

# Another Counterexample to Little Finitistic Dimension Conjecture

## Introduction

This note gives a characteristic-zero construction of an algebra of infinite little finitistic dimension, following OpenAI [OAI]. Tang [Tang] gives refinements of OpenAI's construction and an extension to arbitrary fields. The argument of Section 3 is adapted from the square-zero-extension step in OpenAI [OAI], expressed here using a square-zero monad; see also [Tang, §6]. Related homological-dimension formulas for trivial extensions were established earlier by Minamoto and Yamaura [MY].

Sections 1–2 provide a different source of the required nilpotence. Section 1 uses the theory of homogenized enveloping algebras of Le Bruyn and Smith [LS] and Le Bruyn, Smith, and Van den Bergh [LSV], together with the graded Beilinson theorem of Minamoto and Mori [MM]. Section 2 constructs the endofunctor explicitly and computes its iterates using the classical Clebsch–Gordan decomposition and the eigenvalues of the Casimir operator.

## Construction

Let  $k$  be a field of characteristic zero, and let

$$\mathrm{Cat}_k = \mathrm{Mod}_{\mathrm{Perf}(k)}(\mathrm{Cat}^{\mathrm{perf}})$$

be the category of small idempotent-complete stable  $k$ -linear categories and exact  $k$ -linear functors. A quotient means a quotient in  $\mathrm{Cat}_k$ . All categories are  $\infty$ -categories, and  $\mathrm{Hom}$  denotes the  $k$ -module of maps.

## 1 A smooth proper category from filtered $\mathfrak{gl}_2$ -modules

Give  $U(\mathfrak{gl}_2)$  its Poincaré–Birkhoff–Witt filtration  $F_\bullet$ , where  $F_d U(\mathfrak{gl}_2)$  is spanned by products of at most  $d$  elements of  $\mathfrak{gl}_2$ , and put

$$B = \mathrm{Rees}_F U(\mathfrak{gl}_2) = \bigoplus_{d \geq 0} F_d U(\mathfrak{gl}_2) s^d \subseteq U(\mathfrak{gl}_2)[s].$$

Write  $\mathrm{Mod}^{\mathbf{Z}}(B)$  for the stable category of graded left  $B$ -modules, and  $\mathrm{Perf}^{\mathbf{Z}}(B)$  for its compact objects. Let

$$\mathrm{Perf}_{\mathrm{fd}}^{\mathbf{Z}}(B) = \left\{ M \in \mathrm{Perf}^{\mathbf{Z}}(B) : \bigoplus_{r \in \mathbf{Z}} M_r \in \mathrm{Perf}(k) \right\}.$$

Thus finite-dimensionality is total over the grading and homotopy degrees. Define

$$\boxed{\mathcal{C} = \mathrm{Perf}^{\mathbf{Z}}(B) / \mathrm{Perf}_{\mathrm{fd}}^{\mathbf{Z}}(B)}, \quad \mathcal{P} = \mathrm{Ind}(\mathcal{C}).$$

We use  $M(r)_d = M_{d+r}$  for grading shifts and  $[r]$  for suspension.

For computations, write  $\mathfrak{gl}_2 = \mathfrak{sl}_2 \oplus kt$ , with  $t$  the identity matrix, and choose the standard generators  $E, F, H$  of  $\mathfrak{sl}_2$ . We use the same letters for their degree-one lifts in  $B$ . The algebra  $B$  is generated by  $E, F, H, t, s$ , with  $s, t$  central and relations

$$HE - EH = 2sE, \quad HF - FH = -2sF, \quad EF - FE = sH. \quad (1)$$

The Poincaré–Birkhoff–Witt theorem gives that  $B$  is a domain,  $s$  is regular, and  $B/(s) = k[E, F, H, t]$ . The homogenization of  $U(\mathfrak{sl}_2)$  is noetherian, Koszul, and Artin–Schelter regular of dimension and Gorenstein parameter four [LS, §2] and [LSV, Corollary 2.7 and Remark 2.8]. Adjoining the central degree-one variable  $t$  gives these properties for  $B$  with dimension and parameter five.

In particular, the  $k(r)$  are compact, where  $k = B/B_{\geq 1}$ , and

$$\mathrm{Perf}_{\mathrm{fd}}^{\mathbf{Z}}(B) = \mathrm{thick}\{k(r) : r \in \mathbf{Z}\}, \quad \mathcal{P} \simeq \mathrm{Mod}^{\mathbf{Z}}(B) / \mathrm{Loc}\{k(r) : r \in \mathbf{Z}\}.$$

Here thick denotes closure under finite stable operations and retracts, and Loc also allows all colimits. When working in  $\mathcal{P}$ , we use the same notation for a graded  $B$ -module and its image in the quotient. The quotient functor preserves compact objects.

The graded Beilinson theorem gives a compact generator

$$p = \bigoplus_{i=0}^4 B(i)$$

of  $\mathcal{P}$ , with

$$\mathrm{Hom}_{\mathcal{C}}(B(i), B(j)) \simeq B_{j-i} \quad (j - i \geq -4), \quad (2)$$

for  $i, j \in \mathbf{Z}$ , where the right side is discrete and  $B_d = 0$  for  $d < 0$  [MM, Propositions 4.3–4.4 and Theorem 4.12]. Thus  $\mathrm{End}_{\mathcal{C}}(p)$  is a finite-dimensional directed algebra of finite global dimension, and  $\mathcal{C}$  is smooth and proper over  $k$ .

For  $n \geq 0$ , let  $V_n = \mathrm{Sym}^n(k^2)$  with its standard  $\mathfrak{gl}_2$ -action  $\rho_n$ ; in particular,  $t$  acts by  $n$ . For  $m \geq 0$ , define

$$M_m = V_m \otimes_k k[s],$$

with  $V_m$  in degree zero and

$$a_{M_m} = s\rho_m(a) \quad (a \in \mathfrak{gl}_2).$$

Thus  $t_{M_m} = ms$ . These finitely generated graded  $B$ -modules are perfect, so  $M_m \in \mathcal{C}$ . Multiplication by  $s$  gives an equivalence in  $\mathcal{C}$

$$s : M_m(-1) \xrightarrow{\sim} M_m, \quad (3)$$

since its cofiber before taking the quotient is a finite direct sum of copies of  $k$ .

The  $s$ -local object  $M_m[s^{-1}] = V_m \otimes_k k[s, s^{-1}]$  is right orthogonal to every  $k(r)$ , and  $M_m[s^{-1}] \simeq M_m$  in  $\mathcal{P}$  by (3). Consequently,

$$\mathrm{Hom}_{\mathcal{P}}(B(i), M_m) \simeq \mathrm{Hom}_{\mathrm{Mod}^{\mathbf{Z}}(B)}(B(i), M_m[s^{-1}]) \simeq V_m,$$

and hence

$$\mathrm{Hom}_{\mathcal{C}}(p, M_m) \simeq V_m^{\oplus 5}. \quad (4)$$

In particular,  $M_m \neq 0$ , and these mapping  $k$ -modules are discrete and finite-dimensional.

## 2 The Casimir operator gives unbounded nilpotence orders

Tensoring with the dual standard representation  $V_1^\vee$  defines an endofunctor  $F$  on graded  $B$ -modules. Explicitly, its underlying graded  $k$ -module is  $V_1^\vee \otimes_k M$ , with

$$\begin{aligned} a_{F(M)} &= \rho_1^\vee(a) \otimes s_M + 1 \otimes a_M \quad (a \in \mathfrak{gl}_2), \\ s_{F(M)} &= 1 \otimes s_M. \end{aligned} \tag{5}$$

These actions satisfy (1). Since  $t$  acts by  $-1$  on  $V_1^\vee$ , the central parameter transforms by

$$t_{F(M)} = 1 \otimes (t_M - s_M).$$

The functor  $F$  is exact, preserves colimits, and sends  $k(r)$  to  $k(r)^{\oplus 2}$ , so it descends to  $\mathcal{P}$ .

The graded  $B$ -linear map

$$B \otimes_k V_1^\vee \longrightarrow F(B), \quad b \otimes v \longmapsto b \cdot (v \otimes 1)$$

is an isomorphism modulo  $s$ . Both modules are bounded below in degree and  $s$ -torsion-free, so induction on degree proves that the map is an isomorphism. Consequently,  $F(B(j)) \simeq B(j)^{\oplus 2}$ , and  $F$  preserves compact objects.

The element

$$u = H^2 + 2sH + 4FE - t(t + 2s) \in B_2$$

is central. Its image modulo  $s$  is  $H^2 + 4FE - t^2 \neq 0$ , so  $u$  is a non-zero-divisor. Define an exact colimit-preserving endofunctor of  $\mathcal{P}$  by

$$K(X) = \text{cofib}(u : F(X) \longrightarrow F(X)(2)). \tag{6}$$

It preserves compact objects and therefore restricts to  $\mathcal{C}$ . For  $0 \leq i, j \leq 4$ , formula (2) gives

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(B(i), KB(j)) &\simeq \text{cofib}(u : B_{j-i}^{\oplus 2} \longrightarrow B_{j+2-i}^{\oplus 2}) \\ &\simeq (B/(u))_{j+2-i}^{\oplus 2}. \end{aligned} \tag{7}$$

In particular,  $\text{Hom}_{\mathcal{C}}(p, Kp)$  is discrete and finite-dimensional.

On  $V_n$ , the Casimir  $H^2 + 2H + 4FE$  acts by  $n(n+2)$ . For  $m \geq 1$ , the Clebsch–Gordan decomposition, as an  $\mathfrak{sl}_2$ -representation, is

$$V_1^\vee \otimes_k V_m \simeq V_{m+1} \oplus V_{m-1}.$$

On  $F(M_m)$ , the parameter  $t$  acts by  $(m-1)s$ , so  $u$  acts on these two summands by

| summand                | $u$                                        |
|------------------------|--------------------------------------------|
| $V_{m+1} \otimes k[s]$ | $((m+1)(m+3) - (m-1)(m+1))s^2 = 4(m+1)s^2$ |
| $V_{m-1} \otimes k[s]$ | $0$ .                                      |

The first cofiber is finite-dimensional and vanishes in  $\mathcal{P}$ . The second summand is  $M_{m-1}$ , so its cofiber becomes  $M_{m-1}(2) \oplus M_{m-1}[1]$ , which is  $M_{m-1} \oplus M_{m-1}[1]$  by (3). For  $m = 0$ , only the first summand occurs, with  $u = 4s^2$ . Thus

$$K(M_0) = 0, \quad K(M_m) \simeq M_{m-1} \otimes_k (k \oplus k[1]) \quad (m \geq 1). \tag{8}$$

For  $0 \leq j \leq m$ , formula (8) gives

$$K^j M_m \simeq M_{m-j} \otimes_k (k \oplus k[1])^{\otimes j}. \tag{9}$$

These objects are compact, and

$$K^{m+1} M_m = 0, \quad K^m M_m \neq 0. \tag{10}$$

### 3 Transforming nilpotence to projective dimension using a square-zero monad

Form the square-zero monad

$$T = \text{id}_{\mathcal{P}} \oplus K$$

and its category of modules  $\mathcal{D} = \text{Mod}_T(\mathcal{P})$ . Write  $\text{Free}$  and  $U$  for the free and forgetful functors, and let  $z : \mathcal{P} \rightarrow \mathcal{D}$  equip objects with the zero action. Put  $L_m = zM_m$ . For every  $y \in \mathcal{P}$ ,

$$\text{Hom}_{\mathcal{D}}(L_m, zy) \simeq \bigoplus_{j=0}^m \text{Hom}_{\mathcal{P}}(K^j M_m, y)[-j]. \quad (11)$$

Moreover,  $L_m$  is compact.

Indeed, for  $x \in \mathcal{P}$ , there is a cofiber sequence

$$z(Kx) \longrightarrow \text{Free}(x) \longrightarrow zx. \quad (12)$$

The forgetful map

$$U_* : \text{Hom}_{\mathcal{D}}(zx, zy) \longrightarrow \text{Hom}_{\mathcal{P}}(x, y)$$

has a section induced by  $z$ , since  $Uz \simeq \text{id}_{\mathcal{P}}$ . Consequently, (12) gives

$$\text{Hom}_{\mathcal{D}}(zx, zy) \simeq \text{Hom}_{\mathcal{P}}(x, y) \oplus \text{Hom}_{\mathcal{D}}(zKx, zy)[-1].$$

Since  $K^{m+1}M_m = 0$ , iteration proves (11). Since  $U$  preserves filtered colimits,  $\text{Free}$  preserves compact objects; backward induction in (12), using the compactness of the  $K^j M_m$ , proves that  $L_m$  is compact. The object  $g = \text{Free}(p)$  is a compact generator of  $\mathcal{D}$ , with an equivalence of  $k$ -modules

$$\text{End}_{\mathcal{D}}(g) \simeq \text{End}_{\mathcal{C}}(p) \oplus \text{Hom}_{\mathcal{C}}(p, Kp). \quad (13)$$

Thus  $\text{End}_{\mathcal{D}}(g)$  is discrete and finite-dimensional by (2) and (7). Moreover,

$$\text{Hom}_{\mathcal{D}}(g, L_m) \simeq V_m^{\oplus 5}$$

is discrete and finite-dimensional by (4).

Equation (9) exhibits  $M_0[m]$  as a direct summand of  $K^m M_m$ . Hence (11), with  $y = M_0$ , exhibits

$$\text{Hom}_{\mathcal{C}}(M_0[m], M_0)[-m] \simeq \text{End}_{\mathcal{C}}(M_0)[-2m]$$

as a direct summand of  $\text{Hom}_{\mathcal{D}}(L_m, L_0)$ . Since  $M_0 \neq 0$ , its identity gives

$$\pi_{-2m} \text{Hom}_{\mathcal{D}}(L_m, L_0) \neq 0.$$

Finally, set  $R = \text{End}_{\mathcal{D}}(g)^{\text{op}}$  and identify  $\mathcal{D} \simeq \text{Mod}(R)$ . Then  $R$  is a discrete finite-dimensional  $k$ -algebra, and the  $L_m$  are discrete finite-dimensional perfect left  $R$ -modules satisfying

$$\boxed{2m \leq \text{pd}_R(L_m) < \infty.}$$

In particular, the little finitistic dimension of  $R$  is infinite.

## References

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