$ is $V_\rho \otimes_K M$ with $M = \text{Hom}_{K[G]}(V_\rho, H_1(J(C_0), \mathbb{Q})_\varepsilon)$, a six-dimensional K -space. The lift T of the G -invariant form E is G -invariant by uniqueness of the lift, and ρ is absolutely irreducible with Schur index one, so $T = h_V \otimes \psi$ for a unique skew-Hermitian ψ on M .

Step 3: the factor. Choose $v \in V_\rho$ with $h_V(v, v) \neq 0$ and let e be the h_V -orthogonal projector onto Kv ; it is fixed by the Rosati involution $g \mapsto g^{-1}$ and lies in $\mathbb{Q}[G]_\varepsilon \cong M_4(K)$. Then $H_1(A_\rho, \mathbb{Q}) = Kv \otimes M$, and since e is Rosati-fixed, the restricted polarization is the restriction of E . Hence $T|_{A_\rho} = h_V(v, v) \psi$ under $m \mapsto v \otimes m$, with $h_V(v, v) \in \mathbb{Q}_{>0}$.

Step 4: parabolic cohomology. Since π is finite and C_0 is normal, $\pi_* \mathbb{Q}_{C_0} = j_* \mathcal{R}$ with $\mathcal{R} = \pi_* \mathbb{Q}|_U$, a local system with fibre $\mathbb{Q}[G]$, and $H^1(C_0, \mathbb{Q}) = H^1(\mathbb{P}^1, j_* \mathcal{R}) = H_{\text{par}}^1(U, \mathcal{R})$, the image of $H_c^1(U, \mathcal{R})$ in $H^1(U, \mathcal{R})$. The cup product on C_0 corresponds to the cup product with coefficient pairing $\mathcal{R} \otimes \mathcal{R} \rightarrow \mathbb{Q}$, multiplication followed by the sum over the fibre. The functor $\text{Hom}_G(V_\rho, -)$ is exact in characteristic zero and commutes with j_* and with cohomology, so

$$H^1(C_0, K)_\rho = V_\rho \otimes_K H_{\text{par}}^1(U, L'), \quad L' = \text{Hom}_G(V_\rho, \mathcal{R} \otimes K),$$

where L' has rank four and monodromy $\rho \circ \mu$ or its complex conjugate, μ being the monodromy of π ; so L' is $\varphi^* L$ or its conjugate. The fibre pairing on $\mathcal{R} \otimes K$ restricts on $V_\rho \otimes L'$ to $h_V \otimes h_{L'}$ for a flat positive definite Hermitian form $h_{L'}$ on L' , unique up to $\mathbb{Q}_{>0}$ since L' is absolutely irreducible. Let

$$\mathcal{S}(a, b) = \int_U h_{L'}(a \cup b) \quad (a, b \in H_{\text{par}}^1(U, L'))$$

be the parabolic cup product: a nondegenerate K -sesquilinear skew-Hermitian form. The polarization E identifies $H_1(A_\rho, \mathbb{Q})$ with $H^1(A_\rho, \mathbb{Q})$, through $x \mapsto E(x, \cdot)$. Under this identification, and Poincaré duality on the curve C_0 , which identifies $H_1(J(C_0), \mathbb{Q}) = H_1(C_0, \mathbb{Q})$ with $H^1(C_0, \mathbb{Q})$ compatibly with G , Steps 2–4 identify M with $H_{\text{par}}^1(U, L')$, or its conjugate, and ψ with $c\mathcal{S}$, or its conjugate, for some $c \in \mathbb{Q}^\times$. The sign of c depends on the orientation convention for the loops (Lemma 6.2); only $c \in \mathbb{Q}^\times$ is used below.

Step 5: normalization. Put $h = \delta \mathcal{S}$. This is a nondegenerate *Hermitian* form on the six-dimensional K -space $H_{\text{par}}^1(U, L')$. By Steps 1–4,

$$H|_{A_p} = 2\delta T|_{A_p} = 2h_V(v, v)c \cdot h \quad \text{or its complex conjugate.}$$

Scaling a six-dimensional Hermitian form by $r \in \mathbb{Q}^\times$ multiplies its determinant by $r^6 = \text{Nm}(r^3)$, and $\det \bar{h} = \det h$. So disc $H|_{A_p} = \det h$ in $\mathbb{Q}^\times / \text{Nm } K^\times$, and $H|_{A_p}$ and h have the same signature up to the order of the pair. The scalar δ that converts $\mathcal{S}$ into h is absorbed in the definition of h ; it does not appear as a factor $\delta^6 = -27$, which would change the class.

Step 6: computation. Since φ is finite, $H_{\text{par}}^1(\mathbb{P}_s^1, \varphi^* L) = H_{\text{par}}^1(\mathbb{P}_z^1, L \otimes \varphi_* K)$, a parabolic cohomology on $\mathbb{P}_z^1$ minus six points $(0, 5, -7, \infty$ and the two other critical values of φ), with monodromy $(A_j \otimes S_j)_j$. Here (A_j) is the monodromy tuple of L , fixed up to simultaneous conjugation by rigidity (Section 5), and (S_j) is the permutation monodromy of φ . The permutation tuples of the exact passport $([4, 2], [2], [2], [3, 3])$ with product one that generate a transitive group fall into exactly four classes under simultaneous conjugation by S_6 : an exhaustive enumeration finds 2880 labelled tuples, all with trivial stabilizer, so the classes have size 720. For each class, certificate D computes H_{par}^1 as Z^1/B^1 in group cohomology of the free group, the form $\mathcal{S}$ by the standard cocycle formula, and the Hermitian matrix of $h = \delta \mathcal{S}$ exactly over K . In every case $\dim H_{\text{par}}^1 = 6$, the signature of h is $(3, 3)$ (counted exactly from its rational characteristic polynomial), and

$$\det h = -\frac{2}{81} = -2 \cdot \text{Nm}(\frac{1}{9}).$$

The Hilbert symbols $(-2, -3)_v$ are nontrivial exactly at $v \in \{2, \infty\}$, so -2 is not a norm. Also $-2/(-1) = 2$ is not a norm, since 2 is inert in K and has odd valuation. So the class is -2 , not the split class -1 .

The form $\mathcal{S}$ is computed by the cocycle formula of Lemma 6.2 below.

An independent test. Certificate D also contains a test of Steps 2–5 on small groups. For random G -covers with explicit generating tuples, it computes van Geemen’s H directly on $e \cdot H^1(C, \mathbb{Q})$ (with E the cup form and K acting through the centre) and compares the class of $\det H$ with that of $\det h$ computed from the parabolic side. They agree in every case; the ratios that occur, $27 = \text{Nm}(3\delta)$ and $108 = \text{Nm}(6\delta)$, are norms. $\square$

The cocycle formula used in Step 6 is the following. Let W be a local system of K -vector spaces on $\mathbb{P}^1$ minus n points, with a flat Hermitian form h and unitary monodromy $T_1, \dots, T_n$ generating a *finite* group, along standard loops $\ell_1, \dots, \ell_n$ with $\ell_1 \cdots \ell_n = 1$. Put $Q_i = T_1 \cdots T_{i-1}$ and $M_i = 1 - T_i$. A parabolic 1-cocycle is determined by vectors $a_1, \dots, a_n$ with $\sum_i Q_i M_i a_i = 0$: its value on ℓ_i is $c_i = M_i a_i$. Put $F_i = \sum_{j < i} Q_j c_j$.

Lemma 6.2. *For parabolic cocycles $a = (a_i)$ and $b = (b_i)$, the cup product of their classes in H_{par}^1 , with the fibre pairing h and evaluated on the*

fundamental class of $\mathbb{P}^1$ oriented so that the loops ℓ_i run clockwise, is

$$\mathcal{S}(a, b) = \sum_{i=1}^n h(Q_i^{-1}F_i^a + a_i, c_i^b).$$

With the opposite orientation it is $-\mathcal{S}(a, b)$. The form vanishes on coboundaries and is skew-Hermitian.

Proof. Choose a Δ -complex structure on $\mathbb{P}^1$ with vertices at the punctures and a base point b_0 . Edges $e_0, \dots, e_n$ run from b_0 , and auxiliary edges f_i and g_i meet the i -th puncture. Since the monodromy group is finite, there is a finite Galois cover $\hat{C}$, branched over the punctures, on which W becomes trivial; lift the Δ -complex to it. On $\hat{C}$, the Alexander–Whitney cup product of 1-cochains followed by h , evaluated on $[\hat{C}]$, is the cup form, by the simplicial cup product on a Δ -complex [8, §3.2]. For equivariant cochains every lift of a triangle contributes the same term, so dividing by the degree gives a sum over the triangles of $\mathbb{P}^1$:

$$\mathcal{S}_{\text{simp}}(f, f') = \sum_i h(v_{e_i}^f - v_{e_{i-1}}^f, v_{g_i}^{f'}),$$

where v_ϵ^f is the value of f on the chosen lift of the edge ϵ . The value of f on the lift of ℓ_i is $z_i = Q_i^{-1}(v_{e_{i-1}} - v_{e_i})$. The cocycle conditions on the two triangles at the i -th puncture give $z_i = M_i Q_i^{-1} v_{f_i}$, so $a_i = Q_i^{-1} v_{f_i}$ is a parabolic cocycle in the sense above, and $v_{e_{i-1}} = v_{e_0} - F_i$. Substituting into $\mathcal{S}_{\text{simp}}$ and telescoping gives

$$\mathcal{S}_{\text{simp}}(f, f') = - \sum_i [h(c_i, a'_i) + h(Q_i c_i, F'_i)] + h\left(\sum_i Q_i c_i, v_{e_0}'\right),$$

whose last term vanishes by the cocycle relation; by unitarity of Q_i , the remaining sum is $-\overline{\mathcal{S}(b, a)}$. Also by unitarity, $h(c_i, c'_i) = h(a_i, c'_i) + h(c_i, a'_i)$, and telescoping $h(F_{i+1}, F'_{i+1}) - h(F_i, F'_i)$ shows that $\mathcal{S}(a, b) + \overline{\mathcal{S}(b, a)} = 0$. The two together give $\mathcal{S}_{\text{simp}} = \mathcal{S}$. A coboundary u has $z(\gamma) = (\rho(\gamma) - 1)u$, and it pairs to zero by the same telescoping. $\square$

In Step 6 the monodromy of $L \otimes \varphi_* K$ lies in the finite group generated by $\rho(G_{32})$ and the permutation matrices of S_6 , so the lemma applies. Certificate E of Appendix C checks Lemma 6.2 independently of this derivation. It builds the Δ -complex and the twisted cochains for the rank-24 local system $L \otimes \varphi_* K$ of Step 6, computes the simplicial cup form with every transport, and compares it entry by entry with $\mathcal{S}$ as computed by certificate D's own code. For all four permutation classes the two forms agree, with $c = 1$. Certificate E also checks the simplicial model itself on fourteen covers that it builds explicitly, over $\mathbb{Z}/3$, S_3 , Q_8 and the Heisenberg group of order 27. On each, H^1 is free of rank $2g$ with a unimodular cup form, and the base formula equals the cup form restricted to the isotypic part.

7. SPREADING

Lemma 7.1. *Let $\mathcal{W} = \bigwedge_K^6 R^1 f_* \mathbb{Q} \subset R^6 f_* \mathbb{Q}$ on M_n . The set*

$$M_n^{\text{alg}} = \{t \in M_n : \mathcal{W}_t \text{ is spanned by classes of algebraic cycles on } \mathcal{A}_t\}$$

is a countable union of closed algebraic subsets of M_n .

Proof. The K -action is by flat endomorphisms, so $\mathcal{W}$ is a rank-two rational sub-local system. Fix a relatively ample bundle on the projective abelian scheme $\mathcal{A} \rightarrow M_n$. The relative Hilbert schemes of subschemes of dimension three of the fibres, over the countably many Hilbert polynomials, are projective over M_n , so they have countably many irreducible components, each with closed image [25, §3.3.2]. Every algebraic class in $H^6(\mathcal{A}_t, \mathbb{Q})$ is a rational combination of classes of integral closed subschemes of dimension three, and a rational combination with negative coefficients is a difference of two with positive ones; so it suffices to consider finite tuples of effective cycles. For a finite tuple $(\mathcal{H}_1, \dots, \mathcal{H}_k)$ of components, let T run through the irreducible components of the fibre product $\mathcal{H}_1 \times_{M_n} \dots \times_{M_n} \mathcal{H}_k$, and let $\tilde{T} \rightarrow T$ be a resolution, with $g: \tilde{T} \rightarrow M_n$ the composite. The pulled-back universal subschemes $\mathcal{Z}_1, \dots, \mathcal{Z}_k \subset \mathcal{A} \times_{M_n} \tilde{T}$ are flat over $\tilde{T}$, since flatness is preserved by base change. So the fibre classes $[\mathcal{Z}_{l,u}]$ ($u \in \tilde{T}$) are flat sections of the pulled-back local system $g^* R^6 f_* \mathbb{Q}$, and their rational span $\mathcal{F} \subseteq g^* R^6 f_* \mathbb{Q}$ is a sub-local system of constant rank [25, p. 94]. On the connected $\tilde{T}$, the condition $g^* \mathcal{W} \subseteq \mathcal{F}$ is locally constant, so it holds everywhere or nowhere. If $\mathcal{W}_t$ is algebraic, choose effective cycles $Z_1, \dots, Z_k$ on $\mathcal{A}_t$ whose classes span a space containing $\mathcal{W}_t$. The point $(Z_1, \dots, Z_k)$ of the fibre product lies on some component T , and on that T the condition holds at a point, hence everywhere; so t lies in the image of a T on which the condition holds. Conversely each such T has algebraic Weil planes over its whole image, which is closed. There are countably many T [25, proof of Theorem 7.19], [7, proof of Proposition 40]. $\square$

Lemma 7.2. *If M_n^{alg} contains a nonempty open subset of M_n in the Euclidean topology, then $M_n^{\text{alg}} = M_n$.*

Proof. An open subset U is a Baire space, and it is the countable union of the closed sets $U \cap Z_j$, Z_j as in Lemma 7.1. So some Z_j has nonempty interior; a proper closed algebraic subset of the irreducible variety M_n has empty interior, so $Z_j = M_n$ [25, proof of Theorem 8.26]. $\square$

Proof of Theorem 1.1. Let $C_1 \in S^\circ$ be the point of Proposition 5.1(2) and let D be a small polydisc around it in the configuration space of the branch points, over which the balanced G_{32} covers form an analytic family. Over the simply connected D , the groups $H_1(A_\rho(C), \mathbb{Q})$ with their K -action and Hermitian forms form a trivial flat bundle. By Proposition 6.1 and Landherr's theorem there is a K -isometry $m: (H_1(A_\rho(C_1), \mathbb{Q}), H) \rightarrow (V, H)$, which we

fix once and transport over D by flatness; this marking is fixed throughout. Sending C to the complex structure of $H_1(A_\rho(C), \mathbb{R})$, transported by m , gives a holomorphic period map $\Phi: D \rightarrow \mathcal{H}_3$, which by Corollary 5.2 is a submersion at C_1 . Shrinking D , $\Phi(D)$ contains a nonempty open ball $\mathcal{B} \subset \mathcal{H}_3$. The quotient map $\mathcal{H}_3 \rightarrow M_n = \Gamma(n) \backslash \mathcal{H}_3$ is open, since $\Gamma(n)$ acts properly discontinuously and freely, so U , the image of $\mathcal{B}$, is open in M_n .

Let $C \in D$ with $\Phi(C) = z \in \mathcal{B}$, and let A_z be the fibre of $\mathcal{A}$ over the image of z , with $H_1(A_z, \mathbb{Z}) = \Lambda$ and the complex structure z . The lattice $\Lambda_C = m(H_1(A_\rho(C), \mathbb{Z}))$ is commensurable with Λ , both being lattices in V stable under an order of K . Choose $N \geq 1$ with $N\Lambda_C \subseteq \Lambda$. Multiplication by N on V is K -linear, respects the complex structure z , and maps Λ_C into Λ , so it defines a K -linear isogeny $A_\rho(C) \rightarrow A_z$; it multiplies the polarization by N^2 , which changes neither the Weil plane nor the discriminant class. By Proposition 4.4 and Lemma 2.2, $W_K(A_z)$ is algebraic. Thus $U \subseteq M_n^{\text{alg}}$, and $M_n^{\text{alg}} = M_n$ by Lemma 7.2. Every polarized sixfold of Weil type for K with discriminant -2 is K -isogenous to a fibre of $\mathcal{A}$ (Lemma 2.2), which proves the theorem. $\square$

Remark 7.3. The argument needs algebraicity on an open set, not at one point; a countable Zariski-dense set of points would not suffice. It uses no deformation of cycles. Deligne's Principle B propagates the property of being absolutely Hodge along a connected base and would not suffice either.

8. TYPE III SIXFOLDS

Theorem 1.1 applies to every polarized sixfold of Weil type for K with discriminant -2 , including those with extra endomorphisms. We record one consequence for abelian sixfolds of Albert type III.

Let A be a simple complex abelian sixfold with $\text{End}^0(A) = D$, a totally definite quaternion algebra over $\mathbb{Q}$. Then $H_1(A, \mathbb{Q}) \cong D^3$ as a left D -module, and the polarization is $E(x, y) = \text{Trd}_{D/\mathbb{Q}} T(x, y)$ for a nondegenerate skew-Hermitian form T on D^3 , with $T(ax, by) = aT(x, y)\bar{b}$, where $b \mapsto \bar{b}$ is the canonical involution [6, §9.6]. Every such T is diagonal in a suitable D -basis, with entries pure quaternions $\lambda_1, \lambda_2, \lambda_3$ (a Gram-Schmidt argument, since $T(x, x) \neq 0$ for some x), and $\text{disc}(T) = \lambda_1^2 \lambda_2^2 \lambda_3^2 \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2}$ is independent of the choice, since a change of basis by P multiplies it by $\text{Nrd}(P)^2$.

Suppose $K = \mathbb{Q}(u) \subset D$ with $u^2 = -3$. Choose a pure quaternion j with $ju = -uj$, and put $b = j^2 < 0$, so that $D = (-3, b)_\mathbb{Q}$ and $D = K \oplus Kj$.

Lemma 8.1. *With the K -action by left multiplication, (A, K, E) is a polarized sixfold of Weil type for K , and its Hermitian discriminant is $-b \text{disc}(T)$ in $\mathbb{Q}^\times / \text{Nm } K^\times$.*

Proof. The Rosati involution of E is the canonical involution of D , which restricts to complex conjugation on K , so E is compatible with K . Since

j anticommutes with u and commutes with the complex structure, it exchanges the two eigenspaces of u on $H^{1,0}(A)$, so both embeddings of K occur with multiplicity three.

For $q \in D$ write $q = q_1 + q_2 j$ with $q_1, q_2 \in K$, and put $\text{pr}(q) = q_1$. Then $T_K = \text{pr} \circ T$ is K -sesquilinear and skew-Hermitian, and $\text{Tr}_{K/\mathbb{Q}} T_K = \text{Trd} \circ T = E$, because $q_2 j$ is a pure quaternion. Here T_K is linear in its first argument, whereas van Geemen's form $H(x, y) = E(x, \iota(u)y) + uE(x, y)$ is linear in its second. Writing $t = T_K(x, y)$, we have $T_K(x, uy) = -ut$, so $H(x, y) = \text{Tr}_{K/\mathbb{Q}}(-ut) + u \text{Tr}_{K/\mathbb{Q}}(t) = 2u\bar{t}$; that is, $H = 2u \overline{T_K}$. Since ${}^t \overline{T_K} = -T_K$ and the size is six, $\det \overline{T_K} = \det(-T_K) = \det T_K$, so $\det T_K$ is rational and equals $\det \overline{T_K}$. Hence $\det H = (2u)^6 \det T_K = -27 \cdot 64 \det T_K$, which is $-\det T_K$ in $\mathbb{Q}^\times / \text{Nm } K^\times$, since $27 = \text{Nm}(3u)$ and $64 = \text{Nm}(8)$.

On the block De with $T(e, e) = \lambda = \ell_1 u + \ell_2 j + \ell_3 u j$ ($\ell_i \in \mathbb{Q}$), take the K -basis (e, je) . Using $jz = \bar{z}j$ for $z \in K$, a direct computation gives the Gram matrix

$$T_K = \begin{pmatrix} \ell_1 u & -b(\ell_2 + \ell_3 u) \\ b(\ell_2 - \ell_3 u) & b \ell_1 u \end{pmatrix}, \quad \det = -3b\ell_1^2 + b^2\ell_2^2 + 3b^2\ell_3^2 = b\lambda^2,$$

since $\lambda^2 = -3\ell_1^2 + b\ell_2^2 + 3b\ell_3^2$. Taking the three blocks gives $\det T_K = b^3 \text{disc}(T)$, so $\text{disc } H = -b^3 \text{disc}(T) = -b \text{disc}(T)$ in $\mathbb{Q}^\times / \text{Nm } K^\times$, since $b^2 = \text{Nm}(b)$. $\square$

Corollary 8.2. *In this situation, suppose that $-b \text{disc}(T) = -2$ in $\mathbb{Q}^\times / \text{Nm } K^\times$. Then $W_K(A)$ is spanned by algebraic classes. Moreover, the Hodge conjecture holds for every power of the general member of the effective PEL family of type III defined by (D, T) .*

Proof. The first statement is Theorem 1.1 applied via Lemma 8.1. The field K is a maximal commutative subfield of D , so the second statement follows from Abdulali's theorem [1, Theorem 4.1]: for the general member of an effective PEL family of type III, the algebraicity of the Weil classes of a maximal subfield of D implies the Hodge conjecture for every power. Here "general" means that the Hodge group equals that of the generic fibre [1, §2.5]. $\square$

Remark 8.3. Corollary 8.2 reaches families that the split theorems do not. Take $D = (-1, -1)_{\mathbb{Q}}$, which is ramified exactly at 2 and ∞ . It contains K , since 2 is inert in K , and $D \cong (-3, -2)_{\mathbb{Q}}$, so we may take $b = -2$. Let i, j be the standard generators of $(-3, -2)_{\mathbb{Q}}$ and $k = ij$, and take $T = \text{diag}(i, j, 3j + 2k)$. The squares of the entries are $-3, -2$ and $-2 \cdot 9 - 6 \cdot 4 = -42$, so $\text{disc}(T) = -252 = -7 \cdot 6^2$, and $-b \text{disc}(T) = -504 = -2 \cdot 252$. Here $252 = \text{Nm}(3 + 9u)$, so the class is -2 .

No imaginary quadratic subfield $K' = \mathbb{Q}(u') \subset D$, with $u'^2 = -d'$, gives a split class. Write $D = (-d', b')_{\mathbb{Q}}$ as above. By Lemma 8.1, the class for K' (computed in the same way) is $-b' \text{disc}(T)$, and it is split exactly when $b' \text{disc}(T) \in \text{Nm } K'^\times$. That is the case exactly when the quaternion algebras $(-d', \text{disc}(T))_{\mathbb{Q}}$ and $(-d', b')_{\mathbb{Q}} = D$ coincide. The first contains

$\mathbb{Q}(\sqrt{\text{disc } T}) = \mathbb{Q}(\sqrt{-7})$, in which 2 splits, and so it cannot be D . Hence neither Markman's theorem nor Schoen's split case applies to this family; the author's survey [21] records it as having no known route.

9. POWERS

Theorem 9.1. *Let A be as in Theorem 1.1, with special Mumford–Tate group SU_H . Then for every $m \geq 1$, every Hodge class on A^m is algebraic.*

The proof reduces to a lemma in invariant theory. Let W be a complex vector space of dimension N , and U, U' complex vector spaces of dimension m . Put

$$\Lambda = \bigwedge(W \otimes U) \otimes \bigwedge(W^* \otimes U'),$$

with the natural actions of $GL(W)$ and $GL(U) \times GL(U')$, which commute. For $u \in U, u' \in U'$ let $\theta(u, u') = \sum_i (w_i \otimes u) \wedge (w_i^* \otimes u')$ for dual bases $(w_i), (w_i^*)$. For $a \in U$ let $\det(W \otimes a) \in \bigwedge^N(W \otimes U)$ be the image of a generator of $\bigwedge^N W$ under $w \mapsto w \otimes a$, and define $\det(W^* \otimes a')$ for $a' \in U'$ in the same way.

Lemma 9.2. *The algebra $\Lambda^{SL(W)}$ is generated by the elements $\theta(u, u'), \det(W \otimes a)$ and $\det(W^* \otimes a')$, for $u, a \in U$ and $u', a' \in U'$.*

Proof. Let $G \subseteq \Lambda^{SL(W)}$ be the subalgebra they generate. It is stable under $GL(U) \times GL(U')$, because θ is bilinear and the $\det(W \otimes a)$ span the $GL(U)$ -stable subspace $\text{Sym}^N U \subseteq \bigwedge^N(W \otimes U)$.

By the skew Cauchy formula, $\Lambda = \bigoplus_{\lambda, \mu} S_\lambda W \otimes S_\mu W^* \otimes S_{\lambda'} U \otimes S_{\mu'} U'$, the sum over partitions λ, μ in the $N \times m$ box. The space $(S_\lambda W \otimes S_\mu W^*)^{SL(W)}$ is one-dimensional when $\lambda - \mu = (k^N)$ for some $k \in \mathbb{Z}$, and zero otherwise. Hence $\Lambda^{SL(W)} = \bigoplus_k R_k$, where for $k \geq 0$

$$R_k = \bigoplus_\mu V_{k, \mu}, \quad V_{k, \mu} \cong S_{(\mu + (k^N))'} U \otimes S_{\mu'} U',$$

the sum over μ with $\mu_1 + k \leq m$; for $k < 0$ the same holds with the roles of (W, U) and (W^*, U') exchanged. The $V_{k, \mu}$ are irreducible and pairwise non-isomorphic $GL(U) \times GL(U')$ -modules.

The case $k = 0$. Here $R_0 = \Lambda^{GL(W)}$, which lives in bidegrees (p, p) . Fix $p \leq m$ and bases $u_1, \dots, u_m$ of U and $u'_1, \dots, u'_m$ of U' . The weight space of Λ of weight $(e_1 + \dots + e_p; e'_1 + \dots + e'_p)$ is $GL(W)$ -isomorphic to $W^{\otimes p} \otimes W^{*\otimes p}$. Its invariants are spanned by the complete contractions [2, Ch. II, §3], which correspond, up to sign, to the products $\prod_j \theta(u_j, u'_{\sigma(j)})$, $\sigma \in S_p$. Both $G \cap R_0$ and R_0 are polynomial $GL(U) \times GL(U')$ -modules, and the category of such modules is semisimple. So if $G \cap R_0 \neq R_0$ in bidegree (p, p) , it has a nonzero complement there. Every irreducible polynomial module of bidegree (p, p) with $p \leq m$ has a nonzero weight space of weight $(e_1 + \dots + e_p; e'_1 + \dots + e'_p)$, so the complement would meet that weight space, which lies in G . Hence $G \cap R_0 = R_0$ in bidegree (p, p) . For $p > m$, choose surjections $\mathbb{C}^p \rightarrow U$ and $\mathbb{C}^p \rightarrow U'$. The induced $GL(W)$ -equivariant algebra

map from the algebra for $(\mathbb{C}^p, \mathbb{C}^p)$ onto Λ is surjective on $GL(W)$ -invariants, since $GL(W)$ is reductive, and it maps θ to θ . Hence $R_0 \subseteq G$.

The case $k \geq 1$. Fix μ with $\mu_1 + k \leq m$. The element $d = \det(W \otimes u_1) \wedge \cdots \wedge \det(W \otimes u_k)$ lies in G , is nonzero, and spans a $GL(W)$ -line of character $\det^k$. Let $U_2 = \langle u_{k+1}, \dots, u_m \rangle$. Since $\dim U_2 = m - k \geq \mu_1$, the $GL(W)$ -invariants of $\bigwedge(W \otimes U_2) \otimes \bigwedge(W^* \otimes U')$ have a nonzero component z of type $S_\mu W \otimes S_\mu W^*$, and $z \in G$ by the case $k = 0$. The elements d and z are products of disjoint sets of the generators $w_i \otimes u_j$ (with $j \leq k$ and $j > k$ respectively) and of the $w_i^* \otimes u_j'$, so $d \wedge z \neq 0$. It is of type $S_{\mu+(k^N)} W \otimes S_\mu W^*$, so it is a nonzero element of $V_{k,\mu} \cap G$. By irreducibility, $V_{k,\mu} \subseteq G$.

The case $k \leq -1$. This is the same argument with W^* , U' and $\det(W^* \otimes a')$ in place of W , U and $\det(W \otimes a)$. $\square$

Proof of Theorem 9.1. Let $V = H^1(A, \mathbb{Q})$, and let $W, W' \subset V_{\mathbb{C}}$ be the eigenspaces on which K acts through the identity embedding and its conjugate, so that $V_{\mathbb{C}} = W \oplus W'$. The polarization identifies W' with W^* as $SU_H(\mathbb{C})$ -modules, and $SU_H(\mathbb{C}) = SL(W)$ [23, §5.2, Theorem 6.11]. The Hodge group of A^m is SU_H acting diagonally on $H^1(A^m, \mathbb{Q}) = V^{\oplus m}$, because the Hodge structure of A^m is the m -fold sum of that of A . Writing $H^1(A^m, \mathbb{C}) = (W \otimes U) \oplus (W' \otimes U')$ with $U = U' = \mathbb{C}^m$ (the coordinates of A^m , kept separate on the two eigenspaces), we get

$$B^*(A^m) \otimes \mathbb{C} = \Lambda^{SL(W)} \quad (N = 6).$$

In degree two, $\Lambda^{SL(W)}$ is spanned by the $\theta(u, u')$, since $\bigwedge^2(W \otimes U)^{SL(W)} = 0$ for $N > 2$. These are therefore Hodge classes of type $(1, 1)$, hence in the complex span of divisor classes by the Lefschetz $(1, 1)$ theorem.

For $a = (a_1, \dots, a_m) \in \mathcal{O}_K^m$, the homomorphism $f_a: A^m \rightarrow A$, $(x_i) \mapsto \sum_i \iota(a_i)x_i$, acts on H^1 by $W \rightarrow W \otimes U$, $w \mapsto w \otimes \tilde{a}$, and $W' \rightarrow W' \otimes U'$, $w' \mapsto w' \otimes \tilde{a}$. Here $\tilde{a}$ is the vector of images of the a_i under the identity embedding. Hence f_a^* maps the two lines spanning $W_K(A) \otimes \mathbb{C}$ to $\det(W \otimes \tilde{a})$ and $\det(W^* \otimes \tilde{a})$, which are algebraic by Theorem 1.1.

The span of these is everything. The map $a \mapsto \det(W \otimes a)$ identifies the span of all $\det(W \otimes a)$, $a \in U$, with $\text{Sym}^N U$: it is the image of the Veronese map $a \mapsto a^N$. A linear form on $\text{Sym}^N U$ that vanishes on all $\tilde{a}^N$, $a \in \mathcal{O}_K^m$, is a homogeneous polynomial of degree N vanishing on the Zariski-dense set $\{\tilde{a}\}$, so it is zero. Hence the $\det(W \otimes \tilde{a})$ span $\text{Sym}^N U$. Conjugation permutes $\mathcal{O}_K^m$, so the same holds for W^* and U' .

By Lemma 9.2, the complex span $\mathcal{C}_{\mathbb{C}}$ of products of algebraic classes contains $B^*(A^m) \otimes \mathbb{C}$. Let $\mathcal{C} \subseteq B^*(A^m)$ be the $\mathbb{Q}$ -span of algebraic classes. Then $B^*(A^m) \otimes_{\mathbb{Q}} \mathbb{C} \subseteq \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{C}$, so $\dim_{\mathbb{Q}}(B^*(A^m) + \mathcal{C}) = \dim_{\mathbb{Q}} \mathcal{C}$ by faithful flatness of $\mathbb{C}$ over $\mathbb{Q}$, and therefore $B^*(A^m) \subseteq \mathcal{C}$. $\square$

Remark 9.3. Lemma 9.2 uses nothing specific to M . Theorem 9.1 therefore holds for any abelian variety of Weil type with special Mumford–Tate group SU_H whose Weil plane is algebraic. The members of M with a smaller Hodge

group carry further Hodge classes on their powers. For them, the determinant criterion of [16, Theorem 2.5] and its treatment of special members in [16, §3] apply with Theorem 1.1 as the input: Theorem 1.1 supplies exactly the algebraic Weil plane at every member, which is the hypothesis used there. This gives the Hodge conjecture on every power of every member of M , at the cost of relying on that manuscript, which we have checked but which has not been refereed. We do not use [15].

10. REMARKS

- (1) The same method applies to any reflection-group family whose isotypic factor is of Weil type, provided the lattice condition of Proposition 4.3 holds and the family dominates. For G_{25} , whose family has dimension eight, the lattice condition holds, but the family cannot dominate a nine-dimensional moduli space.
- (2) For the powers A^m see Section 9.

APPENDIX A. THE SUPPLIER WITNESS

The table lists the 51 suppliers (H_i, χ_i) of the identity $e_\rho = \sum_i n_i a(\chi_i)$ of Proposition 4.3, computed in one GAP session from a product-one generating tuple of the balanced class datum (Section 3), with exact records for every row. The columns are: the coefficient n_i (so $\sum_i |n_i| = 56$ and $\max_i |n_i| = 3$); the order of H_i and its identifier in the SmallGroups library (a dash where $|H_i|$ is outside the library's identification range); the order m of χ_i ; the genus q of C/H_i ; the number s of branch points of $C/\ker \chi_i \rightarrow C/H_i$; $h = 2q - 2 + s$; $d = \gcd(\alpha, m)$ for ramified rows; the branch exponents, written a^k for the exponent a occurring k times; and the type: E for an unramified supplier, S for a ramified one whose exponents are simple in Schoen's sense (Section 4). Every row is admissible: h is even, and every ramified row is simple, including the 27 rows with $d > 1$. The coefficients of the quaternionic relations $2e_q$ are all zero. The table identifies each supplier only up to isomorphism; certificate A of Appendix C gives each H_i by generators in fixed generators of G_{32} , each χ_i by its values, and each vector $a(\chi_i)$, and checks the identity integrally.

i	n_i	$ H_i $	id	m	q	s	h	d	exponents	type
1	-1	155520	—	3	0	12	10	1	$1^6, 2^6$	S
2	-1	3456	—	6	108	12	226	3	3^{12}	S
3	1	4320	—	6	115	12	240	3	3^{12}	S
4	1	1728	(1728, 32236)	3	221	108	548	1	$1^{54}, 2^{54}$	S
5	-1	486	(486, 115)	3	905	288	2096	1	$1^{144}, 2^{144}$	S
6	1	144	(144, 156)	6	3505	12	7020	3	3^{12}	S
7	1	1944	(1944, 3782)	3	229	96	552	1	$1^{48}, 2^{48}$	S
8	-1	648	(648, 660)	6	685	288	1656	2	$2^{144}, 4^{144}$	S
9	-2	486	(486, 210)	6	913	96	1920	2	$2^{48}, 4^{48}$	S
10	-3	486	(486, 210)	6	913	384	2208	2	$2^{192}, 4^{192}$	S

i	n_i	$ H_i $	id	m	q	s	h	d	exponents	type
11	-1	486	(486, 210)	6	913	96	1920	2	$2^{48}, 4^{48}$	S
12	-2	486	(486, 210)	6	913	384	2208	2	$2^{192}, 4^{192}$	S
13	-1	24	(24, 9)	12	21025	72	42120	6	6^{72}	S
14	-1	24	(24, 9)	12	21025	72	42120	6	6^{72}	S
15	-1	96	(96, 166)	12	5257	18	10530	6	6^{18}	S
16	1	15	(15, 1)	15	33697	0	67392	-	-	E
17	-1	1296	(1296, 3142)	6	339	144	820	2	$2^{72}, 4^{72}$	S
18	-1	36	(36, 14)	6	14017	96	28128	3	3^{96}	S
19	-1	1152	(1152, 153391)	6	436	12	882	3	3^{12}	S
20	1	144	(144, 180)	6	3397	324	7116	2	$2^{162}, 4^{162}$	S
21	1	972	(972, 616)	3	449	192	1088	1	$1^{96}, 2^{96}$	S
22	-1	576	(576, 5498)	6	871	0	1740	-	-	E
23	1	108	(108, 45)	3	4385	432	9200	1	$1^{216}, 2^{216}$	S
24	-1	1152	(1152, 157114)	6	430	36	894	3	3^{36}	S
25	-1	576	(576, 5498)	6	871	12	1752	3	3^{12}	S
26	-1	576	(576, 7427)	6	769	324	1860	2	$2^{162}, 4^{162}$	S
27	-1	486	(486, 210)	3	913	96	1920	1	$1^{48}, 2^{48}$	S
28	-1	486	(486, 210)	6	913	288	2112	2	$2^{144}, 4^{144}$	S
29	-1	486	(486, 210)	3	913	384	2208	1	$1^{192}, 2^{192}$	S
30	-1	324	(324, 68)	6	1537	48	3120	3	3^{48}	S
31	1	18	(18, 5)	6	28057	96	56208	3	3^{96}	S
32	-1	648	(648, 643)	3	685	288	1656	1	$1^{144}, 2^{144}$	S
33	-1	432	(432, 698)	3	989	108	2084	1	$1^{54}, 2^{54}$	S
34	-1	480	(480, 1046)	15	1045	0	2088	-	-	E
35	-1	432	(432, 698)	3	989	540	2516	1	$1^{270}, 2^{270}$	S
36	-1	384	(384, 5496)	6	1306	36	2646	3	3^{36}	S
37	-1	144	(144, 156)	6	3073	12	6156	3	3^{12}	S
38	-1	144	(144, 156)	3	3505	0	7008	-	-	E
39	1	144	(144, 156)	6	3505	12	7020	3	3^{12}	S
40	1	144	(144, 156)	3	3397	324	7116	1	$1^{162}, 2^{162}$	S
41	1	54	(54, 11)	6	9361	0	18720	-	-	E
42	1	576	(576, 7427)	3	769	324	1860	1	$1^{162}, 2^{162}$	S
43	-1	432	(432, 698)	6	989	4	1980	3	3^4	S
44	-1	432	(432, 698)	3	989	108	2084	1	$1^{54}, 2^{54}$	S
45	-1	486	(486, 115)	3	905	384	2192	1	$1^{192}, 2^{192}$	S
46	-2	432	(432, 577)	6	1159	48	2364	3	3^{48}	S
47	-1	432	(432, 698)	3	989	432	2408	1	$1^{216}, 2^{216}$	S
48	-1	384	(384, 5496)	6	1306	0	2610	-	-	E
49	-1	324	(324, 156)	12	1369	288	3024	4	$4^{144}, 8^{144}$	S
50	1	243	(243, 51)	3	1825	576	4224	1	$1^{288}, 2^{288}$	S
51	-1	216	(216, 134)	3	1979	1080	5036	1	$1^{540}, 2^{540}$	S

APPENDIX B. THE RANK CERTIFICATE

The base system. The residues of Section 5 are, with $B_5 = u_1 v_1^T$ and $B_{-7} = u_2 v_2^T$,

$$\begin{aligned} u_1 &= (1, 1, 0, 1), \\ v_1 &= \left(\frac{119}{648}, \frac{28}{135}, -\frac{41472}{289}, -\frac{187}{3240} \right), \\ u_2 &= \left(-\frac{5598720}{14161}, 0, 1, \frac{5598720}{22253} \right), \\ v_2 &= \left(-\frac{2648107}{3627970560}, -\frac{1090397}{755827200}, \frac{56}{135}, -\frac{2648107}{18139852800} \right). \end{aligned}$$

One checks directly that $\text{tr } B_5 = \frac{1}{3}$, $\text{tr } B_{-7} = \frac{2}{3}$, that $B_\infty = -(B_0 + B_5 + B_{-7})$ is diagonalizable with eigenvalues $\pm \frac{2}{3}$, each of multiplicity two, and that $1, B_0, B_5, B_{-7}$ generate $M_4(\mathbb{Q})$ as an algebra. No two eigenvalues of a residue differ by a nonzero integer, so each local monodromy is conjugate to $\exp(2\pi i B)$ for the residue B , which gives the four local classes listed in Section 5.

The pullback. On $\mathbb{P}_s^1$ the system $\varphi^* L$ has thirteen genuine poles: $s = 0$ with residue $2B_0$ (eigenvalues $-\frac{3}{2}, -1, -1, \frac{3}{2}$, monodromy $x^2 \in \mathcal{J}$), the six roots of $s^2(s-3)^4 - 5(s-1)^3(s+2)^3$ (residue conjugate to B_5), and the six roots of $s^2(s-3)^4 + 7(s-1)^3(s+2)^3$ (residue conjugate to B_{-7}). It also has three apparent singularities with integral residues: $s = 3$ (residue $4B_0$, eigenvalues $-3, -2, -2, 3$) and $s = 1, -2$ (residues $3B_\infty$, eigenvalues $-2, -2, 2, 2$). Over the critical values of φ outside $\{0, 5, -7, \infty\}$ the system is unramified. The derivative is

$$\varphi'(s) = \frac{3s(s-3)^3(5s^2-3s+4)}{(s-1)^4(s+2)^4},$$

and the critical values of the quadratic factor are the roots of $729z^2 - 3674z + 5120$, which are not real.

The deformation and the certificate. The velocities v_i of the thirteen genuine poles are small rationals drawn by a seeded pseudo-random generator (seed 1); the apparent singularities move with the flow. The Schlesinger system is integrated as a Taylor series in τ in 800-bit complex ball arithmetic to order 70. A Cauchy majorant built from coarse rational inequalities (initial norm below 926, the scalar comparison equation $w' \leq 166w^2$ on $|\tau| \leq 10^{-6}$) bounds the remainder at $\tau = 10^{-8}$ by $2 \cdot 10^{-139}$ on every residue entry. At C_1 :

- the residue eigenvalues and their multiplicities are those at C_0 , because the Schlesinger flow preserves adjoint orbits; the logarithmic obstructions at the apparent singularities and all forbidden polar coefficients are enclosed by balls containing 0, consistent with the exact log-freeness that isomonodromy preserves;
- the linear systems defining A (36 unknowns) and B (48 unknowns) have kernels of dimension exactly three; a square pivot block of size

- 33, resp. 45, of each is certified invertible in ball arithmetic, which by Lemma B.1 encloses a basis of A , resp. B ;
- the matrix of $\mu(C_1)$ in the residue coordinates of Lemma B.2, of size 9×13 , is enclosed entrywise, and the 9×9 minor on nine of the thirteen residue columns is enclosed in a ball of radius about $4.3 \cdot 10^{-177}$ around a point of absolute value $2.603433 \cdot 10^{-80}$.

A second direction (seed 2) gives a nonzero minor of absolute value about $4.1 \cdot 10^{-81}$. The scripts and their transcripts are certificate C of Appendix C.

The spaces A and B . Let P be the set of the sixteen poles of the system at C_1 : the thirteen genuine poles and the three apparent singularities. The point $s = \infty$ is not a pole. Let Y be a fundamental solution. As at the base point, A is identified with the space of $r \in \mathbb{C}(s)^4$ such that $Y^{-1}r ds$ is holomorphic on C , and B with the space of $q \in \mathbb{C}(s)^4$ such that $Y^T q ds$ is holomorphic on C . Concretely, at $p \in P$ let the residue be $A_p = P_p \Lambda_p P_p^{-1}$ with $\Lambda_p = \text{diag}(\lambda_{p,1}, \dots, \lambda_{p,4})$, and let $Y = P_p H_p (s - p) (s - p)^{\Lambda_p}$ be a local solution with H_p holomorphic and $H_p(0) = 1$. Any eigenbasis P_p may be used. At the apparent points some eigenvalues differ by integers; the logarithmic obstructions there vanish, and the resonant entries of the Taylor coefficients of H_p may be chosen to be 0. The conditions are

$$\text{ord}_p [(P_p H_p)^{-1} r]_j \geq \lfloor \lambda_{p,j} \rfloor, \quad \text{ord}_p [(P_p H_p)^T q]_j \geq \lfloor -\lambda_{p,j} \rfloor,$$

at each $p \in P$ and $j = 1, \dots, 4$; holomorphy at the other points of $\mathbb{P}^1$; and $\text{ord}_\infty r, \text{ord}_\infty q \geq 2$, since ds has a double pole at ∞ . Put $M_p = \max(0, -\min_j \lfloor \lambda_{p,j} \rfloor)$ for A , and the same with $-\lambda_{p,j}$ for B .

Lemma B.1. (1) *Every $r \in A$ is uniquely of the form*

$$r = \sum_p \sum_{m=1}^{M_p} c_{p,m} (s - p)^{-m}, \quad c_{p,m} \in \mathbb{C}^4,$$

the sum running over the $\mathcal{J}$ -pole (at $s = 0$ when $\tau = 0$) and the three apparent singularities; this is $4 \sum_p M_p = 36$ unknowns. Every $q \in B$ is uniquely of the same form plus $\sum_p b_p w_p / (s - p)$ over the twelve reflection poles, where w_p spans the line of w with $w^T A_p = \lambda_{p,4} w^T$ and $\lambda_{p,4} \in \{\frac{1}{3}, \frac{2}{3}\}$ is the nonzero eigenvalue; this is $36 + 12 = 48$ unknowns.

- (2) *A , resp. B , is the kernel of the exact linear map Φ from the space of unknowns to the Laurent coefficients that the conditions above require to vanish, together with the coefficient of s^{-1} at ∞ . The kernel has dimension exactly 3.*
- (3) *Let n be the number of unknowns, and suppose that a square submatrix $S = \Phi_{R,F'}$ of size $n - 3$, on a row set R and the complement F' of three columns F , is certified invertible in ball arithmetic. For $f \in F$ let x_f have coordinate 1 at f , 0 at the other two columns of*

F , and $-S^{-1}\Phi_{R,f}$ on F' . Then $\{x_f\}$ is a basis of $\ker \Phi$, and the ball vectors computed by the same formula enclose it.

Proof. (1) Away from P , Y is holomorphic and invertible, so r is holomorphic there. At $p \in P$, $P_p H_p$ is holomorphic and invertible, so $\text{ord}_p r \geq \min_j [\lambda_{p,j}] \geq -M_p$. At the twelve reflection poles the eigenvalues are $(0, 0, 0, \frac{1}{3})$ or $(0, 0, 0, \frac{2}{3})$, so $M_p = 0$ for A . Since r vanishes at ∞ , it is the sum of its principal parts, which gives the form; the coefficients are unique because principal parts at distinct points are independent. For q at a reflection pole, $[-\lambda_{p,j}] = (0, 0, 0, -1)$, so q has at most a simple pole with coefficient c , and the conditions for $j = 1, 2, 3$ in order -1 say that $c^T v = 0$ for every v with $A_p v = 0$, since $H_p(0) = 1$. The annihilator of that three-dimensional space is the line of w_p , because $\lambda_{p,4} w_p^T v = w_p^T A_p v = 0$.

(2) Each condition is the vanishing of finitely many Laurent coefficients, from order $-\max_p M_p = -3$ up to $[\pm\lambda_{p,j}] - 1 \leq 2$. These involve the Taylor coefficients of H_p only to order 5, and the certificate computes them to order 9, so every row of Φ is an exact linear functional, enclosed by its ball. By (1), $\ker \Phi$ maps isomorphically onto A , resp. B . The cover C_1 is a balanced G_{32} cover (Section 5), so $\dim A = \dim B = 3$ by Chevalley–Weil.

(3) The vectors x_f lie in the kernel K_R of the rows R of Φ by construction, and are independent, since their coordinates on F are the unit vectors. Since S is invertible, the rows R have rank $n-3$, so $\dim K_R = 3$ and K_R is spanned by the x_f . Now $\ker \Phi \subseteq K_R$ and both have dimension 3, so $\ker \Phi = K_R$. The exact S and $\Phi_{R,f}$ lie in their balls, and ball inversion encloses the inverse of every matrix in the ball, so the computed vectors enclose the exact x_f . $\square$

The eigenbases P_p are enclosed by the same argument, with $\Phi = A_p - \lambda$ and kernel dimension the multiplicity of λ . The multiplicity is exact, because the Schlesinger flow preserves adjoint orbits.

Lemma B.2. *Let $S \subset \mathbb{C}$ be a set of 13 points, so that $\infty \notin S$. The map sending $Q = g(s) ds^2 \in H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(S))$ to its residues $(\text{Res}_p g)_{p \in S} \in \mathbb{C}^{13}$ is injective. Its image is the 10-dimensional subspace cut out by $\sum_p \rho_p p^k = 0$ for $k = 0, 1, 2$.*

Proof. The function g has at most simple poles on S and is otherwise holomorphic on $\mathbb{C}$. At ∞ , with $s = 1/t$, $ds^2 = t^{-4} dt^2$, so $g = O(s^{-4})$. Hence g has no polynomial part, and $g = \sum_p \rho_p / (s - p)$ with $\rho_p = \text{Res}_p g$, which proves injectivity. Expanding, $g = \sum_{k \geq 0} (\sum_p \rho_p p^k) s^{-k-1}$, so $g = O(s^{-4})$ is exactly the three relations. The image has dimension $13 - 3 = 10 = h^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(S))$. $\square$

So the rank of $\mu(C_1)$ is the rank of the 9×13 matrix of residues of $q_b^T r_a ds^2$ at the thirteen genuine poles, for the enclosed bases r_a of A and q_b of B . The certificate also encloses the polar coefficients of $q_b^T r_a$ of order at most -2 at the genuine poles and at most -1 at the apparent ones. These

vanish exactly, because μ lands in $H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(S))$, and every enclosing ball contains 0, which is a consistency check.

APPENDIX C. THE CERTIFICATES

Each certificate is a script with its exact inputs, run by one command, with a transcript whose last line states the result. Tool versions and SHA-256 hashes of the outputs are listed in the bundle's README.

- A. The lattice identity (GAP).:** Fixes G_{32} by explicit generators in $GL_4(K)$; constructs the product-one generating tuple of the balanced datum; for each of the 51 suppliers, gives H_i by generators, χ_i by its values, the branch exponents recomputed from double cosets, the simplicity check, and the vector $a(\chi_i)$; checks that every irreducible character of G_{32} is K -valued and that $a_t(\chi_i) = \pm a(\chi_i)$ for every t ; and checks $\sum_i n_i a(\chi_i) = e_\rho$ in $\mathbb{Z}^{\text{pairs}} \oplus (\mathbb{Z}/2)^{\text{quat}}$. This search script is the discovery step. The certificate proper is a second GAP script that reads the fixed 51-row witness (each H_i by words in the fixed generators, each χ_i by its values on them) and checks it without searching and without reference to any ordering of the irreducible characters: for every non-real irreducible σ with $m_\sigma > 0$ it computes $\sum_i n_i (\langle \sigma|_{H_i}, \chi_i \rangle - \langle \bar{\sigma}|_{H_i}, \chi_i \rangle)$ and requires +1 at $\sigma = \rho$, -1 at $\sigma = \bar{\rho}$ and 0 otherwise, and for every quaternionic σ with $m_\sigma > 0$ it requires $\sum_i n_i \langle \sigma|_{H_i}, \chi_i \rangle$ to be even. An independent script, not using GAP, rechecks the identity and the simplicity from the exported vectors.
- B. The monodromy (Sage).:** The tuple (r_1, r_2, x, y) as explicit matrices with its classes, product, generation and irreducibility, and the dimensions $6 + 6 + 10 + 8$ of the local classes; the passport and the permutation monodromy of φ ; the classes of the thirteen nontrivial local monodromies of φ^*L , derived from the ramification indices and $x^4 = y^3 = 1$; and the computable inputs of the surjectivity argument of Section 5: $|G_{32}| = 155520$, the indices 27, 36, 40, 40, 45 of the maximal subgroups of $PSp_4(3)$, that the derived subgroup of G_{32} has index 3 and equals $\ker \det \rho$, and that a reflection in $\mathcal{R}_1$ has determinant ω .
- C. The rank (Sage with Arb).:** The exact base-point computation of Proposition 5.1(1), and the outward-rounded ball computation of Proposition 5.1(2), with the velocities and the tail bound, for two directions.
- D. The discriminant (Sage).:** The four permutation classes; for each, the Hermitian matrix of $h = \delta\mathcal{S}$ over K , its characteristic polynomial, signature and determinant; and the small-group test of the dictionary.
- E. The cocycle pairing (Sage).:** An independent check of Lemma 6.2 for the system of Step 6 of Proposition 6.1. It

builds a Δ -complex on $\mathbb{P}_z^1$, the twisted simplicial cochains of $L \otimes \varphi_* K$ and the simplicial cup form, and compares it entry by entry with certificate D's form for all four permutation classes. It also checks the simplicial model on fourteen explicitly built small covers.

The bundle is the directory `weil-nonsplit-g32-review`, distributed with this draft; on publication it will be deposited permanently. The tools are GAP 4.14.0 and SageMath 10.7, whose ball arithmetic is FLINT/Arb. All commands were rerun from a clean copy on 14 October 2026; the full hashes, wall times and memory use are in the bundle's rerun record. The commands, run from the root of the bundle, and the first sixteen hex digits of the SHA-256 hashes of their outputs are:

- A The certificate: `gap -q certificate/A/verify_fixed_witness.g`, reading `witness51.json` (1b8a06d826887eeb) after the consistency check `python3 certificate/A/extract_witness51.py --check`: transcript 083398d5f8363771. The discovery run that produced the witness: `gap -q certificate/A/lattice_cert.g`: transcript be1aace01749be4d, `lattice.json` 187f24e91e22f2f0; a rerun may find a different valid witness. The independent check `python3 certificate/A/verify_lattice.py certificate/A/lattice.json` does not use GAP.
- B `sage certificate/B/monodromy_cert.sage`: 2f42184288042fe0.
- C `sage certificate/C/base_point_cert.sage`: 15cbc69e5d6e6188; `sage certificate/C/rank_cert.sage` 800 70 1/100000000
1: 1b19e85c8afd67c8; the same with final argument 2:
7074bfeee457aaa2.
- D `sage certificate/D/disc_cert.sage`: transcript
c9d0f22cacbc57a1, `disc.json` 3f338fa6d2ef9836; `sage`
`certificate/D/dictcheck_cert.sage`: dbb0c3324651dce8.
- E `sage certificate/E/pairing_cert.sage`: b2663eb87e2e0cd9;
`sage certificate/E/pairing_smallcases.sage`:
87ba3467f0bbe23c.

Every transcript ends with a line stating its result. Certificate C reproduces the minor of Proposition 5.1 as $2.603433 \cdot 10^{-80}$ with radius below $3.5 \cdot 10^{-177}$ for the first direction, and $4.135429 \cdot 10^{-81}$ for the second, with the tail bounded by coarse rational inequalities alone. Certificate D gives, for all four permutation classes, the same Hermitian matrix $h = \delta S$, with characteristic polynomial

$$\begin{aligned} x^6 - \frac{4}{3}x^5 - \frac{26}{9}x^4 + \frac{41}{27}x^3 + \frac{19}{27}x^2 - \frac{22}{81}x - \frac{2}{81} \\ = \left(x^2 - \frac{2}{9}\right) \left(x^4 - \frac{4}{3}x^3 - \frac{8}{3}x^2 + \frac{11}{9}x + \frac{1}{9}\right). \end{aligned}$$

Its six roots are real: approximately -1.32577 , -0.47140 , -0.07814 , 0.47140 , 0.47388 and 2.26336 . So the signature is $(3, 3)$ (certificate D counts it exactly in the real algebraic numbers), and the determinant is $-\frac{2}{81}$.

What the certificates do not prove. Certificate E checks the cocycle formula, not the identifications of Steps 2–4 of Proposition 6.1, which are proved in the text; it uses certificate D’s ordering of the six loops. The surjectivity of the pulled-back monodromy is proved in Section 5, not computed; certificate B checks only its group-theoretic inputs. Rigidity is used only for the four-point system L , where it is cited from Katz; the thirteen-point system φ^*L is not rigid, and no rigidity statement about it is used. The exact preservation of the apparent singularities along the isomonodromic flow is the argument of Section 5; the certificate checks only its consistency. The dictionary between A , B and rational vector functions, and the completeness of the ansatz, are Lemma B.1, not computations.

REFERENCES

- [1] S. Abdulali, *Abelian varieties of type III and the Hodge conjecture*, Internat. J. Math. 10 (1999), no. 6, 667–675.
- [2] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of Algebraic Curves I*, Grundlehren der mathematischen Wissenschaften 267, Springer, 1985.
- [3] V. Alexeev, R. Donagi, G. Farkas, E. Izadi and A. Ortega, *The uniformization of the moduli space of principally polarized abelian 6-folds*, arXiv:1507.05710.
- [4] V. Alexeev, R. Donagi, G. Farkas, E. Izadi and A. Ortega, *Hodge classes on the moduli space of $W(E_6)$ -covers and the geometry of $\mathcal{A}_6$* , arXiv:2107.09691.
- [5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford University Press, 1985; the group $U_4(2) \cong PSp_4(3)$, whose maximal subgroups have indices 27, 36, 40, 40 and 45.
- [6] C. Birkenhake and H. Lange, *Complex Abelian Varieties*, second edition, Grundlehren der mathematischen Wissenschaften 302, Springer, 2004.
- [7] F. Charles and C. Schnell, *Notes on absolute Hodge classes*, arXiv:1101.3647.
- [8] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
- [9] P. Frediani, A. Ghigi and M. Penegini, *Shimura varieties in the Torelli locus via Galois coverings*, arXiv:1402.0973.
- [10] E. Markman, *Cycles on abelian $2n$ -folds of Weil type from secant sheaves on abelian n -folds*, arXiv:2502.03415.
- [11] N. M. Katz, *Rigid Local Systems*, Annals of Mathematics Studies 139, Princeton University Press, 1996.
- [12] K. Koike, *Algebraicity of some Weil Hodge classes*, arXiv:math/0211304.
- [13] E. Markman, *Secant sheaves and Weil classes on abelian varieties*, arXiv:2509.23403.
- [14] J. S. Milne, *Lefschetz classes on abelian varieties*, Duke Math. J. 96 (1999), no. 3, 639–675.
- [15] OpenAI, *Algebraicity of Weil classes on split abelian eightfolds*, manuscript dated 18 September 2026, <https://github.com/openai/math>.
- [16] OpenAI, *Weil classes and Hodge classes on abelian powers*, manuscript dated 30 September 2026, <https://github.com/openai/math>.
- [17] OpenAI, *Abelian covers, Gale correspondences, and the Hodge conjecture for powers*, manuscript dated 30 September 2026, <https://github.com/openai/math>.
- [18] D. Patel and Y. Zhang, *Algebraicity of Hodge classes on some generalized Prym varieties*, arXiv:2506.13729.

- [19] C. Schoen, *Hodge classes on self-products of a variety with an automorphism*, Compositio Math. 65 (1988), 3–32.
- [20] C. Schoen, *Addendum to: Hodge classes on self-products of a variety with an automorphism*, Compositio Math. 114 (1998), 329–336.
- [21] B. Severn, *The Hodge conjecture for abelian varieties of dimension six to eight: a status overview*, draft, 2026.
- [22] B. Severn, *Algebraicity of Weil classes on Pryms of balanced cyclic covers*, draft, 2026.
- [23] B. van Geemen, *An introduction to the Hodge conjecture for abelian varieties*, in: Algebraic Cycles and Hodge Theory (Torino, 1993), Lecture Notes in Math. 1594, Springer, 1994, 233–252.
- [24] B. van Geemen and A. Verra, *Quaternionic Pryms and Hodge classes*, arXiv:math/0103111.
- [25] C. Voisin, *Hodge Theory and Complex Algebraic Geometry II*, Cambridge Studies in Advanced Mathematics 77, Cambridge University Press, 2003.
- [26] A. Weil, *Abelian varieties and the Hodge ring*, in: Collected Papers III, Springer, 1980, 421–429.

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