

An Arbor Polytope with Nonreal Ehrhart Roots

Yuran Xiu
IIIS, Tsinghua University
xiuyr23@mails.tsinghua.edu.cn

14 October 2026

Abstract

We construct a singleton-labeled arbor on 243 vertices whose Ehrhart polynomial has a nonreal root. This disproves the conjecture that the Ehrhart roots of every arbor polytope are real and lie in $[-1, 0]$. The polynomial factors as $(m+1)^{230}(7m+2)^8 f(m)/6144$, where f is an explicit quintic. We derive this identity by counting lattice points for every nonnegative integer dilation, using three moments of subtree deficits. A negative value of $f'^2 - ff''$ at $-3/10$ gives an exact nonreal-root certificate. The lattice-count identity and the resulting counterexample have been formalized and checked in Lean.

1 Introduction and the counterexample

Chapoton introduced polytopes associated with arbors, rooted trees whose vertices carry disjoint nonempty sets of labels [1]. The real-rootedness conjecture for their Ehrhart polynomials is also stated as Conjecture 1.2 by Athanasiadis and Chapoton [2]. We disprove it within the subclass in which each vertex has one label.

For a rooted tree T , write T_v for the set consisting of v and all its descendants. The singleton-labeled arbor polytope is

$$Q_T = \left\{ x \in \mathbb{R}_{\geq 0}^{V(T)} : \sum_{u \in T_v} x_u \leq |T_v| \quad (v \in V(T)) \right\}. \quad (1)$$

This is a lattice polytope [1]. Its Ehrhart polynomial E_T is characterized by $E_T(m) = |mQ_T \cap \mathbb{Z}^{V(T)}|$ for every integer $m \geq 0$. The tree used here is described without any implicit identifications: its root a has one child b , and b has eleven children. One of these children is the center of a star with 180 leaves; each of the other ten is the center of a star with five leaves. Every vertex receives a distinct singleton label. Thus T has $2 + 181 + 10 \cdot 6 = 243$ vertices and 230 leaves.

Theorem 1.1. *For this tree T ,*

$$E_T(m) = \frac{(m+1)^{230}(7m+2)^8}{6144} f(m), \quad (2)$$

where

$$\begin{aligned} f(m) = & 24 + 5852m + 400056m^2 + 8533751m^3 \\ & + 44368122m^4 + 67184474m^5. \end{aligned} \quad (3)$$

The polynomial f , and hence E_T , has a nonreal complex root.

The theorem concerns the Ehrhart polynomial itself. It makes no claim about the real-rootedness of the h^* -polynomial and no minimality claim for the dimension 243. Nonreal Ehrhart roots for general lattice polytopes were already known. For example, Higashitani [3] constructed counterexamples to a conjectured strip bound on the real parts of Ehrhart roots. The issue here is the more restrictive arbor family and its proposed real-rootedness property.

2 Lattice points and subtree deficits

Fix an integer $m \geq 0$. For a lattice point in a subtree with N vertices, define its *deficit* to be mN minus the sum of its coordinates. This is a nonnegative integer by the subtree constraint. For a finite collection of configurations with deficits d , let M_0 denote its cardinality and, for $r \geq 1$, let $M_r = \sum d^r$. For disjoint subtrees the configurations are independent and deficits add.

Lemma 2.1. *Suppose the configurations in all child subtrees of a new singleton root have total deficit s . There is exactly one choice of the root coordinate for each new deficit $d \in \{0, \dots, m + s\}$.*

Proof. The root coordinate is $m + s - d$. Its nonnegativity and the new subtree inequality are equivalent to $0 \leq d \leq m + s$. The child inequalities are unchanged, so this correspondence is a bijection. $\square$

A leaf has deficits $0, \dots, m$. Put $h = m + 1$. Dividing its first three raw moments by h gives

$$\mu_1 = \frac{m}{2}, \quad \mu_2 = \frac{m(2m+1)}{6}, \quad \mu_3 = \frac{m^2(m+1)}{4}. \quad (4)$$

These follow from the sums of the first three powers. If S is the sum of k leaf deficits, its moments divided by the number h^k of leaf configurations are

$$\begin{aligned} s_1 &= k\mu_1, \\ s_2 &= k\mu_2 + k(k-1)\mu_1^2, \\ s_3 &= k\mu_3 + 3k(k-1)\mu_1\mu_2 + k(k-1)(k-2)\mu_1^3. \end{aligned} \quad (5)$$

Indeed, expand S^r and separate tuples of indices according to their equalities. The formulas include $k = 0, 1, 2$ with the vanishing factors interpreted in the usual way.

Let $\text{Star}(k)$ consist of a singleton root and k singleton leaves. Denote its cardinality, first deficit moment, and second deficit moment divided by h^k by P_k, D_k, W_k , respectively.

Lemma 2.2. *For every $m \geq 0$ and $k \geq 0$,*

$$P_k = h + s_1 = 1 + \frac{k+2}{2}m, \quad (6)$$

$$D_k = \frac{m(m+1) + (2m+1)s_1 + s_2}{2}, \quad (7)$$

$$\begin{aligned} W_k &= \frac{1}{6}(2m^3 + 3m^2 + m + (6m^2 + 6m + 1)s_1 \\ &\quad + (6m + 3)s_2 + 2s_3). \end{aligned} \quad (8)$$

Proof. For fixed leaf deficits of sum S , Lemma 2.1 says that the new deficit ranges from 0 to $m + S$. Its cardinality, sum, and sum of squares are, respectively,

$$m + S + 1, \quad \frac{(m+S)(m+S+1)}{2}, \quad \frac{(m+S)(m+S+1)(2m+2S+1)}{6}.$$

Sum over the h^k leaf configurations, expand, and apply (5). In particular, W_k is a raw second moment, rather than a binomial moment or a variance. $\square$

3 The Ehrhart polynomial of the witness

Write the coordinates of T as $a, b, u, y_i, v_j, z_{j,i}$, where $1 \leq i \leq 180$ for the leaves below u , and $1 \leq j \leq 10, 1 \leq i \leq 5$ for the leaves below v_j . Besides nonnegativity, the conditions in mQ_T are exactly

$$\begin{aligned} y_i &\leq m, & z_{j,i} &\leq m, \\ u + \sum_{i=1}^{180} y_i &\leq 181m, & v_j + \sum_{i=1}^5 z_{j,i} &\leq 6m, \\ b + u + \sum_i y_i + \sum_j (v_j + \sum_i z_{j,i}) &\leq 242m, \\ a + b + u + \sum_i y_i + \sum_j (v_j + \sum_i z_{j,i}) &\leq 243m. \end{aligned}$$

Fix configurations in the eleven stars and let s be the sum of their deficits. The remaining coordinates satisfy $0 \leq b \leq m + s$ and $0 \leq a \leq 2m + s - b$. Their number is

$$\sum_{b=0}^{m+s} (2m + s - b + 1) = \frac{(m + s + 1)(3m + s + 2)}{2}. \quad (9)$$

The bounds are valid also at $m = 0$.

Abbreviate

$$P = P_{180}, \quad D = D_{180}, \quad W = W_{180}, \quad Q = P_5, \quad d = D_5, \quad w = W_5.$$

The combined moments of the eleven stars, after extracting the common factor $h^{230}Q^8$, are

$$\begin{aligned} N_0 &= PQ^2, \\ N_1 &= DQ^2 + 10PQd, \\ N_2 &= WQ^2 + 10PQw + 90Pd^2 + 20DdQ. \end{aligned} \quad (10)$$

For instance, the factor 90 counts ordered pairs of distinct small stars; the factor 20 counts the two orders of a large–small pair. All three identities are polynomial factorizations, so no division at a possibly vanishing value of Q is required.

Summing (9) using (10) gives

$$E_T(m) = \frac{h^{230}Q^8}{2} ((m + 1)(3m + 2)N_0 + (4m + 3)N_1 + N_2). \quad (11)$$

Substitution from Lemma 2.2, followed by expansion, yields (2)–(3); here $Q = (7m + 2)/2$. This proves the identity by counting every dilation, not by interpolating a finite list of values. Alternatively, the count and the right side agree for infinitely many integers, which identifies the Ehrhart polynomial uniquely. As checks on normalization, the right side has degree 243 and constant term $2^8 \cdot 24/6144 = 1$.

4 An exact obstruction to real-rootedness

For a real polynomial p , put $\mathcal{L}(p) = (p')^2 - pp''$.

Lemma 4.1 (Laguerre inequality). *If a nonzero real polynomial has only real complex roots, then $\mathcal{L}(p)(x) \geq 0$ for every $x \in \mathbb{R}$.*

Proof. The fundamental theorem of algebra factors p into a real constant and real linear factors, with multiplicities. The product identity

$$\mathcal{L}(pq) = q^2\mathcal{L}(p) + p^2\mathcal{L}(q)$$

follows by differentiation. A constant has $\mathcal{L} = 0$, and a monic linear polynomial has $\mathcal{L} = 1$. Induction proves the assertion, including evaluations at repeated roots; no division by $p(x)$ is used. $\square$

For the quintic in (3), exact rational arithmetic gives

$$\begin{aligned} f(-3/10) &= -716031/50000, \\ f'(-3/10) &= -854809/1000, \\ f''(-3/10) &= -2922684, \end{aligned}$$

and consequently

$$\mathcal{L}(f)(-3/10) = -\frac{41123948517599}{1000000} < 0. \quad (12)$$

Lemma 4.1 implies that f has a nonreal root. Equation (2) makes this a root of E_T , proving Theorem 1.1 and disproving the arbor real-rootedness conjecture. No numerical approximation to a root is needed.

5 Formal verification and reproducibility

The accompanying project `arbor-ehrhart-counterexample` formalizes the actual rooted tree, its descendant inequalities, and a bijection with the configurations counted above. In particular, `ArborCounterexample.E_counts_all_dilations` counts every dilation; `ArborCounterexample.ehrhart_unique` identifies the polynomial; and `ArborCounterexample.complete_counterexample` combines the count with the nonreal-root obstruction. The universal conclusion is `ArborCounterexample.not_all_arbor_ehrhart_roots_real`.

The seven-module verification uses Lean 4.27.0 and mathlib revision `a3a10db0e9d66acbebf76c5e6a135066525ac900`. With the dependencies in `lake-manifest.json` provisioned, run `python3 verify.py`; use `--package-root` for another dependency location. Reproduction instructions and successful logs are in `REPRODUCE.txt` and `verification/result.json`. The final transitive axiom lists contain only `propext`, `Classical.choice`, and `Quot.sound`; there are no proof placeholders, additional mathematical axioms, or native-computation certificates. The compiler and pinned library proof objects are trusted. Lean checks the formal declarations, not the manuscript’s prose or bibliographic claims.

References

- [1] F. Chapoton, *On posets and polytopes attached to arbors*, [arXiv:2503.04247v2](#) (2025).
- [2] C. A. Athanasiadis and F. Chapoton, *Polytopes and posets associated to preorders*, [arXiv:2605.26916v1](#) (2026).
- [3] A. Higashitani, *Counterexamples of the conjecture on roots of Ehrhart polynomials*, [arXiv:1106.4633](#) (2011).