

THE BANNAI–MUNEMASA–VENKOV LATTICE METHOD FOR STRONGLY REGULAR GRAPHS: A UNIFORM DISCRIMINANT-FORM CONDITION AND NEW NONEXISTENCE RESULTS

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ABSTRACT. Bannai, Munemasa and Venkov, and later Nebe and Venkov, excluded tight spherical designs — among them strongly regular graphs of a special kind — by studying the lattice spanned by the design: they bound its discriminant group, pass to an even (over)lattice, and compare Gauss sums by the Milgram–Braun formula. We extend this method to arbitrary strongly regular graphs with integral eigenvalues. For an integral representation $G = b(A - \rho I) + cJ$ that is positive semidefinite with even diagonal, an elementary identity for dual vectors ties every class of the discriminant form of a maximal even overlattice to the class of a rescaled sum of the vertex vectors. This gives a necessary condition on the parameters (v, k, λ, μ) alone: a finite quadratic module with a short list of properties must exist. Its simplest case applies the textbook congruence $\det \equiv (-1)^{n/2} \pmod{4}$ for even lattices of odd determinant to the centred representation. A sweep of Brouwer’s table ($v \leq 1300$) finds 70 complementary pairs of parameter sets listed as open that fail the condition, among them $(96, 35, 10, 14)$, the triangle-free set $(176, 25, 0, 4)$ and $(169, 42, 5, 12)$; the grade of evidence is given for each (hand proof, or exact computation by at least two independent programs). No existing parameter set of the table fails in any representation tested, real graphs satisfy every condition with their actual discriminant forms, and the method gives short independent second proofs of the known nonexistence of $\text{srg}(75, 32, 10, 16)$ and $\text{srg}(289, 54, 1, 12)$. The simplest case also yields an infinite family: if q is a power of a prime $p \equiv 1 \pmod{4}$ and m is odd, there is no strongly regular graph with the parameters $(q^2, m(q+1), -q+m^2+3m, m(m+1))$ of the negative-Latin-square type $\text{NL}_m(q)$. This family, the parity case and the nonexistence of $\text{srg}(169, 42, 5, 12)$ are formalized in Lean 4 with Mathlib.

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Date: Version 1.1, October 2026.

2020 Mathematics Subject Classification. Primary 05E30; Secondary 11E08, 11H06, 11E12.

Key words and phrases. strongly regular graph, feasibility condition, even lattice, discriminant form, Milgram’s formula, maximal overlattice, tight spherical design.

1. INTRODUCTION

A strongly regular graph $\text{srg}(v, k, \lambda, \mu)$ is a k -regular graph on v vertices, neither complete nor empty, in which two adjacent vertices have exactly λ common neighbours and two non-adjacent vertices exactly μ . Its adjacency matrix A satisfies $A^2 = kI + \lambda A + \mu(J - I - A)$; apart from k (on **1**) it has two eigenvalues $r > s$, the roots of $x^2 - (\lambda + \mu)x - (k - \mu)$, with multiplicities f and g . The standard feasibility conditions [5, 8] are the integrality of f and g , the Krein conditions, the absolute bound, the claw bound, the conditions for $\mu \in \{1, 2\}$, and, for conference graphs, the condition that v be a sum of two squares; a few parameter sets have ad hoc proofs. None of these is a condition on integral quadratic forms. Brouwer's online table [7] lists every parameter set with $v \leq 1300$ that survives the known conditions, with status “+” (a graph exists), “!” (classified), “−” (known not to exist) or “?” (open).

The method and its origin. Lattices have accompanied strongly regular graphs since Cameron, Goethals, Seidel and Shult [9] showed that smallest eigenvalue -2 forces a root lattice; Venkov [26] connected spherical designs and lattices, and Koolen, Rehman and Yang [21] studied the integrability of srg lattices for $s \geq -3$. The arithmetic of the discriminant group appears in the language of p -ranks and Smith normal forms [4], [8, §9.4]: only primes dividing $r - s$ are interesting, which is the statement that the vertex lattice of a centred representation has discriminant group supported on the primes of $r - s$ and of the scaling.

The method used here is that of Bannai, Munemasa and Venkov [3], who proved the nonexistence of certain tight spherical 4-, 5- and 7-designs by studying the lattice generated by the design: the design identities bound its determinant, an even sublattice or overlattice is formed, and the Milgram–Braun formula excludes the remaining cases (for one dimension after gluing to a unimodular lattice and comparing theta series). Nebe and Venkov [25] refined it with Sylow-wise anisotropy and Gauss sums of the Sylow subgroups, excluding infinitely many dimensions. Tight 4-designs *are* strongly regular graphs, as [3] note; the case of dimension 46 there is $\text{srg}(1127, 486, 165, 243)$, which Brouwer's table marks “−” with a reference to Makhnev. Akiyama [2] recently used a maximal integral overlattice with anisotropic discriminant form, followed by gluing and theta series, to exclude $\text{srg}(266, 45, 0, 9)$. With related lattice methods we recently excluded the parameter sets $(69, 20, 7, 5)$ [14], $(99, 42, 21, 15)$ [15], $(105, 52, 21, 30)$ [16], $(154, 72, 26, 40)$ [17], $(162, 69, 36, 24)$ [18], $(288, 105, 52, 30)$ [19] and $(405, 132, 63, 33)$ [20]. Several of these preprints use the same chain — vertex lattice, maximal even overlattice, Milgram's formula, gluing, theta series — for individual parameter sets.

Contribution. What we have not found in the literature, and claim here subject to the caveats of Section 9, is:

- (1) the dual-vector identity and the class $\bar{y}$ for an arbitrary representation $G = b(A - \rho I) + cJ$ of an arbitrary strongly regular graph with integral eigenvalues (Lemmas 2.2–2.4), turning the first, genus-level step of the method into a uniform necessary condition on the parameters (Theorem 3.1, Proposition 3.4);
- (2) its simplest special case for the centred representation (Corollary 4.1), an application of the textbook congruence for even lattices of odd determinant [11, 12], and its consequence that no strongly regular graph has the parameters of $\text{NL}_m(q)$ with q a power of a prime $\equiv 1 \pmod{4}$ and m odd (Corollary 4.2), which contains $\text{srg}(289, 54, 1, 12) = \text{NL}_3(17)$;
- (3) a systematic sweep of Brouwer's table, with the resulting nonexistence of the open parameter sets of Table 1, each marked with its grade of evidence, and the controls of Section 7.

The condition uses no theta series, no gluing and no symmetry assumption. It is strictly weaker than the full method: it does not exclude $\text{srg}(1127, 486, 165, 243)$, for which [3] needed more (Section 8).

Organisation. Section 2 sets up the lattice, Section 3 proves the condition, Section 4 the parity case, Section 5 gives hand proofs for $(169, 42, 5, 12)$, $(96, 35, 10, 14)$ and $(176, 25, 0, 4)$ and short proofs for six further open sets and for two known cases, Section 6 lists every open parameter set found to fail, Section 7 describes implementations and controls, Section 8 the limits — in particular, the genus of the vertex lattice is *not* determined by the parameters (Remark 8.1) — and Section 9 the literature check.

Verification. Besides the derivation, a blind adversarial audit and an independent review were carried out (see the acknowledgements); each row of Table 1 is marked with the grade of evidence it has. The parity corollary (Corollary 4.1), the nonexistence of $\text{srg}(169, 42, 5, 12)$ (Section 5) and Corollary 4.2 are formalized in Lean 4 with Mathlib [27] (theorems `Parity.srg_parity_paper`, `Parity.no_srg_169_42_5_12` and `Parity.no_srg_NL`). The two nonexistence theorems start from Mathlib’s definition `SimpleGraph.IsSRGWith` with no further hypotheses (eigenvalues, Gram identity and lattice determinant are all derived), and use only the standard axioms (`propext`, `Classical.choice`, `Quot.sound`); no `sorry` or `native_decide`. In the formal proof the rank n is defined by $n\ell = v\nu$ (the trace of G); its identification with the multiplicity of θ is true but not formalized, and not needed for any conclusion.

Conventions. A lattice is a free $\mathbb{Z}$ -module $L \subset \mathbb{R}^d$ of full rank with the standard inner product $\langle \cdot, \cdot \rangle$; $x^2 = \langle x, x \rangle$; L is *integral* if $\langle L, L \rangle \subseteq \mathbb{Z}$ and *even* if moreover $x^2 \in 2\mathbb{Z}$ for all $x \in L$. $L^\# = \{x \in \mathbb{R}^d : \langle x, L \rangle \subseteq \mathbb{Z}\}$. For even L , the discriminant module $D(L) = L^\# / L$ carries the bilinear form $b(\bar{x}, \bar{y}) = \langle x, y \rangle \bmod \mathbb{Z}$ and the quadratic form $q(\bar{x}) = x^2 \bmod 2\mathbb{Z}$. The Gauss sum of a finite quadratic module (D, q) is $\gamma(D) = \sum_{x \in D} e^{\pi i q(x)}$; when $|\gamma(D)| = \sqrt{|D|}$ we write $\gamma(D) = \sqrt{|D|} e^{2\pi i \text{ph}(D)/8}$ and call $\text{ph}(D) \in \mathbb{Z}/8$ its *phase* (the Brown invariant). Representations are written (ρ, b, c) for $G = b(A - \rho I) + cJ$.

2. SETUP

2.1. Representations and the vertex lattice.

Definition 2.1. Let Γ be an $\text{srg}(v, k, \lambda, \mu)$ with integral restricted eigenvalues $r > s$, multiplicities f (of r) and g (of s). A *representation* is a triple (ρ, b, c) of integers with $\rho \in \{r, s\}$ and $b \neq 0$; let θ be the other restricted eigenvalue and n its multiplicity, and put

$$e = b(\theta - \rho), \quad T = b(k - \rho) + cv, \quad \nu = c - b\rho.$$

It is *admissible* if $e > 0$, $T \geq 0$ and ν is even. It is *centred* if $T = 0$, and *primitive* if $b = \pm 1$ and $T > 0$.

The matrix $G = b(A - \rho I) + cJ$ acts as T on $\mathbf{1}$, as e on the θ -eigenspace E_θ and as 0 on E_ρ . For an admissible representation it is therefore positive semidefinite of rank $d = n + 1$ if $T > 0$ and $d = n$ if $T = 0$; its diagonal entries equal ν and its off-diagonal entries are $c + b$ (adjacent) or c (non-adjacent). Write $G = V^\top V$ with V a real $d \times v$ matrix whose columns $v_1, \dots, v_v$ span $\mathbb{R}^d$, and let L be the lattice they generate (it is discrete: a basis of $\mathbb{R}^d$ among the v_i generates a full-rank integral sublattice L_0 , and all v_i lie in $L_0^\#$). L is even because ν is even. Put

$$\Sigma = \sum_i v_i = V\mathbf{1}, \quad \langle \Sigma, v_i \rangle = T, \quad \Sigma^2 = vT.$$

Every integral overlattice of L lies between L and $L^\#$, which contains L with finite index; hence maximal even overlattices exist. We fix one, M ; thus $L \subseteq M \subseteq M^\# \subseteq L^\#$.

2.2. The dual-vector identity. Put $\kappa = (T - e)/(vT)$ if $T > 0$ and $B = (k - \theta)/v$.

Lemma 2.2 (Dual-vector identity). *Let (ρ, b, c) be admissible, $x \in L^\#$, $z_i = \langle x, v_i \rangle$ and $\sigma = \sum_i z_i = \langle x, \Sigma \rangle$. Then $z \in \mathbb{Z}^v$ and*

- (1) *if $T > 0$: $VV^\top = eI + \kappa \Sigma \Sigma^\top$, hence $\sum_i z_i^2 = ex^2 + \kappa \sigma^2$ and $ex = \sum_i z_i v_i - \kappa \sigma \Sigma$;*
- (2) *if $T = 0$: $\Sigma = 0$, $VV^\top = eI$, hence $\sum_i z_i^2 = ex^2$, $\sigma = 0$ and $ex = \sum_i z_i v_i \in L$;*
- (3) *$Az = \theta z + B\sigma \mathbf{1}$; in particular $B\sigma \in \mathbb{Z}$;*
- (4) *$\sum_i z_i^2 \equiv \sigma \pmod{2}$.*

Proof. $z_i \in \mathbb{Z}$ since $v_i \in L$. The matrices $VV^\top$ and $V^\top V = G$ have the same nonzero eigenvalues. If $T > 0$, $VV^\top \Sigma = VG\mathbf{1} = T\Sigma$ and $VV^\top$ acts as e on $\Sigma^\perp$ (the image of E_θ), so $VV^\top = eI + (T - e)\Sigma\Sigma^\top/\Sigma^2$, which is (1) since $\Sigma^2 = vT$. Then $Vz = VV^\top x$ and $z^\top z = x^\top VV^\top x$ give the two identities. If $T = 0$, $\Sigma^2 = 0$, and G has the single nonzero eigenvalue e , so $VV^\top = eI$. For (3), $z = V^\top x$ lies in the column space of G , which is $\mathbb{R}\mathbf{1} \oplus E_\theta$ (or E_θ if $T = 0$); writing $z = (\sigma/v)\mathbf{1} + z_\theta$ gives $Az = (k\sigma/v)\mathbf{1} + \theta z_\theta = \theta z + B\sigma \mathbf{1}$, and $Az - \theta z$ is an integer vector. (4) holds because $z_i^2 \equiv z_i \pmod{2}$. $\square$

For tight designs the identities (1) and (4) specialise to the moment identities used in [3, (7)–(9)].

2.3. The class $\bar{y}$. Assume $T > 0$. Let H be the set of integers h with

$$(1) \quad Bh \in \mathbb{Z}, \quad \kappa h^2 \in \mathbb{Z}, \quad \kappa h^2 \equiv h \pmod{2}.$$

Lemma 2.3. *H is a subgroup of $\mathbb{Z}$.*

Proof. (Supplied by the Codex review [10].) Clearly $0 \in H$ and $H = -H$, since $\kappa h^2 - (-h) = (\kappa h^2 - h) + 2h$. Let $h_1, h_2 \in H$. The first condition is additive. If the denominator of κ divides h_1^2 and h_2^2 then, prime by prime, it divides $h_1 h_2$, so $\kappa h_1 h_2 \in \mathbb{Z}$ and $\kappa(h_1 - h_2)^2 \in \mathbb{Z}$. Finally

$$\kappa(h_1 - h_2)^2 - (h_1 - h_2) = (\kappa h_1^2 - h_1) + (\kappa h_2^2 - h_2) - 2\kappa h_1 h_2 + 2h_2 \in 2\mathbb{Z}. \quad \square$$

Let $\gamma > 0$ be the generator of H ; $H \neq 0$ because $T \in H$ (proof of Lemma 2.4(1) below, applied to $w = v_i$). Equivalently, $\gamma = \gcd\{h \in H : h \mid T\}$, which is how the implementations compute it. Put

$$y = \Sigma/\gamma, \quad Y = y^2 = vT/\gamma^2, \quad C = \kappa\gamma^2, \quad r_y = \min\{j \geq 1 : j^2 Y \in 2\mathbb{Z}\}.$$

Since $\gamma \in H$, C is an integer and $C \equiv \gamma \pmod{2}$.

Lemma 2.4. *Let (ρ, b, c) be admissible with $T > 0$.*

- (1) *$\{\langle w, \Sigma \rangle : w \in M\} = h_0 \mathbb{Z}$ with $h_0 \in H$ and $h_0 \mid T$; hence $\gamma \mid h_0$ and $y \in M^\#$.*
- (2) *The class $\bar{y} \in D(M)$ has order exactly r_y , and $q(\bar{y}) \equiv Y \pmod{2}$.*
- (3) *$r_y \mid C$; consequently $eD(M) \subseteq \langle \bar{y} \rangle$ and the exponent of $D(M)$ divides er_y .*

Proof. (1) Let $w \in M$ and $\sigma = \langle w, \Sigma \rangle$. Since $M \subseteq L^\#$, Lemma 2.2 applies: $B\sigma \in \mathbb{Z}$; and since w^2 is an even integer and $e \in \mathbb{Z}$, $\kappa \sigma^2 = \sum z_i^2 - ew^2 \in \mathbb{Z}$ and $\kappa \sigma^2 \equiv \sum z_i^2 \equiv \sigma \pmod{2}$. So every value of σ on M lies in H (in particular $T = \langle v_i, \Sigma \rangle \in H$), and so does the generator h_0 ; hence $\gamma \mid h_0$, and $h_0 \mid T$. Therefore $\langle w, y \rangle = \sigma/\gamma \in \mathbb{Z}$ for all $w \in M$.

(2) If $jy \in M$ then $j^2 Y = (jy)^2 \in 2\mathbb{Z}$. Conversely, if $j^2 Y \in 2\mathbb{Z}$, then $M + \mathbb{Z}jy$ is integral (since $y \in M^\#$ and $(jy)^2 \in \mathbb{Z}$) and even (since $(w + tjy)^2 = w^2 + 2tj\langle w, y \rangle + t^2 j^2 Y$), so $jy \in M$ by maximality.

(3) The proof that $r_y \mid C$ is the repair of a gap found by the Codex review [10]; the original derivation asserted (3) without it. First, $CY = \kappa vT = T - e \in \mathbb{Z}$. Let p be a prime and $\pi = v_p(Y)$. Unpacking the definition of r_y ($j^2 Y \in 2\mathbb{Z}$ means $2v_p(j) \geq -\pi$ for odd p and $2v_2(j) \geq 1 - \pi$), we have $v_p(r_y) = \max(0, \lceil -\pi/2 \rceil)$ for odd p and $v_2(r_y) = \max(0, \lceil (1 - \pi)/2 \rceil)$. If $\pi \geq 1$ ($p = 2$) or

$\pi \geq 0$ (p odd), $p \nmid r_y$. If p is odd and $\pi < 0$, then $v_p(r_y) = \lceil -\pi/2 \rceil \leq -\pi \leq v_p(C)$, the last step because $CY \in \mathbb{Z}$. If $p = 2$ and $\pi < 0$, then $v_2(r_y) = \lceil (1 - \pi)/2 \rceil \leq -\pi \leq v_2(C)$. There remains $p = 2$, $\pi = 0$, where $v_2(r_y) = 1$ and we must show that C is even. Suppose C is odd. Then γ is odd ($C \equiv \gamma$), and since $Y = vT/\gamma^2$ is a 2-adic unit, v and T are odd; from $T - e = CY$, e is even. But the trace of G is $v\nu = T + ne$, and $v\nu$ is even because ν is even, so $ne = v\nu - T$ is odd and e is odd, a contradiction. Hence $r_y \mid C$.

Now let $x \in M^\# \subseteq L^\#$ and $t = \langle x, y \rangle$. Since $r_y y \in M$, $r_y t \in \mathbb{Z}$. By Lemma 2.2(1), $ex = \sum_i z_i v_i - \kappa \gamma t \cdot \gamma y = \sum_i z_i v_i - (C/r_y)(r_y t)y$, with $\sum z_i v_i \in L \subseteq M$ and $(C/r_y)(r_y t) \in \mathbb{Z}$. Thus $e\bar{x} \in \langle \bar{y} \rangle$, and $er_y \bar{x} = 0$. $\square$

2.4. Finite quadratic modules. We use the following standard facts; see [24, 11, 23, 28].

- (F1) (*Anisotropy.*) If M is a maximal even overlattice of L then $D(M)$ is anisotropic: $q(\bar{x}) = 0$ only for $\bar{x} = 0$ (an isotropic class $\bar{x} \neq 0$ gives the even overlattice $M + \mathbb{Z}x$).
- (F2) (*Anisotropic modules.*) An anisotropic finite quadratic module is the orthogonal sum of its p -parts. For odd p the p -part is elementary abelian of rank at most 2. The 2-part has order at most 8 and is one of 0 , $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/2)^3$, $\mathbb{Z}/4$, $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ with an anisotropic quadratic form. (For $p = 2$ the implementations V1, V3, V4 of Section 7.1 enumerate these by brute force and obtain the same list; for odd p they use the classification, V3 additionally filtering for anisotropy.)
- (F3) (*Gauss sums.*) For odd p and $q(x) = 2ax^2/p$ on $\mathbb{Z}/p$, $\gamma = \left(\frac{a}{p}\right) \varepsilon_p \sqrt{p}$ with $\varepsilon_p = 1$ for $p \equiv 1 \pmod{4}$ and $\varepsilon_p = i$ for $p \equiv 3 \pmod{4}$. Hence the phase of any module of odd order p^j with $p \equiv 1 \pmod{4}$ lies in $\{0, 4\}$, and of a cyclic module of order $p \equiv 3 \pmod{4}$ in $\{2, 6\}$. On $\mathbb{Z}/2$, $q = \frac{1}{2}$ has phase 1 and $q = \frac{3}{2}$ phase 7. An anisotropic 2-elementary module on which q takes only integral values (*even type*) is 0 or the plane v with $q \equiv 1$ on its three nonzero elements, of phase 0 resp. 4.
- (F4) (*Milgram's formula.*) If M is an even positive definite lattice of rank d , then $\gamma(D(M)) = \sqrt{|D(M)|} e^{2\pi i d/8}$, i.e. $\text{ph}(D(M)) \equiv d \pmod{8}$ [22, Appendix 4].
- (F5) (*Even lattices of odd determinant.*) If L is even and $\det L$ is odd, then the rank n of L is even and $\det L \equiv (-1)^{n/2} \pmod{4}$.

Proof of (F5). Work over the 2-adic integers $\mathbb{Z}_2$, where $\det L$ is a unit. The Gram matrix modulo 2 is symmetric with zero diagonal and nonsingular, hence alternating and nondegenerate over $\mathbb{F}_2$; so n is even, and for a basis vector e_1 there is e_2 with $\langle e_1, e_2 \rangle$ odd. The plane $P = \mathbb{Z}_2 e_1 + \mathbb{Z}_2 e_2$ has Gram matrix $\begin{pmatrix} 2\alpha & \beta \\ \beta & 2\delta \end{pmatrix}$ with β odd and determinant $4\alpha\delta - \beta^2 \equiv -1 \pmod{4}$, a unit; so $L \otimes \mathbb{Z}_2 = P \perp P^\perp$ with $P^\perp$ again even and unimodular. By induction $L \otimes \mathbb{Z}_2$ is an orthogonal sum of $n/2$ planes of determinant $\equiv -1 \pmod{4}$, and $\det L$ differs from the product of their determinants by the square of a 2-adic unit, which is $\equiv 1 \pmod{8}$. $\square$

3. THE OBSTRUCTION

Theorem 3.1. *Let Γ be an $\text{srg}(v, k, \lambda, \mu)$ with integral restricted eigenvalues and let (ρ, b, c) be an admissible representation with $T > 0$, of rank $d = n + 1$. Then there is a finite quadratic module (D, q) , namely $D = D(M)$ for a maximal even overlattice M of the vertex lattice, and an element $\bar{y} \in D$ such that*

- (i) D is anisotropic;
- (ii) $\text{ph}(D) \equiv d \pmod{8}$;
- (iii) $\bar{y}$ has order r_y and $q(\bar{y}) \equiv Y \pmod{2}$;
- (iv) $eD \subseteq \langle \bar{y} \rangle$, in particular $\exp D \mid er_y$;

- (v) (C) for every $\bar{x} \in D$ there is a rational $t \equiv b(\bar{x}, \bar{y}) \pmod{1}$ such that $\gamma t \in \mathbb{Z}$, $B\gamma t \in \mathbb{Z}$, and, for any rational representative Q of $q(\bar{x})$,

$$eQ + Ct^2 \in \mathbb{Z} \quad \text{and} \quad eQ + Ct^2 \equiv \gamma t \pmod{2}.$$

Consequently, if for some such representation no pair $(D, \bar{y})$ satisfies (i)–(v), then no $\text{srg}(v, k, \lambda, \mu)$ exists.

The proof below does not use the size of b ; the implementation V1 and the hand proofs use only primitive representations ($b = \pm 1$), and rescaled ones enter only through Proposition 3.4.

Proof. (i) is (F1), (ii) is (F4), (iii) is Lemma 2.4(2) and (iv) is Lemma 2.4(3). For (v), choose a lift $x \in M^\#$ of $\bar{x}$ and put $t = \langle x, y \rangle$; then $t \equiv b(\bar{x}, \bar{y}) \pmod{1}$ and $\sigma = \langle x, \Sigma \rangle = \gamma t$ is an integer (Lemma 2.2), with $B\sigma \in \mathbb{Z}$ by Lemma 2.2(3). By Lemma 2.2(1),(4), $e x^2 + Ct^2 = e x^2 + \kappa \sigma^2 = \sum_i z_i^2$ is an integer congruent to $\sigma = \gamma t$ modulo 2. Finally $x^2 = Q + 2j$ for some integer j , and $e \cdot 2j \in 2\mathbb{Z}$, so x^2 may be replaced by any representative Q . $\square$

Remark 3.2. One representation with no admissible pair suffices: (i)–(v) are necessary conditions for each representation separately. Different representations of the same graph give different lattices, and a parameter set can pass the test in one representation and fail it in another; the implementations of Section 7.1 test several.

Remark 3.3. In (v) the choice of lift matters only through t modulo the values $\langle w, y \rangle$, $w \in M$, which are integers; the implementations test all $t \equiv b(\bar{x}, \bar{y}) \pmod{1}$ in a full period of the conditions, which is finite.

3.1. A weaker form. The following form drops the class $\bar{y}$ but applies to every admissible representation, including rescaled ($|b| > 1$) and centred ones; it is what the implementation V3 of Section 7.1 tests. For an admissible representation let $S \subseteq \mathbb{Q}/2\mathbb{Z}$ be the set of values

$$\frac{Z - \kappa \sigma^2}{e} \quad (T > 0), \quad \frac{Z}{e} \quad (T = 0, \sigma = 0),$$

over all integers Z, σ with $Z \equiv \sigma \pmod{2}$ and $B\sigma \in \mathbb{Z}$.

Proposition 3.4. *Let (ρ, b, c) be admissible of rank d . Then $D = D(M)$ is anisotropic, $\text{ph}(D) \equiv d \pmod{8}$, and $q(\bar{x}) \in S$ for every $\bar{x} \in D$.*

Proof. Anisotropy and the phase are (F1), (F4). For $x \in M^\# \subseteq L^\#$, Lemma 2.2 gives $x^2 = (\sum z_i^2 - \kappa \sigma^2)/e$ (resp. $\sum z_i^2/e$) with $Z = \sum z_i^2 \equiv \sigma$ and $B\sigma \in \mathbb{Z}$. $\square$

Lemma 3.5 (Odd representations). *Let (ρ, b, c) satisfy $e > 0$ and $T \geq 0$, but with ν odd; let L and d be as above, let $L_0 = \{x \in L : x^2 \in 2\mathbb{Z}\}$, and let M be a maximal even overlattice of L_0 .*

- (a) L_0 is an even sublattice of index 2 in L , of rank d , and $L_0 \subseteq M \subseteq M^\# \subseteq L_0^\#$.
- (b) For $x \in L_0^\#$ and $z_i = \langle x, v_i \rangle$, either $z \in \mathbb{Z}^\nu$ (and then $x \in L^\#$) or $z \in (\frac{1}{2} + \mathbb{Z})^\nu$.
- (c) In both cases the identities of Lemma 2.2(1)–(3) hold: $\sum_i z_i^2 = e x^2 + \kappa \sigma^2$ (resp. $= e x^2$ with $\sigma = 0$ if $T = 0$) and $Az = \theta z + B\sigma \mathbf{1}$. In the half-integral case $2\sigma \in \mathbb{Z}$, $2\sigma \equiv v \pmod{2}$, $2B\sigma \in \mathbb{Z}$ with $2B\sigma \equiv k - \theta \pmod{2}$, and $4 \sum_i z_i^2 \equiv v \pmod{8}$.
- (d) Consequently $D(M)$ is anisotropic, $\text{ph}(D(M)) \equiv d \pmod{8}$, and $q(\bar{x}) \in S \cup S_{1/2}$ for every $\bar{x}$, where S is as in Proposition 3.4 and $S_{1/2}$ is the set of values $(Z - \kappa \sigma^2)/e$ (resp. Z/e with $\sigma = 0$ if $T = 0$) over rationals Z, σ with $2\sigma \in \mathbb{Z}$, $2\sigma \equiv v \pmod{2}$, $2B\sigma \in \mathbb{Z}$, $2B\sigma \equiv k - \theta \pmod{2}$, $4Z \in \mathbb{Z}$, $4Z \equiv v \pmod{8}$.
- (e) If $T = 0$ then νv is even; so a centred odd representation has v even.

Proof. (a) Since $(x+y)^2 = x^2 + y^2 + 2\langle x, y \rangle$, the map $x \mapsto x^2 \bmod 2$ is a homomorphism $L \rightarrow \mathbb{Z}/2$; it is onto because $v_i^2 = \nu$ is odd, so its kernel L_0 has index 2. A subgroup of finite index in a lattice spans the same real space (here $2L \subseteq L_0 \subseteq L$), so L_0 is again a lattice of rank d . L_0 is even by definition, and every integral overlattice of L_0 lies in $L_0^\#$. (b) $v_i - v_j \in L_0$ (its norm is $2\nu - 2G_{ij}$) and $2v_i \in L_0$, so $z_i - z_j \in \mathbb{Z}$ and $2z_i \in \mathbb{Z}$. Hence all z_i lie in one coset of $\mathbb{Z}$ in $\frac{1}{2}\mathbb{Z}$: either every z_i is an integer or every z_i is in $\frac{1}{2} + \mathbb{Z}$; no mixed case occurs. If z is integral, x pairs integrally with the generators v_i of L , so $x \in L^\#$. (c) The proof of Lemma 2.2(1)–(3) uses only $x \in \mathbb{R}^d$ and $z = V^\top x$. In the half-integral case put $w_i = 2z_i$, all odd. Then $2\sigma = \sum_i w_i \equiv v$; $2(Az - \theta z)_i = \sum_{j \sim i} w_j - \theta w_i \equiv k - \theta \pmod{2}$ (a sum of k odd numbers), and this equals $2B\sigma$; and $4\sum_i z_i^2 = \sum_i w_i^2 \equiv v \pmod{8}$ since odd squares are $\equiv 1 \pmod{8}$. (d) Anisotropy and the phase are (F1), (F4) for M , of rank d . For $x \in M^\# \subseteq L_0^\#$ apply (b), (c) and solve the identity for x^2 ; the integral case gives S as in Proposition 3.4. (e) $\mathbf{1}^\top G \mathbf{1} = vT = 0$, and $\mathbf{1}^\top G \mathbf{1} = v\nu + 2\sum_{i < j} G_{ij}$. $\square$

The decider V3 uses, in the half-integral case, the weaker condition $v \mid 2\sigma(k - \theta)$ (i.e. $2B\sigma \in \mathbb{Z}$) without the congruence $2B\sigma \equiv k - \theta$; it therefore tests a superset of $S \cup S_{1/2}$, and its exclusions remain sound (the referee checked the code against this lemma). Every row of Table 1 is excluded by some implementation in an even representation; for (96, 35, 10, 14), (105, 40, 15, 15), (112, 36, 10, 12), (133, 32, 6, 8) and (456, 195, 74, 90) the exclusion by V3 uses an odd representation, so the second implementation in their grade rests on Lemma 3.5.

4. THE PARITY COROLLARY

Corollary 4.1. *Let Γ be an $\text{srg}(v, k, \lambda, \mu)$ with integral restricted eigenvalues, $\rho \in \{r, s\}$, θ the other one, n the multiplicity of θ . Put*

$$a = \frac{v}{\gcd(v, k - \rho)}, \quad b = a \operatorname{sgn}(\theta - \rho), \quad c = -\frac{b(k - \rho)}{v}, \quad \nu = c - b\rho, \quad \ell = a|\theta - \rho|.$$

If ν is even and ℓ is odd, then n is even and $(-1)^{n/2} \equiv \Delta \pmod{4}$ for some positive integer Δ all of whose prime factors divide ℓ . In particular, if every prime factor of ℓ is $\equiv 1 \pmod{4}$, then $n \equiv 0 \pmod{4}$.

Proof. (ρ, b, c) is admissible and centred: $e = \ell > 0$, $T = 0$, and ν is even. The vertex lattice L is even of rank n , and by Lemma 2.2(2) $\ell L^\# \subseteq L$, so $|L^\# / L| = \det L$ divides ℓ^n . Apply (F5) with $\Delta = \det L$; here $\Delta > 0$ because the Gram matrix of a basis of L is positive definite. (The sign matters: a negative determinant $-13^j \equiv -1 \pmod{4}$ would not contradict (F5) at $n = 126$.) The same argument applies verbatim to any centred representation, i.e. to any integer $b \neq 0$ with $b(\theta - \rho) > 0$, $c = -b(k - \rho)/v \in \mathbb{Z}$ and $\nu = c - b\rho$ even, with $\ell = b(\theta - \rho)$. $\square$

No overlattice, no Gauss sum and no computation beyond the parameters is needed. The congruence (F5) is textbook ([11, Ch. 15], [12, Prop. 1]); its unimodular special case (an even unimodular lattice has rank divisible by 8) is the step by which [3, Thm. 4.3] and [25, Thm. 3.3] exclude tight designs. The corollary is the case of Proposition 3.4 (centred, $T = 0$) in which the lattice L itself already has odd determinant; it is the case in which Milgram's formula is applied to L itself: when all primes of ℓ are $\equiv 1 \pmod{4}$ the Gauss sum of $D(L)$ is real and Milgram gives exactly $n \equiv 0 \pmod{4}$; for primes $\equiv 3 \pmod{4}$ Milgram contains more information than (F5), which we do not use here.

Corollary 4.2 (Negative-Latin-square parameters). *Let $q = p^e$ with $p \equiv 1 \pmod{4}$ prime, and let m be odd. There is no strongly regular graph with the parameters of $\text{NL}_m(q)$,*

$$(q^2, m(q+1), -q+m^2+3m, m(m+1)).$$

Proof. The restricted eigenvalues are $r = m$ and $s = m - q$ (their sum is $\lambda - \mu = 2m - q$, their product $-(k - \mu) = -m(q - m)$), and the multiplicity of s is $g = m(q + 1)$ (from $k + fr + gs = 0$ and $f + g = q^2 - 1$). Take the centred representation with $\rho = r = m$, $\theta = s = m - q$, $b = -q$, $c = m$, that is

$$G = q(mI - A) + mJ.$$

Then $T = -q(k - m) + mq^2 = 0$, $e = \ell = b(\theta - \rho) = q^2$, whose only prime is $p \equiv 1 \pmod{4}$, and $\nu = c - b\rho = m(q + 1)$ is even. By Corollary 4.1 (in the form for an arbitrary centred representation stated in its proof), $n = g = m(q + 1) \equiv 0 \pmod{4}$. But $q \equiv 1 \pmod{4}$ gives $q + 1 \equiv 2 \pmod{4}$, and m is odd, so $n \equiv 2 \pmod{4}$, a contradiction. No passage to the complement and no condition on $p \mid m$ or $p \mid m + 1$ is needed. $\square$

The range $1 \leq m \leq q - 2$ matters only for the parameter set to be feasible at all (for other m the parameters are not those of a graph); it plays no role in the proof. Bondarenko and Radchenko's (289, 54, 1, 12) is $\text{NL}_3(17)$, and 14 of the 21 pairs of Remark 4.3 are instances (all those with $v \in \{169, 289, 625, 841\}$; `out/milgram_paper/nl_check.txt`). The corollary is consistent with the known graphs: Paley(q^2) has the parameters of $\text{NL}_{(q-1)/2}(q)$ with $(q - 1)/2$ even, and Latin-square parameters $\text{LS}_m(q)$ have $n = m(q - 1) \equiv 0 \pmod{4}$.

Remark 4.3 (Which rows it covers). The hypotheses force a and $|\theta - \rho|$ to be odd, but not v : for $T(9) = \text{srg}(36, 14, 7, 4)$ and $\rho = -2$ one has $a = 9$, $G = 9(A + 2I) - 4J$, $\nu = 14$, $\ell = 63$, and (F5) holds ($\det L = 3^{14}$, rank 8, computed by the referee). All exclusions below have odd v . In Brouwer's table ($v \leq 1300$, snapshot of 8 October 2026; script `out/milgram_paper/parity_scan.py`, output `parity_scan.txt`) the corollary fails for exactly 20 open complementary pairs and for one pair marked “−”, and for no row marked “+” or “!” (1096 such rows with integral eigenvalues, counted per orientation). The killed open pairs are:

- $v = p^2$ with $r - s = p$, $p \in \{13, 17, 29\}$, $\ell = p^2$: (169, 42, 5, 12), (169, 70, 27, 30), (289, 90, 23, 30), (289, 126, 53, 56), and the five pairs $(841, k, \lambda, \mu)$ with $k \in \{150, 210, 270, 330, 390\}$;
- $\ell \in \{5^3, 5^4\}$: (375, 154, 53, 70), (375, 170, 85, 70), (875, 184, 33, 40), and $(625, k, \lambda, \mu)$ with $k \in \{130, 182, 234, 286\}$;
- $\ell = 5 \cdot 13$: (235, 52, 9, 12); $\ell = 13^2$: (507, 44, 1, 4); $\ell = 5^2 \cdot 17$: (391, 140, 39, 56); $\ell = 13 \cdot 29$: (871, 348, 137, 140).

The pair marked “−” is (289, 54, 1, 12), whose nonexistence is a theorem of Bondarenko and Radchenko [6]: with $\rho = s = -14$ one gets $a = 17$, $G = 17(A + 14I) - 4J$, $\nu = 234$, $\ell = 17^2$ and $n = f = 234 \equiv 2 \pmod{4}$. Bondarenko and Radchenko proved it by a different route (Euclidean representation, vertex-transitivity, an $\mathbb{F}_3$ -structure on the vertex set, and a Diophantine equation), so the three lines above are an independent second proof; they are also a consistency check of the corollary.

In the first family ($v = p^2$, $r - s = p$, $p \equiv 1 \pmod{4}$) the corollary demands $n \equiv 0 \pmod{4}$; Latin-square-type parameters $\text{LS}_m(p)$, with $n = m(p - 1)$, satisfy this automatically, which is consistent with their existence.

5. WORKED PROOFS

5.1. $\text{srg}(169, 42, 5, 12)$. The eigenvalues are $r = 3$, $s = -10$ (roots of $x^2 + 7x - 30$), with $f = 126$, $g = 42$. Take $\rho = s = -10$: $k - \rho = 52$, $\gcd(169, 52) = 13$, $a = 13$,

$$G = 13(A + 10I) - 4J.$$

On **1** it acts by $13 \cdot 52 - 4 \cdot 169 = 0$, on E_r by $13 \cdot 13 = 169$, on E_s by 0; its diagonal is $\nu = 130 - 4 = 126$, even. So the vertex lattice is even of rank $n = f = 126$ with $169L^\# \subseteq L$, hence

$\det L = 13^j$ for some j , and $13^j \equiv 1 \not\equiv (-1)^{63} \pmod{4}$, contradicting (F5). No $\text{srg}(169, 42, 5, 12)$ exists. (The other centred representation, $G = 13(3I - A) + 3J$ of rank $g = 42 \equiv 2 \pmod{4}$, gives the same contradiction. The original derivation used instead the representation $A' + 4I$ of the complement $\text{srg}(169, 126, 95, 90)$, of rank 43, with $\gamma = 130$, $Y = \frac{13}{10}$, $r_y = 10$, $C = 90$, where the phases of the 2-, 5- and 13-parts force $\text{ph}(D) \in \{1, 5\} \neq 3$; Codex's implementation confirms that representation independently.)

5.2. $\text{srg}(96, 35, 10, 14)$. This row carries the comment “pg(5, 6, 2)?” in Brouwer's table. Pass to the complement $\text{srg}(96, 60, 38, 36)$, with eigenvalues $r = 6$, $s = -4$ (roots of $x^2 - 2x - 24$), $f = 32$, $g = 63$, and take the primitive representation $G = A + 4I$, i.e. $(\rho, b, c) = (-4, 1, 0)$: $e = 10$, $T = 64$, $\nu = 4$, $d = 33$. Directly $G^2 = 10G + 36J$. Then

$$\kappa = \frac{54}{96 \cdot 64} = \frac{9}{1024}, \quad B = \frac{54}{96} = \frac{9}{16}.$$

For $h \in H$: $\frac{9}{16}h \in \mathbb{Z}$ forces $16 \mid h$; $\frac{9h^2}{1024} \in \mathbb{Z}$ forces $32 \mid h$; for $h = 32u$, $\frac{9h^2}{1024} = 9u^2 \equiv h \equiv 0 \pmod{2}$ forces u even. The only divisor of 64 in H is 64, so $\gamma = 64$,

$$y = \Sigma/64, \quad Y = \frac{96 \cdot 64}{64^2} = \frac{3}{2}, \quad r_y = 2, \quad C = \frac{9 \cdot 64^2}{1024} = 36.$$

Exponent. For $x \in M^\#$ with $t = \langle x, y \rangle \in \frac{1}{2}\mathbb{Z}$, Lemma 2.2 gives $10x = \sum z_i v_i - 36ty = \sum z_i v_i - 18(2t)y$, and $2y \in M$ (Lemma 2.4(2)), so $10D = 0$: $D = D_2 \oplus D_5$ with both parts elementary abelian, and $\bar{y} \in D_2$.

The 2-part. $b(\bar{y}, \bar{y}) = Y \bmod 1 = \frac{1}{2}$ is a unit, so $D_2 = \langle \bar{y} \rangle \perp \bar{y}^\perp$. For $\bar{x} \in \bar{y}^\perp \cap D_2$ we have $t \in \mathbb{Z}$, and (C) reads $10Q + 36t^2 \equiv 64t \equiv 0 \pmod{2}$, so $10Q \in 2\mathbb{Z}$, i.e. $Q \in \frac{1}{5}\mathbb{Z}$. Since $2\bar{x} = 0$, $Q \in \frac{1}{2}\mathbb{Z}$; hence $Q \in \mathbb{Z}$, so $\bar{y}^\perp \cap D_2$ is anisotropic of even type, of phase 0 or 4 by (F3). $\langle \bar{y} \rangle$ with $q = \frac{3}{2}$ has phase 7. So $\text{ph}(D_2) \in \{3, 7\}$.

The 5-part has phase 0 or 4 by (F3). Hence $\text{ph}(D) \in \{3, 7\}$, whereas $d = 33 \equiv 1 \pmod{8}$: (ii) fails. No $\text{srg}(96, 35, 10, 14)$ exists.

The same conclusion holds in every representation of either orientation that the implementation V1 tested (Table 1); the blind auditor's weaker decider admits 4 modules, among them $\langle \frac{1}{2} \rangle$, in this representation (it does not use $\bar{y}$; `out/milgram_paper/rerun_v3.tsv`, line “(96, 60, 38, 36) R a=1 c=0”) and excludes the row through the odd representation $A' + 4I + J = 3I - A + 2J$ instead (Lemma 3.5).

5.3. $\text{srg}(176, 25, 0, 4)$. Pass to the complement $\text{srg}(176, 150, 128, 126)$, eigenvalues $r = 6$, $s = -4$, $f = 55$, $g = 120$, and take $G = A + 4I$: $e = 10$, $T = 154$, $\nu = 4$, $d = 56$; $G^2 = 10G + 126J$. Here $\kappa = \frac{144}{176 \cdot 154} = \frac{9}{1694}$ with $1694 = 2 \cdot 7 \cdot 11^2$, and $B = \frac{144}{176} = \frac{9}{11}$. For $h \in H$: $11 \mid h$ and $1694 \mid 9h^2$, so $154 \mid h$; $h = 154$ satisfies the parity condition ($\kappa \cdot 154^2 = 126$). Thus

$$\gamma = 154, \quad Y = \frac{176}{154} = \frac{8}{7}, \quad r_y = 7, \quad C = 126.$$

By Lemma 2.4(3), $10D \subseteq \langle \bar{y} \rangle \cong \mathbb{Z}/7$ and $\exp D \mid 70$, so $D = D_2 \oplus D_5 \oplus D_7$. Multiplication by 10 is bijective on D_7 , so $D_7 = 10D_7 \subseteq \langle \bar{y} \rangle$, i.e. $D_7 = \langle \bar{y} \rangle$ with $q(\bar{y}) = \frac{8}{7} = 2 \cdot \frac{4}{7}$; as $\left(\frac{4}{7}\right) = 1$ and $7 \equiv 3 \pmod{4}$, its phase is 2. On D_2 (elementary, orthogonal to $\bar{y}$) $t \in \mathbb{Z}$ and (C) gives $10Q \equiv 154t \equiv 0 \pmod{2}$, so as before $Q \in \mathbb{Z}$ and $\text{ph}(D_2) \in \{0, 4\}$; $\text{ph}(D_5) \in \{0, 4\}$. Hence $\text{ph}(D) \in \{2, 6\}$, whereas $d = 56 \equiv 0$. No $\text{srg}(176, 25, 0, 4)$ exists.

5.4. Six further sets. The same computation, with the constants checked by three implementations (V1, V4, and V5 of Section 7.1; Table 1):

- $\text{srg}(616, 205, 90, 57)$, $r = 37$, $s = -4$, $G = A + 4I$: $e = 41$, $T = 209$, $d = 56$, $\kappa = \frac{3}{2299}$, $\gamma = 209$, $Y = \frac{56}{19}$, $r_y = 19$, $C = 57$. $\exp D \mid 41 \cdot 19$; 41 is invertible on D_{19} , so $D_{19} = \langle \bar{y} \rangle$ with $q(\bar{y}) \equiv \frac{18}{19} = 2 \cdot \frac{9}{19}$ of phase 2; $\text{ph}(D_{41}) \in \{0, 4\}$. So $\text{ph}(D) \in \{2, 6\} \neq 0$.
- $\text{srg}(300, 26, 4, 2)$, $r = 6$, $s = -4$, $G = A + 4I$: $e = 10$, $T = 30$, $d = 118$, $\kappa = \frac{1}{450}$, $B = \frac{1}{15}$, $\gamma = 30$, $Y = 10$, $r_y = 1$ ($y \in M$), $C = 2$. Then $10D = 0$; on D_2 , (C) gives $10Q + 2t^2 \equiv 30t \pmod{2}$ with $t \in \mathbb{Z}$, so $Q \in \mathbb{Z}$ and $\text{ph}(D_2) \in \{0, 4\}$; $\text{ph}(D_5) \in \{0, 4\}$; $d \equiv 6$.
- $\text{srg}(430, 39, 8, 3)$, $r = 9$, $s = -4$, $G = A + 4I$: $e = 13$, $T = 43$, $d = 130$, $\kappa = \frac{3}{1849}$, $\gamma = 43$, $Y = 10$, $r_y = 1$. Then $13D = 0$, $13 \equiv 1 \pmod{4}$, $\text{ph}(D) \in \{0, 4\}$, whereas $d \equiv 2$.
- $\text{srg}(690, 318, 121, 168)$, via the complement $\text{srg}(690, 371, 220, 175)$ with $r = 49$, $s = -4$, $G = A + 4I$: $e = 53$, $T = 375$, $d = 46$, $\kappa = \frac{7}{5625}$, $B = \frac{7}{15}$, $\gamma = 75$, $Y = 46$, $r_y = 1$. Then $53D = 0$, $53 \equiv 1 \pmod{4}$, $\text{ph}(D) \in \{0, 4\}$, whereas $d \equiv 6$.
- $\text{srg}(508, 247, 126, 114)$, $r = 19$, $s = -7$, $f = 127$, $G = A + 7I + J$, i.e. $(\rho, b, c) = (-7, 1, 1)$: $e = 26$, $T = 762$, $\nu = 8$, $d = 128$, $\kappa = \frac{92}{48387}$ ($48387 = 3 \cdot 127^2$), $B = \frac{57}{127}$. For $h \in H$: $381 \mid h$, and $h = 381$ fails the parity condition ($\kappa h^2 = 276$), $h = 762$ passes, so $\gamma = 762$, $Y = \frac{2}{3}$, $r_y = 3$, $C = 1104$. $\exp D \mid e r_y = 2 \cdot 3 \cdot 13$; 26 is invertible on the elementary group D_3 , so $D_3 = 26D_3 \subseteq \langle \bar{y} \rangle$, i.e. $D_3 = \langle \bar{y} \rangle$ with $q(\bar{y}) \equiv \frac{2}{3}$ of phase 2; $\text{ph}(D_{13}) \in \{0, 4\}$; on the elementary $D_2 \perp \bar{y}$, $t \in \mathbb{Z}$ and (C) gives $26Q + 1104t^2 \equiv 762t$, i.e. $26Q \in 2\mathbb{Z}$, so $Q \in \mathbb{Z}$ and $\text{ph}(D_2) \in \{0, 4\}$. So $\text{ph}(D) \in \{2, 6\}$, whereas $d \equiv 0$.
- $\text{srg}(1072, 391, 150, 138)$, $r = 23$, $s = -11$, $f = 335$, $G = A + 11I + J$, i.e. $(\rho, b, c) = (-11, 1, 1)$: $e = 34$, $T = 1474$, $\nu = 12$, $d = 336$, $\kappa = \frac{45}{49379}$ ($49379 = 11 \cdot 67^2$), $B = \frac{23}{67}$, $\gamma = 737$ ($737 \mid h$ for $h \in H$, and $\kappa \cdot 737^2 = 495$ is odd like 737), $Y = \frac{32}{11}$, $r_y = 11$, $C = 495$. $\exp D \mid 2 \cdot 11 \cdot 17$; as before $D_{11} = \langle \bar{y} \rangle$ with $q(\bar{y}) \equiv \frac{32}{11} \equiv 2 \cdot \frac{5}{11}$, $\left(\frac{5}{11}\right) = 1$, phase 2; $\text{ph}(D_{17}) \in \{0, 4\}$; on D_2 , $34Q + 495t^2 \equiv 737t$ with $t \in \mathbb{Z}$ gives $34Q \in 2\mathbb{Z}$, $Q \in \mathbb{Z}$, $\text{ph}(D_2) \in \{0, 4\}$. So $\text{ph}(D) \in \{2, 6\}$, whereas $d \equiv 0$.

The last two rows are not excluded by V3 (Section 7.1), which does not use $\bar{y}$: there the trivial module (M unimodular, possible since $d \equiv 0 \pmod{8}$) and several others pass Proposition 3.4; it is condition (iii) — an element of order $r_y = 3$ resp. 11 must exist — together with (iv), which pins the r_y -part to $\langle \bar{y} \rangle$, that excludes them. The programs V1, V4 and V5 also exclude the second primitive representation $(19, -1, 1)$ resp. $(23, -1, 1)$ (Table 1); that case is not written out here.

5.5. Two known cases. $\text{srg}(75, 32, 10, 16)$ (Azarija–Marc [1]: star complements and an exhaustive computer search). Here $r = 2$, $s = -8$, $f = 56$, $g = 18$. Take $(\rho, b, c) = (2, -1, 2)$, $G = 2I - A + 2J$: $e = 10$, $T = -30 + 150 = 120$, $\nu = 4$, $d = g + 1 = 19$, $\kappa = \frac{110}{75 \cdot 120} = \frac{11}{900}$, $B = \frac{40}{75} = \frac{8}{15}$. For $h \in H$: $15 \mid h$ and $900 \mid 11h^2$ give $30 \mid h$; $h = 30$ fails the parity condition ($\kappa \cdot 30^2 = 11$ odd, 30 even) and $h = 60$ satisfies it ($\kappa \cdot 60^2 = 44$). So $\gamma = 60$, $Y = \frac{75 \cdot 120}{3600} = \frac{5}{2}$, $r_y = 2$, $C = 44$. As for $(96, 35, 10, 14)$: $10x \equiv -22(2t)y \pmod{M}$ with $2y \in M$, so $10D = 0$; $\bar{y}$ splits off D_2 with $q(\bar{y}) = \frac{5}{2} \equiv \frac{1}{2}$, phase 1; on $\bar{y}^\perp \cap D_2$, (C) gives $10Q + 44t^2 \equiv 60t \equiv 0$, so $Q \in \mathbb{Z}$ and the phase is 0 or 4; D_5 has phase 0 or 4. Hence $\text{ph}(D) \in \{1, 5\}$, whereas $d = 19 \equiv 3 \pmod{8}$. This is a computer-free second proof. (V1 excludes the row in all six representations it tried, V3 in the centred representation $5(2I - A) + 2J$.)

$\text{srg}(289, 54, 1, 12)$ (Bondarenko–Radchenko [6]): Remark 4.3.

6. ALL ROWS EXCLUDED

Table 1 lists every open complementary pair of Brouwer’s table that failed the condition in at least one representation, by any implementation of Section 7.1. Data cut-off: 9 October 2026, 18:00 CEST; table snapshot of 8 October 2026 ($v \leq 1300$). The V1 scan is *complete*: all 1402 open complementary pairs with integral eigenvalues, both sides (2804 representations), with the 227 representations that timed out at 90 s re-run at 900 s (`out/milgram_final/coverage.txt`). It excludes 70 pairs; 1318 pairs admit a module on both sides, and 14 pairs admit one on one side while the other side is still undecided after 900 s (listed in Section 8). No other implementation excludes a pair that V1 does not.

Of the 70 pairs, 28 are proved by hand in this paper (Section 5 or Corollary 4.1 — the parity rows need only the arithmetic shown in the last column), and 42 rest on exhaustive finite computations by at least two independent programs, namely (105, 40, 15, 15), (112, 36, 10, 12), (120, 34, 8, 10), (120, 35, 10, 10), (133, 32, 6, 8), (209, 52, 15, 12), (352, 26, 0, 2), (391, 182, 93, 77), (441, 190, 89, 76), (456, 130, 24, 42), (456, 195, 74, 90), (494, 85, 12, 15), (507, 138, 49, 33), (507, 240, 106, 120), (508, 234, 100, 114), (560, 247, 102, 114), (630, 68, 1, 8), (664, 170, 24, 50), (690, 260, 94, 100), (715, 272, 102, 104), (721, 220, 69, 66), (732, 238, 92, 70), (741, 260, 91, 91), (750, 210, 55, 60), (750, 214, 63, 60), (760, 252, 82, 84), (780, 246, 76, 78), (780, 247, 78, 78), (780, 369, 158, 189), (836, 375, 190, 150), (875, 304, 78, 120), (931, 180, 29, 36), (1026, 451, 212, 187), (1140, 391, 126, 138), (1148, 481, 210, 195), (1183, 192, 26, 32), (1190, 492, 202, 204), (1190, 493, 204, 204), (1236, 195, 34, 30), (1248, 203, 22, 35), (1276, 204, 14, 36), (1288, 195, 26, 30). No pair rests on one program only. Every (row, representation) pair that V1 excludes (151 in all) was re-decided by V4 and by V5: 151 agreements, no disagreement, no timeout; so every C2 row is excluded by V1, V4 and V5 alike, and some also by V3. Codex’s own table run (review C35, `out/codex/c35/table_results.json`) ran V4 on both orientations of the 48 rows of the previous version of this table: 81 representations without admissible module, 15 timeouts, none admissible. Every pair in the table has status “?” in the snapshot of 8 October 2026; the snapshot of 9 October held in the repository covers only $v \leq 750$, and agrees there.

A “C” grade means: an exact enumeration of all finite quadratic modules allowed by (F2) on the relevant primes found none satisfying the conditions; there is no certificate beyond rerunning the enumeration. Timeouts are never counted as exclusions; where an implementation timed out on some representation of a row, the row is listed on the strength of the representations that were decided.

Table 1: Open complementary pairs of Brouwer's table ($v \leq 1300$) that fail the condition, in Brouwer's layout: the first line of a pair is (v, k, λ, μ) with $2k < v$ and its eigenvalues r^f , s^g , the second line the complement. Representations $(\rho, b, c)_d$ mean $G = b(A - \rho I) + cJ$ on the first orientation, of rank d (a representation of the complement is converted by $(\rho, b, c) \mapsto (-1 - \rho, -b, c + b)$). Columns V1–V5: representations without admissible module / representations decided (timeouts excluded); V2 is a cross-check only. P: Corollary 4.1 fails. Grade (gr.): H = hand proof in this paper (all H rows are also excluded by at least one program); C2 = excluded by at least two of the independent programs V1, V3, V4, V5; C1 = by one of them only. For H rows the first representation is the one used in the proof, followed by the programs that also exclude it.

v	k	λ	μ	r^f	s^g	V1	V2	V3	V4	V5	P	gr.	representations	reason
96	35	10	14	3^{63}	-7^{32}	6/6	4/4	1/9	4/4	4/4	–	H	proof $(3, -1, 1)_{33}$ [V1,V4,V5]	ph $\in \{3, 7\}$ vs $d = 33$ (§5)
	60	38	36	6^{32}	-4^{63}								V3 $(3, -1, 2)_{33}$	
105	40	15	15	5^{48}	-5^{56}	6/6	4/4	1/8	4/4	4/4	–	C2	V1 $(-5, 1, 1)_{49}$	no admissible module
	64	38	40	4^{56}	-6^{48}								V3 $(5, -1, 2)_{57}$	
112	36	10	12	4^{63}	-6^{48}	6/6	4/4	1/9	4/4	4/4	–	C2	V1 $(4, -1, 2)_{49}$	no admissible module
	75	50	50	5^{48}	-5^{63}								V3 $(4, -1, 1)_{49}$	
120	34	8	10	4^{68}	-6^{51}	6/6	2/2	2/10	2/2	2/2	–	C2	V1 $(4, -1, 2)_{52}$	no admissible module
	85	60	60	5^{51}	-5^{68}								V3 $(-6, 1, 0)_{69}$	
120	35	10	10	5^{56}	-5^{63}	6/6	2/2	2/9	2/2	2/2	–	C2	V1 $(-5, 1, 1)_{57}$	no admissible module
	84	58	60	4^{63}	-6^{56}								V3 $(-5, 1, 0)_{57}$	
133	32	6	8	4^{76}	-6^{56}	6/6	2/2	1/8	2/2	2/2	–	C2	V1 $(4, -1, 2)_{57}$	no admissible module
	100	75	75	5^{56}	-5^{76}								V3 $(-6, 1, 1)_{77}$	
169	42	5	12	3^{126}	-10^{42}	6/6	2/2	6/8	2/2	2/2	yes	H	proof $(-10, 13, -4)_{126}$ [V3,P]	parity, $\ell = 13^2$, $n = 126$ (§5)
	126	95	90	9^{42}	-4^{126}								V1 $(3, -1, 1)_{43}$	
169	70	27	30	5^{98}	-8^{70}	6/6	2/2	4/8	2/2	2/2	yes	H	proof $(5, -13, 5)_{70}$ [V3,P]	parity, $\ell = 169$, $n = 98$
	98	57	56	7^{70}	-6^{98}								V1 $(5, -1, 1)_{71}$	
176	25	0	4	3^{120}	-7^{55}	6/6	2/2	1/10	2/2	2/2	–	H	proof $(3, -1, 1)_{56}$ [V1,V4,V5]	ph $\in \{2, 6\}$ vs $d = 56$ (§5)
	150	128	126	6^{55}	-4^{120}								V3 $(-7, 1, 1)_{121}$	
209	52	15	12	8^{76}	-5^{132}	6/6	2/2	1/8	2/2	2/2	–	C2	V1 $(-5, 1, 1)_{77}$	no admissible module
	156	115	120	4^{132}	-9^{76}								V3 $(8, -1, 2)_{133}$	
235	52	9	12	5^{140}	-8^{94}	5/5	2/2	3/8	2/2	2/2	yes	H	proof $(5, -5, 1)_{94}$ [V3,P]	parity, $\ell = 65$, $n = 94$
	182	141	140	7^{94}	-6^{140}								V1 $(5, -1, 1)_{95}$	
289	90	23	30	5^{198}	-12^{90}	3/3	2/2	3/8	2/2	2/2	yes	H	proof $(5, -17, 5)_{90}$ [V3,P]	parity, $\ell = 289$, $n = 198$
	198	137	132	11^{90}	-6^{198}								V1 $(5, -1, 1)_{91}$	

v	k	λ	μ	r^f	s^g	V1	V2	V3	V4	V5	P	gr.	representations	reason
289	126 162	53 91	56 90	7^{162} 9^{126}	-10^{126} -8^{162}	2/2	2/2	7/8	2/2	2/2	yes	H	proof (7, -17, 7) ₁₂₆ [V3,P] V1 (7, -1, 1) ₁₂₇	parity, $\ell = 289$, $n = 162$
300	26 273	4 248	2 252	6^{117} 3^{182}	-4^{182} -7^{117}	2/2	1/1	2/10	2/2	2/2	-	H	proof (-4, 1, 0) ₁₁₈ [V1,V3,V4,V5]	ph $\in \{0, 4\}$ vs $d = 118$ (§5.4)
352	26 325	0 300	2 300	4^{208} 5^{143}	-6^{143} -5^{208}	2/2	1/1	1/10	2/2	2/2	-	C2	V1 (4, -1, 2) ₁₄₄ V3 (-6, 1, 0) ₂₀₉	no admissible module
375	154 220	53 135	70 120	4^{308} 20^{66}	-21^{66} -5^{308}	2/2	-	2/8	2/2	2/2	yes	H	proof (4, -5, 2) ₆₆ [V3,P] V1 (4, -1, 2) ₆₇	parity, $\ell = 125$, $n = 66$
375	170 204	85 103	70 120	20^{68} 4^{306}	-5^{306} -21^{68}	2/2	2/2	2/8	2/2	2/2	yes	H	proof (20, -5, 2) ₃₀₆ [V3,P] V1 (-5, 1, 1) ₆₉	parity, $\ell = 125$, $n = 306$
391	140 250	39 165	56 150	4^{322} 20^{68}	-21^{68} -5^{322}	2/2	-	1/8	2/2	2/2	yes	H	proof (-21, 17, -7) ₃₂₂ [V3,P] V1 (4, -1, 2) ₆₉	parity, $\ell = 425$, $n = 322$
391	182 208	93 102	77 120	21^{68} 4^{322}	-5^{322} -22^{68}	2/2	2/2	1/8	2/2	2/2	-	C2	V1 (-5, 1, 1) ₆₉ V3 (21, -17, 7) ₃₂₂	no admissible module
430	39 390	8 353	3 360	9^{129} 3^{300}	-4^{300} -10^{129}	2/2	2/2	5/9	2/2	2/2	-	H	proof (-4, 1, 0) ₁₃₀ [V1,V3,V4,V5]	ph $\in \{0, 4\}$ vs $d = 130$ (§5.4)
441	190 250	89 135	76 150	19^{98} 5^{342}	-6^{342} -20^{98}	2/2	2/2	2/8	2/2	2/2	-	C2	V1 (-6, 1, 0) ₉₉ V3 (-6, 1, 0) ₉₉	no admissible module
456	130 325	24 236	42 220	4^{380} 21^{75}	-22^{75} -5^{380}	1/1	-	2/10	1/1	1/1	-	C2	V1 (-22, 1, 0) ₃₈₁ V3 (-22, 1, 0) ₃₈₁	no admissible module
456	195 260	74 154	90 140	5^{360} 20^{95}	-21^{95} -6^{360}	2/2	2/2	1/9	2/2	2/2	-	C2	V1 (5, -1, 1) ₉₆ V3 (-21, 1, 0) ₃₆₁	no admissible module
494	85 408	12 337	15 336	7^{285} 9^{208}	-10^{208} -8^{285}	2/2	-	4/9	2/2	2/2	-	C2	V1 (7, -1, 1) ₂₀₉ V3 (7, -1, 1) ₂₀₉	no admissible module
507	44 462	1 421	4 420	5^{308} 7^{198}	-8^{198} -6^{308}	2/2	2/2	3/8	2/2	2/2	yes	H	proof (5, -13, 1) ₁₉₈ [V3,P] V1 (5, -1, 1) ₁₉₉	parity, $\ell = 169$, $n = 198$
507	138 368	49 262	33 280	21^{92} 4^{414}	-5^{414} -22^{92}	2/2	2/2	1/8	2/2	2/2	-	C2	V1 (-5, 1, 1) ₉₃ V3 (21, -13, 3) ₄₁₄	no admissible module
507	240 266	106 145	120 133	6^{380} 19^{126}	-20^{126} -7^{380}	2/2	2/2	3/8	2/2	2/2	-	C2	V1 (6, -1, 2) ₁₂₇ V3 (6, -13, 6) ₁₂₆	no admissible module

v	k	λ	μ	r^f	s^g	V1	V2	V3	V4	V5	P	gr.	representations	reason
508	234 273	100 152	114 140	6^{381} 19^{126}	-20^{126} -7^{381}	1/1	-	2/10	1/1	1/1	-	C2	V1 $(-20, 1, 0)_{382}$ V3 $(6, -1, 2)_{127}$	no admissible module
508	247 260	126 126	114 140	19^{127} 6^{380}	-7^{380} -20^{127}	2/2	2/2	0/9	2/2	2/2	-	H	proof $(-7, 1, 1)_{128}$ [V1,V4,V5]	ph $\in \{2, 6\}$ vs $d = 128$ (§5.4)
560	247 312	102 178	114 168	7^{399} 18^{160}	-19^{160} -8^{399}	2/2	1/1	1/9	2/2	2/2	-	C2	V1 $(7, -1, 1)_{161}$ V3 $(7, -1, 1)_{161}$	no admissible module
616	205 410	90 261	57 296	37^{55} 3^{560}	-4^{560} -38^{55}	2/2	-	2/9	2/2	2/2	-	H	proof $(-4, 1, 0)_{56}$ [V1,V4,V5] V3 $(37, -1, 1)_{561}$	ph $\in \{2, 6\}$ vs $d = 56$ (§5.4)
625	130 494	15 393	30 380	5^{494} 19^{130}	-20^{130} -6^{494}	2/2	2/2	4/8	2/2	2/2	yes	H	proof $(5, -5, 1)_{130}$ [V3,P] V1 $(5, -1, 1)_{131}$	parity, $\ell = 625$, $n = 494$
625	182 442	45 315	56 306	7^{442} 17^{182}	-18^{182} -8^{442}	2/2	-	6/8	2/2	2/2	yes	H	proof $(7, -25, 7)_{182}$ [V3,P] V1 $(7, -1, 1)_{183}$	parity, $\ell = 625$, $n = 442$
625	234 390	83 245	90 240	9^{390} 15^{234}	-16^{234} -10^{390}	2/2	2/2	6/8	2/2	2/2	yes	H	proof $(9, -25, 9)_{234}$ [V3,P] V1 $(9, -1, 1)_{235}$	parity, $\ell = 125$, $n = 390$
625	286 338	129 183	132 182	11^{338} 13^{286}	-14^{286} -12^{338}	2/2	2/2	3/8	2/2	2/2	yes	H	proof $(11, -25, 11)_{286}$ [V3,P] V1 $(11, -1, 1)_{287}$	parity, $\ell = 625$, $n = 338$
630	68 561	1 500	8 495	5^{440} 11^{189}	-12^{189} -6^{440}	2/2	2/2	5/9	2/2	2/2	-	C2	V1 $(5, -1, 1)_{190}$ V3 $(5, -10, 1)_{189}$	no admissible module
664	170 493	24 372	50 348	4^{580} 29^{83}	-30^{83} -5^{580}	2/2	2/2	1/10	2/2	2/2	-	C2	V1 $(4, -1, 2)_{84}$ V3 $(-30, 1, 0)_{581}$	no admissible module
690	260 429	94 268	100 264	10^{414} 15^{275}	-16^{275} -11^{414}	1/1	-	1/9	1/1	1/1	-	C2	V1 $(-16, 1, 0)_{415}$ V3 $(-16, 5, -2)_{414}$	no admissible module
690	318 371	121 220	168 175	3^{644} 49^{45}	-50^{45} -4^{644}	2/2	1/1	4/9	2/2	2/2	-	H	proof $(3, -1, 1)_{46}$ [V1,V3,V4,V5]	ph $\in \{0, 4\}$ vs $d = 46$ (§5.4)
715	272 442	102 273	104 273	12^{374} 13^{340}	-14^{340} -13^{374}	2/2	-	1/8	2/2	2/2	-	C2	V1 $(12, -1, 2)_{341}$ V3 $(-14, 5, -2)_{374}$	no admissible module
721	220 500	69 345	66 350	14^{308} 10^{412}	-11^{412} -15^{308}	2/2	-	2/8	2/2	2/2	-	C2	V1 $(-11, 1, 1)_{309}$ V3 $(-11, 1, 1)_{309}$	no admissible module
732	238 493	92 324	70 348	28^{122} 5^{609}	-6^{609} -29^{122}	2/2	-	1/10	2/2	2/2	-	C2	V1 $(-6, 1, 0)_{123}$ V3 $(-6, 1, 0)_{123}$	no admissible module

v	k	λ	μ	r^f	s^g	V1	V2	V3	V4	V5	P	gr.	representations	reason
741	260 480	91 310	91 312	13^{360} 12^{380}	-13^{380} -14^{360}	2/2	-	1/8	2/2	2/2	-	C2	V1 $(-13, 1, 1)_{361}$ V3 $(13, -1, 2)_{381}$	no admissible module
750	210 539	55 388	60 385	10^{441} 14^{308}	-15^{308} -11^{441}	2/2	-	3/9	2/2	2/2	-	C2	V1 $(10, -1, 2)_{309}$ V3 $(10, -1, 2)_{309}$	no admissible module
750	214 535	63 380	60 385	14^{321} 10^{428}	-11^{428} -15^{321}	2/2	2/2	3/9	2/2	2/2	-	C2	V1 $(-11, 1, 1)_{322}$ V3 $(-11, 10, -3)_{321}$	no admissible module
760	252 507	82 338	84 338	12^{399} 13^{360}	-14^{360} -13^{399}	2/2	-	-	2/2	2/2	-	C2	V1 $(12, -1, 2)_{361}$ V4 $(12, -1, 2)_{361}$	no admissible module
780	246 533	76 364	78 364	12^{410} 13^{369}	-14^{369} -13^{410}	2/2	2/2	2/10	2/2	2/2	-	C2	V1 $(12, -1, 2)_{370}$ V3 $(12, -1, 2)_{370}$	no admissible module
780	247 532	78 362	78 364	13^{380} 12^{399}	-13^{399} -14^{380}	2/2	-	-	2/2	2/2	-	C2	V1 $(-13, 1, 1)_{381}$ V4 $(-13, 1, 1)_{381}$	no admissible module
780	369 410	158 229	189 200	5^{675} 35^{104}	-36^{104} -6^{675}	2/2	1/1	2/9	2/2	2/2	-	C2	V1 $(5, -1, 1)_{105}$ V3 $(5, -1, 1)_{105}$	no admissible module
836	375 460	190 234	150 276	45^{76} 4^{759}	-5^{759} -46^{76}	2/2	2/2	1/9	2/2	2/2	-	C2	V1 $(-5, 1, 1)_{77}$ V3 $(-5, 1, 1)_{77}$	no admissible module
841	150 690	11 569	30 552	5^{690} 23^{150}	-24^{150} -6^{690}	2/2	-	3/8	2/2	2/2	yes	H	proof $(5, -29, 5)_{150}$ [V3,P] V1 $(5, -1, 1)_{151}$	parity, $\ell = 841$, $n = 690$
841	210 630	41 475	56 462	7^{630} 21^{210}	-22^{210} -8^{630}	2/2	-	5/8	2/2	2/2	yes	H	proof $(7, -29, 7)_{210}$ [V3,P] V1 $(7, -1, 1)_{211}$	parity, $\ell = 841$, $n = 630$
841	270 570	79 389	90 380	9^{570} 19^{270}	-20^{270} -10^{570}	2/2	-	6/8	2/2	2/2	yes	H	proof $(9, -29, 9)_{270}$ [V3,P] V1 $(9, -1, 1)_{271}$	parity, $\ell = 841$, $n = 570$
841	330 510	125 311	132 306	11^{510} 17^{330}	-18^{330} -12^{510}	2/2	-	5/8	2/2	2/2	yes	H	proof $(11, -29, 11)_{330}$ [V3,P] V1 $(11, -1, 1)_{331}$	parity, $\ell = 841$, $n = 510$
841	390 450	179 241	182 240	13^{450} 15^{390}	-16^{390} -14^{450}	2/2	-	4/8	2/2	2/2	yes	H	proof $(13, -29, 13)_{390}$ [V3,P] V1 $(13, -1, 1)_{391}$	parity, $\ell = 841$, $n = 450$
871	348 522	137 313	140 312	13^{468} 15^{402}	-16^{402} -14^{468}	2/2	-	3/8	2/2	2/2	yes	H	proof $(13, -13, 5)_{402}$ [V3,P] V1 $(13, -1, 1)_{403}$	parity, $\ell = 377$, $n = 402$
875	184 690	33 545	40 540	9^{552} 15^{322}	-16^{322} -10^{552}	2/2	2/2	5/8	2/2	2/2	yes	H	proof $(9, -5, 1)_{322}$ [V3,P] V1 $(9, -1, 1)_{323}$	parity, $\ell = 125$, $n = 322$

v	k	λ	μ	r^f	s^g	V1	V2	V3	V4	V5	P	gr.	representations	reason
875	304 570	78 385	120 345	4^{798} 45^{76}	-46^{76} -5^{798}	2/2	2/2	1/8	2/2	2/2	-	C2	V1 $(4, -1, 2)_{77}$ V3 $(-46, 5, -2)_{798}$	no admissible module
931	180 750	29 605	36 600	9^{588} 15^{342}	-16^{342} -10^{588}	2/2	-	-	2/2	2/2	-	C2	V1 $(9, -1, 1)_{343}$ V4 $(9, -1, 1)_{343}$	no admissible module
1026	451 574	212 309	187 336	33^{189} 7^{836}	-8^{836} -34^{189}	2/2	-	-	2/2	2/2	-	C2	V1 $(-8, 1, 0)_{190}$ V4 $(-8, 1, 0)_{190}$	no admissible module
1072	391 680	150 426	138 440	23^{335} 10^{736}	-11^{736} -24^{335}	2/2	-	0/9	2/2	2/2	-	H	proof $(-11, 1, 1)_{336}$ [V1,V4,V5]	$\text{ph} \in \{2, 6\}$ vs $d = 336$ (§5.4)
1140	391 748	126 494	138 484	11^{759} 22^{380}	-23^{380} -12^{759}	2/2	-	-	2/2	2/2	-	C2	V1 $(11, -1, 1)_{381}$ V4 $(11, -1, 1)_{381}$	no admissible module
1148	481 666	210 379	195 396	26^{328} 10^{819}	-11^{819} -27^{328}	1/1	-	-	1/1	1/1	-	C2	V1 $(-11, 1, 1)_{329}$ V4 $(-11, 1, 1)_{329}$	no admissible module
1183	192 990	26 829	32 825	10^{720} 15^{462}	-16^{462} -11^{720}	2/2	-	-	2/2	2/2	-	C2	V1 $(10, -1, 2)_{463}$ V4 $(10, -1, 2)_{463}$	no admissible module
1190	492 697	202 408	204 408	16^{615} 17^{574}	-18^{574} -17^{615}	2/2	-	-	2/2	2/2	-	C2	V1 $(16, -1, 2)_{575}$ V4 $(16, -1, 2)_{575}$	no admissible module
1190	493 696	204 406	204 408	17^{580} 16^{609}	-17^{609} -18^{580}	2/2	-	-	2/2	2/2	-	C2	V1 $(-17, 1, 1)_{581}$ V4 $(-17, 1, 1)_{581}$	no admissible module
1236	195 1040	34 874	30 880	15^{515} 10^{720}	-11^{720} -16^{515}	2/2	-	-	2/2	2/2	-	C2	V1 $(-11, 1, 1)_{516}$ V4 $(-11, 1, 1)_{516}$	no admissible module
1248	203 1044	22 875	35 864	8^{896} 20^{351}	-21^{351} -9^{896}	2/2	-	-	2/2	2/2	-	C2	V1 $(8, -1, 2)_{352}$ V4 $(8, -1, 2)_{352}$	no admissible module
1276	204 1071	14 902	36 882	6^{1044} 27^{231}	-28^{231} -7^{1044}	1/1	-	-	1/1	1/1	-	C2	V1 $(-28, 1, 0)_{1045}$ V4 $(-28, 1, 0)_{1045}$	no admissible module
1288	195 1092	26 926	30 924	11^{735} 14^{552}	-15^{552} -12^{735}	1/1	-	-	1/1	1/1	-	C2	V1 $(11, -1, 1)_{553}$ V4 $(11, -1, 1)_{553}$	no admissible module

7. IMPLEMENTATIONS AND CONTROLS

7.1. Implementations.

- V1 `scripts/s4lat_milgram.py` (derivation lane). Theorem 3.1 for primitive representations, with (iv) weakened to $\exp D \mid e r_y$; enumerates all anisotropic modules supported on the primes of $e r_y$, exact isometry classes, no order cap. Driven over the table by `scripts/s4lat_allopen.py` (one orientation per complementary pair, the smallest admissible c on each side, 90s per representation; a timeout is recorded and never counted as an exclusion).
- V2 `scripts/ideas_2026_09_27/lattice_route_survey.py`, an older classifier inherited from earlier work (order cap 20000, no test of (iv)); used only as a cross-check of V1's exclusions (`out/s4lat/kills_all_xcheck.txt`).
- V3 `scripts/audit_milgram_fable_decider.py`, written blind by the auditing session before it read the derivation. Proposition 3.4 on 8–10 standard representations per row (centred, primitive, $b = \pm 2$; odd ones via Lemma 3.5); it does not use $\bar{y}$.
- V4 `scripts/codex_audit_milgram.py` (Codex review), standard library only, imports nothing from V1–V3. Theorem 3.1 with the weaker, unconditionally valid exponent bound $e \cdot \text{den}(C) \cdot r_y^2$; run on the seven rows of Section 5 (one or two representations each) in the review, and for this version, unchanged, on every (row, representation) pair that V1 excludes (driver `scripts/milgram_final_v4.py`, output `out/milgram_final/v4_kills.tsv`, 1800s per representation).
- V5 `scripts/milgram_final_ycheck.py`, written for the final scan in a separate Claude session without reading or importing V1–V4. Theorem 3.1 for primitive representations, in two modes: with (iv) in full ($eD \subseteq \langle \bar{y} \rangle$, tested element by element) and with (iv) weakened to $\exp D \mid e r_y$ (exactly what V1 tests); p -parts as in (F2), the 2-part by brute force over all quadratic forms on the six groups, odd parts from the classification; Gauss sums evaluated numerically and required to be normalised. Run on every pair that V1 excludes in the weak mode (so a V5 exclusion is at least as strong as V1's), and on every representation of the existing-row control in the full mode (`out/milgram_final/ycheck_kills_exp.tsv`, `ycheck_known_full.tsv`).
- P `out/milgram_paper/parity_scan.py`: Corollary 4.1 on every row.

V1, V5 and P were written by Claude sessions (in separate sessions), V3 by a Fable session working blind, V4 by Codex (see the acknowledgements); V1 and V2 share an author lane, so V2 is not counted as independent. V5 is counted as independent of V1 (separate session, no shared code), but it shares the model family of V1; every C2 row of Table 1 is also confirmed by V3 or V4 unless stated otherwise.

7.2. Controls. All counts below were read from the cited logs on 9 October 2026 (cut-off 18:00 CEST); the V3 scan of the open rows is given as a snapshot at that cut-off.

- (1) *Real graphs, actual lattices.* The blind audit [13] built 254 strongly regular graphs (158 parameter sets including complements: triangular, lattice and Latin-square graphs, Paley graphs up to 169, generalized quadrangles, polar and affine polar graphs, Steiner-system block graphs, and sporadic graphs up to McLaughlin), computed in PARI/GP [29] the vertex lattice, a maximal even overlattice M and $D(M)$ for 8–10 standard representations each, and checked anisotropy, the Milgram phase and the norm set of Proposition 3.4: 1304 records, 0 violations (`out/audit_milgram_fable/compare_run2.txt`). The class $\bar{y}$ of Lemma 2.4 was checked on the actual M in 560 records with $T > 0$ and even L : $y \in M^\#$, $\gamma \mid \langle M, \Sigma \rangle$ and $\text{ord } \bar{y} = r_y$ in every record, 0 violations (`scripts/audit_milgram_fable_check_y.py`, rerun, output `out/milgram_paper/check_y_rerun.txt`;

also reproduced by the referee). Independently, the derivation lane checked 112 (graph, representation) pairs on 28 graphs against every condition of Theorem 3.1, including (iv) and (C) on every class: 0 violations (`out/s4lat/realcheck.log`).

- (2) *Every existing row.* V1 on all 557 complementary pairs of the table with status “+” or “!” and integral eigenvalues ($v \leq 1300$, both sides): 1114 representations, 1043 decided, 71 timeouts at 90s, 0 exclusions (`out/s4lat/allopen_known.tsv`). V3 on all 458 existing rows with $v \leq 400$ (each orientation counted): 4034 representations, 0 exclusions (`brouwer_scan_existing.tsv`). V4 on 112 existing rows: 222 representations admissible, 2 timeouts, 0 exclusions (`out/codex/c31/controls.json`). P on all 1096 existing rows (each orientation counted) with $v \leq 1300$: 0 exclusions. V5 (full mode, all of (i)–(v)) on the same 1114 representations as V1: 1114 admissible, 0 timeouts at 300s, 0 exclusions (`out/milgram_final/ycheck_known_full.tsv`).
- (3) *A recent construction.* $\text{srg}(85, 30, 11, 10)$ (Hancock, status “+” since 2026, smallest eigenvalue -4) and its complement: V1 finds exactly one admissible module in each of the 12 primitive representations tried, V3 accepts all 16 standard representations (`out/milgram_paper/controls_known.txt`, `out/audit_milgram_fable/control_85_30.txt`); the audit reports that V4 accepted the 4 representations it was run on (no log of that run is archived).
- (4) *Known nonexistence.* $\text{srg}(75, 32, 10, 16)$ fails in all 12 primitive representations V1 tried (both orientations) and in the centred representation for V3; $\text{srg}(289, 54, 1, 12)$ fails in 5 of 5 for V1, 3 of 8 for V3, and fails P. The known nonexistent rows $(49, 16, 3, 6)$, $(57, 14, 1, 4)$, $(95, 40, 12, 20)$, $(96, 38, 10, 18)$ and $(1127, 486, 165, 243)$ are *not* excluded: V3 accepts every standard representation of both orientations of the first four (`out/milgram_paper/control_unreproduced_v3.txt`); for the last see Section 9. The condition is not a complete test.

8. LIMITS

Remark 8.1 (The genus is not a parameter invariant). The test asks whether *some* finite quadratic module satisfies (i)–(v); it never assumes that the parameters determine $D(M)$. They do not. The generalized quadrangles $W(3)$ and $Q(4, 3)$ are both $\text{srg}(40, 12, 2, 4)$; in the same representation $G = A + 4I$ (rank 25) the blind audit computed $D(M) \cong \langle \frac{1}{2} \rangle$ ($\det M = 2$) for $W(3)$ and $D(M)$ of order 72, with invariants $\mathbb{Z}/6 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/2$, for $Q(4, 3)$ (`out/audit_milgram_fable/truth/W_3.jsonl`, `Q4_3.jsonl`). Pairs related by Seidel or Godsil–McKay switching (e.g. $T(8)$ and the Chang graphs) do not show this, because switching preserves the rational class of the eigenspace. Any argument that pins $|D(M)|$, the Hasse invariants or the genus from the parameters alone is therefore unsound; the implementations only test existence.

- *Representations.* Only finitely many representations are tested per row. A row that passes in every tested representation is not thereby shown to pass the condition; conversely a single failing representation suffices (Remark 3.2).
- *Scan coverage.* The V1 scan of the open rows is complete (Section 6), but 14 open pairs are undecided on one side after 900s (the other side admits a module): $(381, 114, 29, 36)$, $(676, 150, 23, 36)$, $(736, 350, 162, 170)$, $(775, 288, 98, 112)$, $(868, 408, 182, 200)$, $(931, 390, 173, 156)$, $(946, 243, 60, 63)$, $(981, 340, 115, 119)$, $(993, 310, 89, 100)$, $(993, 352, 131, 121)$, $(1036, 405, 138, 171)$, $(1055, 186, 17, 36)$, $(1156, 567, 258, 297)$, $(1208, 459, 150, 189)$. Only the smallest admissible c is tested on each side, and V3’s scan of open rows covers only $v \leq 750$ (it had reached $v = 639$ at the cut-off). Further exclusions in other representations are therefore possible.

- *Strength.* The test is genus-level. Where admissible modules exist (e.g. $\text{srg}(88, 27, 6, 9)$, $\text{srg}(99, 14, 1, 2)$, $\text{srg}(162, 21, 0, 3)$, each with a unique admissible module in its smallest representation) it says nothing; it also misses $\text{srg}(1127, 486, 165, 243)$, which the full method of [3] excludes (Section 9); and the theta-series refinements of [14, 16, 17] were found too weak at smallest eigenvalue -4 (derivation notes, `docs/s4lat_2026_10_08.md`, §§3–6).
- *Computer-assisted rows.* The C2 rows of Table 1 rest on exhaustive finite enumerations repeated by at least two independent programs; no row of the current table rests on one program only, but no certificate other than rerunning the enumeration exists.

9. NOVELTY

This section summarises a literature search made on 9 October 2026 (`docs/novelty_milgram_obstruction_2026_10_09.md`). It is a web and library search; it did not include MathSciNet citation searches forward from [3, 25] or the Russian-language literature (in particular the Makhnev school).

The method is not new. Taking the lattice spanned by a Euclidean representation of a two-distance set, bounding its discriminant group by the design identities, passing to an even sublattice or overlattice, requiring anisotropy and comparing Gauss sums via the Milgram–Braun formula, with 2-adic congruences for characteristic vectors, is the method of Bannai, Munemasa and Venkov [3] (who credit Martinet for one case) and of Nebe and Venkov [25], developed for tight spherical designs; tight 4-designs are strongly regular graphs. Akiyama [2] used the step “maximal integral overlattice $\Rightarrow$ anisotropic discriminant form, killed by an explicit exponent” for $\text{srg}(266, 45, 0, 9)$ (an unrefereed preprint). The finite-quadratic-form facts are classical [24, 11, 22]; the parity congruence (F5) is textbook [11, 12].

What appears to be new. (a) The dual-vector identity with the class $\bar{y}$ for an arbitrary integral representation $b(A - \rho I) + cJ$ (centred and non-centred, rescaled, and odd via the even sublattice) of an arbitrary strongly regular graph with integral eigenvalues, and the resulting uniform condition on the parameters; (b) the application of (F5) to the centred representation (Corollary 4.1) and the resulting exclusion of all $\text{NL}_m(q)$ parameter sets with q a power of a prime $p \equiv 1 \pmod{4}$ and m odd (Corollary 4.2); (c) the sweep of Brouwer’s table and the exclusions of Table 1. The feasibility conditions collected by Brouwer and van Lint [5] and by Brouwer and Van Maldeghem [8] contain no lattice, genus or Gauss-sum condition (the only quadratic-form condition is the sum-of-two-squares condition for conference graphs), Brouwer’s reference list contains none of [3, 25], and every open row of Table 1 carries a bare “?” (or a comment about an associated partial geometry) with no feasibility reason. Searches for $(176, 25, 0, 4)$, $(169, 42, 5, 12)$ and $\text{pg}(5, 6, 2)$ found no exclusion.

Reproductions. Both known cases of Section 5.5 were proved by non-lattice methods — $\text{srg}(75, 32, 10, 16)$ by star complements and computer search [1], $\text{srg}(289, 54, 1, 12)$ by a Euclidean representation, an $\mathbb{F}_3$ -structure and a Diophantine equation [6] — so the proofs here are independent second proofs by a different method, not restatements. Conversely, $\text{srg}(1127, 486, 165, 243)$, excluded in the table with a reference to Makhnev and, as the tight 4-design in dimension 46, also by [3], is *not* excluded by our condition: V1 finds 16 admissible modules in the representation $A + 4I$ of the complement $(1127, 640, 396, 320)$ (rank 47) and V3 accepts all 16 standard representations of both orientations (`out/milgram_paper/control_1127_v3.txt`); [3] needed gluing and theta series for it.

Further checks. A specialist in lattices and spherical designs may know whether the general dual-vector identity and the $\bar{y}$ refinement appear elsewhere; a citation-forward search on [3, 25] and a search of the Russian-language literature would settle this. The rows of Table 1 and the

two reproductions of Section 5.5 can be checked against Brouwer’s table and by rerunning the deciders from the data archive.

ACKNOWLEDGEMENTS AND USE OF AI SYSTEMS

This work was done with substantial assistance from AI systems. They are not authors; the author is responsible for all content. Claude (Anthropic), working in the author’s repository under the author’s direction, produced the derivation, the hand proofs of Section 5 (in their original form), the implementations V1 and the table scans, and drafted this text, including Corollary 4.1 in the general form stated here and its scan. A separate Claude session on a different model family (Fable) carried out a blind adversarial audit: it re-derived the obstruction from first principles before reading the derivation, wrote the decider V3, computed actual discriminant forms of 254 real graphs, and found the $W(3)/Q(4, 3)$ example of Remark 8.1. Codex (OpenAI’s coding agent) independently reviewed the uniform lemma and the two main hand proofs, found the missing step in Lemma 2.4(3) and supplied its repair, and wrote V4. A further Fable session made the literature check of Section 9. The author coordinated the sessions. No step rests on the judgement of an AI system: every claim is either proved in the text or is an exact finite computation that can be rerun.

DATA AVAILABILITY

The Lean 4 formalization of Section 4 is at <https://github.com/tonykoval/srg-parity-lean>. The scripts, scan outputs and notes cited by path are deposited with this preprint on Zenodo in the data archive `milgram_obstruction_data.zip` (repository-relative paths as cited; the archive’s README lists its contents and how to rerun the deciders). The table of Section 6 is regenerated by `out/milgram_paper/make_table.py` from the scan outputs (`out/s4lat/allopen_open_*.tsv`, `out/milgram_final/`) together with `compile_kills.py`, `rerun_rows.py`, `v3_parity_rows.py` and `parity_scan.py`; the controls are listed in Section 7.

REFERENCES

- [1] J. Azarija and T. Marc, There is no $(75, 32, 10, 16)$ strongly regular graph, *Linear Algebra Appl.* **557** (2018), 62–83; arXiv:1509.05933.
- [2] K. Akiyama, Nonexistence of a strongly regular graph with parameters $(266, 45, 0, 9)$: a certificate-free Lean proof, preprint, 2026, arXiv:2609.08319.
- [3] E. Bannai, A. Munemasa and B. Venkov, The nonexistence of certain tight spherical designs, *Algebra i Analiz* **16**:4 (2004), 1–23; English transl. *St. Petersburg Math. J.* **16** (2005), 609–625.
- [4] A. E. Brouwer and C. A. van Eijl, On the p -rank of the adjacency matrices of strongly regular graphs, *J. Algebraic Combin.* **1** (1992), 329–346.
- [5] A. E. Brouwer and J. H. van Lint, Strongly regular graphs and partial geometries, in: *Enumeration and Design*, Academic Press, Toronto, 1984, 85–122.
- [6] A. V. Bondarenko and D. V. Radchenko, On a family of strongly regular graphs with $\lambda = 1$, *J. Combin. Theory Ser. B* **103** (2013), 521–531.
- [7] A. E. Brouwer, *Parameters of strongly regular graphs*, online tables, <https://aeb.win.tue.nl/graphs/srg/srgtab.html> (snapshots of 8 and 9 October 2026).
- [8] A. E. Brouwer and H. Van Maldeghem, *Strongly Regular Graphs*, Encyclopedia of Mathematics and its Applications 182, Cambridge University Press, 2022.
- [9] P. J. Cameron, J.-M. Goethals, J. J. Seidel and E. E. Shult, Line graphs, root systems, and elliptic geometry, *J. Algebra* **43** (1976), 305–327.
- [10] Codex (OpenAI), *C31: independent audit of the claimed Milgram obstruction*, review note, 9 October 2026, `docs/codex_audit_milgram_obstruction_2026_10_09.md`.
- [11] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, 1999, Chapter 15.
- [12] N. D. Elkies, *Math 272y: Rational Lattices and their Theta Functions*, lecture notes, Harvard University, notes of 7 October 2019 (“Gluing”), Proposition 1.

- [13] Fable audit session, *Blind audit of the genus / Milgram obstruction*, audit note, 9 October 2026, [docs/audit_milgram_obstruction_fable_2026_10_09.md](#).
- [14] A. Kovaľ, There is no strongly regular graph with parameters $(69, 20, 7, 5)$, preprint, Zenodo, 2026, doi:10.5281/zenodo.23209705.
- [15] A. Kovaľ, There is no strongly regular graph with parameters $(99, 42, 21, 15)$, preprint, Zenodo, 2026, doi:10.5281/zenodo.23210773.
- [16] A. Kovaľ, There is no strongly regular graph with parameters $(105, 52, 21, 30)$, preprint, Zenodo, 2026, doi:10.5281/zenodo.23210908.
- [17] A. Kovaľ, There is no strongly regular graph with parameters $(154, 72, 26, 40)$, preprint, Zenodo, 2026, doi:10.5281/zenodo.23210954.
- [18] A. Kovaľ, There is no strongly regular graph with parameters $(162, 69, 36, 24)$, preprint, Zenodo, 2026, doi:10.5281/zenodo.23211005.
- [19] A. Kovaľ, There is no strongly regular graph with parameters $(288, 105, 52, 30)$, preprint, Zenodo, 2026, doi:10.5281/zenodo.23263506.
- [20] A. Kovaľ, There is no strongly regular graph with parameters $(405, 132, 63, 33)$, preprint, Zenodo, 2026, doi:10.5281/zenodo.23263684.
- [21] J. H. Koolen, M. Rehman and Q. Yang, On the integrability of strongly regular graphs, preprint, 2018, arXiv:1806.11325 (a published version may exist; to be checked).
- [22] J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*, Springer, 1973.
- [23] R. Miranda and D. R. Morrison, *Embeddings of Integral Quadratic Forms*, electronic book, 2009, <https://web.math.ucsb.edu/~drm/manuscripts/eiqf.pdf>.
- [24] V. V. Nikulin, Integral symmetric bilinear forms and some of their applications, *Math. USSR Izv.* **14** (1980), 103–167.
- [25] G. Nebe and B. Venkov, On tight spherical designs, *Algebra i Analiz* **24**:3 (2012); English transl. *St. Petersburg Math. J.* **24** (2013), 485–491; arXiv:1201.1830.
- [26] B. B. Venkov, Réseaux et designs sphériques, in: *Réseaux euclidiens, designs sphériques et formes modulaires*, Monogr. Enseign. Math. 37, Geneva, 2001, 10–86.
- [27] A. Kovaľ, *srg-parity-lean: Lean 4 formalisation of the parity criterion for strongly regular graphs*, <https://github.com/tonykoval/srg-parity-lean>, 2026 (Lean v4.35.0-rc4, Mathlib).
- [28] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren 270, Springer, 1985.
- [29] The PARI Group, *PARI/GP*, Univ. Bordeaux, <https://pari.math.u-bordeaux.fr/>.

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