

Finite Analytic Characters and Tannakian Source Reconstruction: A Finite Tensor–Quiver Framework for Beyond Endoscopy

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Abstract

This paper organizes finite tensor–quiver reconstruction for reductive source data. For a fixed reductive Zariski-closed subgroup H of a reductive complex group G and a faithful finite-dimensional representation, the standard mixed-tensor reconstruction theorem and Noetherianity imply that finitely many H -invariant tensors have common stabilizer H . Classical quiver invariant theory then implies that finitely many traces of oriented cycles separate closed orbits, including semisimple realizations.

For $\mathrm{GL}_2/\mathbf{Q}$, we give a restricted application of Cheng’s symmetric-square limiting distribution. After a finite-level cuspidal block has been specified and a target constituent selected by an interpolating Hecke test function built from that block’s Hecke characters, ratios of trace-formula distributions recover the first two Satake power sums at auxiliary unramified places, provided Cheng’s hypotheses hold and the full selected coefficient is nonzero. This is post-selection recovery, not a method for identifying the target constituent, and applies only to the nonzero-coefficient visible subset.

Keywords: Beyond Endoscopy; automorphic representations; Langlands program; trace formulas; Tannakian categories; invariant theory; algebraic groups; quiver representations; Hecke algebras; spectral localization.

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1 Introduction

Beyond Endoscopy seeks to extract functorial information from the automorphic spectrum using trace formulas and distinguished spectral contributions. This paper proposes a finite tensorial refinement: instead of constructing infinitely many functorial lifts or a universal matrix-valued object, one fixes a finite tensor presentation determining the algebraic source and seeks to realize finitely many scalar cycle traces analytically.

Let $\rho : \Gamma_F \rightarrow G^\vee$ be a semisimple parameter and set $H_\rho = \overline{\rho(\Gamma_F)}^{\text{Zar}} \subset G^\vee$. Recovery of a semisimple parameter from trace data is distinct from recovery of its algebraic envelope inside the dual group; the latter is tensorial. The algebraic steps use finite invariant-tensor certificates and quiver invariant theory. The analytic steps concern which spectral atom has been identified and whether finitely many analytic observables reproduce its tensor character.

The algebraic ingredients are established results rather than new invariant-theory theorems: mixed-tensor reconstruction follows from Tannakian theory, with Noetherianity making the certificate finite, while the cycle-trace assertion follows from the Le Bruyn–Procesi theorem. The purpose here is to place these ingredients in one finite presentation and to separate the resulting algebraic reconstruction from the distinct analytic problem of identifying a spectral atom.

The paper’s specific automorphic calculation is limited. For $\text{GL}_2/\mathbf{Q}$, the tensor-square identities translate the first two Satake power sums into local tensor characters. Using Cheng’s residue-weighted distribution, an interpolating Hecke test function can then recover these power sums only after the target has been selected using its Hecke character on a finite-level block. The interpolator encodes that prior label; the calculation is not a label-free selector, and its nonzero-coefficient support need not be the full cuspidal spectrum. The universal analytic character conjecture is stated as a research program, not as a proved theorem.

For context, Altuğ’s work develops the GL_2 Beyond Endoscopy method for the standard representation, including trace-formula and isolation analysis in the unramified setting [5, 6]. Cheng’s ramified standard-representation analysis establishes a limit-form formula through cancellation estimates [8]; his ramified simple-trace-formula treatment for the symmetric-square representation gives the residue-weighted distribution used here and relates it to detection of dihedral forms [7]. Warade has independently developed a Jacquet–Zagier approach to the Beyond Endoscopy trace formula for symmetric powers of GL_2 [9]. Deng and Wu treat a different problem: deriving the symmetric-square L -function functional equation from level-one Petersson and Kuznetsov trace formulas [12]. The present paper does not derive these trace formulas or prove a new functorial lift; its automorphic result is the finite-block, post-selection ratio identity obtained by inserting an interpolator into Cheng’s symmetric-square formula.

2 Finite tensor reconstruction

Let k be an algebraically closed field of characteristic zero, let G be a reductive algebraic group over k , let $H \subseteq G$ be a reductive Zariski-closed subgroup, and let V be a faithful finite-dimensional representation of G . Put

$$T(V) = \bigoplus_{a,b \geq 0} V^{\otimes a} \otimes (V^\vee)^{\otimes b},$$

and for $n \geq 0$ let $T^{\leq n}(V)$ be the finite-dimensional sum over $a + b \leq n$.

Theorem 2.1 (Finite tensor reconstruction). *Define*

$$H_n = \{g \in G : g \cdot x = x \text{ for every } x \in T^{\leq n}(V)^H\}.$$

Then there exists N such that $H_N = H$. In particular, if $\tau_1, \dots, \tau_r$ is a basis of $T^{\leq N}(V)^H$, then

$$H = \bigcap_{j=1}^r \text{Stab}_G(\tau_j).$$

Proof. Each H_n is Zariski closed, being the common stabilizer of finitely many tensors, and $H \subseteq H_n$ for all n . The invariant-tensor reconstruction theorem for reductive subgroups in characteristic zero gives

$$H = \{g \in G : g \cdot x = x \text{ for every } x \in T(V)^H\};$$

this is the mixed-tensor form of algebraic Tannaka reconstruction [14]; compare also Schrijver's complex/unitary formulation [18]. By the definition of the H_n , it follows that

$$\bigcap_{n \geq 0} H_n = H.$$

Since G is affine, its Zariski topology is Noetherian. Hence the descending chain

$$H_0 \supseteq H_1 \supseteq H_2 \supseteq \dots$$

stabilizes. Its stable value is its intersection, hence is H . $\square$

The theorem is pointwise in the fixed subgroup H ; no uniform tensor-degree bound is asserted. The statement justifies the phrase *finite tensor certificate*. It should not be confused with a finite rigid tensor category. Tensor closure normally produces infinitely many objects. The finite object is a finite presentation of the relevant tensor system whose stabilizer already equals H .

3 Polarized observables and Reynolds projectors

Assume now that the base field is $\mathbf{C}$ (or extend scalars to $\mathbf{C}$ for this discussion) and that H is reductive. For a finite-dimensional G -representation W , averaging over a maximal compact subgroup K_H gives the Reynolds projector

$$P_H^W = \int_{K_H} \rho_W(h) dh, \quad (P_H^W)^2 = P_H^W, \quad \text{im } P_H^W = W^H.$$

Its matrix coefficients are scalar observables. For bases $u_1, \dots, u_d$ and dual bases $v_1^\vee, \dots, v_d^\vee$, the quantities

$$v_i^\vee P_H^W u_j$$

determine P_H^W .

Reynolds projectors are useful motivation for polarized analytic observables, but they do not by themselves recover every intertwiner. Indeed, if $f : W \rightarrow W'$ is H -equivariant then

$$f P_H^W = P_H^{W'} f,$$

but the converse can fail for an arbitrary linear map. The correct tensorial identity is

$$\text{Hom}_H(W, W') \cong (W^\vee \otimes W')^H.$$

Thus the tensor category is recovered from invariant tensors in mixed tensor powers, while polarized traces explain how finite-dimensional operator data can be accessed through scalar observables.

4 Why scalar moments fail

For a finite subgroup H and a representation V , consider the scalar moments

$$M_k(H) = \dim(V^{\otimes k})^H.$$

Even the complete sequence $(M_k)_{k \geq 0}$ need not distinguish source subgroups.

Proposition 4.1 (S_4 obstruction). *Let $G = S_4$, let V be its standard three-dimensional representation, and let $H_1 = C_4$, $H_2 = V_4$. Then*

$$M_k(H_1) = M_k(H_2) = \frac{3^k + 3(-1)^k}{4}$$

for all $k \geq 0$, while

$$\dim(\wedge^3 V)^{C_4} = 0, \quad \dim(\wedge^3 V)^{V_4} = 1.$$

Proof. The standard character takes the value 3 at the identity and -1 at the three nonidentity elements of both subgroups, giving the displayed moment formula. On the other hand $\wedge^3 V \simeq \text{sgn}$, and C_4 contains odd permutations whereas V_4 consists of even permutations. $\square$

Principle 4.2 (Channel principle). Scalar invariant dimensions are an incomplete decategorification of tensor information. Source detection must retain enough tensor channels to distinguish invariant morphisms that scalar moments identify.

The obstruction is deliberately modest. It does not say that no scalar family can ever distinguish a particular pair of sources. It shows only that one natural infinite scalarization, namely all tensor-power invariant dimensions in one faithful representation, can be fundamentally incomplete.

5 Finite tensor presentations and quivers

Fix a finite tensor presentation consisting of finitely many tensor objects $W_1, \dots, W_m$ and finitely many distinguished structural morphisms, together with relations encoding composition, duality, evaluation, coevaluation, contractions, permutation maps, and selected tensor decompositions. Choose bases $W_i \simeq k^{d_i}$. The presentation can be encoded by a finite quiver with relations

$$Q = (Q_0, Q_1, R),$$

where path-algebra relations are used whenever the presentation is a representation of an associative algebra. More general tensor-compatibility constraints may also be imposed as polynomial equations; in that case, let X denote the resulting closed G_d -stable realization space. A realization is a tuple of matrices

$$A_a \in \text{Mat}_{d_{t(a)} \times d_{s(a)}}(k)$$

subject to the corresponding polynomial equations, including the equations from R and any additional tensor-compatibility constraints. The base-change group

$$G_d = \prod_{i \in Q_0} \text{GL}(d_i)$$

action is

$$A_a \longmapsto g_{t(a)} A_a g_{s(a)}^{-1}.$$

For an oriented cycle $c = a_\ell \cdots a_1$, put

$$A(c) = A_{a_\ell} \cdots A_{a_1}, \quad \tau_c(A) = \text{Tr}(A(c)).$$

For source reconstruction, the presentation must be *anchored*: one vertex represents the chosen faithful representation V , a unit vertex represents the tensor unit, and the finite invariant tensors from Theorem 2.1 are included as distinguished arrows from the unit into the corresponding mixed-tensor vertices. The chosen evaluation, coevaluation, permutation, contraction, and tensor-structure maps, together with their relations, must force changes of basis at mixed-tensor vertices to be induced by the change of basis at V . The common stabilizer at the anchored faithful vertex is then the finite tensor-certificate stabilizer H . An arbitrary unanchored quiver isomorphism, by itself, does not identify an embedded subgroup of G .

6 Finite quiver reconstruction

Theorem 6.1 (Finite quiver character reconstruction). *Let Q be a finite quiver with dimension vector d , and let $X \subseteq \text{Rep}(Q, d)$ be any closed G_d -stable subvariety defined by polynomial relations. Then $k[X]^{G_d}$ is generated by restrictions of traces of oriented cycles. Consequently, there are finitely many cycles $c_1, \dots, c_M$ whose traces separate closed G_d -orbits in X . In particular, if $A_1, A_2 \in X$ have closed G_d -orbits and*

$$\tau_{c_j}(A_1) = \tau_{c_j}(A_2) \quad (1 \leq j \leq M),$$

then A_1 and A_2 are isomorphic as quiver representations. When $X = \text{Rep}(A, d)$ for a quotient $A = kQ/I$ by path-algebra relations, closed orbits are precisely semisimple A -modules; in that case, the statement applies in particular to semisimple modules.

Proof. The Le Bruyn–Procesi theorem states that the invariant ring of the unconstrained quiver representation variety is generated by traces of oriented cycles [15]. Passing to the closed subvariety X amounts to quotienting the coordinate ring by a G_d -stable ideal. Since G_d is reductive in characteristic zero, taking invariants is exact on rational representations, so the restricted cycle traces generate $k[X]^{G_d}$. Finite generation gives a finite subfamily of these trace functions that generates the invariant algebra and hence separates closed orbits in the affine quotient. If two points have closed orbits with the same invariant values, those orbits coincide, so the points are isomorphic as quiver representations. For representation varieties of associative algebras, the standard closed-orbit criterion identifies closed orbits with semisimple modules. $\square$

The algebraic compression is therefore

$$\begin{aligned} \text{finite tensor presentation} &\longrightarrow \text{finite quiver with relations} \\ &\longrightarrow \text{finite cycle character} \longrightarrow \text{semisimple realization.} \end{aligned}$$

No infinite tensor tower is needed once a finite tensor certificate has been fixed.

7 Finite spectral localization versus finite-packet interpolation

The phrase “finite Hecke localization” admits two logically different uses. For a prescribed finite packet, one may separate the packet by Hecke characters. For a genuine spectral projector in a trace formula, the entire spectral support of that trace formula must already be known to be finite, or an independent global projector must be available. We distinguish the two.

Proposition 7.1 (Finite-packet Hecke interpolation). *Let $\Pi = \{\pi_1, \dots, \pi_m\}$ be a finite set of automorphic representations and suppose that the associated characters of a commutative Hecke algebra are pairwise distinct. In the semisimple quotient of the finite-dimensional Hecke image, there are pairwise orthogonal idempotents e_{π_i} satisfying*

$$\lambda_{\pi_j}(e_{\pi_i}) = \delta_{ij}.$$

Proof. The distinct characters give distinct maximal ideals in the finite-dimensional commutative algebra. After semisimplification, the Chinese remainder theorem identifies the algebra with a product of copies of $\mathbf{C}$, and the coordinate idempotents give the claim. $\square$

This proposition by itself does not annihilate representations outside Π . We therefore use the terminology *finite-packet Hecke interpolation* when only a prescribed finite set is involved, *finite spectral localization* when complete spectral support has been proved finite, and *global Hecke projector* only when vanishing on the complete global spectral support has been established.

8 The finite analytic realization

Let Q be a finite tensor presentation. For each vertex $i \in Q_0$, suppose a genuine spectral localization supplies a projector $e_{\pi,i}$ on its channel space. For each arrow $a \in Q_1$, suppose an analytic construction supplies a transfer, Hankel, Hecke, or structural operator T_a from the channel space at $s(a)$ to that at $t(a)$ (with transfer/Hankel examples provided by Sakellaridis [16, 17]). Define the localized arrow by

$$A_{\text{an},\pi,a} = e_{\pi,t(a)} T_a e_{\pi,s(a)}.$$

For the finite quiver formalism to apply, one must additionally specify finite-dimensional channel spaces (or finite-dimensional quotients) on which these localized operators are represented by matrices of the prescribed dimensions. For a closed cycle $c = a_\ell \cdots a_1$, set

$$A_{\text{an},\pi}(c) = A_{\text{an},\pi,a_\ell} \cdots A_{\text{an},\pi,a_1}, \quad t_{\text{an},\pi}(c) = \text{Tr}(A_{\text{an},\pi}(c)).$$

When such finite-dimensional channel spaces are supplied and the prescribed quiver relations hold, the cycle traces give a finite analytic realization of the tensor presentation. A trace-formula identity or spectral localization alone does not establish this finite-dimensional factorization; it is part of the analytic input required by Theorem 10.1 and Conjecture 17.1.

No claim is made that the underlying trace-formula spaces form a neutral Tannakian category. The Tannakian language is used only for the algebraic endpoint, where the standard formalism applies [14]. The analytic object is an auxiliary finite-dimensional realization whose existence must be verified in the chosen analytic setting.

9 Observable tensor symmetry

Definition 9.1 (Observable tensor category). Let $\mathcal{C}_{\text{obs}}$ be the *full rigid symmetric tensor subcategory* of $\text{Rep}(G^\vee)$ generated by the representations accessible to the analytic probe. Thus all $G^\vee$ -equivariant tensor morphisms among the generated objects are retained; the finite presentation used below is only a finite certificate for this full tensor structure. Let

$$\omega_{\text{obs}} : \mathcal{C}_{\text{obs}} \rightarrow \text{Vect}_{\mathbf{C}}$$

be the restricted forgetful fiber functor.

Definition 9.2 (Observable kernel). Define

$$K_{\text{obs}} = \ker(G^\vee \rightarrow \text{Aut}^\otimes(\omega_{\text{obs}})).$$

If $W_1, \dots, W_r$ generate the accessible representation system, then

$$K_{\text{obs}} = \bigcap_{i=1}^r \ker(G^\vee \rightarrow \text{GL}(W_i)).$$

For this full rigid tensor subcategory, the observable group is the image of $G^\vee$ in $\text{Aut}^\otimes(\omega_{\text{obs}})$, equivalently $G^\vee/K_{\text{obs}}$.

For any algebraic subgroup $H \subseteq G^\vee$, write

$$K_{\text{obs}}(H) := H \cap K_{\text{obs}},$$

so the image of H in the ambient observable group is $H/K_{\text{obs}}(H)$.

For source-specific Tannakian reconstruction, distinguish $\mathcal{C}_{\text{obs}}$ from the full rigid symmetric tensor subcategory $\mathcal{C}_{\text{obs},H} \subseteq \text{Rep}(H)$ generated by the restricted representations $W_i|_H$. Thus all H -equivariant morphisms among these generated objects are retained; the finite anchored presentation is a certificate for this full tensor structure. Let $\omega_{\text{obs},H}$ be its restricted forgetful fiber functor. Its Tannaka group is the image of H , namely

$$\text{Aut}^\otimes(\omega_{\text{obs},H}) \simeq H/K_{\text{obs}}(H).$$

This source-specific group is not generally the same as $\text{Aut}^\otimes(\omega_{\text{obs}}) \simeq G^\vee/K_{\text{obs}}$: the latter retains only $G^\vee$ -equivariant morphisms, whereas $\mathcal{C}_{\text{obs},H}$ retains the full H -equivariant tensor structure needed to recover the source envelope.

If one of the accessible representations is faithful on the envelope H_ρ , then $K_{\text{obs}}(H_\rho) = 1$. A nontrivial kernel occurs when the accessible tensor system factors through a quotient, for example an adjoint or projective quotient. The quotient therefore belongs to the envelope actually being reconstructed and should not automatically be imposed on the ambient dual group.

10 Finite analytic-character reconstruction

Theorem 10.1 (Finite analytic-character reconstruction). *Let $\rho : \Gamma_F \rightarrow G^\vee$ be semisimple, put $H_\rho = \overline{\rho(\Gamma_F)}^{\text{Zar}}$, and choose a finite tensor presentation of the tensor system generated by H_ρ . Encode it by a finite quiver with relations Q_ρ , and let A_ρ be the arithmetic realization.*

Let X be the relevant closed G_d -stable realization space from Section 6. Suppose a localized analytic construction produces a point $A_{\text{an},\pi} \in X$ and the arithmetic construction gives $A_\rho \in X$, and assume that both points have closed G_d -orbits in X (as holds for semisimple modules in the path-algebra quotient setting). Let $c_1, \dots, c_M$ be finite cycle coordinates separating closed orbits in X , and assume

$$\text{Tr}(A_{\text{an},\pi}(c_j)) = \text{Tr}(A_\rho(c_j)) \quad (1 \leq j \leq M).$$

Then $A_{\text{an},\pi} \simeq A_\rho$ as quiver realizations with closed orbits in X . Their stabilizers are conjugate in the quiver base-change group. Identifying this stabilizer as a specified embedded subgroup of $G^\vee$ requires the isomorphism to be compatible with the anchored ambient tensor generators and structural maps described in Section 5, not merely an arbitrary quiver-representation isomorphism. If, in addition, the comparison is induced by a tensor equivalence from the source-specific category $\mathcal{C}_{\text{obs},H_\rho}$ to the analytic category, carrying $\omega_{\text{obs},H_\rho}$ to the analytic forgetful fiber functor, then the analytic Tannakian automorphism group is abstractly isomorphic to $H_\rho/K_{\text{obs}}(H_\rho)$. Equality of the finite cycle traces alone does not construct this tensor equivalence.

Proof. Equality of the finitely many separating cycle traces gives equality in the affine invariant quotient. By the explicit hypothesis, both orbits are closed, so they lie in the same closed orbit and the realizations are isomorphic. Isomorphic quiver realizations have conjugate stabilizers under the corresponding base change. The identification with the prescribed ambient subgroup of $G^\vee$ additionally requires compatibility with the anchored tensor generators and structural maps. If the source-specific categories are tensor-equivalent and the equivalence transports the fiber functor as specified in the theorem, the Tannakian groups are isomorphic; the arithmetic one is the image of H_ρ on the retained system, namely $H_\rho/K_{\text{obs}}(H_\rho)$. $\square$

Remark 10.2. The theorem is unconditional as an algebraic reconstruction statement once the specified closed realization space, closed-orbit hypotheses, and cycle equalities are given. Its automorphic content is conditional on the existence of the semisimple analytic realization and the finite cycle-character equalities. The Tannakian group identification additionally uses the fullness and compatible fiber-functor hypotheses above.

Parameter versus envelope. For GL_n in characteristic zero, traces determine a semisimple representation up to isomorphism by Brauer–Nesbitt. Recovering the algebraic envelope inside a general dual group is different: it requires the mixed-tensor character, not only a pseudocharacter of the parameter.

11 The GL_2 tensor laboratory

Let

$$G^\vee = \text{GL}_2(\mathbf{C}), \quad S = \text{Std}.$$

The basic decompositions are

$$S \otimes S \simeq \text{Sym}^2 S \oplus \det, \quad S \otimes S^\vee \simeq \mathbf{1} \oplus \text{Ad}^0 S.$$

Let τ be the transposition on $S \otimes S$, $\tau(v \otimes w) = w \otimes v$. For any $A \in \text{GL}_2(\mathbf{C})$,

$$\text{Tr}(A \otimes A) = \text{Tr}(A)^2, \quad \text{Tr}(\tau(A \otimes A)) = \text{Tr}(A^2).$$

Therefore

$$\text{Tr}(\text{Sym}^2 A) = \frac{\text{Tr}(A)^2 + \text{Tr}(A^2)}{2}, \quad \det(A) = \frac{\text{Tr}(A)^2 - \text{Tr}(A^2)}{2}.$$

Writing $t = \text{Tr}(A)$ and $d = \det(A)$, Cayley–Hamilton gives

$$\text{Tr}(A^2) = t^2 - 2d, \quad \text{Tr}(\text{Sym}^2 A) = t^2 - d.$$

The adjoint character is

$$\chi_{\text{Ad}^0}(A) = \text{Tr}(A) \text{Tr}(A^{-1}) - 1 = \frac{\text{Tr}(A)^2}{\det(A)} - 1.$$

Moreover

$$\text{Ad}^0(\rho) \simeq \text{Sym}^2(\rho) \otimes \det(\rho)^{-1},$$

so the adjoint channel is insensitive to a central twist. For irreducible two-dimensional parameters, isomorphism of the projectivizations implies twist-equivalence by a character: after conjugation, write $\rho_1 \simeq \rho_2 \otimes \chi$. If their determinants agree, then equality of determinants forces $\chi^2 = 1$. Thus

the adjoint channel detects the projective parameter up to twist, while determinant data constrain the residual central ambiguity.

A convenient finite GL_2 tensor spine is

$$\mathrm{Std}, \quad \det, \quad \mathrm{Std}^\vee, \quad \mathrm{Sym}^2 \mathrm{Std}, \quad \mathrm{Ad}^0(\mathrm{Std}),$$

with the displayed decompositions and the associated evaluation, coevaluation, permutation, and projection maps.

11.1 The local state and its projective coordinate

At a selected maximal-level unramified place v , write the semisimple Satake parameter as A_v and set

$$p_{1,v} = \mathrm{Tr}(A_v), \quad p_{2,v} = \mathrm{Tr}(A_v^2), \quad d_v = \det(A_v) = \frac{p_{1,v}^2 - p_{2,v}}{2}.$$

Because $A_v \in \mathrm{GL}_2(\mathbf{C})$, $d_v \neq 0$. Thus the two channels determine its characteristic polynomial

$$X^2 - p_{1,v}X + d_v$$

and its semisimple conjugacy class. If $c_{m,v} = \mathrm{Tr}(\mathrm{Sym}^m A_v)$, the Clebsch–Gordan recurrence gives

$$c_{0,v} = 1, \quad c_{1,v} = p_{1,v}, \quad c_{m+1,v} = p_{1,v}c_{m,v} - d_v c_{m-1,v},$$

so

$$\sum_{m \geq 0} c_{m,v} u^m = \frac{1}{1 - p_{1,v}u + d_v u^2}.$$

Equivalently, in the spherical Hecke algebra with $\mathrm{Sat}(T_v) = x_1 + x_2$ and $\mathrm{Sat}(Z_v) = x_1 x_2$, the elements

$$H_{0,v} = 1, \quad H_{1,v} = T_v, \quad H_{m+1,v} = T_v H_{m,v} - Z_v H_{m-1,v}$$

satisfy $\mathrm{Sat}(H_{m,v}) = \mathrm{Tr}(\mathrm{Sym}^m A_v)$. This is a same-place algebraic recurrence. It does not assert the existence of a global automorphic symmetric-power lift. On the finite visible support, each fixed $H_{m,v}$ can be evaluated by the Hecke-insertion formula whenever the hypotheses and nonzero denominator of Section 12 hold.

The projective adjoint coordinate is

$$y_v = \chi_{\mathrm{Ad}^0}(A_v) = \frac{p_{1,v}^2}{d_v} - 1 = \frac{p_{1,v}^2 + p_{2,v}}{p_{1,v}^2 - p_{2,v}}.$$

It is invariant under scalar twists of A_v . The irreducible algebraic representations of $\mathrm{PGL}_2(\mathbf{C})$ are

$$W_r = \mathrm{Sym}^{2r}(\mathrm{Std}) \otimes \det^{-r}, \quad r \geq 0,$$

and their characters are polynomials in y_v : if $P_r(y) = \chi_{W_r}(y)$, then

$$P_0(y) = 1, \quad P_1(y) = y, \quad P_{r+1}(y) = (y - 1)P_r(y) - P_{r-1}(y).$$

Consequently the local state $(p_{1,v}, p_{2,v})$ determines every algebraic projective character at that place.

There is an important limit to this conclusion. The value y_v is generally complex for an arbitrary Satake parameter; it need not lie in $[-1, 3]$. A positive trace measure of the kind used in compact projective source geometry requires a separately specified compact subgroup $K \subset \mathrm{PGL}_2(\mathbf{C})$, or a

global projective parameter whose image has relatively compact closure. After conjugation such a compact group lies in $\mathrm{PU}(2) \simeq \mathrm{SO}(3)$, where the adjoint trace lies in $[-1, 3]$. A finite projective image is one sufficient case. The compact topological closure of a projective image and the Zariski envelope H_ρ used in this paper are different constructions and should not be identified without an additional hypothesis. Thus the result here supplies the local algebraic coordinates for source geometry, but does not by itself construct a global source measure or prove compactness of the projective image.

12 Finite-level visible-spectrum realization

We now give the main analytic statement. Fixing finite level and a discrete-series archimedean type gives a finite set of global cuspidal constituents. In this GL_2 setting the required archimedean isolation is established below by choosing a very-cuspidal pseudo-coefficient and using the generic unitary dual of $\mathrm{GL}_2(\mathbf{R})$. This removes the separate global-support isolation hypothesis for the stated discrete-series setting; it does not provide an analogous result for arbitrary real reductive groups or arbitrary archimedean targets.

Let

$$G = \mathrm{GL}_2/\mathbf{Q}, \quad S = \{\infty, q_1, \dots, q_r\},$$

with $2 \in S$ and with the two distinguished places required by Cheng's Assumption 1.1. Enlarge S , if necessary, to include every finite place at which K_f is non-maximal; further unrestricted auxiliary finite places may be added without changing the distinguished roles. The cuspidal spectral expansion is taken on the norm-one quotient $G(\mathbf{Q}) \backslash G(\mathbf{A})^1$ used in Cheng's formula, where $G(\mathbf{A})^1$ is the kernel of the adelic modulus character. We label its constituents by the corresponding global cuspidal automorphic representations of $G(\mathbf{A})$. The term discrete series for $G(\mathbf{R}) = \mathrm{GL}_2(\mathbf{R})$ is understood in the usual sense of square-integrability modulo center.

Fix a compact-open subgroup

$$K_f = \prod_p K_p \subset G(\mathbf{A}_f)$$

and an irreducible discrete-series representation π_∞^0 of $G(\mathbf{R})$. Define the finite-level block of global isomorphism classes

$$\Sigma(K_f, \pi_\infty^0) = \{\sigma \text{ cuspidal automorphic representation} : \sigma_f^{K_f} \neq 0, \sigma_\infty \simeq \pi_\infty^0\}.$$

For the stated $\mathrm{GL}_2/\mathbf{Q}$ setting, the block is finite by fixed-level, fixed-weight finiteness for cuspidal automorphic forms. To see the reduction, choose a congruence subgroup $K(N)$ contained in K_f . The fixed discrete-series type determines the weight and parity, while the central characters compatible with K_f form a finite set because their conductors are bounded by the fixed level. Every member of the block therefore contributes to one of finitely many finite-dimensional classical cusp-form spaces of the prescribed weight, level, and central character [13]. Hence only finitely many global isomorphism classes occur. We do not use Shin–Templier's field-of-rationality finiteness theorem to justify this assertion: their result includes a bounded-field-of-rationality hypothesis, which is a distinct statement [26]. This is a finiteness statement about isomorphism classes, not an identification of a global automorphic representation with a finite-dimensional vector space.

12.1 Archimedean isolation

Let $Z_+ = \{xI_2 : x > 0\}$ denote the positive real scalar center, as in Cheng's test-function space. Since the spectral expansion is on $G(\mathbf{A})^1$, a global automorphic constituent is unchanged by twisting

an extension to $G(\mathbf{A})$ by $|\det|_{\mathbf{A}}^{is}$. We use this freedom to normalize every archimedean representative so that its central character is trivial on Z_+ . Indeed, if a unitary central character has the form $\omega_\infty(x) = \text{sgn}(x)^\epsilon |x|^{it}$, replacing π_∞ by $\pi_\infty \otimes |\det|^{-it/2}$ changes its central character on xI_2 to $\text{sgn}(x)^\epsilon$. This makes the local representation and test function well-defined on $Z_+ \backslash G(\mathbf{R})$ without changing the constituent on $G(\mathbf{A})^1$. We make this normalization consistently throughout the finite-level block and the Hecke-character comparisons.

Definition 12.1 (Archimedean trace isolation). A Cheng-admissible f_∞ trace-isolates π_∞^0 on the relevant cuspidal spectrum if

$$\text{Tr } \pi_\infty^0(f_\infty) = 1$$

and

$$\text{Tr } \sigma_\infty(f_\infty) = 0$$

for every cuspidal automorphic representation σ occurring in Cheng's spectral expansion with $\sigma_\infty \not\sim \pi_\infty^0$.

Proposition 12.2 (Archimedean trace isolation for GL_2). *Let $G = \text{GL}_2/\mathbf{Q}$ and let π_∞^0 be an irreducible discrete-series representation in the central-character normalization compatible with Cheng's quotient. There exists a function f_∞ in the required central-character test-function space on $Z_+ \backslash G(\mathbf{R})$ that is admissible at infinity for Cheng and trace-isolates π_∞^0 on the global cuspidal spectrum.*

Proof. The discrete-series pseudo-coefficient theorem, together with Labesse's very-cuspidal refinement, gives a compactly supported-modulo-center pseudo-coefficient whose trace is 1 on π_∞^0 , 0 on every inequivalent irreducible tempered representation, and whose two-sided unipotent integrals vanish [2, 3]. The normalization above makes the function descend to $Z_+ \backslash G(\mathbf{R})$. If needed, project to the required central-parity block for $Z(\mathbf{R})/Z_+ \simeq \{\pm I\}$ by averaging over the two central classes; this preserves compact support modulo center, the trace identity on the target, and very-cuspidality. Thus we may choose f_∞ with

$$\text{Tr } \pi_\infty^0(f_\infty) = 1, \quad \text{Tr } \tau(f_\infty) = 0$$

for every other irreducible tempered representation τ in the same central-character block, and with

$$\int_{\mathbf{R}} f_\infty(gn(x)h) dx = 0 \quad (g, h \in G(\mathbf{R})), \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

This is precisely Cheng's supercuspidality condition at ∞ . The pseudo-coefficient is compactly supported modulo the center, which is stronger than the required compact-support condition on its orbital integrals modulo Z_+ .

Now let σ be a cuspidal automorphic representation contributing to Cheng's spectral expansion. Such a σ is unitary and globally generic [10, 11]; hence its local component σ_∞ is generic. If σ_∞ is tempered, the pseudo-coefficient identity gives the required trace 1 for the target and 0 for every other tempered type. If σ_∞ is non-tempered, the classification of the unitary generic dual of $\text{GL}_2(\mathbf{R})$ [4] identifies it with an irreducible complementary-series principal series induced from the proper Borel subgroup. The very-cuspidal property implies that the trace on such a parabolically induced representation is zero. At reducibility parameters, non-tempered one-dimensional constituents are nongeneric and therefore cannot occur as local components of cuspidal automorphic representations;

generic constituents at those parameters are tempered and are already covered by the pseudo-coefficient identity. Consequently

$$\mathrm{Tr} \sigma_\infty(f_\infty) = \begin{cases} 1, & \sigma_\infty \simeq \pi_\infty^0, \\ 0, & \sigma_\infty \not\simeq \pi_\infty^0. \end{cases}$$

This proves the claimed trace isolation and Cheng-admissibility. $\square$

Remark 12.3. The proposition proves the archimedean isolation needed in this paper's stated $\mathrm{GL}_2/\mathbf{Q}$ setting, for a discrete-series-modulo-center target and after the explicit normalization of the local extension. It uses the very-cuspidal pseudo-coefficient theorem and local genericity of cuspidal GL_2 representations. It does not prove an isolation theorem for arbitrary real reductive groups, arbitrary archimedean types, or non-generic automorphic spectra.

12.2 Finite-level compatibility

Choose the fixed finite local test functions in Cheng's construction to be bi- $K_{f,w}$ -invariant. At almost every finite place put

$$K_p = G(\mathbf{Z}_p).$$

At a distinguished finite place q_1 where Cheng requires a supercuspidal or elliptic-supported test function, do not impose maximal spherical level unless that is independently compatible with the chosen function. Instead choose K_{q_1} small enough that the fixed local test function is bi- K_{q_1} -invariant. Since compactly supported locally constant functions are bi-invariant under some compact-open subgroup, this causes no formal obstruction.

Outside S , Cheng's n -dependent local functions are spherical [7]. Hence they are bi-invariant under $G(\mathbf{Z}_p)$ and their variation with n changes Hecke eigenvalues without enlarging the fixed-level spectral space.

Proposition 12.4 (Complete spectral support). *Under the preceding finite-level compatibility hypotheses, with f_∞ chosen as in Proposition 12.2, every cuspidal representation contributing nontrivially to Cheng's spectral expansion belongs to $\Sigma(K_f, \pi_\infty^0)$. In particular, the complete cuspidal spectral support of the chosen Cheng distribution is finite.*

Proof. Suppose σ contributes nontrivially. Proposition 12.2 gives $\sigma_\infty \simeq \pi_\infty^0$. At a finite place w where f_w is bi- $K_{f,w}$ -invariant, writing $e_{K_{f,w}}$ for the normalized characteristic function of $K_{f,w}$ gives

$$\sigma_w(f_w) = \sigma_w(e_{K_{f,w}}) \sigma_w(f_w) \sigma_w(e_{K_{f,w}}).$$

Because a nonzero local trace implies that the operator $\sigma_w(f_w)$ is nonzero, the displayed factorization forces $\sigma_w^{K_{f,w}} \neq 0$. At every place outside S , Cheng's varying function is spherical, so $K_{f,w} = G(\mathbf{Z}_w)$ is compatible. Hence $\sigma_f^{K_f} \neq 0$ and $\sigma \in \Sigma(K_f, \pi_\infty^0)$. $\square$

12.3 Hecke separation on the finite-level block

Let $\Sigma = \Sigma(K_f, \pi_\infty^0)$ be the finite-level block defined above. It contains the complete cuspidal support of the chosen distribution, but may be strictly larger than that support because some members of the block can have zero coefficient for the chosen test data. For every pair $\sigma_i \neq \sigma_j$, strong multiplicity one supplies a finite place [10, 11] where their unramified local Hecke characters differ. Since Σ is finite, the union of the pairwise separating places is finite. Because K_f is maximal at almost every

place, these separating places may be chosen among maximal-level unramified places: otherwise two representations would agree at all but finitely many places and hence be globally isomorphic.

Choose maximal-level spherical Hecke elements $t_1, \dots, t_s$ at those places such that

$$\Phi(\sigma) = (\lambda_\sigma(t_1), \dots, \lambda_\sigma(t_s))$$

is injective on Σ . Multivariable interpolation gives, for each $\pi \in \Sigma$, a polynomial P_π with

$$P_\pi(\Phi(\pi)) = 1, \quad P_\pi(\Phi(\sigma)) = 0 \quad (\sigma \neq \pi).$$

Set

$$e_\pi = P_\pi(t_1, \dots, t_s).$$

This is a finite linear combination of genuine compactly supported spherical Hecke test functions, so it remains admissible in Cheng's framework by linearity.

Definition 12.5 (Finite visible Hecke quotient). Let $\mathcal{H}_\Sigma^{\text{sph}}$ denote the commutative spherical Hecke algebra generated by the maximal-level unramified local factors used in the interpolation on this finite-level block, and let

$$\mathcal{H}_\Sigma = \mathcal{H}_\Sigma^{\text{sph}} / \text{Ann}(\Sigma)$$

be its quotient by the annihilator of the finite-level block. The joint Hecke-character map identifies

$$\mathcal{H}_\Sigma \simeq \prod_{\sigma \in \Sigma} \mathbf{C},$$

provided the chosen finite Hecke coordinates separate the elements of the finite-level block Σ . This quotient acts on the entire block, even when the support of a particular trace-formula distribution is a proper subset.

Proposition 12.6 (Primitive idempotents on the finite Hecke quotient). *For every $\pi \in \Sigma$ there is an admissible finite Hecke test function e_π whose image in $\mathcal{H}_\Sigma$ is the primitive idempotent corresponding to the factor indexed by π . Equivalently,*

$$\lambda_\sigma(e_\pi) = \delta_{\sigma, \pi} \quad (\sigma \in \Sigma).$$

No assertion is made that e_π is idempotent in the full spherical Hecke algebra.

Proof. The finite set Σ is separated by finitely many joint spherical Hecke coordinates. Polynomial interpolation therefore produces an element acting by 1 on π and by 0 on every other $\sigma \in \Sigma$. Its class in $\mathcal{H}_\Sigma$ is exactly the corresponding primitive idempotent under

$$\mathcal{H}_\Sigma \simeq \prod_{\sigma \in \Sigma} \mathbf{C}.$$

The interpolating element is a finite linear combination of compactly supported spherical Hecke functions, hence remains an admissible test function by linearity. $\square$

13 Cheng's limiting distribution and Hecke insertion

For an admissible S -test function F_S , define the limiting functional

$$D_{\text{Ch}, S}(F_S) = \lim_{X \rightarrow \infty} \frac{1}{X} \sum_{\substack{n < X \\ (n, S)=1}} I_{\text{cusp}}(F^{n^2}),$$

whenever the limit is supplied by Cheng's theorem. Under his hypothesis $\varrho < 1/8$, Theorem 1.2 gives the corresponding asymptotic with a power-saving error, and Corollary 1.5 gives the spectral expansion [7]. For general background on absolute convergence on the spectral side of trace formulas, see Finis–Lapid–Müller [24]:

$$D_{\text{Ch},S}(F_S) = \sum_{\sigma} m_{\sigma} R_S(\sigma) \prod_{u \in S} \text{Tr } \sigma_u(F_u),$$

where

$$R_S(\sigma) = \text{Res}_{s=1} \frac{L^S(s, \sigma, \text{Sym}^2)}{L^S(2s, \chi_{\sigma}^2)}.$$

Proposition 13.1 (Hecke insertion). *Let $v \in S$ be an unrestricted finite auxiliary place and let h_v be a spherical Hecke function. Then*

$$D_{\text{Ch},S}(h_v \otimes F_{S \setminus \{v\}}) = \sum_{\sigma} m_{\sigma} R_S(\sigma) \lambda_{\sigma_v}(h_v) \prod_{u \neq v} \text{Tr } \sigma_u(F_u).$$

More generally, for finitely many auxiliary places $v_1, \dots, v_r$,

$$D_{\text{Ch},S} \left(\bigotimes_i h_{v_i} \otimes F_{S \setminus \{v_1, \dots, v_r\}} \right) = \sum_{\sigma} W_S(\sigma) \prod_i \lambda_{\sigma_{v_i}}(h_{v_i}),$$

with the corresponding definition of $W_S(\sigma)$.

Proof. This is the spectral formula with the corresponding local test-function components replaced. For unramified spherical σ_v , the trace of h_v is its spherical-Hecke character. Finite linear combinations follow by linearity. $\square$

14 The tensor-flip Hecke element

At a maximal unramified place v , define intrinsically

$$h_{\tau,v} = \text{Sat}^{-1}(x_1^2 + x_2^2).$$

This normalized Satake definition is primary. Under the convention

$$\text{Sat}(T_v) = x_1 + x_2, \quad \text{Sat}(Z_v) = x_1 x_2,$$

we have

$$h_{\tau,v} = T_v^2 - 2Z_v.$$

If U_v denotes the complete degree-two spherical Hecke element with

$$\text{Sat}(U_v) = x_1^2 + x_1 x_2 + x_2^2,$$

then $h_{\tau,v} = U_v - Z_v$.

If

$$A_{\pi,v} = \text{diag}(\alpha_{\pi,v}, \beta_{\pi,v}),$$

then

$$\lambda_{\pi_v}(h_{\tau,v}) = \alpha_{\pi,v}^2 + \beta_{\pi,v}^2.$$

This agrees with Cheng's local coefficient identity

$$A_{\pi}(p) = a_{\pi}(p^2) - \chi_{\pi}(p) = \alpha_p^2 + \beta_p^2$$

in his Section 10 / Proposition 10.1 [7].

Lemma 14.1 (Tensor-flip identity). *For every finite-dimensional endomorphism A ,*

$$\mathrm{Tr}(\tau(A \otimes A)) = \mathrm{Tr}(A^2).$$

Proof. In a basis e_i with dual basis $e_i^\vee$,

$$\mathrm{Tr}(\tau(A \otimes A)) = \sum_i e_i^\vee(\tau(Ae_i \otimes Ae_i)) = \sum_i e_i^\vee(A^2 e_i) = \mathrm{Tr}(A^2).$$

□

Theorem 14.2 (Finite visible-spectrum analytic certification of the local symmetric-cube character). *Let $G = \mathrm{GL}_2/\mathbf{Q}$, let*

$$S_0 = \{\infty, q_1, \dots, q_r\}$$

be a finite set containing the places required in Cheng's Assumption 1.1, the place 2, and all finite places at which K_f is non-maximal, and assume $\varrho < 1/8$. Fix a compact-open K_f and an irreducible discrete-series π_∞^0 , assume the finite-level compatibility hypotheses above, and choose f_∞ as in Proposition 12.2. Let

$$\Sigma = \{\sigma \text{ cuspidal} : \sigma_f^{K_f} \neq 0, \sigma_\infty \simeq \pi_\infty^0\}.$$

Then Σ is finite and contains the complete cuspidal spectral support for each local test datum satisfying the stated finite-level compatibility hypotheses and using the archimedean isolator from Proposition 12.2. Choose a finite set T of maximal-level unramified auxiliary places, disjoint from S_0 , that separates the elements of Σ by spherical Hecke characters, and let e_π be an interpolating test function supported at T with

$$\lambda_\sigma(e_\pi) = \delta_{\sigma, \pi} \quad (\sigma \in \Sigma).$$

Finally choose a further maximal-level unramified place $v \notin S_0 \cup T$ and put $S' = S_0 \cup T \cup \{v\}$. Apply Cheng's limiting formula with the enlarged set S' and the corresponding modified S' -local test data. At v , let e_{K_v} be the normalized spherical unit. For brevity, $D_{\mathrm{Ch}, S'}(e_\pi)$ denotes the functional evaluated on the fixed test components at S_0 , the interpolator e_π at the places in T , and e_{K_v} at v ; $D_{\mathrm{Ch}, S'}(e_\pi h_{j,v})$ replaces only the v -component by $h_{j,v}$. Expressions involving e_π are extended by linearity from pure tensors.

The effective selected coefficient is

$$W_{S'}^{\mathrm{sel}}(\pi) := m_\pi R_{S'}(\pi) \left(\prod_{u \in S_0} \mathrm{Tr} \pi_u(f_u) \right) \lambda_\pi(e_\pi) \lambda_{\pi_v}(e_{K_v}).$$

Here $\lambda_\pi(e_\pi) = 1$ and $\lambda_{\pi_v}(e_{K_v}) = 1$ by normalization. This is the coefficient of π after the interpolator is inserted, rather than a product of local traces for a generally non-factorizable interpolator. Assume this coefficient is nonzero. Define

$$h_{1,v} = T_v, \quad h_{2,v} = T_v^2 - 2Z_v$$

by

$$\mathrm{Sat}(h_{1,v}) = x_1 + x_2, \quad \mathrm{Sat}(h_{2,v}) = x_1^2 + x_2^2.$$

Then

$$\boxed{\frac{D_{\mathrm{Ch}, S'}(e_\pi h_{1,v})}{D_{\mathrm{Ch}, S'}(e_\pi)} = p_1(A_{\pi, v}) = \alpha_v + \beta_v,}$$

$$\frac{D_{\text{Ch},S'}(e_\pi h_{2,v})}{D_{\text{Ch},S'}(e_\pi)} = p_2(A_{\pi,v}) = \alpha_v^2 + \beta_v^2.$$

Consequently

$$d_v = \det A_{\pi,v} = \frac{p_1(A_{\pi,v})^2 - p_2(A_{\pi,v})}{2},$$

so the semisimple local conjugacy class of $A_{\pi,v}$ is determined by the extracted analytic pair. Its symmetric-cube realization has character

$$\text{Tr}(\text{Sym}^3 A_{\pi,v}) = p_1(A_{\pi,v})p_2(A_{\pi,v}).$$

Furthermore,

$$p_3(A_{\pi,v}) = -\frac{1}{2}p_1(A_{\pi,v})^3 + \frac{3}{2}p_1(A_{\pi,v})p_2(A_{\pi,v}),$$

so the complete S_3 -cycle character of the local symmetric-cube tensor construction is determined by the same finite certificate.

Proof. The finiteness and complete-support assertion follow from the preceding fixed-level argument. Pairwise strong multiplicity one gives, for each distinct pair in the finite set Σ , an unramified place at which their spherical Hecke characters differ [10, 11]; a finite union of such places yields the separating set T . Multivariable interpolation gives e_π with the stated eigenvalues, and its class in the finite Hecke quotient is the primitive idempotent.

We now reapply Cheng's theorem with the enlarged set $S' = S_0 \cup T \cup \{v\}$. The spectral formula gives

$$D_{\text{Ch},S'}(e_\pi) = W_{S'}^{\text{sel}}(\pi),$$

because every other member of Σ is annihilated by the Hecke interpolator. Replacing only the v -component by $h_{j,v}$ and using the spectral insertion identity gives

$$D_{\text{Ch},S'}(e_\pi h_{j,v}) = W_{S'}^{\text{sel}}(\pi) \lambda_{\pi_v}(h_{j,v}), \quad j = 1, 2.$$

Since $W_{S'}^{\text{sel}}(\pi) \neq 0$, division yields the displayed identities. The normalized Satake definitions give the eigenvalues $\alpha_v + \beta_v$ and $\alpha_v^2 + \beta_v^2$. The remaining formulas are elementary identities for a semisimple 2×2 matrix, and the S_3 permutation-cycle formula then determines the remaining local cycle traces. $\square$

Remark 14.3. The theorem concerns only the finite Cheng-visible subset under the stated finite-level compatibility conditions, the proved archimedean pseudo-coefficient construction, and the nonvanishing condition on $W_{S'}^{\text{sel}}(\pi)$. In Cheng's unitary automorphic setting, $L^{S'}(2s, \chi_\pi^2)$ is regular and nonzero at $s = 1$, so a nonzero residue in $R_{S'}(\pi)$ forces $L^{S'}(s, \pi, \text{Sym}^2)$ to have a pole at $s = 1$. Cheng describes this residue-weighted limit as a method for detecting dihedral forms [7]. Thus the theorem concerns the residue-detected part of the cuspidal spectrum with the required local trace factors nonzero; it does not establish the certificate for a general cuspidal representation or automatically for every dihedral representation.

Corollary 14.4 (Finite visible-spectrum symmetric-cube tensor certification). *For the nonzero-weight subset $\Sigma_{\text{Ch}}(S', \mathbf{f}_0)$ associated to the enlarged set and fixed test data of Theorem 14.2, the preceding theorem supplies the exact primitive analytic channels p_1 and p_2 at the selected auxiliary place v . The identities*

$$\det A_{\pi,v} = \frac{p_1(A_{\pi,v})^2 - p_2(A_{\pi,v})}{2}, \quad \text{Tr}(\text{Sym}^3 A_{\pi,v}) = p_1(A_{\pi,v})p_2(A_{\pi,v})$$

and

$$p_3(A_{\pi,v}) = -\frac{1}{2}p_1(A_{\pi,v})^3 + \frac{3}{2}p_1(A_{\pi,v})p_2(A_{\pi,v})$$

therefore determine the complete local S_3 -cycle character of the symmetric-cube tensor construction. Under the explicit hypotheses of Theorem 14.2, this is a finite visible-spectrum certificate; it is not an unrestricted global analytic character.

14.1 The Cheng-visible set

For an enlarged set $S' = S_0 \cup T \cup U$ and fixed test components $\mathbf{f}_0 = (f_u)_{u \in S_0}$, define the coefficient before inserting the target interpolator by

$$W_{S', \mathbf{f}_0}^0(\pi) = m_\pi R_{S'}(\pi) \prod_{u \in S_0} \text{Tr } \pi_u(f_u), \quad \Sigma_{\text{Ch}}(S', \mathbf{f}_0) = \{\pi \in \Sigma : W_{S', \mathbf{f}_0}^0(\pi) \neq 0\}.$$

For each target π in this set, choose its interpolator e_π on T and the normalized spherical units at the auxiliary places in U . These extra factors have eigenvalue 1 on the target. Thus the effective selected coefficient in Theorem 14.2 equals $W_{S', \mathbf{f}_0}^0(\pi)$, while the interpolator kills all other members of the complete finite support in the trace-formula evaluation. The visible set depends on the enlarged set and the fixed test data; it is not asserted to be all of Σ . In particular, this construction does not establish that every cuspidal representation has nonzero Cheng weight. Multiplicity one for GL_2 is not needed for the ratio: any nonzero multiplicity appears in numerator and denominator and cancels.

14.2 The finite Hecke quotient as the natural intermediate object

The quotient $\mathcal{H}_\Sigma$ is the correct algebraic home for finite spectral selection on the finite-level block. The full spherical Hecke algebra need not contain a global idempotent attached to π . What is true, and what the theorem uses, is the finite semisimple quotient

$$\mathcal{H}_\Sigma = \mathcal{H}_\Sigma^{\text{sph}} / \text{Ann}(\Sigma) \simeq \prod_{\sigma \in \Sigma} \mathbf{C}.$$

The element e_π is a lift to the full Hecke algebra of the primitive idempotent of this quotient. Thus $\mathcal{H}_\Sigma$ is the finite selection algebra; additional local character values are recovered by insertion after the target has been selected. For a fixed target and enlarged set S' as in Theorem 14.2, the trace-formula ratio

$$\lambda_{\pi_v}(h_v) = \frac{D_{\text{Ch}, S'}(e_\pi h_v)}{D_{\text{Ch}, S'}(e_\pi)}$$

recovers the local spherical Hecke character at v for every admissible auxiliary spherical h_v for which the insertion is defined and the denominator is nonzero. This formulation removes any implication that the unrestricted automorphic spectrum admits an exact point projector in the full Hecke algebra.

15 Finite visible Hecke characters

Corollary 15.1 (Multi-place visible Hecke character). *Under the hypotheses of Theorem 14.2, let $U = \{v_1, \dots, v_r\}$ be a finite set of maximal-level unramified auxiliary places disjoint from $S_0 \cup T$,*

and put $S_U = S_0 \cup T \cup U$. For each i , let $e_{K_{v_i}}$ denote the normalized spherical unit, and define

$$W_{S_U}^{\text{sel}}(\pi) = m_\pi R_{S_U}(\pi) \left(\prod_{u \in S_0} \text{Tr } \pi_u(f_u) \right) \lambda_\pi(e_\pi) \prod_{i=1}^r \lambda_{\pi_{v_i}}(e_{K_{v_i}}).$$

Reapply Cheng's limiting formula with the enlarged set S_U , interpreting $D_{\text{Ch}, S_U}(e_\pi)$ as using the fixed data at S_0 , the interpolator at T , and the spherical units at every place of U . Assume $W_{S_U}^{\text{sel}}(\pi) \neq 0$. Then, for $h_i \in \mathcal{H}(G(\mathbf{Q}_{v_i}), G(\mathbf{Z}_{v_i}))$,

$$\boxed{\frac{D_{\text{Ch}, S_U}(e_\pi \prod_{i=1}^r h_i)}{D_{\text{Ch}, S_U}(e_\pi)} = \prod_{i=1}^r \lambda_{\pi_{v_i}}(h_i)}.$$

Proof. Apply the multi-place Hecke-insertion identity to the enlarged set S_U and use the projector relation $\lambda_\sigma(e_\pi) = \delta_{\sigma, \pi}$ on the complete finite support. $\square$

In particular, the Satake multipliers

$$x_1 + x_2, \quad x_1 x_2, \quad x_1^2 + x_2^2, \quad x_1^2 + x_1 x_2 + x_2^2,$$

and

$$(x_1 + x_2)(x_1^{-1} + x_2^{-1}) - 1$$

recover the local characters of

$$\text{Std}, \quad \det, \quad \psi^2(\text{Std}), \quad \text{Sym}^2(\text{Std}), \quad \text{Ad}^0(\text{Std})$$

on the visible spectrum, where ψ^2 denotes the second Adams operation in the representation ring (a virtual character, not in general an actual representation). We call this a *finite visible Hecke character*; it is not a full analytic Tannakian character.

16 Four levels of spectral reconstruction

The previous sections isolate the finite tensor target and the finite visible-spectrum experiment. The remaining logical structure is best expressed as a four-level hierarchy. The key distinction is between the existence of spectral atoms, their character functions, the problem of identifying a prescribed atom, and the finite tensor reconstruction that follows once the atom or parameter has been identified.

16.1 Level I: finite spectral-algebra reconstruction

Let Π_{cusp} denote the set of cuspidal automorphic representations under consideration and let $\mathcal{A}$ be a commutative algebra of spectral observables. For $h \in \mathcal{A}$, write

$$b_h(\pi) = \lambda_\pi(h).$$

Define the reduced global observable algebra by

$$B_{\text{glob}}^{\text{red}} = B_{\text{cusp}} = \mathbf{C}[b_h : h \in \mathcal{A}] \subseteq \text{Fun}(\Pi_{\text{cusp}}, \mathbf{C}).$$

Every cuspidal representation gives an evaluation character

$$\text{ev}_\pi : B_{\text{glob}}^{\text{red}} \rightarrow \mathbf{C}.$$

For a finite cuspidal block $\Sigma \subset \Pi_{\text{cusp}}$, let

$$B_{\Sigma}^{\text{red}} = \text{im}(B_{\text{cusp}} \rightarrow \text{Fun}(\Sigma, \mathbf{C})).$$

If the observables separate Σ , then

$$B_{\Sigma}^{\text{red}} \simeq \prod_{\pi \in \Sigma} \mathbf{C}.$$

Thus the global algebra need not be finite-dimensional: its finite restrictions are the finite spectral algebras relevant to the reconstruction problem.

Suppose $\mathcal{A}$ is a finitely generated commutative $*$ -algebra of normal, mutually commuting observables, with evaluations that respect the involution, and define the exact finite-block moment functional by

$$M_{\Sigma}(PP^*) = \sum_{\pi \in \Sigma} m_{\pi} |P(\lambda_{\pi})|^2.$$

Then

$$I_{\Sigma} = \{P \in \mathcal{A} : M_{\Sigma}(PP^*) = 0\} = \bigcap_{\pi \in \Sigma} \mathfrak{m}_{\pi},$$

provided the chosen observables separate Σ .

Theorem 16.1 (Canonical finite spectral-algebra reconstruction). *Let Σ be a finite cuspidal spectral block and let $\mathcal{A}$ be a finitely generated commutative $*$ -algebra of normal, mutually commuting spectral observables separating the members of Σ , whose evaluation characters respect the involution. If the exact cuspidal spectral moments $M_{\Sigma}(PP^*)$ are known for $P \in \mathcal{A}$, then*

$$B_{\Sigma}^{\text{red}} \simeq \mathcal{A}/I_{\Sigma} \simeq \prod_{\pi \in \Sigma} \mathbf{C},$$

and its primitive characters and primitive idempotents are determined by those moments.

Proof. Since each $m_{\pi} > 0$, the equality $M_{\Sigma}(PP^*) = 0$ is equivalent to $P(\lambda_{\pi}) = 0$ for every $\pi \in \Sigma$. Hence $I_{\Sigma} = \bigcap_{\pi \in \Sigma} \mathfrak{m}_{\pi}$. The Chinese remainder theorem yields the product decomposition, whose coordinate evaluations and coordinate functions are precisely the primitive characters and primitive idempotents. $\square$

This theorem is an exact algebraic reconstruction result whose input is the full finite-block moment functional, evaluated on every PP^* with P in the chosen algebra. The finite-dimensional output is reconstructed from that functional; the theorem does not claim that a finite list of moment values alone suffices. It does not identify a named atom by itself.

16.2 Level II: labeled unrestricted spectral character

Suppose an atom x_{π} is externally identified. Its restrictions

$$\text{ev}_{\pi, \Sigma} : B_{\Sigma}^{\text{red}} \rightarrow \mathbf{C}$$

are compatible under restriction along an exhaustion by finite spectral blocks. The resulting global evaluation gives

$$\text{ev}_{\pi} : B_{\text{glob}}^{\text{red}} \rightarrow \mathbf{C}, \quad \lambda_{\pi} = \text{ev}_{\pi}|_{\mathcal{A}}.$$

Thus evaluation on the chosen observable algebra is formal once the spectral point has been identified. When $\mathcal{A}$ contains the full Hecke algebra under discussion, this is the full Hecke character; for a smaller $\mathcal{A}$, it recovers only the character restricted to that observable algebra.

Within this inverse-limit construction, the finite-to-global passage contains no additional analytic selection problem once the compatible atom has been identified. The same point is selected at every stage, so the finite evaluations glue to the global character.

16.3 Level III-A/B: identifiability and construction

Let D be an external description of a target, and let $\text{Obs}(D)$ be the permitted spectral functions constructed from the geometric input and D , without querying the full Hecke character. Define

$$\Phi_D(x) = (F(x))_{F \in \text{Obs}(D)}, \quad x \equiv_D y \iff \Phi_D(x) = \Phi_D(y).$$

A canonical selector can identify x_π only if its fiber is a singleton:

$$[x_\pi]_{\equiv_D} = \{x_\pi\}.$$

This is necessary, not an algorithm. Conversely, once an independent construction actually produces x_π , evaluation gives the character restricted to the chosen observable algebra, and finite-block reconstruction or a BPLZZ-type isolation theorem may be used downstream under its own hypotheses. Atom selection is logically prior to character evaluation.

16.4 Level III-C: independent global parameters

A particularly strong and reusable constructive mechanism is an independently constructed global parameter.

Theorem 16.2 (Independent-parameter Level III criterion). *Let $\mathcal{C}$ be a class of cuspidal automorphic representations and suppose that an independently constructed parameter assignment*

$$\mathcal{A} : \mathcal{C} \rightarrow \mathcal{P}$$

associates to each $\pi \in \mathcal{C}$ an object P_π satisfying:

1. P_π is constructed without using the complete Hecke character λ_π ;
2. for almost all places v , the local parameter attached to P_π agrees with the unramified local Langlands parameter of π_v ;
3. the automorphic class has a uniqueness theorem: if $\pi', \pi \in \mathcal{C}$ have the same local parameters for almost all v , then $\pi' = \pi$.

Then P_π identifies the unique atom x_π among candidates in $\mathcal{C}$ and hence determines λ_π within that class. To claim uniqueness in a larger spectrum, hypothesis (3) must apply to all competing representations in that larger spectrum.

Proof. If π' is compatible with the same P_π , hypothesis (2) gives equality of the local parameters at almost all places. Hypothesis (3) gives $\pi' = \pi$. Thus the parameter identifies one atom within the candidate class $\mathcal{C}$, and Level II gives its evaluation character there. $\square$

For GL_n , strong multiplicity one supplies the uniqueness hypothesis in the cuspidal setting [10, 11]. Whenever an independently constructed Galois representation or other global parameter with the required local-global compatibility is available, Theorem 16.2 gives a genuine Level-III mechanism.

16.5 Level III-D: universal atom identification

The universal problem is now isolated precisely:

$$\boxed{\forall \pi \in \Pi_{\text{cusp}}, \quad \exists D_\pi \text{ independently constructible and atom-identifying.}}$$

Here “atom-identifying” means that the full permitted observable profile has a singleton fiber at x_π and that a constructive procedure produces that point without inserting λ_π or an equivalent exact projector.

Theorem 16.3 (No-go for invariant Level-III data). *Let a group Γ act on the observable cuspidal spectrum and suppose both the geometric input and the external datum D are Γ -invariant. If*

$$\Phi_D(\gamma x) = \Phi_D(x)$$

for all $\gamma \in \Gamma$ and some γ satisfies $\gamma x_\pi \neq x_\pi$, then no canonical selector based only on D and the permitted observables can identify x_π .

Proof. The observable profile is constant on the Γ -orbit, so any selector based only on that profile can at most select the orbit. It cannot canonically choose one member of a nontrivial orbit. $\square$

The theorem applies to arbitrary observable algebras, not just finite-dimensional moment systems.

16.5.1 Examples of residual ambiguity

For GL_2 , if $\chi^2 = 1$, then $\text{Sym}^2(\pi \otimes \chi) \simeq \text{Sym}^2(\pi)$. Thus observables generated only by symmetric-square data cannot separate a quadratic-twist orbit; adding an odd channel, such as the standard or symmetric-cube channel, can break this particular symmetry. More generally, finitely many local Hecke values do not determine an unrestricted cuspidal representation: strong multiplicity one is an almost-all-places theorem [10, 11]. Packet-level data may identify a packet without identifying a member. BPLZZ-type isolation applies after a spectral component has been specified, while Sauvageot-type density gives approximation rather than exact singleton evaluation [23, 25]. None of these tools supplies the missing universal selector. If the permitted profile is invariant under a nontrivial action on distinct atoms, no deterministic procedure using only that profile can select one atom canonically.

16.6 Arithmetic completion and the tensor endpoint

An independently constructed compatible parameter determines its trace words, which specialize to Hecke/Satake characteristic-polynomial data at unramified places. The finite tensor and quiver results then give a finite certificate for any fixed faithful representation and tensor presentation.

Theorem 16.4 (Arithmetic completion from an independent parameter). *Let π lie in an arithmetic class for which a compatible semisimple Galois representation ρ_π is independently available. Assume its Frobenius characteristic polynomials agree with those of the automorphic local parameters at almost all places. Then the global parameter determines the semisimple character data of π , and every fixed finite mixed-tensor trace word in ρ_π is canonically determined. In particular, for GL_n in a class where strong multiplicity one identifies the automorphic representation from almost-all local parameters, ρ_π is a Level-III external identifier.*

Proof. Compatibility identifies the unramified parameters with Frobenius characteristic polynomials, and the global representation determines all its trace words. For GL_n , strong multiplicity one gives uniqueness from almost-all local parameters [10, 11]. $\square$

Remark 16.5. This is an arithmetic completion theorem. It does not assert that the trace formula constructs ρ_π ; its role is to explain exactly how an independently supplied global parameter closes the atom-identification and tensor-coherence stages.

Theorem 16.6 (Finite Tannakian certificate from the global parameter). *Assume a semisimple global parameter ρ_π is known, and put $H_\pi = \overline{\rho_\pi(\Gamma_F)}^{\text{Zar}}$. Fix a faithful finite-dimensional representation of its dual group together with a finite tensor certificate and a finite separating cycle family for the associated quiver presentation. Then the finite semisimple quiver point is determined by the mixed invariant-tensor data of ρ_π . For the source-specific category $\mathcal{C}_{\text{obs}, H_\pi} \subseteq \text{Rep}(H_\pi)$, generated by the retained representations and their H_π -equivariant structural morphisms, the Tannaka group is*

$$\text{Aut}^\otimes(\omega_{\text{obs}, H_\pi}) \simeq H_\pi / K_{\text{obs}}(H_\pi).$$

In the symmetric-cube laboratory, retaining the standard representation together with Sym^3 removes the central μ_3 kernel that occurs for Sym^3 alone. The Tannakian group assertion is understood for the full rigid tensor subcategory generated by the retained representations, as in Section 9.

Remark 16.7. This is a finite algebraic completion of the parameter. It does not prove analytic realization of the finite tensor cycles and does not solve universal Level III-D.

17 The finite analytic character conjecture

The algebraic reconstruction theorem identifies the exact analytic input needed beyond the proved finite-level example. The conjecture below concerns the finite tensor character after a chosen spectral point has been identified; it is therefore a Level-II-to-Tannakian statement, not the label-free Level-III selection problem.

Conjecture 17.1 (Finite Analytic Character Conjecture). *Fix a class $\mathcal{C}$ by explicit bounds on the dual-group rank, dimensions of the chosen representations, tensor degree and quiver size, ramification, and local test-function complexity. For every $\pi \in \mathcal{C}$ with a semisimple parameter ρ_π and with its spectral atom already identified, there exist a finite tensor quiver with relations Q_π , a finite separating family of cycles $C_\pi = \{c_1, \dots, c_M\}$, and finitely many Hecke-localized transfer, Hankel, and structural constructions such that the associated analytic cycle character satisfies*

$$t_{\text{an}, \pi}(c_j) = t_{\rho_\pi}(c_j) \quad (1 \leq j \leq M).$$

The quiver, cycles, and constructions may depend on π ; no uniform bound on the number of cycles or a uniform construction across $\mathcal{C}$ is asserted here.

Corollary 17.2 (Conditional Langlands-envelope reconstruction). *Assume the Finite Analytic Character Conjecture for π and a tensor equivalence, preserving fiber functors, between the analytic category and the source-specific category $\mathcal{C}_{\text{obs}, H_{\rho_\pi}}$ from Section 9. Then*

$$\text{Aut}^\otimes(\omega_{\text{an}, \pi}) \simeq H_{\rho_\pi} / K_{\text{obs}}(H_{\rho_\pi}).$$

The conjecture is finite twice over: finitely many tensor generators and finitely many scalar cycle coordinates. Its intended content excludes constructions whose coefficients or test functions are defined by querying the full Hecke character λ_π or by inserting an equivalent exact point projector. The admissible class of transfer, Hankel, and structural constructions, and the permitted dependence on the identified atom, must be fixed explicitly in any application; until that class is specified, Conjecture 17.1 should be read as a research schema rather than a fully formal existence statement. It is weaker than demanding a universal matrix-valued trace formula or all possible functorial lifts, and it begins after atom identification rather than solving Level III-D.

18 The first concrete analytic target

The first tensor cycle is the transposition on $\text{Std}^{\otimes 2}$, with $\text{Tr}(\tau(A_v \otimes A_v)) = \text{Tr}(A_v^2)$. Theorem 14.2 realizes this identity on the finite Cheng-visible subset after the stated localization and nonvanishing conditions. Removing those restrictions and realizing a richer character on a larger spectrum remain open.

19 Arithmetic examples of independent atom identification

Section 16 isolates universal atom identification as the open problem. The following examples illustrate the established Level-III mechanism: an independently constructed arithmetic object supplies compatible almost-all local parameters, which identify an automorphic representation when a uniqueness theorem applies.

19.1 Elliptic curves over $\mathbf{Q}$

Let $E/\mathbf{Q}$ be an elliptic curve. For $p \nmid N_E \ell$, choose the Frobenius convention for which the Tate-module representation has characteristic polynomial $T^2 - a_p(E)T + p$, where $a_p(E) = p + 1 - \#E(\mathbf{F}_p)$. Equivalently,

$$\det(1 - X\rho_{E,\ell}(\text{Frob}_p)) = 1 - a_p(E)X + pX^2.$$

The roots have product p and sum $a_p(E)$; the corresponding unitary-normalized Satake parameters are obtained by dividing them by $p^{1/2}$. Modularity and local-global compatibility identify these characteristic polynomials with those of the associated weight-two automorphic representation at almost all primes [1]. After fixing an embedding of the coefficient field into $\mathbf{C}$, strong multiplicity one identifies the automorphic representation [10, 11].

19.2 Regular-algebraic Hilbert modular forms

In the regular-algebraic Hilbert modular cases where compatible Galois representations and local-global compatibility are established, the same argument identifies the automorphic representation from almost-all local parameters. The cited constructions of Harris–Lan–Taylor–Thorne and Scholze, together with Varma’s local-global compatibility result, apply only in their stated settings [19, 22, 21]. Newton–Thorne’s symmetric-power functoriality result concerns the classes covered by their theorem [20]; none of these results supplies a universal parametrization for arbitrary cuspidal representations.

20 Research program

The next analytic step is to realize more separating cycles beyond the GL_2 tensor flip and to establish how transfer, Hankel, and structural operators interact after localization. Local identities such as

$$\text{Tr}(A^2) = \text{Tr}(A)^2 - 2\det(A), \quad \chi_{\text{Sym}^2}(A) = \text{Tr}(A)^2 - \det(A)$$

provide finite targets. For a distinct trace-formula derivation of the symmetric-square functional equation in the level-one setting, see Deng and Wu [12].

A second direction is to compare exact finite-block moments and analytic cycle traces with their arithmetic counterparts. The moment functional reconstructs the reduced algebra once the finite

block and separating observables are specified; the unresolved task is to identify the target atom without using its full Hecke character as input.

Finally, source detection for general reductive dual groups should use faithful representations, finite tensor certificates, and finite quiver cycle coordinates. The analytic challenge is to realize a separating family of those coordinates and determine when it detects a proper source subgroup through additional invariant tensors or relations.

21 Status and limitations

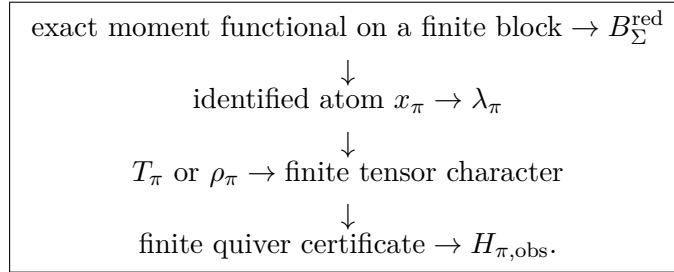
The four-level hierarchy separates finite-block algebra reconstruction and conditional arithmetic completion from the unresolved universal, label-free identification of a prescribed atom.

Status	Statements
Proved algebraically / spectrally	Finite tensor certificates; finite quiver cycle reconstruction; finite-block algebra reconstruction from the exact moment functional.
Proved from Cheng and standard local representation theory, under the stated restrictions	Fixed-level compatibility, a constructed very-cuspidal archimedean pseudo-coefficient, nonzero full Cheng weight, and required local trace nonvanishing; finite visible support, finite-quotient idempotents, Hecke insertion, and visible-spectrum p_1, p_2 recovery. The certificate is not a result for the general cuspidal spectrum.
Available after an external spectral label	Compatible character on the chosen observable algebra; arithmetic completion in settings with an available compatible Galois representation and uniqueness theorem.
Downstream isolation technology	BPLZZ-type isolation after the spectral character has been supplied; it is not label-free identification.
Plausible but unproved	An intrinsic finite-rank spectral quotient; conversion of Cheng's $a_\pi(n^2)$ distribution into a full $A_\pi(n)$ distribution.
Genuinely open	Universal label-free atom identification; full-spectrum tensor-flip realization; intrinsic global localization; global trace-formula realization of $\text{Sym}^2 - \det$; prime/logarithmic-derivative realization of the symmetric-square L -function ratio.

No general Langlands correspondence, universal Beyond Endoscopy trace formula, or reconstruction of arbitrary non-semisimple representations is claimed. The GL_2 certificate is restricted to the visible subset satisfying finite-level compatibility and nonvanishing of the Cheng weight and local factors; Cheng describes the underlying residue-weighted formula as a method for detecting dihedral forms [7].

22 Conclusion

The paper’s reconstruction strategy separates four arrows:



The algebraic steps are finite once the source and tensor presentation are fixed. The proved analytic application is narrower: on the finite support where the Cheng weight and required local factors are nonzero, fixed-level compatibility, archimedean pseudo-coefficient isolation, and Hecke interpolation recover local tensor channels. It does not cover arbitrary GL_2 cusp forms or automatically every dihedral representation.

The remaining universal question is whether independently defined data can identify an arbitrary prescribed cuspidal atom without encoding its full Hecke character. The framework clarifies the input needed for such a result but does not solve that open problem. In the GL_2 case, the channels recovered here also determine (t_v, d_v) and the projective adjoint coordinate y_v place by place. These identities prepare a source-side analysis of compact projective subgroups, but the passage from local coordinates to a global compact source and its positive Haar trace measure requires separate global hypotheses; it is not claimed as part of the present theorem.

Provenance and AI Use

This manuscript was predominantly written with substantial assistance from OpenAI’s GPT 5.6 Luna and GPT 6, under the direction of Shawn Schafer. The human contributor directed the AI-assisted research workflow and used these systems to conduct literature research, investigate and test mathematical claims and calculations, organize arguments, and draft and revise the manuscript and its $\text{\LaTeX}$ source. AI tools contributed substantially to both the mathematical development and the written exposition.

The current version has undergone an AI-assisted mathematical and source audit, including checks of theorem statements and proof dependencies, verification of the displayed local algebraic identities, comparison with relevant Beyond Endoscopy literature available through October 2026, and compilation, citation, and cross-reference checks. No proof assistant was used for formal verification, and this audit is not a machine-checked proof certificate. Shawn Schafer is the human contributor responsible for the submitted version, its claims and citations, and the accuracy of this provenance statement. No AI system is listed as an author or contributor.

Tools used: OpenAI GPT 5.6 Luna; OpenAI GPT 6.

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