

An unavoidable set of divisors for the nearest integers to $\xi(3/2)^n$

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ABSTRACT. We prove that for every real $\xi \neq 0$ the sequence of nearest integers $[\xi(3/2)^n + 1/2]$, $n \in \mathbb{N}$, contains infinitely many elements divisible by 2 or by 5. In particular, for $\xi > 0$ infinitely many of these nearest integers are composite. This is the nearest-integer counterpart of the theorem of Forman and Shapiro on $[(3/2)^n]$, for which Dubickas and Novikas give the unavoidable set $\{2, 5, 7, 11\}$; for nearest integers two primes suffice. The proof follows the method of Dubickas and Novikas: if 2 and 5 divide only finitely many terms, the sequence of operations is forced to become $(A^2B^2)^\infty$, which is impossible by their Lemma 2. All results are machine-verified in Lean 4.

1. INTRODUCTION

Let ξ be a real number and $a > 1$ a rational noninteger number. Forman and Shapiro [FS67] proved that $[(3/2)^n]$ and $[(4/3)^n]$ are composite for infinitely many n , and Dubickas and Novikas [DN05] sharpened this by exhibiting explicit finite *unavoidable* sets of divisors: by [DN05, Thm. 1], for every $\xi > 0$ the set $[\xi(3/2)^n]$, $n \in \mathbb{N}$, contains infinitely many elements divisible by at least one of 2, 5, 7, 11. Following [DN05] we write $[z + 1/2]$ for the nearest integer to z . For nearest integers, [DN05, Thm. 4] treats the cases $a = 7$, $5/3$ and $7/5$; the case $a = 3/2$ does not appear there, nor in the account of shifted integer parts in [Nov12, §1.6]. We settle it.

Theorem 1.1. *Let $\xi \neq 0$ be a real number. Then the set $[\xi(3/2)^n + 1/2]$, $n \in \mathbb{N}$, contains infinitely many elements divisible by 2 or 5.*

Corollary 1.2. *Let $\xi > 0$ be a real number. Then the set $[\xi(3/2)^n + 1/2]$, $n \in \mathbb{N}$, contains infinitely many composite numbers.*

Proof. Since $\xi(3/2)^n \rightarrow \infty$ and $|[z + 1/2] - z| \leq 1/2$, we have $[\xi(3/2)^n + 1/2] > 5$ for all sufficiently large n . Such an element that is divisible by 2 or 5 is composite, so the claim follows from Theorem 1.1. $\square$

Thus $\{2, 5\}$ is an unavoidable set for nearest integers at $a = 3/2$. It is a proper subset of the set $\{2, 5, 7, 11\}$ of [DN05, Thm. 1] for integer parts. The reason is visible in Lemma 2.3 below: for the shift $1/2$ the two operations

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divide $[0, 1)$ into two halves, and the intervals in Lemma 2.3 show at once that neither A^3 nor B^3 can occur. For integer parts ($\nu = 0$), by contrast, B can be repeated arbitrarily often.

Remark 1.3. (i) Theorem 1.1 asserts nothing about the gaps between consecutive n with $\gcd([\xi(3/2)^n + 1/2], 10) > 1$. The argument below leaves the periodic pattern $(A^2B^2)^\infty$ to be excluded by Lemma 2.1. Lemma 2.1 is a statement about the whole tail of the sequence, not about a window of bounded length.¹

(ii) Computations, which are not part of the formal verification, suggest that the method cannot be run with fewer primes. With a single prime it fails, and among the pairs $\{2, \ell\}$ with $\ell \leq 19$ only $\ell = 5$ gives a contradiction.

2. PRELIMINARIES

Following [DN05, §2] with $a = 3/2$ and shift $\nu = 1/2$, put, for $n \in \mathbb{N}$,

$$x_n = [\xi(3/2)^n + 1/2], \quad y_n = \{\xi(3/2)^n + 1/2\} \in [0, 1), \quad s_n = 2x_{n+1} - 3x_n$$

From $\frac{3}{2}(x_n + y_n - \frac{1}{2}) = x_{n+1} + y_{n+1} - \frac{1}{2}$ we obtain

$$(1) \quad s_n = 3y_n - 2y_{n+1} - \frac{1}{2},$$

so $s_n \in S(3/2, 1/2) = \{-2, -1, 0, 1, 2\}$. The only external input is Lemma 2 of [DN05].

Lemma 2.1 (Dubickas–Novikas [DN05, Lemma 2]). *Let $a = b/c > 1$ with $b > c > 1$ coprime integers, let $\xi \neq 0$ and ν be real, and let $s_n = cx_{n+1} - bx_n$ with $x_n = [\xi a^n + \nu]$. Then the sequence $s_0, s_1, s_2, \dots$ is not eventually periodic.*

Remark 2.2. In [DN05] the lemma is stated for $\xi > 0$. The proof given there uses only $\xi \neq 0$, and so does ours. Write $e_n = x_n - \xi a^n = \nu - y_n$. If s has period ℓ from some point on, then along each residue class modulo ℓ the bounded sequence e_n satisfies an affine recursion with multiplier $a^\ell > 1$, so it is constant. Hence y_n is eventually periodic. Then the integers $x_{n+\ell} - x_n = \xi a^n(a^\ell - 1)$, for all large n , force $\xi = 0$ by coprimality of b and c , as in [DN05, Lemma 1].

Lemma 2.3. *If x_n is odd, then $s_n \in \{-1, 1\}$. More precisely, exactly one of the following two operations takes place:*

$$\begin{aligned} A: \quad x &\mapsto (3x - 1)/2, \quad y \mapsto (6y + 1)/4, \quad [0, 1/2) \rightarrow [1/4, 1); \\ B: \quad x &\mapsto (3x + 1)/2, \quad y \mapsto (6y - 3)/4, \quad [1/2, 1) \rightarrow [0, 3/4). \end{aligned}$$

The operation is A (that is, $s_n = -1$) if and only if $y_n < 1/2$.

¹Computations, which are not part of the formal verification, exhibit values of ξ for which the first 9, 33, 129 and 513 nearest integers are all prime to 10.

Proof. From $2x_{n+1} = 3x_n + s_n$ we get $s_n \equiv x_n \pmod{2}$, so $s_n = \pm 1$. By (1), $s_n = 1$ requires $y_n \geq 1/2$ (since $y_{n+1} \geq 0$) and $s_n = -1$ requires $y_n < 1/2$ (since $y_{n+1} < 1$). The formulas for x_{n+1} and y_{n+1} follow from the definition of s_n and (1). $\square$

From now on assume, contrary to Theorem 1.1, that there is N such that x_n is divisible neither by 2 nor by 5 for every $n \geq N$. Since $2^{-1} \equiv 3 \pmod{5}$, the operations act on residues modulo 5 as $r \mapsto 4r + 3s$. In the notation of [DN05, §2], where $(k|$ means that residue k is mapped to 0,

$$A = (1)(34)(2|, \quad B = (12)(4)(3| \pmod{5}.$$

Hence the classes $\{1, 2\}$ and $\{3, 4\}$ are invariant. Since no x_n with $n \geq N$ is divisible by 5, Lemma 2.3 gives the following window constraints for $n \geq N$:

$$(2) \quad x_n \equiv 2 \pmod{5} \implies y_n \geq 1/2, \quad x_n \equiv 3 \pmod{5} \implies y_n < 1/2.$$

3. PROOF OF THEOREM 1.1

The three lemmas below hold under the standing assumption that x_n is prime to 10 for all $n \geq N$. Each applies to every $n \geq N$.

Lemma 3.1. *If $y_n < 3/8$ and $x_n \equiv 1$ or $4 \pmod{5}$, then $5/54 \leq y_n < 1/6$.*

Proof. Suppose $1/6 \leq y_n < 3/8$. Then the operation is A and $y_{n+1} \in [1/2, 13/16)$, so the next operation is B . If $x_n \equiv 1$, the residues are $1 \rightarrow 1 \rightarrow 2$ and $y_{n+2} < 15/32 < 1/2$, contradicting (2). If $x_n \equiv 4$, then $x_{n+1} \equiv 3$ while $y_{n+1} \geq 1/2$, again contradicting (2).

Suppose $0 \leq y_n < 5/54$. The operations are forced to be AAB , with $y_{n+1} \in [1/4, 7/18)$, $y_{n+2} \in [5/8, 5/6)$ and $y_{n+3} \in [3/16, 1/2)$. If $x_n \equiv 1$, the residues are $1 \rightarrow 1 \rightarrow 1 \rightarrow 2$, and $y_{n+3} < 1/2$ contradicts (2). If $x_n \equiv 4$, the residues are $4 \rightarrow 3 \rightarrow 4 \rightarrow 4$. The next operation is then A , which gives $x_{n+4} \equiv 3$ and $y_{n+4} \geq 17/32 > 1/2$, a contradiction. $\square$

Lemma 3.2. *If $5/54 \leq y_n < 1/6$ and $x_n \equiv 1$ or $4 \pmod{5}$, then the next four operations are A^2B^2 , that is, $(s_n, s_{n+1}, s_{n+2}, s_{n+3}) = (-1, -1, 1, 1)$. Moreover $y_{n+4} < 3/8$ and $x_{n+4} \equiv x_n \pmod{5}$.*

Proof. By Lemma 2.3 the values $y_{n+1} \in [7/18, 1/2)$, $y_{n+2} \in [5/6, 1)$, $y_{n+3} \in [1/2, 3/4)$ and $y_{n+4} \in [0, 3/8)$ are successively forced. The residues run through $1 \rightarrow 1 \rightarrow 1 \rightarrow 2 \rightarrow 1$ or $4 \rightarrow 3 \rightarrow 4 \rightarrow 4 \rightarrow 4$. $\square$

Lemma 3.3. *There is $M \geq N$ with $y_M < 3/8$ and $x_M \equiv 1$ or $4 \pmod{5}$.*

Proof. If $x_N \equiv 2$ (resp. 3), then (2) forces the operation B (resp. A), and $x_{N+1} \equiv 1$ (resp. 4). So we may start at a residue 1 or 4. From residue 1 the possible patterns are B , AB and AAB . Each one reaches residue 2 with $y < 3/4$, and one forced B then lands in $[0, 3/8)$ at residue 1. From residue 4 the possible patterns are B and AAB , the latter because after A the residue 3 forces another A by (2). If this does not already give $y < 3/8$, the orbit is at residue 4 with $y \in [3/8, 3/4)$. There A is excluded, since it

would lead to residue 3 with $y \geq 13/16$. So one more B lands in $[0, 3/8)$ at residue 4. In every case $M \leq N + 5$. $\square$

Proof of Theorem 1.1. Take M as in Lemma 3.3. Lemmas 3.1 and 3.2, applied alternately at $M, M+4, M+8, \dots$, show by induction that the sequence of operations from M on is $(A^2B^2)^\infty$. So s_n is periodic with period 4 for $n \geq M$, which contradicts Lemma 2.1. Hence for every N there is $n \geq N$ with $2 \mid x_n$ or $5 \mid x_n$. $\square$

Remark 3.4. As in [DN05, §§3–6], the proof combines arithmetic modulo the primes of $\mathcal{P}$ with interval arithmetic on the fractional parts y_n . Here the interaction is unusually tight. Modulo 5, each operation is forbidden at exactly one residue (A at 2, B at 3). Since the operation is a function of y_n , each forbidden residue becomes a window constraint (2) on y_n .

APPENDIX A. FORMAL VERIFICATION

All statements of this note are proved in Lean 4 with Mathlib in the files `DubC2/NearestStep.lean`, `DubC2/ProductState.lean`, `DubC2/NearestCert.lean` and `DubC2/Nearest.lean`, with Lemma 2.1 in `Z32/DubickasWord.lean`, available in the Github repository <https://github.com/rwst/lean-code>.

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