

Small witnesses for eventual Erdős–Straus binomial divisibility*

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Abstract

Erdős and Straus asked whether, for each fixed $n \geq 2$, every sufficiently large integer m admits $1 \leq k \leq m - n$ with $\binom{n+k}{n} \mid \binom{m+k}{k}$. We prove that it does; previously this was known only for a set of m of density one. The witness can be taken with $k = O_{n,\varepsilon}(m^\varepsilon)$ for every $\varepsilon > 0$: a factorial identity reduces the problem to finding a small value of a fixed polynomial coprime to $\binom{m}{n}$, which an elementary sieve provides. Conversely, the least witness is at least logarithmic in m along an infinite sequence.

1 Introduction

1.1 Background and the source problem

For positive integers n, k , write $A(n, k) = (n+1) \cdots (n+k)$. Comparing two such products is easy in size but difficult in divisibility: increasing every factor does not ensure that the larger product contains the prime powers of the smaller one. Erdős and Straus asked how much freedom in the common length k suffices to obtain $A(n, k) \mid A(m, k)$ [1]. Since

$$\frac{A(m, k)}{A(n, k)} = \frac{\binom{m+k}{k}}{\binom{n+k}{k}},$$

their question has a binomial formulation. The restriction $k \leq m - n$ keeps the two intervals disjoint. Without it, the factorial identity $A(m, k)/A(n, k) = A(n+k, m-n)/A(n, m-n)$ provides easy, potentially very large witnesses. The following is the fixed- n question in [1, §1, p. 66, (1.4)]; it is also restated in [2, Appendix C.5.1].

Question 1.1 (Erdős–Straus, [1, §1, (1.4)]). For a fixed integer $n \geq 2$, does every sufficiently large integer m admit an integer k satisfying

$$1 \leq k \leq m - n, \quad \binom{n+k}{n} \mid \binom{m+k}{k} ? \tag{1}$$

If not, what is the density of the integers admitting such a witness?

Definition 1.2. For $n \geq 2$, let G_n be the set of integers $m > n$ for which (1) has a solution. The natural density of a set $S \subseteq \mathbb{N}$, when it exists, is $\lim_{X \rightarrow \infty} |S \cap [1, X]|/X$. Its natural lower density is $\liminf_{X \rightarrow \infty} |S \cap [1, X]|/X$.

There are two substantially different parameter regimes. When m/n is bounded and k is at least a fixed positive multiple of n , Erdős and Straus prove that no solutions exist once n is large enough [1, Theorem 2.1]. The question above instead keeps n fixed and lets m tend to infinity. The best previous result in this regime is the density-one theorem of Zheng, Zhang, Avigad, Tetali and Welleck [2, Theorem C.5.1], whose authors state that “whether all large m lie in G_n remains open” [2, Remark C.5.2] and note that their proof gives no bound for the least admissible k .

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1.2 Main results

We answer the first part of the question affirmatively: for every fixed $n \geq 2$, the set G_n contains every sufficiently large integer, so the density question becomes moot. More precisely:

1. Proposition 3.1 gives, for every $m > n$, a positive k with $\binom{n+k}{n} \mid \binom{m+k}{k}$ and $k \leq n!((n+1)^{2\omega(\binom{m}{n})} + 1)$, where ω counts distinct prime factors.
2. Theorem 3.2 gives an explicit threshold above which this witness lies in $[1, m - n]$, and Corollary 3.4 shows that it is $O_{n,\varepsilon}(m^\varepsilon)$ for every $\varepsilon > 0$.
3. Theorem 4.6 shows that the least witness is at least $(\log m)/\log 16 - n + 2$ along an infinite sequence of m , while it equals one on another infinite sequence. Thus no fixed finite list of lengths serves all large m .

The asymptotic assertions keep n fixed; their constants need not be uniform as n grows. Since the constructed k is a positive multiple of $n!$ and is $o(m)$, it eventually satisfies even $1 < k < m - n$, so endpoint conventions do not matter. We make no assertion about the boundary case $k = m - n$ discussed below.

1.3 Related work

The Erdős–Straus problem and its boundary case. The immediate predecessor is [2, Appendix C.5]. There one chooses a length k for which $\binom{n+k}{n}$ has no small prime factor and then identifies a periodic set of m for which that length works; varying this construction makes the density of the covered set tend to one. An infinite exceptional set can have density zero, so this cannot by itself establish eventual existence. The argument below instead starts with an arbitrary m and chooses k to avoid the prime factors of an integer determined by that m . The boundary specialization $m = n + k$ asks, for each n , for a length k with $\binom{n+k}{n} \mid \binom{n+2k}{k}$. Ulas, with an appendix by Schinzel, studies this question and gives computational witnesses through $n = 20$ [3, Question 1.1 and §2]. It is Erdős Problem 389 [4]: in its notation, a block $a(a+1)\cdots(a+k-1)$ must divide the immediately following block $(a+k)\cdots(a+2k-1)$, and $a = n+1$ gives precisely $m = n+k$. The freedom to choose a short k for a prescribed large m is essential to our argument, which therefore does not resolve Problem 389. Further divisibility questions of this kind are collected by Erdős and Graham [7].

Arithmetic of products of consecutive integers. The study of prime factors of blocks of consecutive integers goes back to the Sylvester–Schur theorem that a product of k consecutive integers exceeding k has a prime factor larger than k ; Erdős gave an elementary proof [5]. Erdős and Selfridge proved that such a product is never a perfect power [6]. These results concern the prime factorization of a single block, while (1) compares two blocks of the same length.

Divisibility of binomial coefficients. Prime-power divisibility of binomial coefficients is classically expressed through base- p digits: Kummer’s theorem counts carries [8], while Lucas’s theorem gives binomial residues digit by digit; see Granville [14, §1] for a survey. Erdős, Graham, Ruzsa and Straus used such digit arguments to study which small primes divide $\binom{2M}{M}$ [9]. Harborth proved that $\binom{M}{j}$ is divisible by $M(M-1)\cdots(M-h+1)$, for fixed h , for almost all entries of Pascal’s triangle [10]; see also the discussion following (5) in [11]. Pomerance proves that, for each fixed positive j , $M+j$ divides $\binom{2M}{M}$ for a set of M of density one [11, Theorem 2], and later obtains density-one divisibility $\binom{M+j}{j} \mid \binom{2M}{M}$ simultaneously for $1 \leq j \leq \exp(0.8\sqrt{\log M})$ [12, Theorems 1 and 2]. Sothanaphan’s account of the proof of Erdős Problem 728 gives another recent use of central-binomial divisibility, producing factorial divisibilities with logarithmic gaps [13, Theorem 1]. These results select suitable values of the row parameter and have a different dividend from (1); they do not supply a witness for each prescribed sufficiently large m . The existence proof below avoids a simultaneous comparison of all prime-power valuations altogether: coprimality lets a factorial identity do that work.

Sieving a polynomial. The remaining sieve problem resembles Jacobsthal's problem of finding a short interval containing an integer coprime to a prescribed integer, studied by Erdős and Iwaniec [15, 16]. For integers with at most r distinct prime factors, a recent OpenAI preprint reports that every interval of length $O(r^2/(\log \log(3r))^2)$ contains a coprime integer [17, Theorem 1.1]. That problem excludes one residue class per prime; the polynomial below may exclude several. Kalmynin and Konyagin study a polynomial analogue, giving lower bounds for the length of a shifted polynomial sequence containing no value coprime to a primorial [18, Theorem 1]; this is complementary to finding a small permitted argument of a fixed polynomial. No Jacobsthal estimate is used as an input: a direct inclusion–exclusion bound suffices for the fixed- n conclusion, and no novelty is claimed for the sieve itself.

1.4 Proof ideas

The useful cancellation identity is

$$\binom{m+k}{k} \binom{m}{n} = \binom{n+k}{n} \binom{m+k}{n+k}.$$

Thus it is enough to make the prospective divisor coprime to $N = \binom{m}{n}$. This is only a sufficient condition, but it replaces all valuation inequalities by avoidance of finitely many primes. Taking $k = n!t$ clears the denominators in

$$\binom{n+k}{n} = F_n(t), \quad F_n(t) = \prod_{j=1}^n \left(1 + \frac{n!}{j} t\right) \in \mathbb{Z}[t].$$

The constant term $F_n(0) = 1$ guarantees that no prime forbids every residue class. The degree bounds the number of forbidden classes by n . If $r = \omega(N)$, inclusion–exclusion counts the permitted integers up to T with main term at least $T/(n+1)^r$ and error at most $(n+1)^r$. Taking $T > (n+1)^{2r}$ therefore works regardless of the sizes of the primes. For example, at $n = 2$ the progression is $k = 2t$ and the polynomial is $(1+2t)(1+t)$.

An exponential bound in r suffices because $C^{\omega(N)} = O_{C,\delta}(N^\delta)$ for every fixed $C > 1$ and $\delta > 0$, whereas $N \leq m^n$. The resulting witness is smaller than every fixed positive power of m , and consequently fits in the permitted interval. The lower bound reverses the arithmetic strategy. When m is a multiple of a sufficiently large prime power p^a and $k < p^a$, $\binom{m+k}{k} \equiv 1 \pmod{p}$. Choosing m as the square of $\text{lcm}(1, \dots, x)$ enforces this simultaneously for all primes up to x , excluding every $k \leq x - n$. Since this m grows at most exponentially in x , the excluded lengths have logarithmic size in m .

Organization. Section 2 proves the coprimality criterion and the polynomial sieve, Section 3 derives eventual existence and the subpolynomial bound, and Section 4 proves the lower bound for the least witness.

2 A coprimality criterion and a polynomial sieve

For a positive integer N , let $\omega(N)$ denote the number of distinct prime divisors, with $\omega(1) = 0$. Let $\pi(x)$ count the primes at most x .

Lemma 2.1. *Let $m \geq n \geq 1$ and $k \geq 0$ be integers. If*

$$\gcd\left(\binom{n+k}{n}, \binom{m}{n}\right) = 1,$$

then $\binom{n+k}{n} \mid \binom{m+k}{k}$.

Proof. The factorial definitions give the identity

$$\binom{m+k}{k} \binom{m}{n} = \frac{(m+k)!}{k!n!(m-n)!} = \binom{n+k}{n} \binom{m+k}{n+k}. \quad (2)$$

The last factor is an integer. Thus $\binom{n+k}{n}$ divides the product on the left. Coprimality with its second factor permits cancellation by Euclid's lemma. $\square$

The next lemma avoids the prime factors of an arbitrary integer without requiring any estimate for the distribution of primes.

Lemma 2.2 (Polynomial avoidance). *Let $F \in \mathbb{Z}[t]$ have degree $d \geq 1$ and $F(0) = 1$. For each positive integer N , there is an integer t such that*

$$1 \leq t \leq (d+1)^{2\omega(N)} + 1, \quad \gcd(F(t), N) = 1. \quad (3)$$

Proof. Let $\mathcal{P}$ be the set of primes dividing N , and put $r = |\mathcal{P}|$. For $p \in \mathcal{P}$, let $\rho(p)$ be the number of roots of F in $\mathbb{Z}/p\mathbb{Z}$. The reduced polynomial is nonzero because its constant term is one. A polynomial of degree at most d over a field has at most d roots, by successive division by its distinct linear root factors. Hence $\rho(p) \leq d$, and the fact that zero is not a root also gives $\rho(p) \leq p-1$. Consequently

$$1 - \frac{\rho(p)}{p} \geq \frac{1}{d+1}. \quad (4)$$

Indeed, if $p \leq d+1$, the left side is at least $1/p$; if $p > d+1$, it is at least $1 - d/p > 1/(d+1)$.

For $A \subseteq \mathcal{P}$, put $q_A = \prod_{p \in A} p$ and $\rho_A = \prod_{p \in A} \rho(p)$, using empty products equal to one. The Chinese remainder theorem implies that $q_A \mid F(t)$ holds in exactly ρ_A residue classes modulo q_A . A residue class contains T/q_A integers in $[1, T]$ with error of absolute value at most one. Thus, for positive integer T ,

$$\#\{1 \leq t \leq T : q_A \mid F(t)\} = \frac{\rho_A T}{q_A} + E_A, \quad |E_A| \leq \rho_A.$$

For $A = \emptyset$ the count is exactly T , so $E_A = 0$ is allowed. Inclusion-exclusion now yields

$$\begin{aligned} S(T) &:= \#\{1 \leq t \leq T : \gcd(F(t), N) = 1\} \\ &= T \prod_{p \in \mathcal{P}} \left(1 - \frac{\rho(p)}{p}\right) + E, \quad |E| \leq \prod_{p \in \mathcal{P}} (1 + \rho(p)) \leq (d+1)^r. \end{aligned} \quad (5)$$

Together with (4), this gives $S(T) \geq T/(d+1)^r - (d+1)^r$. At $T = (d+1)^{2r} + 1$ the right side is $(d+1)^{-r} > 0$. The integer count $S(T)$ is therefore positive. Empty products give the same conclusion when $N = 1$. $\square$

Lemma 2.3. *For $C > 1$, $\delta > 0$ and positive integer N ,*

$$C^{\omega(N)} \leq C^{\pi(C^{1/\delta})} N^\delta. \quad (6)$$

Proof. Set $R = C^{1/\delta}$. In the product $C^{\omega(N)} = \prod_{p \mid N} C$, primes at most R contribute at most $C^{\pi(R)}$. For each remaining prime, $C < p^\delta$. The product of any subset of the distinct prime divisors of N is at most N , so their contribution is at most N^δ . $\square$

3 Eventual existence and small witnesses

The arithmetic progression $k = n!t$ makes the divisor in (1) an integral polynomial with constant term one. This permits Lemma 2.2 to be applied with $N = \binom{m}{n}$.

Proposition 3.1. *For all integers $m > n \geq 2$, there exists a positive multiple k of $n!$ such that*

$$\binom{n+k}{n} \mid \binom{m+k}{k}, \quad k \leq n! \left((n+1)^{2\omega(\binom{m}{n})} + 1 \right). \quad (7)$$

This assertion alone imposes no upper bound $m - n$ on k .

Proof. Define

$$F_n(t) = \prod_{j=1}^n \left(1 + \frac{n!}{j} t \right). \quad (8)$$

Each j divides $n!$, so $F_n \in \mathbb{Z}[t]$, its degree is n , and $F_n(0) = 1$. For nonnegative integer t , multiplication of the factors gives

$$F_n(t) = \frac{(n!t+1) \cdots (n!t+n)}{n!} = \binom{n+n!t}{n}.$$

Lemma 2.2 produces a positive t at most $(n+1)^{2\omega(\binom{m}{n})} + 1$ for which $F_n(t)$ is coprime to $\binom{m}{n}$. Put $k = n!t$ and apply Lemma 2.1. $\square$

Theorem 3.2. *For $n \geq 2$, define*

$$B_n = (n+1)^{4n}, \quad D_n = (n+1)^{2\pi(B_n)}, \quad M_n = \max\{4(n!D_n)^2, 2(n+n!)\}. \quad (9)$$

Every integer $m \geq M_n$ belongs to G_n .

Proof. The threshold exceeds n . Set $N = \binom{m}{n}$. Apply Lemma 2.3 with $C = (n+1)^2$ and $\delta = 1/(2n)$. Since $C^{1/\delta} = B_n$, it gives $(n+1)^{2\omega(N)} \leq D_n N^{1/(2n)}$. Also $N \leq m^n$. The witness furnished by Proposition 3.1 therefore satisfies

$$k \leq n!D_n\sqrt{m} + n!. \quad (10)$$

The first condition in (9) implies $n!D_n\sqrt{m} \leq m/2$, and the second gives $n! \leq m/2 - n$. Substitution in (10) proves $k \leq m - n$. Positivity and divisibility follow from Proposition 3.1. $\square$

Remark 3.3. The word “eventual” is necessary: $m = n+1$ never belongs to G_n . The only permitted value is $k = 1$, which would require $n+1$ to divide $n+2$. This fails for $n \geq 2$. The threshold in Theorem 3.2 gives a finite interval containing all exceptions; it does not identify those exceptions.

Corollary 3.4. *For fixed $n \geq 2$ and $\varepsilon > 0$, a constant $K_{n,\varepsilon}$ exists such that each $m > n$ has a positive divisibility witness $k \leq K_{n,\varepsilon}m^\varepsilon$. If $m \geq M_n$, the witness can simultaneously be chosen in $[1, m - n]$.*

Proof. In Lemma 2.3, take $C = (n+1)^2$ and $\delta = \varepsilon/n$. Writing $A_{n,\varepsilon} = (n+1)^{2\pi((n+1)^{2n/\varepsilon})}$ gives

$$(n+1)^{2\omega(\binom{m}{n})} \leq A_{n,\varepsilon} \binom{m}{n}^{\varepsilon/n} \leq A_{n,\varepsilon} m^\varepsilon.$$

Every witness selected by the construction in Proposition 3.1 obeys this bound and the bound used in Theorem 3.2. Thus one can take $K_{n,\varepsilon} = n!(A_{n,\varepsilon} + 1)$, with both conclusions holding for the same selected witness. $\square$

Example 3.5. For $n = 2$, the polynomial is $F_2(t) = (1+2t)(1+t)$. At $m = 10$, $\binom{10}{2} = 45$. The choices $t = 1, 2$ give values 6, 15, which are not coprime to 45; $t = 3$ gives 28, which is. Thus $k = 6 \leq 10 - 2$ is an admissible witness, and indeed $\binom{8}{2} = 28$ divides $\binom{16}{6} = 8008$. The sieve guarantees the existence of such a choice; its general upper bound is not intended to be close to the first successful t .

4 Lower bounds for the least witness

Definition 4.1. For integers $m > n \geq 2$, define

$$\kappa_n(m) = \min \left\{ k \in \mathbb{Z}_{\geq 1} : \binom{n+k}{n} \mid \binom{m+k}{k} \right\}.$$

Proposition 3.1 ensures that this minimum exists. For $m \geq M_n$, it is also the least witness in the original permitted interval. Write $L_s = \text{lcm}(1, \dots, s)$, and use natural logarithms. For a prime p , $v_p(r)$ denotes the exponent of p in a positive integer r .

Lemma 4.2. *If p is prime, $p^a \mid m$ with $a \geq 1$, and $0 \leq k < p^a$, then $\binom{m+k}{k} \equiv 1 \pmod{p}$.*

Proof. The case $k = 0$ is immediate. For $1 \leq j \leq k$, write $j = p^{b_j} u_j$ with $p \nmid u_j$. Since $j < p^a$, we have $b_j < a$, and m/p^{b_j} is divisible by p . The integer products

$$U = \prod_{j=1}^k u_j, \quad V = \prod_{j=1}^k \left(u_j + \frac{m}{p^{b_j}} \right)$$

satisfy $U \binom{m+k}{k} = V$ by the product formula for the binomial coefficient. Each factor of V equals the corresponding factor of U modulo p . Thus $V \equiv U \pmod{p}$, and cancellation of the nonzero residue U gives the conclusion. $\square$

Lemma 4.3. *For every integer $s \geq 1$, $L_s \leq 4^{s-1}$.*

Proof. For $s \geq 2$, put $u = \lceil s/2 \rceil$ and $v = \lfloor s/2 \rfloor$. First, L_s/L_u divides $\binom{s}{v}$. To prove this, let p divide that quotient and let a be maximal with $p^a \leq s$. Then $p^a > u$ and $p^{a-1} \leq s/p \leq s/2 \leq u$. Hence p occurs in L_s/L_u to exponent exactly one. Counting multiples of p^j in each factorial gives $v_p(r!) = \sum_{j \geq 1} \lfloor r/p^j \rfloor$, and hence

$$v_p \left(\binom{s}{v} \right) = \sum_{j \geq 1} \left(\left\lfloor \frac{s}{p^j} \right\rfloor - \left\lfloor \frac{u}{p^j} \right\rfloor - \left\lfloor \frac{v}{p^j} \right\rfloor \right).$$

Each term is nonnegative since $s = u + v$. The term at $j = a$ is one, because $u, v < p^a \leq s < 2p^a$. Thus every prime factor of the squarefree quotient L_s/L_u divides $\binom{s}{v}$, proving the assertion.

We now induct on s , starting with $L_1 = 1$. If $s = 2v$, then $u = v$, so the asserted divisibility and the binomial theorem imply $L_s \leq 4^{v-1} 2^{2v} = 4^{s-1}$. If $s = 2v + 1 \geq 3$, then $u = v + 1$. The two equal central coefficients $\binom{s}{v}$ and $\binom{s}{v+1}$ have sum at most 2^s , so $\binom{s}{v} \leq 2^{s-1}$. Consequently $L_s \leq 4^{u-1} 2^{s-1} = 4^{s-1}$, completing the induction. $\square$

Proposition 4.4. *Fix integers $n \geq 2$ and $K \geq 1$. Every positive multiple m of L_{K+n}^2 satisfies $m > n$ and $\kappa_n(m) > K$. In particular, the set of such $m > n$ with $\kappa_n(m) > K$ has natural lower density at least L_{K+n}^{-2} .*

Proof. Put $x = K + n$. Since $x \mid L_x$, we have $m \geq x^2 > n$. For a prime $p \leq x$, let $e \geq 1$ satisfy $p^e \leq x < p^{e+1}$. Then $p^{2e} \mid m$ and $p^{2e} \geq p^{e+1} > x$. Lemma 4.2 therefore gives $\binom{m+k}{k} \equiv 1 \pmod{p}$ for every $1 \leq k \leq K$. On the other hand, $\binom{n+k}{n} > 1$ and every one of its prime factors is at most $n + k \leq x$, as follows from $\binom{n+k}{n} n! k! = (n+k)!$. The two binomial coefficients are coprime; the first cannot divide the second. This excludes every $k \leq K$. Finally, the number of positive multiples of L_x^2 up to X is $\lfloor X/L_x^2 \rfloor$. Dividing by X and taking the lower limit proves the density claim. $\square$

Remark 4.5. For a fixed K , every sufficiently large member of the progression in Proposition 4.4 nevertheless belongs to G_n by Theorem 3.2. Its admissible witnesses then lie above K . Thus eventual existence requires the witness length to vary with m .

Theorem 4.6. For fixed $n \geq 2$ and integers $x \geq n + 1$, let $m_x = L_x^2$. Then $m_x \rightarrow \infty$ and

$$\kappa_n(m_x) \geq x - n + 1 \geq \frac{\log m_x}{\log 16} - n + 2. \quad (11)$$

Consequently

$$\limsup_{m \rightarrow \infty} \frac{\kappa_n(m)}{\log m} \geq \frac{1}{\log 16}, \quad \liminf_{m \rightarrow \infty} \frac{\kappa_n(m)}{\log m} = 0.$$

Both limits range over integers $m > n$. Thus a uniform $o(\log m)$ upper bound for positive witnesses is impossible.

Proof. Since $L_x \geq x$, the integers m_x tend to infinity. Proposition 4.4 with $K = x - n$ gives $\kappa_n(m_x) > x - n$. Integrality gives the first inequality in (11). Lemma 4.3 implies $m_x \leq 16^{x-1}$, and hence $x \geq 1 + \log m_x / \log 16$. This proves the second inequality. Division by $\log m_x$ gives the stated lower bound for the limit superior. The sequence m_x need not be strictly increasing; its divergence to infinity is all that this limit argument uses.

For the limit inferior, choose $m = (n + 1)q - 1$ with integer $q \geq 2$. Then $m > n$, $1 \leq m - n$, and

$$\binom{n+1}{n} = n + 1 \mid m + 1 = \binom{m+1}{1}.$$

Thus $\kappa_n(m) = 1$ on an infinite progression. The corresponding ratios tend to zero, and all ratios are nonnegative, proving the limit-inferior assertion. A uniform $o(\log m)$ bound would contradict the positive limit superior. Finally, $m_x \geq M_n$ for sufficiently large x , so these lower bounds then concern witnesses that also exist in $[1, m_x - n]$. $\square$

5 Concluding remarks

The bounds leave a considerable gap between logarithmic growth along a sequence and a sub-polynomial uniform upper bound. Determining the maximal order of the least witness remains a quantitative problem. The explicit threshold in (9) is a consequence of a coarse prime-factor estimate and is not asserted to be close to optimal. Since the constructed witness is a positive multiple of $n!$ and is $o(m)$, it also satisfies $1 < k < m - n$ for sufficiently large m . Thus endpoint conventions in the original interval do not affect the eventual conclusion.

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