

MARKED POSITIVE HANDLESIDE PENTAGONS IN COMPRESSION BODIES: RELATIVE COHERENCE

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ABSTRACT. Let $W : X \rightarrow Y_{\text{geom}}$ be a completed oriented compression body with $N \geq 3$ ordered product 1-handles and a supplied protected three-handle sector. We realize one positive handleslide pentagon by five diffeomorphisms with compatible complete endpoint markings. An isotopy fixed on the entire protected boundary collar identifies the two-edge and three-edge route composites, so the comparison takes place at the level of marked maps rather than homology matrices.

The construction fixes the marked disk systems at all five vertices before choosing any edge map, so incident edges use the same endpoint data. The cut-ball maps intertwine all fifteen disk pairings on common full-normal intervals and therefore descend smoothly through the paired faces. Extending these quotient maps by the identity gives global representatives. The same maps satisfy the finite compatibility conditions, including at $N = 3$, when the unused-handle domains are empty but remain part of the typed construction.

The resulting boundary maps preserve the embedded meridian and product-framed dual-longitude rows, so their homological actions have blocks M_e and M_e^{-T} with no mixed shear. They also preserve the same rational disk-kernel Lagrangian. Walker's forward multiplication law lifts the relative isotopy to a literal weight-zero equality of mapping cylinders, including the seam and route weights.

1. INTRODUCTION

Consider a positive three-disk handleslide pentagon in a completed compression body. On homology, its two paths are represented by products of elementary matrices, and the two products agree. Does this agreement lift to equality of fully marked relative mapping cylinders? The arithmetic alone does not supply the actual diffeomorphisms, compatible endpoint markings, or relative isotopy needed to compare the composites.

For one positive pentagon in a supplied standard three-handle sector of a completed compression body, we construct a two-edge route and a three-edge route whose composites are isotopic relative to the entire protected marking. The resulting marked mapping cylinders satisfy a literal weight-zero Walker equality. The construction retains signed cut faces with full normal collars and embedded meridian and dual-longitude rows. Their matrix identity is an exact integral coordinate check for the geometric relation.

The handleslide pentagon itself is known: Hatcher–Thurston cut-system relations contain pentagon cells [HT80], and the Heegaard-diagram complex of Juhász–Thurston–Zemke contains the directly relevant three-curve handleslide pentagon [JTZ21, arXiv v4, Appendix A, Definition A.3(6) and Figure 22]. The priority-sensitive content here is only the printed refinement—one supplied protected sector, coordinated selected representatives, complete retained markings, an isotopy fixed on the full protected collar, and the resulting literal Walker relation. Without those qualifiers, the headline reduces to known move-complex coherence. In the sutured compression-body setting, Qin proves simple connectivity after attaching six kinds of triangle, square, and pentagon cells to a complex of isotopy classes of cut-systems [Qin25, Theorem 1.1, Definitions 2.5 and 4.2, and the opening of §4]. Those vertices are isotopy classes, whereas here

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the selected representatives carry the prescribed full-collar data. The full-collar isotopy of the selected complete-marked routes is the narrower question addressed here.

1.1. Coordinating the geometric representatives. The pentagon has five vertices and five directed edges: a two-edge route and a three-edge route with common endpoints. It is tempting to realize each edge separately by a standard handleslide map and compare the route matrices afterward. That procedure leaves three gaps.

First, an endpoint shared by two edges can acquire two unrelated cut-ball charts, collar depths, or boundary parameterizations. Equality of endpoint surfaces then does not mean equality of the complete endpoint germs needed for composition. Second, agreement of a boundary map with disk pairings at normal coordinate zero is insufficient for smooth quotient descent: both directed pairing identities must hold on one full-normal interval, with all jets. Third, the cut-ball map, its quotient descent, and its global extension have different domains and roles. In particular, the full two-sided collar of the exterior attachment disk exists only after descent and global extension; it cannot be assigned retroactively to a cut-level map.

The proof resolves all three gaps by reversing the order of construction. All five vertices are built first, with one common positive collar depth and one literal collection of endpoint data at every shared endpoint. Only then are the ten incidence-labelled endpoint markings and charts and the five edge representatives selected. This temporal order is part of the theorem, rather than an organizational preference.

1.2. From the supplied sector to the Walker relation. Fix $g \geq 1$, $N \geq 3$, a completed oriented compression body $W : X \rightarrow Y_{\text{geom}}$, three ordered distinguished handles, an exterior completed row, and the fully protected standard sector specified in Definition 2.1. The sector includes literal product coordinates, positive carrier push-offs, three compatible zero-framed band germs, simultaneous smoothing data, reserve regions, and one common exterior parameterization. Thus the theorem does not search an arbitrary compression body for standard position.

The proof of Theorem 4.1 begins with five independently constructed vertex disk systems and their cut balls. Ten endpoint markings and charts are coordinated before the five cut maps $\tilde{F}_e$ are chosen. Each cut map carries the actual face, ribbon, spoke, detector, meridian, and dual-longitude data and satisfies all fifteen full-normal pairing identities.

These identities let every $\tilde{F}_e$ descend to a smooth map $f_{e,U}$ of the active quotient sector and then extend by the coordinate identity to a global map F_e of W . The construction keeps these three stages distinct, so a cut-level assertion is not confused with a global one. The same maps satisfy the finite compatibility conditions, including the five unused-handle records whose domains are empty when $N = 3$. Their geometric proof is given in Section 11.

Because the maps transport embedded meridian and product-framed dual-longitude rows before homology is read off, their active homology blocks are M_e and M_e^{-T} , with both mixed blocks zero. Every map and its inverse preserve the same rational disk-kernel Lagrangian. A cut-level rigidity path is then kept fixed on a full protected collar, descended through the pairings for every parameter value, and extended globally. It gives a relative isotopy between the two route composites.

The relative isotopy first yields equality of the marked mapping classes. Walker's forward composition law is then applied in its Lagrangian-weighted form [GM13]. Since the common Lagrangian is preserved in both directions, each ordered seam Maslov index is zero, so the two products of weight-zero mapping cylinders are literally equal.

The local analytic-topological ingredients are used with their relative conditions exposed. Relative smoothing of a doubled three-manifold homeomorphism uses the classical relative homeomorphism-approximation theorem, in the form stated by Pardon [Par21, arXiv v2, Theorem 4.6]; the rel-boundary ball rigidity step uses the Smale-conjecture consequence proved by Hatcher [Hat83]; and the passage from fixed boundary and normal-tube data to a smooth fixed-germ isotopy uses Cerf's relative results [Cer61]. These results enter at specifically marked steps. They do not replace the explicit endpoint, pairing, or collar constructions.

1.3. Homology as an exact coordinate check. Let S_v record the three meridian descendants at a vertex and put $M_e = S_w S_v^{-1}$ for an edge $e : v \rightarrow w$. Direct multiplication gives the same matrix along the two-edge and three-edge routes, and the inverse-transpose products agree as well. We record this calculation early because it is a useful typing check: for example, the two edges involving the letter B have different source roles and different matrices. We do not use the common product to infer a mapping-class relation. A homological representation need not detect the geometry with complete marking, the required full-collar behavior, or a relative isotopy. In the proof, the direction of implication is therefore

$$\begin{aligned} & \text{embedded marked geometry} \longrightarrow \text{relative isotopy} \\ & \longrightarrow \text{mapping-class equality} \longrightarrow \text{weight-zero Walker equality,} \end{aligned}$$

while the matrix arithmetic remains an independent exact check.

1.4. Exact scope. For every supplied protected sector the theorem constructs one coordinated system. It asserts neither uniqueness nor relations among several pentagons, and it does not by itself imply stabilization, global Heegaard invariance, or an operator-level comparison. The precise boundary is stated in Section 15.

1.5. Organization. Sections 2–4 fix the input, type the pentagon, and state the witness theorem. The middle sections follow the construction diagram: vertex data, endpoint charts, relative extensions, cut maps, descent, compatibility, and homology. Sections 13 and 14 prove the relative isotopy and weight-zero Walker equality. The mathematical appendices retain the local cut, collar, and positive-slide arguments needed by those proofs.

2. COMPRESSION BODIES AND SUPPLIED PROTECTED SECTORS

Before a route can be compared, the local geometry on which it acts must be part of the input. This section identifies the completed compression body, the protected three-handle sector, and the precise meaning of “supplied.”

The theorem is local in a precise sense. Its input is not an arbitrary completed compression body in which a suitable three-handle configuration must somehow be found. Rather, the configuration, all of its product coordinates, and the room needed for the subsequent constructions are supplied. This distinction is the first scope boundary of the result.

2.1. The completed compression body. Fix integers $g \geq 1$ and $N \geq 3$. We write

$$(1) \quad W : X \longrightarrow Y_{\text{geom}}$$

for a compact oriented compression body obtained from $X \times [0, 1]$, where X is a closed oriented surface of genus g , by attaching N ordered product 1-handles to the top. The arrow in (1) records the incoming and outgoing boundaries; it is not a map of surfaces. The outgoing boundary is denoted $\partial_+ W = Y_{\text{geom}}$.

A *completed row* consists of an ordered complete co-core system together with the data that will be held fixed under relative diffeomorphisms: signed cut-face charts, meridian and product-framed dual-longitude germs, ribbon and spoke germs, detector arcs, inward and two-sided normal collars, and an outgoing parameterization. Choose three pairwise distinct indices

$$i, j, k \in \{1, \dots, N\}.$$

The *exterior completed row* $\mathfrak{C}_{\text{ext}}$ contains all old data and the co-cores D_t , with all their marking data, for $t \notin \{i, j, k\}$. It also includes a fixed identification of the old homology summand and a rational disk-kernel Lagrangian

$$(2) \quad \lambda_g \subset H_1(X; \mathbb{Q}).$$

Every construction below is pointwise the identity on this exterior row. When $N = 3$, the collection of unused co-cores is empty; all statements involving it are then read as empty sums or vacuous families.

2.2. The protected three-handle sector.

Definition 2.1 (Supplied protected sector). A fully protected standard sector $\mathcal{L}_{ijk}$ consists of the following mutually compatible smooth, oriented, product-parameterized data.

(i) A top product collar

$$(3) \quad \chi : X \times [1 - \tau, 1] \hookrightarrow W$$

and a closed oriented disk $Q \subset X \times \{1\}$, equipped with a fixed disk chart and a common normal-height coordinate.

(ii) Six mutually disjoint attaching feet $q_r^\pm \in \text{int } Q$, $r \in \{i, j, k\}$, and three oriented, product-framed handle embeddings

$$(4) \quad \iota_r : [-2, 2] \times D^2 \hookrightarrow W, \quad D_r = \iota_r(\{0\} \times D^2).$$

The positive normal to D_r is $d\iota_r(\partial_x)$. The input contains full co-core collars and fixed meridian and dual-longitude product germs.

(iii) Separated height slabs, a number $\varepsilon > 0$, and the mutually disjoint product push-offs

$$(5) \quad i_0; \quad j_0, j_\varepsilon; \quad k_0, k_\varepsilon, k_{2\varepsilon}.$$

Every push-off carries its orientation, positive normal, parameterized full collar, and product-zero framing. A nonzero subscript records an auxiliary parallel copy, not an additional disk in a completed row.

(iv) Three oriented product-zero-framed rectangular band germs, with their full collars and smoothing boxes:

$$(6) \quad \begin{array}{c|c|c} \text{band} & \text{moving attachment} & \text{positive carrier attachment} \\ \hline \beta_0 & j_0 & k_\varepsilon \\ \alpha & i_0 & j_\varepsilon \\ \beta_1 & j_\varepsilon & k_{2\varepsilon}. \end{array}$$

Each band germ is supplied by a rectangular $\bar{\gamma}$ in the high boundary whose interior is pushed slightly inward through the frozen high-boundary product collar, with its attaching short sides fixed. Its product-zero framing is the transported constant normal of that rectangle. Thus these are boundary-parallel product bands in the specified charts, not arbitrary embedded framed rectangles. There is one product-normal translation ν_ε , defined on the complete relevant band and smoothing germs, such that

$$(7) \quad \nu_\varepsilon(j_0) = j_\varepsilon, \quad \nu_\varepsilon(k_\varepsilon) = k_{2\varepsilon}, \quad \nu_\varepsilon(\beta_0) = \beta_1$$

to every order in the product coordinates. Thus β_1 is the literal normal translate of β_0 , not merely an independently chosen parallel band.

(v) One simultaneous oriented corner-rounding rule for the four descendant disks

$$(8) \quad \begin{aligned} J &= j_0 \#_{\beta_0} k_\varepsilon, \\ J^+ &= j_\varepsilon \#_{\beta_1} k_{2\varepsilon}, \\ I_A &= i_0 \#_{\alpha} j_\varepsilon, \\ I_\star &= \text{smooth}(i_0 - \alpha - j_\varepsilon - \beta_1 - k_{2\varepsilon}) \\ &= i_0 \#_{\alpha} J^+ = (I_A) \#_{\beta_1} k_{2\varepsilon}. \end{aligned}$$

The two descriptions of $I_\star$ specify the same embedded disk with the same full product germ. Moreover $J^+ = \nu_\varepsilon(J)$ with its entire product germ. Retain a closed subdisk $\Omega_j \in I_A$, with a full collar, in the realized j_ε -lobe. The fixed parameterizations of I_A and $I_\star$ agree on Ω_j ; every face sector, ribbon attachment, and spoke germ belonging to that lobe lies there away from the β_1 -attachment. Thus the AAC moving-face map carries this attached j -portion and its complete marking rather than treating it as an omitted spectator.

(vi) The rounded active sector

$$(9) \quad U_{ijk} = \text{Round} \left(\chi(Q \times [1 - \tau, 1]) \cup \bigcup_{r \in \{i, j, k\}} \iota_r([-2, 2] \times D^2) \right),$$

its unique exterior interface disk E , and a fixed bicollar $E \times [-\rho, \rho]$. The symbol E will always denote this global interface. The edge-local interface disks introduced in Section 6 are denoted E_e and are different objects.

- (vii) The exterior attachment $(\mathfrak{C}_{\text{ext}}, \lambda_g)$ and one common exterior parameterization to a fixed abstract outgoing surface Y . It is literally product-coordinate identity on the old region and all unused handles, and its support is disjoint from the active data above.
- (viii) Reserve neighborhoods and positive separation constants for the seven cut-face sites, six branch corridors, one detector corridor, the five possible internal-interface sites, and a complementary-disk annulus with an inward collar. These are capacity and clearance data. They do not themselves constitute the faces, ribbons, pairings, atlases, or amalgamations constructed later.

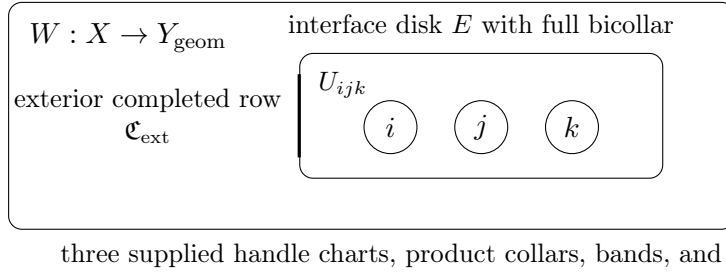


FIGURE 1. Schematic incidence data for the supplied protected sector. The diagram records the active three-handle region, the exterior row, and their single collared interface. It does not prescribe an embedding, page orientation, band placement, framing arrow, or collar width beyond Definition 2.1.

The disjointness in this definition includes the interiors, complete collars, and smoothing neighborhoods of the three bands, except at their displayed attachments on distinct parallel disk copies. In particular,

$$(10) \quad j_0 \cap j_\varepsilon = \emptyset, \quad k_0 \cap k_\varepsilon = k_0 \cap k_{2\varepsilon} = k_\varepsilon \cap k_{2\varepsilon} = \emptyset,$$

and

$$(11) \quad \alpha \cap (\beta_0 \cup \beta_1) = \emptyset, \quad \beta_0 \cap \beta_1 = \emptyset$$

away from those prescribed attachments. These geometric separations, not identities in homology, are what make the five disk rows below embedded.

2.3. The local positive two-site theorem used as input. The five-edge argument does not manufacture the elementary slide class from a homology matrix. It uses the following local theorem, stated here in the exact form consumed below and proved in Section F.

Theorem 2.2 (Positive two-site slide with complete markings). *Let D_1, D_2 be disjoint oriented co-core disks in a frozen oriented two-handle sector. Write $m_r = \partial D_r$, and equip this old ordered disk system with a complete local marking: signed disk-face charts with full normal collars, actual embedded meridian and product-framed dual-longitude rows, ribbon and detector germs, and one fixed exterior interface disk E_γ with its product neighborhood. The old dual longitudes ℓ_1, ℓ_2 meet the meridians geometrically by*

$$(12) \quad \begin{aligned} m_r \pitchfork \ell_r &= \{x_r\} \text{ with positive sign,} \\ m_1 \cap m_2 &= \ell_1 \cap \ell_2 = \emptyset, \quad m_r \cap \ell_s = \emptyset \quad (r \neq s). \end{aligned}$$

In particular, $m_r \cdot \ell_s = \delta_{rs}$.

Fix a constant-height positive product push-off in the supplied co-core collar, D_2^{po} with

$$D_2^{\text{po}} \cap (D_1 \cup D_2) = \emptyset,$$

a positive product-zero-framed band γ from D_1 to D_2^{po} , supplied as a high-boundary rectangle $\bar{\gamma}$ whose interior is pushed slightly inward through the frozen product collar while its attaching short sides remain fixed, and the slid disk system

$$D_1^\gamma = D_1 \natural_\gamma D_2^{\text{po}}, \quad D_2^\gamma = D_2.$$

The attaching rectangles miss x_1, x_2 . The band interior and its smoothing box are disjoint from the two co-cores and every old protected marking except at the prescribed attaching rectangles. Fix the prescribed embedded product-zero-framed dual band $\gamma^\vee$: it is the narrow parallel rectangle routed on the specified side of the boundary trace of γ , from the ℓ_2 -side to a positive push-off of the ℓ_1 -side, with its second short side oriented oppositely to ℓ_1 . Its interior misses the four old curves except at those prescribed ends. Set geometrically

$$(13) \quad m_1^\gamma = \partial D_1^\gamma, \quad m_2^\gamma = m_2, \quad \ell_1^\gamma = \ell_1, \quad \ell_2^\gamma = \ell_2 \natural_{\gamma^\vee} (-\ell_1)^+.$$

Assume that the fixed exterior interface E_γ cuts off a closed product-ended two-handle subsector containing the two co-cores, this push-off, both prescribed bands, their smoothing boxes, and all local protected marking germs. Its product neighborhood is disjoint from those germs. Thus E_γ is part of the supplied relative boundary, not a support disk selected by the proof. Then the slid ordered disk system admits a complete local marking whose actual meridian and product-framed dual-longitude rows are precisely those in (13), while retaining the fixed exterior-interface germ. The slid disk system is complete in the two-handle sector. Cutting the two completely marked systems gives marked balls

$$B_-^{\text{LS}}, \quad B_+^{\text{LS}}.$$

Moreover, there is an orientation-preserving diffeomorphism

$$\phi_\gamma^{\text{cut}} : B_-^{\text{LS}} \longrightarrow B_+^{\text{LS}}$$

with the following properties.

- (i) Its boundary map carries the complete old cut-cellulation to the complete slid cut-cellulation. It transports the actual embedded meridian and dual-longitude rows, every marked ribbon and detector, and is the coordinate identity on E_γ and its product germ.
- (ii) For each of the two cut disks it intertwines the minus-to-plus pairing and its inverse on a common full-normal collar. In the paired coordinates the normal variable is preserved, so the two descended expressions agree with all jets after regluing.
- (iii) On a full outer collar it has the prescribed product formula. Its relative class is fixed as follows. The complete cellwise boundary reference, denoted B_0 , is first smoothed without changing any protected germ. A collar-product radial ball extension is then smoothed relative to that full collar. The resulting diffeomorphism lies in the topological relative class B_5 of the radial extension: if p, q are the two inner-ball maps with the same boundary trace, their quotient fixes the inner sphere and the explicit Alexander family

$$(14) \quad H_s(x) = \begin{cases} s q^{-1} p(x/s), & |x| \leq s, \\ x, & s \leq |x| \leq 1, \end{cases} \quad 0 < s \leq 1, \quad H_0 = \text{id},$$

The path H_s runs from id to $q^{-1}p$; hence $q \circ H_s$, extended by the identity on the protected collar, is the required relative topological comparison from q to p . No uniqueness of a smooth representative in this class is asserted.

- (iv) Regluing the two paired face systems and extending by the identity outside the local sector gives a positive relative sector diffeomorphism. The four rows in (13) are embedded with their complete product framings and satisfy

$$m_r^\gamma \cdot \ell_s^\gamma = \delta_{rs}, \quad m_r^\gamma \cdot m_s^\gamma = 0, \quad \ell_r^\gamma \cdot \ell_s^\gamma = 0.$$

Setwise, the two distinct meridians and the two distinct longitudes are disjoint, and

$$m_r^\gamma \cap \ell_s^\gamma = \emptyset \quad (r \neq s).$$

The diffeomorphism carries the actual embedded framed rows

$$m_1 \mapsto m_1^\gamma, \quad m_2 \mapsto m_2^\gamma, \quad \ell_1 \mapsto \ell_1^\gamma, \quad \ell_2 \mapsto \ell_2^\gamma,$$

where

$$(15) \quad [m_1^\gamma] = [m_1] + [m_2], \quad [m_2^\gamma] = [m_2], \quad [\ell_1^\gamma] = [\ell_1], \quad [\ell_2^\gamma] = [\ell_2] - [\ell_1].$$

These are consequences of the embedded marked rows, not definitions of them.

In the present paper, Theorem 2.2 is used only for the two endpoint balls, their complete boundary dictionaries and pairings, the nonempty $B0/B5$ relative class, and the embedded row transport. Its local sector is inserted into each of the five typed roles in Equation (21); all old, unused-handle, and spectator data are added independently and are fixed by the identity. No local Walker conclusion is imported into the pentagon proof.

Definition 2.3 (Protected input tuple). The universal datum of the main theorem is the nine-component tuple

$$(16) \quad \mathcal{T} = (g, N, W : X \rightarrow Y_{\text{geom}}, i, j, k, \mathfrak{C}_{\text{ext}}, \lambda_g, \mathcal{L}_{ijk}),$$

where $g, N, W, i, j, k, \mathfrak{C}_{\text{ext}}, \lambda_g$ are as above and $\mathcal{L}_{ijk}$ is one supplied sector in the sense of Definition 2.1.

Remark 2.4 (Exact hypothesis boundary). Existence of $\mathcal{L}_{ijk}$ is a hypothesis. No argument in this paper extracts such a disk Q , the six feet, the three compatible bands, or their reserve system from an arbitrary completed chart. The theorem is uniform in $g \geq 1$ and $N \geq 3$ only because everything outside this one supplied sector is extended by the literal identity. It does not assert a simultaneous family of protected sectors or any uniqueness of the supplied localization.

For each vertex v we will construct a completed row $\mathfrak{C}_v$ by adjoining one ordered active triple to $\mathfrak{C}_{\text{ext}}$ and constructing its marking independently. The vertex rows, charts, collars, and outgoing parameterizations are fixed before any edge map is chosen; none is defined by transport along an incoming edge. Thus incident edges will meet at the same already specified endpoint data.

3. THE FIVE-EDGE HANDLESIDE PENTAGON

The pentagon records two histories of the same three interacting slides. Its vertices carry different disk rows, so the first task is to specify which disk, band, and direction belongs to each edge before any comparison of routes is attempted.

We next fix the directed graph, the five disk rows, and the role of every disk at every edge. This typing is part of the geometry: the two edges whose names contain B do not have the same moving disk or the same homology matrix.

3.1. Vertices, words, and route order. For current descendants p and q , write

$$p \leftarrow q$$

for the supplied positive, product-zero-framed slide of p over the specified positive push-off of q . Set

$$(17) \quad A = (i \leftarrow j), \quad B = (j \leftarrow k), \quad C = (i \leftarrow k).$$

Words are chronological from left to right. Diffeomorphisms, as usual, compose right factor first.

The vertex and directed edge sets are

$$(18) \quad V = \{0, B, A, AC, \star\}, \quad E_5 = \{0B, B\star, 0A, AAC, AC\star\}.$$

The graph has a two-edge upper route and a three-edge lower route, as shown in Figure 2.

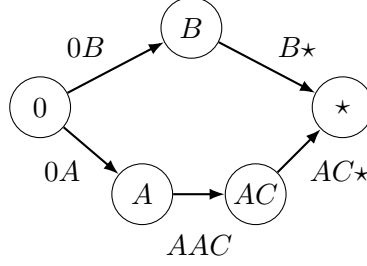


FIGURE 2. The directed two-versus-three handleslide pentagon.

Theorem A (Protected-sector geometric pentagon). For every supplied protected three-handle sector, the five slides can be chosen with compatible complete endpoint markings so that the two route composites are isotopic through maps fixed on the protected collar. Their boundary maps have no meridian–longitude shear, preserve one rational Lagrangian, and determine the same weight-zero Walker cylinder. The formal witness package is stated in Theorem 4.1; the relative isotopy and Walker equality are isolated again in Theorems 13.5 and 14.1.

The figure records only the route skeleton. To compare the two composites, we must also record the actual complete disk row carried at each vertex. The five rows below are constructed independently from the supplied descendants; an edge label does not mean that its map has already been applied to the row at the preceding vertex. This keeps the graph of route choices separate from the endpoint data on which the later edge maps act. The active disk rows at the five vertices are defined directly from the supplied descendants:

$$(19) \quad \begin{array}{c|ccc} v & D_i^v & D_j^v & D_k^v \\ \hline 0 & i_0 & j_0 & k_0 \\ B & i_0 & J & k_0 \\ A & I_A & j_0 & k_0 \\ AC & I_\star & j_0 & k_0 \\ \star & I_\star & J & k_0. \end{array}$$

Section 5 proves that these are five complete, independently constructed disk systems; no row in (19) is obtained by applying an edge map to a previous row.

For later use, if F_e denotes the eventual global representative of the edge e , define

$$(20) \quad F_L = F_{B\star} \circ F_{0B}, \quad F_R = F_{AC\star} \circ F_{AAC} \circ F_{0A}.$$

Equation (20) is notation only at this stage. Neither a representative nor a relation between the two composites has yet been chosen.

3.2. The five typed positive slides. For an edge $e : v \rightarrow w$, let $d_{e,1}^v, d_{e,2}^v, d_{e,s}^v$ denote respectively its moving, carrier, and spectator disks at the source. The five local instances are exactly

$$(21) \quad \begin{array}{c|ccc|cc} e & d_{e,1}^v & d_{e,2}^v & \text{positive carrier push-off} & \text{band} & d_{e,s}^v \\ \hline 0B & j_0 & k_0 & k_\varepsilon & \beta_0 & i_0 \\ B\star & i_0 & J & J^+ & \alpha & k_0 \\ 0A & i_0 & j_0 & j_\varepsilon & \alpha & k_0 \\ AAC & I_A & k_0 & k_{2\varepsilon} & \beta_1 & j_0 \\ AC\star & j_0 & k_0 & k_\varepsilon & \beta_0 & I_\star. \end{array}$$

Their target descendants are

$$(22) \quad \begin{array}{c|ccc} e & d_{e,1}^w & d_{e,2}^w & d_{e,s}^w \\ \hline 0B & J & k_0 & i_0 \\ B\star & I_\star & J & k_0 \\ 0A & I_A & j_0 & k_0 \\ AAC & I_\star & k_0 & j_0 \\ AC\star & J & k_0 & I_\star. \end{array}$$

Three points in these tables prevent common but invalid simplifications. At $B\star$, the band α meets the j_ε -lobe of the complete push-off J^+ , rather than an elementary j -disk substituted for J . At AAC , the entire disk I_A , including its attached j -portion and complete marking, is the mover. At $AC\star$, the spectator is the actual terminal disk $I_\star$, not its ancestral i_0 -piece.

Equivalently, forgetting descendants but retaining pair type, the five roles are

$$(23) \quad \begin{array}{c|ccccc} e & 0B & B\star & 0A & AAC & AC\star \\ \hline \text{active pair; spectator} & (j, k; i) & (i, J; k) & (i, j; k) & (I_A, k; j) & (j, k; I_\star) \\ \text{model pair} & (j, k) & (i, j) & (i, j) & (i, k) & (j, k). \end{array}$$

The model-pair row controls the three common footprint disks constructed in Section 6; the first row controls the literal endpoint-specific marking at each incidence.

3.3. The smallest case $N = 3$. When $N = 3$, the active handles (i, j, k) are all the product handles, so the five rows above can be read without a spectator region of unused handles. Along the upper route, $0B$ slides j_0 over k_0 along β_0 to produce J , and $B\star$ then slides i_0 over the complete push-off J^+ along α to produce $I_\star$. Along the lower route, $0A$ first produces I_A by sliding i_0 over j_0 ; AAC moves the whole marked disk I_A over k_0 along β_1 to produce $I_\star$; and $AC\star$ produces J while the actual terminal disk $I_\star$ is the spectator. Thus both routes end at the same independently fixed row $(I_\star, J, k_0)$, but they reach it through different typed edge data.

The later construction assigns endpoint charts, full-normal collars, cut maps, quotient descents, and global extensions to these five edges. In the five-edge finite compatibility enumeration (denoted PC5; see Section 11), the five unused-handle G8 records are still present as typed slots, but their domains are empty at $N = 3$; every other protected product domain remains nonempty. The common coordinate product displayed below is therefore a useful check of this smallest route, not a substitute for the relative isotopy proved in Section 13.

3.4. The integral coordinate check. The matrices record the meridian descendants and product-framed dual action of the five endpoint rows. They check the typed edge data; the geometric construction and relative isotopy remain to be proved. In the ordered original meridian basis (m_i, m_j, m_k) , let the columns of S_v be the meridian descendants at v . Then

$$(24) \quad \begin{aligned} S_0 &= I, & S_B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, & S_A &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ S_{AC} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, & S_\star &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

For $e : v \rightarrow w$, put $M_e = S_w S_v^{-1}$. The typed matrices are

$$(25) \quad \begin{aligned} M_{0B} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, & M_{B\star} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\ M_{0A} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & M_{AAC} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\ M_{AC\star} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

In particular, $M_{0B} \neq M_{B\star}$; there is no untyped B -matrix that can be reused at both edges.

The formal product-framed dual block is $L_e = M_e^{-T}$:

$$(26) \quad \begin{aligned} L_{0B} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, & L_{B\star} &= \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ L_{0A} &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & L_{AAC} &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ L_{AC\star} &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Direct multiplication gives

$$(27) \quad M_{B\star} M_{0B} = M_{AC\star} M_{AAC} M_{0A} = S_\star,$$

$$(28) \quad L_{B\star} L_{0B} = L_{AC\star} L_{AAC} L_{0A} = S_\star^{-T} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 3.1 (What the matrices do not prove). Equations (27) and (28) prove only finite arithmetic. They do not construct any edge diffeomorphism, show that its full action is $\text{diag}(M_e, M_e^{-T})$, rule out mixed meridian-longitude shear, or produce a relative isotopy between F_L and F_R . Those are geometric conclusions of Sections 12 and 13. In particular, mapping-class equality will not be inferred from the common matrix product.

4. MAIN THEOREM AND CONSTRUCTION ORDER

The edge maps must fit together at their endpoints as marked maps. Choosing each slide representative separately would not ensure that two incident edges use the same disk charts, collars, or spectator data. We therefore choose all five vertex systems before the endpoint charts and edge maps. The theorem constructs one such marked system and compares its two routes.

Whenever a signed bicollar appears below, its negative normal coordinate is on the local side and its positive normal coordinate is on the spectator side. The later collar construction gives the corresponding product coordinates.

Full-normal pairing identities allow the cut maps to descend and extend to global maps. Transport of the embedded meridian and dual-longitude rows then gives their homology blocks and the common preserved Lagrangian. The relative isotopy compares the two composites with their complete protected collar fixed, and the preserved Lagrangian makes the Walker seam corrections vanish. Each step concerns the same selected maps and endpoint data.

Let

$$(4.1) \quad \mathcal{T} = (g, N, W : X \longrightarrow Y_{\text{geom}}, i, j, k, \mathfrak{C}_{\text{ext}}, \lambda_g, \mathcal{L}_{ijk})$$

be a supplied protected-sector tuple in the sense of Section 2. Thus $g \geq 1$, $N \geq 3$, the indices $i, j, k \in \{1, \dots, N\}$ are pairwise distinct, and $\mathcal{L}_{ijk}$ contains the complete protected local geometry rather than merely its homology classes. Recall the vertex and edge sets

$$(4.2) \quad V = \{0, B, A, AC, \star\}, \quad E_5 = \{0B, B\star, 0A, AAC, AC\star\}.$$

Theorem 4.1 (Positive handleslide pentagon with complete markings). *For every tuple $\mathcal{T}$ in (4.1), there is a coordinated system with the following properties.*

(i) One vertex-first package. *There is first one simultaneous choice*

$$(4.3) \quad \mathbf{V} = \left(\mathfrak{C}_v, K_v, \mathcal{Q}_v, q_v^P, q_v^{(3)}, \rho_v^{(3)}, \rho_v, p_v, \bar{z}_v, z_v : D^2 \times [0, u_0] \longrightarrow K_v \right)_{v \in V}$$

with one positive, vertex-only depth u_0 . Every edge incident to a vertex uses the literal corresponding member of $\mathbf{V}$; there are no edge-indexed replacements. On the boundary of the disk collar,

$$(4.4) \quad p_v \rho_v z_v(x, 0) = \bar{z}_v(x).$$

The trace map ρ_v in (4.4) is essential for the types to match.

(ii) Ten endpoint atlases followed by five edge triples. *Only after $\mathbf{V}$ has been fixed are there a common depth $0 < u_{\text{cmp}} < u_0$, ten incidence-labelled spectator atlases, and five coherent triples. The incidence set is*

$$\begin{aligned} \mathcal{I}_{10} = \{ & (0B, 0), (0B, B), (B\star, B), (B\star, \star), (0A, 0), (0A, A), \\ & (AAC, A), (AAC, AC), (AC\star, AC), (AC\star, \star) \}. \end{aligned}$$

For each $(e, u) \in \mathcal{I}_{10}$ there is an atlas $\Xi_{e,u}^z$. Equal atlas values do not erase incidence labels. For every edge $e : v \rightarrow w$ there is then one triple

$$(4.5) \quad \tilde{F}_e : K_v \longrightarrow K_w, \quad f_{e,U} : U_{ijk,v} \longrightarrow U_{ijk,w}, \quad F_e : (W, \mathfrak{C}_v) \longrightarrow (W, \mathfrak{C}_w),$$

of orientation-preserving diffeomorphisms. The first is the cut-ball map, the second its quotient descent, and the third its global extension. They obey

$$(4.6) \quad \tilde{F}_e z_v = z_w \quad \text{on } D^2 \times [0, u_{\text{cmp}}],$$

$$(4.7) \quad f_{e,U} \rho_v^{(3)} = \rho_w^{(3)} \tilde{F}_e,$$

$$(4.8) \quad F_e|_{U_{ijk,v}} = f_{e,U}, \quad F_e|_{W \setminus \text{int } U_{ijk,v}} = \text{id},$$

$$(4.9) \quad (F_e|_{\partial_+ W}) \rho_v = \rho_w (\tilde{F}_e|_{\partial K_v}).$$

The map $f_{e,U}$ is the unique map induced by $\tilde{F}_e$, and its smooth inverse is induced by $(\tilde{F}_e)^{-1}$.

(iii) All-jet stabilization and finite compatibility. *For each edge, the local and spectator pieces meet along an internal disk E_e , which is not the global attachment disk E . Their extensions agree to every order on an open signed bicollar of E_e . All fifteen full-normal pairing conjugacies, in both directed forms, hold before quotient descent. After descent and global extension, the same three-stage map system satisfies the complete stabilization and finite-compatibility conditions.*

More precisely, the finite compatibility package has 45 ordered group headers and 185 typed records, with per-edge group sizes

$$(4.10) \quad (7, 6, 2, 12, 1, 1, 3, 4, 1).$$

At $N = 3$ the five unused-handle records remain in this typed census but their domains are empty. Write $\varepsilon_{\text{pair}}^{\text{src}} > 0$ for the common full-normal pairing depth obtained during the vertex construction. The package then defines the common subordinate depths

$$(4.11) \quad u_{\text{pair}} = \frac{1}{2} \min\{u_{\text{cmp}}, \varepsilon_{\text{pair}}^{\text{src}}\}, \quad \varepsilon_{\text{PC5,work}} = \frac{1}{2} u_{\text{cmp}}, \quad \varepsilon_{\text{PC5,pair}} = \frac{1}{2} u_{\text{pair}}.$$

In particular, no source depth is silently reused as a later compatibility depth.

(iv) Domains and roles of the three map stages. *The three maps in (4.5) form one system*

$$(4.12) \quad \widetilde{F}_e \longmapsto f_{e,U} \longmapsto F_e.$$

Cut-level face, ribbon, spoke, pairing, and internal-interface assertions belong to $\widetilde{F}_e$. Quotient descent belongs to $f_{e,U}$. The two-sided global- E collar, the old region, and unused handles belong only to F_e , after descent and exterior extension. With these domains fixed, the two cut routes use the same vertex data and their discrepancy is the identity on one complete protected boundary collar.

(v) Homology and the common Lagrangian. *Put*

$$(4.13) \quad U = \text{span}_{\mathbb{Q}}\{m_i, m_j, m_k\}, \quad \Lambda = \lambda_g^Y \oplus \bigoplus_{t \notin \{i,j,k\}} \mathbb{Q}[m_t] \oplus U.$$

For $e : v \rightarrow w$, let

$$(4.14) \quad h_e = p_w(F_e|_{\partial_+ W})p_v^{-1} : Y \longrightarrow Y.$$

The same global maps F_e transport the embedded meridian and product-framed dual-longitude rows. Relative to the resulting splitting,

$$(4.15) \quad h_{e*} = I_{\text{ext}} \oplus \begin{pmatrix} M_e & 0_3 \\ 0_3 & M_e^{-T} \end{pmatrix}.$$

Both zero mixed blocks are geometric conclusions; neither is inferred from symplecticity. The actual maps and their actual inverses satisfy

$$(4.16) \quad h_{e*}\Lambda = \Lambda = h_{e*}^{-1}\Lambda.$$

When $N = 3$, the unused-handle sum in (4.13) is empty.

(vi) Relative pentagon and mapping classes. *In right-factor-first order, set*

$$(4.17) \quad F_L = F_{B*} \circ F_{0B}, \quad F_R = F_{AC*} \circ F_{AAC} \circ F_{0A}.$$

There is an isotopy $\mathcal{H}_s$, fixed on the prescribed complete marking and collar, with

$$(4.18) \quad \mathcal{H}_0 = F_R^{-1} \circ F_L, \quad \mathcal{H}_1 = \text{id}_W.$$

Consequently $\Gamma_s = F_R \circ \mathcal{H}_s$ has $\Gamma_0 = F_L$ and $\Gamma_1 = F_R$. For the common background marking $\mathfrak{C}_{\text{bg}}$,

$$(4.19) \quad [h_{B*} \circ h_{0B}] = [h_{AC*} \circ h_{AAC} \circ h_{0A}] \quad \text{in } \pi_0 \text{Diff}^+(Y, \mathfrak{C}_{\text{bg}}).$$

This equality is induced by the relative isotopy, not by the equality of the corresponding integer matrices.

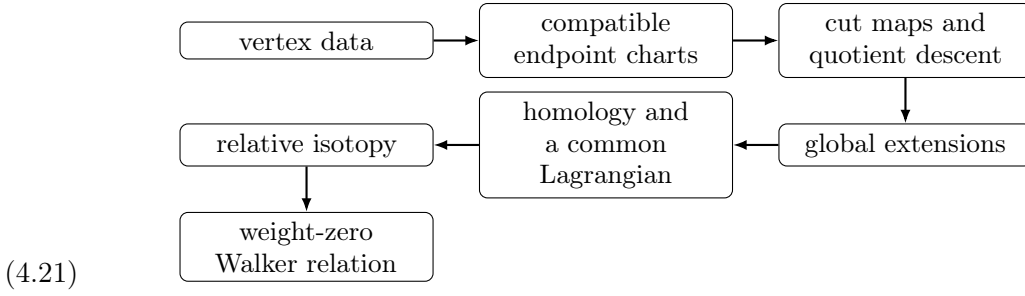
(vii) Literal zero-weight Walker lift. *For the Walker mapping cylinders $H_e = C([h_e], 0)$, forward composition and the two-sided equality (4.16) give*

$$(4.20) \quad H_{B*}H_{0B} = H_{AC*}H_{AAC}H_{0A} = C([h_{B*} \circ h_{0B}], 0).$$

Every constituent weight, all three ordered Maslov seam corrections, and both route weights are literally zero.

The theorem is existential, not canonical: it produces one coordinated system for every supplied protected sector. It neither asserts uniqueness nor compares arbitrary choices. The precise external boundary is collected in Section 15.

4.1. **Construction order.** For later reference, the reader-facing construction order is



The endpoint-extension equation is therefore asserted only after the representative has been selected. Likewise, the edge-dependent collar widths restrict u_{cmp} ; they do not redefine u_0 , any z_v , or the domain $D^2 \times [0, u_0]$.

5. VERTEX-FIRST CUT GEOMETRY

At a shared endpoint, composition requires one marked cut ball and one set of collars for both incident edges. We therefore construct the complete vertex rows and their cut balls before choosing any edge map. The same vertex collars will later coordinate the local and spectator extensions.

Throughout this section the protected input $\mathcal{T}$ is fixed, but no edge representative is chosen. Every object is constructed once for each $v \in V$, directly from the row in (19). Every incident edge uses this same vertex datum.

5.1. **Complete rows and relative ball identifications.** For $v \in V$, cut the active sector along its three displayed row disks:

$$(29) \quad K_v = U_{ijk} \setminus (D_i^v \cup D_j^v \cup D_k^v).$$

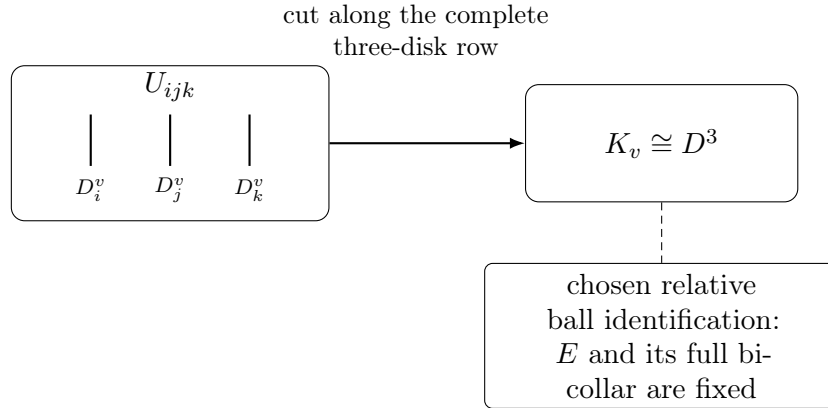


FIGURE 3. The vertex-first cut-ball construction. Every edge incident to v uses this same complete row and the same relative ball identification. The drawing records the cutting operation and fixed bicollar only; the disk shapes are schematic.

Proposition 5.1 (Five complete vertex rows). *The disks in each row of (19) are embedded, pairwise disjoint, oriented, and product framed. The cut manifold K_v is an oriented smooth 3-ball. Moreover, the ball identification may be chosen to fix the global interface disk E and its full bicollar. After adjoining the unchanged exterior row, each active triple therefore gives a complete global row $\mathfrak{C}_v$.*

Band separation and relative ball gluing. The height and band separations first make the composite disks embedded and product framed. Cutting the central co-cores then reduces each row to the same three elementary handles, while the prescribed cancellation order keeps the terminal overlapping lobes distinct. Because every move misses the interface disk and its bicollar, the relative ball identifications glue to one complete global row.

Proof. The height separations (10), band separations (11), and disjoint smoothing boxes show directly that every band sum in (8) is embedded. The band orientations are the positive band-sum orientations, and the simultaneous quadrant rounding makes the product normals on disk pieces and bands agree. Thus each composite disk has one global product-normal collar. This proves embeddedness, disjointness, orientation, and framing without using homology.

Let

$$(30) \quad C = \text{Round}(\chi(Q \times [1 - \tau, 1])).$$

It is a ball, and U_{ijk} is obtained from it by attaching the three product 1-handles. Cutting a handle along its middle co-core leaves two half-handles, each a ball attached to the current ball along a disk; absorbing the half-handle is a boundary-collar compression. The row $v = 0$ is therefore a complete co-core system.

For the four composite rows, after the carrier co-core has been cut, each band lobe is boundary parallel in its supplied product box. Sliding it through the cleared half-handle retracts the composite disk to its root while preserving the entire normal germ. Applying this elementary cancellation in the exact orders printed in Table 1 reduces every row to the three elementary co-cores. At the terminal vertex the k_ε -lobe of J is nearer and is removed first; only then may the $k_{2\varepsilon}$ -lobe of $I_\star$ pass through the cleared region. No simultaneous cancellation or disjoint corridor for those two lobes is assumed. All isotopies are supported in the supplied handle, band, and smoothing reserves, which miss E , its bicollar, the old region, and the unused handles. This proves the relative ball assertion.

Finally cut the frozen exterior disks first. The active remainder meets the exterior cut product only along E , and the five relative identifications agree there. Replacing one K_v by another therefore does not alter the exterior cut product, proving completeness of the global row. $\square$

5.2. Direct dual longitudes. The dual curves must be constructed geometrically; their homology classes alone do not supply embedded product-framed representatives.

Lemma 5.2 (Direct dual-track update). *Suppose a positive product-zero-framed slide replaces a moving disk D_a by $D'_a = D_a \# D_b^+$ and fixes the carrier D_b . If λ_a, λ_b are disjoint product-framed duals before the slide, then in a boundary-parallel track beside the meridian band one may take*

$$(31) \quad \lambda'_a = \lambda_a, \quad \lambda'_b = \lambda_b \natural (-\lambda_a)^+.$$

After a product-germ-preserving bigon push, these are embedded disjoint duals for D'_a, D_b .

Proof. Before the bigon push, D'_a meets λ'_b in two points. The carrier lobe D_b^+ meets the λ_b -piece positively, while the D_a -piece meets $(-\lambda_a)^+$ negatively. Their joining arcs bound the plumbing rectangle, whose interior misses all four objects. Push the longitude side across this bigon while keeping the ambient normal coordinate fixed. The two intersections cancel; the positive diagonal intersections $D'_a \cdot \lambda'_a = D_b \cdot \lambda'_b = 1$ remain, and all spectator germs are unchanged. $\square$

Choose the three joining rectangles in distinct reserved tracks and set

$$(32) \quad \begin{aligned} L_{ji} &= \ell_j \natural_{\alpha^\vee} (-\ell_i)^+, \\ K_{ki} &= \ell_k \natural_{\beta_{ki}^\vee} (-\ell_i)^+, \\ K_{kj} &= \ell_k \natural_{\beta_{kj}^\vee} (-\ell_j)^+. \end{aligned}$$

Repeated application of Lemma 5.2, still before any edge, gives

$$(33) \quad \begin{array}{c|ccc} v & \lambda_i^v & \lambda_j^v & \lambda_k^v \\ \hline 0 & \ell_i & \ell_j & \ell_k \\ B & \ell_i & \ell_j & K_{kj} \\ A & \ell_i & L_{ji} & \ell_k \\ AC & \ell_i & L_{ji} & K_{ki} \\ \star & \ell_i & L_{ji} & K_{kj}. \end{array}$$

The rectangles and their bigon boxes are disjoint, so the simultaneous rows are embedded and product framed. Put $m_s^v := \partial D_s^v$ on the outgoing boundary. More strongly,

$$(34) \quad m_s^v \cdot \lambda_r^v = \delta_{sr}, \quad m_s^v \cap \lambda_r^v = \emptyset \quad (s \neq r),$$

where the diagonal intersection is one positive transverse point in the original handle chart. The identities

$$(35) \quad [L_{ji}] = [\ell_j] - [\ell_i], \quad [K_{ki}] = [\ell_k] - [\ell_i], \quad [K_{kj}] = [\ell_k] - [\ell_j]$$

are checks on these already constructed curves, not substitutes for the track construction.

After cutting, the diagonal intersection opens each λ_r^v into an oriented arc

$$(36) \quad c_{r,v} : D_{r,v}^- \longrightarrow D_{r,v}^+.$$

Its product neighborhood is the framed dual ribbon used in the cut-sphere marking.

5.3. The marked cut sphere. Let $S = S_{Q0}^2$ denote the abstract sphere cellulated in Section A. It has seven core faces

$$(37) \quad E, \quad i^-, i^+, \quad j^-, j^+, \quad k^-, k^+,$$

three spoke–dual branches $E \rightarrow r^- \rightarrow r^+$, one complementary 24-gon Z_{mod} , and cell count

$$(38) \quad (V_{\text{cell}}, E_{\text{cell}}, F_{\text{cell}}) = (48, 84, 38).$$

Proposition 5.3 (Five direct smooth realizations). *For each $v \in V$ there is an orientation-preserving diffeomorphism*

$$(39) \quad b_v : S \xrightarrow{\cong} \partial K_v$$

which sends every indexed face, ribbon, spoke, detector, and attachment sector to its directly constructed counterpart on ∂K_v . The five maps are chosen independently from the vertex rows and obey one common signed-face convention.

The fat-tree disk and its complementary disk. The seven cut faces and three dual arcs first form a realized fat tree on each cut sphere. The prescribed spoke slots make its regular neighbourhood a disk, leaving one complementary disk; product charts identify the common ribbon data, and an endpoint-flat boundary interpolation extends that identification over the complement. The common detector and face convention are therefore part of the smooth realization, not consequences of the cell count.

Proof. On ∂K_v , begin with the seven realized cut faces and the three disjoint dual arcs (36). They form four forest components. Choose one spoke $s_{r,v} : E \rightarrow D_{r,v}^-$ at a time, attaching on E in the cyclic slots

$$(40) \quad p_i = 1, \quad p_j = 5, \quad p_k = 3.$$

Each spoke joins two distinct components in the connected complement, so no cycle is created. The resulting regular neighborhood of the connected fat tree is a disk; its complement in the sphere ∂K_v is a second disk Z_v . Thickening the six tree edges gives the spoke and dual-ribbon rectangles. The supplied E -collar also carries one common detector with zero turning in both frame directions.

Product-coordinate maps identify the regular neighborhood of the abstract fat tree with the realized one and agree on complete overlap strips. To extend over the complementary disk, choose polar charts in which the boundary germ has the form

$$(41) \quad (r, [\theta]) \longmapsto (r, [f_v(\theta)]), \quad f_v(\theta + 1) = f_v(\theta) + 1, \quad f'_v(\theta) > 0.$$

With $f_{v,s} = (1-s)\text{id} + sf_v$ and an endpoint-flat radial cutoff η , use

$$(42) \quad (r, [\theta]) \longmapsto (r, [f_{v,\eta(r)}(\theta)]).$$

It is the identity near the polar center, hence smooth there, and for $r > 0$ its angular derivative is $(1 - \eta(r)) + \eta(r)f'_v(\theta) > 0$. It is therefore a positive disk diffeomorphism agreeing to all orders with the boundary map on a full annulus. Gluing gives b_v . The explicit face list in Section A proves the full incidence statement; a cell count alone would not. $\square$

For each r , fix an orientation-reversing $j_r : D^2 \rightarrow D^2$ and positive face charts $\kappa_{r,v}^{\sigma,\partial} : D^2 \rightarrow D_{r,v}^\sigma$ so that

$$(43) \quad \kappa_{r,v}^{+, \partial} = \gamma_{r,v} \kappa_{r,v}^{-, \partial} j_r^{-1},$$

where $\gamma_{r,v}$ is the canonical identification of the two cut copies. The surface pairings are

$$(44) \quad \begin{aligned} \pi_{r,v}^{-+, \partial} \kappa_{r,v}^{-, \partial}(z) &= \kappa_{r,v}^{+, \partial}(j_r z), \\ \pi_{r,v}^{+, \partial} &= (\pi_{r,v}^{-+, \partial})^{-1}. \end{aligned}$$

They are smooth orientation-reversing surface maps. They are not cellular: a negative face has four sectors and a positive face has two.

The inherited two-sided face collars have a uniform positive width d_{inh} . In their local doubles define

$$(45) \quad \pi_{r,v}^{-+, \text{full}}(\kappa_{r,v}^{-, \partial}(z), u) = (\kappa_{r,v}^{+, \partial}(j_r z), u), \quad |u| < d_{\text{inh}}.$$

The normal coordinate is preserved for both signs. Negative u belongs to the reflected double and is not claimed to be a point of K_v .

5.4. One relative inward collar at each vertex.

Lemma 5.4 (Relative collar extension). *Let M be compact with boundary, let $O_0 \Subset O_1 \subset \partial M$, and suppose that*

$$c^{\text{loc}} : O_1 \times [0, a) \longrightarrow M$$

is a prescribed inward collar. Then, for some $b > 0$, it extends to a global inward collar

$$c : \partial M \times [0, b) \longrightarrow M$$

which equals c^{loc} literally on $O_0 \times [0, b)$.

Proof. Push ∂_u forward by c^{loc} to a field X_{loc} . Choose $O_0 \Subset O_{1/2} \Subset O_1$, a smaller depth $a_0 < a$, any inward field Y near ∂M , and a cutoff ρ equal to one on $c^{\text{loc}}(\overline{O_{1/2}} \times [0, a_0])$, with support in the image of the local collar. Extend ρX_{loc} by zero and put

$$(46) \quad X = \rho X_{\text{loc}} + (1 - \rho)Y.$$

The inward half-space is convex, so X is strictly inward along the boundary. Compactness and the inverse function theorem give a common short time for which its flow is a global collar. On O_0 , uniqueness of integral curves makes that flow equal to the prescribed local flow. $\square$

The compatibility required for this application is ambient compatibility, not a consequence of the boundary diffeomorphism b_v . Write each inherited ambient product germ as

$$g_\alpha : B_\alpha \times [0, a_\alpha) \longrightarrow K_v.$$

Every nonempty old-old overlap is a frozen handle, band, or smoothing product box, and its transition fixes the inward variable to all orders. The local cancellation isotopies and the direct-dual bigon pushes are also the identity in that variable. At a global E -attachment the detector and initial spoke rectangle inherit the same inward variable from the fixed E -bicollar.

The remaining auxiliary spoke and annular regions are chosen only afterward and have no competing old ambient germ. These cases are exhaustive because the other full local boxes are disjoint. Hence, after one finite shrink, every overlap has the literal form

$$(47) \quad g_\alpha(x, t) = g_\beta(h_{\beta\alpha}(x), t),$$

and the fields $(g_\alpha)_*(\partial_t)$ glue to one field X_v^{fix} on the union of the inherited product germs.

Choose positive-depth unions $\overline{G}_v^{\text{keep}} \subseteq G_v^{\text{work}}$ inside those germs, with the keep set containing every overlap on which literal equality is required. On the newly chosen realized fat-tree and protected-annulus pieces choose any smooth inward field Y_v , and let χ_v^{keep} be one on $\overline{G}_v^{\text{keep}}$ and supported in G_v^{work} . Extending $\chi_v^{\text{keep}} X_v^{\text{fix}}$ by zero, set

$$(48) \quad X_v^{\text{pre}} = \chi_v^{\text{keep}} X_v^{\text{fix}} + (1 - \chi_v^{\text{keep}}) Y_v.$$

The inward cone is convex, so a common short-time flow gives a local collar equal to every old product germ on a smaller positive-depth neighborhood; its restrictions define the previously unassigned auxiliary spoke and annular germs. The inward face-pairing domains and images use this same cut-face collar. Only the nonnegative halves of the two-sided pairings lie in the realized cut ball; their negative halves remain in the corresponding local doubles and are not asserted to be restrictions of an inward collar. Thus all realized cut-face, E -interface, band, ribbon, spoke, detector, pairing, annular, and complete marking germs are restrictions of this one local collar, and the relative collar-extension lemma applies.

Taking the minimum over the five resulting global collars gives a common width $b_{\text{col}} > 0$. Let $d_{\text{germ}} > 0$ be the minimum positive depth on which the local collars agree literally with the finitely many inherited germs, decreased if necessary to lie below their common inherited depth. After restricting the inherited full-normal face collars to one common positive depth, denote that source-backed pairing depth by $\varepsilon_{\text{pair}}^{\text{src}} > 0$. Choose, before every edge,

$$(49) \quad \begin{aligned} 0 < c_P \ll a \ll h_0 \ll h, \quad 3h < \min\{b_{\text{col}}, d_{\text{germ}}\}, \\ 0 < \epsilon_c < \varepsilon_{\text{pair}}^{\text{src}} < \delta_{\text{rig}} < \epsilon_{\text{work}} < c_P, \end{aligned}$$

and restrict the resulting collars to

$$(50) \quad c_v : \partial K_v \times [0, 3h] \longrightarrow K_v.$$

On every protected germ c_v is the already prescribed product collar, literally and to all orders.

In particular, define the actual inward face pairing throughout $[0, c_P)$ by

$$(51) \quad \begin{aligned} \Pi_{r,v}^{-+, (3)}(c_v(\kappa_{r,v}^{-, \partial}(z), u)) &= c_v(\kappa_{r,v}^{+, \partial}(j_r z), u), \\ \Pi_{r,v}^{+-, (3)} &= (\Pi_{r,v}^{-+, (3)})^{-1}. \end{aligned}$$

On $0 \leq u \leq \epsilon_c$, this is the inward restriction of (45). Thus regluing uses the same inward coordinate as every other protected marking. The surface equation (44) alone would not supply this ambient compatibility.

5.5. The pre-edge quotient and disk collar. Choose a positive disk chart $\zeta_{\text{mod}} : D^2 \rightarrow Z_{\text{mod}}$. Using its polar radius, choose nested protected regions

$$(52) \quad \mathcal{P}_{\text{mod}} \subseteq \mathcal{U}_{\text{mod}} \subseteq W \subset S, \quad \iota_0(x) = (x, 0),$$

where W omits a closed disk around the polar center and contains the full protected annulus. We use the direct specialization

$$(53) \quad b_v^\partial = b_v, \quad \xi_v = (b_v|_{Z_{\text{mod}}})\zeta_{\text{mod}}, \quad G_v = \text{id}_{\mathbb{R}};$$

thus no endpoint gauge is solved from an incident edge.

Define the protected and thickened vertex embeddings by

$$(54) \quad q_v^P(x, t) = c_v(b_v(x), t), \quad (x, t) \in W \times [0, c_P),$$

$$(55) \quad q_v^{(3)}(x, t) = c_v(b_v^\partial(x), t), \quad (x, t) \in \mathcal{U}_{\text{mod}} \times [0, c_P),$$

$$(56) \quad q_v = q_v^{(3)} \iota_0|_{\mathcal{P}_{\text{mod}}}.$$

The two ambient maps agree on their common domain.

Reglue the three signed face pairs using (51). The resulting typed quotient is

$$(57) \quad \rho_v^{(3)} : K_v \longrightarrow U_{ijk,v} := K_v / \langle \Pi_{i,v}^{(3)}, \Pi_{j,v}^{(3)}, \Pi_{k,v}^{(3)} \rangle, \quad \rho_v = \rho_v^{(3)}|_{\partial K_v}.$$

This is vertex data only; it asserts no compatibility with a future edge map.

To define the outgoing parameterization, construct the same five marked rows directly in a fixed abstract copy of the three-handle sector in Y . Let $\widehat{b}_v : S \rightarrow \widehat{S}_v$ be its cut-sphere realization and $\widehat{\rho}_v : \widehat{S}_v \rightarrow Y$ its regluing quotient. On the exposed sector set

$$(58) \quad p_v(\rho_v b_v(x)) = \widehat{\rho}_v \widehat{b}_v(x), \quad x \in S,$$

and use the supplied common exterior parameterization elsewhere. The face-pairing conjugacies make (58) independent of the chosen representative of a quotient class. Since they hold on open face collars, the descended map is smooth; the two clauses agree to all orders on the global E -bicollar. Thus

$$p_v : \partial_+(W, \mathfrak{C}_v) \xrightarrow{\cong} Y$$

is one direct vertex parameterization. In particular, $p_\star$ is not defined by either route.

Set

$$(59) \quad \bar{z}_v = p_v \rho_v b_v \zeta_{\text{mod}} : D^2 \longrightarrow Y.$$

Finally choose, once from the common vertex depth c_P ,

$$(60) \quad u_0 = \frac{1}{2} c_P > 0, \quad z_v(x, t) = c_v(b_v \zeta_{\text{mod}}(x), t), \quad (x, t) \in D^2 \times [0, u_0].$$

No later edge-dependent shrink changes u_0 , z_v , or its domain. At $t = 0$, the definitions give the correctly typed boundary identity

$$(61) \quad \boxed{p_v \rho_v z_v(x, 0) = \bar{z}_v(x).}$$

The factor ρ_v cannot be omitted: $z_v(x, 0)$ lies on the cut sphere, whereas p_v is defined after regluing.

Collect the cellulation, pairings, collars, and quotient data into $\mathcal{Q}_v$. These choices give one common set of endpoint data for the later edge constructions.

Proposition 5.5 (Simultaneous pre-edge vertex package). *For the fixed tuple $\mathcal{T}$, there is one simultaneous choice*

$$(62) \quad \left(\mathfrak{C}_v, K_v, \mathcal{Q}_v, q_v^P, q_v, q_v^{(3)}, \rho_v^{(3)}, \rho_v, p_v, \bar{z}_v, z_v : D^2 \times [0, u_0] \rightarrow K_v \right)_{v \in V}$$

with one common $u_0 > 0$. Every component is fixed before all edge footprints, endpoint atlases, and edge representatives.

The parameterizations in this simultaneous construction identify the supplied exterior data with one common abstract background marking $\mathfrak{C}_{\text{bg}}$ on Y . It contains exactly the old, unused-handle, and global- E data induced by $\mathfrak{C}_{\text{ext}}$, and no varying active vertex row. This is the relative structure used when the two outgoing routes are compared in Section 13.

5.6. The frozen exterior splitting. The vertex construction also fixes, independently of all edges,

$$(63) \quad \iota_{\text{old}} : H_1(X; \mathbb{Q}) \hookrightarrow H_1(Y; \mathbb{Q}), \quad H_{\text{old}} = \text{im } \iota_{\text{old}}, \quad \lambda_g^Y = \iota_{\text{old}}(\lambda_g).$$

For $\mathcal{I}_{\text{unused}} = \{1, \dots, N\} \setminus \{i, j, k\}$, put

$$(64) \quad \begin{aligned} H_{\text{ext}} &= H_{\text{old}} \oplus \bigoplus_{t \in \mathcal{I}_{\text{unused}}} (\mathbb{Q}[m_t] \oplus \mathbb{Q}[\ell_t]), \\ I_{\text{ext}} &= \text{id}_{H_{\text{ext}}}, \\ \Lambda_{\text{ext}} &= \lambda_g^Y \oplus \bigoplus_{t \in \mathcal{I}_{\text{unused}}} \mathbb{Q}[m_t]. \end{aligned}$$

Writing m_r, λ_r for the p_0 -images of the active framed row, set

$$(65) \quad U = \text{span}_{\mathbb{Q}}\{[m_i], [m_j], [m_k]\}, \quad U^\vee = \text{span}_{\mathbb{Q}}\{[\lambda_i], [\lambda_j], [\lambda_k]\}.$$

Then

$$(66) \quad H_1(Y; \mathbb{Q}) = H_{\text{ext}} \oplus (U \oplus U^\vee), \quad \Lambda = \Lambda_{\text{ext}} \oplus U.$$

When $N = 3$, the unused sums in (64) are empty, and $\dim \Lambda = g + 3 = g + N$. The later edge actions are compared in this fixed homology splitting; preservation of Λ is proved in Section 12.

6. COMMON CHARTS AND ENDPOINT MARKINGS

The vertex data in (62) are now fixed. An endpoint chart-and-marking system gives a local coordinate model in which incident edge maps can later be compared. Three common pair-type footprints and five rounded local models supply the shared geometry; ten incidence-labelled endpoint-native atlases retain which endpoint each chart describes. These charts are chosen before the edge representatives, so none can be tailored to an equation involving an unselected cut map.

For each pair type, the common disk must contain the active footprints from every incidence of that type while avoiding their spectator markings. At the exceptional $AC\star$ ribbon it must also contain a reserved neighborhood for the longitude correction, called the *exceptional-ribbon reserve* below. The rounded model over this disk gives compatible local coordinates at both ends. The endpoint charts and extensions are chosen before an edge representative; the subsequent boundary isotopy, collar extension, and smoothing place that representative in the prescribed relative class.

6.1. Incidence-labelled support and marking footprints. Let

$$(67) \quad \mathcal{I}_{10} = \{(e, u) : e \in E_5, u \text{ is an endpoint of } e\}.$$

The incidence label is retained even if two atlases later have the same underlying value. For each $(e, u) \in \mathcal{I}_{10}$, distinguish the following three objects.

- (a) The *named native carrier* $\mathcal{S}_{e,u}^{\text{nat}}$: the mover meridian tube, the slide band, the carrier longitude and push-off, the listed smoothing boxes, and the direct-dual ribbons belonging to that typed local slide. For the two $AC\star$ incidences it also contains the complete exceptional-ribbon reserve and its product sweep.
- (b) The compact *active base footprint* $G_{e,u}^{\text{mc}} \subset \partial K_u$, prescribed by the endpoint role of the later transported named carrier.
- (c) The *actual spectator-marking footprint* $\mathfrak{M}_{e,u}^{\text{sp}} \subset \partial K_u$, including its complementary disk, branch, full collar, ribbon, spoke, pairing, eight-letter boundary word, detector, and protected-annulus footprint.

The first object is a native three-dimensional set and is not yet a subset of a realized c_u -chart. It is also not asserted to be the full support of any eventual representative. The latter two are compact subsets of the same concrete boundary sphere, so their separation is well typed. Pull them back before choosing a model footprint:

$$(68) \quad \overline{G}_{e,u}^{\text{mc}} = b_u^{-1}(G_{e,u}^{\text{mc}}), \quad \overline{\mathfrak{M}}_{e,u}^{\text{sp}} = b_u^{-1}(\mathfrak{M}_{e,u}^{\text{sp}}) \subset S.$$

There is one deliberate exposition-order exception. The exceptional-ribbon reserve is a pre-edge choice: its row-typed forbidden sets, positive clearances, swept strip, and the exceptional-ribbon reserve disk A_{jk}^{ear} are constructed using only the fixed vertex maps and native carrier data. The coordinate construction is printed at the start of Section 7, where the exceptional ribbon is used, but it is logically prior to the common-disk choice below. In particular, that construction does not consume an endpoint chart-and-marking system, an edge representative, or the disk A_{jk} .

The support scales belong to the same pre-edge choice. Let $d_{\text{ret}} > 0$ be the minimum of the finitely many retained positive product-germ depths. The germs and global collars were frozen with depth beyond $3h$; after restricting them to one common depth, choose d_{prod} with

$h < d_{\text{prod}} < 2h$. Once the positive reserve clearances have been formed, make the simultaneous final pre-edge scale choice in (113). The resulting values satisfy

$$(69) \quad 3c_P < n_0 < n_1 < h_0 < h < d_{\text{prod}} < 2h < 3h < \min\{b_{\text{col}}, d_{\text{germ}}, d_{\text{ret}}\}.$$

Thus every scale used by the coordinate construction at the start of Section 7 is fixed before the three common disks. All earlier occurrences of these widths, including $u_0 = c_P/2$, mean their final restricted values; no vertex witness is altered after it is declared final.

Lemma 6.1 (Three common pair-type disks). *There are smooth rounded disks $A_{ij}, A_{ik}, A_{jk} \in \mathcal{W}$, and a smaller closed rounded disk A_{jk}^{ear} , such that*

$$(70) \quad \begin{aligned} & A_{jk}^{\text{ear}} \in \text{int } A_{jk}, \\ & \bigcup_{(e,u) \in \mathcal{I}_{pq}} \overline{G}_{e,u}^{\text{mc}} \in \text{int } A_{pq} \in \mathcal{W} \setminus \bigcup_{(e,u) \in \mathcal{I}_{pq}} \overline{\mathfrak{M}}_{e,u}^{\text{sp}}, \quad pq \in \{ij, ik, jk\}, \end{aligned}$$

where $\mathcal{I}_{pq}$ consists of the incidences having model pair (p, q) . The reserve disk contains, with margin, both $AC\star$ exceptional reserves.

Proof. For the j, k pair, the exceptional reserve is enlarged without rechoice.

For a fixed pair (p, q) , take the finite union of the complete active branch subtrees and the terminal portions of their spokes. Join its two active components by an arc in the annular work collar. Use respectively the gaps between the i, j boundary words, the cyclic wrap between the k, i words, and the gap between the j, k words. The connector misses the complete spectator branch. The resulting spine is a tree, so a sufficiently small rounded regular neighborhood is a disk.

For the j, k pair, begin with the already chosen exceptional reserve tree and its rounded disk A_{jk}^{ear} , then attach the remaining finitely many prescribed subtrees by tree arcs; no cycle is introduced. The pre-edge participant/forbidden construction displayed in Equations (108) and (110) and the margin (114) make the finite active union disjoint from the finite union of the actual spectator markings. Enlarging the tree neighborhood, rather than rechoosing the reserve disk, gives $A_{jk}^{\text{ear}} \in \text{int } A_{jk}$. Normality of the surface and compactness then give the two strict containments in (70) and a positive margin. The choices for the three pair types are independent; the disks are not required to be mutually disjoint. $\square$

For an incidence, put $A(e) = A_{pq}$ according to its model pair and

$$(71) \quad A_{e,u} = b_u(A(e)).$$

The full role table is

	incidence	$A(e)$	mover/carrier	spectator	exceptional-ribbon reserve
	$(0B, 0)$	A_{jk}	j_0, k_0	i_0	---
	$(0B, B)$	A_{jk}	J, k_0	i_0	---
	$(B\star, B)$	A_{ij}	i_0, J	k_0	---
	$(B\star, \star)$	A_{ij}	$I_\star, J$	k_0	---
(72)	$(0A, 0)$	A_{ij}	i_0, j_0	k_0	---
	$(0A, A)$	A_{ij}	I_A, j_0	k_0	---
	(AAC, A)	A_{ik}	I_A, k_0	j_0	---
	(AAC, AC)	A_{ik}	$I_\star, k_0$	j_0	---
	$(AC\star, AC)$	A_{jk}	j_0, k_0	$I_\star$	inside
	$(AC\star, \star)$	A_{jk}	J, k_0	$I_\star$	inside.

Every row separately satisfies

$$(73) \quad \overline{G}_{e,u}^{\text{mc}} \in \text{int } A(e), \quad G_{e,u}^{\text{mc}} \in \text{int}_{\partial K_u} A_{e,u}, \quad \text{dist}(A_{e,u}, \mathfrak{M}_{e,u}^{\text{sp}}) > 0.$$

These are own-incidence statements realized by three common finite-union choices. They do not claim that $A_{e,u}$ avoids spectator markings belonging to unrelated pair types.

Retain the source working domain

$$(74) \quad D^{\text{work}} = \mathcal{W} \times [0, \epsilon_{\text{work}}), \quad 0 < \epsilon_{\text{work}} < c_P < h_0.$$

This width is part of the pre-edge data. It is not a later comparison, pairing, or finite-compatibility depth.

Lemma 6.2 (Exceptional-ribbon reserve clearance). *At $u = AC, \star$, the full exceptional $AC\star$ longitude-correction sweep lies in the rounded endpoint-specific local ball based on A_{jk} and misses its side and cap.*

Proof. The surface margin separates the sweep from the local ball's side, and the product-depth bound separates it from the cap. Both bounds use the same vertex collar. This containment is established before an edge representative is selected; it does not construct the corrected row or define a cross-endpoint edge map.

The direct vertex specialization gives, on the reserve disk,

$$(75) \quad b_u^\partial|_{A_{jk}^{\text{ear}}} = b_u|_{A_{jk}^{\text{ear}}}.$$

The already fixed collar c_u supplies the common product coordinate there through depth $2h$; no second “vertex” collar is introduced. Recall the pre-edge scales $n_0, n_1, d_{\text{prod}}$ and d_{ret} fixed in (69). The closed sweep has base in $b_u(A_{jk}^{\text{ear}})$ and depth at most n_1 . The surface margin $b_u(A_{jk}^{\text{ear}}) \subseteq \text{int } b_u(A_{jk})$, the common product coordinate (75), and $n_1 < h < 2h$ keep it away from both the side over ∂A_{jk} and the cap at depth $2h$.

The boundary and ambient separations used here are the finite positive minima (110) and (112); the actual side/cap identification is (115). $\square$

The three atlas layers now have separate jobs. The common pair-type disks provide one base region for all incidences with the same local pair; the rounded local model turns that region into a common protected three-ball; and the ten endpoint-native systems retain which edge endpoint is being described. The first two layers make comparison possible, but they do not select an edge map. The endpoint-native layer is therefore the last chart datum before the relative extension theorem, not a hidden transport rule between vertices.

6.2. The common rounded local model. For each edge form, in the abstract collared ball,

$$(76) \quad K_{e,\text{mod}}^{\text{loc}} = \text{Round}(A(e) \times [0, 2h]),$$

using one rounding profile at both endpoints. Its internal interface is

$$(77) \quad E_{e,\text{mod}} = \text{Round}(A(e) \times \{2h\} \cup \partial A(e) \times [0, 2h]).$$

This disk is the edge-local interface E_e ; it is not the global exterior disk E . The common rounding fixes a signed bicollar

$$(78) \quad c_{e,\text{mod}} : D^2 \times (-h_0, h_0) \longrightarrow B_{\text{mod}}^3,$$

negative on the local side and positive on the spectator side.

Collapse E_e to the central fat vertex and retain the two active branches. For the model pair (p, q) the resulting local fat tree is

$$(79) \quad E_e \xrightarrow{s_p} p^- \xrightarrow{c_p} p^+, \quad E_e \xrightarrow{s_q} q^- \xrightarrow{c_q} q^+.$$

Give E_e, p^-, q^- four sectors and p^+, q^+ two sectors. Its regular neighborhood is a disk, and the complement in the local boundary sphere is one disk Z_e^{loc} . With

$$(80) \quad B_r = f_{E_e, r}^{-1} r_{s_r} f_{r^-, 1}^{-1} r_{c_r} f_{r^+, 1}^{-1} l_{c_r}^{-1} f_{r^-, 0}^{-1} l_{s_r}^{-1},$$

the oriented complementary boundary is $B_p B_q$. For A_{ik} , the connector crosses the cyclic wrap gap, so this word is read from the start of B_i ; $B_k B_i$ is the same cyclic word after an eight-letter rotation.

Relabelling the standard two-site cellulation by

$$(81) \quad E_\gamma \mapsto E_e, \quad 1^\pm \mapsto p^\pm, \quad 2^\pm \mapsto q^\pm$$

maps every cell and cyclic boundary word, not merely the cell counts. The independent cell count is

$$(82) \quad \begin{aligned} V_{\text{loc}} &= 2(4 + 4 + 2 + 4 + 2) = 32, \\ E_{\text{loc}} &= 3(4 + 4 + 2 + 4 + 2) + 2 \cdot 4 = 56, \\ F_{\text{loc}} &= 5 + (4 + 4 + 2 + 4 + 2) + 4 + 1 = 26. \end{aligned}$$

Choose a positive chart

$$(83) \quad \zeta_e^{\text{loc}} : D^2 \longrightarrow Z_e^{\text{loc}}$$

whose full outer annulus is radial and agrees with all sixteen indexed edge parameters. It is common to both endpoints and is chosen before their endpoint-specific charts.

6.3. The common spectator model and signed cover. Let $L_{e,\text{mod}}^{\text{sp}}$ be the closure of the complement of $K_{e,\text{mod}}^{\text{loc}}$ in the model ball. Its protected fat graph has vertex disks

$$(84) \quad E, \quad E_e, \quad s^-, \quad s^+,$$

and edges

$$(85) \quad \tau_p, \tau_q : E \rightarrow E_e, \quad s_s : E \rightarrow s^-, \quad c_s : s^- \rightarrow s^+.$$

At E use the inherited rotation (i, j, k) , replacing each active spoke by its continuation τ_r ; at E_e use (τ_p, τ_q) ; at s^- use (s_s, c_s) ; and at s^+ use the one-element rotation. If α_{he} exchanges the two half-edges of every edge and σ_{rot} is this rotation, then the boundary permutation $\beta_{\text{bd}} = \sigma_{\text{rot}} \alpha_{\text{he}}$ has the following two orbits:

	e	six-half-edge orbit	two-half-edge orbit
(86)	$0B, AC\star$	$((c_i, i^+), (s_i, i^-), (\tau_j, E), (\tau_k, E_e), (s_i, E), (c_i, i^-))$	$((\tau_j, E_e), (\tau_k, E))$
	$B\star, 0A$	$((c_k, k^+), (s_k, k^-), (\tau_i, E), (\tau_j, E_e), (s_k, E), (c_k, k^-))$	$((\tau_i, E_e), (\tau_j, E))$
	AAC	$((c_j, j^+), (s_j, j^-), (\tau_k, E), (\tau_i, E_e), (s_j, E), (c_j, j^-))$	$((\tau_i, E), (\tau_k, E_e))$

Thus the regular neighborhood P_e^{sp} of the rank-one fat graph is an annulus and its complement consists of exactly two disks

$$Z_e^{\text{sp},+}, \quad Z_e^{\text{sp},-}.$$

Their signs are fixed by the two sides of the oriented global word $B_i B_j B_k$ and are never exchanged at an endpoint. Expanding the half-edge intervals gives the check

$$(87) \quad \begin{aligned} V_{\text{sp}} &= 2(6 + 4 + 4 + 2) = 32, \\ E_{\text{sp}} &= 3(6 + 4 + 4 + 2) + 2 \cdot 4 = 56, \\ F_{\text{sp}} &= 4 + (6 + 4 + 4 + 2) + 4 + 2 = 26. \end{aligned}$$

At this stage the two labels and their full radial annular germs are fixed, while the inner disk fills remain available for the relative band choice in Section 8. This is essential at the polar centers: a boundary-circle agreement alone does not determine a smooth positive disk chart.

At an endpoint u , bind the corresponding endpoint-specific halves by

$$(88) \quad K_{e,u}^{\text{loc}} := \text{Round}(c_u(A_{e,u} \times [0, 2h])), \quad L_{e,u}^{\text{sp}} := \overline{K_u} \setminus \overline{K_{e,u}^{\text{loc}}}.$$

They meet exactly in the rounded realized copy of $E_{e,\text{mod}}$.

The common rounding also yields an endpoint-independent smooth separator

$$(89) \quad \rho_e : D^{\text{work}} \longrightarrow \mathbb{R}$$

which is negative on the local side, zero exactly on the rounded interface, and positive on the spectator side. It is the signed normal coordinate in (78) and is extended by negative and positive constants of absolute value at least h_0 away from that collar. Define the relatively closed halves

$$(90) \quad \mathcal{D}_{e,\text{loc}}^{\text{work}} = \{\rho_e \leq 0\}, \quad \mathcal{D}_{e,\text{sp}}^{\text{work}} = \{\rho_e \geq 0\}.$$

Then

$$(91) \quad D^{\text{work}} = \mathcal{D}_{e,\text{loc}}^{\text{work}} \cup \mathcal{D}_{e,\text{sp}}^{\text{work}}, \quad \mathcal{D}_{e,\text{loc}}^{\text{work}} \cap \mathcal{D}_{e,\text{sp}}^{\text{work}} = \rho_e^{-1}(0),$$

and, at either endpoint,

$$(92) \quad q_u^P(\mathcal{D}_{e,\text{loc}}^{\text{work}}) \subset K_{e,u}^{\text{loc}}, \quad q_u^P(\mathcal{D}_{e,\text{sp}}^{\text{work}}) \subset L_{e,u}^{\text{sp}}.$$

For a fixed $0 < \delta_e < h_0$, only the chart extensions have the open domains

$$(93) \quad \mathcal{O}_{e,\text{loc}} = \{\rho_e < \delta_e\}, \quad \mathcal{O}_{e,\text{sp}} = \{\rho_e > -\delta_e\}.$$

The closed halves in (90) are not silently replaced by these larger open sets.

6.4. Ten endpoint-native local chart systems. For $e : v \rightarrow w$, let $B_{e,u}^{\text{LS}}$ be the endpoint ball furnished by Theorem 2.2 after the typed substitution in Equation (21). Its indexed boundary cellulation is the one relabelled in (81). The cellwise product charts and the positive polar fill on its unique complementary disk give, independently at the two endpoints,

$$(94) \quad \widehat{h}_{e,u} : \partial K_{e,\text{mod}}^{\text{loc}} \xrightarrow{\cong} \partial B_{e,u}^{\text{LS}}, \quad (e, u) \in \mathcal{I}_{10}.$$

Every indexed face, ribbon, interface germ, and complete complementary disk is included. There is no source-to-target transition in the definition of either endpoint chart-and-marking system.

Independently, push the rounded local model through the already fixed vertex chart:

$$(95) \quad G_{e,u}^{\text{loc}}(x, t) = c_u(b_u(x), t) \quad \text{on} \quad \text{Round}(A(e) \times [0, 2h]), \quad g_{e,u}^{\text{loc}} = G_{e,u}^{\text{loc}}|_{\partial K_{e,\text{mod}}^{\text{loc}}}.$$

The common rounding makes $G_{e,u}^{\text{loc}}$ an orientation-preserving ball diffeomorphism onto $K_{e,u}^{\text{loc}}$. It is constructed from b_u, c_u , not from a desired edge equation.

Before any ball extension, choose for each edge one native collar width and one compression

$$(96) \quad \ell_e > 0, \quad \lambda_e : [0, h_0] \longrightarrow [0, \ell_e], \quad \lambda_e(0) = 0, \quad \lambda_e'(t) > 0, \quad \lambda_e(t) = t \text{ near } 0,$$

with $\lambda_e([0, \epsilon_{\text{work}}]) \subseteq [0, \ell_e]$. The same (ℓ_e, λ_e) is used at both endpoints. The actual endpoint-only boundary identification is

$$(97) \quad \widehat{a}_{e,u}^{\text{loc}} = g_{e,u}^{\text{loc}}(\widehat{h}_{e,u})^{-1} : \partial B_{e,u}^{\text{LS}} \longrightarrow \partial K_{e,u}^{\text{loc}}.$$

In full disk charts it is the coordinate identity on the protected region, the complete complementary disk, and their full annular overlap.

There is one exceptional endpoint decoration. At the $AC\star$ source, the native direct rows contain L_{ji}, K_{ki} , so the raw carrier longitude is

$$(98) \quad \widetilde{K} = K_{ki} \natural_{\beta_{AC\star}^\vee} (-L_{ji})^+.$$

At the target the directly constructed terminal row already contains K_{kj} . The product-normal ear cancellation carries $\widetilde{K}$ to K_{kj} , relative to its common tail and all other marking germs. By Lemma 6.2, its whole sweep stays in the target local ball. Use the terminal time of this isotopy only to decorate the target component of $\widehat{a}_{AC\star,\star}^{\text{loc}}$. It neither constructs K_{kj} nor defines a cross-endpoint edge map, and no decoration is made at the source.

Lemma 6.3 (Relative local-ball extension). *For every incidence there is an orientation-preserving diffeomorphism*

$$(99) \quad \widehat{\Theta}_{e,u} : B_{e,u}^{\text{LS}} \xrightarrow{\cong} K_{e,u}^{\text{loc}}$$

extending (97) and satisfying

$$(100) \quad \widehat{\Theta}_{e,u}(\widehat{h}_{e,u}(x), \lambda_e(t)) = G_{e,u}^{\text{loc}}(x, t)$$

on the full prescribed outer collar. On the signed internal interface it is the coordinate identity, with all jets.

Proof. The boundary map is already a product map on mutually compatible open collars of every protected face, the complete complementary disk, and the signed internal interface. Extend it by the displayed product formula over a smaller boundary collar. Removing that collar leaves two smooth inner balls and an orientation-preserving sphere map that is fixed on the induced protected disks. Radial Alexander extension gives a homeomorphism of the inner balls which is already the displayed smooth product map on a full boundary collar. The remaining step is a

relative smooth extension, and we record why the relative clause applies rather than treating the Alexander formula as smooth at its cone point.

Double the two inner balls and the homeomorphism across that product collar. This gives a homeomorphism between smooth closed three-manifolds which is already a diffeomorphism on a full open two-sided neighborhood of the equator. Let K be the equator together with a closed thinner two-sided collar containing every preassigned protected germ. The classical relative homeomorphism-approximation theorem, in the form stated by Pardon [Par21, Theorem 4.6], supplies a diffeomorphism of the two doubles equal to the doubled homeomorphism on a neighborhood of K .

The equator separates each double into two connected components. On its fixed collar the approximating diffeomorphism is the side-preserving product map, so the connected image of the selected source side lies in the selected target side. The same argument for the inverse gives the reverse inclusion. Restriction to the closures therefore gives the required smooth inner-ball extension. Its determinant is positive on the unchanged collar and hence everywhere on the connected ball.

Gluing this extension to the original outer-collar formula produces $\widehat{\Theta}_{e,u}$; equality on an open product overlap gives agreement of every derivative. The exceptional target uses the relative product isotopy above, whose support has already been proved to miss the side and cap, as one of the protected germs held fixed by the relative smoothing. $\square$

Only after $\widehat{\Theta}_{e,u}$ exists define the native shallow embedding on the closed local half by

$$(101) \quad \widehat{q}_{e,u}^{\text{LS}}(x, t) = (\widehat{h}_{e,u}(x), \lambda_e(t)), \quad (x, t) \in \mathcal{D}_{e,\text{loc}}^{\text{work}}.$$

Because $G_{e,u}^{\text{loc}} = q_u^P$ on that half, (100) gives

$$(102) \quad \boxed{\widehat{\Theta}_{e,u}^{-1} q_u^P = \widehat{q}_{e,u}^{\text{LS}} \quad \text{on } \mathcal{D}_{e,\text{loc}}^{\text{work}}.}$$

The ball extension therefore identifies the native shallow collar with the already fixed vertex collar. This equation follows from the extension; it does not define either endpoint chart.

The realized image of the named native carrier is now well typed:

$$(103) \quad \mathcal{S}_{e,u}^{\text{phys}} := \widehat{\Theta}_{e,u}(\mathcal{S}_{e,u}^{\text{nat}}) \subset c_u(A_{e,u} \times [0, h]) \Subset K_{e,u}^{\text{loc}}.$$

For the eight ordinary incidences it lies more sharply in $c_u(G_{e,u}^{\text{mc}} \times [0, h_0])$; the two exceptional incidences use Lemma 6.2. This is a statement about the named carrier, not about arbitrary interior smoothing support of a future representative.

Finally, and only after the endpoint data at both ends in (99)–(102) are in place, define the native boundary transition

$$(104) \quad \widehat{\phi}_e^\partial = \widehat{h}_{e,w}(\widehat{h}_{e,v})^{-1} : \partial B_{e,v}^{\text{LS}} \longrightarrow \partial B_{e,w}^{\text{LS}}.$$

Equation (104), the ten atlases, and the deterministic extension recipe form an *uninstantiated compatible-extension schema*. They do not select or name a cut representative.

Remark 6.4 (What remains deliberately unfrozen). The final spectator ball maps and the theorem's ten atlases $\Xi_{e,u}^z$ are not the maps $\widehat{\Theta}_{e,u}$. Their two complementary disk charts must first be chosen relative to actual overlap bands and compatible inward collars; that construction occurs in Section 8. Only after all ten final endpoint atlases have been frozen may the compatible-extension schema select the five cut representatives. This preserves both the vertex-first quantifier order and the distinction between the local interface E_e and the global interface E .

7. COMPATIBLE ENDPOINT EXTENSIONS AND THE EXCEPTIONAL-RIBBON RESERVE

Independent fillings of the two endpoint disks need not agree in the local model. The endpoint transition incorporates both fillings, and its relative extension will transport the complete marking without requiring their gauges to coincide. The exceptional-ribbon reserve must also be fixed before that transition is used to select an edge representative.

The vertex construction of Section 5 is complete before the present section begins. In particular, the five cut balls K_u , the maps q_u^P , the vertex collars

$$z_u : D^2 \times [0, u_0] \longrightarrow K_u,$$

and the complete markings are already fixed. For expository continuity, the first subsection spells out the pre-edge exceptional-reserve choice announced before Lemma 6.1. Its logical inputs are only these vertex data and the named native carrier; it precedes the choice of A_{jk} and all endpoint chart-and-marking systems. From the second subsection onward, the common local models and ten incidence labels constructed in Section 6 are fixed, and nothing below alters them.

The reserve keeps the longitude correction inside the common local model. The independent endpoint charts then determine the boundary transition and its relative extension recipe. Selection of the five representatives waits until the ten spectator atlases of Section 8 are fixed, so both halves of every cut ball already have compatible interface germs.

7.1. The exceptional $AC\star$ ribbon. For an edge $e : v \rightarrow w$, retain the named native carrier of Section 6 under the notation

$$\mathcal{S}_{e,u}^{\text{LS,geom}} := \mathcal{S}_{e,u}^{\text{nat}}.$$

It consists of the mover meridian tube, the slide band, the carrier longitude and its push-off, the listed smoothing boxes, and the direct-dual ribbons, with their prescribed product germs. This notation does not mean the support of an arbitrary interior smoothing of a cut-ball representative. At the two $AC\star$ incidences it also contains the complete source reserve or terminal raw-to-target sweep.

The terminal longitude is not defined by an edge map. Before any edge representative is chosen, the direct dual construction fixes

$$L_{ji} = \ell_j \natural_{\alpha^\vee} (-\ell_i)^+, \quad K_{ki} = \ell_k \natural_{\beta_{ki}^\vee} (-\ell_i)^+, \quad K_{kj} = \ell_k \natural_{\beta_{kj}^\vee} (-\ell_j)^+.$$

The raw $AC\star$ update is

$$(105) \quad \tilde{K} = K_{ki} \natural_{\beta_{AC\star}^\vee} (-L_{ji})^+.$$

The exceptional reserve is built inside the direct-dual track and then enlarged, without rechoice, to the common disk A_{jk} in Lemma 6.1. Let $V_5 = \{0, B, A, AC, \star\}$, and let $\mathcal{U}_{\text{DQ}}^{\text{top}}$ be the frozen open direct-dual longitude reserve containing the cancelling $i^{-1}i$ -track. Before any edge, choose closed plumbing boxes

$$(106) \quad R^{\text{top}} \Subset \text{int } R^{+, \text{top}}, \quad \overline{R^{+, \text{top}}} \Subset \mathcal{U}_{\text{DQ}}^{\text{top}}.$$

The larger box still misses every meridian disk and meridian-band germ. Cutting therefore carries the boxes, without splitting them, to typed patches

$$(107) \quad R_v \Subset \text{int } R_v^+ \subset \partial K_v, \quad v \in V_5.$$

Thus every avoidance assertion below is row-typed inside a single ∂K_v . At the terminal row $R_\star$ is the rectangle called R below; $R_\star^+$ is only its support buffer and does not enlarge the moving ear or weaken literal equality off R .

Remove the participating subribbons before forming a forbidden set. The row-typed participant union $\mathfrak{P}_v^\partial$ is

v	$\mathfrak{P}_v^\partial$
$0, A$	$\emptyset$,
B	the designated K_{kj} -subribbon in R_B ,
AC	the exact K_{ki} - and $(-L_{ji})^+$ -subribbons and the $\beta_{AC\star}^\vee$ joining strip used by (105),
$\star$	the raw and terminal subribbons and their complete swept surface strip.

(108)

These are alternatives in different cut spheres. In particular, the participating pieces of K_{ki} and L_{ji} at AC are allowed to enter R_{AC} ; their common portions outside that box remain fixed.

For each row let $\mathfrak{F}_v^\partial$ be the union of the following closed boundary footprints after deleting $\mathfrak{P}_v^\partial$:

(F1) the global E -footprint; the complete disk footprints of

$$i_0, j_0, k_0, j_\varepsilon, k_\varepsilon, k_{2\varepsilon}, J, J^+, I_A, I_\star;$$

the three meridian bands α, β_0, β_1 ; every smoothing box; and all six signed face-collar footprints $D_{r,v}^\pm$, $r \in \{i, j, k\}$;

(F2) every complete cut-dual ribbon not in $\mathfrak{P}_v^\partial$, so at $\star$ this includes the fixed ℓ_i - and L_{ji} -ribbons, while at AC it excludes precisely the participating pieces in the raw word;

(F3) all fifteen auxiliary spoke ribbons, their attachment sectors on global E and the signed faces, and their reserved ribbon tubes;

(F4) all five detector germs with both framing vectors, every two-sided signed-face pairing footprint, and every protected complementary-disk annulus;

(F5) the global E -bicollar footprint and every nonparticipating interface-side reserve among

$$E_{0B,0}, E_{0B,B}, E_{B\star,B}, E_{B\star,\star}, E_{0A,0}, E_{0A,A}, \\ E_{AAC,A}, E_{AAC,AC}, E_{AC\star,AC}, E_{AC\star,\star}.$$

At this stage these are only boundary or side-reserve roles: no future local-base interior, depth cap, or endpoint witness is being imported.

For each nonempty participant row choose a closed rounded reserve

$$(109) \quad \mathcal{S}_{\text{res},v}^\partial \subseteq \text{int } R_v^+$$

containing exactly the participating source strips at AC , the designated strip at B , and at $\star$ both endpoint strips, all intermediate strips, and the common tails. The common tails may meet ∂R_v , but the whole reserve lies compactly in R_v^+ . The routing is chosen jointly inside the open direct-dual reserve so these sets miss $\mathfrak{F}_v^\partial$. Fixed boundary metrics and the finite number of rows give the typed clearance

$$(110) \quad \delta_\partial = \min_{\mathcal{S}_{\text{res},v}^\partial \neq \emptyset} \text{dist}_{\partial K_v}(\mathcal{S}_{\text{res},v}^\partial, \mathfrak{F}_v^\partial) > 0.$$

Both arguments of each distance lie in the same boundary sphere.

Use the retained depth d_{prod} from (69), which lies below every relevant positive product-germ depth. In the terminal cut ball set

$$(111) \quad \widehat{\mathcal{S}}_{\text{res}} = c_\star(\mathcal{S}_{\text{res},\star}^\partial \times [0, d_{\text{prod}}]),$$

and let $\widehat{\mathfrak{F}}_\star^+$ be the finite union of the closed depth- d_{prod} positive inward halves of all nonparticipating terminal germs in (F1)–(F5). Product separation and (110) give

$$(112) \quad \delta_{\text{amb}} = \text{dist}_{K_\star}(\widehat{\mathcal{S}}_{\text{res}}, \widehat{\mathfrak{F}}_\star^+) > 0.$$

The negative halves of the two-sided pairings stay in their local doubles; their boundary traces occur in $\mathfrak{F}_\star^\partial$, while their realized positive halves occur in $\widehat{\mathfrak{F}}_\star^+$. Thus no negative germ is placed illegally inside $K_\star$.

Now make the simultaneous final pre-edge scale choice announced before Lemma 6.1. First choose the coordinate width a and the moving surface-ribbon radii inside the open reserve so that the ensuing compact clearances remain positive. Then choose the final restrictions of h_0, n_0, n_1, c_P , together with the subordinate $\epsilon_{\text{work}}, u_0, \delta_{\text{rig}}$, the restricted source pairing depth $\epsilon_{\text{pair}}^{\text{src}}$, and ϵ_c , so that

$$(113) \quad 3c_P < n_0 < n_1 < h_0, \quad u_0 = c_P/2, \quad 0 < \epsilon_c < \epsilon_{\text{pair}}^{\text{src}} < \delta_{\text{rig}} < \epsilon_{\text{work}} < c_P, \\ 20h_0 < \min\{\delta_\partial, \delta_{\text{amb}}, d_{\text{prod}}\},$$

every moving surface-ribbon radius is less than $\delta_\partial/20$.

The open reserve permits a to be chosen sufficiently small that these nested inequalities are compatible with $c_P \ll a \ll h_0$. This is one initial finite choice, made before the common disks

and endpoint chart-and-marking systems; it is not a later change to the already fixed vertex data. It preserves (49) and (69).

For completeness, let $N_{\text{sweep}}^\partial$ denote the compact terminal surface strip swept by the explicit family ρ_t below, including its common tails. The simultaneous direct-dual choice puts

$$b_{AC}^{-1}(\mathcal{S}_{\text{res},AC}^\partial) \quad \text{and} \quad b_\star^{-1}(N_{\text{sweep}}^\partial)$$

in the same protected j/k -arm of W . Choose the truncated j - and k -branch trees, their spokes stopped before global E , and a protected connector through both compact sets while avoiding the i -branch. The choices are made jointly so that, after retracting the ribbon reserves to their core tracks, the resulting spine is a tree and the unretracted union has a disk regular neighborhood. A slightly enlarged closed rounded such neighborhood is a disk

$$(114) \quad A_{jk}^{\text{ear}} \subseteq W, \quad b_{AC}^{-1}(\mathcal{S}_{\text{res},AC}^\partial) \cup b_\star^{-1}(N_{\text{sweep}}^\partial) \subseteq \text{int } A_{jk}^{\text{ear}}.$$

This construction uses only the common pre-edge surface and the already fixed maps $b_{AC}, b_\star$. Put

$$(115) \quad \begin{aligned} A_{jk,v}^{\text{ear}} &= b_v(A_{jk}^{\text{ear}}), \\ \widehat{\mathcal{S}}_{AC}^{\text{ear}} &= c_{AC}(\mathcal{S}_{\text{res},AC}^\partial \times [0, n_1]), \\ E_{AC\star,v}^{\text{ear}} &= c_v(A_{jk,v}^{\text{ear}} \times \{2h\}) \cup c_v(\partial A_{jk,v}^{\text{ear}} \times [0, 2h]), \quad v \in \{AC, \star\}. \end{aligned}$$

Include $\partial A_{jk,v}^{\text{ear}}$ in the corresponding interface-side forbidden footprint in (F5), and include

$$c_\star(\partial A_{jk,\star}^{\text{ear}} \times [0, d_{\text{prod}}]) \subset \widehat{\mathfrak{F}}_\star^+.$$

Take the minima in (110) and (112) after these additions. The compact margin in (114) keeps both endpoint reserves away from these sides, and $n_1 < h < d_{\text{prod}} < 2h$ keeps the moving support away from the caps. After this pre-edge disk is fixed, Lemmas 6.1 and 6.2 enlarge it to the actual endpoint base and bind it to the endpoint-specific local ball by

$$A_{jk}^{\text{ear}} \subseteq \text{int } A_{jk}, \quad b_v|_{A_{jk}^{\text{ear}}} = b_v|_{A_{jk}^{\text{ear}}}$$

and the already fixed collar c_v supplies the common product coordinate through depth $2h$. Thus the surface reserve and its inward sweep lie in the same endpoint-specific local ball.

Lemma 7.1 (Exceptional-ribbon reserve and isotopy). *The fixed direct-dual choices come with the closed rounded disk A_{jk}^{ear} of Lemma 6.1 and admit an ambient isotopy $\widehat{H}_t$ such that:*

- (i) $A_{jk}^{\text{ear}} \subseteq \text{int } A_{jk}$, and the source and terminal ribbon sweeps have compact base projection in A_{jk}^{ear} ;
- (ii) $\widetilde{K}$ and K_{kj} , including their parameterized ribbon and inward-normal germs, agree literally outside one closed rectangle R ;
- (iii) on the whole protected ribbon germ,

$$(116) \quad \widehat{H}_t \widehat{\rho}_0(r, s, n) = \widehat{\rho}_t(r, s, n), \quad |r| \leq r_1, \quad 0 \leq n \leq n_0,$$

so the longitudinal parameter s , surface-normal parameter r , and inward-normal parameter n retain their meanings throughout;

- (iv) $\widehat{H}_1 \widehat{\rho}_0 = \widehat{\rho}_{kj}$, its support is contained at depth $< n_1 < h_0 < h$, and it misses the side and cap of the actual $AC\star$ local ball as well as every nonparticipating protected germ.

The target K_{kj} is fixed before $\widehat{H}_t$; the arrow is from the raw ribbon to that pre-existing target ribbon, never conversely.

Bigon interpolation in the reserved disk. The raw and terminal ribbons have common tails, so the discrepancy is localized to one empty bigon in the reserved disk. An endpoint-flat interpolation of that bigon lifts through the fixed collar to an ambient flow; the depth cutoff preserves the full ribbon germ, while the clearance inequalities keep the motion away from the side, cap, and nonparticipating data. The final endpoint binding therefore changes only the exceptional raw-ribbon component.

Proof. A boundary ribbon isotopy. The oriented traversal of (105) is

$$\ell_k \longrightarrow (-\ell_i)^+ \longrightarrow (\ell_i)^+ \longrightarrow (-\ell_j)^+.$$

The two middle tracks are adjacent and oppositely oriented. Route the raw and terminal alternatives on the same ℓ_k -tail and $(-\ell_j)^+$ -tail, and place the intervening empty bigon in a rectangle

$$R = [-2, 2]_x \times [7a/2, 11a/2]_u.$$

With the endpoint-flat function

$$\psi(s) = \begin{cases} \exp\left(1 - \frac{1}{1-s^2}\right), & |s| < 1, \\ 0, & |s| \geq 1, \end{cases}$$

the middle arcs

$$\gamma_t(s) = \left(-1 + 2(1-t)\psi(s), \frac{9a}{2} + \frac{a}{2}s\right), \quad -1 \leq s \leq 1,$$

are embedded, agree to every order at their endpoints, and interpolate from the reserve arc to the straight terminal arc. Thickening them by

$$\rho_t(r, s) = \left(-1 + 2(1-t)\psi(s) + r, \frac{9a}{2} + \frac{a}{2}s\right)$$

gives orientation-preserving ribbon charts because

$$\det \frac{\partial(x, u)}{\partial(r, s)} = \frac{a}{2} > 0.$$

More precisely, if T_{tail}^∂ is the union of the two common closed tail strips, then

$$(117) \quad N_{\text{sweep}}^\partial = T_{\text{tail}}^\partial \cup \bigcup_{0 \leq t \leq 1} \rho_t([-r_1, r_1] \times [-1, 1]) \subseteq \mathcal{S}_{\text{res}, \star}^\partial.$$

It is compact, and (110) separates it from every nonparticipating disk, ribbon, spoke, detector, pairing, annular, and global-interface footprint. Its positive product lift lies in $\widehat{\mathcal{S}}_{\text{res}}$, so (112) gives the corresponding ambient separation.

Extension through the fixed collar. On the moving ribbon tube use the time-dependent field $V_t(\rho_t(r, s)) = -2\psi(s)\partial_x$, cut off before the boundary of the reserve. Its flow H_t satisfies $H_t\rho_0 = \rho_t$ by uniqueness of integral curves. Lift it through the fixed endpoint collar $c_\star$, with a depth cutoff equal to one on $[0, n_0]$ and zero from n_1 onward. The lifted field has no ∂_n -component, so its flow gives (116) and preserves both framing vectors. Flatness of ψ gives literal agreement of all tangential and normal derivatives with the fixed tails.

Comparison with the fixed endpoint map. Finally, (114) puts both endpoint sweep preimages compactly in the interior of the pre-edge reserve disk. Its boundary was included in the forbidden set before the final clearance minima were taken. The endpoint binding displayed above agrees with the fixed vertex maps on that disk and uses their common collar through depth $2h$. The strict inequalities

$$3c_P < n_0 < n_1 < h_0 < h < 2h$$

together with (113) therefore place the closed source and terminal sweeps away from the actual side and cap and from all nonparticipating protected germs. This is the required binding to A_{jk} , rather than an assumption imported from the preliminary sharp-row construction. $\square$

7.2. Independent native atlases. Recall from (67) that

$$\mathcal{I}_{10} = \{(e, u) : e \in E_5, u \text{ is an endpoint of } e\}.$$

The common local boundary model has the finite face list

$$E_e, p^-, p^+, q^-, q^+; \quad C_1, \dots, C_{16}; \quad R_{s_p}, R_{c_p}, R_{s_q}, R_{c_q}; \quad Z_e.$$

Here p, q are the moving and carrier roles and Z_e is the unique unmarked complementary disk.

Proposition 7.2 (Independent endpoint chart-and-marking systems). *For every $(e, u) \in \mathcal{I}_{10}$, the atlas already fixed in (94) is an orientation-preserving full boundary atlas*

$$(118) \quad \widehat{h}_{e,u} : \partial K_{e,\text{mod}}^{\text{loc}} \longrightarrow \partial B_{e,u}^{\text{LS}}.$$

The ten atlases were fixed independently of all edge representatives and of the atlas at the other endpoint. They intertwine the two abstract pairings with the endpoint pairings, use the common oriented interface coordinate, and contain a full positive disk chart

$$\widehat{\zeta}_{e,u} : D^2 \longrightarrow Z_{e,u}^{\text{LS}}$$

whose germ is fixed on the whole overlap annulus. Their residual freedom is exactly

$$(119) \quad \prod_{(e,u) \in \mathcal{I}_{10}} \text{Diff}^+(D^2 \text{ rel } \mathcal{A}_{e,\text{mod}}).$$

Proof. Its construction maps the first five faces, sixteen collar rectangles, and four ribbon rectangles through the corresponding endpoint's own fixed product charts. On overlaps, the two incident faces use the same tangent parameter and opposite boundary-side designation. The signed pairings are tied to the same abstract involutions, and the interface charts use one oriented disk coordinate. Thus these maps form a smooth protected atlas rather than a list of unrelated face maps.

Concatenate the sixteen oriented edge parameters to $\theta \in \mathbb{R}/\mathbb{Z}$, assigning the m -th edge the interval $[(m-1)/16, m/16]$. Together with the common radial variable this gives the full annular chart. Fill the remaining disk independently at each endpoint, relative to an open neighborhood of that annulus. All other charts were already fixed, so the transition between two possible fillings is precisely a factor in (119). $\square$

For the two $AC\star$ incidences, the source atlas contains the direct L_{ji} - and K_{ki} -rows and hence the raw ribbon $\widehat{K}$. The target atlas contains the independently fixed K_{kj} -row. The map $\widehat{H}_1$ from Lemma 7.1 is used only when the target endpoint is identified with its endpoint-specific local ball.

7.3. The common template and quotient-compatible endpoint extensions. Using the rounded endpoint chart already fixed at endpoint u , retain

$$G_{e,u}^{\text{loc}}(x, t) = c_u(b_u(x), t), \quad g_{e,u}^{\text{loc}} = G_{e,u}^{\text{loc}}|_{\partial K_{e,\text{mod}}^{\text{loc}}}.$$

The pair already fixed, before either endpoint extension and before any representative, in (96) is

$$(120) \quad \ell_e > 0, \quad \lambda_e : [0, h_0] \longrightarrow [0, \ell_e], \quad \lambda_e(0) = 0, \quad \lambda'_e > 0, \quad \lambda_e(t) = t \text{ near } 0,$$

with $\lambda_e([0, \epsilon_{\text{work}}]) \subseteq [0, \ell_e]$. The same pair (ℓ_e, λ_e) is used at both endpoints.

Proposition 7.3 (Endpoint extension and native shallow identity). *The endpoint-only boundary map of (97) is*

$$\widehat{a}_{e,u}^{\text{loc}} = g_{e,u}^{\text{loc}}(\widehat{h}_{e,u})^{-1}.$$

The orientation-preserving diffeomorphism fixed in Lemma 6.3,

$$\widehat{\Theta}_{e,u} : B_{e,u}^{\text{LS}} \longrightarrow K_{e,u}^{\text{loc}},$$

has the full collar formula

$$(121) \quad \widehat{\Theta}_{e,u}(\widehat{h}_{e,u}(x), \lambda_e(t)) = G_{e,u}^{\text{loc}}(x, t)$$

and is coordinate identity on the signed local interface, with all jets. After $\widehat{\Theta}_{e,u}$ is fixed, the shallow map of (101) is

$$\widehat{q}_{e,u}^{\text{LS}}(x, t) = (\widehat{h}_{e,u}(x), \lambda_e(t))$$

on the common local shallow domain. Then

$$(122) \quad \widehat{\Theta}_{e,u}^{-1} q_u^P = \widehat{q}_{e,u}^{\text{LS}}.$$

Moreover, the transported named carrier satisfies

$$(123) \quad \mathcal{S}_{e,u}^{\text{LS,phys}} := \widehat{\Theta}_{e,u}(\mathcal{S}_{e,u}^{\text{LS,geom}}) \subset c_u(A_{e,u} \times [0, h)) \Subset K_{e,u}^{\text{loc}}.$$

Proof. In the two full atlases, $\widehat{a}_{e,u}^{\text{loc}}$ is coordinate identity on the protected region, the full complementary disk, their whole annular overlap, and the signed interface. At the exceptional target, compose only the raw-ribbon component with $\widehat{H}_1$; by Lemma 7.1 this changes no other endpoint datum. The hypotheses of the already established relative local-ball extension Lemma 6.3 are therefore satisfied endpoint by endpoint. Its conclusion is exactly (121).

The definition of $\widehat{q}_{e,u}^{\text{LS}}$ is made only after that extension exists. On the common local shallow half, $G_{e,u}^{\text{loc}} = q_u^P$, and (122) follows by applying $\widehat{\Theta}_{e,u}^{-1}$ to (121). The identity thus uses only the data at this endpoint, independently of the opposite endpoint and any edge representative.

The named carrier has pre-edge base closure in $\text{int } A_{e,u}$ and native depth $< h_0 < h$. Formula (121) therefore gives (123). For the exceptional incidences the side-and-cap control comes from Lemma 7.1. No statement about the support of the later interior smoothing is made. $\square$

7.4. The uninstantiated extension schema. Only after the two endpoint constructions in Proposition 7.3 exist is the boundary transition already fixed in (104) retained as

$$(124) \quad \widehat{\phi}_e^\partial := \widehat{h}_{e,w}(\widehat{h}_{e,v})^{-1} : \partial B_{e,v}^{\text{LS}} \longrightarrow \partial B_{e,w}^{\text{LS}}.$$

It sends every protected row by the common label dictionary and absorbs the two independent complementary-disk gauges. Indeed, if $\widehat{\zeta}_{e,u} = \zeta_{e,u}^0 \psi_{e,u}$, then its disk row is

$$\zeta_{e,w}^0 \psi_{e,w} \psi_{e,v}^{-1} (\zeta_{e,v}^0)^{-1}.$$

Thus neither gauge is synchronized with the other.

Proposition 7.4 (Compatible endpoint-extension schema). *The boundary map (124) is orientation preserving, intertwines separately the two source active pairings with their corresponding target pairings, is the identity in the signed interface coordinate, and lies in the boundary relative class of the positive two-site slide from Theorem 2.2. It determines a recipe for choosing an orientation-preserving cut-ball representative in that theorem's B0/B5 class, with the same full collar formula and all marked product germs.*

Relative Alexander isotopy on the remaining disk. The two endpoint chart-and-marking systems already agree on the protected cellulation and annulus, so their quotient is confined to the remaining disk. A relative Alexander isotopy identifies that disk map with the positive local boundary class; product extension across a collar and relative approximation then produce the smooth cut-ball representative while preserving every prescribed germ.

Proof. Smoothness, orientation, pairing compatibility, and the interface formula follow directly from the two independently constructed atlases. On the protected cellulation the map is the cellular B0 dictionary. On the complementary disk, its quotient with the B0 boundary reference fixes the whole overlap annulus. The explicit radial Alexander isotopy on the inner disk therefore gives a topological isotopy relative to the protected region; no smooth relative isotopy is asserted.

Extend the boundary map as a product on a full collar and radially extend the remaining inner sphere map. Denote the resulting ball homeomorphism by $\widehat{\Phi}_e^A$, and double it across the boundary. The doubled map is already smooth on a full two-sided equatorial neighborhood. Let K_e be the equator together with a closed thinner two-sided collar containing all prescribed product germs. Applied directly to this doubled homeomorphism, the classical relative homeomorphism-approximation theorem, in the form stated by Pardon [Par21, Theorem 4.6], gives a diffeomorphism of the source double to the target double which equals the doubled map on a neighborhood of K_e .

The equator separates both doubles. Since the new diffeomorphism is the side-preserving product map near the equator, connectedness sends the chosen source side into the chosen target side; applying the same argument to the inverse proves equality of those sides. Restriction gives a temporary smoothed recipe output $\widehat{\Phi}_e^{\text{sm}}$ with all prescribed product germs. Its collar determinant is positive, hence so is its determinant everywhere by connectedness.

Here is the full three-dimensional B0/B5 comparison. The maps $\widehat{\Phi}_e^A$ and $\widehat{\Phi}_e^{\text{sm}}$ agree on the same full boundary collar. After deleting a thinner open collar, identify the remaining source with the standard closed ball and form

$$Q_e = (\widehat{\Phi}_e^{\text{sm}})^{-1} \widehat{\Phi}_e^A.$$

Its restriction to the inner ball fixes the boundary pointwise. For $0 < s \leq 1$, the dimension-three Alexander formula

$$\mathcal{H}_{e,s}(x) = \begin{cases} sQ_e(x/s), & |x| \leq s, \\ x, & s \leq |x| \leq 1, \end{cases} \quad \mathcal{H}_{e,0}(x) = x$$

is a jointly continuous isotopy from the identity to Q_e , with inverse given by the same formula using Q_e^{-1} . Extend it by the identity over the deleted collar. Then $\widehat{\Phi}_e^{\text{sm}} \mathcal{H}_{e,s}$ is a topological isotopy, relative to the full prescribed collar, from the smoothed recipe output to $\widehat{\Phi}_e^A$.

The two-dimensional Alexander path constructed above compares the boundary of $\widehat{\Phi}_e^A$ with the B0 cellular boundary map while fixing the protected region, the complete overlap annulus, both pairing equations, and the signed interface germ. At each parameter extend that boundary path as a product on the same collar and use the radial Alexander extension on the complementary inner ball. The collar and radial formulas agree at their seam and vary continuously, giving a topological ball isotopy, relative to all those protected data, from $\widehat{\Phi}_e^A$ to the B0 ball extension. Concatenating the two ball isotopies proves that the smoothed recipe output lies in exactly the specified B0/B5 relative class. Only a topological relative isotopy is used; no smooth relative-isotopy uniqueness is asserted. $\square$

Remark 7.5 (No representative has yet been selected). At this stage (124) and Proposition 7.4 specify a deterministic construction recipe only. The symbol $\widehat{\phi}_e^{\text{cut}}$ is not bound until all ten spectator atlases have been constructed in Proposition 8.5. Consequently no shallow, pairing, or interface equation involving that representative is used in this section.

8. RELATIVE BANDS, DISKS, AND DECORATED-BALL EXTENSION

The spectator map must match the vertex disk collar in an inner region and the local construction in an outer region. Agreement on their middle seam does not determine the transition on the complete protected annulus required for relative ball extension. Starting from the local endpoint maps of Proposition 7.3, we choose the actual annular overlap bands, then compatible full-disk charts and inward collars. Their full-annulus agreement supplies the relative extension hypothesis, allowing extension over the spectator halves while retaining all protected boundary data.

The capital letters used below name the four disk rows, not additional edge maps. The short lower and upper rows are $J = j_0 \#_{\beta_0} k_\epsilon$ and $J^+ = j_\epsilon \#_{\beta_1} k_{2\epsilon}$; the two longer rows are $I_A = i_0 \#_\alpha j_\epsilon$ and $I_\star = i_0 \#_\alpha J^+$. Likewise, E denotes the global pentagon interface, while E_e is the disk interface belonging to the single edge e . The several collar depths introduced in this section refine the vertex hierarchy already fixed in (49) and, at the exceptional edge, the chain in (69); they do not define independent geometric layers.

8.1. Relative overlap bands. Retain the positive disk chart

$$\zeta_{\text{mod}} : D^2 \xrightarrow{\cong} Z_{\text{mod}} \subset S$$

and let $r : D^2 \rightarrow [0, 1]$ be the polar radius. On the model surface put

$$\bar{r}(y) = \begin{cases} r(\zeta_{\text{mod}}^{-1}y), & y \in Z_{\text{mod}}, \\ 1, & y \notin \text{int } Z_{\text{mod}}. \end{cases}$$

This is a continuous surface function; no derivative of $\bar{r}$ will be taken. The radial pre-edge choice of the work surface is recorded, without changing it, by

$$(125) \quad \begin{aligned} &0 < r_{\text{work}} < r_{\text{keep}} < r_P < r_A < 1, \\ &W = \{y \in S : \bar{r}(y) > r_{\text{work}}\}, \quad \mathcal{A}_{\text{mod}}^{\text{work}} = \{r > r_{\text{work}}\}, \quad \mathcal{A}_{\text{mod}}^{\text{open}} = \{r > r_{\text{keep}}\}, \\ &\mathcal{A}_{\text{mod}} = \{r \geq r_A\}, \quad \mathcal{A}_{\text{mod}} \subset \mathcal{A}_{\text{mod}}^{\text{open}} \Subset \mathcal{A}_{\text{mod}}^{\text{work}}, \quad \zeta_{\text{mod}}(\mathcal{A}_{\text{mod}}^{\text{open}}) \subset W \cap Z_{\text{mod}}. \end{aligned}$$

The full protected annulus in (125) is different from the artificial inner/outer seam introduced below.

For each edge, let K_e^S be the nonempty compact base trace of its rounded local support and spectator spine, excluding the still-free complementary-disk fills and annular neighborhoods. The footprints have already been chosen in the open work surface, so

$$\alpha = \min_{e \in E_5} \min_{y \in K_e^S} \bar{r}(y) > r_{\text{work}}.$$

Choose, before filling any complementary disk,

$$(126) \quad r_{\text{work}} < r_0 < r_- < r_+ < r_1 < \alpha, \quad r_c = \frac{1}{2}(r_{\text{work}} + r_0),$$

then choose

$$0 < \varepsilon_{01} < \frac{1}{2} \min\{r_0 - r_{\text{work}}, r_- - r_0, r_+ - r_-, r_1 - r_+, 1 - r_1\}$$

and the open annular germ

$$\mathcal{N}_{01}^D = \{r_0 - \varepsilon_{01} < r < r_1 + \varepsilon_{01}\}$$

whose closure therefore retains a positive margin from every adjacent radius. Write

$$\mathcal{A}_{01}^D = \{r_0 \leq r \leq r_1\}, \quad C_a^D = \{r \leq a\}.$$

Put

$$r_{cc} = \frac{1}{2}(r_{\text{work}} + r_c), \quad D_c = C_{r_{cc}}^D, \quad D_c^{++} = C_{r_c}^D.$$

Both are new witnesses; D_c^{++} is the larger safe core disk.

Proposition 8.1 (Relative band and disk data). *For every edge e there are:*

(i) *a label $\sigma_e \in \{+, -\}$ and a positive full-disk chart*

$$j_e^c : C_{r_1}^D \xrightarrow{\cong} Z_e^{\text{sp}, \sigma_e};$$

(ii) *a common pre-extension shallow embedding*

$$q_e^{\text{sp}} : \mathcal{D}_{e, \text{sp}}^{\text{work}} \hookrightarrow L_{e, \text{mod}}^{\text{sp}};$$

(iii) *for each endpoint u , a pre-extension product map $\hat{a}_{e, u}^{\text{sp}}$ on an open germ containing $\mathcal{A}_{01}^D \times [0, d_{\text{ann}})$;*

(iv) *realized endpoint disk charts with the same label σ_e , satisfying*

$$b_u \zeta_{\text{mod}}(C_{r_1}^D) = Z_{e, u}^{\text{sp}, \sigma_e}, \quad b_u \zeta_{\text{mod}}(\partial C_{r_1}^D) = \partial Z_{e, u}^{\text{sp}, \sigma_e}.$$

They satisfy

$$(127) \quad P_e^{\text{sp}} \cap Z_e^{\text{sp}, \sigma_e} = j_e^c(\mathcal{A}_{01}^D),$$

$$(128) \quad \hat{a}_{e, u}^{\text{sp}} q_e^{\text{sp}}(\zeta_{\text{mod}}(x), t) = q_u^P(\zeta_{\text{mod}}(x), t) = c_u(b_u \zeta_{\text{mod}}(x), t)$$

for $x \in \mathcal{A}_{01}^D$ and $0 \leq t < d_{\text{ann}}$, with all three formulas defined on the surrounding open product germ. Moreover,

$$j_e^c(D_c^{++}) \Subset \text{int } Z_e^{\text{sp}, \sigma_e} \quad \text{and} \quad j_e^c(D_c^{++}) \cap P_e^{\text{sp}} = \emptyset,$$

and the same safe disk misses the rounded local support.

Proof. The closed disk $C_{r_1}^D$ misses the already fixed spine and rounded local support. Its connected image lies in one of the two labeled spectator complementary disks; this defines σ_e . Relatively refine the still-free annular neighborhood so that its inner boundary is the image of $r = r_0$, extend that coordinate across both boundary circles on $\mathcal{N}_{01}^D$, and fill positively across the disk. The compact separation from the support gives the assertion about D_c^{++} .

The pre-extension shallow model supplies the common spectator half $\mathcal{D}_{e,sp}^{\text{work}}$, including the outer-work part, the portions of both annular disk germs that lie in the work surface, and the signed internal-interface half. Bind q_e^{sp} there using j_e^c on the work-surface part of the selected disk. The complete disks are chart domains, not an enlargement of this shallow domain.

Compactness gives a common positive d_{ann} for which the whole annulus, not just its middle band, has a product neighborhood. Shrink it, if necessary, so that

$$d_{\text{ann}} < \min\{c_P, \epsilon_{\text{work}}, \epsilon_{\text{pair}}^{\text{src}}\}.$$

Equation (128) then defines $\hat{a}_{e,u}^{\text{sp}}$ from the pre-edge product coordinates. The endpoint's own b_u, c_u fills the realized disk compatibly. No post-extension atlas or edge representative occurs. $\square$

For the later inner/outer proof, keep disk and surface types separate:

$$\begin{aligned} C_+^D &= \{r < r_+\}, & \mathcal{A}_{-+}^D &= \{r_- < r < r_+\}, \\ C_+^S &= \zeta_{\text{mod}}(C_+^D), & \mathcal{W}_- &= \{y \in \mathcal{W} : \bar{r}(y) > r_-\}, \\ (129) \quad \mathcal{A}_{-+}^S &= \zeta_{\text{mod}}(\mathcal{A}_{-+}^D) = (C_+^S \cap \mathcal{W}) \cap \mathcal{W}_-. \end{aligned}$$

Then

$$(130) \quad \mathcal{W} = (C_+^S \cap \mathcal{W}) \cup \mathcal{W}_-, \quad D^2 = C_+^D \cup \zeta_{\text{mod}}^{-1}(\mathcal{W}_- \cap Z_{\text{mod}}).$$

8.2. A relative decorated-ball extension.

Lemma 8.2 (Relative decorated-ball extension). *Let B_0, B_1 be compact smooth oriented 3-balls. Suppose*

$$\partial B_i = P_i \cup \bigcup_{\lambda \in \Lambda} Z_{i,\lambda}, \quad P_i \cap Z_{i,\lambda} = A_{i,\lambda},$$

where Λ is finite, the $Z_{i,\lambda}$ are pairwise disjoint closed disks, and the $A_{i,\lambda}$ are full closed annuli. Let $a : P_0 \rightarrow P_1$ be an orientation-preserving diffeomorphism in compatible product charts, preserving every label and acting as the identity in a specified interface product coordinate.

Assume full positive disk charts $\zeta_{i,\lambda} : D^2 \rightarrow Z_{i,\lambda}$ have already been chosen and that, on every complete overlap annulus,

$$(131) \quad \zeta_{1,\lambda}^{-1} a \zeta_{0,\lambda}(r, [\theta]) = (r, [f_\lambda(\theta)]), \quad f_\lambda(\theta + 1) = f_\lambda(\theta) + 1, \quad f'_\lambda(\theta) > 0.$$

Then a extends to an orientation-preserving diffeomorphism

$$T : B_0 \longrightarrow B_1$$

with the prescribed full outer-collar formula and identity interface product germ, hence with equality of every jet on every protected germ. If a finite union of compatible open product germs is specified in advance, the extension may be required to agree there as well.

Radial interpolation and collar extension. The annular normal form is first interpolated radially between the prescribed boundary map and the identity near each disk centre. Endpoint flatness gives agreement of all tangential and normal derivatives on the full annuli, after which collars and the doubled-ball approximation extend the boundary map without disturbing the protected germs. The full disk charts are therefore essential hypotheses of the extension.

Proof. Choose an endpoint-flat radial cutoff η , equal to zero near the center and one on a neighborhood of each full annulus, and set

$$f_{\lambda,s}(\theta) = (1-s)\theta + s f_\lambda(\theta).$$

Since $\partial_\theta f_{\lambda,s} = (1-s) + s f'_\lambda(\theta) > 0$, the formula

$$a_{Z,\lambda} \zeta_{0,\lambda}(r, [\theta]) = \zeta_{1,\lambda}(r, [f_{\lambda,\eta(r)}(\theta)])$$

is a positive disk diffeomorphism, coordinate identity near the center, and equal to a on the entire annulus. Endpoint flatness gives agreement of all tangential and normal derivatives at both radial seams. The disk maps and a therefore glue to an orientation-preserving sphere diffeomorphism a_∂ .

Choose inward collars extending all named normal coordinates. On a full outer subcollar use

$$c_0(x, t) \longmapsto c_1(a_\partial(x), t).$$

After removing a smaller open collar, the remaining pieces are closed balls. Write $\psi_\partial : S^2 \rightarrow S^2$ for the induced inner-sphere homeomorphism. In fixed ball coordinates extend it by the radial Alexander formula $ru \mapsto r\psi_\partial(u)$. This gives a ball homeomorphism already smooth on a full boundary collar.

Double the balls and this map. The double is smooth on a full two-sided equatorial neighborhood. Let K be the equator together with a closed thinner two-sided collar containing all preassigned compatible germs. The classical relative homeomorphism-approximation theorem, in the form stated by Pardon [Par21, Theorem 4.6], applied to the doubled homeomorphism itself, supplies a diffeomorphism of the source double to the target double which equals the doubled map on a neighborhood of K . The equator separates both doubles, and the fixed product germ identifies the selected source and target sides. Connectedness puts the image of the selected source side inside the selected target side; the same argument for the inverse proves surjectivity onto it. Restriction to their closures is the required T . Its determinant is positive on the unchanged collar and hence on the connected ball. $\square$

Remark 8.3. The lemma requires the full disk charts and (131) as hypotheses. It does not assert that an arbitrary annulus map has that form in independently chosen disk charts. This is why Proposition 8.1 precedes every spectator extension.

8.3. Compatible core and full-disk collars. On the product neighborhood from Proposition 8.1, let

$$X_e^{\text{fix}} = (q_e^{\text{sp}})_*(\partial_t).$$

Choose $\overline{G_e^{\text{keep}}} \Subset G_e^{\text{work}}$ inside the existing spectator collar, with the keep set containing the entire closed annular overlap through positive depth. Extend the fixed field to a strictly inward field Y_e near the filled disk and use a cutoff χ_e to form

$$X_e^{\text{pre}} = \chi_e X_e^{\text{fix}} + (1 - \chi_e)Y_e.$$

The inward cone is convex. A uniform short-time flow $\Phi_{e,t}$ therefore defines, after shrinking $d_e^c < d_{\text{ann}}$,

$$(132) \quad \iota_{e,\text{loc}}^c(x, t) = \Phi_{e,t}(j_e^c(x)), \quad (x, t) \in C_{r_1}^D \times [0, d_e^c].$$

Uniqueness of integral curves gives, on the complete annulus,

$$(133) \quad \iota_{e,\text{loc}}^c(x, t) = q_e^{\text{sp}}(\zeta_{\text{mod}}(x), t).$$

Because the cutoff is one on an open neighborhood of the closed annulus, this is an all-jet equality at both boundary circles.

Let

$$\zeta_e^{\text{sp}, \sigma_e} : D^2 \longrightarrow Z_e^{\text{sp}, \sigma_e}, \quad \zeta_{e,u}^{\text{sp}, \sigma_e} : D^2 \longrightarrow Z_{e,u}^{\text{sp}, \sigma_e}$$

denote the retained pre-extension positive disk charts, and set

$$(134) \quad \gamma_{e,u}^{\text{pre}} = \zeta_{e,u}^{\text{sp}, \sigma_e}(\zeta_e^{\text{sp}, \sigma_e})^{-1}.$$

Write $a_{e,u}^{\text{sp}}$ for the boundary trace of $\widehat{a}_{e,u}^{\text{sp}}$ on the open annular germ, and put

$$d_{\text{seam}} = \min\{d_{\text{ann}}, d_e^c\}.$$

Proposition 8.4 (Positive spectator disk comparison). *For every incidence (e, u) there is a positive full-disk diffeomorphism*

$$\gamma_{e,u}^z : Z_e^{\text{sp}, \sigma_e} \longrightarrow Z_{e,u}^{\text{sp}, \sigma_e}$$

which equals $b_u \zeta_{\text{mod}}(j_e^c)^{-1}$ on $\mathcal{U}_{\text{in}} = j_e^c(C_+^D)$ and equals $\gamma_{e,u}^{\text{pre}}$ on

$$\mathcal{U}_{\text{out}} = Z_e^{\text{sp}, \sigma_e} \setminus j_e^c(C_{r_-}^D).$$

On the whole protected annulus it agrees with $a_{e,u}^{\text{sp}}$, and its product lift satisfies

$$(135) \quad \hat{\gamma}_{e,u}^z q_e^{\text{sp}}(\zeta_{\text{mod}}(x), t) = \hat{a}_{e,u}^{\text{sp}} q_e^{\text{sp}}(\zeta_{\text{mod}}(x), t) = q_u^P(\zeta_{\text{mod}}(x), t)$$

on an open germ containing $\mathcal{A}_{01}^D \times [0, d_{\text{seam}})$, with all jets. If

$$\zeta_{e,u}^{z, \sigma_e} := \gamma_{e,u}^z \zeta_e^{\text{sp}, \sigma_e},$$

then the disk-extension transition is, on the entire overlap annulus,

$$(136) \quad a_{e,u}^{\text{sp}} = \zeta_{e,u}^{z, \sigma_e} (\zeta_e^{\text{sp}, \sigma_e})^{-1}.$$

Proof. Use the protected formula on $j_e^c(C_+^D)$ and the retained pre-extension disk map outside $j_e^c(C_{r_-}^D)$. On their open overlap the operative chain is

$$(137) \quad \gamma_{e,u}^{\text{pre}} j_e^c(x) = a_{e,u}^{\text{sp}} j_e^c(x) = b_u \zeta_{\text{mod}}(x) = \gamma_{e,u}^z j_e^c(x), \quad x \in \mathcal{A}_{-+}^D.$$

This is a restriction of the pre-extension product identity (128), together with (133), and therefore holds on an open germ, not merely on its central circle. The formulas glue to a positive local diffeomorphism of the closed disk. It is proper, has boundary degree +1, and hence has disk degree +1. A proper local diffeomorphism of a disk of degree one is a one-sheeted covering, so it is a diffeomorphism.

With endpoints included by continuity, the complete annular transition is the exhaustive three-piece formula

$$\gamma_{e,u}^z j_e^c(x) = a_{e,u}^{\text{sp}} j_e^c(x) = \begin{cases} b_u \zeta_{\text{mod}}(x), & r_0 \leq r(x) \leq r_-, \\ b_u \zeta_{\text{mod}}(x) = \gamma_{e,u}^{\text{pre}} j_e^c(x), & r_- \leq r(x) \leq r_+, \\ \gamma_{e,u}^{\text{pre}} j_e^c(x), & r_+ \leq r(x) \leq r_1. \end{cases}$$

Each row preserves the same inward coordinate on a surrounding open product germ. This proves (135) on the entire annulus, including every tangent, normal, and mixed derivative. Equation (136) is then the definition of $\zeta_{e,u}^{z, \sigma_e}$ rewritten in invariantly typed form. $\square$

8.4. Five model collars and ten realized endpoint collars. For a model edge e , let $\mathcal{O}_{e, \text{germ}}^{\text{mod}}$ be the open union of the following restricted product germs, and let $\mathcal{O}_{0,e}^{\text{mod}}$ contain the closures of their required base loci:

- (M1) the core collar $\iota_{e, \text{loc}}^c$ only over the disk-typed inner region C_+^D ;
- (M2) the outer-work collar q_e^{sp} only over $\mathcal{W}_- = \{y \in \mathcal{W} : \bar{r}(y) > r_-\}$;
- (M3) the complete protected product germ on $P_e^{\text{sp}} \cap Z_e^{\text{sp}, \sigma_e}$;
- (M4) the retained outer and residual disk annuli; and
- (M5) every protected, pairing, ribbon, spoke, detector, and signed E_e internal-interface product germ.

The restrictions in (M1)–(M2) are part of the construction: neither prescription is extended across the other one's exclusive region.

There are three potentially nonformal overlaps. Core/protected is the region $r_0 \leq r < r_+$, where (133) and (135) give literal equality with all jets. Core/outer is $\mathcal{A}_{-+}^D$, hence is a restriction of that same identity. Protected/outer is the part of the complete annulus with $\bar{r} > r_-$, where both entries are the identical retained q_e^{sp} -product germ. The same equation holds on every triple intersection. All other intersections are unchanged product intersections from the common

endpoint chart-and-marking system. Thus the compatibility check precedes the collar extension. Choose

$$(138) \quad O_{0,e}^{\text{mod}} \Subset O_{1,e}^{\text{mod}} \Subset \mathcal{O}_{e,\text{germ}}^{\text{mod}} \subset \partial L_{e,\text{mod}}^{\text{sp}}.$$

The formulas glue to one local collar on $O_{1,e}^{\text{mod}}$. Apply Lemma B.3 once for each of the five model edges to obtain

$$(139) \quad d_e : \partial L_{e,\text{mod}}^{\text{sp}} \times [0, b_e^{\text{mod}}) \hookrightarrow L_{e,\text{mod}}^{\text{sp}}, \quad b_e^{\text{mod}} > 0.$$

For each realized endpoint incidence (e, u) , take the corresponding restricted list: the z_u -core germ only for $x \in C_+^D$, the realized outer-work germ only over $\mathcal{W}_-$, both complete endpoint disk charts, the complete realized protected-annulus product germ, and all retained protected and interface product germs. The endpoint's fixed b_u, c_u , together with the three-piece identity in the proof of Proposition 8.4, carries each of the three model overlap checks literally to this realized list. Hence no endpoint-specific compatibility is delegated to the collar lemma. Let $O_{0,e,u}$ contain the closures of the corresponding required base loci. If $\mathcal{O}_{e,u,\text{germ}}$ is the open union of these germs, choose

$$(140) \quad O_{0,e,u} \Subset O_{1,e,u} \Subset \mathcal{O}_{e,u,\text{germ}} \subset \partial L_{e,u}^{\text{sp}},$$

and apply the lemma once for each of the ten realized endpoint incidences to obtain

$$(141) \quad d_{e,u} : \partial L_{e,u}^{\text{sp}} \times [0, b_{e,u}) \hookrightarrow L_{e,u}^{\text{sp}}, \quad b_{e,u} > 0.$$

Thus there are exactly five positive model widths and ten positive realized endpoint widths.

Put

$$(142) \quad \iota_e^{\text{core}}(x, t) = d_e(j_e^c(x), t), \quad (x, t) \in C_{r_1}^D \times [0, b_e^{\text{mod}}).$$

The realized relative extension has the corresponding core-collar identity

$$(143) \quad d_{e,u}(\gamma_{e,u}^z j_e^c(x), t) = z_u(x, t), \quad x \in C_+^D, \quad 0 \leq t < \min\{b_e^{\text{mod}}, b_{e,u}\}.$$

Only now choose the common comparison depth

$$(144) \quad u_{\text{cmp}} = \frac{1}{2} \min \left\{ u_0, \varepsilon_{\text{pair}}^{\text{src}}, \epsilon_{\text{work}}, d_{\text{ann}}, \min_e d_e^c, \min_e b_e^{\text{mod}}, \min_{(e,u) \in \mathcal{I}_{10}} b_{e,u} \right\} > 0.$$

In particular $u_{\text{cmp}} < u_0$. This restricts comparison equations; it neither changes the domain nor redefines any already fixed vertex map z_v , written z_u when u is the endpoint in question.

Proposition 8.5 (Selected spectator endpoint chart-and-marking systems). *For every $(e, u) \in \mathcal{I}_{10}$, relative multi-disk extension gives*

$$\Xi_{e,u}^z : L_{e,\text{mod}}^{\text{sp}} \longrightarrow L_{e,u}^{\text{sp}}$$

such that

$$(145) \quad \Xi_{e,u}^z \iota_e^{\text{core}}(x, t) = z_u(x, t), \quad x \in C_+^D, \quad 0 \leq t \leq u_{\text{cmp}},$$

$$(146) \quad \Xi_{e,u}^z q_e^{\text{sp}}(y, t) = q_u^P(y, t), \quad (y, t) \in \mathcal{D}_{e,\text{sp}}^{\text{work}} \cap (\mathcal{W}_- \times [0, u_{\text{cmp}}]).$$

It also preserves, on their complete product domains, the exact labeled list

$$E, \quad \tau_p, \quad \tau_q, \quad s^-, \quad s^+, \quad s_s, \quad c_s, \quad b_{\delta_E},$$

together with both complementary-disk annuli and the signed internal-interface bicollar. These are respectively the global cut attachment, the two common spectator arcs, both spoke pieces, the spectator branch, and the retained ribbon/detector germs. For every such label a ,

$$(147) \quad \Xi_{e,u}^z \chi_{a,e}^{\text{sp}}(x, \mathbf{t}) = \chi_{a,e,u}^{\text{sp}}(x, \mathbf{t}) \quad \text{with all tangent, normal, framing, and jet data.}$$

Proof. Apply Lemma 8.2 to the two full disk charts and the already compatible model and realized collars. Its full-annulus hypothesis is precisely (135); the smaller split seam is not substituted for it. Take as relative fixed set the union of the inner-core and outer-work prescriptions. They agree on the whole surface overlap, while every protected-annulus transition has already been checked on its complete domain. The relative conclusion gives (145)–(147). $\square$

Remark 8.6 (The outer equation is deliberately restricted). Equation (146) is not asserted on all of $\mathcal{D}_{e,\text{sp}}^{\text{work}}$. Extending it beyond $\mathcal{W}_-$ while retaining (145) would prescribe two independent model inputs at points where their equality has not been proved, contradicting injectivity in general.

8.5. Selection of the edge representatives. All ten incidence-labeled maps $\Xi_{e,u}^z$ are now fixed, so the local representatives can be chosen alongside already specified spectator maps. For each edge, apply the recipe of Proposition 7.4. The maps Ξ supply the spectator interface germ; the local extension recipe still depends only on its independently constructed endpoint data.

Proposition 8.7 (Selected naturalized representatives). *For every $e : v \rightarrow w$, there is an orientation-preserving*

$$\hat{\phi}_e^{\text{cut}} : B_{e,v}^{\text{LS}} \longrightarrow B_{e,w}^{\text{LS}}$$

in the positive B0/B5 relative class fixed by Theorem 2.2. It satisfies

$$(148) \quad \hat{\phi}_e^{\text{cut}} \hat{q}_{e,v}^{\text{LS}} = \hat{q}_{e,w}^{\text{LS}}, \quad \hat{\phi}_e^{\text{cut}}(z, t) = (z, t)$$

on the complete local shallow domain and signed local-interface germ, respectively, with all jets. It also intertwines separately the two source active pairings with their corresponding target pairings on the full half-collars.

On the common signed internal bicollar $c_{e,u}^{E_e}$, the local and spectator conjugates have the same product germ:

$$(149) \quad \begin{aligned} \hat{\Theta}_{e,w} \hat{\phi}_e^{\text{cut}} \hat{\Theta}_{e,v}^{-1} c_{e,v}^{E_e}(z, t) &= c_{e,w}^{E_e}(z, t), \\ \Xi_{e,w}^z (\Xi_{e,v}^z)^{-1} c_{e,v}^{E_e}(z, t) &= c_{e,w}^{E_e}(z, t), \quad |t| < \eta, \end{aligned}$$

and every derivative of either coordinate expression minus (z, t) vanishes.

Proof. The compatible-extension schema already fixed the boundary transition and its relative class without choosing an edge map. Instantiate its collar-extension, radial-homeomorphism, and relative-smoothing recipe now. The common template (120) makes the full boundary collar formula leave the inward variable unchanged. Hence the shallow identity in (148) follows from (122); the two pairing equations follow independently from the two abstract pairing intertwining of Proposition 7.2. The signed interface formula is coordinate identity by construction.

Conjugating that interface formula by the endpoint maps $\hat{\Theta}_{e,u}$ gives the first row of (149). The second is (147) for the signed spectator interface. Both statements hold on open product germs, so equality includes every jet and not just the central disk. $\square$

The closed local and spectator halves meet along the internal disk $E_{e,u}$, whereas (149) compares their extensions on a signed open bicollar. Here $E_{e,u}$ is the internal local/spectator interface and is not the global exterior disk E . The next section uses these two already constructed germs to define the cut map. Old regions and unused handles do not belong to either cut-ball half and acquire an identity map only after quotient descent and global extension.

9. EDGE MAPS, FULL-NORMAL PAIRINGS, AND STABILIZATION

For each edge, the local and spectator maps have the same product formula on the internal bicollar, so they glue to a smooth cut-ball map. To reglue the three cut disks, this map must also intertwine their pairings on full normal collars. The five edge constructions give the fifteen pairing conjugacies needed for smooth quotient descent.

Fix a directed edge $e : v \rightarrow w$ of E_5 . All vertex data and all endpoint chart-and-marking systems used below have already been chosen in the order of (4.21). In particular, no construction in this section changes u_0 , z_v , z_w , or either endpoint marking.

9.1. The amalgamated cut map. For each endpoint $a \in \{v, w\}$, the cut ball has the decomposition

$$(150) \quad K_a = K_{e,a}^{\text{loc}} \cup_{E_{e,a}} L_{e,a}^{\text{sp}}, \quad E_{e,a} \neq E.$$

Here $E_{e,a}$ is the internal local-spectator interface, whereas E is the global exterior attachment disk. The two closed halves in (150) meet precisely along $E_{e,a}$. The relative constructions of Sections 7 and 8 also give a signed open bicollar

$$c_{e,a}^{E_e} : E_e \times (-\eta_e, \eta_e) \longrightarrow K_a$$

to which both half-maps extend.

Define the local and spectator maps

$$(151) \quad \mathcal{L}_e = \widehat{\Theta}_{e,w} \widehat{\phi}_e^{\text{cut}} \widehat{\Theta}_{e,v}^{-1}, \quad \mathcal{S}_e = \Xi_{e,w}^z (\Xi_{e,v}^z)^{-1}.$$

The hats in the first formula are essential: the representative $\widehat{\phi}_e^{\text{cut}}$ was selected only after all ten endpoint chart-and-marking systems had been frozen.

Proposition 9.1 (Cut-map gluing). *The formula*

$$(152) \quad \widetilde{F}_e(x) = \begin{cases} \mathcal{L}_e(x), & x \in K_{e,v}^{\text{loc}}, \\ \mathcal{S}_e(x), & x \in L_{e,v}^{\text{sp}} \end{cases}$$

defines an orientation-preserving diffeomorphism $K_v \rightarrow K_w$. On the complete signed interface bicollar,

$$(153) \quad \mathcal{L}_e c_{e,v}^{E_e}(z, t) = c_{e,w}^{E_e}(z, t) = \mathcal{S}_e c_{e,v}^{E_e}(z, t), \quad |t| < \eta_e,$$

and hence the two branches agree to every order. In spectator coordinates $(\Xi_{e,w}^z)^{-1} \mathcal{S}_e \Xi_{e,v}^z = \text{id}$.

Proof. The local and spectator maps have different geometric roles, but their extensions use the same signed coordinate on the internal bicollar. We first check that these two formulas are one smooth map on that open overlap; the same check for their inverses then proves that their union is a diffeomorphism.

Edgewise naturality, transported through the endpoint-specific local charts $\widehat{\Theta}_{e,v}$ and $\widehat{\Theta}_{e,w}$, gives the first equality in (153). The relative decorated-ball extension was fixed on the entire signed bicollar, so its spectator atlas gives the second equality, with the same variables (z, t) . Thus, on an open neighborhood of the common disk, both branches of (152) are the single coordinate map $c_{e,v}^{E_e}(z, t) \mapsto c_{e,w}^{E_e}(z, t)$. The piecewise formula is therefore smooth, not merely continuous across E_e , and all of its jets match there.

Each branch is orientation preserving on its half. Since they use the same signed normal coordinate on the overlap, their orientations agree. The corresponding formula with v, w exchanged and each constituent inverted is a smooth two-sided inverse. This proves the claim. The spectator-coordinate identity follows immediately from the definition of $\mathcal{S}_e$. $\square$

The spectator half in (150) contains the global E -cut attachment and the protected spectator data. It does not contain the quotient exterior, the old region, or any unused handle. Consequently Proposition 9.1 makes no assertion about those global regions.

9.2. Inner and outer collar comparison. Let $C_+^D \subset D^2$ be the buffered inner disk, let $\mathcal{W}_-$ be the outer work region on the protected boundary, and let $\mathcal{A}_+^S$ be their open surface overlap. The relative-band construction gives the two covers

$$(154) \quad \mathcal{W} = (C_+^S \cap \mathcal{W}) \cup \mathcal{W}_-, \quad D^2 = C_+^D \cup \zeta_{\text{mod}}^{-1}(\mathcal{W}_- \cap Z_{\text{mod}}).$$

Lemma 9.2 (Full-disk comparison). *For every $e : v \rightarrow w$,*

$$(155) \quad \widetilde{F}_e q_v^P = q_w^P \quad \text{on } \mathcal{W} \times [0, u_{\text{cmp}}),$$

$$(156) \quad \widetilde{F}_e z_v = z_w \quad \text{on } D^2 \times [0, u_{\text{cmp}}].$$

Proof. The endpoint core equations control the inner region, and the local and restricted spectator equations control the outer region. The two covers above join these comparisons into one formula on the full disk; continuity then includes its closed depth face.

On $C_+^S \cap W$, the inner-core equation for the two endpoint chart-and-marking systems and the spectator branch of (152) give $\tilde{F}_e q_v^P = q_w^P$. On W_- , the local formula and the restricted outer spectator formula give the same equality. The two calculations agree on the open seam $\mathcal{A}_{-+}^S$ by (153); the first cover in (154) proves (155).

On C_+^D , the two endpoint inner-core equations give directly

$$\tilde{F}_e z_v = \Xi_{e,w}^z (\Xi_{e,v}^z)^{-1} z_v = \Xi_{e,w}^z \iota_e^{\text{core}} = z_w.$$

On the second member of the disk cover, $\zeta_{\text{mod}}(x) \in W$, where the vertex construction does identify $q_a^P(\zeta_{\text{mod}}(x), t) = z_a(x, t)$; there (156) follows from (155). The two formulas agree on their open overlap. They hold for $0 \leq t < u_{\text{cmp}}$, and both sides extend continuously to the closed face $t = u_{\text{cmp}}$; equality there follows by taking a sequence with $t \nearrow u_{\text{cmp}}$. $\square$

9.3. The fifteen full-normal pairing conjugacies. For the five edges, the underlying ac-
tion/spectator label triples are

$$(157) \quad \begin{array}{c|ccccc} e & 0B & B\star & 0A & AAC & AC\star \\ \hline (p_e, q_e; s_e) & (j, k; i) & (i, j; k) & (i, j; k) & (i, k; j) & (j, k; i). \end{array}$$

Uppercase curves such as $J, I_A, I_\star$ are transported geometric curves; the pairing labels p_e, q_e, s_e remain elements of $\{i, j, k\}$.

Recall the inherited full-normal pairing depth $\varepsilon_{\text{pair}}^{\text{src}} > 0$ fixed in (49), and put

$$(158) \quad u_{\text{pair}} = \frac{1}{2} \min\{u_{\text{cmp}}, \varepsilon_{\text{pair}}^{\text{src}}\} > 0.$$

For $r \in \{i, j, k\}$ and a vertex a , let $\kappa_{r,a}^{\sigma,(3)}$, $\sigma \in \{-, +\}$, be the positive inward half-collar charts. The supplied pairing and its inverse satisfy

$$(159) \quad \Pi_{r,a}^{-+, (3)} \kappa_{r,a}^{-, (3)}(z, t) = \kappa_{r,a}^{+, (3)}(j_r z, t), \quad \Pi_{r,a}^{+-, (3)} = (\Pi_{r,a}^{-+, (3)})^{-1},$$

for $0 \leq t \leq u_{\text{pair}}$. In particular, the normal coordinate is $t \mapsto t$, not $t \mapsto -t$. Write

$$\tilde{F}_e^{r, \sigma} := \tilde{F}_e|_{\text{im } \kappa_{r,v}^{\sigma, (3)}}$$

for the corresponding signed-face restriction. The source label v in this definition is determined by the directed edge $e : v \rightarrow w$.

Proposition 9.3 (Full-normal pairing conjugacy). *For every edge $e : v \rightarrow w$ and every $r \in \{p_e, q_e, s_e\}$, both directed identities*

$$(160) \quad \tilde{F}_e^{r, +\Pi_{r,v}^{-+, (3)}} = \Pi_{r,w}^{-+, (3)} \tilde{F}_e^{r, -},$$

$$(161) \quad \tilde{F}_e^{r, -\Pi_{r,v}^{+-, (3)}} = \Pi_{r,w}^{+-, (3)} \tilde{F}_e^{r, +}$$

hold on the complete half-collar of depth u_{pair} , with all jets. Thus there are exactly $5 \cdot 3 = 15$ conjugacy pairs.

Proof. For $r = p_e, q_e$, edgewise naturality gives $\tilde{F}_e \kappa_{r,v}^{\sigma, (3)} = \kappa_{r,w}^{\sigma, (3)}$. For $r = s_e$, the same formula follows from the selected spectator atlas. Hence

$$\begin{aligned} \tilde{F}_e^{r, +\Pi_{r,v}^{-+, (3)} \kappa_{r,v}^{-, (3)}}(z, t) &= \tilde{F}_e \kappa_{r,v}^{+, (3)}(j_r z, t) \\ &= \kappa_{r,w}^{+, (3)}(j_r z, t) \\ &= \Pi_{r,w}^{-+, (3)} \kappa_{r,w}^{-, (3)}(z, t) \\ &= \Pi_{r,w}^{-+, (3)} \tilde{F}_e^{r, -} \kappa_{r,v}^{-, (3)}(z, t). \end{aligned}$$

The chart is onto its half-collar image, proving (160). Composing with the two inverse pairing maps gives (161). Every displayed map is productwise in t , so differentiating in the common

product coordinates proves equality of all jets. The role table (157) has three labels in each of five rows, giving the asserted count. $\square$

The common spectator chart also supplies a full product chart $\epsilon_{E,e,a}^{\text{cut}} = \Xi_{e,a}^z \epsilon_E^{\text{sp}}$ for the global- E cut attachment. The spectator-coordinate identity in Proposition 9.1 gives

$$(162) \quad \tilde{F}_e \epsilon_{E,e,v}^{\text{cut}} = \epsilon_{E,e,w}^{\text{cut}}$$

on its complete product domain, with every derivative. This remains a cut-attachment statement; it is not yet an exterior identity.

9.4. Stabilization before descent. For an active label $r \in \{p_e, q_e\}$, the local interface word begins at E_e ,

$$(163) \quad \mathcal{B}_{e,r}^{\text{loc}} = f_{E_e,r}^{-1} r_{s_r} f_{r-,1}^{-1} r_{c_r} f_{r+}^{-1} l_{c_r}^{-1} f_{r-,0}^{-1} l_{s_r}^{-1},$$

For every $r \in \{i, j, k\}$, the global branch word instead begins at E ,

$$(164) \quad \mathcal{B}_r^{\text{glob}} = f_{E,r}^{-1} r_{s_r} f_{r-,1}^{-1} r_{c_r} f_{r+}^{-1} l_{c_r}^{-1} f_{r-,0}^{-1} l_{s_r}^{-1}.$$

These words have the same formal length but different initial objects. Their five amalgamations, including the position of the spectator word, are

e	two local words	spectator insertion	resulting global order
$0B$	$\mathcal{B}_{e,j}^{\text{loc}} \mathcal{B}_{e,k}^{\text{loc}}$	$\mathcal{B}_i^{\text{glob}}$ before	$\mathcal{B}_i^{\text{glob}} \mathcal{B}_j^{\text{glob}} \mathcal{B}_k^{\text{glob}}$
$B\star$	$\mathcal{B}_{e,i}^{\text{loc}} \mathcal{B}_{e,j}^{\text{loc}}$	$\mathcal{B}_k^{\text{glob}}$ after	$\mathcal{B}_i^{\text{glob}} \mathcal{B}_j^{\text{glob}} \mathcal{B}_k^{\text{glob}}$
$0A$	$\mathcal{B}_{e,i}^{\text{loc}} \mathcal{B}_{e,j}^{\text{loc}}$	$\mathcal{B}_k^{\text{glob}}$ after	$\mathcal{B}_i^{\text{glob}} \mathcal{B}_j^{\text{glob}} \mathcal{B}_k^{\text{glob}}$
AAC	$\mathcal{B}_{e,i}^{\text{loc}} \mathcal{B}_{e,k}^{\text{loc}}$	$\mathcal{B}_j^{\text{glob}}$ between	$\mathcal{B}_i^{\text{glob}} \mathcal{B}_j^{\text{glob}} \mathcal{B}_k^{\text{glob}}$
$AC\star$	$\mathcal{B}_{e,j}^{\text{loc}} \mathcal{B}_{e,k}^{\text{loc}}$	$\mathcal{B}_i^{\text{glob}}$ before	$\mathcal{B}_i^{\text{glob}} \mathcal{B}_j^{\text{glob}} \mathcal{B}_k^{\text{glob}}$

Proposition 9.4 (Cut-level stabilization). *The maps $\tilde{F}_e$ preserve the complete cut-level marking: the two active local words, the inserted spectator word, all six framed ribbons, the complete spokes, the zero-turn detector, all forty spectator letter germs, the global- E cut chart, and the fifteen pairing conjugacies. The local rotation census is $(32, 56, 26)$; inserting the spectator annular graph contributes $(16, 28, 12)$, giving*

$$(166) \quad (32, 56, 26) + (16, 28, 12) = (48, 84, 38).$$

No old-region or unused-handle assertion occurs at this stage.

Proof. The common signed interface model identifies the cyclic orders on the two halves with opposite induced orientations. Gluing those cyclic orders, not merely adding Euler characteristics, inserts the rank-one spectator annular graph into the two-active-branch tree. The explicit core, collar, ribbon, and complementary cycles in Equations (228) to (230) and (232) enumerate the inserted vertices, edges, and faces as $(16, 28, 12)$; Proposition A.1 verifies that these cycles exhaust the rounded boundary. This proves (166) and the prescribed twenty-four-letter global rotation word. The relative positions in (165) are exactly the positions at which the spectator cycles enter that enumeration.

For every active incidence, the endpoint charts split the already fixed complete realized spoke uniquely as

$$s_{r,a} = \tau_{e,r,a} * s_{e,r,a}^{\text{loc}}.$$

The spectator coordinate identity transports $\tau_{e,r,v}$ to $\tau_{e,r,w}$, while the local branch transports $s_{e,r,v}^{\text{loc}}$ to $s_{e,r,w}^{\text{loc}}$. Their concatenation therefore transports the complete spoke. The same product-coordinate calculation transports the six ribbon charts, the detector, and each of the eight ordered germs in the spectator word. It also preserves the insertion position of every letter. Repeated incidences merely give different coordinate decompositions of the same vertexwise spoke; they do not reselect it or identify the genuinely edge-dependent pieces. Finally, Proposition 9.3 and (162) supply the pairing and global- E cut clauses. All these equalities hold on full product germs, so all derivatives agree.

For completeness, the incidence-independence just used is literal. Let $\mathcal{R}_6 = \{s_i, c_i, s_j, c_j, s_k, c_k\}$, and let $\mathcal{L}_r$ be the eight ordered letters of $\mathcal{B}_r^{\text{glob}}$. If e, e' meet at a , then on their common product domains the middle symbols in the following display *define* the indicated common vertexwise values:

$$\begin{aligned}
 \rho_{\alpha, e, a}^{(3)} &= \rho_{\alpha, a}^{(3)} = \rho_{\alpha, e', a}^{(3)} & (\alpha \in \mathcal{R}_6), \\
 \epsilon_{E, e, a}^{\text{cut}} &= \epsilon_{E, a}^{\text{cut}} = \epsilon_{E, e', a}^{\text{cut}}, \\
 \delta_{e, a} &= \delta_a = \delta_{e', a}, & b_{\delta_E, e, a} = b_{\delta_E, a} = b_{\delta_E, e', a}, \\
 \chi_{\ell, e, a}^{\text{sp}} &= \chi_{\ell, r, a}^{\text{sp}} = \chi_{\ell, e', a}^{\text{sp}} & (\ell \in \mathcal{L}_r), \\
 \tau_{e, r, a} * s_{e, r, a}^{\text{loc}} &= s_{r, a} = \tau_{e', r, a} * s_{e', r, a}^{\text{loc}} & \text{for a repeated } (r, a).
 \end{aligned}
 \tag{167}$$

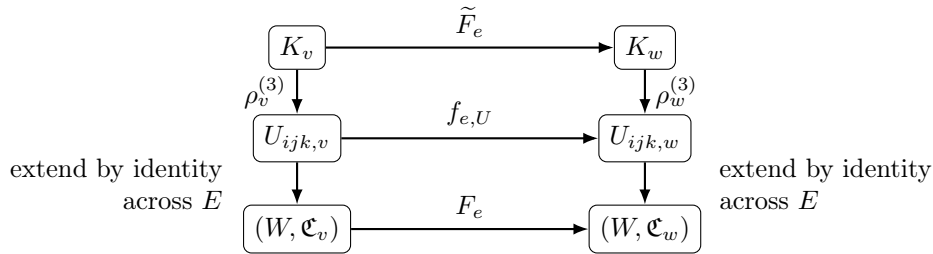
The face, pairing, and protected-collar charts already carry only the vertex label, so no further cross-edge identification is required. Equation (167) identifies the fixed vertexwise objects, not the edge-dependent continuations or local pieces.

The assertion concerns only the two halves of the cut ball and their markings. The old region and unused handles lie outside that ball and therefore cannot enter the conclusion. $\square$

10. QUOTIENT DESCENT AND GLOBAL EXTENSION

The full-normal pairing identities make the cut-ball map smooth after the three disks are reglued. Its identity formula near the exterior attachment then permits extension to the compression body by the identity on the old region and unused handles. These two operations produce the quotient and global maps from the already constructed cut map.

Throughout, $e : v \rightarrow w$ is fixed and the pairing maps are those of Proposition 9.3. We first descend through the central faces, then identify the quotient map on the inside half of the global interface, and finally adjoin the exterior identity. The terminal stabilization statement retains the distinct domains of the three maps and their marking conditions.



full-normal pairing identities make the top square smooth through every seam

FIGURE 4. Cut map, quotient descent, and global extension. The commuting square is (170); smoothness uses the two directed full-normal pairing identities, not only equality on the central faces. The lower arrows record extension by the identity on the fixed exterior.

10.1. Smooth descent through the three paired faces. Let $\sim_a$ be the equivalence relation on K_a generated by the three central signed-face identifications

$$\kappa_{r,a}^{-(3)}(z, 0) \sim_a \Pi_{r,a}^{-+, (3)} \kappa_{r,a}^{-(3)}(z, 0), \quad r \in \{i, j, k\}.
 \tag{168}$$

Only the central faces are identified in the quotient. The positive-depth extensions of the pairing maps are seam charts used to prove smoothness; they are not additional quotient generators. Write

$$\rho_a^{(3)} : K_a \longrightarrow U_{ijk,a} = K_a / \sim_a
 \tag{169}$$

for the quotient. Its exposed-boundary trace is ρ_a .

Proposition 10.1 (Quotient descent). *There is a unique orientation-preserving diffeomorphism $f_{e,U} : U_{ijk,v} \rightarrow U_{ijk,w}$ satisfying*

$$(170) \quad f_{e,U} \rho_v^{(3)} = \rho_w^{(3)} \tilde{F}_e.$$

Its inverse is the descent of $(\tilde{F}_e)^{-1}$, and on the exposed boundary

$$(171) \quad (f_{e,U}|_{\partial_+ U_{ijk,v}}) \rho_v = \rho_w (\tilde{F}_e|_{\partial K_v}).$$

Proof. The pairing conjugacies have two roles: they make the quotient formula independent of the chosen representative and make it smooth across each reglued disk.

Define

$$f_{e,U}([x]_v) = [\tilde{F}_e(x)]_w.$$

It is enough to check a generating pair. If $x = \kappa_{r,v}^{-(3)}(z, 0)$ and $y = \Pi_{r,v}^{+, (3)}(x)$, then (160) gives

$$\tilde{F}_e(y) = \Pi_{r,w}^{+, (3)} \tilde{F}_e(x),$$

so the two images represent the same w -class. The backward identity (161) gives the same conclusion for the opposite directed generator. Thus the definition is independent of the chosen representative. Surjectivity of $\rho_v^{(3)}$ gives uniqueness and (170).

Away from a reglued disk, the quotient chart is an ordinary cut-ball chart, so smoothness follows from Proposition 9.1. Near a reglued disk, use the common full-normal coordinates of (159). The two representatives have the same normal variable, and Proposition 9.3 says that both expressions for the descended map, together with every derivative, coincide. Hence the quotient formula is smooth through the seam. Repeating the argument for $(\tilde{F}_e)^{-1}$ produces a smooth inverse. Because the cut map is positive and the quotient normal is preserved, the descended map is orientation preserving. Restricting (170) to the exposed boundary gives (171). $\square$

10.2. The global E -seam. Let $\mathcal{N}_a^{\text{in}}(E) \subset U_{ijk,a}$ be the image under $\rho_a^{(3)}$ of the inside half of the cut chart $\epsilon_{E,e,a}^{\text{cut}}$. The supplied exterior marking identifies these endpoint neighborhoods, in their complete product coordinates, with one fixed inside-half neighborhood $\mathcal{N}^{\text{in}}(E)$. Descending (162) yields

$$(172) \quad f_{e,U}|_{\mathcal{N}^{\text{in}}(E)} = \text{id}.$$

Let $\mathcal{N}^{\text{out}}(E)$ be the supplied exterior product half and set

$$(173) \quad \mathcal{N}^{\text{two}}(E) = \mathcal{N}^{\text{in}}(E) \cup_E \mathcal{N}^{\text{out}}(E).$$

Also let W_{old} be the fixed old region. For each $t \notin \{i, j, k\}$, let $\mathcal{N}_t$ be the fixed product neighborhood of the t -th unused handle together with its complete marking.

Proposition 10.2 (Global extension and ownership). *The formula*

$$(174) \quad F_e(x) = \begin{cases} f_{e,U}(x), & x \in U_{ijk,v}, \\ x, & x \in W \setminus \text{int } U_{ijk,v} \end{cases}$$

defines an orientation-preserving complete-marked diffeomorphism $(W, \mathfrak{C}_v) \rightarrow (W, \mathfrak{C}_w)$. It satisfies

$$(175) \quad F_e|_{\mathcal{N}^{\text{two}}(E)} = \text{id},$$

$$(176) \quad F_e|_{W_{\text{old}}} = \text{id}, \quad F_e|_{\bigcup_{t \notin \{i,j,k\}} \mathcal{N}_t} = \text{id},$$

$$(177) \quad (F_e|_{\partial_+ W}) \rho_v = \rho_w (\tilde{F}_e|_{\partial K_v}).$$

Thus the three maps occur in the fixed order

$$(178) \quad \tilde{F}_e \longmapsto f_{e,U} \longmapsto F_e.$$

Proof. The quotient map is already the identity on the inside of the E -collar; extending $f_{e,U}$ by the identity on the exterior therefore uses one coordinate formula on the full bicollar.

In the two-sided product coordinate $E \times (-\delta, \delta)$, the first branch of (174) is the identity for the inside sign by (172); the second branch is the identity for the outside sign by definition. At the central face, both expressions and all normal derivatives are those of the same coordinate identity. Consequently they glue to a smooth map on $\mathcal{N}^{\text{two}}(E)$, proving (175). The same argument applied to the piecewise inverse, using the inverse from Proposition 10.1, gives a smooth two-sided inverse on W . Positivity follows from positivity on the protected sector and the identity exterior branch.

The old and unused regions lie in the exterior branch, proving (176). On the outgoing boundary, (174) and (171) give (177). All vertex markings were fixed before the edge maps, and the cut-level marked germs are transported by Proposition 9.4; adjoining the fixed exterior marking therefore equips F_e with the complete marking. Finally, each arrow in (178) is a construction from the preceding map, so none of the three stages can be reselected independently. $\square$

Remark 10.3 (The case $N = 3$). If $N = 3$, the set of indices $t \notin \{i, j, k\}$ is empty and hence $\bigcup_t \mathcal{N}_t = \emptyset$. The last identity in (176) is then a valid empty-domain statement. The old region and the two-sided global- E neighborhood remain present. No argument above requires an unused handle.

10.3. Terminal stabilization.

Corollary 10.4 (Stage-correct stabilization). *For each edge, the cut-level conclusion of Proposition 9.4, the descended global- E identity, and the exterior identities of Proposition 10.2 form one complete stabilization package. Its cut clauses are owned by $\tilde{F}_e$, its quotient clause by $f_{e,U}$, and its global/exterior clauses by F_e . The endpoint parameterization p_v is fixed metadata and is not asserted to be fixed pointwise by F_e .*

Proof. All cut-ball words, ribbons, spokes, detector data, pairings, and the global- E cut trace were proved before descent in Proposition 9.4. Equation (172) adds the descended inside half, and (175) adds the exterior half. Equation (176) is the first statement involving the old or unused regions. These domains exhaust the marking fields attached to the protected sector and its exterior completion, proving the combined claim with the stated ownership. Since p_v is used only to identify the outgoing boundary with the fixed surface Y , no pointwise F_e -invariance of p_v is needed or implied. $\square$

11. FINITE COMPATIBILITY CONDITIONS FOR THE THREE MAP STAGES

The five edge constructions must agree with all protected charts before their route composites can be compared. We collect these agreements for the same cut, quotient, and global maps already constructed; no additional edge representative is chosen.

We first restrict the cut-level chart identities to depths common to their respective categories. The nine families below distinguish faces and ribbons, continuations and spokes, detector and carrier data, pairings, and global identities. This distinction keeps each equation on the domain of the map that satisfies it, especially at the exterior attachment.

11.1. Common subordinate depths. Because there are only five edges and finitely many charts in each marked cut ball, the positive chart widths have positive minima. Choose $\varepsilon_{\text{face}}, \varepsilon_{\text{rib}}, \varepsilon_{\tau}, \varepsilon_{\text{spoke}}, \varepsilon_{\text{det}} > 0$ so that

$$(179) \quad \varepsilon_{\text{face}} \leq u_{\text{pair}}, \quad \max\{\varepsilon_{\text{rib}}, \varepsilon_{\tau}, \varepsilon_{\text{spoke}}, \varepsilon_{\text{det}}\} \leq u_{\text{cmp}},$$

and each is strictly smaller than the minimum width of the charts in its own category. Independently set

$$(180) \quad \varepsilon_{\text{PC5,work}} = \frac{1}{2}u_{\text{cmp}}, \quad \varepsilon_{\text{PC5,pair}} = \frac{1}{2}u_{\text{pair}}.$$

Thus G6 lies strictly inside the comparison depth and G7 lies strictly inside the source-backed pairing depth. Neither number is a renamed source work width.

For compactness, retain the notation $\mathcal{L}_e$ and $\mathcal{S}_e$ of (151). On a signed face put

$$(181) \quad \mathcal{O}_{e,r} = \begin{cases} \mathcal{L}_e, & r = p_e, q_e, \\ \mathcal{S}_e, & r = s_e. \end{cases}$$

11.2. The nine compatibility families.

Proposition 11.1 (Finite compatibility scheme). *For every edge $e : v \rightarrow w$, the following nine families hold on the displayed complete product domains, with equality of all jets.*

G1 Faces, seven records. For $r = p_e, q_e, s_e$ and $\sigma \in \{-, +\}$,

$$(182) \quad \mathcal{O}_{e,r} \kappa_{r,v}^{\sigma,(3)}(z, u) = \kappa_{r,w}^{\sigma,(3)}(z, u), \quad 0 \leq u \leq \varepsilon_{\text{face}}.$$

The seventh record is the global- E cut equation (162) on its complete cut half-germ.

G2 Ribbons, six records. For $a \in \{s_i, c_i, s_j, c_j, s_k, c_k\}$,

$$(183) \quad \tilde{F}_e \rho_{a,e,v}^{(3)}(x, t, u) = \rho_{a,e,w}^{(3)}(x, t, u), \quad 0 \leq u \leq \varepsilon_{\text{rib}},$$

and the differential fixes the ordered frame $(\partial_x, \partial_t, \partial_u)$. Active carrier ribbons are local-owned, active spoke ribbons use the two branches and are split-owned, and the two spectator ribbons are spectator-owned. In the terminal $AC\star$ carrier row, the target chart contains the fixed product-normal exceptional-ribbon comparison.

G3 Active continuations, two records. For $r = p_e, q_e$,

$$(184) \quad (\Xi_{e,v}^z)^{-1} \tau_{e,r,v} = \tau_{e,r}^{\text{sp}} = (\Xi_{e,w}^z)^{-1} \tau_{e,r,w}, \quad \mathcal{S}_e \tau_{e,r,v} = \tau_{e,r,w}.$$

This is equality in common spectator coordinates, not equality of arcs in different realized balls. It holds on the complete continuation product germ through depth ε_τ , with its full frame and all jets.

G4 Spokes and the spectator word, twelve records. For each active $r = p_e, q_e$ and $a \in \{v, w\}$,

$$(185) \quad s_{r,a} = \tau_{e,r,a} * s_{e,r,a}^{\text{loc}}, \quad \tilde{F}_e(s_{r,v}) = s_{r,w}.$$

The two continuations are spectator-owned and the two local segments are local-owned. The spectator spoke, the complete spectator branch, and its eight ordered letter germs are spectator-owned and are carried productwise with their positions unchanged. Thus the count is $2 + 1 + 1 + 8 = 12$, not twelve further records in addition to the eight letters. Every one of these statements is on its full product germ through depth $\varepsilon_{\text{spoke}}$.

G5 Detector, one record. On the complete detector chart,

$$(186) \quad \mathcal{S}_e \delta_v(\theta, n, u) = \delta_w(\theta, n, u), \quad d\mathcal{S}_e(\partial_\theta, \partial_n, \partial_u) = (\partial_\theta, \partial_n, \partial_u),$$

the core is transported, and the turn remains zero. The displayed chart includes the closed parameter face $0 \leq u \leq \varepsilon_{\text{det}}$.

G6 Full protected carrier, one record. Let $\mathcal{A}_{\text{mod}}^{\text{open}}$ be the fixed nonempty relative-open neighborhood of the entire closed protected annulus. Then

$$(187) \quad \tilde{F}_e q_v^P(\zeta_{\text{mod}}(x), u) = q_w^P(\zeta_{\text{mod}}(x), u), \quad x \in \mathcal{A}_{\text{mod}}^{\text{open}}, \quad 0 \leq u \leq \varepsilon_{\text{PC5,work}}.$$

This record is split-owned: its local and spectator restrictions agree on a nonempty open overlap to every order.

G7 Pairings, three records. For $r = i, j, k$,

$$(188) \quad \begin{aligned} \tilde{F}_e^{r,+} \Pi_{r,v}^{+, (3)} \kappa_{r,v}^{-, (3)}(z, u) &= \Pi_{r,w}^{+, (3)} \tilde{F}_e^{r,-} \kappa_{r,v}^{-, (3)}(z, u) \\ &= \kappa_{r,w}^{+, (3)}(j_r z, u), \quad 0 \leq u \leq \varepsilon_{\text{PC5,pair}}. \end{aligned}$$

The normal remains $u \mapsto u$. The two active records are local-owned and the spectator record is spectator-owned.

G8 Stage-typed global data, four records. *The global- E cut trace is owned by $\tilde{F}_e$, through $\mathcal{S}_e$. Only the global map owns the remaining three records:*

$$(189) \quad F_e|_{\mathcal{N}^{\text{two}}(E)} = \text{id}, \quad F_e|_{W_{\text{old}}} = \text{id}, \quad F_e|_{\bigcup_{t \notin \{i,j,k\}} \mathcal{N}_t} = \text{id}.$$

No Ξ^z - or $\tilde{F}_e$ -formula is asserted on the old or unused regions.

G9 Typed role transport, one record. *On the complete named-carrier product germ, the edge map carries the source active curve to the prescribed target curve, fixes its indicated spectator, and preserves the carrier and product normal. The five rows are*

0B $j_0 \rightarrow J$ over k_0, k_ε via β_0 ; spectator i_0 .

B $\star$ $i_0 \rightarrow I_\star$ over complete J, J^+ via α , where $J^+ = j_\varepsilon \#_{\beta_1} k_{2\varepsilon}$; α ends on the j_ε lobe; spectator k_0 .

0A $i_0 \rightarrow I_A$ over j_0, j_ε via α ; spectator k_0 .

AAC The whole I_A , including Ω_j and its j_ε lobe and marking, maps to $I_\star$ over $k_0, k_{2\varepsilon}$ via β_1 ; spectator j_0 .

AC $\star$ $j_0 \rightarrow J$ over k_0, k_ε via β_0 ; actual spectator $I_\star$.

where the last row includes the product-normal exceptional-ribbon comparison $\tilde{K} \rightarrow K_{kj}$, with $\tilde{K}$ as in (105), and leaves the pre-edge $q_\star, p_\star$ unchanged. These records are owned by the edge construction itself; they do not enlarge the representative's support beyond the named carrier germ.

Proof. G1 is the union of the six signed-face consequences used in Proposition 9.3 and the cut equation (162). G2–G5 are the product-germ statements proved in Proposition 9.4; the local/spectator decomposition of the active spokes gives precisely the asserted split ownership. G6 is (155) restricted to the fixed full-carrier germ and to the strictly smaller depth in (180); (156) supplies its closed disk-collar extension. G7 is Proposition 9.3 restricted to the subordinate pairing depth. G8 is Proposition 10.2 together with the separately proved cut trace. The named-carrier construction and the exceptional-ribbon lemma give the five G9 rows, including the product normal in the terminal row. These derivations also establish the owners displayed above. In particular, no family chooses a new map. The literal shared-incidence equalities needed when two successive edges meet are (167); they identify the frozen vertex charts, never the genuinely edge-dependent continuation pieces. $\square$

The per-edge sizes are

$$(7, 6, 2, 12, 1, 1, 3, 4, 1),$$

so five edges give 45 group headers and

$$(190) \quad 5(7 + 6 + 2 + 12 + 1 + 1 + 3 + 4 + 1) = 185$$

typed records. When $N = 3$, the five unused-handle G8 records remain included in this finite compatibility enumeration, but their domains are empty; the other records retain their stated nonempty product domains.

11.3. Shared vertex data and the radial collar. Set, in right-factor-first order,

$$(191) \quad \tilde{F}_L = \tilde{F}_{B\star} \circ \tilde{F}_{0B}, \quad \tilde{F}_R = \tilde{F}_{AC\star} \circ \tilde{F}_{AAC} \circ \tilde{F}_{0A}, \quad \mathcal{Q}_{\text{cut}} = \tilde{F}_R^{-1} \circ \tilde{F}_L.$$

Corollary 11.2 (Cut-owned radial compatibility). *The two cut routes in (191) use the literal same vertex package at every common endpoint. On every protected chart listed in G1–G7 and on the cut trace in G8, their coordinate expressions agree. Hence $\mathcal{Q}_{\text{cut}}$ is the identity on those complete product germs.*

Proof. At an intermediate vertex, the outgoing chart of one edge and the incoming chart of the next are restrictions of the same previously chosen vertex chart. Thus composing the edge equations telescopes: both routes carry each protected coordinate at vertex 0 to the same coordinate at vertex $\star$. Applying $\tilde{F}_R^{-1}$ gives the identity. For the radial variables, the transition reparameterizations are the literal identity, and G6 supplies a full open annular germ rather than only its boundary circle. The all-jet overlap statements in Proposition 11.1 make these

identities compatible on chart intersections. The conclusion is cut-owned; G8's old and unused rows are not subsets of K_0 and are not part of it. $\square$

12. HOMOLOGY, NO SHEAR, AND THE COMMON LAGRANGIAN

The geometric maps now have a common boundary surface on which their action can be read. This section records the embedded meridian and dual-longitude transports, computes the blocks of the homology action, and extracts the Lagrangian data needed for the Walker lift.

The homological calculation uses the global maps F_e from Proposition 10.2. A cut map or quotient map has neither the domain nor the exterior data required in this section.

For each edge $e : v \rightarrow w$, define

$$(192) \quad h_e = p_w(F_e|_{\partial_+ W})p_v^{-1} : Y \longrightarrow Y.$$

Let

$$U = \text{span}_{\mathbb{Q}}\{m_i, m_j, m_k\}, \quad U^\vee = \text{span}_{\mathbb{Q}}\{\lambda_i, \lambda_j, \lambda_k\}.$$

The vertex parameterizations send the embedded meridian and product-framed dual-longitude rows to the corresponding fixed homology labels on Y .

12.1. Embedded transports.

Proposition 12.1 (Geometric row transport). *The same global maps F_e used in (192) induce the following transports of embedded product-framed rows:*

$$(193) \quad \begin{array}{c|cc} e & \text{meridians} & \text{dual longitudes} \\ \hline 0B & (i_0, j_0, k_0) \mapsto (i_0, J, k_0) & (\ell_i, \ell_j, \ell_k) \mapsto (\ell_i, \ell_j, K_{kj}) \\ B\star & (i_0, J, k_0) \mapsto (I_\star, J, k_0) & (\ell_i, \ell_j, K_{kj}) \mapsto (\ell_i, L_{ji}, K_{kj}) \\ 0A & (i_0, j_0, k_0) \mapsto (I_A, j_0, k_0) & (\ell_i, \ell_j, \ell_k) \mapsto (\ell_i, L_{ji}, \ell_k) \\ AAC & (I_A, j_0, k_0) \mapsto (I_\star, j_0, k_0) & (\ell_i, L_{ji}, \ell_k) \mapsto (\ell_i, L_{ji}, K_{ki}) \\ AC\star & (I_\star, j_0, k_0) \mapsto (I_\star, J, k_0) & (\ell_i, L_{ji}, K_{ki}) \mapsto (\ell_i, L_{ji}, K_{kj}). \end{array}$$

The literal spectator pairs, in the same edge order, are

$$(194) \quad (i_0, \ell_i), \quad (k_0, K_{kj}), \quad (k_0, \ell_k), \quad (j_0, L_{ji}), \quad (I_\star, \ell_i).$$

Every meridian image has zero dual-longitude component, and every dual-longitude image has zero meridian component.

Proof. On each active local branch, the positive two-site theorem Theorem 2.2 and the selected compatible representative transport the two embedded mover/carrier rows. On the spectator branch, the coordinate identity transports the pair in (194) with its complete product framing. Proposition 9.3 identifies the two signed representatives with the same normal coordinate, so these cut-level transports descend to embedded curves in the quotient. The boundary identity (177) and the parameterizations p_v, p_w then give the displayed global transports. In the terminal $AC\star$ row, the exceptional-ribbon isotopy identifies the raw product-framed $\tilde{K}$ ribbon of (105) with K_{kj} ; its product normal and zero turn are part of that construction.

Let P_U and $P_{U^\vee}$ denote the projections associated with the active splitting $U \oplus U^\vee$. The meridian half of (193) consists of boundaries of the transported embedded meridian disks. The transporting annuli join such disk boundaries without entering a longitude track, and therefore

$$P_{U^\vee} h_{e*}|_U = 0.$$

Independently, every dual-longitude row is carried by its displayed product-framed annulus. These annuli have zero turn in the fixed product normal and introduce no winding about a meridian core; the terminal exceptional-ribbon reserve is compared to K_{kj} by a product-normal isotopy with the same zero-turn property. Consequently their classes are linear combinations only of the displayed dual-longitude rows, so

$$P_U h_{e*}|_{U^\vee} = 0.$$

The two zero mixed components are thus conclusions about two distinct families of embedded annuli, proved before any matrix is introduced. $\square$

12.2. The block action. The exterior identity in (176) gives the splitting

$$(195) \quad H_1(Y; \mathbb{Q}) = H_{\text{old}} \oplus \bigoplus_{t \notin \{i,j,k\}} (\mathbb{Q}m_t \oplus \mathbb{Q}\ell_t) \oplus (U \oplus U^\vee).$$

Corollary 12.2 (No-shear block form).

In the ordered active basis $(m_i, m_j, m_k, \lambda_i, \lambda_j, \lambda_k)$,

$$(196) \quad h_{e*} = I_{\text{ext}} \oplus \begin{pmatrix} M_e & 0_3 \\ 0_3 & N_e \end{pmatrix}, \quad N_e = M_e^{-T}.$$

The five keyed matrices and their route products are retained as repository reproducibility data; the public argument uses only the block identities displayed here.

Proof. The two geometric zero statements in Proposition 12.1 give the off-diagonal zero blocks. Reading the meridian coefficients in (193) gives M_e , and reading the independently transported dual-longitude coefficients gives N_e . Direct calculation gives $N_e = M_e^{-T}$. The old and unused summands are fixed by the actual global map, giving I_{ext} . No symplecticity argument is used to create either zero block. $\square$

The matrix arithmetic also gives

$$(197) \quad M_{B*}M_{0B} = M_{AC*}M_{AAC}M_{0A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

This is a consistency check and a homological consequence of the embedded rows. It will not be used to produce the relative isotopy in Section 13.

12.3. The common Lagrangian. Set

$$(198) \quad \Lambda = \lambda_g^Y \oplus \bigoplus_{t \notin \{i,j,k\}} \mathbb{Q}[m_t] \oplus U.$$

The old summand is Lagrangian by hypothesis, the unused meridians are the disk-kernel summands of the unused handles, and U is the active meridian summand. Hence Λ is a rational Lagrangian in $H_1(Y; \mathbb{Q})$.

Proposition 12.3 (Two-sided preservation). *For every edge e , the actual global map and its actual inverse satisfy*

$$(199) \quad h_{e*}\Lambda = \Lambda = h_{e*}^{-1}\Lambda.$$

Proof. The global map fixes λ_g^Y and every unused meridian. By Corollary 12.2, it sends U into U by the unimodular matrix M_e ; hence it sends U onto U . Therefore $h_{e*}\Lambda = \Lambda$. The inverse of F_e constructed in Proposition 10.2 induces the actual inverse h_{e*}^{-1} . Applying h_{e*}^{-1} to the established equality gives $h_{e*}^{-1}\Lambda = \Lambda$. This proves both directions and does not infer invariance merely from dimensions. $\square$

If $N = 3$, the unused direct sums in (195) and (198) are empty. The proof then reduces verbatim to the old and active summands; no additional case is needed.

13. FULL-COLLAR RIGIDITY AND THE RELATIVE PENTAGON

Both route composites now exist at the cut, quotient, and global levels. To compare them, we will contract the cut-level discrepancy while fixing one complete protected boundary collar, then descend and extend the entire path. All compositions in this section are right-factor-first.

The three map levels are the ones shown in Figure 4: $\tilde{F}_e$ is the cut-ball map, $f_{e,U}$ its quotient on the protected sector, and F_e the resulting global map. The exterior attachment E and the edge-local disk interface E_e retain their distinct roles during this passage.

Besides the cut routes of (191), put

$$(200) \quad f_{L,U} = f_{B\star,U} \circ f_{0B,U}, \quad f_{R,U} = f_{AC\star,U} \circ f_{AAC,U} \circ f_{0A,U},$$

$$(201) \quad F_L = F_{B\star} \circ F_{0B}, \quad F_R = F_{AC\star} \circ F_{AAC} \circ F_{0A},$$

and define

$$(202) \quad \mathcal{Q}_U = f_{R,U}^{-1} \circ f_{L,U}, \quad \mathcal{Q}_F = F_R^{-1} \circ F_L.$$

The chart agreements of the preceding section are the starting point, but relative three-ball rigidity needs identity on an open neighborhood of a collar of the entire boundary. We obtain that collar from the work-surface and complementary-disk charts. Because it contains every pairing seam and the exterior attachment, the contraction descends through the paired faces and extends by the identity over the exterior. Restriction and conjugation then give the mapping-class pentagon fixed on the complete marking. The homology and matrix calculations play no role in constructing this relative isotopy.

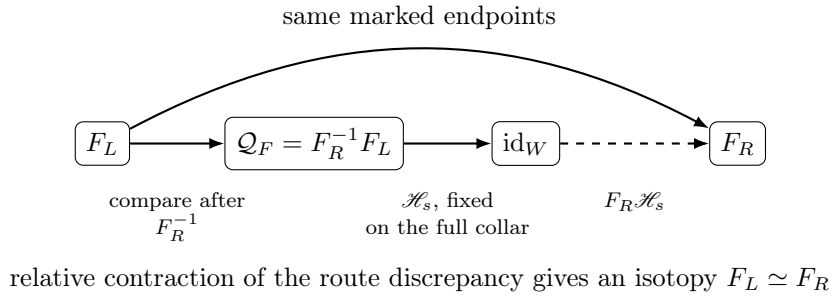


FIGURE 5. The route-isotopy mechanism. The cut-level discrepancy is first shown to be the identity on one complete boundary collar, then contracted relative to that collar; the entire path descends and extends to W . No mapping-class or homology equality is used to manufacture the path.

Lemma 13.1 (Compatibility of the three route levels). *The discrepancies in (191) and (202) satisfy*

$$(203) \quad \rho_0^{(3)} \mathcal{Q}_{\text{cut}} = \mathcal{Q}_U \rho_0^{(3)}, \quad \mathcal{Q}_F|_{U_{ijk,0}} = \mathcal{Q}_U, \quad \mathcal{Q}_F = F_R^{-1} \circ F_L.$$

Proof. Compose (170) first along the left route and then along the right route. Applying the inverse of the right identity to the left identity gives the first equation in (203). Uniqueness in Proposition 10.1 shows that the quotient of the cut discrepancy is exactly $\mathcal{Q}_U$. The global maps on either route are the corresponding quotient composites on the protected sector and the identity on the same exterior complement. Their inverse-composite is therefore $\mathcal{Q}_U$ on $U_{ijk,0}$ and the identity outside, proving the remaining assertions. $\square$

13.1. Coverage by one full boundary collar. The radial statement in Corollary 11.2 gives identity on every listed protected germ. To apply relative three-ball rigidity, one must show that these germs form a collar of the entire boundary, rather than a collection of unrelated neighborhoods.

Proposition 13.2 (Full-collar coverage). *There is a closed full boundary collar*

$$(204) \quad c_0 : \partial K_0 \times [0, \delta] \longrightarrow K_0, \quad u_{\text{pair}} < \delta < u_{\text{cmp}},$$

such that

$$(205) \quad \mathcal{Q}_{\text{cut}}|_{\text{im } c_0} = \text{id}.$$

The collar is the restriction of the vertex-first collar fixed in (50). Its boundary base therefore contains every boundary face and attaching stratum, including all pairing faces, ribbons, spokes, the detector, the complementary 24-gon, and the global-E cut attachment. The edge-dependent E_e -bicollars are internal local-spectator gluing charts, not boundary strata of K_0 , and are not

asserted to lie in this collar. Moreover, $\text{im } c_0$ is contained in an open collar on which $\mathcal{Q}_{\text{cut}}$ is the identity.

Proof. The full vertex collar

$$c_0 : \partial K_0 \times [0, 3h] \longrightarrow K_0$$

was constructed before any edge. Because $u_{\text{pair}} < u_{\text{cmp}}$, choose δ strictly between them and restrict this already fixed collar. The boundary base is covered by the two vertex charts

$$(206) \quad q_0^P(W \times [0, \delta]) \quad \text{and} \quad z_0(D^2 \times [0, \delta]).$$

Indeed, the first covers the work-surface part and the second covers the entire complementary disk, including its polar center; their overlap is the protected annular product germ. This is a cover of the full cut-boundary cellulation, not a list of selected subgerms.

For every directed edge, (155) gives the literal endpoint transition on the first member of (206), and (156) gives it on the second. Composing these identities along the two-edge and three-edge routes makes both route maps carry each chart at vertex 0 to the same chart at vertex $\star$. Consequently their discrepancy is the identity on both members of the cover, and hence on all of $\text{im } c_0$. Since $\delta > u_{\text{pair}}$, the prescribed pairing half-collars needed for quotient descent are contained in this fixed collar.

Finally choose δ' with $\delta < \delta' < u_{\text{cmp}}$. The same two endpoint identities hold for every depth below δ' , so the closed collar of depth δ lies inside an open identity collar. This proves (205) with the exact relative buffer required by three-ball rigidity. $\square$

No preliminary collar-supported correction is needed, since the discrepancy already fixes a full boundary collar. We may therefore take

$$(207) \quad \mathcal{Q}_t^{\text{prelim}} = \mathcal{Q}_{\text{cut}}, \quad \mathcal{Q}_0^{\text{prelim}} = \mathcal{Q}_1^{\text{prelim}} = \mathcal{Q}' = \mathcal{Q}_{\text{cut}}.$$

13.2. Relative rigidity and descent of the entire path.

Lemma 13.3 (Prescribed-collar three-ball rigidity). *Let $q \in \text{Diff}^+(D^3)$, and let C be a specified closed full boundary collar. If q is the identity on an open neighborhood of C , then there is a jointly smooth path*

$$q_s \in \text{Diff}^+(D^3), \quad 0 \leq s \leq 1, \quad q_0 = q, \quad q_1 = \text{id},$$

which is pointwise the identity on the same C for every s .

Proof. The open identity neighborhood lets us place the inner boundary strictly beyond the prescribed collar. This leaves room to contract the inner-ball map while fixing a neighborhood of that boundary and to extend the resulting path smoothly by the identity.

Write $C = c(\partial D^3 \times [0, a])$. Because q is the identity on an open neighborhood of C , extend the same collar coordinate to some $a_1 > a$ so that

$$C_1 = c(\partial D^3 \times [0, a_1])$$

still has an open identity neighborhood. The closure $B = \overline{D^3 \setminus C_1}$ is a three-ball, and $q|_B$ is the identity on a neighborhood of ∂B .

Hatcher's contractibility theorem for $\text{Diff}(D^3 \text{ rel } \partial D^3)$ [Hat83, Appendix, p. 604] first places $q|_B$ and id_B in the same rel-boundary component. Cerf's fixed-tube versus fixed-infinite-jet comparison, followed by his fixed-infinite-jet versus pointwise-boundary comparison [Cer61, II.4.2.3, Corollary 3; II.5.2.1, Proposition 9 and Corollary 2], shows that they are already in the same component of the subgroup fixed on one prescribed boundary neighborhood. Finally, Cerf's local smooth endpoint section and a finite subdivision of the parameter interval turn that path into a jointly smooth one, constant near its time endpoints [Cer61, Lemma 1.5.3]. Extend every member by the identity across C_1 . Since the inner path is constant on a neighborhood of ∂B , the extension is jointly smooth in (s, x) and is pointwise the identity on the original collar C . $\square$

The open-collar clause of Proposition 13.2 supplies the hypothesis of Lemma 13.3 for $\mathcal{Q}_{\text{cut}}$. We obtain

$$(208) \quad \widetilde{\mathcal{H}}_s : K_0 \longrightarrow K_0, \quad \widetilde{\mathcal{H}}_0 = \mathcal{Q}_{\text{cut}}, \quad \widetilde{\mathcal{H}}_1 = \text{id}_{K_0}, \quad \widetilde{\mathcal{H}}_s|_{\text{im } c_0} = \text{id}.$$

Proposition 13.4 (Parametric quotient descent and global gluing). *The entire path (208) descends and extends to a jointly smooth path $\mathcal{H}_s \in \text{Diff}^+(W)$, fixed on the complete marking and exterior data, with*

$$(209) \quad \mathcal{H}_0 = F_R^{-1} \circ F_L, \quad \mathcal{H}_1 = \text{id}_W.$$

Proof. First descend through the three paired faces. If $x \sim_0 y$, then either $x = y$ or a chain of the three pairing generators relates them. By (168), every generator is a pair of central-face representatives, and (208) fixes both representatives. Moreover, its entire normal seam germ through depth u_{pair} lies in $\text{im } c_0$ because $\delta > u_{\text{pair}}$. Thus $\widetilde{\mathcal{H}}_s(x) \sim_0 \widetilde{\mathcal{H}}_s(y)$ for every s . Hence

$$(210) \quad \mathcal{H}_s^U([x]) = [\widetilde{\mathcal{H}}_s(x)]$$

is well defined on $U_{ijk,0}$.

Away from the reglued faces, joint smoothness in (s, x) follows from that of (208). In a reglued full-normal chart, the map in (210) is the literal identity for every s , on both representatives and in the common normal coordinate. The two quotient expressions and all derivatives, including derivatives in s , therefore agree across each seam. Thus $(s, [x]) \mapsto \mathcal{H}_s^U([x])$ is jointly smooth. Applying the same construction to the smoothly parameterized inverse family $\widetilde{\mathcal{H}}_s^{-1}$ gives the inverse of $\mathcal{H}_s^U$, so every member is an orientation-preserving diffeomorphism.

The collar $\text{im } c_0$ contains the complete global- E cut attachment. Its quotient image is the inside product half $\mathcal{N}^{\text{in}}(E)$, on which every $\mathcal{H}_s^U$ is the identity. Define on W

$$(211) \quad \mathcal{H}_s(x) = \begin{cases} \mathcal{H}_s^U(x), & x \in U_{ijk,0}, \\ x, & x \in W \setminus \text{int } U_{ijk,0}. \end{cases}$$

In the fixed two-sided coordinate $E \times (-\delta, \delta)$, both branches of (211) are the identity for all s . They therefore glue with all space and parameter derivatives. The piecewise inverse glues in the same way, proving that (211) is a jointly smooth path in $\text{Diff}^+(W)$. It is the identity on the old region and every unused handle; if $N = 3$, the unused union is empty and this clause is vacuous.

At $s = 0$, Lemma 13.1 identifies the quotient of $\widetilde{\mathcal{H}}_0 = \mathcal{Q}_{\text{cut}}$ with $\mathcal{Q}_U$, and its global extension with $\mathcal{Q}_F = F_R^{-1} \circ F_L$. At $s = 1$, both branches in (211) are the identity. This proves (209). The path fixes all protected marking germs because they lie in the full collar, and it fixes all exterior marking germs by the second branch. $\square$

13.3. The relative and mapping-class pentagons.

Theorem 13.5 (Relative handleslide pentagon). *Set*

$$(212) \quad \Gamma_s = F_R \circ \mathcal{H}_s.$$

Then Γ_s is a complete-marked relative isotopy with

$$(213) \quad \Gamma_0 = F_L, \quad \Gamma_1 = F_R.$$

Consequently, for the common background marking $\mathfrak{C}_{\text{bg}}$,

$$(214) \quad [h_{B\star} \circ h_{0B}] = [h_{AC\star} \circ h_{AAC} \circ h_{0A}] \quad \text{in } \pi_0 \text{Diff}^+(Y, \mathfrak{C}_{\text{bg}}).$$

Proof. Using (209) and associativity,

$$\Gamma_0 = F_R \circ (F_R^{-1} \circ F_L) = F_L, \quad \Gamma_1 = F_R.$$

The fixation statement in Proposition 13.4 shows that this isotopy preserves the complete relative marking, not only the underlying manifold. Restrict to the outgoing boundary and conjugate the two endpoints by the fixed parameterizations $p_0, p_\star$. Because those parameterizations identify the same background marking, the boundary isotopy lies in $\text{Diff}^+(Y, \mathfrak{C}_{\text{bg}})$ and yields (214). No homology or matrix identity is used in this implication. $\square$

14. LIFTING THE PENTAGON TO WALKER'S EXTENDED COBORDISM CATEGORY

The relative isotopy gives equality of marked mapping classes, but the paper's category also remembers an integer weight. This section checks the Maslov seams in the fixed Lagrangian and lifts the geometric pentagon to weight-zero Walker cylinders.

Equip Y with the rational Lagrangian Λ of (198). For a mapping class f and an integer m , write $C(f, m)$ for its extended mapping cylinder of weight m . With the right-factor-first convention, forward multiplication is

$$(215) \quad C(g, n)C(f, m) = C(g \circ f, n + m - \mu(f_*\Lambda, \Lambda, g_*^{-1}\Lambda)).$$

The minus sign and the order of the three Lagrangians are the convention used here; see [GM13, arXiv v3, Section 3, equation (2)]. We use only this forward multiplication formula.

Put

$$(216) \quad H_e = C([h_e], 0).$$

By Proposition 12.3, the three seams encountered in forward multiplication are

$$(217) \quad \mu(h_{0B*}\Lambda, \Lambda, h_{B**}^{-1}\Lambda) = 0,$$

$$(218) \quad \mu(h_{0A*}\Lambda, \Lambda, h_{AAC*}^{-1}\Lambda) = 0,$$

$$(219) \quad \mu((h_{AAC} \circ h_{0A})_*\Lambda, \Lambda, h_{AC**}^{-1}\Lambda) = 0.$$

Indeed, each of the three entries in every displayed Maslov index is literally Λ : the first entries use forward preservation, the third entries use inverse preservation, and the composite in (219) preserves Λ because each of its factors does. A Maslov index with two equal entries is zero.

Theorem 14.1 (Weight-zero Walker pentagon). *All constituent, seam, and route weights are zero, and*

$$(220) \quad H_{B*}H_{0B} = H_{AC*}H_{AAC}H_{0A} = C([h_{B*} \circ h_{0B}], 0).$$

Proof. Apply (215) first to the left route. Using (217),

$$\begin{aligned} H_{B*}H_{0B} &= C([h_{B*}], 0)C([h_{0B}], 0) \\ &= C([h_{B*} \circ h_{0B}], 0 + 0 - 0) \\ &= C([h_{B*} \circ h_{0B}], 0). \end{aligned}$$

For the right route, first multiply the two rightmost factors. Equation (218) gives

$$H_{AAC}H_{0A} = C([h_{AAC} \circ h_{0A}], 0).$$

Now multiply on the left by H_{AC*} . The seam is exactly (219), so

$$\begin{aligned} H_{AC*}H_{AAC}H_{0A} &= C([h_{AC*}], 0)C([h_{AAC} \circ h_{0A}], 0) \\ &= C([h_{AC*} \circ h_{AAC} \circ h_{0A}], 0 + 0 - 0) \\ &= C([h_{AC*} \circ h_{AAC} \circ h_{0A}], 0). \end{aligned}$$

Only after these forward multiplications do we use the mapping-class equality (214). It identifies the two underlying mapping classes, while the computations above identify their weights as the same literal integer 0. This proves (220). $\square$

No converse to (215), injectivity of the cylinder assignment, pair-determination theorem, projective quotient, or weight-forgetting argument is used.

15. BOUNDARY OF THE THEOREM

The theorem constructs one fully marked geometric pentagon and its literal weight-zero Walker lift. The remaining limitations distinguish that result from uniqueness, higher coherence, and downstream quantum comparisons.

15.1. Existence, not choice independence. The quantifier is

(15.1) for every supplied protected sector, there exists one coordinated system.

There is no uniqueness assertion. The proof does not compare two arbitrary coordinated systems, and therefore does not imply independence of arbitrary bands, disk charts, endpoint gauges, collar widths, or representatives of the proved local slide class. Equal endpoint chart-system values also do not identify their labelled incidences.

15.2. No higher or global conclusion. The five-edge relation is one pentagon in one supplied three-handle sector. Nothing here constructs a simultaneous family of such pentagons or verifies relations among pentagons. Consequently, the theorem does not by itself establish higher coherence, coherence for an arbitrary sequence of handleslides, stabilization invariance, or global Heegaard invariance.

Similarly, the Walker conclusion is the literal equality of the two weight-zero route products in the stated convention. The proof uses only the forward multiplication law. It contains no converse, injectivity, or pair-determination claim, and it does not pass to a projective quotient or forget the weights.

The theorem is therefore not a TQFT-functoriality theorem and makes no claim about WRT invariants, Habiro completions, or $\widehat{Z}$. Such conclusions would require additional comparison and global-coherence input not supplied here.

15.3. The downstream comparison remains separate. An operator-level pentagon built from these geometric representatives also requires a compatible comparison between a reduced skein module and the fixed-level TQFT state space. This is the input denoted COMP-Q in the comparison formulation. Its construction is separate from the geometric theorem proved here. The present result supplies the complete-marked geometric relation to be transported; it makes no stronger functorial conclusion.

15.4. Geometric output. The reusable output is the relative route equality, together with the complete marking, common Lagrangian, and zero Walker weights that make it literal.

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APPENDIX A. FIVE CUT SCHEDULES AND COMPLETE CELLULATIONS

This appendix records the finite disk-and-cell data used in Section 5. The purpose of printing the complete lists is twofold: the terminal nested cancellation cannot be replaced by a simultaneous isotopy, and the cut-sphere realization requires the actual oriented face cycles rather than an Euler-characteristic count.

A.1. The product-germ cancellation mechanism. Consider a product 1-handle $[-2, 2] \times D^2$ with middle co-core $D_s = \{0\} \times D^2$. Suppose that a second disk has a lobe on a positive parallel copy D_s^+ , joined to its root disk by a product-zero band in a supplied rectangular box. Cutting along D_s turns the handle into two half-handles. In the half-handle containing D_s^+ , the lobe and its band are boundary parallel: in product coordinates their union is the graph of an embedded strip whose short side lies on the new cut face. Compress this strip monotonically toward that short side. Extending the compression by the identity in the product-normal variable produces an ambient isotopy of the already cut manifold. It removes the lobe and retracts the disk to its root while preserving its full normal germ, the global E -bicollar, and every disjoint marking germ.

If two lobes occupy ordered heights in the same handle, this mechanism is applied from the nearer lobe to the farther one. The first isotopy clears the track used by the second. Nothing in the construction asserts that the two sweeps have disjoint simultaneous supports.

A.2. The five disk rows and cut schedules. For reference, the row disks are

$$(221) \quad \begin{array}{c|ccc} v & D_i^v & D_j^v & D_k^v \\ \hline 0 & i_0 & j_0 & k_0 \\ B & i_0 & J & k_0 \\ A & I_A & j_0 & k_0 \\ AC & I_\star & j_0 & k_0 \\ \star & I_\star & J & k_0, \end{array} \quad \begin{array}{l} J = j_0 \#_{\beta_0} k_\varepsilon, \\ I_A = i_0 \#_\alpha j_\varepsilon, \\ I_\star = i_0 \#_\alpha J^+ = (I_A) \#_{\beta_1} k_{2\varepsilon}. \end{array}$$

In a schedule below, cut D means cut the actual disk in the displayed row; a following arrow describes its image after the earlier cuts.

TABLE 1. The five product-germ-preserving cut schedules.

vertex	operations in the already cut manifold
0	cut k_0 ; then cut j_0 ; then cut i_0 .
B	cut k_0 ; $J \rightsquigarrow j_0$ by removing its k_ε -lobe; cut J (now represented by j_0); cut i_0 .
A	cut j_0 ; $I_A \rightsquigarrow i_0$ by removing its j_ε -lobe; cut I_A (now represented by i_0); cut k_0 .
AC	cut k_0 ; $I_\star \rightsquigarrow I_A$ by removing the $k_{2\varepsilon}$ -lobe and adjacent β_1 -strips; cut j_0 ; $I_A \rightsquigarrow i_0$; cut $I_\star$ (now represented by i_0).
$\star$	cut k_0 ; first remove the nearer k_ε -lobe so $J \rightsquigarrow j_0$; then move the farther $k_{2\varepsilon}$ -lobe of $I_\star$ through the cleared region so $I_\star \rightsquigarrow I_A$; cut J (now represented by j_0); remove the j_ε -lobe so $I_A \rightsquigarrow i_0$; cut $I_\star$ (now represented by i_0).

Each arrow is the product-germ cancellation described above. Its support lies in the supplied handle, band, and smoothing boxes and is disjoint from E , the exterior row, and the unused handles. At AC the β_1 -lobe is removed before the α -lobe. At $\star$ the k_ε -lobe is removed before the $k_{2\varepsilon}$ -lobe; the order is part of the schedule.

After these cancellations every row reduces to the three elementary co-cores. Cutting them leaves the rounded central ball

$$C = \text{Round}(\chi(Q \times [1 - \tau, 1]))$$

together with six half-handles attached along boundary disks. Absorbing the half-handles by boundary-collar compression identifies the remainder with D^3 . Because every support misses the global interface, the identifications

$$(222) \quad \Phi_v : K_v \xrightarrow{\cong} D^3, \quad v \in V,$$

are orientation preserving and fixed on the same E -bicollar.

A.3. The abstract oriented cut-sphere cellulation. The model has seven core disk faces

$$(223) \quad E, \quad i^-, i^+, \quad j^-, j^+, \quad k^-, k^+,$$

and three branches

$$(224) \quad E \xrightarrow{s_r} r^- \xrightarrow{c_r} r^+, \quad r = i, j, k.$$

Use six sectors on E , four on every r^- , and two on every r^+ . For a face f with n_f sectors, introduce vertices $x_{f,a}, y_{f,a}$ and oriented edges

$$(225) \quad \begin{array}{l} a_{f,a} : x_{f,a} \longrightarrow x_{f,a+1}, \\ b_{f,a} : y_{f,a} \longrightarrow y_{f,a+1}, \\ d_{f,a} : x_{f,a} \longrightarrow y_{f,a}, \end{array} \quad a \in \mathbb{Z}/n_f.$$

The spoke attachment slots on E are

$$(226) \quad p_i = 1, \quad p_j = 5, \quad p_k = 3.$$

For each r , orient the four long sides of the spoke and dual ribbons by the endpoint table

	side	initial endpoint	terminal endpoint
(227)	l_{s_r}	y_{E,p_r}	$y_{r^-,2}$
	r_{s_r}	y_{E,p_r+1}	$y_{r^-,1}$
	l_{c_r}	$y_{r^-,3}$	$y_{r^+,2}$
	r_{c_r}	$y_{r^-,4}$	$y_{r^+,1}$

where the E -indices are cyclic.

The seven oriented core faces are

$$(228) \quad \partial C_f = a_{f,1} a_{f,2} \cdots a_{f,n_f}.$$

The n_f collar quadrilaterals at f are

$$(229) \quad \partial A_{f,a} = d_{f,a} b_{f,a} d_{f,a+1}^{-1} a_{f,a}^{-1}.$$

For every $r = i, j, k$, the two ribbon quadrilaterals have boundary

$$(230) \quad \begin{aligned} \partial R_{s_r} &= l_{s_r} b_{r^-,1}^{-1} r_{s_r}^{-1} b_{E,p_r}^{-1}, \\ \partial R_{c_r} &= l_{c_r} b_{r^+,1}^{-1} r_{c_r}^{-1} b_{r^-,3}^{-1}. \end{aligned}$$

Thus the six ribbons consume, on their non-collar sides,

$$(231) \quad b_{E,p_r}, \quad b_{r^-,1}, \quad b_{r^-,3}, \quad b_{r^+,1} \quad (r = i, j, k).$$

On the two-sector face r^+ , the edges $b_{r^+,1}$ and $b_{r^+,2}$ have the same unordered endpoints but are distinct parallel 1-cells. The ribbon uses the first and the complementary face uses the second.

The remaining outer edges form the following simple oriented 24-vertex cycle:

$$(232) \quad \begin{aligned} & y_{E,3} \rightarrow y_{E,2} \rightarrow y_{i^-,1} \rightarrow y_{i^-,4} \rightarrow y_{i^+,1} \rightarrow y_{i^+,2} \rightarrow y_{i^-,3} \rightarrow y_{i^-,2} \\ & \rightarrow y_{E,1} \rightarrow y_{E,6} \rightarrow y_{j^-,1} \rightarrow y_{j^-,4} \rightarrow y_{j^+,1} \rightarrow y_{j^+,2} \rightarrow y_{j^-,3} \rightarrow y_{j^-,2} \\ & \rightarrow y_{E,5} \rightarrow y_{E,4} \rightarrow y_{k^-,1} \rightarrow y_{k^-,4} \rightarrow y_{k^+,1} \rightarrow y_{k^+,2} \rightarrow y_{k^-,3} \rightarrow y_{k^-,2} \rightarrow y_{E,3}. \end{aligned}$$

Every arrow is the unique incident oriented 1-cell or its inverse, except at a positive face, where the indicated arrow is $b_{r^+,2}$. Attaching one 24-gon along (232) completes the face list.

Proposition A.1 (The finite model is a marked sphere). *The cycles (228)–(232) define a connected closed oriented cellulation of S^2 with*

$$(233) \quad (V, E, F) = (48, 84, 38).$$

The 24-gon is the unique complementary disk.

Proof. The incidence and link checks establish a connected closed oriented surface; only then does the Euler count identify it as a sphere. The fat-tree neighborhood identifies the 24-gon as its complementary disk.

There are

$$\begin{aligned} V &= 2(6 + 3 \cdot 4 + 3 \cdot 2) = 48, \\ E &= 3(6 + 3 \cdot 4 + 3 \cdot 2) + 3 \cdot 4 = 84, \\ F &= 7 + (6 + 3 \cdot 4 + 3 \cdot 2) + 6 + 1 = 38. \end{aligned}$$

The printed incidence list places every 1-cell in exactly two face cycles with opposite induced orientations. Its one-skeleton is connected. Each x -vertex has a 3-cycle link and each y -vertex a 4-cycle link, including the explicitly distinguished parallel edge at r^+ . Hence the result is a connected closed orientable 2-manifold. Its Euler characteristic is $48 - 84 + 38 = 2$, so it is a sphere. Since all other faces lie in the regular neighborhood of the protected fat tree, the last face is its unique complementary disk. $\square$

A.4. Realized fat trees and all five vertex models. At vertex v , denote the seven realized faces by

$$E, \quad D_{i,v}^-, D_{i,v}^+, \quad D_{j,v}^-, D_{j,v}^+, \quad D_{k,v}^-, D_{k,v}^+.$$

The cut dual longitudes provide the three arcs $c_{r,v} : D_{r,v}^- \rightarrow D_{r,v}^+$. Their union with the seven face disks has four components. Choose disjoint spokes

$$(234) \quad s_{r,v} : E \longrightarrow D_{r,v}^-, \quad r = i, j, k,$$

one at a time in the supplied corridors, attaching to E in the slots (226). Each joins two different components, so

$$(235) \quad T_v = (\text{seven face disks}) \cup \bigcup_{r=i,j,k} (s_{r,v} \cup c_{r,v})$$

is a connected fat tree. Its closed regular neighborhood is a disk and the closure of its complement in ∂K_v is a disk Z_v . This proves the complementary-disk statement geometrically, before any Euler-characteristic check.

Map the abstract faces, branches, ribbons, detector, and collar rectangles to these realized objects in their product coordinates. The coordinate maps agree on full overlap strips and give a positive diffeomorphism on a regular neighborhood of the tree. In polar charts on the two complementary disks, its boundary germ is

$$(r, [\theta]) \longmapsto (r, [f_v(\theta)]), \quad f_v(\theta + 1) = f_v(\theta) + 1, \quad f'_v(\theta) > 0.$$

For an endpoint-flat cutoff $\eta(r)$, extend it by

$$(236) \quad (r, [\theta]) \longmapsto (r, [(1 - \eta(r))\theta + \eta(r)f_v(\theta)]).$$

Near $r = 0$ this is the identity, so there is no polar-center singularity. Away from the center its angular derivative is positive. It agrees with the tree-neighborhood map on an open outer annulus with all derivatives. The resulting

$$b_v : S_{Q0}^2 \xrightarrow{\cong} \partial K_v$$

realizes every one of the 38 indexed faces. This construction is performed independently at all five vertices.

A.5. Signed faces and full normal pairings. For each r , fix an orientation-reversing disk diffeomorphism $j_r : D^2 \rightarrow D^2$. Choose a positive chart $\kappa_{r,v}^{-,\partial} : D^2 \rightarrow D_{r,v}^-$, and define

$$(237) \quad \kappa_{r,v}^{+,\partial} = \gamma_{r,v} \kappa_{r,v}^{-,\partial} j_r^{-1},$$

where $\gamma_{r,v}$ is the canonical identification of the two copies of the same row disk. Then

$$(238) \quad \begin{aligned} \pi_{r,v}^{-+,\partial} \kappa_{r,v}^{-,\partial}(z) &= \kappa_{r,v}^{+,\partial}(j_r z), \\ \pi_{r,v}^{+-,\partial} &= (\pi_{r,v}^{-+,\partial})^{-1}. \end{aligned}$$

The canonical identification and j_r each reverse the relevant disk orientation, so both face charts are positive while the pairing is orientation reversing. It is a smooth pairing, not a cellular one: four displayed sectors on r^- cannot be paired cellwise with two on r^+ .

Let

$$\nu_{r,v}^\sigma : D_{r,v}^\sigma \times [0, d_{r,v}) \longrightarrow K_v$$

be the inherited inward collar. In its local double, use the formula

$$(239) \quad \pi_{r,v}^{-+,\text{full}}(\kappa_{r,v}^{-,\partial}(z), u) = (\kappa_{r,v}^{+,\partial}(j_r z), u), \quad |u| < d_{r,v}.$$

It reverses the surface orientation and preserves the complete normal variable. The negative half belongs to the reflected local double; only the nonnegative half lies in the realized cut ball. Taking the minimum over the three labels and five vertices gives one positive width for all fifteen undirected pairings and their directed inverses.

APPENDIX B. COLLARS AND ALL-JET COMPATIBILITY

This appendix records the differential-topological mechanisms used in Sections 7 and 8. Its first purpose is to distinguish compatibility on an open product germ from equality on a central face. Its second is to retain a *specified full collar* through the relative rigidity argument; replacing that collar by a smaller unspecified one would not suffice for the marked construction.

Compatible inward germs produce one collar, and signed bicollars make the local and spectator formulas agree on an open neighborhood. The subsequent rigidity argument retains that specified collar by comparing the relevant path components and then choosing a jointly smooth isotopy. Thus the same protected neighborhood survives both gluing and contraction.

B.1. Germs and jets.

Definition B.1 (Agreement of all tangential and normal derivatives). Let $c_i : Z \times (-a, a) \rightarrow M_i$ be smooth product charts and let $F, G : M_0 \rightarrow M_1$ be smooth wherever the displayed expressions are defined. Put

$$\tilde{F} = c_1^{-1} F c_0, \quad \tilde{G} = c_1^{-1} G c_0.$$

We say that F and G agree with all jets along Z if, for every multi-index α and $m \geq 0$,

$$\partial_z^\alpha \partial_t^m [\tilde{F} - \tilde{G}](z, 0) = 0.$$

We say that they agree on an open product germ if there is an open neighborhood $\mathcal{U} \subset Z \times (-a, a)$ of $Z \times \{0\}$ on which $\tilde{F} = \tilde{G}$. Open-germ equality implies all-jet agreement. The converse is false for general smooth maps, and is never used here: every gluing argument below proves the stronger open-germ equality.

Remark B.2. Equality of values on $Z \times \{0\}$, even with matching first normal derivative, does not imply smoothness of a piecewise-defined map across Z . Likewise, equality on the artificial annulus $\mathcal{A}_+^D$ does not verify the disk-extension hypothesis on the whole protected annulus $\mathcal{A}_{01}^D$.

B.2. Relative extension of compatible collar germs.

Lemma B.3 (Relative collar extension). *Let M be a compact smooth manifold with boundary and let $O_0 \Subset O_1 \subset \partial M$. Suppose a finite collection of smooth collar embeddings is prescribed on open subsets of O_1 , all pointing strictly inward and agreeing on every overlap. Then there is $b > 0$ and a collar*

$$d : \partial M \times [0, b) \longrightarrow M$$

which agrees with every prescription on O_0 through a positive depth. If the prescribed germs agree on open neighborhoods of the closures of their domains, the resulting agreement includes all jets at the domain boundaries.

Proof. After shrinking the finitely many collar depths, push forward ∂_t on the complete prescribed product germs:

$$X_{\text{fix}}(c_\nu(x, t)) = (c_\nu)_*(\partial_t).$$

The overlap hypotheses make this a single smooth field on an open neighborhood of the prescribed product images, not merely a field on their central faces. On the boundary it points strictly inward. Choose any smooth field Y , defined near the boundary and strictly inward there. Extend X_{fix} slightly beyond the smaller prescribed images. With a cutoff χ , equal to one on those images over O_0 and supported over O_1 , put

$$X = \chi X_{\text{fix}} + (1 - \chi)Y.$$

The cone of inward-pointing vectors is convex, so X remains strictly inward. Extend it to a neighborhood of the boundary. Compactness and the inverse function theorem give a uniform $b > 0$ for which the positive-time flow of X is a collar embedding.

On the region where $\chi = 1$, the new flow and each prescribed collar solve the same autonomous initial-value problem throughout the retained product germ. Uniqueness of integral curves makes them equal there. The last assertion follows because this is open-germ equality in the sense of Definition B.1. $\square$

In Section 8, this lemma is applied five times to the model spectator balls and ten times to the realized spectator balls. The compatibility checks precede those applications. Taking a minimum of the resulting fifteen positive widths is legitimate only afterward, which is why u_{cmp} is defined at (144) rather than at the vertex stage.

B.3. Signed internal bicollars. Let $E_{e,u}$ be the internal disk separating the local and spectator halves of K_u . Orient its disk factor by the boundary orientation of the local half. The spectator boundary orientation is then the opposite one. A signed bicollar is written

$$c_{e,u}^{E_e} : E_e \times (-\eta, \eta) \longrightarrow K_u,$$

with negative coordinate on the local side and positive coordinate on the spectator side. If $s_{\text{loc}}, s_{\text{sp}} \geq 0$ are the two inward variables, the endpoint maps have the formulas

$$(240) \quad \widehat{\Theta}_{e,u}(z, s_{\text{loc}}) = c_{e,u}^{E_e}(z, -s_{\text{loc}}),$$

$$(241) \quad \Xi_{e,u}^z(z, s_{\text{sp}}) = c_{e,u}^{E_e}(z, +s_{\text{sp}}).$$

Both are orientation preserving because the induced orientations on the two copies of E_e are opposite.

Lemma B.4 (All-jet two-sided gluing). *Let*

$$K_u = K_{e,u}^{\text{loc}} \cup_{E_{e,u}} L_{e,u}^{\text{sp}}$$

and suppose smooth positive diffeomorphisms on the two closed halves extend to one signed bicollar and both have coordinate expression

$$(z, t) \longmapsto (z, t).$$

Then their union is a smooth orientation-preserving diffeomorphism. Its inverse is obtained by gluing the inverse half-maps.

Proof. The two maps agree on an open neighborhood of the central disk in the signed coordinate, not only on the disk itself. Their union is therefore locally one smooth map there and is smooth away from the disk by hypothesis. The same argument applies to the inverse formulas. Positivity holds on either half and hence everywhere by connectedness. $\square$

For the maps prepared in Proposition 8.7, (149) gives more explicitly, for every multi-index α and $m \geq 0$,

$$\begin{aligned} \partial_z^\alpha \partial_t^m \left[(c_{e,w}^{E_e})^{-1} \widehat{\Theta}_{e,w} \widehat{\phi}_e^{\text{cut}} \widehat{\Theta}_{e,v}^{-1} c_{e,v}^{E_e}(z, t) - (z, t) \right] &= 0, \\ \partial_z^\alpha \partial_t^m \left[(c_{e,w}^{E_e})^{-1} \Xi_{e,w}^z (\Xi_{e,v}^z)^{-1} c_{e,v}^{E_e}(z, t) - (z, t) \right] &= 0. \end{aligned}$$

These equations are the differential content of the corrected local/spectator gluing. The disk E_e here is internal to the cut ball; it is not the global exterior disk E .

B.4. Fixed collar versus fixed boundary. For $M = D^2$ or D^3 , fix a specified smooth closed full collar

$$c : \partial M \times [0, a] \longrightarrow M, \quad C = c(\partial M \times [0, a]).$$

Define, with the compact-open C^∞ -topology,

$$\begin{aligned} G_C &= \text{Diff}^\infty(M \text{ rel } C), \\ G_J &= \{g \in \text{Diff}^\infty(M) : j_{\partial M}^\infty g = j_{\partial M}^\infty \text{id}\}, \\ G_\partial &= \text{Diff}^\infty(M \text{ rel } \partial M). \end{aligned}$$

Lemma B.5 (Retention of a specified full collar). *The inclusion $G_C \hookrightarrow G_\partial$ induces a bijection on path components.*

Proof. The specified collar is compared to pointwise boundary fixing in two stages: first fixed-tube data are compared with infinite-jet data, then infinite-jet data are compared with boundary data. Keeping both stages visible is what allows the original full collar, rather than an unspecified thinner one, to remain fixed in the relative mapping class.

Equip a slightly larger collar with a product metric. The prescribed u -lines are normal geodesics, and a radial contraction supported in the outer buffer makes the specified C , not a smaller replacement, an interior normal tube in Cerf's sense. Cerf's fixed-tube versus fixed-infinite-jet comparison gives

$$\pi_0(G_C) \xrightarrow{\cong} \pi_0(G_J),$$

and his fixed-infinite-jet versus pointwise-boundary comparison gives

$$\pi_0(G_J) \xrightarrow{\cong} \pi_0(G_\partial)$$

[Cer61, II.4.2.3, Corollary 3; II.5.2.1, Proposition 9 and Corollary 2]. These are two distinct steps; the second may not be omitted. Their composition proves the claim. $\square$

Lemma B.6 (Jointly smooth representative of a path). *Let $A \subset M$ be fixed. Suppose*

$$g_0, g_1 \in \text{Diff}^\infty(M \text{ rel } A)$$

lie in the same path component for the C^∞ -topology, then they are joined by a jointly smooth isotopy through maps fixed on A . The isotopy can be made constant in time near both endpoints.

Proof. Cerf's local smooth endpoint section supplies a neighborhood V of the identity and a smoothly parameterized isotopy from id to every element of V [Cer61, Lemma 1.5.3]. Given a continuous path γ from g_0 to g_1 , compactness of its parameter interval and the topological-group structure give a subdivision $0 = t_0 < \dots < t_m = 1$ for which

$$d_i = \gamma(t_{i+1})\gamma(t_i)^{-1} \in V.$$

Compose the local isotopy for d_i with $\gamma(t_i)$, and reparameterize time by a smooth function flat at zero and one. Concatenation is then jointly smooth at every subdivision point and fixes A throughout. $\square$

Theorem B.7 (Relative three-ball rigidity). *Let $h \in \text{Diff}^+(D^3)$ be the identity on one specified closed full boundary collar C . There is a jointly smooth isotopy*

$$H_s \in \text{Diff}^+(D^3), \quad H_0 = h, \quad H_1 = \text{id}, \quad H_s|_C = \text{id}$$

for every $s \in [0, 1]$.

Proof. Hatcher's equivalent form of the Smale conjecture gives

$$\text{Diff}^\infty(D^3 \text{ rel } \partial D^3) \simeq *$$

[Hat83, Appendix, p. 604]. Hence h and id have the same class in $\pi_0(G_\partial)$. By Lemma B.5, they already have the same class in $\pi_0(G_C)$. Apply Lemma B.6 with $A = C$. $\square$

Remark B.8 (Exact scope). The theorem retains the originally specified collar through the entire path. It is not a uniqueness theorem for relative diffeomorphisms and does not identify two maps from their homology actions. In the main proof it is applied only after the complete protected collar has been shown fixed by the route defect.

APPENDIX F. THE LOCAL POSITIVE TWO-SITE SLIDE

This appendix proves the local input stated in Theorem 2.2. The point of the construction is to specify a relative diffeomorphism from geometric data, rather than to infer one from the four homology classes in (15). We therefore keep separate the small ball which constructs the slid disk, the larger two-handle sector which supports the map, and the two balls obtained by cutting that sector. Write $W_{\mathcal{C}}$ for the frozen oriented compression-body sector with the old complete marking, and $W_{\mathcal{C}\gamma}$ for the same sector with the slid complete marking.

Fix throughout the exact push-off, band, smoothing boxes, dual band, and exterior-interface germ supplied in Theorem 2.2. Every coordinate chosen below is adapted to those objects. In particular, no occurrence of $D_2^{\text{po}}, \gamma, \gamma^\vee$, or E_γ denotes a newly selected replacement.

The construction has five geometric transitions. First the prescribed band sum produces the slid disk inside a small regular neighborhood. Next the old and slid disk systems are shown to cut the same product sector, while a larger sector supplies the two cut balls on which the map will live. The finite boundary dictionary then records the actual faces, ribbons, pairings, and detector collar. Finally, the cellular boundary map is smoothed relative to those data, extended across the cut balls, and reglued. Keeping these roles separate is essential: the small neighborhood constructs the disk, but the larger cut-ball extension is what supplies the relative diffeomorphism.

F.1. The product model and the small slide ball. Let Q be a coordinate disk in the high boundary which contains the four supplied handle feet and the prescribed joining-band neighborhoods in its interior. This only restricts the ambient chart. Write the two oriented handles as

$$(242) \quad H_r = [-2, 2] \times D^2, \quad \{-2, 2\} \times D^2 \subset \text{int}(Q), \quad r = 1, 2,$$

with orientation $dt \wedge dx \wedge dy$. Their co-cores and chosen positive normals are

$$(243) \quad D_r = \{0\} \times D^2, \quad n_r = \partial_t.$$

The product coordinates on the lateral annuli $[-2, 2] \times \partial D^2$ induce the framings used below.

Let $\varepsilon > 0$ be the constant height of the prescribed positive product push-off. After rescaling the handle coordinate, without moving the push-off, assume $\varepsilon < 1/10$ and write

$$(244) \quad D_2^{\text{po}} = \{\varepsilon\} \times D^2 \subset H_2.$$

Thus D_2^{po} is the push-off in the direction n_2 , and it is disjoint from $D_1 \cup D_2$. Let a_γ be the oriented core of the prescribed band, including its short lateral-annulus segments. The hypotheses of Theorem 2.2 put its first endpoint on $m_1 = \partial D_1$, its second on ∂D_2^{po} , away from the two diagonal dual-row points, and its interior away from both boundary circles and every other protected marking.

Parametrize the already prescribed high-boundary rectangle by $\bar{\gamma} = [-1, 1] \times [0, 1]$ with its product orientation and its short sides at $v = 0, 1$. The supplied boundary-parallel push-in datum is a smooth graph in the frozen inward collar: for its recorded function b_γ , that same prescribed band is

$$(245) \quad (u, v) \mapsto (\bar{\gamma}(u, v), s = b_\gamma(u, v)), \quad 0 \leq b_\gamma < \varepsilon/4, \quad b_\gamma(u, 0) = b_\gamma(u, 1) = 0.$$

The graph is unchanged at the attaching short sides, and its push-in datum carries the constant product normal of the boundary rectangle to the supplied framing. Thus (245) records the fixed band γ , rather than constructing a new one. Its supplied orientation makes the boundary band sum positive.

Lemma F.1 (Small slide ball). *The regular neighborhood*

$$B_\gamma^{\text{sl}} = \overline{N(D_1 \cup D_2^{\text{po}} \cup a_\gamma)}$$

is a closed 3-ball. Its three inward disk faces are product-parallel copies of D_1 , D_2^{po} , and the smoothed oriented disk

$$D_1^\gamma = D_1 \natural_\gamma D_2^{\text{po}}.$$

The last disk is embedded, its boundary is the positive band sum, and it is disjoint from D_2 . We put $D_2^\gamma = D_2$.

Proof. The two disks are disjoint, and the interior of a_γ misses both. The regular-neighborhood disk-slide construction [GL15, Section 4, disk-slide construction] identifies their neighborhood with a 3-ball; the complement of its high-boundary pair of pants consists of the three stated disk faces. The third is obtained by deleting the two attaching intervals and adjoining the long sides of the rectangle (245).

In each prescribed band-corner smoothing box take adapted oriented quadrant coordinates $(r, s) \in [0, 1]^2$ in which its fixed rounding is represented by $r^2 + s^2 = \eta^2$, $r, s \geq 0$, for one common recorded η . The four supplied smoothing boxes are disjoint. The result is a disk, because it is a boundary connected sum of two disks, and is embedded, because the disk pieces, band interior, and smoothing boxes are mutually disjoint. Its boundary orientation is the chosen positive band-sum orientation, so

$$(246) \quad [\partial D_1^\gamma] = [m_1] + [m_2].$$

Finally, take the regular neighborhood smaller than the positive distance from $D_1 \cup D_2^{\text{po}} \cup a_\gamma$ to D_2 . This proves the asserted disjointness. Notice that the frontier of this small ball contains both an old and a slid disk face. Thus this ball constructs D_1^γ , but it cannot be the support of a map which is fixed on its whole frontier. Only the stated local regular-neighborhood conclusion is used; the subsequent absolute minimal-disk-system assertion in the cited source is not imported. $\square$

Lemma F.2 (Completeness of the two disk systems). *The ordered systems (D_1, D_2) and (D_1^γ, D_2^γ) are complete relative co-core systems for the same two-handle compression body. Cutting the entire compression body along either pair produces the underlying product compression body, not a 3-ball.*

Proof. Cutting H_r at $t = 0$ gives the half-handles $[-2, 0] \times D^2$ and $[0, 2] \times D^2$. Each is absorbed into the product collar on its attaching foot. Cutting both handles therefore recovers the product region, relative to the incoming boundary and the complement of the four foot collars.

For the slid pair, first cut along D_2 . The strip between $D_2 = \{0\} \times D^2$ and $D_2^{\text{po}} = \{\varepsilon\} \times D^2$ then becomes a boundary product $D^2 \times [0, \varepsilon]$. Push the D_2^{po} -lobe of D_1^γ , together with the adjacent half of the band, monotonically through this product. At time u , move the slice $D^2 \times \{\varepsilon(1 - u)\}$ toward the attaching short side. At $u = 1$ the lobe and the two adjacent band strips cancel, leaving D_1 . The four corner roundings move inside their disjoint quadrant boxes, so this is a proper isotopy with no self-intersection. It is fixed off B_γ^{sl} and on the incoming boundary. Cutting the remaining disk therefore gives the same product region as the old pair. $\square$

F.2. The larger sector and the two cut balls. Let C_γ be the closed product-ended piece cut off by the supplied interface E_γ . In an adapted product coordinate, rescaled without moving the interface, write $C_\gamma = Q \times [1 - \tau, 1]$ with $0 < \tau < 1/10$. Its rounded frontier against the unchanged product region is precisely the prescribed disk

$$(247) \quad E_\gamma = (Q \times \{1 - \tau\}) \cup (\partial Q \times [1 - \tau, 1]).$$

Let U_γ be obtained from C_γ by adjoining both complete handles. The interface hypothesis places the prescribed bands strictly inside this sector. Shrinking only the auxiliary regular neighborhood if necessary gives $B_\gamma^{\text{sl}} \subset U_\gamma$.

Lemma F.3 (Sector and cut balls). *The sector U_γ is a genus-two handlebody and meets the closure of its complement exactly in E_γ . Its exposed boundary*

$$P = \overline{\partial U_\gamma \setminus E_\gamma}$$

is a genus-two surface with one boundary component $\delta = \partial E_\gamma$. Both cut manifolds

$$(248) \quad K_{\mathfrak{C}} = U_\gamma \setminus (D_1 \cup D_2), \quad K_{\mathfrak{C}^\gamma} = U_\gamma \setminus (D_1^\gamma \cup D_2^\gamma)$$

are 3-balls. Their boundaries contain, respectively,

$$D_1^-, D_1^+, D_2^-, D_2^+, \quad (D_1^\gamma)^-, (D_1^\gamma)^+, (D_2^\gamma)^-, (D_2^\gamma)^+,$$

and both contain the same interface disk E_γ .

Proof. The piece C_γ is a 0-handle. Attaching the two products $[-2, 2] \times D^2$ along their pairs of feet adds two 1-handles, so U_γ is a genus-two handlebody. Its explicit product description shows that its only interface with the complement is (247). Removing this disk from the boundary of a genus-two handlebody gives the stated surface P , together with the directed collar of δ inherited from the rounded coordinates.

After cutting the old disks, C_γ has four product half-handles attached along boundary disks. Attaching a ball to a ball along a disk is again a ball: in collar coordinates the attached piece is compressed by $(z, s) \mapsto (z, s/2)$. Applying this four times proves that $K_{\mathcal{C}}$ is a ball and identifies the four signed faces. For the slid cut, first cut D_2 and use the inverse-lobe isotopy from Lemma F.2; it takes D_1^γ to D_1 in the cut sector and fixes E_γ . The remaining cut is then the same four-half-handle construction. This proves independently that the second cut manifold is a ball. $\square$

The balls in (248), rather than the small ball B_γ^{sl} , are the domain and target of the extension.

F.3. The embedded dual row. In P , retain the prescribed dual band $\gamma^\vee$: the narrow boundary rectangle parallel to the old product band, running from the ℓ_2 -side to a positive push-off of the ℓ_1 -side on the specified side of $\bar{\gamma}$, with its second short side oriented oppositely to ℓ_1 . Its supplied interior is disjoint from all four old curves away from its ends. Define

$$(249) \quad \ell_1^\gamma = \ell_1, \quad \ell_2^\gamma = \ell_2 \sharp_{\gamma^\vee} (-\ell_1)^+.$$

Lemma F.4 (Geometric rows and signs). *The four slid curves are embedded, oriented, and product framed. Their oriented classes are*

$$(250) \quad \begin{aligned} [m_1^\gamma] &= [m_1] + [m_2], & [\ell_1^\gamma] &= [\ell_1], \\ [m_2^\gamma] &= [m_2], & [\ell_2^\gamma] &= [\ell_2] - [\ell_1]. \end{aligned}$$

After the prescribed relative bigon removal, the actual intersections are

$$(251) \quad m_r^\gamma \cdot \ell_s^\gamma = \delta_{rs}, \quad m_r^\gamma \cap \ell_s^\gamma = \emptyset \quad (r \neq s), \quad m_1^\gamma \cap m_2^\gamma = \emptyset, \quad \ell_1^\gamma \cap \ell_2^\gamma = \emptyset.$$

Cutting along either disk system turns each longitude into an oriented arc from the negative to the positive copy of its disk face; its product framing becomes a ribbon neighborhood.

Proof. The boundary components of the meridian pair of pants are $m_1, m_2, -m_1^\gamma$; those of the longitude pair of pants are $\ell_2, -\ell_1, -\ell_2^\gamma$. Their oriented boundary relations give (250), including its two independent signs.

Retain the original positive points $m_1 \cap \ell_1$ and $m_2 \cap \ell_2$ outside the band neighborhoods. Initially m_1^γ meets ℓ_2^γ twice: the m_2 -lobe meets the ℓ_2 -piece positively, while the m_1 -piece meets the $(-\ell_1)^+$ -piece negatively. Parallel routing of the two bands makes these points the corners of an embedded bigon disjoint from every other curve. Pushing the longitude side across this bigon removes both points and preserves the product framing. All other cross pairs are disjoint by construction, proving (251) geometrically. Cutting at a longitude-co-core crossing separates a small interval into endpoints on the two signed faces; cutting its framed normal interval at the same time gives the stated ribbon. $\square$

F.4. A complete dictionary on the two cut spheres. We now make the boundary map finite and explicit. Write $\mathbf{e}_\gamma$ for the cell vertex representing the local interface disk E_γ ; this is not the global interface symbol E of Definition 2.1. On the old sphere use the five disk vertices

$$\mathcal{V}^o = \{\mathbf{e}_\gamma, 1-, 1+, 2-, 2+\};$$

on the slid sphere use

$$\mathcal{V}^\gamma = \{\mathbf{e}_\gamma, 1\gamma-, 1\gamma+, 2\gamma-, 2\gamma+\}.$$

Let $\nu : \mathcal{V}^o \rightarrow \mathcal{V}^\gamma$ be the displayed index-preserving bijection. The oriented fat-tree edges are

$$(252) \quad s_1 : \mathbf{e}_\gamma \longrightarrow 1-, \quad c_1 : 1- \longrightarrow 1+, \quad s_2 : \mathbf{e}_\gamma \longrightarrow 2-, \quad c_2 : 2- \longrightarrow 2+,$$

and the target has the correspondingly decorated four edges. The edges c_1, c_2 are the actual dual arcs from Lemma F.4; s_1, s_2 are auxiliary arcs chosen in the complement of their ribbons.

Put

$$n_{\mathbf{e}_\gamma} = n_{1-} = n_{2-} = 4, \quad n_{1+} = n_{2+} = 2.$$

For every $v \in \mathcal{V}^o$ and $1 \leq j \leq n_v$, with j read cyclically, take vertices $x_{v,j}^o, y_{v,j}^o$, inner edges

$$\iota_{v,j}^o : x_{v,j}^o \longrightarrow x_{v,j+1}^o,$$

outer edges

$$\omega_{v,j}^o : y_{v,j}^o \longrightarrow y_{v,j+1}^o,$$

and radial edges

$$\rho_{v,j}^o : x_{v,j}^o \longrightarrow y_{v,j}^o.$$

Use the same definitions with superscript γ and vertex $\nu(v)$ on the target sphere. This is a complete list of the thirty-two vertices and the forty-eight inner, outer, and radial edges on either sphere.

The attachment names for the outer edges are fixed by

$$(253) \quad \begin{array}{c|cccc} v & \omega_{v,1} & \omega_{v,2} & \omega_{v,3} & \omega_{v,4} \\ \hline \mathbf{e}_\gamma & a_{\mathbf{e}_\gamma, s_1} & f_{\mathbf{e}_\gamma, 0} & a_{\mathbf{e}_\gamma, s_2} & f_{\mathbf{e}_\gamma, 1} \\ 1- & a_{1-, s_1} & f_{1-, 0} & a_{1-, c_1} & f_{1-, 1} \\ 2- & a_{2-, s_2} & f_{2-, 0} & a_{2-, c_2} & f_{2-, 1} \end{array} \quad \begin{array}{c|cc} v & \omega_{v,1} & \omega_{v,2} \\ \hline 1+ & a_{1+, c_1} & f_{1+} \\ 2+ & a_{2+, c_2} & f_{2+} \end{array}.$$

The target table is obtained by applying ν to every non- $\mathbf{e}_\gamma$ vertex and decorating each tree-edge name. For every tree edge e , add two long edges l_e, r_e , directed from its initial to its terminal vertex. These eight long edges bring the total to fifty-six.

The following formulas list all twenty-six faces and all of their cyclic words. They apply once to the old data and once to the decorated target data:

$$(254) \quad F_v : \iota_{v,1} \iota_{v,2} \cdots \iota_{v,n_v}, \quad v \in \mathcal{V}^o,$$

$$(255) \quad C_{v,j} : \rho_{v,j} \omega_{v,j} \rho_{v,j+1}^{-1} \iota_{v,j}^{-1}, \quad v \in \mathcal{V}^o, \ 1 \leq j \leq n_v,$$

$$(256) \quad R_e : l_e a_{\text{ter}(e),e}^{-1} r_e^{-1} a_{\text{ini}(e),e}^{-1}, \quad e \in \{s_1, c_1, s_2, c_2\}.$$

Here the five F_v are the interface disk and four signed disk faces, the sixteen $C_{v,j}$ are their full collar sectors, and the four R_e are the tree ribbons. The remaining face Z has the following complete sixteen-edge boundary word:

$$(257) \quad (f_{\mathbf{e}_\gamma, 0})^{-1} r_{s_1} (f_{1-, 1})^{-1} r_{c_1} (f_{1+})^{-1} l_{c_1}^{-1} (f_{1-, 0})^{-1} l_{s_1}^{-1} \\ \cdot (f_{\mathbf{e}_\gamma, 1})^{-1} r_{s_2} (f_{2-, 1})^{-1} r_{c_2} (f_{2+})^{-1} l_{c_2}^{-1} (f_{2-, 0})^{-1} l_{s_2}^{-1}.$$

For the target, decorate this word as above. Its successive source vertices are

$$(258) \quad y_{\mathbf{e}_\gamma, 3}, y_{\mathbf{e}_\gamma, 2}, y_{1-, 1}, y_{1-, 4}, y_{1+, 1}, y_{1+, 2}, y_{1-, 3}, y_{1-, 2}, y_{\mathbf{e}_\gamma, 1}, \\ y_{\mathbf{e}_\gamma, 4}, y_{2-, 1}, y_{2-, 4}, y_{2+, 1}, y_{2+, 2}, y_{2-, 3}, y_{2-, 2}, y_{\mathbf{e}_\gamma, 3}.$$

Thus every consecutive pair, including the last and first edges, shares the indicated endpoint.

For clarity, the source–target product dictionary is:

faces	target	prescribed map	pairing or role
$F_{\mathbf{e}_\gamma}^o$	$F_{\mathbf{e}_\gamma}^\gamma$	identity in the fixed disk chart $d_{\mathbf{e}_\gamma}$ and its normal collar	interface
F_{r-}^o	$F_{r\gamma-}^\gamma$	$d_{r-}^o \times t$ to $d_{r\gamma-}^\gamma \times t$	paired with F_{r+}
F_{r+}^o	$F_{r\gamma+}^\gamma$	$d_{r+}^o \times t$ to $d_{r\gamma+}^\gamma \times t$	paired with F_{r-}
$C_{v,j}^o$	$C_{\nu(v),j}^\gamma$	$\kappa_{\nu(v),j}^\gamma \circ \kappa_{v,j}^o$	pointwise product map on the entire collar
R_e^o	R_e^γ	$\varrho_e^\gamma \circ \varrho_e^{o-1}$	actual dual ribbon for $e = c_r$, auxiliary ribbon for $e = s_r$
Z^o	Z^γ	indexed affine map after coning	sole unmarked complementary disk

In the two signed-face rows $r = 1, 2$; in the collar row $v \in \mathcal{V}^o$ and $1 \leq j \leq n_v$; and in the ribbon row $e \in \{s_1, c_1, s_2, c_2\}$. These displayed finite ranges are part of the dictionary.

Every inner edge occurs once in a core word and inversely in a collar word. Every radial edge occurs oppositely in two adjacent collar words; every attachment edge occurs in a collar and inversely in a ribbon; every free outer edge occurs in a collar and inversely in Z ; and every long

edge occurs in a ribbon and inversely in Z . Thus each of the fifty-six edges occurs twice with opposite orientations. There are thirty-two vertices and twenty-six faces, and

$$(259) \quad 32 - 56 + 26 = 2.$$

The word (257) is the boundary walk of a regular neighborhood of the tree (252). At each vertex, the core, collar, ribbon, and complementary corners occur once in the displayed cyclic order. Hence every vertex link is a circle. The connected complex is therefore a closed oriented surface of Euler characteristic two, so it is the whole cut sphere; in particular Z is a disk and no face is missing.

Put an interior vertex z^o in Z^o and z^γ in Z^γ , and cone the sixteen edges in (257). Denote the resulting triangles by T_j^o and T_j^γ , $1 \leq j \leq 16$, in boundary order. There are no other complementary triangles.

The four e_γ -collar rectangles carry coordinates

$$(260) \quad (\theta, u) \in (\mathbb{R}/\mathbb{Z}) \times [-1, 1], \quad \theta = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$$

on their radial edges. Choose the trace of the detector b_δ to be $\rho_{e_\gamma,1}$, directed from $u = -1$ on the E_γ -side to $u = 1$ on the P -side. The union $C_{e_\gamma,4} \cup C_{e_\gamma,1}$ is its framed ribbon. Its two outside radial edges record the normal and product framing, and its incident inner and outer edges record both endpoint and tangent germs. The rectangles R_{c_1}, R_{c_2} are precisely the two geometric dual-arc ribbons. Thus the cellulation contains every signed disk face, dual arc, framing ribbon, product collar, detector trace, and auxiliary cutting arc.

Proposition F.5 (Complete cut-sphere map). *There is an orientation-preserving cellular homeomorphism*

$$\phi_\gamma^{\text{PL}} : (\partial K_{\mathfrak{C}}, \mathcal{Q}_{\mathfrak{C}}) \longrightarrow (\partial K_{\mathfrak{C}^\gamma}, \mathcal{Q}_{\mathfrak{C}^\gamma})$$

which is the identity on E_γ and its product germ, maps every signed old disk face and its complete normal collar to the corresponding slid face, maps the actual old dual arcs and ribbons to the actual slid ones, and is the prescribed pointwise product map on every collar. It intertwines each source minus-to-plus pairing with its corresponding target pairing, and therefore also intertwines their inverses. On the δ -collar it is $(\theta, u) \mapsto (\theta, u)$, so the detector has zero turns.

Proof. The complete vertex and edge assignment is, for every displayed finite index,

$$\begin{array}{lll} x_{v,j}^o \longmapsto x_{\nu(v),j}^\gamma & y_{v,j}^o \longmapsto y_{\nu(v),j}^\gamma & \\ \iota_{v,j}^o \longmapsto \iota_{\nu(v),j}^\gamma & \omega_{v,j}^o \longmapsto \omega_{\nu(v),j}^\gamma & \rho_{v,j}^o \longmapsto \rho_{\nu(v),j}^\gamma, \\ l_e^o \longmapsto l_e^\gamma & r_e^o \longmapsto r_e^\gamma. & \end{array}$$

Map the five disk cores, sixteen collar rectangles, and four ribbons by the product maps in the dictionary. Send z^o to z^γ and map T_j^o affinely and orientation preservingly to T_j^γ , for each $1 \leq j \leq 16$. The cyclic words show that adjacent formulas agree on every common edge. The map is bijective on vertices, edges, and faces and preserves every face orientation, so it is an orientation-preserving homeomorphism of the whole spheres.

Let j_r^o and j_r^γ be the orientation-reversing pairings of the two signed copies of the r -th disk. The paired disk maps use the same normal-side coordinate, whence

$$(261) \quad \phi_\gamma^{\text{PL}} j_r^o(z) = j_r^\gamma \phi_\gamma^{\text{PL}}(z), \quad r = 1, 2.$$

The e_γ -rows are literal identities, and the c_r -rows are the actual geometric ribbons of Lemma F.4. Finally, (260) is preserved pointwise, with no term $\theta \mapsto \theta + qu$. Thus both endpoint germs, the directed tangent, the chosen normal, and the two framing edges of b_δ are transported with $q = 0$. $\square$

F.5. A smooth boundary representative. The cellular map fixes the incidence and relative class, but agreement of affine values along a cell edge is not yet agreement of smooth germs. We next construct the smooth boundary map which will be extended over the balls.

The smooth representative is built in the same order as the cellular one. First choose compatible vertex and edge charts, then use the prescribed product maps on the protected faces

and ribbons. The only free piece is the complementary disk: an interpolation of the boundary circle map fills it while remaining the identity near its center. The resulting sphere map has the same full annular germ as the cellular map, and the Alexander family records that the two fillings lie in the same topological relative class.

Let $\mathcal{P}_\partial^\circ$ be a closed regular neighborhood of the first twenty-five faces in the dictionary: it contains the interface and its collar, all signed disk faces and collars, all four ribbons, the inherited site collars, the directed δ -collar, and the full framed detector trace. Its complementary face is the single disk Z_0° . Round the finitely many corners of the common boundary inside the prescribed face collars and adjoin to Z_0° a smaller closed product collar of that rounded boundary. Denote the enlarged disk by Z° and the overlap annulus by

$$A^\circ = \mathcal{P}_\partial^\circ \cap Z^\circ.$$

Make the identical indexed construction in the target. Thus the two pieces cover each sphere and overlap in one full closed annulus, not merely in a circle.

At every vertex in a prescribed product chart, use that chart and its indexed target chart. At every remaining vertex choose a small oriented smooth disk chart, and choose the target chart simultaneously so that the local indexed map is the coordinate identity. Make these vertex disks disjoint. For each open edge, remove the charted endpoint intervals and choose a two-sided tubular chart

$$(s, n) \in [a, b] \times [-\varepsilon_e, \varepsilon_e]$$

whose s -coordinate agrees with the endpoint charts. On a frozen edge this is the prescribed product chart; on any other edge choose the target chart by transporting the source chart through its indexed parameterization. Shrinking first the finitely many vertex disks and then the edge strips makes every overlap the same coordinate expression $(s, n) \mapsto (s, n)$. The one-skeleton map is therefore smooth with one two-sided germ at every vertex and edge.

On the five core faces use the frozen disk parameterizations, on the sixteen collar faces their pointwise product parameterizations, and on the four ribbon faces their oriented rectangle parameterizations. Their boundary variables are exactly the variables just fixed on the one-skeleton. Adjacent face germs hence agree on a full two-sided strip. For a paired pair of signed disk faces, use the same normal variable and define the positive-face map from the negative-face map by conjugating with j_r° and j_r^γ . This gives a smooth orientation-preserving diffeomorphism

$$\phi_{\gamma, \mathcal{P}}^{\text{sm}} : \mathcal{P}_\partial^\circ \longrightarrow \mathcal{P}_\partial^\gamma$$

which preserves every protected disk, ribbon, collar, tangent, normal, and framing in its actual product coordinates.

It remains to fill the sole disk Z° . Choose the source and target smooth disk parameterizations simultaneously with their collars. For some $0 < r_A < 1$, the overlap annuli are $[r_A, 1] \times \mathbb{R}/\mathbb{Z}$, and the protected map has the form

$$(r, \theta) \longmapsto (r, f(\theta)).$$

Here the degree-one circle diffeomorphism has a smooth lift satisfying

$$f(\theta + 1) = f(\theta) + 1, \quad f'(\theta) > 0.$$

For $0 \leq s \leq 1$, put

$$f_s(\theta) = (1 - s)\theta + sf(\theta).$$

Then $\partial_\theta f_s = (1 - s) + sf'(\theta) > 0$, so each f_s is a circle diffeomorphism. Choose $0 < r_0 < r_1 < r_A$ and an endpoint-flat smooth function $\eta : [0, 1] \rightarrow [0, 1]$ which is zero on $[0, r_0]$ and one on $[r_1, 1]$. Define

$$(262) \quad \phi_{\gamma, Z}^{\text{sm}}(r, \theta) = (r, f_{\eta(r)}(\theta)).$$

It is the coordinate identity near the center and therefore smooth there. It equals the protected product map on the whole overlap annulus. For $r > 0$, its angular Jacobian is $(1 - \eta(r)) + \eta(r)f'(\theta) > 0$; it preserves r , and hence is an orientation-preserving disk diffeomorphism. Its inverse is obtained radius by radius from the smooth inverses of the circle maps $f_{\eta(r)}$; it is the

identity near the center and the fixed product inverse near the annulus. Endpoint flatness gives agreement to all orders at the two radial seams.

The protected and complementary formulas glue to an orientation-preserving sphere diffeomorphism

$$(263) \quad \phi_\gamma^{\text{sm}} : \partial K_{\mathfrak{C}} \longrightarrow \partial K_{\mathfrak{C}^\gamma}.$$

It represents the cellular relative class. Indeed, on the inner disk bounded by A° , let p be the cellular filling and q the smooth filling; they have the same boundary map. Put $h = q^{-1}p$ and define

$$(264) \quad H_s(x) = \begin{cases} sh(x/s), & |x| \leq s, \\ x, & s \leq |x| \leq 1, \end{cases} \quad 0 < s \leq 1, \quad H_0 = \text{id}.$$

The two expressions agree at $|x| = s$, and the inverse uses h^{-1} . At $s = 0$, the inner image has norm at most s , proving joint continuity. Thus $q \circ H_s$ is an isotopy from q to p . Extending H_s by the identity on the full annulus and the protected region compares the two sphere maps without changing any marked germ or either pairing relation.

F.6. Ball extension and relative smoothing. The full boundary collar is extended by the product formula first. Radial extension across the inner 3-ball supplies a homeomorphism with the required collar behavior, but its cone point is not asserted to be smooth. Doubling across the boundary and applying relative homeomorphism approximation away from a thinner two-sided collar produces the smooth cut map while preserving all marked germs and the relative class.

Choose a sufficiently small number $\epsilon_{\text{col}} > 0$ and full inward smooth collars

$$c^\circ : \partial K_{\mathfrak{C}} \times [0, 3\epsilon_{\text{col}}] \longrightarrow K_{\mathfrak{C}}, \quad c^\gamma : \partial K_{\mathfrak{C}^\gamma} \times [0, 3\epsilon_{\text{col}}] \longrightarrow K_{\mathfrak{C}^\gamma},$$

extending every fixed face-normal coordinate. On $0 \leq t \leq 2\epsilon_{\text{col}}$, prescribe

$$(265) \quad c^\circ(x, t) \longmapsto c^\gamma(\phi_\gamma^{\text{sm}}(x), t).$$

The complements of the open collars $t < 2\epsilon_{\text{col}}$ are closed balls. Parameterize them by standard balls so that the boundary coordinates are the $t = 2\epsilon_{\text{col}}$ copies above. If $\psi_\partial : S^2 \rightarrow S^2$ is the resulting inner boundary map, extend it radially by

$$(266) \quad \mathcal{A}_{\psi_\partial}(ru) = r\psi_\partial(u), \quad 0 \leq r \leq 1, \quad u \in S^2,$$

with $\mathcal{A}_{\psi_\partial}(0) = 0$. This is continuous at the center, and its inverse is $\mathcal{A}_{\psi_\partial^{-1}}$. It agrees with (265) at the inner seam. The two formulas therefore give a homeomorphism

$$(267) \quad \Phi_\gamma^A : K_{\mathfrak{C}} \longrightarrow K_{\mathfrak{C}^\gamma}$$

which is a smooth orientation-preserving diffeomorphism on the full outer collar $0 \leq t < \epsilon_{\text{col}}$. We do not claim that the radial formula is smooth at its cone point.

Double the two cut balls along their boundaries. The signed collar coordinate identifies a neighborhood of each equator with $S^2 \times (-\epsilon_{\text{col}}, \epsilon_{\text{col}})$, and the doubled map $\widehat{\Phi}_\gamma^A$ has the smooth product formula

$$(x, t) \longmapsto (\phi_\gamma^{\text{sm}}(x), t)$$

there. Let K be the equator together with the closed thinner two-sided collar

$$S^2 \times [-\epsilon_{\text{col}}/2, \epsilon_{\text{col}}/2].$$

This closed set lies inside the open region on which $\widehat{\Phi}_\gamma^A$ is already a diffeomorphism. The classical relative homeomorphism-approximation theorem, in the form stated by Pardon [Par21, Theorem 4.6], applied directly to $\widehat{\Phi}_\gamma^A$, supplies a diffeomorphism $\widehat{\Psi}_\gamma$ of the source double to the target double which equals $\widehat{\Phi}_\gamma^A$ on a neighborhood of K .

The equator separates each double into two connected components. A point of the selected source side near the equator maps by the fixed product formula to the correspondingly labelled target side. Connectedness puts the image of the whole source side in that target side; applying

the same argument to $\widehat{\Psi}_\gamma^{-1}$ proves surjectivity onto it. Restricting to the closures of the chosen sides therefore gives a diffeomorphism

$$(268) \quad \widetilde{\Phi}_\gamma : K_{\mathfrak{C}} \longrightarrow K_{\mathfrak{C}^\gamma}$$

which agrees with (265) on a full boundary collar.

The smooth cut map lies in the intended topological relative class. In dimension $n = 3$, apply the same formula

$$(269) \quad H_s(x) = \begin{cases} sh(x/s), & |x| \leq s, \\ x, & s \leq |x| \leq 1, \end{cases} \quad 0 < s \leq 1, \quad H_0 = \text{id},$$

to the quotient $h = q^{-1}p$, where p is the inner part of Φ_γ^A and q is the inner part of $\widetilde{\Phi}_\gamma$. The maps agree on the full outer collar, so the quotient is the identity there and preserves the complementary inner ball. Indeed, a point of the collar is already its own unique preimage; if an inner point entered the collar, bijectivity would give that collar point a second preimage. Formula (269), extended by the identity on the collar, is an isotopy from the identity to $q^{-1}p$. Consequently $q \circ H_s$ is a topological isotopy relative to the whole collar from q to p . The seam, inverse, and continuity checks are exactly those following (264). This comparison is topological; no smooth relative isotopy is claimed.

F.7. Pairing, regluing, and the asserted local map. The common paired normal coordinates let the smooth cut map descend through the two disk identifications with equality of all normal jets. Its identity on the supplied interface collar then allows extension by the identity to the larger sector and the ambient compression body. The transported signed rows and their product framings give the geometric intersection and homology relations required by the local input.

The construction of the smooth sphere map with common paired normal coordinates gives two separate identities

$$(270) \quad \phi_\gamma^{\text{sm}} j_r^{\text{o}}(z) = j_r^\gamma \phi_\gamma^{\text{sm}}(z), \quad r = 1, 2.$$

Since $\widetilde{\Phi}_\gamma$ is exactly the product map on the full boundary collar, it satisfies

$$(271) \quad \widetilde{\Phi}_\gamma c^{\text{o}}(j_r^{\text{o}}(z), t) = c^\gamma(j_r^\gamma \phi_\gamma^{\text{sm}}(z), t), \quad r = 1, 2.$$

It therefore descends through both source disk-face identifications to the two target identifications. The normal coordinate t is unchanged, so the two descended expressions agree to all orders across each reglued disk. We obtain a diffeomorphism of the two copies of U_γ . It is the identity on E_γ and a product neighborhood thereof, so extending it by the identity outside U_γ is smooth. Denote the resulting map by

$$(272) \quad F_\gamma : W_{\mathfrak{C}} \longrightarrow W_{\mathfrak{C}^\gamma}.$$

The determinant of $\widetilde{\Phi}_\gamma$ has constant sign on the connected cut ball. On the outer collar its tangential determinant is positive and its normal derivative is +1. Hence the cut map, its regluing, and the extended sector map are orientation preserving. Set

$$(273) \quad B_-^{\text{LS}} = K_{\mathfrak{C}}, \quad B_+^{\text{LS}} = K_{\mathfrak{C}^\gamma}, \quad \phi_\gamma^{\text{cut}} = \widetilde{\Phi}_\gamma.$$

The signed disk rows reglue to the actual meridians, and the actual dual-arc ribbons reglue to the product-framed longitudes. Consequently the extended relative sector diffeomorphism carries

$$m_1 \mapsto m_1^\gamma, \quad m_2 \mapsto m_2^\gamma, \quad \ell_1 \mapsto \ell_1^\gamma, \quad \ell_2 \mapsto \ell_2^\gamma,$$

and Lemma F.4 gives both their embedded intersection pattern and the four class identities (15). The complete boundary dictionary, (270), the full collar formula, and the three-dimensional comparison prove all four parts of Theorem 2.2.

Remark F.6 (Exact limitations). The construction fixes one positive slide of D_1 over the chosen positive push-off of D_2 , for the specified band, dual band, complete cellulation, and collar data. It does not prove an opposite-sign slide, a slide of D_2 over D_1 , independence of arbitrary band choices, or uniqueness of a relative diffeomorphism from its homology action. The small ball

B_γ^{sl} constructs the disk but is not the support of the map. The Alexander comparisons are topological relative isotopies, not smooth ones. No compatibility among several slides, higher coherence, or Walker-theoretic conclusion is part of this local argument.

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