

Unipotent tensor categories and pronilpotent Lie quasibialgebras

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Abstract

We construct a bijection between tensor equivalence classes of unipotent tensor categories over $\mathbb{C}$ and continuous Lie quasibialgebras with linearly compact pronilpotent underlying Lie algebra, modulo continuous isomorphisms and classical twists. The bijection is defined using a fixed universal quantization functor and solves the problem raised in Remark 4.6 of Etingof–Gelaki [1]. The proof combines reconstruction with filtered dequantization modulo twists and convergence of universal formulas. A category admits a tensor fiber functor precisely when a classical twist removes its curvature.

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1 Introduction

A tensor category is *unipotent* if its only simple object is the unit object $\mathbf{1}$. Throughout this paper, tensor categories are essentially small, rigid, abelian, and $\mathbb{C}$ -linear, with finite-dimensional morphism spaces, finite-length objects, and $\text{End}(\mathbf{1}) = \mathbb{C}$. Tensor products are exact in each variable. For a unipotent category, a composition-series argument gives

$$\text{length}(X \otimes Y) = \text{length}(X) \text{length}(Y).$$

Etingof-Gelaki classified coconnected Hopf algebras over $\mathbb{C}$ by quantizing Lie bialgebras on linearly compact pronilpotent Lie algebras [1, Theorem 4.2]. Here coconnected means that the coradical is spanned by the unit. In categorical terms, their result classifies unipotent tensor categories equipped with a tensor fiber functor to Vec_{fd} , the category of finite-dimensional complex vector spaces [1, Remark 4.4]. A chosen tensor fiber functor reconstructs the Hopf algebra. Without such a functor, the natural reconstruction has an associator, and changes of the comparison maps for tensor products act by twists. The corresponding classical data are therefore Lie quasibialgebras modulo isomorphisms and classical twists.

We prove this extension, anticipated in [1, Remark 4.6]. A linearly compact Lie algebra is *pronilpotent* if it is an inverse limit of finite-dimensional nilpotent Lie algebras, with the inverse-limit topology. The classical data are such a Lie algebra $\mathfrak{g}$, a continuous cobracket $\delta : \mathfrak{g} \rightarrow \widehat{\wedge}^2 \mathfrak{g}$, and a curvature tensor $\varphi \in \widehat{\wedge}^3 \mathfrak{g}$, satisfying the Lie quasibialgebra identities in Section 2. Tensor and exterior powers of linearly compact spaces are completed. The finite-dimensional Lie quotients used to define pronilpotence are not required to be preserved by δ .

Theorem 1.1 (Classification). *Fix a universal quantization functor Q for Lie quasibialgebras, compatible with classical twists, whose restriction to Lie bialgebras is a fixed Etingof–Kazhdan quantization functor. There is a bijection*

$$\begin{aligned} & \{\text{unipotent tensor categories over } \mathbb{C}\} / \text{tensor equivalence} \\ & \longleftrightarrow \\ & \left\{ \begin{array}{l} \text{continuous Lie quasibialgebras } (\mathfrak{g}, \delta, \varphi), \\ \mathfrak{g} \text{ pronilpotent and linearly compact} \end{array} \right\} / (\text{isomorphisms and twists}). \end{aligned} \tag{1}$$

The category associated to $(\mathfrak{g}, \delta, \varphi)$ is

$$\mathcal{C}_Q(\mathfrak{g}, \delta, \varphi) = \text{Rep}_{\text{fd}}^{\text{cont}} B_Q(\mathfrak{g}, \delta, \varphi),$$

where B_Q is the complete local quasi-Hopf algebra obtained by evaluating the universal quantization formulas in the unipotent topology, without retaining a formal parameter.

The parametrization depends on Q . Equivalent or universally twist-equivalent choices give the same tensor categories; no independence of arbitrary choices is asserted. Only the underlying Lie algebra is required to be pronilpotent. The cobracket need not be conilpotent, and the double need not be nilpotent.

The role of the curvature is expressed by the following consequence. We use the Schouten bracket on the exterior algebra and write d_δ for the degree-one derivation extending δ .

Corollary 1.2 (Fiber-functor criterion). *The category $\mathcal{C}_Q(\mathfrak{g}, \delta, \varphi)$ admits a tensor fiber functor to Vec_{fd} if and only if there is an $f \in \widehat{\wedge}^2 \mathfrak{g}$ such that*

$$\varphi + d_\delta f + \frac{1}{2}[f, f] = 0. \tag{2}$$

The proof of Corollary 1.2 is given in Section 6. In finite dimension, a Lie quasibialgebra modulo twists is equivalently a Manin pair $(\mathfrak{d}, \mathfrak{g})$: a quadratic Lie algebra with a Lagrangian Lie subalgebra. In this description $\mathfrak{g}$ is nilpotent, and the criterion says that it admits a Lagrangian Lie-subalgebra complement in $\mathfrak{d}$.

Nonzero curvature can either be removable or essential. Twisting the trivial structure on the Heisenberg Lie algebra produces nonzero curvature without changing the tensor category (Example 2.2). In contrast, for an abelian $\mathfrak{g}$ with $\delta = 0$, no twist changes φ ; any nonzero element of $\hat{\wedge}^3 \mathfrak{g}$ gives a category without a tensor fiber functor (Section 6.2). The pronilpotent generality is also necessary: Section 6.4 gives a finitely tensor-generated category whose classical Lie algebra is infinite-dimensional.

Strategy of proof. The proof proceeds through three reductions. Section 3 constructs a quasi-fiber functor and reconstructs a complete local quasi-Hopf algebra B , up to continuous isomorphism and normalized twist. The associated graded for its closed augmentation filtration is cocommutative. PBW then presents B as a positive perturbation of a symmetric bialgebra: noncommutative terms decrease the number of primitive factors, while other corrections raise the augmentation weight.

Theorem 4.1 is the central technical result. It classifies these perturbations by positive Lie quasibialgebra structures modulo twists. Universal quantization is supplied by Enriquez–Halbout [3]; Sakáloš–Ševera give an alternative construction [4]. To obtain a classification, we prove both that every perturbation is a quantization and that quantum equivalence implies classical equivalence. A filtered contraction constructs a slice, and a comparison of universal relation ideals identifies its full equations. Finally, Section 5 proves convergence of the universal formulas and identifies the auxiliary weight filtration with the closed augmentation filtration. These results give the existence and uniqueness assertions of Theorem 1.1. Section 6 proves the consequences and examples stated above.

2 Lie quasibialgebras and twists

2.1 Topological conventions

A linearly compact vector space is an inverse limit of finite-dimensional vector spaces over the discrete field $\mathbb{C}$. We use completed tensor products, completed quotient exterior powers, and continuous linear maps in this setting. Quotients by closed subspaces retain their quotient topologies; in particular, bounded-weight quotients and homogeneous pieces need not be discrete or finite-dimensional. The closed filtrations used below control convergence; they need not be neighborhood bases. Example 5.3 illustrates the distinction.

We reserve F for a quasi-fiber functor, $\Gamma^r \mathfrak{g}$ for the closed lower-central filtration, and W^r for weight filtrations. On a reconstructed algebra B , the weight filtration is $W^r B = \overline{I}^r$, where I is the augmentation ideal. A *complete local algebra* means a linearly compact inverse limit of finite-dimensional augmented local algebras with residue field $\mathbb{C}$.

2.2 Classical structures

Let $[\ , \]$ denote the Schouten bracket on $\hat{\wedge}^\bullet \mathfrak{g}$, with exterior degree minus one. Our sign convention is that for $a \in \mathfrak{g}$ the bracket $[a, -]$ is the adjoint action of a extended as a derivation of the exterior algebra, and that the graded antisymmetry $[u, v] = -(-1)^{(|u|-1)(|v|-1)}[v, u]$ holds; in particular

$[f, x] = -[x, f] = -x \cdot f$ for $f \in \widehat{\wedge}^2 \mathfrak{g}$ and $x \in \mathfrak{g}$. A continuous map

$$\delta : \mathfrak{g} \longrightarrow \widehat{\wedge}^2 \mathfrak{g}$$

extends uniquely to a continuous derivation d_δ of the exterior algebra, of degree one.

Definition 2.1. A Lie quasibialgebra is a Lie algebra $\mathfrak{g}$, a map δ as above, and an element $\varphi \in \widehat{\wedge}^3 \mathfrak{g}$, called its *curvature*, such that d_δ is a derivation of the Schouten bracket and

$$d_\delta^2 = [\varphi, -], \quad d_\delta \varphi = 0. \quad (3)$$

Thus δ is a Lie-algebra 1-cocycle for the adjoint action on $\widehat{\wedge}^2 \mathfrak{g}$. Equations (3) express the quasi-co-Jacobi identity and its compatibility condition. This convention fixes the signs in the twist formula below; replacing the twist parameter by its negative converts to another common convention.

For $f \in \widehat{\wedge}^2 \mathfrak{g}$, define

$$\begin{aligned} d_{\delta f} &= d_\delta + [f, -], \\ \varphi^f &= \varphi + d_\delta f + \frac{1}{2}[f, f]. \end{aligned} \quad (4)$$

The graded Jacobi identity gives

$$(d_\delta + [f, -])^2 = [\varphi + d_\delta f + \frac{1}{2}[f, f], -], \quad (d_\delta + [f, -])(\varphi + d_\delta f + \frac{1}{2}[f, f]) = 0.$$

So every f defines a classical twist. There is no Maurer–Cartan equation on f unless one requires the new φ to vanish. Successive twists add: twisting by f and then by f' is twisting by $f + f'$.

Consequently, the right-hand side of (1) can also be written

$$\coprod_{[\mathfrak{g}]} \frac{\{(\delta, \varphi) \text{ satisfying Definition 2.1}\}}{\text{Aut}_{\text{cont}}(\mathfrak{g}) \ltimes \widehat{\wedge}^2 \mathfrak{g}}. \quad (5)$$

2.3 Twists and isomorphisms

Example 2.2. Let $\mathfrak{g}$ be the Heisenberg Lie algebra with basis x, y, z and $[x, y] = z$. Start with $(\delta, \varphi) = (0, 0)$ and twist by $f = x \wedge y$. In the convention above, $\delta^f(x) = [f, x] = -x \cdot f$, so

$$\delta^f(x) = -x \wedge z \neq 0, \quad \varphi^f = \frac{1}{2}[f, f] = x \wedge y \wedge z.$$

The original and twisted quasibialgebras are not isomorphic, since one cobracket vanishes and the other does not. Their quantizations nevertheless have tensor-equivalent representation categories. The curvature here is removed by the inverse twist. This contrasts with the abelian example in Section 6.2.

3 Reconstruction and the PBW reduction

We first reduce the categorical problem to filtered quasibialgebras. Reconstruction provides a complete local quasi-Hopf algebra. Its closed augmentation filtration and a PBW refinement then put it in the form required by Theorem 4.1.

3.1 Quasi-fiber functors

A *quasi-fiber functor* here means an exact faithful linear functor

$$F : \mathcal{C} \longrightarrow \text{Vec}_{\text{fd}}, \quad F(\mathbf{1}) = \mathbb{C},$$

together with unital natural isomorphisms

$$J_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y),$$

which satisfy the unit constraints but need not satisfy the associativity constraint. When they also satisfy associativity, F is a tensor fiber functor.

Lemma 3.1. *Every unipotent tensor category admits a quasi-fiber functor. Its underlying exact functor, with $F(\mathbf{1}) = \mathbb{C}$, is unique up to natural isomorphism, and*

$$\dim F(X) = \text{length}(X).$$

Proof. The locally finite Grothendieck category $\text{Ind}(\mathcal{C})$ has an injective envelope E of $\mathbf{1}$. Its socle is $\mathbf{1}$. Set

$$F(X) = \text{Hom}(X, E)^*.$$

Injectivity of E gives exactness. A composition series gives $\dim \text{Hom}(X, E) = \text{length}(X)$, since $\text{Hom}(\mathbf{1}, E) = \mathbb{C}$. Thus F is finite-dimensional on $\mathcal{C}$, and it is faithful because it is exact and kills no simple object.

For uniqueness, extend another such exact functor F' to an exact colimit-preserving functor on $\text{Ind}(\mathcal{C})$. Its right adjoint sends $\mathbb{C}$ to an injective object E' with

$$\text{Hom}(X, E') \cong F'(X)^*.$$

The socle of E' is $\mathbf{1}$. Every nonzero subobject of an ind-object contains a simple subobject, so this socle is essential. Hence E' is an injective envelope of $\mathbf{1}$ and is isomorphic to E .

It remains to construct J ; equality of dimensions alone would not suffice. The exact functor F reconstructs $\mathcal{C}$ as finite-dimensional comodules over a coalgebra H with coradical $\mathbb{C}$. Finite-dimensional subcoalgebras give finite full abelian subcategories equivalent to module categories of finite-dimensional local algebras.

Consider two such subcategories, with algebras A and D . The bifunctor

$$T(M, N) = F(M \otimes N)$$

is right exact in both variables. By the elementary two-variable module description of right-exact functors, it has the form

$$T(M, N) \cong P \otimes_{A \otimes D} (M \otimes N)$$

for a finite-dimensional right $(A \otimes D)$ -module P . Exactness in each variable and $T(\mathbb{C}, \mathbb{C}) = \mathbb{C}$ imply

$$\dim P = (\dim A)(\dim D), \quad P/P \text{rad}(A \otimes D) \cong \mathbb{C}.$$

Nakayama's lemma makes P cyclic. A surjection $A \otimes D \twoheadrightarrow P$ is an isomorphism by the dimension equality. Thus T extends to an exact functor on the product abelian category.

Here the product category is

$$\mathcal{C} \boxtimes \mathcal{C} := \text{Comod}_{\text{fd}}(H \otimes H).$$

Every object is supported on $H_1 \otimes H_2$ for finite-dimensional subcoalgebras $H_1, H_2 \subset H$: its finitely many matrix coefficients lie in such a tensor product. On the corresponding module category, the preceding construction is the right-exact extension of the given bifunctor. For inclusions of the finite subcoalgebras, these extensions agree by uniqueness of the right-exact extension from pairs of modules. The comparison isomorphisms satisfy the cocycle condition by the same uniqueness. Thus the extensions glue to an exact functor on $\mathcal{C} \boxtimes \mathcal{C}$.

This product is again a length category with one simple object, $\mathbf{1} \boxtimes \mathbf{1}$. The uniqueness proof above uses only these abelian-category properties. It therefore identifies the glued functor with the external product of the two forgetful functors, whose value on (X, Y) is $F(X) \otimes F(Y)$. Restricting the resulting isomorphism to pairs gives J . The unit constraints allow J to be normalized on $\mathbf{1}$. $\square$

3.2 Duals and reconstruction

For an infinite-dimensional coquasi-bialgebra, rigidity supplies a preantipode. To reconstruct a coquasi-antipode, we also identify the underlying spaces of categorical duals with ordinary vector-space duals. Uniqueness of the normalized exact functor supplies this identification.

Lemma 3.2 (Compatibility with duals). *Let $F : \mathcal{C} \rightarrow \text{Vec}_{\text{fd}}$ be a normalized exact faithful linear functor on a unipotent tensor category. For any choice of right duals $X \mapsto X^\vee$ (with evaluation $X \otimes X^\vee \rightarrow \mathbf{1}$, as in [7]; since $\mathcal{C}$ is rigid, the same argument applies verbatim to left duals), there is a natural isomorphism*

$$\nu_X : F(X^\vee) \xrightarrow{\sim} F(X)^*.$$

Consequently, if F is equipped with a quasi-monoidal structure, its reconstructed complete local quasibialgebra has a continuous quasi-antipode.

Proof. The functor

$$G(X) = F(X^\vee)^*$$

is covariant, exact, faithful, and satisfies $G(\mathbf{1}) \cong \mathbb{C}$: categorical duality is an exact contravariant equivalence and finite-dimensional vector-space duality is exact. By the uniqueness part of Lemma 3.1, G is naturally isomorphic to F . Dualizing and using finite-dimensional biduality gives ν .

Transport the reconstructed coaction on $F(X^\vee)$ through ν_X . Thus the categorical dual of every reconstructed comodule is carried by the ordinary vector-space dual. Saracco's reconstruction theorem for coquasi-Hopf algebras, [7, Remark 2.17 and Theorem 2.19], then gives a coquasi-antipode on the continuous dual coalgebra. Dualizing gives a continuous quasi-antipode on the linearly compact algebra reconstructed from F .

Equivalently, one may describe the antipode component directly. If $b \in \text{End}(F)$, then

$$S(b)_X = (\nu_X b_{X^\vee} \nu_X^{-1})^*$$

under the canonical identification $F(X)^{**} = F(X)$. Transporting evaluation and coevaluation through F , its tensorator, and ν supplies the distinguished elements α, β ; the duality identities are precisely the quasi-antipode identities. Continuity is immediate from the formula: the component $S(b)_X$ depends only on $b_{X^\vee}$. $\square$

Let

$$B = \text{End}(F), \quad \varepsilon : B \longrightarrow \text{End}(F(\mathbf{1})) = \mathbb{C}, \quad I = \ker \varepsilon.$$

The topology on B is pointwise convergence on finite-dimensional values of F . It is a linearly compact algebra, an inverse limit of finite-dimensional local algebras. Reconstruction identifies

$$\mathcal{C} \simeq \text{Rep}_{\text{fd}}^{\text{cont}} B.$$

The maps J give a continuous coproduct and an associator

$$\Delta : B \longrightarrow B \hat{\otimes} B, \quad \Phi \in (B^{\hat{\otimes} 3})^\times,$$

with

$$(\text{id} \otimes \Delta)\Delta(b) = \Phi(\Delta \otimes \text{id})\Delta(b)\Phi^{-1}. \quad (6)$$

The pentagon is the quasibialgebra pentagon, and Lemma 3.2 supplies a continuous quasi-antipode. Unit normalization gives

$$\Phi - 1 \in I^{\hat{\otimes} 3}. \quad (7)$$

Changing J changes this quasi-Hopf algebra by a normalized twist

$$T \in 1 + I \hat{\otimes} I,$$

with the convention

$$\begin{aligned} \Delta^T(b) &= T\Delta(b)T^{-1}, \\ \Phi^T &= (1 \otimes T)(\text{id} \otimes \Delta)(T) \Phi (\Delta \otimes \text{id})(T^{-1})(T^{-1} \otimes 1). \end{aligned} \quad (8)$$

Lemma 3.1 implies that tensor-equivalent unipotent categories yield complete local quasi-Hopf algebras related by a continuous isomorphism and such a twist. Conversely, these operations give tensor equivalences of representation categories.

3.3 The associated graded and PBW coordinates

Put $W^r B = \overline{I}^r$ for $r \geq 0$. These closed subspaces form a separated complete filtration in the sense of Section 2.1. Every finite-dimensional continuous quotient algebra is local, so sufficiently large $W^r B$ acts by zero on every finite-dimensional continuous module.

Lemma 3.3. *The associated graded algebra $\text{gr}_W B$ is a connected graded cocommutative Hopf algebra. It is the graded completed enveloping algebra of its positively graded Lie algebra of primitive elements.*

Proof. The counit gives, for $b \in I$,

$$\Delta(b) = b \otimes 1 + 1 \otimes b \pmod{I \hat{\otimes} I}.$$

Since Δ is an algebra homomorphism, it preserves total I -adic degree. The associated graded is generated, with closure in each homogeneous component, by degree one. Its coproduct is cocommutative on these generators and hence everywhere.

The associator satisfies (7). In (6), conjugation by Φ therefore has strictly positive total filtration degree, so the associated graded coproduct is coassociative. Connectedness gives an antipode by the reduced-coproduct recursion.

Finally, the characteristic-zero connected cocommutative Hopf-algebra argument applies degree by degree: the convolution logarithm of the identity projects to primitives, and PBW symmetrization identifies the symmetric coalgebra on the primitive space with the Hopf algebra. These formulas are finite in each positive degree. They therefore apply also when the homogeneous spaces are linearly compact. $\square$

Set

$$V = \prod_{r \geq 1} V_r, \quad V_r = (\text{Prim}(\text{gr}_W B))_r.$$

Give V the filtration

$$W^r V = \prod_{s \geq r} V_s.$$

Choose continuous splittings of the filtration and use PBW to identify the underlying vector space of B with the completed $S(V)$. There are two degrees on a PBW monomial: its augmentation weight w and its number ℓ of primitive factors. Refine the filtration by the lexicographic degree

$$(w, -\ell). \tag{9}$$

For fixed $w \leq N$, one has $0 \leq \ell \leq w$, so there are only finitely many levels.

This refinement makes B a positive perturbation of the commutative and cocommutative symmetric bialgebra on V . On the associated graded for the first weight, the multiplication is the PBW multiplication of $U(\text{Prim}(\text{gr}_W B))$. Its noncommutative terms decrease the number of primitive factors, so they strictly increase $(w, -\ell)$. All other corrections to multiplication and coproduct increase the first weight. The leading coproduct, in PBW coordinates, is the standard symmetric coproduct. The associator differs from 1 by positive weight.

For example, a weight-preserving bracket sends two primitive factors to one, so its degree difference is $(0, 1)$. A cobracket raising weight by one sends one primitive factor to two, so its difference is $(1, -1)$. Both differences are positive in lexicographic order. Section 4.1 formalizes this completion and proves the finiteness needed for the dequantization argument.

4 Filtered dequantization modulo twists

We now solve the perturbation problem obtained in Section 3.3. Fix a universal quantization Q compatible with classical twists. Enriquez–Halbout construct such functors and show that they can extend a prescribed Lie bialgebra quantization [3, Theorem 2.1 and Proposition 5.1]. The construction of Sakáloš–Ševera uses the double and a deformation of a quasi-monoidal functor [4, Sections 2 and 13].

The inversion step adapts the recursive inversion of a map with identity leading term in Enriquez–Etingof [5, Section 4.3, Proposition 4.1]. There the semiclassical prop morphism is first proved surjective in [5, Lemma 4.2], and quantization makes its reduction an isomorphism; Hensel’s lemma then inverts the quantization morphism. Here the relation-ideal identity (38) supplies the corresponding comparison after passing to a gauge slice. Equivalences are handled separately, including twists and inner coboundaries.

4.1 Positive order and the dequantization theorem

Let $V = \prod_{r \geq 1} V_r$ be a positively weighted linearly compact space, and put $W^r V = \prod_{s \geq r} V_s$. We work on the completed symmetric space

$$A = \hat{S}(V) = \prod_{w \geq 0} S(V)_w, \quad \bar{A} = \prod_{w > 0} S(V)_w,$$

where $S(V)_w$ denotes the completed homogeneous space of total weight w , with the topology of Section 2.1. Here $\bar{A}$ is the augmentation ideal. We denote the standard product and primitive

coproduct by m_0 and Δ_0 . A PBW monomial has weight w and symmetric degree ℓ . A homogeneous component of a multilinear map has degree difference

$$(w_{\text{out}} - w_{\text{in}}, \ell_{\text{in}} - \ell_{\text{out}}). \quad (10)$$

We call it *positive* when this pair is positive in lexicographic order. All maps considered below are continuous. A positive map can have infinitely many homogeneous components, but all its nonzero components must be positive in this sense.

The symmetric-bialgebra operations preserve both degrees. Composing or tensoring maps adds their degree differences. In a bounded total-weight range $w \leq N$, the possible degrees $(w, -\ell)$ form a finite ordered set, since $0 \leq \ell \leq w$. Consequently, an expression with sufficiently many positive perturbation vertices vanishes in that range. This observation applies also when the vertices act on different tensor factors: process the vertices in a topological ordering of the diagram and follow the total degree of its intermediate strands. Each positive vertex strictly increases that degree.

For later use, this finiteness has a uniform bound. On the range $w \leq N$ put

$$\rho_N(w, \ell) = (N + 1)w - \ell.$$

An unperturbed symmetric-bialgebra operation preserves this rank. A positive vertex raises it by at least one. Indeed, if the weight is unchanged then positivity strictly decreases ℓ ; if the weight rises, then throughout the range $w \leq N$ one has

$$(N + 1)\Delta w + \ell_{\text{in}} - \ell_{\text{out}} \geq (N + 1) - N = 1.$$

Moreover $0 \leq \rho_N(w, \ell) \leq N(N + 1)$. Hence a diagram containing more than $N(N + 1)$ positive vertices cannot contribute to output weight at most N . This bound applies equally to compositions and to vertices acting on different tensor factors, and makes stabilization of all recursive inversions below explicit.

All layer-by-layer arguments below are carried out in bounded total weight, using the fixed projections and homotopy constructed next. These constructions commute with truncation, so the finite-range arguments assemble. A global least positive degree difference is not assumed.

A *positive change of variables* on V is a continuous linear automorphism u with $u - \text{id}$ positive. On A , it is a continuous linear automorphism with this property that fixes 1 and preserves the augmentation. A normalized quantum twist is an invertible $T \in 1 + \bar{A} \hat{\otimes} \bar{A}$; it is positive when $T - 1$, viewed as a map from $\mathbb{C}$, is positive. The same convention applies to a classical bivector twist. A normalized quantum perturbation has the standard unit and counit and positive corrections to m_0, Δ_0 , and 1.

The following theorem is stated for this weight/PBW completion. The filtration-preserving contraction needed in its proof is constructed in Lemma 4.2.

Theorem 4.1 (Filtered dequantization). *Let Q be a universal Lie quasibialgebra quantization compatible with twists and normalized as in Section 4.2. For the completion above, the following hold.*

(i) *Every continuous normalized positive quasibialgebra perturbation of $\hat{S}(V)$ is related, by a positive change of variables and a positive normalized twist, to $Q(\mu, \delta, \varphi)$ for a continuous positive Lie quasibialgebra structure on V .*

(ii) *Two such quantizations are related by positive quantum changes of variables and twists if and only if their classical structures are related by positive linear changes of variables and classical twists.*

The constructions are continuous and locally finite in bounded total weight.

The theorem classifies equivalence classes of structures; it does not identify all automorphism groups. Its proof has three ingredients. The contraction in Section 4.3 produces a slice. The relation-ideal comparison in Section 4.4 identifies the slice equations. The normalized twist comparison in Section 4.5 then reflects equivalences. The rank bound above makes the recursions converge.

4.2 Universal formulas and normalizations

We take the unit and counit of the quantized symmetric algebra to be the standard ones, using a preliminary universal change of variables if necessary. Exterior powers mean the quotients by the adjacent symmetric relations, with closure in the linearly compact case. Over $\mathbb{C}$, normalized antisymmetrization identifies this quotient with the alternating tensors. Write

$$a_r : \wedge^r V \longrightarrow V^{\otimes r}, \quad a_r(v_1 \wedge \cdots \wedge v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)},$$

and let $q_r : V^{\otimes r} \rightarrow \wedge^r V$ be the quotient map, so that $q_r a_r = \text{id}$. Write $\pi_1 : S(V) \rightarrow V$ for projection to symmetric degree one.

Universal expressions are linear combinations of directed acyclic diagrams. Their structure vertices are μ , δ , and φ , of input/output arities $(2, 1)$, $(1, 2)$, and $(0, 3)$. Composition joins strands, and tensor product juxtaposes diagrams. The *vertex degree* counts structure vertices; the symmetric-bialgebra maps and permutations have degree zero. A prop ideal generated by specified relations consists of sums of diagrams with an insertion of one of those relations, and its completed version allows convergent such sums.

The variable t below records vertex degree, independently of the weight/PBW filtration. Write the first-order terms as

$$m = m_0 + t m_1 + O(t^2), \quad \Delta = \Delta_0 + t \Delta_1 + O(t^2), \quad \Phi = 1 + t \Phi_1 + O(t^2).$$

In the conventions of Definition 2.1 and (6), its classical limit is normalized by

$$\begin{aligned} \mu(v, w) &= \pi_1(m_1(v, w) - m_1(w, v)), \\ \delta(v) &= q_2 \pi_1^{\otimes 2} \Delta_1(v), \\ \varphi &= -q_3 \pi_1^{\otimes 3} \Phi_1. \end{aligned} \tag{11}$$

The minus sign in the last line belongs to our convention $d_\delta^2 = [\varphi, -]$; it is not an extra restriction on quantization. The generators and normalizations in [3] are transported to these conventions. The first-order coordinate on each cohomology summand is fixed by these formulas.

4.3 A continuous filtered contraction

We give a self-contained calculation, including the chain homotopies. Initially $A = S(V)$ is algebraic, with augmentation ideal $\bar{A}$, primitive coproduct Δ_0 , and antipode S_0 . Put

$$\mathcal{B}^n = \bigoplus_{\substack{p \geq 0, q \geq 1 \\ p+q=n+2}} C^{p,q}, \quad C^{p,q} = \text{Hom}(\bar{A}^{\otimes p}, \bar{A}^{\otimes q}). \tag{12}$$

Here $\bar{A}^{\otimes 0} = \mathbb{C}$. Extend a cochain to $A^{\otimes p}$ by making it vanish whenever an input is 1. Its output is annihilated by the counit in every slot. The differential is the normalized Hochschild/co-Hochschild differential, with total convention

$$d = d_v + (-1)^q d_h \quad \text{on } C^{p,q}.$$

The formulas for both components are given below. In particular,

$$\begin{aligned}\mathcal{B}^{-1} &= \bar{A}, \\ \mathcal{B}^0 &= C^{1,1} \oplus C^{0,2}, \\ \mathcal{B}^1 &= C^{2,1} \oplus C^{1,2} \oplus C^{0,3}, \\ \mathcal{B}^2 &= C^{3,1} \oplus C^{2,2} \oplus C^{1,3} \oplus C^{0,4}.\end{aligned}$$

Degree one parametrizes multiplication, coproduct, and associator. Degree zero parametrizes changes of variables and twists. Degree -1 supplies their inner coboundaries. The input column $p = 0$ is therefore retained throughout.

Lemma 4.2 (Filtered normalized contraction). *There is a natural deformation retract*

$$(\mathcal{B}, d) \xrightleftharpoons[i]{p_H} (\mathcal{L}, 0), \quad p_H i = \text{id}, \quad dh + hd = \text{id} - ip_H, \quad (13)$$

where

$$\mathcal{L}^n = \bigoplus_{\substack{p \geq 0, \ q \geq 1 \\ p+q=n+2}} \text{Hom}(\wedge^p V, \wedge^q V). \quad (14)$$

One can arrange $h^2 = hi = p_H h = 0$. The maps preserve the degree differences (10). They extend continuously to the nonnegative-order, and hence to the positive-order, cochains in the weight/PBW completion. On universal expressions they preserve vertex degree and completed prop ideals of relations among the structure maps.

Proof. 1. Identification of the complex. Write $\Delta_q = \Delta_0^{(q-1)}$. Before restriction to normalized inputs,

$$\begin{aligned}(d_h f)(a_0, \dots, a_p) &= \Delta_q(a_0) f(a_1, \dots, a_p) \\ &\quad + \sum_{j=0}^{p-1} (-1)^{j+1} f(a_0, \dots, a_j a_{j+1}, \dots, a_p) \\ &\quad + (-1)^{p+1} f(a_0, \dots, a_{p-1}) \Delta_q(a_p),\end{aligned} \quad (15)$$

$$\begin{aligned}(d_v f)(a_1, \dots, a_p) &= \sum (a_{1(1)} \cdots a_{p(1)}) \otimes f(a_{1(2)}, \dots, a_{p(2)}) \\ &\quad + \sum_{j=1}^q (-1)^j \Delta_{0,j} f(a_1, \dots, a_p) \\ &\quad + (-1)^{q+1} \sum f(a_{1(1)}, \dots, a_{p(1)}) \otimes (a_{1(2)} \cdots a_{p(2)}).\end{aligned} \quad (16)$$

An empty product is 1, so these formulas include $p = 0$. The subscript j on a coproduct means that it is applied in output slot j .

For maps $A^{\otimes p} \rightarrow A^{\otimes q}$ use convolution, and set

$$\eta_{p,q} = \Delta_q m_0^{(p)}, \quad \eta_{0,q}(1) = 1^{\otimes q}.$$

Its convolution inverse is obtained by applying S_0 to the inputs. Define

$$\mathbb{T}_{p,q}(f) = \eta_{p,q}^{*-1} * f. \quad (17)$$

Thus

$$(\mathbb{T}_{p,q} f)(a_1, \dots, a_p) = \sum \Delta_q \left(S_0(a_{1(1)} \cdots a_{p(1)}) \right) f(a_{1(2)}, \dots, a_{p(2)}). \quad (18)$$

The inverse is convolution with $\eta_{p,q}$. Input normalization is preserved because $\Delta_0(1) = 1 \otimes 1$; output normalization is preserved because each partial counit is an algebra homomorphism. For $p = 0$, T is the identity.

We check conjugation face by face. The two outer terms of (15) become, respectively,

$$\varepsilon(a_0)(\mathsf{T}f)(a_1, \dots, a_p), \quad (\mathsf{T}f)(a_0, \dots, a_{p-1})\varepsilon(a_p).$$

For the first identity, the factors from the additional input cancel by $\sum S_0(a_{0(1)})a_{0(2)} = \varepsilon(a_0)1$; the last identity uses the right-handed version. An interior face commutes with T because multiplication is a coalgebra map and $\eta_{p+1,q} = \eta_{p,q} \circ m_{0,j}$. Thus the transformed horizontal differential is the bar differential with trivial coefficient actions.

For the vertical differential, denote the first and last unsigned terms in (16) by λf and ρf . The same antipode cancellation, now in the first or last output, gives

$$\mathsf{T}_{p,q+1}(\lambda f) = 1 \otimes \mathsf{T}_{p,q}f, \quad \mathsf{T}_{p,q+1}(\rho f) = (\mathsf{T}_{p,q}f) \otimes 1.$$

Here coassociativity and cocommutativity permit the Sweedler factors to be regrouped before cancellation. Each interior face satisfies

$$\mathsf{T}_{p,q+1}(\Delta_{0,j}f) = \Delta_{0,j}(\mathsf{T}_{p,q}f),$$

since $\Delta_{0,j}$ is an algebra map and $\eta_{p,q+1} = \Delta_{0,j}\eta_{p,q}$. These identities hold on arbitrary inputs, not only on primitives.

Let BA be the reduced bar complex, with $(BA)_p = \bar{A}^{\otimes p}$, $(BA)_0 = \mathbb{C}$, and differential

$$d_{\text{bar}}[a_1 | \dots | a_p] = \sum_{j=1}^{p-1} (-1)^j [a_1 | \dots | a_j a_{j+1} | \dots | a_p]. \quad (19)$$

Let ΩA be the reduced cobar complex, with $(\Omega A)^q = \bar{A}^{\otimes q}$ and

$$d_{\Omega} = \sum_{j=1}^q (-1)^j \text{id}^{\otimes(j-1)} \otimes \bar{\Delta}_0 \otimes \text{id}^{\otimes(q-j)}, \quad \bar{\Delta}_0(a) = \Delta_0(a) - 1 \otimes a - a \otimes 1. \quad (20)$$

The cancellations just proved identify our complex with $\text{Hom}(BA, \Omega A)$, with the degree-zero cobar summand omitted, and differential

$$Df = d_{\Omega}f + (-1)^q f d_{\text{bar}}, \quad f : (BA)_p \longrightarrow (\Omega A)^q. \quad (21)$$

Associativity and coassociativity imply that the differentials in (19) and (20) square to zero: the two orders of merging, or splitting, two places have opposite signs. Postcomposition and precomposition commute, so the sign in (21) gives $D^2 = 0$. The omitted cobar summand is isolated: its differential is zero and there is no incoming differential.

2. The bar homology, directly from resolutions. Suppose first that V is finite-dimensional. The augmented free bar resolution has terms $P_p = A \otimes \bar{A}^{\otimes p}$ and differential

$$\begin{aligned} \partial_P(a_0[a_1 | \dots | a_p]) &= a_0 a_1 [a_2 | \dots | a_p] \\ &\quad + \sum_{j=1}^{p-1} (-1)^j a_0 [a_1 | \dots | a_j a_{j+1} | \dots | a_p], \end{aligned}$$

with augmentation $P_0 = A \xrightarrow{\varepsilon} \mathbb{C}$. Put $\bar{a} = a - \varepsilon(a)1$. A contracting homotopy of this augmented complex, as a complex of vector spaces, is

$$s_P(a_0[a_1|\cdots|a_p]) = 1[\bar{a}_0|a_1|\cdots|a_p], \quad s_P(c) = c1 \quad (c \in \mathbb{C}).$$

Indeed, consecutive interior terms of $\partial_P s_P + s_P \partial_P$ cancel. The remaining terms are $\bar{a}_0[a_1|\cdots|a_p]$ and $\varepsilon(a_0)1[a_1|\cdots|a_p]$, whose sum is the original tensor. This also checks degree zero with the augmentation included.

The Koszul complex has terms $K_p = A \otimes \wedge^p V$ and differential

$$\partial_K(a \otimes v_1 \wedge \cdots \wedge v_p) = \sum_{j=1}^p (-1)^{j-1} a v_j \otimes v_1 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_p.$$

It too resolves the augmentation module. Here is an explicit verification. Choose a basis $x_1, \dots, x_d$ of V . On $S^m V \otimes \wedge^p V$ with $m + p > 0$, set

$$s_K(a \otimes \omega) = \frac{1}{m+p} \sum_{i=1}^d \frac{\partial a}{\partial x_i} \otimes (x_i \wedge \omega). \quad (22)$$

Set $s_K = 0$ on constants, and include $\mathbb{C} \rightarrow K_0$ as the homotopy in augmentation degree. If ι_i is contraction on $\wedge^\bullet V$ by the dual basis vector, then

$$\partial_K = \sum_i x_i \iota_i, \quad \left\{ \partial_K, \sum_i \partial_{x_i} (x_i \wedge -) \right\} = \sum_i x_i \partial_{x_i} + \sum_i (x_i \wedge -) \iota_i.$$

The right-hand side acts by $m + p$. This proves $\partial_K s_K + s_K \partial_K = \text{id}$ in positive total degree. The derivative in (22) is the natural map $S^m V \rightarrow S^{m-1} V \otimes V$, so this formula does not depend on the basis.

There is an A -linear comparison map $j : K \rightarrow P$,

$$j_p(a \otimes v_1 \wedge \cdots \wedge v_p) = \sum_{\sigma \in S_p} \text{sgn}(\sigma) a[v_{\sigma(1)}|\cdots|v_{\sigma(p)}]. \quad (23)$$

Its interior bar terms cancel in pairs by swapping the two factors that are multiplied. The first bar terms, grouped by the first factor, are exactly $j_{p-1} \partial_K$. Thus j is a chain map lifting the identity of $\mathbb{C}$.

For completeness, such a comparison of free resolutions is a homotopy equivalence for an elementary reason. Construct a reverse map degree by degree on the free generators. The image of the differential of a generator is a cycle in the target; the target contraction just constructed supplies a lift of that cycle. Extend the lift A -linearly. This gives a chain map lifting the identity. If r is either composite minus the identity, construct an A -linear homotopy H recursively: on a free generator e in degree p ,

$$H_p(e) = s(r_p(e) - H_{p-1} \partial(e)).$$

The argument of s is a cycle, by the identity already proved in degree $p - 1$; hence $\partial H + H \partial = r$. In degree zero it has zero augmentation. Here s is s_P or s_K , and the formula is extended A -linearly from the generators. This proves the asserted homotopy equivalence without invoking a cohomology formula.

Tensoring over A with the augmentation module identifies P with BA and K with $(\wedge^\bullet V, 0)$. The resulting homotopy equivalence gives

$$H_p(BA) = \wedge^p V, \quad (24)$$

represented by the unnormalized antisymmetrization in (23). Its internal symmetric degree is p . The maps preserve internal degrees and the multidegree assigned to the basis vectors. For arbitrary algebraic V , every chain lies in the bar complex of a finite-dimensional subspace, so the same conclusion follows by taking the filtered union.

3. *Natural finite contractions and the cobar complex.* We now choose homotopies that are natural in V and preserve every grading. This is stronger than the existence of arbitrary vector-space splittings. For $n > 0$, let $\mathcal{P}_{n,p}$ be the complex vector space with basis the ordered partitions

$$[B_1 | \cdots | B_p], \quad B_1 \sqcup \cdots \sqcup B_p = \{1, \dots, n\}, \quad B_i \neq \emptyset.$$

It is zero outside $1 \leq p \leq n$. Its differential b merges adjacent blocks with the signs in (19). The symmetric group S_n acts by relabeling. This is the multilinear component of the bar complex for $S(\mathbb{C}^n)$. By (24), its homology is the sign representation in degree n , spanned by

$$z_n = \sum_{\sigma \in S_n} \text{sgn}(\sigma) [\{\sigma(1)\} | \cdots | \{\sigma(n)\}].$$

There is no homology in other degrees.

Give the ordered-partition basis the standard inner product, and write b^t for the transpose matrix. This auxiliary inner product is on the finite rational complex, not on V . Let Π_n be zero outside degree n and, in degree n , let

$$\Pi_n = \frac{1}{n!} z_n z_n^t.$$

The Laplacian $L_n = bb^t + b^t b$ has kernel $\ker b \cap \ker b^t$, which maps isomorphically to homology. Indeed, an orthogonal decomposition of $\ker b$ into $\text{im } b$ and its orthogonal complement gives this identification. Thus its kernel is exactly the image of Π_n . Define the rational matrices

$$G_n = (L_n + \Pi_n)^{-1}(\text{id} - \Pi_n), \quad h_{\text{bar}} = b^t G_n, \quad h_\Omega = b G_n. \quad (25)$$

The inverse exists: L_n is positive definite on the orthogonal complement of its kernel, and Π_n is the identity on the kernel. All its entries are rational, and all these matrices commute with S_n .

Since b and b^t commute with L_n and annihilate Π_n , they commute with G_n . Consequently

$$bh_{\text{bar}} + h_{\text{bar}}b = \text{id} - \Pi_n, \quad h_{\text{bar}}^2 = 0, \quad h_{\text{bar}}\Pi_n = \Pi_n h_{\text{bar}} = 0.$$

For the cochain differential b^t the analogous identities hold with $h_\Omega = bG_n$. These are explicit finite-matrix formulas for all the homotopies; no equivariant choice of a complement remains unspecified.

To pass to V , use the natural identification

$$\mathcal{P}_{n,p} \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \cong \bigoplus_{\substack{n_1 + \cdots + n_p = n \\ n_i > 0}} S^{n_1} V \otimes \cdots \otimes S^{n_p} V. \quad (26)$$

Here the right group action is obtained by inverting the relabeling action. A partition tensored with $v_1 \otimes \cdots \otimes v_n$ maps to $\prod_{i \in B_1} v_i \otimes \cdots \otimes \prod_{i \in B_p} v_i$. The stabilizer of a partition permutes entries within its blocks, so this map is an isomorphism on each orbit. Under it, b becomes the reduced bar differential.

Under the same identification, b^t becomes the reduced cobar differential. In fact, transposing a merger splits a block into all ordered pairs of nonempty subsets, with sign $(-1)^j$ in slot j . This is exactly the reduced primitive coproduct of the corresponding monomial. When some v_i coincide,

the repeated subsets give the required binomial coefficients. Thus the second contraction in (25) proves directly that

$$H^q(\Omega A) = \wedge^q V. \quad (27)$$

For $n = 0$ the homotopy is zero: retain the isolated degree-zero bar summand and omit the isolated degree-zero cobar summand as above.

Here are the inclusions and projections, with their normalizations. On bar homology take

$$\begin{aligned} i_{\text{bar}}(v_1 \wedge \cdots \wedge v_p) &= \sum_{\sigma \in S_p} \text{sgn}(\sigma) [v_{\sigma(1)} | \cdots | v_{\sigma(p)}], \\ p_{\text{bar}} &= \frac{1}{p!} q_p \pi_1^{\otimes p}. \end{aligned}$$

In degree zero both are the identity of $\mathbb{C}$. On cobar cohomology take

$$i_\Omega = a_q, \quad p_\Omega = q_q \pi_1^{\otimes q} \quad (q \geq 1).$$

The projections vanish unless total symmetric degree equals homological or cohomological degree. Both composites ip are the projections induced by Π_n , and $pi = \text{id}$. In particular, all the side conditions hold with these specified normalizations.

As a low-degree check, for $u, v \in V$ the formulas give

$$\begin{aligned} h_{\text{bar}}[uv] &= -\frac{1}{2}([u|v] + [v|u]), & d_{\text{bar}}[u|v] &= -[uv], \\ h_\Omega[u|v] &= -\frac{1}{2}[uv], & d_\Omega[uv] &= -[u|v] - [v|u]. \end{aligned}$$

Thus on $V \otimes V$, $dh + hd$ is the symmetric projection, complementary to $a_2 q_2$. In particular, $d_\Omega[v^2] = -2[v|v]$; the multiplicity must not be discarded.

4. *The contraction on cochains and its signs.* Put $P_{\text{bar}} = i_{\text{bar}} p_{\text{bar}}$ and $P_\Omega = i_\Omega p_\Omega$. For $f : (BA)_p \rightarrow (\Omega A)^q$ define

$$\begin{aligned} \tilde{p}(f) &= p_\Omega f i_{\text{bar}}, & \tilde{i}(c) &= i_\Omega c p_{\text{bar}}, \\ \tilde{h}(f) &= h_\Omega f + (-1)^q P_\Omega f h_{\text{bar}}. \end{aligned} \quad (28)$$

The first summand of $\tilde{h}$ satisfies

$$D(h_\Omega f) + h_\Omega(Df) = (\text{id} - P_\Omega)f;$$

the terms involving $f d_{\text{bar}}$ cancel because its two output degrees differ by one. For $s(f) = (-1)^q P_\Omega f h_{\text{bar}}$, the identities $d_\Omega P_\Omega = P_\Omega d_\Omega = 0$ give

$$Ds(f) + s(Df) = P_\Omega f (h_{\text{bar}} d_{\text{bar}} + d_{\text{bar}} h_{\text{bar}}) = P_\Omega f (\text{id} - P_{\text{bar}}).$$

Adding gives

$$D\tilde{h} + \tilde{h}D = \text{id} - \tilde{i}\tilde{p}, \quad \tilde{p}\tilde{i} = \text{id}.$$

The side conditions on the two contractions give $\tilde{h}^2 = \tilde{h}\tilde{i} = \tilde{p}\tilde{h} = 0$; in the square, the two mixed terms vanish because $h_\Omega P_\Omega = P_\Omega h_\Omega = 0$. The target is (14), with zero differential and the indicated shift $p + q = n + 2$.

Now set

$$p_H = \tilde{p}\mathbb{T}, \quad i = \mathbb{T}^{-1}\tilde{i}, \quad h = \mathbb{T}^{-1}\tilde{h}\mathbb{T}.$$

These give (13). Fixed scalar changes of coordinates on the cohomology summands conjugate i, p_H and leave ip_H, h unchanged. We use the conventions in (11), in particular the minus sign on the curvature summand. With them the representatives of μ, δ, φ have primitive parts

$$m_1(v, w) = \frac{1}{2}\mu(v, w), \quad \Delta_1(v) = a_2\delta(v), \quad \Phi_1 = -a_3\varphi.$$

Convolution with η extends the first two by the biderivation and Δ_0 -derivation rules, respectively. This verifies the representatives used in the quadratic calculation below. The construction also includes $H^{-1}(\mathcal{B}) = V$ and the bivector summand of H^0 ; neither is removed by normalization.

5. Completion, continuity, and universal ideals. The matrices in (25) act only on the finite ordered-partition spaces. After (26), they are finite Schur operations on V . They preserve internal symmetric degree and every weight grading of V . Their extensions to completed tensor powers of linearly compact spaces are therefore continuous; no inner product or basis of such a space is used.

The bar and cobar homotopies preserve the total weight and symmetric degree on their respective sides. In (18), the part of an input removed by its coproduct is multiplied back into the output. Consequently both T and its inverse preserve the differences of these two degrees, as does the cochain homotopy. This proves preservation of (10).

For continuity of convolution in the completed setting, fix output weight at most N . A nonnegative-order cochain receives only input weights at most N in such a component. There are finitely many symmetric degrees and weight distributions in this range, and each Schur operation and coproduct component is a finite continuous operation. All the formulas above are therefore finite sums on each bounded-weight test. They are compatible as N increases. The quotient pieces keep their original linearly compact topologies; no finite-dimensional quotient is required to be preserved by the cochain. This proves the completed contraction, and the rank bound of Section 4.1 applies to its subsequent positive-order recursions.

Finally, the operations just constructed use only permutations, symmetric and exterior projectors, homogeneous projections, the symmetric-bialgebra maps, and rational coefficients from finite matrices. They introduce no μ, δ, φ vertices. Applied to a diagram containing an insertion of a classical relation, they give linear combinations of diagrams still containing that insertion. Hence they preserve vertex degree and the relation prop ideal. The bounded-weight continuity just proved gives the same assertion for its completion. This proves all parts of the lemma. $\square$

The finite-dimensional, untruncated Gerstenhaber–Schack cohomology calculation also appears in [6, Proposition 3.11]. The argument above proves the specific normalized and completed version used here, independently of that reference. In particular, it supplies the homotopies and their filtration properties, not only a cohomology isomorphism.

From now on, $\mathcal{B}$ and $\mathcal{L}$ denote the continuous completed versions of these complexes. We use the retract on their positive-order subcomplexes; a subscript > 0 denotes that part. Their cohomology coordinates are continuous maps between completed exterior powers. In degree zero they are $(\text{End}_{\text{cont}}(V) \oplus \widehat{\wedge}^2 V)_{>0}$.

4.4 The quadratic obstruction equations

For a triple $\gamma = (\mu, \delta, \varphi)$, do not yet impose any identities. Extend μ as a skew biderivation of the exterior algebra, and extend δ as an exterior derivation. Let $\mathcal{R}(\gamma)$ be the following four homogeneous

quadratic expressions:

$$\begin{aligned}
R_{3,1}(x, y, z) &= [[x, y], z] + [[y, z], x] + [[z, x], y], \\
R_{2,2}(x, y) &= \delta([x, y]) - [x, \delta(y)] + [y, \delta(x)], \\
R_{1,3}(x) &= d_\delta^2(x) - [\varphi, x], \\
R_{0,4} &= d_\delta \varphi.
\end{aligned} \tag{29}$$

The subscripts record input/output arities. These four expressions generate the classical relation prop ideal J . Thus $\mathcal{R}(\gamma) = 0$ is precisely the Lie quasibialgebra condition, equivalently the big-bracket Maurer–Cartan equation in the conventions above.

Write x for a perturbation of $(m_0, \Delta_0, 1)$. Use polynomial quantum identity defects: associativity, multiplicativity of the coproduct, quasi-coassociativity written as

$$((\text{id} \otimes \Delta)\Delta)\Phi - \Phi((\Delta \otimes \text{id})\Delta),$$

and the pentagon. Products have fixed parentheses before associativity is imposed. Choose the multiplicativity defect to be $(m \otimes m)(\text{id} \otimes \tau \otimes \text{id})(\Delta \otimes \Delta) - \Delta m$. With the corresponding signs the linearization is

$$\mathcal{E}(x) = dx + N(x), \quad N(x) = O(x^2) \in \mathcal{B}^2. \tag{30}$$

The quadratic projection of these defects is (29). For precision, use the normalized first-order representatives

$$m_1(a, b) = \frac{1}{2}\{a, b\}_\mu, \quad \Delta_1(v) = a_2\delta(v), \quad \Phi_1 = -a_3\varphi,$$

where $\{, \}_\mu$ is the biderivation of $S(V)$ extending μ , and Δ_1 is extended as a Δ_0 -derivation. These representatives are obtained from the bar/cobar retract above. For a normalized cochain c of input/output arity (p, q) , write

$$P_{p,q}(c) = q_q \pi_1^{\otimes q} c \circ (p!a_p), \quad a_0 = \text{id}_\mathbb{C},$$

where a_p is followed by the primitive inclusions into $A^{\otimes p}$. Thus $P_{p,q}$ uses unnormalized antisymmetrization on the primitive inputs and quotient exterior projection on the primitive outputs. Let $\mathcal{E}_{p,q}^{(2)}(\gamma)$ be the quadratic term of the corresponding quantum defect evaluated on these first-order representatives. The four raw projections, before changing the signs of the cohomology coordinates, are

$$\begin{aligned}
P_{3,1}(\mathcal{E}_{3,1}^{(2)}(\gamma)) &= R_{3,1}(\gamma), & P_{2,2}(\mathcal{E}_{2,2}^{(2)}(\gamma)) &= -R_{2,2}(\gamma), \\
P_{1,3}(\mathcal{E}_{1,3}^{(2)}(\gamma)) &= -R_{1,3}(\gamma), & P_{0,4}(\mathcal{E}_{0,4}^{(2)}(\gamma)) &= R_{0,4}(\gamma).
\end{aligned} \tag{31}$$

Accordingly, in degree two we use signs $+, -, -, +$ on the exterior projections of arities $(3, 1), (2, 2), (1, 3), (0, 4)$, respectively.

For example, on a primitive input v , the primitive alternating part of the quasi-coassociativity defect is

$$-d_\delta^2(v) + [\varphi, v].$$

Indeed, $q_3((\text{id} \otimes \Delta_1)\Delta_1 - (\Delta_1 \otimes \text{id})\Delta_1)(v) = -d_\delta^2(v)$, while the commutator with $-a_3\varphi$ contributes $[\varphi, v]$. This checks both the relative coefficient and the sign of the curvature term. The other components follow by the same primitive-input computation. This is the normalized version of the semiclassical computation in [3, Proposition 2.1].

Consequently, with these fixed identifications of the cohomology summands,

$$p_H \mathcal{E}(i(\gamma)) = \mathcal{R}(\gamma) + O(\gamma^3). \tag{32}$$

The next lemma upgrades this quadratic identification to a comparison of completed relation ideals once ideal membership is established.

Lemma 4.3 (Comparison of quadratic relation ideals). *Let $R = (R_j)$ be homogeneous quadratic equations in formal variables γ , and let J be their completed ideal. Suppose*

$$\vartheta(\gamma) = \gamma + O(\gamma^2), \quad K(\vartheta(\gamma)) \in J, \quad K(\vartheta(\gamma)) = R(\gamma) + O(\gamma^3).$$

Assume that all expressions are locally finite in the completion in use. Then

$$K(\vartheta(\gamma)) = (\text{id} + M_+(\gamma))R(\gamma), \tag{33}$$

where M_+ has positive order. The two vectors of equations generate the same completed ideal after the coordinate change ϑ .

The same statement holds for a homogeneous prop ideal. In that case the entries of M_+ are operators that compose and tensor a relation component with other operations, rather than scalar coefficients.

Proof. Subtract R . Its homogeneous component of degree $r \geq 3$ lies in the degree- r part of J . Since J is generated in degree two, this component is a sum of expressions containing one distinguished relation and $r - 2$ additional variable occurrences. Choose such an expression in each degree. They assemble into M_+ , and

$$(\text{id} + M_+)^{-1} = \sum_{k \geq 0} (-M_+)^k$$

converges. The same proof in a prop uses diagrams with one distinguished relation vertex. It requires neither a regular sequence of relations nor an identification of ideals with their radicals. $\square$

4.5 Normalization of comparison twists

Reflection of equivalences requires the quantum comparison to induce the identity on the bivector part of degree-zero cohomology. We verify that normalization here. Let $f \in \hat{\Lambda}^2 V$, assigned vertex degree one together with μ, δ, φ . The universal equivalence between $Q(\gamma)$ and $Q(\gamma^f)$ supplied by [3, Proposition 5.1] can be normalized to be the identity at $f = 0$: compose it with the inverse of its specialization there. Its comparison twist and change of variables then have first terms

$$T_Q(\gamma, f) = 1 + a_2(f) + O((\gamma, f)^2), \quad A_Q(\gamma, f) = \text{id} + O((\gamma, f)^2). \tag{34}$$

Here the first formula fixes the twist direction as in (8).

To verify the coefficient, at vertex degree one the only possible normalized twist depending on f is $c a_2(f)$. A change of variables depending only linearly on f vanishes: it would multiply an alternating pair into a symmetric output. Compare the terms involving one μ and one f in the coproduct identity. Their primitive alternating part is

$$\delta^f(x) - \delta(x) = c[f, x].$$

A degree-two change of variables contributes a cohomological boundary and disappears under this projection. The classical twist formula requires $[f, x]$ with coefficient one. This operation is nonzero universally, and there is no relation involving just one μ and one free f that could change its coefficient. Hence $c = 1$. Comparing the terms with one δ and one f in the associator identity gives, with (11), the term $d_\delta f$ in (4). Thus (34) induces the identity on the bivector part of degree-zero cohomology.

4.6 Comparison of equations and equivalences

In the proof, $\mathcal{E}$ denotes the full quantum identity defects, and $\mathcal{R}$ the four classical relations. The contraction defines a slice with cohomology coordinate z and a uniquely determined complementary coordinate $y(z)$. The function $\mathcal{K}(z)$ is the cohomology part of the defects on this slice. Finally, ϑ is the coordinate change obtained by putting a universal quantization into the slice. The decisive step is to compare $\mathcal{K} \circ \vartheta$ with $\mathcal{R}$ as systems of equations.

Proof of Theorem 4.1. All references to a first or unmatched filtration layer are made in a fixed bounded-weight range, as in Section 4.1.

1. *A slice.* The contraction gives a continuous splitting

$$\mathcal{B}^n = B^n \oplus H^n \oplus E^n, \quad B^n = \text{im } d, \quad H^n = i(\mathcal{L}^n), \quad E^n = \text{im } h,$$

where $d : E^n \rightarrow B^{n+1}$ is an isomorphism with inverse h . Write $p_{\text{bdry}} = dh$ for the boundary projection. A normalized change of variables and twist has leading action da for $a \in \mathcal{B}^0$. At each order its E^0 -component therefore removes the B^1 -component of the perturbation. Iterating puts every quantum structure into the slice

$$x = i(z) + y, \quad z \in \mathcal{L}^1, \quad y \in E^1.$$

For a point of this slice, the equations $p_{\text{bdry}}\mathcal{E}(x) = 0$ are equivalent to

$$y = -hN(i(z) + y). \quad (35)$$

They have a unique positive-order solution $y = y(z) = O(z^2)$, determined recursively. Define

$$\mathcal{K}(z) = p_H\mathcal{E}(i(z) + y(z)) = \mathcal{R}(z) + O(z^3). \quad (36)$$

The slice equation kills the boundary component of the quantum defects, and $\mathcal{K}(z)$ is their cohomology component. It remains to prove that $\mathcal{K}(z) = 0$ forces the E^2 -component to vanish. We do this by comparison with the universal classical relation ideal.

2. *Comparing the full equations.* Lift the universal quantization formulas to the completed free prop on μ, δ, φ , before imposing their relations. Let J be the completed homogeneous ideal generated by (29). Every quantum identity defect of the lift lies in J , because the quantization is defined on the quotient prop.

Apply the slice-normalization recursion to these lifted formulas. Off the associative locus, fix parentheses and formal inverses in the gauge formulas; modulo J they are the usual gauge operations. These extensions have the same linearized action, which is all that the recursion needs. Lemma 4.2 shows that each normalization operation preserves J and vertex degree. Hence the normalized lift still satisfies all the quantum identities modulo J .

Its surviving coordinate has the form

$$z = \vartheta(\gamma) = \gamma + O(\gamma^2). \quad (37)$$

The identity linear term is exactly (11). Thus ϑ has a formal inverse. Modulo J , the E^1 -component of the normalized lift solves (35). Uniqueness of that recursion identifies it with $y(\vartheta(\gamma))$ modulo J . Therefore

$$\mathcal{E}(i(\vartheta(\gamma)) + y(\vartheta(\gamma))) \in J, \quad \mathcal{K}(\vartheta(\gamma)) \in J.$$

Its quadratic part is $\mathcal{R}(\gamma)$ by (32). Lemma 4.3 now gives

$$\mathcal{K}(\vartheta(\gamma)) = (\text{id} + M_+(\gamma))\mathcal{R}(\gamma), \quad (38)$$

with M_+ positive. Hence $\text{id} + M_+$ is invertible, and (38) identifies the completed relation ideals after the change of coordinates ϑ . This is an identity of ideals, not only of their quadratic terms. In particular,

$$\mathcal{K}(\vartheta(\gamma)) = 0 \iff \mathcal{R}(\gamma) = 0.$$

Suppose now that $\mathcal{K}(z) = 0$. Put $\gamma = \vartheta^{-1}(z)$. Then $\mathcal{R}(\gamma) = 0$, so $Q(\gamma)$ is an actual quasibialgebra. Its normalized representative has cohomology coordinate z , and its other slice coordinate must be $y(z)$ by (35). Thus *all* identity defects of $i(z) + y(z)$ vanish. This proves sufficiency of the slice equations and assertion (i).

3. *Comparing equivalences.* The relevant degree-zero cohomology is

$$H^0(\mathcal{B}_{>0}) = (\text{End}_{\text{cont}}(V) \oplus \hat{\wedge}^2 V)_{>0}.$$

Functoriality quantizes a linear change of variables. Twist compatibility, with (34), quantizes a bivector twist. After normalizing the endpoints, these comparisons induce the identity on the displayed cohomology. The normalization of the endpoints does not alter this leading class: their change due to the equivalence contains both its parameter and a positive structure perturbation, and is of higher order.

Let x_γ and $x_{\gamma'}$ be the normalized quantizations, and suppose a positive quantum equivalence connects them. At its first unmatched filtration layer, let $a \in \mathcal{B}^0$ be the parameter. Its leading effect on the structure is da ; products of a with a structure perturbation have higher order. Both endpoints have zero B^1 -component. Projecting their difference to B^1 therefore gives

$$da = 0.$$

First remove the $\text{End}_{\text{cont}}(V)$ component of the class of a by a positive linear change of variables and its functorial quantization. Next remove the $\hat{\wedge}^2 V$ component by a classical twist and the quantum comparison (34). Composing the inverses of these comparisons with the given equivalence changes the source only within its classical equivalence class. The residual leading cocycle is a boundary.

These boundaries are realized by actual inner coboundary equivalences, not merely by infinitesimal corrections. For a unit u normalized by $\varepsilon(u) = 1$, put

$$T_u = (u \otimes u)\Delta(u^{-1}), \quad \text{Ad}_u(b) = ubu^{-1}.$$

Direct substitution in (8) gives

$$\begin{aligned} \Delta^{T_u}(\text{Ad}_u b) &= (\text{Ad}_u \otimes \text{Ad}_u)\Delta(b), \\ \Phi^{T_u} &= (\text{Ad}_u)^{\otimes 3}\Phi. \end{aligned} \quad (39)$$

Thus twisting by T_u and transporting back by Ad_u is a self-equivalence. If $u = 1 + v + \dots$, its leading parameter is the boundary of $v \in \mathcal{B}^{-1} = \bar{A}$. This removes the remaining boundary class without changing either normalized structure.

The unmatched order has now increased. Repeat. The positive corrections stabilize in bounded total weight by the rank bound in Section 4.1. Their composition therefore converges to a classical change of variables and one classical twist; successive twists combine by (4), with the intervening linear changes acting on their parameters. The remaining quantum equivalence tends to the identity.

Hence the limiting normalized structures agree, and the invertibility of ϑ implies that the limiting classical source is γ' . This proves the forward implication in assertion (ii). The reverse implication is functoriality and twist compatibility.

4. *Passage to the completion.* Every operation used above preserves degree differences. Every nonconstant term of a recursion contains additional positive perturbation vertices. By Section 4.1, the recursions for gauges, y , ϑ^{-1} , and $(\text{id} + M_+)^{-1}$ stabilize in bounded total weight. Lemma 4.2 proves continuity inside each such range. This justifies every formal manipulation after evaluation on the completed spaces and completes the proof. $\square$

5 Convergence and proof of the classification

We first evaluate the universal quantization on arbitrary pronilpotent classical data. We then apply Theorem 4.1 to the reconstruction in Section 3. Equality of the closed weight and augmentation filtrations connects these two constructions in the uniqueness argument.

5.1 Convergence of universal formulas

Let $(\mathfrak{g}, \delta, \varphi)$ be as in Theorem 1.1. Write $\Gamma^1 \mathfrak{g} = \mathfrak{g}$ and

$$\Gamma^{r+1} \mathfrak{g} = \overline{[\mathfrak{g}, \Gamma^r \mathfrak{g}]}.$$

The 1-cocycle identity for δ implies, by induction, that

$$\delta(\Gamma^r \mathfrak{g}) \subset \sum_{i+j \geq r+1} \Gamma^i \mathfrak{g} \hat{\otimes} \Gamma^j \mathfrak{g}. \quad (40)$$

Sums are understood as closed subspaces. The induction uses $[\Gamma^i \mathfrak{g}, \Gamma^j \mathfrak{g}] \subset \Gamma^{i+j} \mathfrak{g}$; continuity deals with closures.

For an explicit convergence estimate, set $\mathfrak{g}_a = \Gamma^{a+1} \mathfrak{g}$, $a \geq 0$. A bracket raises the sum of these shifted indices by one, a cobracket does not decrease it, and an occurrence of φ creates three outputs of nonnegative index.

Fix a component of a universal formula between tensor products of symmetric powers. Let p and q be the total numbers of primitive input and output strands in its underlying Lie quasibialgebra diagram. These are the total symmetric degrees, not the numbers of algebra inputs and outputs. If the diagram contains a brackets, b cobrackets, and c curvature vertices, then

$$q - p = -a + b + 3c. \quad (41)$$

Project every output to a finite-dimensional nilpotent Lie quotient of class s . A nonzero output must have total shifted index at most $q(s-1)$, so

$$a \leq q(s-1), \quad b + 3c = a + q - p, \quad a + b + c \leq 2q(s-1) + q - p. \quad (42)$$

Thus only finitely many diagrams contribute to fixed input/output symmetric degrees in a fixed finite-dimensional Lie quotient. Also $p \leq a + q \leq qs$, so only finitely many input symmetric degrees can contribute to a fixed output degree.

The same estimate works for twist formulas: if there are d bivector vertices, add $2d$ to the right-hand side of (41). This bounds d as well. No nilpotence of δ is used.

It follows that the universal quantization, its universal normalizing coordinate changes, and the universal comparison twists define continuous operations on

$$B_Q \cong \prod_{n \geq 0} \widehat{S^n \mathfrak{g}}$$

with its completed PBW topology. This is the topology obtained by testing finite-dimensional Lie quotients and bounded symmetric degrees; the displayed identification is a vector-space identification, not an assertion that the quantized multiplication is the ordinary multiplication of $U(\mathfrak{g})$.

Remark 5.1. There is no specialization map $\mathbb{C}[[\hbar]] \rightarrow \mathbb{C}$ with $\hbar \mapsto 1$. Indeed, $\hbar - 1$ is a unit in $\mathbb{C}[[\hbar]]$. Evaluation is justified here only after the coefficientwise local-finiteness argument above. This is the appropriate interpretation of “setting $\hbar = 1$.”

5.2 Locality and rigidity

The augmentation of B_Q is the standard projection to its degree-zero component. Its kernel is topologically nilpotent. Indeed, an N -fold product of augmentation elements has at least N primitive input strands. In a bounded output degree q and a quotient of class s , (42) makes it zero for $N > qs$. Hence every finite-dimensional continuous quotient algebra is local.

There are enough such quotient algebras: the continuous dual of the linearly compact associative algebra B_Q is a coalgebra, and its finite-dimensional subcoalgebras dualize to a cofinal family of finite-dimensional quotient algebras. Thus B_Q is an inverse limit of local algebras.

The following argument establishes rigidity directly from locality.

Lemma 5.2. *The finite-dimensional continuous representations of a complete local quasibialgebra form a rigid category.*

Proof. Let M be such a representation. The functor $N \mapsto N \otimes M$ is represented by the topological bimodule

$$P = B \otimes M, \quad b \cdot (c \otimes m) = \Delta(b)(c \otimes m), \quad (c \otimes m) \cdot b = cb \otimes m.$$

In other words, $P \otimes_B N \cong N \otimes M$. Filter M with trivial one-dimensional successive quotients. The resulting filtration of the left B -module P has successive quotients the regular module B . These extensions split continuously as left modules: a lift of 1 in a quotient B defines a splitting by multiplication. Hence P is a finite-rank free left B -module.

The right adjoint

$$R_M(N) = \text{Hom}_{B, \text{cont}}(P, N)$$

is exact and finite-dimensional on finite-dimensional N . To check continuity of its induced B -action, choose a continuous left B -basis $e_1, \dots, e_r$ of P and write

$$e_i b = \sum_{j=1}^r r_{ij}(b) e_j.$$

The coefficient maps $r_{ij} : B \rightarrow B$ are continuous: they are the coordinate maps of the continuous right action in this left basis. Under $R_M(N) \cong N^{\oplus r}$, the action is

$$(b \cdot f)(e_i) = f(e_i b) = \sum_{j=1}^r r_{ij}(b) f(e_j).$$

Its matrix entries are the continuous maps $B \xrightarrow{r_{ij}} B \rightarrow \text{End}(N)$, so $R_M(N)$ is a continuous B -module. The associativity constraint and the adjunction counit give a natural map

$$N \otimes R_M(\mathbf{1}) \longrightarrow R_M(N).$$

It is an isomorphism for $N = \mathbf{1}$. Both sides are exact in N , so a composition-series induction makes it an isomorphism for every N . Thus the adjunction exhibits $R_M(\mathbf{1})$ as a dual of M . Apply the same argument to $M \otimes N$ to obtain the other dual. $\square$

Thus $\text{Rep}_{\text{fd}}^{\text{cont}} B_Q$ is rigid and unipotent. By Lemma 3.2, its forgetful quasi-fiber functor supplies a continuous quasi-antipode. Hence B_Q is quasi-Hopf and its continuous dual $H_Q = B_Q^\vee$ is a coconnected coquasi-Hopf algebra. The latter has a coassociative coproduct and a quasi-associative multiplication, whereas B_Q has associative multiplication and a quasi-coassociative coproduct.

5.3 The intrinsic augmentation filtration

Give $\mathfrak{g}$ its closed lower-central filtration, with $\mathfrak{g}$ in weight one, and give B_Q the total weight on PBW monomials. Denote this filtration by $W^\bullet B_Q$. Equation (40) says that δ strictly raises total weight; φ also has positive weight. Consequently

$$\text{gr}_W B_Q \cong U(\text{gr}_\Gamma \mathfrak{g})$$

with degreewise completion. To see the identification without imposing an extra normalization on Q , set $\delta = \varphi = 0$ in the weight-preserving part of the universal formulas. A universal expression involving only brackets cannot contribute to the reduced coproduct of a primitive generator: it would have one primitive input and at least two outputs. Similarly, the commutator of two primitive generators can contain only one bracket vertex, and its coefficient is fixed by the classical limit. Thus the generators are primitive with bracket exactly μ . The induced map from the enveloping algebra is an isomorphism by its triangular PBW form.

The Lie algebra $\text{gr}_\Gamma \mathfrak{g}$ is topologically generated by its degree-one part. Hence so is $\text{gr}_W B_Q$. If I_Q is the augmentation ideal, this implies

$$W^r B_Q = \overline{I_Q^r}. \quad (43)$$

Indeed, one inclusion follows from multiplicativity of W ; the other follows by lifting products of degree-one generators successively modulo $W^{r+1}, W^{r+2}, \dots$ and taking closure.

The following example shows why this filtration need not be a neighborhood basis.

Example 5.3. Let $\mathfrak{g} = \prod_{i \geq 1} \mathbb{C} e_i$ with zero bracket, cobracket, and curvature, and with its product topology. Then

$$B = \prod_{n \geq 0} \widehat{S^n \mathfrak{g}}, \quad B/\overline{I^2} \cong \mathbb{C} \oplus \mathfrak{g}$$

as linearly compact vector spaces. This quotient is not discrete, so $\overline{I^2}$ is not open. Nevertheless the closed augmentation and total-weight filtrations agree. Thus equality of these filtrations cannot be used to replace the linearly compact topology by the ordinary adic topology.

The equality is sufficient for the convergence and uniqueness arguments. If $B_Q \rightarrow D$ is a finite-dimensional continuous quotient algebra, the image of I_Q is nilpotent. Hence $\overline{I_Q^N}$ maps to zero for all sufficiently large N . A recursion of increasing augmentation weight therefore stabilizes in every such quotient. Furthermore, a continuous augmented algebra isomorphism preserves the closed augmentation powers, and hence the closed weight filtration. When comparing quantum equivalences we use both this filtration preservation and continuity in the original topology.

5.4 The recovered classical data

Let $B = \text{End}(F)$ be reconstructed from a unipotent category. In the PBW coordinates of Section 3.3, Theorem 4.1 gives continuous maps

$$\mu : \hat{\wedge}^2 V \rightarrow V, \quad \delta : V \rightarrow \hat{\wedge}^2 V, \quad \varphi \in \hat{\wedge}^3 V,$$

satisfying the Lie quasibialgebra identities. Their positivity gives

$$\begin{aligned} \mu(W^i V, W^j V) &\subset W^{i+j} V, \\ \delta(W^r V) &\subset \sum_{i+j \geq r+1} W^i V \hat{\otimes} W^j V, \\ \varphi &\in \hat{\wedge}^3 W^1 V. \end{aligned} \tag{44}$$

These are the degree calculations in Section 3.3: a bracket may preserve total weight, whereas a cobracket must raise it.

Let $\mathfrak{g} = (V, \mu)$. It is linearly compact and pronilpotent. Its quotients by $W^{N+1} V$ are linearly compact nilpotent Lie algebras of class at most N . To see that each is an inverse limit of finite-dimensional nilpotent quotients, let $\mathfrak{a}$ be a linearly compact nilpotent Lie algebra of class s , and let $\lambda \in \mathfrak{a}^\vee$. Continuity of the bracket means that its transpose sends λ to a finite tensor in $\mathfrak{a}^\vee \otimes \mathfrak{a}^\vee$. Hence the span of all coadjoint translates $\text{ad}_x^* \lambda$ is finite-dimensional. Repeating this observation shows that

$$\text{span}\{\lambda, \text{ad}_{x_1}^* \lambda, \dots, \text{ad}_{x_{s-1}}^* \dots \text{ad}_{x_1}^* \lambda : x_1, \dots, x_{s-1} \in \mathfrak{a}\}$$

is finite-dimensional and coadjoint-stable; all expressions of length s vanish. Starting with finitely many functionals gives a finite-dimensional coadjoint-stable space containing them. Its annihilator is an open Lie ideal. Such ideals are cofinal among open subspaces, proving the assertion about $\mathfrak{a}$. Taking the inverse limit also over N proves pronilpotence of $\mathfrak{g}$.

Theorem 4.1 now identifies B , up to continuous isomorphism and normalized twist, with $B_Q(\mathfrak{g}, \delta, \varphi)$.

5.5 Proof of the main theorem

Proof of Theorem 1.1. The convergence and rigidity arguments of this section construct $\mathcal{C}_Q$ for every classical datum. Functoriality and twist compatibility show that it depends only on the classical equivalence class. Conversely, Section 3 reconstructs a complete local quasi-Hopf algebra from any unipotent category, and Section 5.4 supplies its classical representative. This proves surjectivity.

For uniqueness, suppose $(\mathfrak{g}_i, \delta_i, \varphi_i)$, $i = 1, 2$, give tensor-equivalent categories, and put $B_i = B_Q(\mathfrak{g}_i, \delta_i, \varphi_i)$. Reconstruction gives a continuous augmented algebra isomorphism and a normalized twist relating B_1 and B_2 . The isomorphism preserves the closed augmentation powers, hence weight by (43). The twist does not affect the associated graded coproduct, so the graded map is a Hopf algebra isomorphism

$$U(\text{gr}_\Gamma \mathfrak{g}_1) \xrightarrow{\sim} U(\text{gr}_\Gamma \mathfrak{g}_2).$$

It is $U(\alpha)$ for its restriction $\alpha : \text{gr}_\Gamma \mathfrak{g}_1 \rightarrow \text{gr}_\Gamma \mathfrak{g}_2$ to primitives.

Continuous splittings of the lower-central filtrations lift α to a filtered continuous linear isomorphism $u : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. Transport the second classical structure along u ; this does not assert that u intertwines the original brackets. Choose quantum PBW coordinates inducing PBW symmetrization on the associated graded via the enveloping-algebra identifications of Section 5.3. The graded quantum map is now the identity, so its difference from the identity raises weight. These PBW

coordinates themselves differ from the original symmetric coordinates of Q by positive changes: their weight-preserving parts are triangular in symmetric degree with identity leading term.

The normalized twist is positive because its nonconstant part lies in the completed tensor product of the two augmentation ideals. Thus, after transport by u and the positive PBW changes, the quantum equivalence is positive for (9). Theorem 4.1(ii) gives a classical equivalence of the transported structures. Composing with u gives a classical equivalence of the original data.

The constructions are continuous in the original linearly compact topologies. The rank bound of Section 4.1 and the contraction of Lemma 4.2 justify passage to the completion. This proves injectivity and completes the classification. $\square$

6 Fiber functors and examples

6.1 The fiber-functor criterion and Manin pairs

Proof of Corollary 1.2. If (2) holds, the twist is a Lie bialgebra. Our choice of Q quantizes it to a Hopf algebra, whose representation category has the usual forgetful tensor fiber functor. Twist compatibility transports this fiber functor to the original category.

Conversely, a tensor fiber functor reconstructs a cocommuted Hopf algebra on the comodule side. Its Lie-bialgebra dequantization has $\varphi = 0$, by the Hopf classification of [1, Theorem 4.2]. Theorem 1.1 then identifies the original datum with this one up to an isomorphism and a classical twist. Absorb the isomorphism into the notation to obtain (2). $\square$

For finite-dimensional $\mathfrak{g}$, there is also a geometric reformulation. A Lie quasibialgebra determines a quadratic Lie algebra $\mathfrak{d}$ containing $\mathfrak{g}$ as a Lagrangian Lie subalgebra. Choosing an isotropic vector-space complement to $\mathfrak{g}$ gives δ and φ ; changing that complement is a classical twist. Hence the finite-dimensional part of (1) is parametrized by isomorphism classes of Manin pairs

$$(\mathfrak{d}, \mathfrak{g}), \quad \mathfrak{g} \text{ nilpotent.}$$

There is no nilpotence requirement on $\mathfrak{d}$. Criterion (2) says that the Manin pair admits a Lagrangian *Lie-subalgebra* complement, that is, it comes from a Manin triple. The double/complement description is also the starting point of the construction in [4].

6.2 A category without a tensor fiber functor

Let V be a finite-dimensional abelian Lie algebra and let $0 \neq \varphi \in \wedge^3 V$. Set

$$B = \widehat{S(V)}, \quad \Delta(v) = v \otimes 1 + 1 \otimes v, \quad \Phi = \exp(-a_3(\varphi)),$$

with a_3 as in Section 4.2. The sign agrees with (11). The coproduct is coassociative and B is commutative. Since $a_3(\varphi)$ is a co-Hochschild 3-cocycle and all the relevant tensors commute, exponentiating its additive cocycle identity proves the pentagon. The antipode $S(v) = -v$ works with quasi-antipode elements $\alpha = \beta = 1$: the contractions of an alternating three-tensor under multiplication vanish.

The finite-dimensional continuous modules form a unipotent tensor category. It has no tensor fiber functor. In fact, any such functor has an underlying linear functor isomorphic to the forgetful one by Lemma 3.1. Its tensor structure would therefore give a normalized twist T killing Φ . Since B is commutative, taking logarithms of (8) would make $a_3(\varphi)$ a co-Hochschild coboundary. But

$$H_{\text{coHoch}}^3(\widehat{S(V)}) = \wedge^3 V,$$

and this tensor represents a nonzero class. This also agrees directly with (2): here $d_\delta = 0$ and the Schouten bracket vanishes, so no twist changes φ .

Taking $\dim V = 3$ gives such a category. Unlike the Heisenberg curvature in Example 2.2, this curvature cannot be removed by a twist.

6.3 A nonconilpotent cobracket

Let L be any finite-dimensional Lie algebra, for instance $\mathfrak{sl}_2$, and give $\mathfrak{g} = L^*$ the zero Lie bracket. Let δ be the transpose of the bracket of L , and set $\varphi = 0$. Then $\mathfrak{g}$ is nilpotent of class one, although its cobracket is not conilpotent when L is not nilpotent. The corresponding coconnected Hopf algebra on the comodule side is $U(L)$, as in [1, Remark 4.3]. Here conilpotence means that sufficiently long iterates of the cobracket vanish on each input; for finite-dimensional L^* this is equivalent to nilpotence of L . Thus no such condition can be imposed in the classification.

6.4 Finite tensor generation

Finite tensor generation does not force finite-dimensional classical data. We say that an object tensor-generates a category if every object is obtained from it and $\mathbf{1}$ using finite direct sums, tensor products, duals, and subquotients. Closure under arbitrary extensions is not included.

Let L be the free Lie algebra on two generators and let $H = U(L) = \mathbb{C}\langle x, y \rangle$, with x, y primitive. Its finite-dimensional comodule category is unipotent and is tensor-generated by the direct sum of the two two-dimensional extensions whose matrix coefficients are $1, x$ and $1, y$. Indeed, the coefficients of tensor powers contain every word in x, y , so their coefficient coalgebras span H . The coefficient coalgebra of any finite-dimensional comodule M is therefore contained in a finite sum C of these coalgebras. Its coaction embeds M into $C^{\oplus \dim M}$. Each coefficient coalgebra of a tensor power is a quotient of finitely many copies of that tensor power, by the coefficient maps. Hence M is a subquotient of a finite direct sum of tensor powers, as required.

Its classical datum has abelian underlying Lie algebra

$$\mathfrak{g} = L^\vee := \text{Hom}_{\mathbb{C}}(L, \mathbb{C}),$$

the full dual of the discrete vector space L , with its linearly compact topology, not the restricted graded dual. Its cobracket is dual to the bracket of L . The space $\mathfrak{g}$ is infinite-dimensional, and classical twists do not change the underlying Lie algebra. This proves the assertion.

This shows that Remark 4.7 of Etingof–Gelaki [1] is incorrect as stated. The example above disproves its finite-generation/algebraicity assertion for the general nonbraided Hopf classification of [1, Theorem 4.2]. Indeed, the category is finitely tensor-generated, whereas its classical additive progroup G has $\text{Lie}(G) = L^\vee$ and coordinate algebra $\mathcal{O}(G) = S(L)$. Since L is infinite-dimensional, $S(L)$ is not finitely generated, so G is not an algebraic group of finite type. The identification of the classical datum of $U(L)$ is already given in [1, Remark 4.3]; thus this counterexample is entirely within the Hopf case and does not depend on Theorem 1.1.

6.5 Finite tensor categories

A category is *finite as an abelian category* if it is equivalent to the category of finite-dimensional modules over a finite-dimensional algebra. Every finite unipotent tensor category over $\mathbb{C}$ is equivalent to Vec_{fd} by [2, Corollary 4.4.2]. That result holds more generally for finite ring categories; rigidity is not required.

In the present setting there is also a short proof from Lemma 3.3. The reconstructed basic algebra B is finite-dimensional and local, and $\text{gr } B$ is a finite-dimensional enveloping algebra. In characteristic zero its primitive Lie algebra must vanish, by PBW. Hence $B = \mathbb{C}$. This proof uses only the associated graded calculation, not the classification theorem.

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Can you extend the results of <https://arxiv.org/pdf/0708.1487> obtaining a classification of all unipotent tensor categories over $\mathbb{C}$ (i.e., categories whose only simple object is $\mathbf{1}$) in terms of (nilpotent) Lie quasibialgebras. The paper above does the special case of bialgebras using the theory quantization of Lie bialgebras of Etingof-Kazhdan (it classifies coconnected Hopf algebras). The extension should be based on the works <https://arxiv.org/pdf/0804.0496> and <https://arxiv.org/abs/1304.6382>, answering positively the question of Drinfeld on whether every Lie quasibialgebra can be quantized.

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