

A six-spin obstruction to antiferromagnetic biclique maximization*

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Abstract

Sah, Sawhney, Stoner and Zhao conjectured that for every spin model whose interaction matrix has exactly one positive eigenvalue, the partition function of any graph is at most a product of biclique partition functions determined by its edge degrees. We disprove this conjecture with a strictly positive six-spin model on a triangle with three pendant edges; the violation persists on an open set of nonsingular models. The mechanism is a negative mixed covariance trace at vertices of degrees two, three and four: the triangle contributes a favourable cubic term, whereas all biclique corrections start at fourth order.

1 Introduction

1.1 Background and the source problem

The arrangement of edges influences the total weight of spin configurations, even when the vertex degrees are fixed. Independent sets and proper colorings suggest a common extremal picture: after accounting for the degrees, complete bipartite graphs can maximize the count. The antiferromagnetic biclique-maximization conjecture proposes a spectral explanation for this phenomenon. It asks whether allowing at most one positive eigenvalue in the interaction matrix suffices for a universal upper bound. The counterexample below shows that unequal degrees can obstruct this principle, even on a graph with only six vertices.

A finite spin model is a symmetric matrix $H = (H(i, j))_{i, j \in I}$ with nonnegative entries. For a finite simple graph F , its partition function is

$$\text{hom}(F, H) = \sum_{\phi: V(F) \rightarrow I} \prod_{uv \in E(F)} H(\phi(u), \phi(v)). \quad (1)$$

The counting measure on I is used throughout. Loops in the interaction matrix are permitted; the test graphs are loopless. For example, $H = K_q$ counts proper q -colorings, while the two-spin matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ counts independent sets. Both have at most one positive eigenvalue: the spectrum of K_q is $q - 1, -1, \dots, -1$, while the two-spin matrix has negative determinant. The following definition is due to Sah, Sawhney, Stoner and Zhao.

Definition 1.1 ([1, Section 1.4]). A model is *antiferromagnetic* if all eigenvalues other than the largest, counted with multiplicity, are nonpositive. It is *biclique-maximizing* if, for every graph G without isolated vertices,

$$\text{hom}(G, H) \leq \mathcal{B}(G, H) := \prod_{uv \in E(G)} \text{hom}(K_{d_u, d_v}, H)^{1/(d_u d_v)}, \quad d_v = \deg_G(v). \quad (2)$$

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Here $K_{r,s}$ is the complete bipartite graph with class sizes r, s . The factor indexed by uv compares that edge with a biclique whose two vertex degrees are d_u, d_v ; its exponent divides by the number of biclique edges. In a d -regular graph on n vertices, the right-hand side reduces to $\text{hom}(K_{d,d}, H)^{n/(2d)}$. For irregular graphs, the product retains the two endpoint degrees of each edge.

Conjecture 1.2 ([1, Conjecture 1.16]). Every antiferromagnetic model is biclique-maximizing.

The cited numbering is that of the journal version and arXiv v3; the older arXiv v2 labels the same assertion Conjecture 1.15. Lee and Seo [6, Section 1.2, (1.3)] explicitly restate its arbitrary-graph formulation. Neither regularity of the test graph nor equality of the row sums of H is assumed.

1.2 Main results

The construction separates spectral antiferromagnetism from biclique maximization on irregular graphs.

1. Theorem 2.1 gives a fixed six-vertex graph and a six-spin model violating (2) for every $0 < t \leq 1/2$. At $t = 1/2$ all matrix entries are rational. Corollary 4.3 gives an open set of counterexamples with one positive and five strictly negative eigenvalues.
2. Lemma 3.2 and Proposition 4.2 identify the degree-dependent mixed covariance trace responsible for the violation and control all remaining terms by explicit estimates.

The test graph is a triangle with zero, one and two pendant edges attached at its three vertices, respectively. Its core degrees are 2, 3, 4, and the model has unequal row sums. The argument leaves the regular-graph and equal-row-sum restrictions unresolved.

1.3 Related work

Homomorphism inequalities. Partition functions of spin models are homomorphism counts, and inequalities between them form a large area; see Lovász [2] for the general theory. The best-known lower bound is Sidorenko’s conjecture [3], which asserts that for bipartite F the homomorphism density $t(F, J)$ is at least $t(K_2, J)^{|E(F)|}$. The inequality (2) is of the opposite, upper-bound type, and in [1] it is accordingly called a reverse Sidorenko inequality.

Extremal regular graphs for spin models. For independent sets, Kahn [7] proved the upper bound on regular bipartite graphs by an entropy argument. Zhao [8, Theorem 1.2] then reduced the general regular case to the bipartite case. Galvin and Tetali [9, Propositions 1.2 and 1.5] extended the bipartite comparison to homomorphism counts and vertex-weighted models. Sah, Sawhney, Stoner and Zhao [10, Theorems 1.2 and 1.5] then proved the degree-product inequality for independent sets and the hard-core partition function on irregular graphs. Davies, Jenssen, Perkins and Roberts [4] gave a different proof of the regular hard-core bound via the occupancy fraction, which also yields the corresponding result for matchings. The biclique is not a universal maximizer: Cohen, Csikvári, Perkins and Tetali [5] showed that for the Widom–Rowlinson model the maximizer among d -regular graphs is instead the clique K_{d+1} . This model has two positive eigenvalues, which motivates the spectral hypothesis in Conjecture 1.2.

Their reverse Sidorenko theorem [1, Theorems 1.9 and 1.12] establishes (2) for every triangle-free test graph and arbitrary nonnegative models, with an even more general formulation allowing different interactions on different edges. For semiproper colorings, the triangle-free restriction can be removed [1, Theorem 1.7]. Two-spin antiferromagnetic models also satisfy the unrestricted bound [1, Corollary 1.15]. Thus a counterexample must exploit more than the presence of a triangle alone: triangles already cause no difficulty for these particular models.

One route to upper bounds compares a graph with its bipartite double cover $G \times K_2$: each vertex has two copies, and each edge uv gives edges $(u, 0)(v, 1)$ and $(v, 0)(u, 1)$ between the two layers.

Zhao’s swapping method [11, Proposition 4.4] proves $\text{hom}(G, H)^2 \leq \text{hom}(G \times K_2, H)$ for suitable target models. Lee, Oh and Seo [12, Theorem 1.6] prove this comparison for every antiferromagnetic target when the test graph is obtained by replacing vertices of a bipartite graph by cliques, with all edges between two cliques precisely when their base vertices are adjacent. In particular, this settles the comparison for complete test graphs. Their method uses the Lorentzian polynomial theory of Brändén and Huh [13], and their Conjecture 1.5 states the regular-graph restriction of the antiferromagnetic extremal problem. The irregular counterexample here does not decide that restriction.

The lower-bound side of this theory has a different answer. Lee and Seo [6, Theorems 1.1–1.2] prove that antiferromagnetic models are clique-minimizing, including a strengthening with a separate fugacity vector at every vertex. Their homogeneous inequality is

$$\text{hom}(G, H) \geq \prod_{v \in V(G)} \text{hom}(K_{d_v+1}, H)^{1/(d_v+1)}.$$

Thus the spectral condition gives the clique lower bound, while the biclique upper bound fails in the generality of Conjecture 1.2.

There is also a distinct recent development concerning the original Sidorenko lower bound. The September 2026 preprint of OpenAI [14, Theorem 1.1] reports a bipartite graph F on 35 vertices and 66 edges and a simple graph J with $t(F, J) < t(K_2, J)^{66}$, where $t(F, J) = \text{hom}(F, J)/|V(J)|^{|V(F)|}$. That comparison is with the edge density of the target. The present comparison is the degree-dependent biclique upper bound (2) under a spectral assumption on the target model. The two counterexamples address different inequalities.

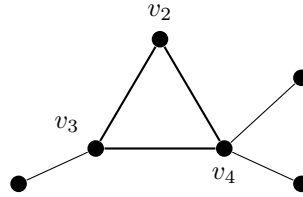


Figure 1: Pendant edges give the triangle vertices degrees 2, 3, 4. They change the spin laws even though their own selected-edge terms vanish in the centered expansion.

1.4 Why unequal degrees create the obstruction

The proof of [1, Proposition 1.10] already compares a cubic triangle correction with fourth-order biclique errors. It uses the two-spin interaction $J_2 + 2\varepsilon I_2$, where J_2 is the all-ones matrix. For $\varepsilon > 0$ its eigenvalues are $2 + 2\varepsilon$ and 2ε , so that example has two positive eigenvalues. The issue here is to obtain the favorable cubic sign while retaining the antiferromagnetic spectral condition.

Begin with a positive rank-one interaction $H_0(i, j) = a_i a_j$. Its partition function factors into one sum at each vertex, and (2) holds with equality. Subtracting tC , where C is a positive semidefinite Gram matrix and $Ca = 0$, preserves exactly one positive eigenvalue for $t > 0$. The perturbed interaction has the form

$$H_t(i, j) = a_i a_j (1 - t \langle x_i, x_j \rangle).$$

After the rank-one factors are removed, the spins at a vertex of degree d have probabilities proportional to a_i^d . Pairing each vector with its negative and giving both signs the same a_i centers every one of these degree-dependent laws.

Expand the normalized partition function by selecting some interaction edges. Any selection with a leaf has zero expectation, since the spin vector at that leaf occurs linearly and is centered. In the chosen graph, the only surviving nonempty selection is the triangle. If Σ_d is the covariance matrix under the degree- d law, its contribution is

$$-t^3 \text{tr}(\Sigma_2 \Sigma_3 \Sigma_4).$$

Each covariance matrix is positive semidefinite, but the trace of their ordered triple product can be negative. The explicit parameters in (4) arrange exactly this sign. In contrast, every nonempty surviving selection in a biclique contains at least four edges, so its correction begins at order t^4 . The positive cubic term therefore wins for small t ; the estimates below prove that it wins on the entire stated interval.

This explains the role of the pendant edges. Their selected-edge contributions vanish, but their effect on the original vertex degrees remains in the three covariance laws. If all three degrees were the same, the cubic trace would instead be $\text{tr}(\Sigma_d^3) \geq 0$. Irregularity changes the sign available to the first nonzero term. The strict gap is quantitatively small; its proof uses exact rational arithmetic and an all-orders bound, rather than a floating-point comparison.

2 The construction

Let G have vertices $v_2, v_3, v_4, \ell_1, \ell_2, \ell_3$ and edges

$$v_2v_3, \quad v_3v_4, \quad v_4v_2, \quad v_3\ell_1, \quad v_4\ell_2, \quad v_4\ell_3. \quad (3)$$

Thus the subscript of each v_d is its degree, and the three ℓ_j have degree one. Set

$$(a_1, a_2, a_3) = (1, 256, 65536), \quad b_1 = (1, 0), \quad b_2 = \frac{1}{4}(1, 1), \quad b_3 = \frac{1}{256}(-1, 2). \quad (4)$$

The spin set is $I = \{1, 2, 3\} \times \{-1, 1\}$. For $t > 0$ define

$$H_t((r, \epsilon), (s, \eta)) = a_r a_s - t\epsilon\eta \langle b_r, b_s \rangle. \quad (5)$$

Theorem 2.1. *For every $0 < t \leq 1/2$, the matrix H_t is strictly positive entrywise and has exactly one positive eigenvalue. Nevertheless,*

$$\text{hom}(G, H_t) > \mathcal{B}(G, H_t).$$

In particular, $H_{1/2}$ is a rational counterexample to Conjecture 1.2.

Lemma 2.2. *The models in Theorem 2.1 have the asserted positivity and inertia. Their row sums are unequal.*

Proof. Let $\mathbf{a} \in \mathbb{R}^I$ have coordinates $a_{(r, \epsilon)} = a_r$, and let C be the Gram matrix of the six vectors ϵb_r . Then

$$H_t = \mathbf{a}\mathbf{a}^\top - tC, \quad C \succeq 0.$$

For each r , the two signs cancel in $\sum_{r, \epsilon} a_r \epsilon b_r$, so $C\mathbf{a} = 0$. Consequently $\mathbf{a}$ is an eigenvector of H_t with positive eigenvalue $\|\mathbf{a}\|^2$. The orthogonal complement of $\mathbf{a}$ is preserved by C ; on that complement $H_t = -tC$ is negative semidefinite. There can therefore be no second positive eigenvalue.

For $i = (r, \epsilon)$ put $x_i = \epsilon b_r / a_r$. The three possible norms are

$$1, \quad \frac{\sqrt{2}}{2^{10}}, \quad \frac{\sqrt{5}}{2^{24}}.$$

Each is at most one. Equation (5) becomes

$$H_t(i, j) = \mathbf{a}_i \mathbf{a}_j (1 - t \langle x_i, x_j \rangle) \geq (1 - t) \mathbf{a}_i \mathbf{a}_j > 0. \quad (6)$$

Sign cancellation also gives $C\mathbf{1} = 0$. Thus $H_t\mathbf{1} = (\sum_i \mathbf{a}_i)\mathbf{a}$, whose coordinates are unequal. $\square$

Remark 2.3. The sign mechanism can already be seen with rank-one matrices. For $u = (1, 0)$, $v = (1, 1)$, and $w = (-1, 2)$, let $U = uu^\top$, $V = vv^\top$, and $W = ww^\top$. Each is positive semidefinite, whereas

$$\text{tr}(UVW) = \langle u, v \rangle \langle v, w \rangle \langle w, u \rangle = 1 \cdot 1 \cdot (-1) = -1.$$

The vectors in (4) use these same three directions. The unequal values of a_r make their relative contributions depend on the degree. The next section makes this dependence explicit and checks the resulting mixed trace exactly.

3 A centered expansion with degree-dependent laws

For integers $d \geq 1$ define

$$s_d = \sum_{i \in I} \mathbf{a}_i^d = 2(1 + 256^d + 65536^d), \quad \mu_d(i) = \frac{\mathbf{a}_i^d}{s_d}. \quad (7)$$

The two signs of each spin have the same probability and opposite x_i , hence $\mathbb{E}_{\mu_d} x = 0$. If F has no isolated vertices, write

$$Z_F(t) = \frac{\text{hom}(F, H_t)}{\prod_{v \in V(F)} s_{\deg_F(v)}}. \quad (8)$$

Lemma 3.1. *Take independent spins X_v with laws $\mu_{\deg_F(v)}$. Then*

$$\begin{aligned} Z_F(t) &= \mathbb{E} \prod_{uv \in E(F)} (1 - t \langle x_{X_u}, x_{X_v} \rangle) = \sum_{A \subseteq E(F)} (-t)^{|A|} \tau_F(A), \\ \tau_F(A) &= \mathbb{E} \prod_{uv \in A} \langle x_{X_u}, x_{X_v} \rangle. \end{aligned} \quad (9)$$

If A has a vertex of selected degree one, then $\tau_F(A) = 0$.

Proof. Factor (6) in every edge of (1). A spin at a vertex of degree d acquires a factor $\mathbf{a}_i^d$, whose sum over spins is s_d . Division by those sums yields the expectation in (9); expansion of its finite product gives the second expression. If a vertex belongs to exactly one edge of A , condition on every other spin. The product is linear in the vector at that vertex and has conditional expectation zero. Its law remains $\mu_{\deg_F(v)}$, determined by the original graph, not by A . $\square$

Lemma 3.2. *Let $\Sigma_d = \mathbb{E}_{\mu_d}(xx^\top)$. For the graph in (3),*

$$Z_G(t) = 1 - t^3 \text{tr}(\Sigma_2 \Sigma_3 \Sigma_4) = 1 + \kappa t^3, \quad 2^{-129} < \kappa \leq 1. \quad (10)$$

Proof. An edge set containing a pendant edge has a selected leaf. A nonempty proper subset of the triangle also has a selected leaf. By Lemma 3.1, the only nonconstant contribution is therefore the whole triangle. For deterministic vectors x, y, z ,

$$\langle x, y \rangle \langle y, z \rangle \langle z, x \rangle = \text{tr}(xx^\top yy^\top zz^\top).$$

Independence gives the first equality in (10).

To evaluate its sign, let

$$T_d = \sum_{r=1}^3 a_r^{d-2} b_r b_r^\top.$$

The two equally weighted signs in (7) give $\Sigma_d = 2T_d/s_d$. With $U = (1, 0)(1, 0)^\top$, $V = (1, 1)(1, 1)^\top$, and $W = (-1, 2)(-1, 2)^\top$, the parameters imply

$$T_2 = U + \frac{V}{16} + \frac{W}{65536}, \quad T_3 = U + 16V + W, \quad T_4 = U + 4096V + 65536W.$$

Equivalently,

$$T_2 = \frac{1}{65536} \begin{pmatrix} 69633 & 4094 \\ 4094 & 4100 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 18 & 14 \\ 14 & 20 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 69633 & -126976 \\ -126976 & 266240 \end{pmatrix}. \quad (11)$$

The product of the numerator matrix of T_2 with T_3 is $\begin{pmatrix} 1310710 & 1056742 \\ 131092 & 139316 \end{pmatrix}$. Consequently

$$\begin{aligned} \text{tr}(T_2 T_3 T_4) &= \frac{1310710 \cdot 69633 - (1056742 + 131092) \cdot 126976 + 139316 \cdot 266240}{65536} \\ &= -\frac{22466248714}{65536} < 0. \end{aligned}$$

Thus $\kappa = -8 \operatorname{tr}(T_2 T_3 T_4) / (s_2 s_3 s_4)$ is positive.

For $d \geq 1$, the inequality $1 + 2^{8d} < 2^{16d}$ yields

$$2^{16d+1} < s_d < 2^{16d+2}. \quad (12)$$

Since $22466248714 > 2^{34}$, the absolute trace exceeds 2^{18} . Hence

$$\kappa > \frac{8 \cdot 2^{18}}{2^{34} 2^{50} 2^{66}} = 2^{-129}.$$

The triangle contraction is an expectation of a product of three inner products of vectors of norm at most one. Its absolute value is at most one, proving $\kappa \leq 1$. $\square$

4 Comparison with the bicliques

The normalization in (8) respects the degree-weighted comparison exactly. Indeed,

$$\operatorname{hom}(K_{r,s}, H_t) = s_s^r s_r^s Z_{K_{r,s}}(t).$$

For each edge uv in the comparison product, its rank-one factor after taking the $1/(d_u d_v)$ power is $s_{d_u}^{1/d_u} s_{d_v}^{1/d_v}$. Each vertex occurs in exactly its degree many such factors. Therefore

$$\prod_{uv \in E(G)} (s_{d_v}^{d_u} s_{d_u}^{d_v})^{1/(d_u d_v)} = \prod_{v \in V(G)} s_{d_v}. \quad (13)$$

Moreover $Z_{K_{1,d}}(t) = 1$: every nonempty edge set of a star has a selected leaf. Thus (2), for the present graph, is equivalent to $Z_G(t) \leq Q(t)$, where

$$Q(t) = Z_{K_{2,3}}(t)^{1/6} Z_{K_{2,4}}(t)^{1/8} Z_{K_{3,4}}(t)^{1/12}. \quad (14)$$

Lemma 4.1. *Put $\rho_d = \mathbb{E}_{\mu_d} \|x\|^2 = \operatorname{tr}(\Sigma_d)$. For $r, s \geq 1$ and $0 \leq t \leq 1$,*

$$|Z_{K_{r,s}}(t) - 1| \leq 2^{rs} t^4 \rho_r^2 \rho_s^2. \quad (15)$$

For $0 \leq t \leq 1/2$, this also bounds $\log Z_{K_{r,s}}(t)$ from above.

Proof. Here $0 \leq \rho_d \leq 1$. By Lemma 3.1, a nonempty contributing edge set A has all incident vertices of selected degree at least two. Simplicity and bipartiteness force at least two incident vertices in each class and at least four selected edges. Independence and Cauchy–Schwarz give

$$|\tau_{K_{r,s}}(A)| \leq \prod_{v \in V(A)} \mathbb{E} \|x_{X_v}\|^{\deg_A(v)}.$$

For each integer $j \geq 2$, the bound $\|x\| \leq 1$ implies $\mathbb{E}_{\mu_d} \|x\|^j \leq \rho_d$. The original vertex laws on the two classes are μ_s and μ_r . There are at least two factors of each kind, so the last display is at most $\rho_r^2 \rho_s^2$. There are at most 2^{rs} edge subsets, and $t^{|A|} \leq t^4$. Summing absolute values proves (15).

When $t \leq 1/2$, every edge weight is positive by Lemma 2.2, so $Z_{K_{r,s}}(t) > 0$. The inequality $\log z \leq z - 1$ for $z > 0$ converts (15) into the asserted logarithmic upper bound. $\square$

Proposition 4.2. *For $0 < t \leq 1/2$,*

$$\log Z_G(t) - \log Q(t) > 255 \cdot 2^{-138} t^3 > 0. \quad (16)$$

Proof. From (11),

$$\mathrm{tr} T_2 = 73733/65536 < 2, \quad \mathrm{tr} T_3 = 38 < 2^6, \quad \mathrm{tr} T_4 = 335873 < 2^{19}.$$

Together with (12) and $\rho_d = 2 \mathrm{tr} T_d / s_d$, these yield

$$\rho_2 < 2^{-31}, \quad \rho_3 < 2^{-42}, \quad \rho_4 < 2^{-45}.$$

For each of the three pairs in (14), it follows that $\rho_r^2 \rho_s^2 < 2^{-146}$. Applying Lemma 4.1 to the three factors gives

$$\begin{aligned} \log Q(t) &\leq \left(\frac{2^6}{6} + \frac{2^8}{8} + \frac{2^{12}}{12} \right) 2^{-146} t^4 \\ &= 384 \cdot 2^{-146} t^4 < 2^{-137} t^4 \leq 2^{-138} t^3. \end{aligned}$$

For $0 \leq y \leq 1$, integration of $1/(1+s) \geq 1/2$ on $[0, y]$ proves $\log(1+y) \geq y/2$. Lemma 3.2 gives $0 < \kappa t^3 \leq 1$, hence

$$\log Z_G(t) = \log(1 + \kappa t^3) \geq \kappa t^3 / 2 > 2^{-130} t^3.$$

Subtracting proves (16). $\square$

Proof of Theorem 2.1. Lemma 2.2 gives the matrix properties. The strict inequality $Z_G(t) > Q(t)$ follows from Proposition 4.2. Restoring the positive factors in (13) proves the strict reversal of the conjectured inequality. All parameters in (4) are rational. Hence the matrix at $t = 1/2$ is rational as well. $\square$

Corollary 4.3. *There is a nonempty open set of strictly positive symmetric six-spin models, each with one positive and five strictly negative eigenvalues, that violate (2) for G .*

Proof. Fix $t = 1/2$ and consider $H_t - \delta I$. Its eigenvalues are those of H_t minus δ . For positive δ smaller than the positive eigenvalue of H_t , it has one positive and five strictly negative eigenvalues. For $\delta < \min_i H_t(i, i)$, all entries remain positive. Every partition function in (2) is a polynomial in the entries of the model. On the strictly positive matrices, its positive fractional powers are continuous. Since Theorem 2.1 gives a strict inequality at $\delta = 0$, that inequality persists for all sufficiently small positive δ . Choose one such δ . Entrywise positivity, the strict eigenvalue signs, and the strict partition-function inequality all persist in a sufficiently small neighborhood in the space of symmetric matrices. This neighborhood proves the assertion. $\square$

Remark 4.4 (Normalization). The sizes of the entries cause no mismatch with bounded-kernel conventions. Multiplication of a model by $c > 0$ multiplies $\mathrm{hom}(G, H)$ by $c^{|E(G)|}$. Each factor indexed by an edge on the right of (2) acquires one factor c , so the same power occurs there. Choosing c small enough gives entries strictly between zero and one and preserves both the violation and the eigenvalue signs. Replacing counting measure on the six spins by uniform probability measure also preserves the comparison: its factors are $6^{-|V(G)|}$ on the left and

$$6^{-\sum_{uv \in E(G)} (1/d_u + 1/d_v)} = 6^{-|V(G)|}$$

on the right. Thus the example applies equally in the graphon normalization of [1].

5 Concluding remarks

Corollary 4.3 shows that the violation is not confined to singular interaction matrices. The normalized gap is small. It persists throughout the stated interval of positive perturbation parameters because the spectral condition does not control mixed-degree triangle contractions. Positivity of the individual covariance matrices does not force positivity of the trace of their

ordered triple product. If two of the matrices coincide, cyclicity of trace reduces the expression to $\text{tr}(AB^2) = \text{tr}(BAB) \geq 0$ for positive-semidefinite A, B . Thus three distinct degree laws are needed for this cubic mechanism. The degrees 2, 3, 4 are the smallest pairwise distinct degrees available on a triangle. This observation explains the choice of graph; it does not establish the conjecture for regular graphs or graphs of maximum degree three.

The test graph has a triangle. It therefore lies outside the triangle-free setting in which [1] proves the biclique inequality for arbitrary models. The example leaves both that theorem and the clique-minimization theorem of [6] intact. It answers the unrestricted antiferromagnetic conjecture by showing that an additional hypothesis, beyond the number of positive eigenvalues, would be necessary.

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