

Stable algebraic families for $\mathbb{P}^2$

Abstract

Let S be a smooth connected complex algebraic curve and let $(f_s)_{s \in S}$ be an algebraic family of endomorphisms of $\mathbb{P}^2$ of degree $d \geq 2$. We prove that if the sum of the equilibrium Lyapunov exponents is harmonic on S , then the family is projectively isotrivial, a family of homogeneous lifts of flexible Lattès maps on $\mathbb{P}^1$, or a family of Lattès maps on $\mathbb{P}^2$. The homogeneous presentation is obtained after a finite base change and projective conjugacy on a nonempty algebraic open. In the two nonisotrivial cases, the Lyapunov sum is $\frac{3}{2} \log d$ and $\log d$, respectively. We also describe a quadratic Lattès family arising from a nonscalar endomorphism of a product of elliptic curves.

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1 Introduction

The stability theory of rational maps on $\mathbb{P}^1$ was developed by Mañé–Sad–Sullivan [1] and Lyubich [2, 3]. Their work relates the persistence of repelling periodic points to holomorphic motion of Julia sets. McMullen [4] proved that a stable algebraic family of rational maps is either isotrivial or a family of flexible Lattès maps. The deformation theory of rational maps was developed further by McMullen–Sullivan [5].

There is an analogous notion of stability for endomorphisms of higher-dimensional projective space. DeMarco [6, 7] introduced the bifurcation current in dimension one, and Bassanelli–Berteloot [8] extended this construction to projective space. The higher-dimensional theory relates the vanishing of this current to the motion of the equilibrium measure and the persistence of repelling Julia cycles; see Berteloot–Bianchi–Dupont [9], Bianchi [10], and Berteloot–Buff [11]. In this paper, we study algebraic families of endomorphisms of $\mathbb{P}^2$ that are stable over their entire parameter curve.

Let S be a smooth connected complex algebraic curve and let

$$f : S \times \mathbb{P}^2 \longrightarrow S \times \mathbb{P}^2, \quad (s, z) \longmapsto (s, f_s(z)),$$

be an algebraic family of endomorphisms of degree $d \geq 2$. Let ω be the Fubini–Study form normalized to have mass one. For each $s \in S$, we write

$$T_s = \lim_{n \rightarrow \infty} d^{-n} (f_s^n)^* \omega, \quad \mu_s = T_s^2, \quad J_s = \text{supp } \mu_s.$$

We denote the two Lyapunov exponents of f_s with respect to μ_s by $\chi_1(f_s)$ and $\chi_2(f_s)$, and set

$$L(f_s) = \chi_1(f_s) + \chi_2(f_s).$$

We say that f is *equilibrium-stable* if $dd^c L = 0$ on S . In particular, stability is imposed on the whole algebraic curve. This global hypothesis will allow us to deduce constancy of algebraic multipliers in Lemma 2.1. Throughout the paper, the Julia set means the small Julia set $J_s = \text{supp } \mu_s$.

Definition 1.1. We say that f is *projectively isotrivial* if, after a finite extension of $\mathbb{C}(S)$, it is conjugate by an element of PGL_3 to an endomorphism defined over $\mathbb{C}$. A *regular polynomial map* of $\mathbb{A}^2$ is a polynomial map that extends to an endomorphism of $\mathbb{P}^2$. Equivalently, its leading binary forms have no common projective zero. A *plane Lattès map* is an endomorphism induced by an expanding affine map of an abelian surface through a finite quotient onto $\mathbb{P}^2$.

Our main result gives the following classification.

Theorem 1.2. *An equilibrium-stable algebraic family as above is projectively isotrivial, or belongs to one of the following two classes:*

- (i) *After finite algebraic base change and projective conjugacy on a nonempty algebraic open, it has the form*

$$f_s[X : Y : Z] = [P_s(X, Y) : Q_s(X, Y) : Z^d], \tag{1.1}$$

where $[P_s : Q_s]$ is a nonisotrivial flexible Lattès family on $\mathbb{P}^1$.

- (ii) *Every f_s is a plane Lattès map. After finite algebraic base change and shrinking there is a simultaneous polarized abelian-surface presentation.*

The conclusions hold for the original map, and the two nonisotrivial classes have, respectively,

$$L(f_s) = \frac{3}{2} \log d \quad \text{and} \quad L(f_s) = \log d.$$

The two nonisotrivial cases are distinguished by their Lyapunov sums. The homogeneous family in (i) is governed by one-dimensional Lattès dynamics at infinity, whereas the family in (ii) comes from an abelian surface. Plane Lattès maps have been classified by Rong [12], using the complex crystallographic groups studied by Kaneko–Tokunaga–Yoshida [13]. Gauthier–Hutz–Kaschner [14] study symmetric products and families of Lattès maps. In Section 8, we write down a degree-two family belonging to Rong’s constructions. This shows that the plane Lattès alternative includes degrees that are not squares.

The stability considered here is stability along the given algebraic family. In the full parameter space, Dujardin [15] proves that the stable locus is not dense in higher dimensions. Biebler [16] proves bifurcation results near plane Lattès maps under the degree and iterate hypotheses of his theorem. Thus a family of Lattès maps may remain stable while lying near maps with different bifurcation behavior.

1.1 Proof strategy

We begin with a repelling periodic point in the small Julia set and study inverse branches that return near this point. This approach is closely related to the homoclinic-return argument of Ji–Xie [17], which gives a new proof of multiplier-spectrum rigidity in dimension one. Their argument uses repelling linearization, nearby periodic points obtained by long inverse prefixes, non-Archimedean degeneration, and monomial rescaling. In dimension two, one must also account for resonances and for the way the return maps act on tangent directions.

The first step is to recover information about the return maps from the multipliers of periodic points. We choose a repelling cycle whose unramified backward orbit has sufficiently large entropy. Long inverse prefixes place each finite collection of returns in a common contracting domain. We then apply the holomorphic trace formula of Bandtlow–Jenkinson [18] and vary the lengths of the prefixes independently. The resulting identities determine weighted characters of the return operators. These characters give simultaneous matrix normalizations on irreducible blocks. To obtain a change of coordinates, we must in addition show that these normalizations preserve the geometric data of the return system.

We carry this out in Section 2. The outcome is a local alternative: the returns preserve a holomorphic foliation, they are affine in suitable coordinates, or the full return system is locally constant. Constant multiplier spectra alone do not give this conclusion. The weighted-character calculation and the reconstruction of the return centers are used to recover identities between the return maps themselves. Hutz [19] studies higher-dimensional periodic multiplier invariants and constructs isospectral families, providing further context for this distinction.

The next step is to pass from these local alternatives to global geometry. In Section 3, we continue the local structures along inverse branches and across critical values. In the rigid case, this produces projective isotriviality. An invariant foliation leads to a pencil or to a regular polynomial iterate, while affine returns lead to a plane Lattès map. The foliation argument uses Favre–Pereira’s classification [20] and the results of Dabija–Jonsson on invariant pencils and algebraic webs [21, 22]. Related configurations occur in the classification of commuting endomorphisms by Dinh–Sibony [23] and Kaufmann [24]. Tapiero [25] constructs invariant foliations under a pluripotential product hypothesis. Here we obtain the required structure from the return identities.

We then study the regular polynomial and pencil cases in Sections 4 and 5. For a regular polynomial family, the formulas of Bedford–Jonsson [26] and Ingram [27] relate the Lyapunov sum to critical escape. Stability forces this escape to vanish at every place of the function field. When the map at infinity is constant, coefficient bounds imply that the family is isotrivial. When it is a flexible Lattès family, we use Tate uniformization and inverse annuli to construct curves with uniformly bounded poles. Riemann–Roch places their coefficients in one finite-dimensional

space, and geometric dynamical Northcott then produces periodic curves. Xie's algebraicity criterion [28] gives the homogeneous form in (i). The pencil case is reduced to one-dimensional stability by studying periodic fibers and their compositions.

Finally, we must choose the geometric structures algebraically as the parameter varies. Section 6 uses countable algebraic incidence conditions to make this choice after a finite base change and to remove the iterates introduced in the geometric classification. In Section 7, the minimum principle for the Lyapunov sum propagates the Lattès property, and we construct a simultaneous abelian-surface presentation on an algebraic open. Section 9 combines these steps to prove Theorem 1.2. The coordinate presentations and abelian schemes are constructed on that open; the intrinsic alternatives and the stated Lyapunov values hold on the original parameter curve.

1.2 Equilibrium measures and stability

We recall the sources of the equilibrium theory used below. Bedford–Taylor [29] developed the Monge–Ampère theory underlying the construction of μ_s . Its dynamical applications were developed by Hubbard–Papadopol [30] and Fornæss–Sibony [31, 32]. Briend–Duval proved the sharp lower bound for Lyapunov exponents and equidistribution of repelling periodic points [33], together with the maximal-entropy and backward-equidistribution results [34, 35]. We use the formulations collected in Dinh–Sibony [36].

For the stability arguments, we use persistence of repelling Julia cycles from Berteloot–Buff [11] and the compact equilibrium web of Berteloot–Bianchi–Dupont [9]. Pham [37] develops bifurcation theory for polynomial-like maps, and Bianchi–Rakhimov [38] study the motion of further invariant measures. Tapiero [39] relates inverse-branch normal forms and minimal Lyapunov exponents to absolute continuity with respect to mixed Green measures. The hypotheses needed from these results are recalled when they enter the argument.

1.3 Notation and conventions

Unless stated otherwise, a holomorphic foliation on a surface is a saturated rank-one subsheaf of the tangent sheaf. We allow isolated singularities and identify two foliations when their meromorphic tangent directions agree. An identity involving infinitely many return germs means an identity for every finite diagram; the neighborhood on which it holds may depend on the diagram.

For an endomorphism g of $\mathbb{P}^2$, we write E_g for the maximal proper algebraic totally invariant set. We use backward equidistribution outside E_g , complete invariance of J_g , continuous Green potentials, and local non-pluripolarity as in [36]. The curve part of E_g consists of at most three lines [40, Proposition 1.1]. Projective conjugacy, finite base change, and restriction to an algebraic open preserve stability. Passing to the n th iterate multiplies the Lyapunov sum by n .

2 The local return reduction

We first study the inverse returns near a repelling periodic point. Our goal is to show that these returns preserve a foliation, are affine, or are locally conjugate as the parameter varies. We initially work with germs at a generic parameter. The continuation of these conjugacies in the parameter, and their descent through the fibers of the global Poincaré map, will be treated separately.

In dimension one, Ji–Xie [17, Sections 2 and 6] use repelling linearization, homoclinic returns, and nearby periodic points in their proof of McMullen's theorem. They combine these

constructions with non-Archimedean degeneration and a monomial rescaling limit. In our setting, the main additional difficulty is to recover coordinate changes in two variables from the multiplier data. We do this by considering the action of the returns on spaces of polynomials of fixed multiplier weight.

2.1 Repelling periodic points and inverse returns

We will use the following results on stable families and repelling periodic points.

1. If the Lyapunov sum of a holomorphic family is pluriharmonic, its repelling periodic points belonging to the support of the equilibrium measure persist as repelling Julia points on the universal parameter cover. This is the weak-stability conclusion of [11, Proposition 1.2 and Theorem 4.1]. The higher-dimensional stability theory was developed in [9]; the bifurcation-current formulation originates in [8].
2. For an endomorphism of $\mathbb{P}^2$ of degree d , sufficiently high iterates have more than d^n disjoint contracting inverse branches on a suitable ball. Their full-shift repeller has an invariant ergodic measure of entropy greater than $n \log d$. Such a measure charges no proper algebraic subset, and its support, including its hyperbolic continuation in a nearby family, is contained in the small Julia set. These are [9, Proposition 3.10, Theorem 3.11, and the proof of Theorem 3.9].
3. There is a proper totally invariant algebraic exceptional set E outside which normalized backward point measures converge to the equilibrium measure. The equilibrium measure gives zero mass to every proper local analytic subset. We use [36, Theorem 1.4.1 and Theorem A.3.5].
4. The convergent Poincaré–Dulac theorem applies to a repelling holomorphic germ: its inverse is an attracting germ in the Poincaré domain. For normal forms of contractions and the finite-jet convergence argument, see [41, Appendix] and [42, Theorem 1.1 and the introductory discussion]. In dimension two, the normal forms and elementary coefficient calculations needed below are described explicitly in the next proof.

Lemma 2.1 (Constant multipliers and normal forms). *Let f be an algebraic family over a smooth connected complex algebraic curve, with pluriharmonic Lyapunov sum. A locally marked repelling Julia cycle has constant multipliers. At a generic parameter, after replacing the map by its return and changing local coordinates holomorphically in the parameter, its germ is a fixed normal form N of one of the following types:*

$$\begin{aligned} N(x, y) &= (\lambda_1 x, \lambda_2 y), \\ N(x, y) &= (\lambda x + y, \lambda y), \\ N(x, y) &= (\lambda x, \lambda^q y + cx^q), \quad q \geq 2, \end{aligned}$$

where every displayed multiplier has modulus greater than one, and c is constant. In the third form, c can be taken to be either zero or one. In the first form we may assume there is no nonlinear coordinate resonance unless it is included in the third form with $c = 0$.

Proof. We first show that the multipliers are constant. After a finite algebraic extension of the parameter function field, both the periodic point and an eigenvalue λ of its return derivative are defined. Persistence implies that $|\lambda| > 1$ at every nonsingular parameter on this marked component. On its smooth compactification, the meromorphic function $1/\lambda$ therefore satisfies $|1/\lambda| < 1$ away from finitely many points. It extends holomorphically across these points and is constant by compactness. Since it is nonzero, λ is constant as well.

We may now put the derivative in a constant diagonal or Jordan form on a neighborhood of a generic parameter. Order the multipliers so that $|\lambda_1| \leq |\lambda_2|$. There is no nonlinear resonance in the first coordinate. In the second coordinate, a resonant monomial involving y and at least one further factor has modulus larger than $|\lambda_2|$. Hence the only possible nonlinear resonance is $\lambda_2 = \lambda_1^q$ for an integer $q \geq 2$.

The convergent Poincaré–Dulac normalization consequently leaves at most one nonlinear term, of the form $c(t)x^q$ in the second coordinate. The coefficient $c(t)$ is holomorphic in the parameter. If it is nonzero, we restrict to a neighborhood on which it does not vanish and rescale y to make it equal to one. If it is identically zero, we retain the semisimple form. We normalize the Jordan coefficient in the same way on a stratum of fixed Jordan type. The homological operators are fixed because the multipliers are constant. Since the original germs vary holomorphically, the normalization and its convergence are locally uniform in the parameter. $\square$

We use the algebraicity of the family in the compactification argument above. For a family defined only on a small analytic disc, persistence need not make the multipliers constant. This distinction also appears in McMullen’s rigidity theorem for algebraic families of rational maps [4, 17].

Let F_t be a fixed iterate and let p_t be its marked repelling fixed point. We write

$$F_t \sigma_t = \sigma_t N, \quad \sigma_t(0) = p_t,$$

on a normal-form chart U , which we choose small enough that $N^{-1}U \Subset U$. We call a germ h_t at 0 an *actual inverse return of depth r* if its image lies in U and

$$F_t^r \sigma_t h_t = \sigma_t. \tag{2.1}$$

Such a return is locally biholomorphic. We consider all unramified returns of every depth, together with their analytic continuations within the chart. We compose returns only when the intermediate centers belong to the required phase domains. Whenever an identity involves the full collection, we mean that it holds as an identity of germs for every finite selection. The parameter and phase neighborhoods may depend on that selection.

Lemma 2.2 (Entrance and center uniqueness). *The marked repeller can be selected so that its unramified backward orbit*

$$\mathcal{A}_t = \{a : F_t^r(a) = p_t, \det D(F_t^r)_a \neq 0 \text{ for some } r \geq 0\}$$

is dense in $J_t = \text{supp } \mu_t$ for every nearby parameter. In every connected coordinate ball meeting J_t , the corresponding centers are a holomorphic uniqueness set, and remain one after deleting a proper local analytic subset. Their holomorphic motions are consistent with every admissible finite composition. Sufficiently long inverse- N prefixes of a fixed finite collection are simultaneous strict contractions on a common ball. Every finite word in these contractions is an actual inverse branch with a repelling Julia fixed point.

Proof. By the second result recalled above, we can choose an iterate $F_0 = f_0^q$ and disjoint contracting inverse branches $g_1, \dots, g_m$ on a ball such that $m > d^q$. Let p_0 be the fixed point of g_1 . The points obtained by applying finite words in the g_i to p_0 are unramified preimages of p_0 . They are dense in the full-shift set K_0 , whose Bernoulli measure has entropy $\log m > q \log d$. These properties persist on the hyperbolic continuation K_t . The same high-entropy result then gives $K_t \subset J_t$.

Fix a nearby parameter. We can choose

$$a \in K_t \setminus (E_{F_t} \cup \text{PostCrit}(F_t) \cup \mathcal{A}_t).$$

Indeed, the first two excluded sets are countable unions of proper algebraic sets, and $\mathcal{A}_t$ is countable. The Bernoulli measure gives them zero mass. There are points $a_j \in \mathcal{A}_t$ tending to a . Let V be a coordinate ball meeting J_t . Since $\mu_t(V) > 0$, backward equidistribution gives a preimage $b \in V$ of a under some F_t^r . Our choice of a makes this preimage unramified, so there is a local inverse u near a . For all sufficiently large j , the point $u(a_j)$ belongs to V and is an unramified backward center of p_t . Hence $\mathcal{A}_t$ is dense in J_t . Complete invariance gives the reverse inclusion $\mathcal{A}_t \subset J_t$.

We next prove the uniqueness assertion. A holomorphic function vanishing on the centers in a connected ball also vanishes on the part of J_t in that ball. If the function were nonzero, its analytic zero set would have positive equilibrium mass, contrary to local non-pluripolarity. The same argument applies after deleting a proper local analytic subset, since such a subset has zero equilibrium mass.

For a center a of unramified return depth r , the implicit function theorem gives a unique holomorphic germ solving $F_t^r(a(t)) = p_t$. If two depths are available, we enlarge them to a common depth. The additional derivatives at p_t are invertible, so uniqueness shows that the two motions agree. If h and k can be composed, their composition satisfies (2.1) with depth equal to the sum of the two depths. Thus

$$a_{h \circ k}(t) = h_t(a_k(t)). \quad (2.2)$$

After shrinking, all the maps in any fixed finite diagram are jointly defined.

It remains to prove the assertion about contracting returns. For each of our normal forms, both N^{-n} and its derivative tend to zero uniformly on compact sets. In the nonlinear case, if we write $\gamma = c\lambda^{-q}$, then

$$N^{-n}(x, y) = (\lambda^{-n}x, \lambda^{-qn}(y - n\gamma x^q)). \quad (2.3)$$

Given a finite list h_i , we can therefore choose the n_i independently sufficiently large that all $N^{-n_i}h_i$ strictly contract the same small ball into itself. Equation (2.1) shows that they are still actual inverse returns. If we iterate any word starting at 0, all successive images belong to J_t by complete invariance. The fixed point of the word lies in J_t because J_t is closed. It is repelling for the corresponding forward iterate. $\square$

We will repeatedly pass from identities at individual centers to joint identities. Suppose that two jointly holomorphic maps on a fixed product neighborhood agree as parameter germs at every fixed phase point of a holomorphic uniqueness set. Each parameter Taylor coefficient of their difference then vanishes on this set, and hence vanishes identically. The maps therefore agree jointly. In particular, the parameter neighborhoods used to mark the individual centers need not have a common radius.

2.2 Recovery of weighted characters

Let N_s be the semisimple part of N . We assign to a monomial $x^a y^b$ the multiplier weight $\chi = \lambda_1^a \lambda_2^b$. Let V_χ be the span of the monomials of weight χ , and let P_χ be the corresponding Taylor projection. Since $|\lambda_i| > 1$, a bound on the modulus of the weight bounds the ordinary degree. Thus each V_χ is finite dimensional. For the pullback $C_h \varphi = \varphi \circ h$, we consider the blocks

$$A_{\chi\psi}(h) = P_\chi C_h|_{V_\psi}.$$

The returns need not fix zero. Consequently these blocks contain information about the values $h(0)$ as well as the derivatives of h . We will recover their traces from the multipliers of the periodic points constructed in Lemma 2.2.

Lemma 2.3 (Weighted character extraction). *For a finite set of weights $\mathcal{F}$, put $W = \bigoplus_{\chi \in \mathcal{F}} V_\chi$ and $T_h = P_W C_h|_W$. There are fixed nilpotents L_χ such that*

$$C_{N^{-n}}|_{V_\chi} = \chi^{-n} \exp(nL_\chi).$$

The trace of every word in a finite collection of the T_h , the weight projectors, and the L_χ extended by zero is independent of the parameter.

Proof. We first describe the nilpotents. They vanish when N is diagonal. For the normal form in (2.3), the weighted degree is $m = a + qb$ and

$$L_m = -\gamma x^q \partial_y|_{V_m}.$$

Here V_m denotes V_{λ^m} . In the linear Jordan case, the unipotent part induces a constant nilpotent derivation on each space of homogeneous polynomials. This proves the formula for $C_{N^{-n}}$ in all cases.

Fix a finite collection of returns h_i . We choose polydiscs $D_r \in D_R$ such that every h_i is holomorphic past $\overline{D_r}$ and maps D_r into a compact subset of D_R . For all sufficiently large n , the map N^{-n} sends D_R into a compact subset of D_r . We work on the Hardy spaces of these two polydiscs. Cauchy estimates show that each $C_{h_i} : H^2(D_R) \rightarrow H^2(D_r)$ is bounded. The Taylor expansion of $C_{N^{-n}} : H^2(D_r) \rightarrow H^2(D_R)$ is a sum of rank-one operators with summable nuclear norms. Thus the products below have well-defined traces, and their finite Taylor truncations converge in trace norm.

We now apply the holomorphic fixed-point trace formula of [18, Theorem 4.2 and formula (11)]; see also Ruelle's work on holomorphic transfer operators [43]. Together with Lemma 2.2, it shows that

$$\mathrm{Tr}(C_{h_1} C_{N^{-n_1}} \cdots C_{h_k} C_{N^{-n_k}}) \quad (2.4)$$

is independent of the parameter whenever the integers $n_1, \dots, n_k$ are independently sufficiently large. Indeed, the trace equals $1/\det(I - Dg(p_g))$, where g is the corresponding contracting word and p_g is its fixed point. Its fixed-point spectrum is constant by Lemma 2.1. Although pullback reverses the order of composition, we may use arbitrary words in either order.

On expanding (2.4), we find that the coefficient of the frequency $\chi_1^{-n_1} \cdots \chi_k^{-n_k}$ is

$$\mathrm{tr}(A_{\chi_k \chi_1}(h_1) e^{n_1 L_{\chi_1}} A_{\chi_1 \chi_2}(h_2) e^{n_2 L_{\chi_2}} \cdots A_{\chi_{k-1} \chi_k}(h_k) e^{n_k L_{\chi_k}}). \quad (2.5)$$

This expression is polynomial in the n_i . There are only finitely many weights of modulus below any fixed bound. We can therefore recover the coefficients by separating the exponential frequencies in decreasing order of modulus. Within a group of equal modulus, finite-difference annihilators separate the distinct frequencies and their polynomial coefficients. Equivalently, we can use the Vandermonde equations for finitely many consecutive shifts. Applying this argument successively to the independent variables n_i recovers every coefficient of (2.5).

To justify this extraction from an infinite expansion, we bound the tails in trace norm. In the nonlinear resonant case, the sum of the nuclear norms of the terms of weight m is at most

$$C^m (1+n)^{m/q} |\lambda|^{-mn},$$

where we have increased C to absorb the polynomial number of monomials. Hence the tail beyond weight M is bounded by a constant times

$$(C(1+n)^{1/q} |\lambda|^{-n})^{M+1}.$$

Compared with the weight- M terms, this bound decays exponentially after multiplication by any fixed polynomial in n . In the Jordan case we use the same estimate with $(1+n)^m$ in place of $(1+n)^{m/q}$. In the diagonal case, the Cauchy bounds grow at most exponentially in the ordinary degree. We absorb this growth by fixing a sufficiently long inverse iterate, and then truncate according to the modulus of the weight. The remaining tail is exponentially smaller than the retained frequencies. These estimates also hold after composition with the other fixed bounded factors. They are uniform on a compact parameter neighborhood and justify the successive coefficient extractions.

The constant coefficients in these polynomials give the cyclic traces of arbitrary products of blocks. The higher coefficients give the same traces with arbitrary powers of the nilpotents inserted. Expanding a word using the weight projectors expresses its trace in terms of these quantities, so its trace is constant. This argument uses products of the compressed substitutions throughout; it does not identify such a product with the compression of a composite. $\square$

Lemma 2.4 (Irreducible character normalization). *Suppose a family of finite-dimensional matrices has constant traces of all words and is absolutely irreducible over the meromorphic parameter-germ field. After shrinking at a regular parameter and allowing a local finite parameter cover, one holomorphic gauge makes the whole collection constant. One finite irreducible subcollection determines a gauge that works for every further matrix in the collection. Already constant auxiliary matrices can remain fixed in this normalization.*

Proof. By Burnside's theorem, we can choose matrix words $W_1, \dots, W_{m^2}$ forming a basis of the full m -by- m matrix algebra. Their trace Gram matrix $\text{tr}(W_i W_j)$ is invertible and constant. Pairing products of basis elements with the same basis shows that the structure constants are constant. The same argument shows that the coordinates of every original matrix are constant. Hence the word algebra is obtained by scalar extension from a matrix algebra over $\mathbb{C}$.

We can construct the conjugating gauge explicitly. Choose a rank-one idempotent in the constant algebra and a holomorphic nonzero vector in its image. Applying the constant matrix units gives a local holomorphic basis wherever its determinant is nonzero. We normalize this basis at the reference parameter. If an auxiliary matrix is already constant, the gauge conjugates it to its reference value and therefore commutes with it.

The same finite word basis suffices for an infinite collection. Indeed, the coordinates of any additional matrix are determined by its traces against the basis, and these are constant as germs. We will also need to choose invariant subspaces in the reducible case. Their invariance is expressed by polynomial equations on a Grassmannian. By Noetherianity, finitely many equations over the meromorphic germ field impose all the required conditions. A solution over the algebraic closure is defined over a finite extension. On a generic local cover, it gives a holomorphic invariant subspace. $\square$

The lemma is an irreducible local version of the trace-character principle for simultaneous matrix invariants; see [44, Theorem 1.3]. In the reducible case, traces generally determine only the semisimplification. We will therefore treat reducible collections by studying their invariant subspaces and the foliations they define.

2.3 Derivative observables and reconstruction

Lemma 2.5 (Propagation of derivative constraints). *If every derivative $Dh(0)$ in the full actual return collection preserves a line $\ell \subset \mathbb{C}^2$, every return preserves the parallel foliation tangent to ℓ on its compatible domain. More generally, let P be a holomorphic submersion with $dP_0(e_x) = 0$. If*

$$d(P \circ h)_0(e_x) = 0 \quad \text{for all } h,$$

then every return preserves the foliation $P = \text{constant}$.

Proof. For the first assertion, we apply the chain rule to obtain

$$Dh(a_k) = D(h \circ k)(0)Dk(0)^{-1}.$$

Both factors on the right preserve ℓ , so $Dh(a_k)$ preserves ℓ at every compatible center. The centers form a uniqueness set, and hence the matrix equations expressing this preservation hold throughout the phase domain.

For the second assertion, the assumption and invertibility imply that $Dk(0)e_x$ spans $\ker dP_{a_k}$. Applying the assumption to $h \circ k$, we find that $d(P \circ h) \wedge dP$ vanishes at each compatible center. It vanishes identically by uniqueness, which proves that h preserves the foliation. Exchanging the two coordinates gives the version with e_y and $P = x$. $\square$

Lemma 2.6 (Transport of a center motion). *Suppose a jointly holomorphic local biholomorphism H_t^q near a center $q = k_0(0)$ follows every sufficiently nearby center, with $H_0^q = \text{id}$. There is a jointly holomorphic family of entire automorphisms Θ_t of the normal-form source, fixing zero, such that*

$$\Theta_0 = \text{id}, \quad \Theta_t N = N \Theta_t, \quad \Theta_t h_0 = h_t \Theta_t$$

as germs for every actual return h . These automorphisms are polynomial for the plane normal forms above.

Proof. We first transport the motion from q to zero. Define $H_t^0 = k_t^{-1} H_t^q k_0$ on a small fixed product neighborhood of zero. The compatibility of the center motions with composition shows that H_t^0 follows all nearby centers and fixes zero. Moreover, applying N^{-1} to the motion of a center gives the motion of its image under N^{-1} . Uniqueness and the parameter Taylor-coefficient argument above therefore give $H_t^0 N^{-1} = N^{-1} H_t^0$ as germs.

We next extend this germ to the whole source. Since every point of $\mathbb{C}^2$ is attracted to zero by N^{-1} , we can define

$$\Theta_t(z) = N^L H_t^0 (N^{-L} z)$$

for all sufficiently large L . The commutation relation makes this expression independent of L . A single L works on each fixed compact set, so the extension is jointly holomorphic. Applying the same construction to the local inverse gives an entire inverse.

Let h be an actual return. Choose L large enough that $g_t = N^{-L} h_t$ maps a neighborhood of zero into the domain where H_t^0 was constructed. The map H_t^0 follows both the input centers and their images. Thus the compatibility of their motions gives $\Theta_t g_0 = g_t \Theta_t$ on a uniqueness set, and hence as germs. Since Θ_t commutes with N , we can cancel N^{-L} to obtain the required identity for h .

It remains to show that the extension is polynomial. We use the equation $\Theta N = N \Theta$. If there is no nonlinear coordinate resonance, all nonlinear Taylor coefficients vanish: in every degree at least two, the homogeneous substitution operators have no coordinate eigenvalue. This includes the Jordan case. In the semisimple resonant case, the weight equations give the form $(ax, by + ex^q)$. In the nonlinear resonant case they give the same form, and comparison of the x^q coefficients in the commutation equation imposes $b = a^q$. No other monomial is allowed by the weight equations. Finally, invertibility implies $ab \neq 0$. $\square$

Lemma 2.7 (Derivative-observable alternatives). *Suppose a coordinate gauge commuting with N makes every $Dh_t(0)$ constant. Then the full collection is affine, preserves a regular local foliation, or satisfies the conclusion of Lemma 2.6.*

Proof. For every compatible finite diagram, the chain rule gives

$$Dh_t(a_k(t)) = D(h_t k_t)(0) Dk_t(0)^{-1}. \quad (2.6)$$

Thus the derivative entries take constant values along each moving center in their domains. We use these entries as functions with which to recover the motion of the centers. Let $\mathcal{O}$ be the field of meromorphic phase germs they generate. Its maximal differential rank is 0, 1, or 2, and is attained by a finite collection. We work at a generic parameter where the ranks of these finite collections are fixed.

Suppose first that this rank is zero. Every derivative entry is then constant in the phase variables, so $h_t(z) = B_h z + a_h(t)$ is affine.

Suppose next that the rank is two. We choose two derivative entries defining a map Φ_t with nonzero Jacobian. By center uniqueness, there is a backward center q in its regular locus. The inverse function theorem defines $H_t^q = \Phi_t^{-1} \Phi_0$ near q . Equation (2.6) and uniqueness of the inverse branch show that this map follows every nearby center. We can therefore apply Lemma 2.6.

We now consider rank one. Choose a nonconstant derivative entry φ , and divide the two coefficients of $d\varphi$ by their common divisor. The resulting saturated holomorphic one-form ω has isolated zeros and defines a meromorphic line field. If $g = N^{-n}k$ and n is sufficiently large, the chain rule gives

$$(Dh) \circ g = D(h \circ g)(Dg)^{-1}.$$

It follows that $\varphi \circ g \in \mathcal{O}$, and hence that $d\varphi \wedge d(\varphi \circ g) = 0$. Since g is locally biholomorphic, this identity implies that it preserves the saturated foliation and its singular set. The same argument applies to every sufficiently large inverse power of N . Taking two consecutive powers, we obtain invariance under N itself.

We claim that the foliation is regular at zero. Center uniqueness gives a return k with $k(0) \neq 0$. If zero were singular, its images $N^{-n}k(0)$ would be singular because the maps $N^{-n}k$ are locally biholomorphic and preserve the foliation. These would be infinitely many nonzero singular points tending to zero, contrary to the isolatedness of the singularities.

We determine this regular N -invariant foliation explicitly. If N is diagonal and nonresonant and the tangent at zero is horizontal, its slope s satisfies $s(Nz) = (\lambda_2/\lambda_1)s(z)$. Any nonzero coefficient of positive degree would give a nonlinear coordinate resonance. Hence $s = 0$. The vertical case is identical. If N is scalar, we first choose the tangent axis; the equation $s(\lambda z) = s(z)$ again implies $s = 0$. For a Jordan form, the only tangent eigenline is horizontal, and the slope satisfies

$$s(Nz) = \frac{\lambda s(z)}{\lambda + s(z)}.$$

If the slope were nonzero, its lowest positive homogeneous term would be fixed by substitution by N . This is impossible because the degree- m substitution operator has eigenvalue $\lambda^m \neq 1$.

If $N = (\lambda x, \lambda^q y)$, a foliation with vertical tangent is vertical everywhere. A foliation with horizontal tangent has slope qax^{q-1} and first integral $y - ax^q$. The shear $(x, y) \mapsto (x, y - ax^q)$ commutes with N and makes this foliation parallel. Finally, suppose that $N = (\lambda x, \lambda^q y + cx^q)$ with $c \neq 0$. A vertical tangent again gives the vertical foliation. A horizontal slope would satisfy

$$s(Nz) - \lambda^{q-1}s(z) = \frac{qc}{\lambda}x^{q-1}.$$

The operator on the left vanishes on the weight- $(q-1)$ space, whereas the right side is nonzero. Thus this last case cannot have a horizontal tangent.

Each return $N^{-n}h$ with sufficiently long prefix preserves the foliation just described on a small neighborhood. This foliation extends to the normal-form source by the displayed first

integral, and its extension is preserved by N . We can therefore cancel the prefix and conclude that each h preserves it on its compatible domain. $\square$

2.4 The semisimple resonant case

We now suppose that $N = (\lambda x, \lambda^q y)$. Let

$$V_1 = \mathbb{C}x, \quad V_q = \text{span}\{x^q, y\}, \quad W = V_1 \oplus V_q, \quad T_h = P_W C_h|_W.$$

The subscripts $1, q$ denote weighted degrees, corresponding to the multiplier weights λ, λ^q . In this subsection, the matrix collection always includes both weight projectors. We first treat the case in which this collection has an invariant subspace.

Lemma 2.8 (Reducible semisimple blocks). *If this collection is reducible, then after a resonant coordinate change it either preserves a regular local foliation or has constant character on the genuine coordinate-function plane. In the latter case it has the alternatives of Lemma 2.7, or a common parallel foliation.*

Proof. Since the collection contains the weight projectors, any invariant subspace splits as $E_1 \oplus E_q$. We consider the possible nonzero proper subspaces in turn. If V_1 is invariant, then $\partial_y u_h(0) = 0$ for every return. Lemma 2.5 gives the vertical foliation.

Suppose next that a line $\mathbb{C}P \subset V_q$ is invariant, where $P = \alpha x^q + \beta y$. The coefficient of x in its pullback must vanish. If $\beta \neq 0$, this is the hypothesis in the second part of Lemma 2.5, so the returns preserve $P = \text{constant}$.

If $\beta = 0$, invariance gives $u_h(0)^{q-1} \partial_x u_h(0) = 0$. At a center a_k off the axis $x = 0$, this implies that $Dk(0)e_x$ is vertical and nonzero. Applying the same identity to $h \circ k$, we obtain

$$(u_h^{q-1} \partial_y u_h)(a_k) = 0.$$

The centers off $x = 0$ still form a uniqueness set. Hence $\partial_y u_h \equiv 0$. For a return k with off-axis center, this contradicts invertibility because $\partial_x u_k(0) = 0$ as well. Thus $\mathbb{C}x^q$ cannot be invariant. The plane V_q cannot be invariant either: its invariance would also imply $\partial_x v_h(0) = 0$, making the first derivative column of an off-axis return zero.

It remains to consider $E = V_1 \oplus \mathbb{C}(\alpha x^q + \beta y)$. If $\beta = 0$, we again obtain the vertical foliation. Otherwise we change coordinates by setting $X = x$ and $Y = \alpha x^q + \beta y$. This change is invertible and commutes with N , and in these coordinates $E = \text{span}\{X, Y\}$. Invariance of E says that the X^q coefficients omitted from this plane vanish in the pullbacks of X and Y . The restricted block is therefore the action on coordinate functions given by $Dh(0)^\top$.

We claim that the traces of words in this restricted collection are constant. Indeed, Lemma 2.3 and Newton's identities imply that every word in the three-dimensional collection has constant characteristic polynomial. The eigenvalues of its restriction form a submultiset of these three fixed eigenvalues. The trace of the restriction is consequently a holomorphic function taking only finitely many values, so it is constant. Restricting commutes with forming words, and taking transposes only reverses their order.

If the derivative collection is reducible, we apply Lemma 2.5. Otherwise Lemma 2.4 gives a gauge making the derivatives constant. The derivative of N^{-1} is already fixed and has distinct eigenvalues, so the gauge is diagonal and commutes with N . We finish by applying Lemma 2.7. $\square$

In the absolutely irreducible case, Lemma 2.4 gives a matrix gauge on W . We must still show that this gauge is induced by a change of the phase coordinates. To do this, we will characterize the line $\mathbb{C}x^q$ using a rank condition. The following coefficient calculation treats the case in which that condition fails to distinguish a finite set of lines.

Lemma 2.9 (Exceptional rank identity). *Write an actual return as $h = (u_h, v_h)$, put $A_h = u_h(0)$ and $p_h = \partial_x u_h(0)$, and define*

$$Q_q(u) = [x^q](u^q - qu(0)^{q-1}u).$$

If every return satisfies $\det Dh(0) = p_h Q_q(u_h)$, the return system preserves either the vertical foliation or the foliation with first integral $y - cx^q/q$ for one constant c in the fixed parameter fiber.

Proof. The assumed identity and invertibility imply that $p_h \neq 0$ for every return h . Let $D_s(x, y) = (sx, s^q y)$ and $H_s = h D_s k$. Since $D_{\lambda^{-n}} = N^{-n}$, the assumed identity applies to H_s when $s = \lambda^{-n}$ and n is sufficiently large. By one-variable holomorphic uniqueness, it holds identically near $s = 0$.

We first assume that $q > 2$. For $2 \leq j \leq q$, set $C_j(k) = [x^j](u_k - A_k)^j$. Choose the outer return h so that $A_h \neq 0$. The determinant of $DH_s(0)$ has order exactly $q + 1$ in s , while p_{H_s} has order exactly one. On the other hand, the coefficient of s^2 in $Q_q(u_{H_s})$ is

$$\binom{q}{2} A_h^{q-2} p_h^2 C_2(k).$$

This coefficient must vanish. Hence $C_2(k) = 0$ for every return k .

We claim that $b_h = [x^2]u_h$ vanishes for every h . Suppose otherwise. We have

$$u_{H_s} - u_{H_s}(0) = s p_h (u_k - A_k) + s^2 b_h (u_k^2 - A_k^2) + O(s^3).$$

The terms involving v_k begin at order s^q . If C_j vanishes for all returns, then the coefficient of s^{j+1} in $C_j(H_s)$ is

$$j p_h^{j-1} b_h (C_{j+1}(k) + 2 A_k C_j(k)).$$

The vertical terms enter only at order s^{q+j-1} . Starting from $C_2 = 0$ and arguing by induction, we would obtain $C_q(k) = p_k^q = 0$. This contradicts $p_k \neq 0$, and proves the claim.

Now choose k with $A_k \neq 0$. Applying the vanishing just proved to H_s gives

$$s^2 p_k^2 \partial_x^2 u_h(s A_k, 0) = O(s^q).$$

It follows that the pure x coefficients of u_h in degrees 2 through $q - 1$ vanish. Thus $Q_q(u_h) = p_h^q$. If $q = 2$, this equality follows directly from the definition of Q_q . We have therefore shown in every case that

$$\det Dh(0) = p_h^{q+1}. \quad (2.7)$$

To recover the foliation, put $r_k = \partial_x v_k(0)/p_k$. Applying (2.7) to H_s and using the chain rule, we obtain

$$\det Dh(s A_k, s^q B_k) = (\partial_x u_h(s A_k, s^q B_k) + s^{q-1} r_k \partial_y u_h(s A_k, s^q B_k))^{q+1},$$

where $B_k = v_k(0)$. Choose the local root $J_h = (\det Dh)^{1/(q+1)}$ satisfying $J_h(0) = p_h$. The two roots in the preceding identity agree at $s = 0$, and hence agree near zero. Comparing the coefficients of s^{q-1} gives

$$A_k^{q-1} E_h = r_k e_h, \quad E_h = [x^{q-1}](J_h(x, 0) - \partial_x u_h(x, 0)), \quad e_h = \partial_y u_h(0).$$

If every e_h vanishes, Lemma 2.5 gives the vertical foliation. Otherwise fix h with $e_h \neq 0$ and set $c = E_h/e_h$. The preceding equality implies $r_k = c A_k^{q-1}$ for every k . Thus $Dk(0)e_x$ spans the direction $(1, cx^{q-1})$ at a_k . Applying this also to $h \circ k$ and using the chain rule shows that the wedge expressing preservation of this direction vanishes at all compatible centers. It vanishes identically by uniqueness. Hence the returns preserve the foliation with first integral $y - cx^q/q$. $\square$

Lemma 2.10 (Geometric semisimple gauge). *If the collection on W is absolutely irreducible, its constant-character gauge can be induced, up to scalar, by a resonant coordinate change commuting with N . In the resulting coordinates every T_h is constant.*

Proof. We first observe that the line $\ell = \mathbb{C}x^q \subset V_q$ satisfies

$$\text{rank}(V_1 \oplus \ell \xrightarrow{T_h} W/\ell) \leq 1, \quad (2.8)$$

for every return h . Indeed, after passage to W/ℓ , the columns corresponding to u_h and u_h^q are proportional first-derivative columns. More generally, let $T = \begin{pmatrix} a & r \\ s & M \end{pmatrix}$ act on $V_1 \oplus V_q$. Testing a vector spanning ℓ against a covector annihilating ℓ shows that (2.8) holds for every line $\ell \subset V_q$ if and only if $aM - sr$ is scalar.

In the basis (x, x^q, y) , the matrix of an actual return is

$$T_h = \begin{pmatrix} p_h & qA_h^{q-1}p_h & \partial_x v_h(0) \\ [x^q]u_h & [x^q]u_h^q & [x^q]v_h \\ \partial_y u_h(0) & qA_h^{q-1}\partial_y u_h(0) & \partial_y v_h(0) \end{pmatrix}.$$

For this matrix, equality of the two diagonal entries of $aM - sr$ is exactly the identity

$$p_h Q_q(u_h) = \det Dh(0).$$

Suppose that (2.8) held for every line and every return. Lemma 2.9 would then give one of two foliations. The vertical foliation makes $\text{span}\{x, x^q\}$ invariant in W . For the other foliation, the first integral $P = y - cx^q/q$ satisfies $P \circ h = \eta_h(P)$. Its weighted projection is $\eta_h'(0)P$, so $\mathbb{C}P$ is invariant in W . Both conclusions contradict absolute irreducibility.

By Lemmas 2.3 and 2.4, we can choose a gauge which is the identity at the reference parameter and makes the collection constant. This gauge fixes both projectors. In the constant representation, the lines satisfying (2.8) for every return form a fixed algebraic subset of $\mathbb{P}(V_q) = \mathbb{P}^1$. We have just shown that this subset is proper, so it is finite. The image of $\mathbb{C}x^q$ under the gauge belongs to this finite set and varies holomorphically. It is therefore constant. Since the gauge is the identity at the reference parameter, it preserves $\mathbb{C}x^q$ and has the form

$$G = \text{diag} \left(\alpha, \begin{pmatrix} \beta & s \\ 0 & \delta \end{pmatrix} \right).$$

We can now realize the gauge by a coordinate change. Locally choose $r^{q-1} = \beta/\alpha$ and put $k = \alpha/r$. Then

$$G = kC_Q|_W, \quad Q(x, y) = (rx, by + ex^q), \quad b = \delta/k, \quad e = s/k.$$

The scalar k does not affect conjugation. Since both Q and Q^{-1} preserve each weight space, conjugating a return by Q conjugates its compressed matrix by $C_Q|_W$. Moreover, $QN = NQ$. This is the required coordinate change. $\square$

Lemma 2.11 (Semisimple full-germ rigidity). *When all T_h are constant, either every return preserves the vertical foliation or every return germ is constant in the parameter. In particular the absolutely irreducible semisimple case is locally rigid.*

Proof. We first recover the first coordinates of the centers. The entries of T_h include $p_h, e_h, qA_h^{q-1}p_h, qA_h^{q-1}e_h$, where $e_h = \partial_y u_h(0)$. Since the first derivative row is nonzero, the constancy of these entries implies that A_h^{q-1} is constant. Holomorphic dependence then makes

A_h itself constant. All entries of the ordinary derivative occur in T_h as well, so $B_h = Dh_t(0)$ is constant for every h .

If every e_h vanishes, Lemma 2.5 gives the vertical foliation. Otherwise choose h_* with $e_* \neq 0$ and define $\Phi_t(x, y) = (x, u_{*,t}(x, y))$. Its derivative at zero is invertible, and its value there is the constant point $(0, A_*)$. Hence $H_t = \Phi_t^{-1}\Phi_0$ is defined near zero and fixes zero. At a nearby center $a_k(t)$, the value of Φ_t is $(A_k, A_{h_* \circ k})$, which is constant. Thus H_t follows the motion of the centers. By Lemma 2.6, it conjugates every return and commutes with N . Since its first coordinate is x , it has the form

$$H_t(x, y) = (x, \beta(t)y + \delta(t)x^q).$$

We differentiate the first component of $H_t h_{*,0} = h_{*,t} H_t$ in y at zero. Since e_* is constant and nonzero, we obtain $e_* = e_* \beta$, and hence $\beta = 1$. Differentiating the conjugacy for an arbitrary return h now gives $DH_t(a_h)B_h = B_h$. It follows that $q\delta A_h^{q-1} = 0$ at every center. By uniqueness, some center lies off $x = 0$, so $\delta = 0$. Thus $H_t = \text{id}$ and all returns are constant. Finally, the vertical case would make $\text{span}\{x, x^q\}$ invariant, so it is excluded under absolute irreducibility. $\square$

2.5 The genuinely nonlinear resonant case

We next assume that $N = (\lambda x, \lambda^q y + cx^q)$ with $c \neq 0$, and put $\gamma = c\lambda^{-q}$. The nonlinear term makes a single inverse orbit off $x = 0$ a uniqueness set. We begin by proving this property.

Lemma 2.12 (A uniqueness orbit). *For $a \neq 0$, the sequence*

$$z_n = N^{-n}(a, b) = (\lambda^{-n}a, \lambda^{-qn}(b - n\gamma a^q))$$

is a holomorphic uniqueness set for germs at zero.

Proof. Let $\varphi = \sum p_{ij}x^i y^j$ be a holomorphic germ at zero. Grouping its terms by weighted degree $m = i + qj$, we obtain

$$\varphi(z_n) = \sum_{m \geq 0} \lambda^{-mn} Q_m(n), \quad Q_m(n) = \sum_{i+qj=m} p_{ij} a^i (b - n\gamma a^q)^j.$$

For a fixed weight, the nonzero factors have distinct polynomial degrees in n . Hence $Q_m = 0$ if and only if every coefficient of weight m vanishes. Suppose that φ is nonzero, and let m_0 be its first nonzero weight. Then Q_{m_0} is a nonzero polynomial. By the Cauchy estimates used in Lemma 2.3, the tail of higher weight, divided by $\lambda^{-m_0 n}$, decays exponentially times a fixed polynomial factor. It cannot cancel $Q_{m_0}(n)$ for every sufficiently large n . Thus a germ vanishing at every z_n must be zero. $\square$

Lemma 2.13 (Nonlinear resonant dichotomy). *The full actual return collection either preserves the vertical foliation or can be made constant by one coordinate change commuting with N .*

Proof. We consider the blocks $M_h = P_q C_h|_{V_q}$ on $V_q = \text{span}\{x^q, y\}$, together with the fixed nilpotent $D = x^q \partial_y$. The only invariant line of D is $\mathbb{C}x^q$. Thus, if the collection is reducible, we have

$$u_h(0)^{q-1} \partial_y u_h(0) = 0 \quad \text{for every } h. \tag{2.9}$$

By center uniqueness, we can choose k with $a = u_k(0) \neq 0$. Equation (2.9) then implies $Dk(0)e_y = \beta e_y$ for some $\beta \neq 0$. Applying the same equation to $hN^{-n}k$ and using the chain rule gives

$$u_h(z_n)^{q-1} \partial_y u_h(z_n) = 0, \quad z_n = N^{-n}k(0).$$

By Lemma 2.12, the germ $u_h^{q-1}\partial_y u_h$ vanishes identically. Since holomorphic germs form an integral domain and u_h cannot be identically zero, we obtain $\partial_y u_h \equiv 0$ for every h . Hence the returns preserve the vertical foliation.

We may now assume that (D, M_h) is irreducible. There is a return k with both $a_k = u_k(0) \neq 0$ and $\partial_y u_k(0) \neq 0$. The off-diagonal blocks of T_k on $W = V_1 \oplus V_q$ are

$$A_{q,1}(k) = \begin{pmatrix} [x^q]u_k \\ \partial_y u_k(0) \end{pmatrix}, \quad A_{1,q}(k) = \begin{pmatrix} qa_k^{q-1}\partial_x u_k(0) & \partial_x v_k(0) \end{pmatrix}.$$

The first block is nonzero by our choice of k , and the second is nonzero by invertibility. Since the collection contains the projectors, irreducibility on V_q and these two nonzero off-diagonal blocks imply irreducibility on W . Lemmas 2.3 and 2.4 therefore give a gauge making all blocks constant and commuting with the projectors and D . Such a gauge has the form $\text{diag}(\alpha, \beta(I + sD))$. Up to an overall scalar, it equals $C_Q|_W$ for

$$Q(x, y) = (rx, r^q(y + sx^q)), \quad r^{q-1} = \beta/\alpha.$$

The coordinate change Q commutes with N and preserves every weight space. After applying it, all T_h , and in particular all $Dh_t(0)$, are constant.

We will use the inverse orbit of $k_t(0)$ to recover the whole return germ. Write $k_t(0) = (a(t), b(t))$ in these coordinates. Since the derivatives of both k and $N^{-1}k$ are constant, $DN^{-1}(k_t(0))$ is constant. Its lower-left entry is a nonzero constant multiple of $a(t)^{q-1}$. Thus $a(t)$ is a nonzero constant, which we denote by a . Relative to the reference parameter 0, define

$$\delta(t) = \frac{b(t) - b(0)}{a^q}, \quad Q_t(x, y) = (x, y + \delta(t)x^q).$$

The shear Q_t commutes with N and identifies the two inverse orbits by $N^{-n}k_t(0) = Q_t N^{-n}k_0(0)$. The derivative of $hN^{-n}k$ at zero is constant. Applying the chain rule and observing that $DN^{-n}(a, b)$ depends only on a , we obtain

$$Dh_t(Q_t z_n) = Dh_0(z_n)$$

for all sufficiently large n . We apply Lemma 2.12 to each entry. If the parameter neighborhoods depend on n , we apply it to the parameter Taylor coefficients. In either case this gives

$$Dh_t(Q_t z) = Dh_0(z). \tag{2.10}$$

We apply the same identity to $N^{-1}h$ and cancel the invertible derivative factors. Comparing the lower-left entries of DN^{-1} gives

$$u_{h,t}(Q_t z)^{q-1} = u_{h,0}(z)^{q-1}.$$

The two functions agree at the reference parameter. Factoring their equal powers and using holomorphic dependence, we find that the root-of-unity factor is constant and equal to one. Thus $u_{h,t} \circ Q_t = u_{h,0}$. Differentiating in x and comparing with the first row of (2.10), we obtain

$$q\delta(t)x^{q-1}\partial_y u_{h,0}(z) = 0.$$

Our chosen return k has nonzero transverse derivative, so $\delta = 0$. Hence $Dh_t = Dh_0$ and the first components agree. It follows that $h_t = h_0 + (0, \eta_h(t))$.

It remains to show that $\eta_h = 0$. Apply the constancy of the first component to $kN^{-n}h$ for sufficiently large n . Since the first component of k is fixed, we obtain

$$u_{k,0}(N^{-n}h_0(z) + (0, \lambda^{-qn}\eta_h(t))) = u_{k,0}(N^{-n}h_0(z)).$$

The derivative of $u_{k,0}$ in y is nonzero near zero. The one-variable inverse function theorem therefore implies $\eta_h(t) = 0$ for nearby parameters. This proves that every return germ is constant. $\square$

2.6 The resulting local alternative

We can now combine the nonresonant and resonant cases. This gives the following alternative for the full collection of inverse returns.

Theorem 2.14 (Local return alternative). *Let f be an equilibrium-stable algebraic curve family of endomorphisms of $\mathbb{P}^2$. At a generic parameter, select and mark the repeller of Lemma 2.2, pass to its fixed return, and use a fixed normal form. After a local finite parameter cover and coordinate changes commuting with that normal form, at least one of the following holds for the full actual unramified return collection:*

1. *It preserves a regular local holomorphic foliation, which can be made parallel by a coordinate change commuting with the normal form.*
2. *The normal form is linear and every return is affine, with constant linear part after normalization.*
3. *One local coordinate change conjugates the full collection to its reference-parameter collection. It commutes with the normal form and extends to a polynomial automorphism of its source.*

The alternatives need not be disjoint. Every identity retains the finite composition-domain and parameter-germ convention fixed above.

Proof. Suppose first that the normal form is linear and has no nonlinear coordinate resonance. The multiplier-weight spaces containing the coordinate functions then sum to the coordinate-function space itself. Lemma 2.3 gives constancy of the traces of derivative words, including when the linear part is scalar or Jordan. If the derivative collection is reducible, Lemma 2.5 gives a parallel foliation. If it is absolutely irreducible, Lemma 2.4 gives a gauge making the derivatives constant. This gauge commutes with the linear normal form because the derivative of the inverse base return is already fixed. We then apply Lemma 2.7.

In the semisimple resonant case, Lemma 2.8 treats all reducible collections. For an irreducible collection, Lemmas 2.10 and 2.11 give conjugacy of the full return germs. In the genuinely nonlinear resonant case, we apply Lemma 2.13. The affine alternative is excluded in these coordinates because the inverse base return is itself nonlinear.

Finally, Lemma 2.7 shows that each resonant first integral can be made parallel by a shear commuting with N . The other foliations found above already have this form. The conjugacy in the third alternative is polynomial by Lemma 2.6, or by the explicit coordinate change used in the nonlinear resonant case. $\square$

For the global argument, we will need this alternative at uncountably many reference parameters. The next proposition keeps the conclusions that are pointwise in the parameter separate from the conclusion giving a conjugacy of families.

Proposition 2.15 (Eligible reference parameters). *There is a nonempty parameter disc U on which the repeller in Lemma 2.2 is marked and its normal form has fixed type, such that every reference parameter $t_0 \in U$ has one of the following conclusions:*

1. *The full actual return system of the reference map F_{t_0} preserves a regular local foliation which is also N -invariant and has the continuation to the normal-form source described in Lemma 2.7: it is parallel after a polynomial coordinate change commuting with N .*
2. *The full actual return system of F_{t_0} is affine in a linear normal form.*

3. *On a parameter germ at t_0 , possibly after a finite local cover, the full actual return collection is conjugate to its reference collection by a holomorphic family of polynomial source automorphisms commuting with the fixed normal form.*

Only the third conclusion asserts a family statement. In particular, there are uncountably many eligible reference parameters. Neither a common foliation for the whole parameter disc nor one domain for all returns is asserted.

Proof. By Lemma 2.2, the chosen repeller and its uniqueness property persist on a parameter disc, and its multipliers are constant. We delete the discrete zero sets of the possible Jordan or nonlinear resonant coefficient and choose a nonempty disc on which the normal-form type is fixed. Lemma 2.1 gives a holomorphic normal-form chart there.

Fix any parameter t_0 in this disc. We consider all actual returns based at t_0 . By the implicit function theorem, every finite selection defines parameter germs at t_0 . Lemma 2.3 therefore applies to this collection even if t_0 is not a generic parameter.

We first make two observations that allow us to work at this specified parameter. Suppose that a collection with constant character is irreducible over the meromorphic parameter-germ field. Choose matrix words forming a basis of the matrix algebra over that field. Their trace Gram determinant is a nonzero constant. Hence the same words remain linearly independent at t_0 . The matrix-unit construction in Lemma 2.4 gives a holomorphic gauge at t_0 itself. In particular, this construction does not require us to discard t_0 because of a pole or a rank drop.

Second, a common invariant subspace can be defined over a finite algebraic extension of the germ field. We realize this extension by a finite local analytic parameter cover. On this cover the Plücker coordinates of the subspace are meromorphic, since the subspace is algebraic over the convergent meromorphic-germ field. Properness of the Grassmannian then extends it across the reference point. For the two-dimensional choices occurring here, we only need properness of $\mathbb{P}^1$. The algebraicity of the Plücker coordinates is essential to this argument; properness alone would not extend an arbitrary holomorphic map from a punctured disc.

The extended subspace has the same dimension and satisfies the closed invariance equations at the reference point. The weight projectors also keep the dimensions of its two weight components fixed. These conclusions apply to each matrix as a germ and do not require a common radius for the infinite collection. Finally, Noetherianity reduces the invariance conditions to finitely many equations over the germ field. Thus one finite local cover suffices for the chosen invariant subspace.

We now return to the normal-form cases. In the nonresonant linear case, a reducible derivative collection has an invariant line at t_0 by the second observation. Lemma 2.5 gives the required foliation of the reference map. In the irreducible case, the first observation gives a gauge making the derivatives constant near t_0 .

We compute the differential rank of the derivative observables on the phase fiber at t_0 . If this rank is zero, every reference return is affine. If it is one, the proof of Lemma 2.7 applies on this fiber: the chain rule gives an invariant saturated foliation, the displaced centers show that it is regular, and the normal-form calculation gives its continuation. If the rank is two, center uniqueness gives a reference backward center at which a pair of observables has nonzero Jacobian. The parameter inverse function theorem reconstructs the motion near this center, and Lemma 2.6 gives the third conclusion as a parameter germ at t_0 . This argument only uses the observable rank at t_0 , so a change in rank at another parameter causes no difficulty.

In the semisimple resonant case, an irreducible weighted collection again admits its holomorphic gauge at t_0 . By Lemma 2.9, the locus of lines satisfying the rank condition in Lemma 2.10 is proper at t_0 . Thus that lemma realizes the gauge by a coordinate change at the reference parameter, and Lemma 2.11 applies.

For a reducible collection, extend the invariant weight subspace as above. An invariant V_1 , or a limiting plane $V_1 \oplus \mathbb{C}x^q$, gives the vertical foliation of the reference map. An invariant line $\mathbb{C}(\alpha x^q + \beta y)$ with $\beta \neq 0$ at t_0 gives a reference foliation by Lemma 2.5. An invariant pure-power line or an invariant V_q is impossible at t_0 , by the invertibility and center-uniqueness arguments in Lemma 2.8.

The remaining case is an invariant plane $V_1 \oplus \mathbb{C}(\alpha x^q + \beta y)$ with $\beta \neq 0$ at t_0 . Its coordinate change is holomorphic and invertible at this parameter. The restricted derivative characters are therefore constant, and we apply the same argument using the observable rank on the reference fiber. If this plane instead limits to the pure-power plane at t_0 , we have already obtained the first conclusion. Thus we need not exclude that parameter.

Finally, consider the genuinely nonlinear resonant case. If the collection is reducible, the fixed nilpotent gives the fixed pure-power line, and hence the vertical foliation of the reference map. If it is irreducible, the constant Gram determinant preserves irreducibility at t_0 . In particular, there is an off-axis return with nonzero transverse derivative at this parameter. Every coordinate change and application of the inverse function theorem in Lemma 2.13 is consequently valid at t_0 . This gives the third conclusion.

We have proved the alternative for every t_0 in the chosen disc. At each reference parameter, we use its own full collection of return germs. The argument therefore requires no joint continuation across U of the infinite collection based at a single parameter. $\square$

Remark 2.16. The third alternative gives a polynomial conjugacy on the source for the local return system. To obtain a global classification, we still have to show that this conjugacy is algebraic in the parameter, that it continues with finite monodromy, and that it respects every fiber of the global Poincaré map. In the foliation alternative, we must instead extend the local foliation across the projective plane. We address these questions in the next section.

3 Global geometric implications of full return data

In this section, we study the global geometry determined by holomorphic return identities. We first use these identities to descend maps and foliations through a Poincaré map. We then show that a rigid return system leads to either a foliation or an isotrivial family, while an affine return system leads to either a foliation or an affine quotient. Throughout the section, we assume the stated identities for the actual return germs, including their branches and domains; equality of their eigenvalues alone is not sufficient.

Let $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ have algebraic degree $d \geq 2$. We write $T = \omega_{\text{FS}} + dd^c g$ for its Green current, with g continuous, $\mu = T^2$, $J = \text{supp } \mu$, and E for its maximal proper algebraic totally invariant set. We recall several properties that we will use. The measure μ does not charge pluripolar sets, is the unique invariant probability measure of entropy $2 \log d$, and satisfies

$$d^{-2n}(f^n)^*\delta_x \longrightarrow \mu \quad (x \notin E).$$

Repelling periodic points in J equidistribute towards μ , and Green potentials are jointly continuous in holomorphic families. We refer to [36] for these facts and to Briend–Duval [33, 34], with the correction [35], for the original equidistribution and uniqueness results. We also use the continuity and non-pluripolarity properties of Bedford–Taylor products [29]. An entire Poincaré map at a repelling Julia point has discrete fibers and its image contains $\mathbb{P}^2 \setminus E$; see [25, Propositions 4.1–4.2]. This global Poincaré construction plays a central role in the rigidity arguments of Dinh–Sibony [23] and Kaufmann [24, Proposition 4.2].

By a singular foliation on a complex surface, we mean a saturated rank-one holomorphic direction field. Locally, we represent it by a holomorphic vector field after dividing out

common divisorial factors. We interpret identities between foliations as identities between their meromorphic tangent directions. In complex dimension two, such a direction field is automatically integrable.

Related geometric rigidity results. The geometric alternatives that arise below also occur in the study of commuting endomorphisms. Dinh–Sibony classify commuting endomorphisms under multiplicative independence of their degrees [23]. When the degrees are multiplicatively dependent, Kaufmann obtains the plane Lattès or homogeneous-lift alternatives for commuting plane endomorphisms with no common iterate [24, Theorem 1.1]. Here we begin with identities among return germs rather than a commuting pair, and use them to construct the global geometry.

3.1 Good backward centers and finite local descent

After replacing f by a fixed iterate when necessary, let $p \in J$ be a repelling fixed point. We define its unramified backward set by

$$A_p = \{a : f^m(a) = p, \det D(f^m)_a \neq 0 \text{ for some } m \geq 0\}.$$

We say that the good-center hypothesis holds if $\overline{A_p}$ contains a compact subset of J carrying a nonatomic probability measure that charges no proper algebraic subset of $\mathbb{P}^2$. Since A_p is countable, we take its closure in this definition. For the argument below, we may equivalently require the compact set to lie in the accumulation set of A_p . We may also use the stronger conclusion of Lemma 2.2, namely backward density and local holomorphic uniqueness, directly wherever this hypothesis occurs.

Our first observation is that the good-center hypothesis supplies enough backward points to determine holomorphic functions near the Julia set.

Lemma 3.1 (Backward uniqueness). *Under the good-center hypothesis, A_p is dense in J . Its intersection with any connected coordinate ball meeting J is a holomorphic uniqueness set. Moreover $p \notin E$.*

Proof. We choose a point a in the compact accumulation set outside A_p , E , and the countable postcritical union. This is possible because the measure in the hypothesis charges neither proper algebraic sets nor countable sets. If a coordinate ball V meets J , backward equidistribution at a provides a preimage $b \in V$ under some f^m . Since a is nonpostcritical, there is a local inverse branch h with $h(a) = b$. We take $a_j \in A_p$ tending to a . For large j , the points $h(a_j)$ belong to $A_p \cap V$. Hence A_p is dense in J .

Now a holomorphic function vanishing on $A_p \cap V$ also vanishes on $J \cap V$. If it were not identically zero, its zero set would be analytic and would have zero μ -mass, contrary to $\mu(V) > 0$. Finally, if $p \in E$, total invariance would give $A_p \subset E$, and density would then give $J \subset E$, a contradiction. $\square$

Let

$$f\Phi = \Phi N, \quad \Phi : \mathbb{C}^2 \longrightarrow \mathbb{P}^2, \quad \Phi(0) = p$$

be the entire Poincaré map, locally invertible at zero, where N is an expanding Poincaré–Dulac polynomial automorphism and $N^{-n} \rightarrow 0$ on compact sets. Its image is exactly $\mathbb{P}^2 \setminus E$. Indeed, the Poincaré theorem recalled above gives one inclusion. For the reverse inclusion, we take the initial chart disjoint from E and use $\Phi(z) = f^n \Phi(N^{-n}z)$ together with total invariance.

To descend a foliation through this map, we will first work on finite proper restrictions. The following trace construction extends a common direction across their branch loci.

Lemma 3.2 (Finite trace extension). *Let $q : V \rightarrow B$ be a finite proper holomorphic map between surface domains, with V contained in a coordinate ball. Suppose a foliation on V has the same pushed-forward direction on every inverse branch over a dense open subset of B . That common direction extends to a singular holomorphic foliation on B .*

Proof. Let v be a holomorphic vector field defining the source foliation. Outside the discriminant, write the inverse branches as $z_j(u)$ and, for a holomorphic source function h , form

$$V_h(u) = \sum_j h(z_j(u)) Dq_{z_j(u)} v(z_j(u)).$$

Properness shows that this vector field is single-valued and locally bounded, so it extends holomorphically through the discriminant. At a regular fiber avoiding the source singularities, we use polynomial interpolation to choose h which kills all but one nonzero summand. Hence some V_h is not identically zero. We saturate its span to obtain a singular foliation. On the given dense open subset, this is the common direction, so the identity theorem shows that different choices give the same foliation. $\square$

We will repeatedly use finite proper local pieces. To construct one for a holomorphic map with discrete fibers between equal-dimensional complex manifolds, we isolate the source point in a small relatively compact neighborhood and shrink the target ball so that its inverse image misses the source boundary. The resulting component is both open and closed in the inverse image of that ball. These restrictions are local; the entire map need not be proper or a covering map. Related constructions appear in [25, §6.3]. In our setting, the full return system will ensure that the branches agree.

Proposition 3.3 (Descent near the nonexceptional Julia set). *Assume the good-center hypothesis.*

1. *Let $\mathcal{H}$ be a regular source foliation preserved by N . Suppose every regular point of $\Phi^{-1}(p)$ pushes $\mathcal{H}$ to the same direction at p . Then $\mathcal{H}$ descends to an invariant singular foliation on a neighborhood of $J \setminus E$.*
2. *Let f_t be a holomorphic family, p_t a marked repelling fixed point, and $R_t : \mathbb{C}^2 \rightarrow \mathbb{P}^2$ jointly holomorphic, with $f_t R_t = R_t N$ and $R_0 = \Phi$. Suppose, as parameter germs,*

$$R_t(z) = p_t \quad \text{for every regular } z \in \Phi^{-1}(p).$$

Then R_t has compatible jointly holomorphic descended germs H_t along $J \setminus E$, with $R_t = H_t \Phi$ and $H_t f = f_t H_t$ there. The fiber identities defining this descent persist under any analytic continuation of R_t .

Proof. Let α, β be two regular inverse branches of Φ over a ball meeting J . We choose $a \in A_p$ in that ball and m such that $f^m(a) = p$ unramified. Then $N^m \alpha(a)$ and $N^m \beta(a)$ are regular lifts of p . Since N preserves the foliation and $D(f^m)_a$ is invertible, the pushed directions agree at a . Lemma 3.1 extends this equality to the ball.

For the second assertion, the two points $R_t \alpha(a)$ and $R_t \beta(a)$ solve $f_t^m(x) = p_t$ and both equal a at $t = 0$. The implicit function theorem therefore identifies them as parameter germs. We expand the difference of the two holomorphic maps in powers of t , using local target coordinates. Each coefficient vanishes on the uniqueness set A_p , and hence vanishes identically. Thus the branches agree jointly on a fixed smaller parameter disc, even though the individual center identities may initially be defined on different discs.

We next consider a critical value and take finite local pieces of Φ . On each connected component of their regular fiber-pair covering, projection to the complement of the discriminant

is surjective. The lifted Julia uniqueness set therefore identifies the two directions or the two maps on that component. For a foliation, we apply Lemma 3.2 to extend the direction. For maps, continuity on all finite fibers gives a continuous descended map. It is holomorphic off the discriminant, and the removable singularity theorem makes it holomorphic across the discriminant as well. The same argument applies with the parameter included.

By uniqueness, different pieces agree near Julia points and glue as germs. A locally finite shrinking gives a neighborhood of $J \setminus E$, without requiring this set to be compact. The source identity gives semiconjugacy. Finally, every equality on a fixed fiber relation is a holomorphic parameter identity, so it persists under analytic continuation. $\square$

An actual inverse-return germ h of depth m in the initial chart satisfies $f^m \Phi h = \Phi$. For every regular $z \in \Phi^{-1}(p)$, we choose n with $N^{-n}z$ in that chart. The inverse Φ -germ taking 0 to z is then $\gamma_z = N^n h$ for an actual return. Consequently, a full return normalization gives the identity

$$R_t \gamma_z = R_t, \quad (3.1)$$

as germs, and not merely at zero. We will use this germ identity to control regularity and monodromy.

3.2 The rigid branch: continuation, monodromy, and isotriviality

We now consider a rigid return system. After a fixed finite algebraic base change, let B be a nonempty algebraic parameter open, let N be fixed and expanding, and let Φ_t be a single-valued jointly holomorphic entire Poincaré family. We assume that, initially at t_0 , a holomorphic gauge $\alpha_t \in G = C(N)$ fixing zero, with $\alpha_{t_0} = \text{id}$, conjugates the full actual return system. We put $R_t = \Phi_t \alpha_t$. Thus, for every reference return h of depth m_h ,

$$f_t^{m_h} R_t h = R_t \quad (3.2)$$

as phase and parameter germs. We also retain the fiber identities of Proposition 3.3, assume that the reference repeller is good, and assume that the family is equilibrium-stable on B . We refer to these assumptions as the rigid hypotheses.

The gauge takes values in the centralizer of the normal form. We first describe this centralizer, so that continuation of the gauge can be reduced to continuation of finitely many coefficients.

Lemma 3.4 (Plane centralizers). *The origin-fixing holomorphic automorphism germs commuting with an expanding plane Poincaré–Dulac normal form are polynomial automorphisms of uniformly bounded degree. Their coefficient closures consist of the following maps, after ordering coordinates appropriately:*

N	coefficient closure of $C(N)$
λid	all linear maps
$(\lambda_1 x, \lambda_2 y)$, nonresonant	(ax, by)
$(\lambda x, \lambda^q y)$, $q \geq 2$	$(ax, by + cx^q)$
$(\lambda x, \lambda^q y + c_N x^q)$, $c_N \neq 0$	$(ax, a^q y + cx^q)$
$(\lambda x + \eta y, \lambda y)$, $\eta \neq 0$	$(ax + by, ay)$

Every rank-two map in this list is an automorphism. Every nonzero rank-one map factors as $P = \iota \sigma$, where $\sigma : \mathbb{C}^2 \rightarrow \mathbb{C}$ is a polynomial submersion and

$$\sigma N = \rho \sigma, \quad N \iota(u) = \iota(\rho u), \quad |\rho| > 1.$$

Proof. We compare homogeneous terms in $AN = NA$. Above a fixed degree, the multiplier weights have modulus larger than either coordinate multiplier, so the homological operator is invertible. After ordering the two multiplier moduli, the only possible nonlinear monomial is x^q in the second coordinate. If the resonant coefficient of N is nonzero, comparison of this coefficient imposes $b = a^q$. For a nontrivial Jordan block, there are no nonlinear resonances, and its linear centralizer is the displayed algebra. This gives the list and the uniform degree bound, also for inverses.

The diagonal coefficients give the rank assertion. For the rank-one factorizations in the resonant row, when $a \neq 0, b = 0$, we take $\sigma = x$; when $a = 0, b \neq 0$, we take $\sigma = by + cx^q$; and when $a = b = 0, c \neq 0$, we take $\sigma = x$ and $\iota(u) = (0, cu^q)$. These choices also cover the nonlinear row. In the linear rows, we factor through a nonzero linear form. Each σ is a polynomial coordinate, and direct substitution gives the required equivariance. $\square$

We next construct a Poincaré family whose normalization is single-valued on an algebraic open. The expanding normal form allows us to determine it from a sufficiently high finite jet.

Lemma 3.5 (Algebraic finite jets give the standard entire family). *Suppose a formal Poincaré conjugacy with fixed expanding N has coefficients in the parameter function field. Then a sufficiently high finite jet determines a single-valued jointly holomorphic entire Poincaré family on a nonempty algebraic open.*

Proof. We normalize the first derivative to the identity, write $D = DN(0)$, and choose an adapted norm and constants $\|D^{-1}\| < \theta < 1$, $M > \|D\|$. Such a norm exists because every eigenvalue of D^{-1} has modulus less than one. We choose m with $M\theta^{m+1} < 1$ and remove the poles and degeneracy locus of the degree- m jet K_t . Then $G_t = K_t^{-1}f_tK_t = N + O(|z|^{m+1})$. On a sufficiently small phase ball over any compact ordinary parameter neighborhood, $\psi = N^{-1}$ contracts by θ and G_t is M -Lipschitz. Hence the operator $U \mapsto G_tU\psi$ is a contraction for the norm

$$\|U - \text{id}\|_{m+1} = \sup_{z \neq 0} \frac{|U(z) - z|}{|z|^{m+1}}.$$

The displacement of the identity is bounded, so there is an invariant norm ball. We shrink the phase ball to keep all compositions defined and obtain a jointly holomorphic fixed point. To see formal uniqueness, the first possible degree- n difference satisfies $P_n = DP_nD^{-1}$, in the composition sense $P_n(z) = DP_n(D^{-1}z)$. The corresponding operator has norm at most $\|D\|\|D^{-1}\|^n < 1$, so this difference vanishes. Thus the local constructions agree on parameter overlaps.

Finally, we extend to the whole source using $f_t^r\Phi_t(N^{-r}z)$. We may choose r uniformly on each compact set in the parameter and source variables, which gives the required joint holomorphy. $\square$

Remark 3.6 (Algebraic normalization). We can choose the fixed normal form of Lemma 2.1 with the algebraic formal data required in Lemma 3.5. First, we mark the periodic point over one finite algebraic extension. Its derivative has constant eigenvalues, so algebraic eigenvectors, or an algebraic Jordan chain, put its first jet in constant normal form after we delete finitely many parameters. At each successive degree, the normalizing equation is a linear homological equation with constant operator and algebraic right side. We choose constant complements to the kernels and set the free normalizing coefficients to zero.

The only nonlinear resonant coefficient is determined by finitely many jets and is therefore algebraic. On its nonzero stratum, an algebraic rescaling makes it one. The higher coefficients are obtained from invertible homological equations and remain in the same function field. Thus the finite jet in Lemma 3.5 has algebraic coefficients and only finitely many poles, and the lemma

constructs the standard entire family on an algebraic open. The local return gauge may still be holomorphic rather than algebraic. By Lemma 3.4, replacing the initial Poincaré normalization by the standard one changes this gauge by a commuting polynomial automorphism.

We will control limits of the gauge by pulling back equilibrium measures. For this, we first record the entropy property of a countable-to-one factor.

Lemma 3.7 (Countable factors preserve entropy). *A countable-to-one measurable factor between standard Borel probability-preserving systems preserves metric entropy, whether or not the transformations are invertible.*

Proof. Let $\pi : (X, m, T) \rightarrow (Y, \nu, S)$ be the factor, m_y its fiber conditional probabilities, and $p(x) = m_{\pi(x)}(\{x\}) > 0$ almost everywhere. Given $\varepsilon > 0$, choose M such that $K_M = \{p \geq e^{-M}\}$ has mass greater than $1 - \varepsilon$. Every fiber meets K_M in at most e^M points. For a partition $\mathcal{A}$ with q atoms and its length- n name partition $\mathcal{A}^n$, splitting each conditional distribution according to K_M gives

$$H_m(\mathcal{A}^n \mid \pi^{-1}\mathcal{B}_Y) \leq \log 2 + M + \varepsilon n \log q.$$

The block entropy inequality is

$$h_m(T, \mathcal{A}) \leq h_\nu(S) + n^{-1}H_m(\mathcal{A}^n \mid \pi^{-1}\mathcal{B}_Y).$$

Indeed, we first condition on the pullback of a finite target partition. We then condition length- nk names on its full name and split them into k blocks of length n . Dividing by nk and letting $k \rightarrow \infty$ gives the block bound for this target partition. Refining the finite target partition gives the displayed inequality. We now let $n \rightarrow \infty$, then $\varepsilon \rightarrow 0$, and take the supremum over $\mathcal{A}$. Monotonicity of entropy under factors gives the reverse inequality. $\square$

Lemma 3.8 (Measure domination and exclusion of rank loss). *Every analytically continued G -valued rigid gauge satisfies*

$$R_t^* \mu_t \geq \nu_0 := \Phi_{t_0}^* \mu_{t_0} \neq 0.$$

A bounded coefficient limit at an ordinary parameter belongs to G .

Proof. The continued fixed fiber identities give a well-defined Borel map H_t on $J_0 \setminus E_0$, with compatible holomorphic germs there, which semiconjugates f_{t_0} to f_t . We claim that its fibers are countable. Indeed, a positive-dimensional local fiber would lift through a finite local piece of Φ_{t_0} to such a fiber of R_t , contrary to discreteness of the Poincaré fibers and invertibility of α_t . Lemma 3.7 and uniqueness of the measure of maximal entropy therefore give $(H_t)_* \mu_{t_0} = \mu_t$. On an injective local chart, we have

$$H_t^* \mu_t(A) = \mu_t(H_t(A)) = \mu_{t_0}(H_t^{-1} H_t(A)) \geq \mu_{t_0}(A).$$

Since the critical sets have zero mass, we may partition into countably many injective charts and pull back through finite local pieces. This proves the inequality on the support of ν_0 ; outside this support it is automatic.

Now let the bounded polynomial gauges tend coefficientwise to A_* . If its linear derivative is invertible, the coefficient recursion for their formal inverses shows that these inverses converge, since their degrees are uniformly bounded. Passing to the limit in both composition identities makes A_* an automorphism. Otherwise, their constant Jacobian determinants tend to zero, so A_* has rank at most one. Joint continuity of the Green potentials and Bedford–Taylor continuity [29, 36] then give

$$(\Phi_t \alpha_t)^* \mu_t \longrightarrow ((\Phi_{t_*} A_*)^* T_{t_*})^2 = 0.$$

To obtain the last equality, we factor locally through one variable on the rank-one locus. The remaining analytic rank-drop locus has zero mass for a Monge–Ampère measure with continuous potential. We have therefore contradicted the fixed nonzero lower bound. $\square$

The preceding lemma rules out loss of invertibility in a bounded limit. We now use an equilibrium web to obtain the coefficient bounds needed along an arbitrary parameter path.

Proposition 3.9 (All-path continuation). *The initial rigid gauge continues as a G -valued holomorphic germ along every path in B .*

Proof. We pass to the simply connected universal cover M of B . By [9, Theorem 1.1], there is an equilibrium web with compact support $\mathcal{K} \subset \text{Hol}(M, \mathbb{P}^2)$, invariant under the graph dynamics. In dimension $k = 2$, this theorem applies to an arbitrary simply connected parameter manifold M .

We claim that p_t is the unique graph of $\mathcal{K}$ through p_{t_0} . Indeed, every forward image of such a graph lies in $\mathcal{K}$ and passes through the repeller at t_0 . Equicontinuity places all their values in one uniformly expanding repeller neighborhood over a smaller parameter disc. Expansion forces the graph to equal p_t on that disc, and hence everywhere. It follows from compactness that all graphs in $\mathcal{K}$ whose initial values tend to p_{t_0} converge to p_t uniformly on compact parameter sets.

For every $x \in J_0 \setminus E_0$, we next show that the initial descended germ $H_t(x)$ is the germ of a graph in $\mathcal{K}$. We approximate x by repelling Julia periodic points q_j . Their global graphs lie in $\mathcal{K}$, and the implicit function theorem identifies them initially with $H_t(q_j)$. On a fixed local x -disc, the identity theorem extends this germ equality, even if its original radius depended on j . Compactness of $\mathcal{K}$ and continuity of the descended map then identify a limiting graph with $H_t(x)$.

We fix a compact path and a relatively compact connected parameter neighborhood O containing it. There is a source ball on which every Φ_t , $t \in O$, is injective, with images containing a uniform ball about p_t . We choose finitely many constant source Julia points s_i sufficiently close to zero that their web graphs $\gamma_i(t)$ stay in these images over O . We may choose them to determine every polynomial vector map of the centralizer degree bound. Indeed, Lemma 3.1 shows that evaluations on the local Julia set separate this finite-dimensional vector space, and finitely many evaluations then suffice. The bounded inverse lifts

$$b_i(t) = (\Phi_t|_{\text{source ball}})^{-1}(\gamma_i(t))$$

are holomorphic throughout O . Applying a fixed linear left inverse of evaluation, we construct a polynomial map $A(t)$ from these values. It initially equals α_t , so the identity theorem extends all commutation, return, and fiber identities throughout O .

Away from the discrete zeros of $\det DA(t)_0$, the formal inverse recursion and the uniform degree bound give a polynomial inverse by the identity theorem. The punctured parameter domain is connected, so these maps continue the original branch. Lemma 3.8 excludes every zero of the determinant. Hence $A(t) \in G$ throughout O , and the gauge continues along the chosen path. $\square$

Lemma 3.10 (Constant discrete monodromy). *The continued gauges have constant right monodromy $\alpha_t^\gamma = \alpha_t A_\gamma$, with a discrete subgroup $\Gamma \subset G$. Every $A \in \Gamma$ preserves ν_0 and satisfies $f_{t_0}^{m_h} \Phi_{t_0} A h = \Phi_{t_0} A$ for every reference return.*

Proof. Let P be the discrete regular source fiber over p_{t_0} . By differentiating (3.1) at zero, we see that every continued R_t is regular at each fixed $z \in P$. We choose finitely many good backward centers s_i determining polynomial maps of the degree bound and integers n_i with

$z_i = N^{n_i} s_i \in P$. Since the maps in G commute with N , the points z_i also determine elements of G .

For a returned branch, we put $A_t = \alpha_t^{-1} \alpha_t^\gamma$. At t_0 , it sends P to P . For fixed $z \in P$, both $A_t z$ and $A_{t_0} z$ lift p_t through R_t and coincide at t_0 . Uniqueness of a local inverse therefore gives $A_t z = A_{t_0} z$. Applying this to the determining points, we conclude that the monodromy is constant. Applying the same argument to the inverse loop shows that it permutes P .

Suppose a sequence of these constants tends to the identity. By discreteness of P , its values at the finitely many z_i eventually equal z_i , so the constants themselves are eventually the identity. Thus Γ is discrete. The return identities persist by holomorphic continuation. Finally, Lemma 3.8 applied to a loop and its inverse gives $A^* \nu_0 \geq \nu_0$ and $(A^{-1})^* \nu_0 \geq \nu_0$. Pulling back the latter inequality gives equality. $\square$

We now distinguish finite monodromy from unbounded monodromy. In the second case, rescaling the monodromy maps produces a rank-one limit, whose fibers give the desired foliation.

Proposition 3.11 (Noncompact monodromy produces a foliation). *Under the rigid hypotheses, if Γ is not relatively compact, every actual reference return preserves one regular polynomial-coordinate foliation $\sigma = \text{constant}$, with $\sigma N = \rho \sigma$, $|\rho| > 1$.*

Proof. We put $S = \Phi_{t_0}^* T_{t_0}$, so $S^2 = \nu_0$ and $N^* S = dS$. A non-relatively-compact subgroup of bounded-degree polynomial automorphisms has unbounded coefficients. Indeed, otherwise its inverses would also be bounded, and simultaneous limits would remain inverse. We choose $A_j \in \Gamma$ with unbounded coefficients and the least $n_j \geq 0$ such that $P_j = N^{-n_j} A_j$ has coefficient norm at most one. Then $n_j \rightarrow \infty$. Minimality and continuity of composition by N keep $\|P_j\|$ bounded away from zero. After passing to a subsequence, we obtain a nonzero polynomial limit P fixing zero and commuting with N .

Continuous potentials give $P_j^* S \rightarrow R = P^* S$ and

$$R^2 = \lim_j d^{-2n_j} A_j^* \nu_0 = 0.$$

We claim that $R \neq 0$. If it vanished, the bounded function $-g \circ \Phi_{t_0} \circ P$ would be plurisubharmonic with curvature $(\Phi_{t_0} P)^* \omega_{\text{FS}}$. Liouville's theorem on complex lines would make this curvature zero, and hence make $\Phi_{t_0} P$ constant. Discreteness of the Poincaré fibers would then make P constant, a contradiction. On the other hand, Lemma 3.4 shows that rank two would make P an automorphism and force $R^2 = P^* \nu_0 \neq 0$. Thus $P = \iota \sigma$ has rank one, with the factorization described in that lemma.

We write $R = \sigma^* \eta$, where $\eta = \iota^* S \neq 0$. The measure η has continuous potentials, hence no atoms, and satisfies $(u \mapsto \rho u)^* \eta = d\eta$. Its support therefore has infinitely many points accumulating at zero.

We fix a return h . By the identity theorem, its holomorphic identity in Lemma 3.10 holds on the entire fixed connected source domain of h , independently of j . Pulling back T_{t_0} , we obtain $h^*(A_j^* S) = d^{-m_h} A_j^* S$ there. We rescale and pass to the limit to obtain $h^* R = d^{-m_h} R$. In coordinates (u, v) with $u = \sigma$, wedging with $i d(\sigma h) \wedge \overline{d\sigma h}$ gives

$$|\partial_v(\sigma h)|^2 \eta(du) dA(v) = 0.$$

We first use holomorphy in v , and then in u , together with the accumulating nonatomic support, to conclude that $\partial_v(\sigma h) = 0$. Hence $\sigma h = \chi_h \circ \sigma$. Since h is locally biholomorphic and σ is a submersion, χ_h' is nonzero after shrinking. $\square$

It remains to treat finite monodromy. After passing to a finite cover, the backward centers will give a Zariski-dense collection of rational preperiodic points over one function field. We can then apply the geometric dynamical Northcott theorem.

Theorem 3.12 (Conditional rigid endpoint). *Under the rigid hypotheses, either every actual reference return preserves a common regular foliation, or the family is projectively isotrivial after finite algebraic base change.*

Proof. Suppose that the foliation alternative does not hold. By Propositions 3.9 and 3.11 and Lemma 3.10, the monodromy is finite. Its finite unramified cover of B compactifies to an algebraic cover, on which R_t is single-valued and jointly holomorphic. For every fixed reference backward center $s = h(0)$, we have $f_t^{m_h}(R_t(s)) = p_t$ throughout the cover. Hence $a_s(t) = R_t(s)$ is a single-valued holomorphic section of a finite algebraic preimage scheme.

This section is algebraic. Indeed, on the normalization of an irreducible component, away from finitely many branch points, it selects a monodromy-invariant sheet of a connected finite cover. That component therefore has degree one. Thus all a_s belong to one function field and are preperiodic.

We claim that these points are Zariski dense over that field. Otherwise, we may normalize the coefficients of a nonzero homogeneous equation to be regular at the reference parameter, with at least one nonzero specialization. Specializing its evaluations at all a_s contradicts local holomorphic uniqueness of the centers.

By [45, Corollary 1.5], the rational height-zero points of a nonisotrivial polarized endomorphism lie in one fixed inverse image of a proper invariant union of periodic isotrivial subvarieties. The union is proper because an isotrivial iterate of the entire map makes the map isotrivial; see Lemma 6.5 and [46, Corollary 16]. This contradicts density. On projective space, isotriviality gives projective conjugacy after finite base change. $\square$

3.3 Global continuation of a Julia-neighborhood foliation

We now extend a foliation from a neighborhood of $\text{supp } \mu$. For comparison, a neighborhood of $\text{supp } T$ enters Tapiero's construction: under an additional pluripotential product structure, he constructs invariant foliations near $\text{supp } T \setminus E$ and applies the extension theorem of Lins Neto when extension across the relevant part of E is available [25, Theorem 1.1 and Corollary 1.2]; see [47]. Starting with the smaller Julia neighborhood, we use the inverse-branch estimates of [36, Proposition 1.4.7 and Lemma 1.4.11]. We continue the foliation along a large majority of the inverse branches. The overlap of any two such majorities will force the resulting germs to agree.

Theorem 3.13 (Majority continuation). *Suppose a singular holomorphic foliation $\mathcal{F}$ is defined on an open neighborhood U of $J \setminus E$ and satisfies $f^*\mathcal{F} = \mathcal{F}$ as germs at every point of $J \setminus E$. Then its germ extends uniquely to an invariant singular foliation on $\mathbb{P}^2 \setminus E$. If E is finite it extends to all of $\mathbb{P}^2$.*

Proof. We put $D = d^2$ and let C_f be the critical divisor with multiplicities. We will use the positive closed current

$$R = \sum_{r \geq 1} D^{-r} (f^r)_* [C_f]$$

to control ramification. It has finite mass because the r th unnormalized mass is $d^r \deg C_f$. The critical chain rule and the finite push-pull identity give

$$D^{-n} (f^n)_* [C_{f^n}] = \sum_{r=1}^n D^{-r} (f^r)_* [C_f] =: R_n \leq R. \quad (3.3)$$

We first construct a germ at a nonpostcritical target x . We choose a nonpostcritical point $a \in J \setminus E$ so that the projective line $L = ax$ is generic for every iterate: $(f^n)^{-1}L$ is smooth,

and L meets each postcritical curve transversely at smooth points and avoids its singular points and rank-zero critical values. These requirements impose countably many proper algebraic conditions on lines through x , so they exclude a set of zero μ -mass for a . Bertini also applies at the base points $f^{-n}(x)$, since x is nonpostcritical.

We define $R|_L$ as the sum of the weighted intersection measures. This is a finite purely atomic measure. The map

$$q_n = f^n|_{(f^n)^{-1}L} : (f^n)^{-1}L \longrightarrow L$$

has degree D^n , and its pushed ramification measure is $D^n R_n|_L$ by (3.3).

We choose an embedded arc from a to x avoiding all atoms, together with fixed simply connected strips $\gamma \subset \Omega_0 \Subset \Omega$ such that $(R|_L)(\Omega) < \varepsilon$. We may also choose their boundaries to avoid the countably many ramification values. By Riemann–Hurwitz for a bordered surface over a disc, a connected component of $q_n^{-1}(\Omega)$ of degree $e \geq 2$ has ramification $r \geq e - 1$, and hence $e \leq 2r$. It follows that at least $(1 - 2\varepsilon)D^n$ components have degree one, giving distinct inverse branches $g_{i,n}$ on Ω .

These are ambient inverse branches along their images. Indeed, if a section passed through an ambient critical point, its derivative would have to map into both TL and the transverse tangent line of the critical-value curve, which is impossible.

The sum of their Fubini–Study areas is at most d^n , the area of the inverse-image curve. Thus all but at most $d^{3n/2}$ of the branches have area at most $d^{-n/2}$. Their diameters on the fixed relatively compact strip Ω_0 tend to zero. To see this, we cover its closure by a finite chain of discs. Local area monotonicity first prevents a small-area image from leaving a fixed coordinate ball, and the holomorphic energy estimate then bounds derivatives on smaller discs. We fix both the domain and the chain before letting n increase. This is the small-area argument underlying the Briend–Duval diameter estimate [33, 36].

We choose a compact core $K \subset J \setminus E$, avoiding the foliation singularities, with $\mu(K) > 1 - \delta$, and an inner neighborhood whose closure is contained in U . Backward equidistribution at a puts at least $(1 - 2\delta)D^n$ preimages in this neighborhood for large n . Combining this with the preceding estimates, we obtain at least $(1 - 2\varepsilon - 2\delta - o(1))D^n$ branches with $g_{i,n}(\Omega_0) \subset U$. In particular, we can retain any fixed fraction $1 - \eta$ by choosing ε, δ first. For each fixed n , the inverse function theorem extends these finitely many branches to a common connected full-dimensional tube V_n around the arc, still with images in U .

At a , every selected inverse orbit lies entirely in $J \setminus E$. Iterating germ invariance, we see that all $g_{i,n}^* \mathcal{F}$ agree with the starting foliation near a , and hence throughout V_n . They continue the same germ along the same arc, so uniqueness of analytic continuation makes the endpoint germ independent of n .

We claim that the endpoint germ is also independent of the arc and starting point. Indeed, two constructions ending at x use more than $D^n/2$ inverse germs for the same sufficiently large n . Their sets intersect, and a common inverse germ pulls back the same source foliation. This proves the claim. More precisely, for any $\eta > 0$, the resulting germ is represented by at least $(1 - \eta)D^n$ depth- n inverse germs for every sufficiently large n .

We next show that these germs vary holomorphically and agree on overlaps. To obtain neighborhoods independent of the depth, we use the fixed-ball inverse-branch theorem [36, Proposition 1.4.7]. Let Y be the reduced critical-value hypersurface, and put

$$R^{\text{DS}} = \sum_{j \geq 0} D^{-j} (f^j)_* [Y].$$

For sufficiently small fixed $c > 0$, Siu’s theorem [48] makes

$$A_c = \{z : \nu(R + R^{\text{DS}}, z) \geq c\}$$

a proper algebraic set. We let $W = \mathbb{P}^2 \setminus (A_c \cup E)$. At each $b \in W$, the inverse-branch theorem supplies one fixed ball $B \subset W$ carrying at least $(1 - \delta)D^n$ shrinking inverse branches for every n . By taking c small, we may make δ as small as required. Equidistribution at b and the compact-core argument allow us to retain at least $(1 - 2\delta)D^n$ branches whose images lie in U .

We fix a nonpostcritical $x_0 \in B$ and compare these branches with its strip construction. At least $(1 - 2\delta - \eta)D^n$ of the ball branches define its canonical germ. Their foliations agree on all of B by the identity theorem, and their germ at x_0 makes them independent of n . At any other nonpostcritical $x \in B$, comparison with its strip construction identifies the same foliation, provided $2\delta + 2\eta < 1$. Hence the ball foliations agree on overlaps and define a foliation on W .

The new foliation agrees with the original one near generic Julia points. Indeed, we may take a short line arc wholly inside an original foliation chart, where the original foliation already continues the starting germ. Thus $f^*\mathcal{F} = \mathcal{F}$ on a nonempty open subset of $W \cap f^{-1}W$, and the identity theorem extends this equality throughout the connected Zariski open set. Iterating first gives $(f^m)^*\mathcal{F} = \mathcal{F}$ wherever all intermediate iterates lie in W . A further application of the identity theorem extends it to $W \cap f^{-m}W$.

It remains to extend across the part outside this Zariski open set. For $x \notin E$, equidistribution supplies $y \in f^{-m}(x) \cap W$. We choose a finite proper local piece $f^m : V \rightarrow B_x$ with $V \subset W$. Away from the discriminants and the relevant algebraic subsets, its inverse branches push the source foliation to the known foliation on $B_x \cap W$. Lemma 3.2 extends this foliation to B_x . Different pieces agree on the dense open W , so they glue on $\mathbb{P}^2 \setminus E$.

At a Julia point of A_c , the original and extended foliations both exist on a small connected ball in $U \setminus E$. Since $\mu(A_c \cup E) = 0$, this ball contains a Julia point of W . The identity theorem therefore identifies the two foliations on the ball. We have extended the original germ at every required point. If E is finite, we extend a meromorphic slope across the missing points by Hartogs and then saturate. $\square$

Corollary 3.14 (Geometric foliation endpoint). *Under the hypotheses of Theorem 3.13, either an iterate of f is regular polynomial in the complement of a totally invariant line, or f preserves a pencil of lines.*

Proof. By [40, Proposition 1.1], the curve part of E , when nonempty, is a union of at most three lines permuted by f . This proposition collects earlier results on exceptional curves, including [49]; see [40]. The sixth iterate makes each line totally invariant. Relative to one such line, its last homogeneous coordinate is a nonzero multiple of Z^{d^6} , so it is a regular polynomial map on the complementary affine plane.

If E has no curve, Theorem 3.13 gives a global invariant foliation. By [20, Theorem C], its degree is zero or one. Every degree-one case in that classification has a totally invariant union of coordinate lines, contrary to finiteness of E . The foliation therefore has degree zero and is a line pencil. Such maps are also covered by the classification of Dabija–Jonsson [21]. In coordinates centered at the pencil center, we have

$$f = [P(X, Y) : Q(X, Y) : R(X, Y, Z)];$$

The forms P, Q cannot have a common projective zero: the corresponding source line would meet $R = 0$ and create a base point. Hence the base rational map has degree d . $\square$

3.4 The full affine return alternative

We now assume that every actual unramified inverse-return germ in the initial Poincaré chart is affine, including the inverse base return. Thus N is linear and expanding. We retain the

good-center hypothesis and set

$$\Gamma = \{T \in \text{Aff}(\mathbb{C}^2) : \Phi T = \Phi\}.$$

The return identities give elements of this deck group. Our first task is to show that the group acts properly on the whole source, so that we can pass from local return maps to a global quotient.

Lemma 3.15 (Deck action and exact index). *The group Γ acts properly discontinuously on all of $\mathbb{C}^2$. It is transitive on every fiber over $J \setminus E$ and satisfies*

$$N\Gamma N^{-1} \subset \Gamma, \quad [\Gamma : N^n \Gamma N^{-n}] = d^{2n}.$$

It induces a finite proper self-map F of degree d^2 on $Q = \mathbb{C}^2/\Gamma$. The induced surjection $\bar{\Phi} : Q \rightarrow \mathbb{P}^2 \setminus E$ has discrete fibers and finite proper local pieces, and $\bar{\Phi}F = f\bar{\Phi}$.

Proof. For regular $z \in \Phi^{-1}(p)$, the germ $N^n h$ constructed above is affine and satisfies $\Phi N^n h = \Phi$. The entire identity theorem therefore extends it to a deck transformation taking 0 to z . Local injectivity at z gives uniqueness. It also shows that Γ is discrete and that its stabilizer at zero is trivial.

Now let z, w be two regular lifts of an unramified backward center a of depth m . We first use transitivity over p to find $T(N^m z) = N^m w$. Then $S = N^{-m} T N^m$ satisfies $f^m \Phi S = f^m \Phi$, and at z the maps before composition with f^m both take the value a . Local injectivity of f^m at a , followed by the entire identity theorem, gives $\Phi S = \Phi$.

We next prove properness. Suppose that distinct $T_j \in \Gamma$ satisfy $x_j \rightarrow x$ and $T_j x_j \rightarrow y$. We take a finite proper local piece V of Φ about y over a target ball B . This piece is open and closed in $\Phi^{-1}(B)$. For a fixed small connected source ball U at x with $\Phi(U) \subset B$, the connected image $T_j(U)$ meets V for large j , so it lies in V . This bounds the affine coefficients. Applying the same argument to T_j^{-1} bounds the inverse coefficients as well. A subsequence then tends to an invertible deck transformation, contrary to discreteness. Hence the action is proper, including at critical source points, and we obtain the normal quotient with its local finite maps.

For arbitrary lifts z, w of $x \in J \setminus E$, we take finite local pieces and approximate x by good centers avoiding their discriminants. The corresponding regular lifts $z_j \rightarrow z, w_j \rightarrow w$ are deck-equivalent. Properness makes a subsequence of their deck transformations constant, which proves transitivity. In particular, every lift of p is regular because it is equivalent to zero. If $f^m(a) = p$, we choose a source lift of a and differentiate $\Phi N^m = f^m \Phi$. This shows that $D(f^m)_a$ is invertible. Thus p is nonpostcritical and all its backward points are unramified.

We write $\Gamma_n = N^n \Gamma N^{-n}$. The right cosets $\Gamma_n T$ correspond to the points $\Phi(N^{-n} T(0)) \in f^{-n}(p)$. Transitivity on the base fiber makes this correspondence surjective. If two values coincide, transitivity over that backward point and the trivial stabilizer at zero identify the cosets. There are d^{2n} values, giving the index formula. Finally, N identifies Q with $\mathbb{C}^2/\Gamma_1$, and the natural map from this quotient to Q is finite proper of the stated degree. $\square$

The remaining distinction is governed by the linear parts of the deck transformations. We show that unbounded linear parts force a common direction, while bounded linear parts allow us to equip the quotient with a flat metric.

Lemma 3.16 (Affine holonomy alternatives). *Let a discrete subgroup $\Gamma \subset \text{Aff}(\mathbb{C}^2)$ act properly on all of $\mathbb{C}^2$, with finite stabilizers allowed, and suppose an expanding linear map N conjugates it into a finite-index subgroup. Then either the linear image of Γ is bounded, or Γ and N preserve a common complex direction. In the second case all maps $N^{-m} T, T \in \Gamma$, preserve the corresponding parallel foliation.*

Proof. We first claim that any subgroup $H \subset \mathrm{GL}_2(\mathbb{C})$ all of whose elements have eigenvalue one is triangularizable. If all determinants are one, we choose a nonidentity unipotent and conjugate it to $I + E_{12}$. Comparing the traces of B and $(I + E_{12})B$ gives $B_{21} = 0$. If there is no such nonidentity element, the subgroup is trivial. If instead some determinant differs from one, we conjugate an element to $\mathrm{diag}(1, a)$, with $a \neq 1$. The eigenvalue-one identities for B and its product with this element give $B_{11} = 1$, $B_{22} = \det B$, and $B_{12}B_{21} = 0$. Nonzero upper and lower off-diagonal entries cannot occur in different elements, since their product would have first diagonal entry different from one. Thus H is triangular in one orientation and is solvable.

A nonabelian free subgroup of Γ would have no nonidentity element with an affine fixed point: such an element has infinite order, whereas properness gives finite stabilizers. Each of its linear parts would therefore have eigenvalue one. The preceding matrix argument and the abelian translation kernel would make the free group solvable, a contradiction. We apply the characteristic-zero Tits alternative [50, Theorem 1] to conclude that Γ is virtually solvable. This form of the theorem applies to arbitrary linear groups and requires no finite-generation hypothesis. Hence its linear Zariski closure G has solvable identity component G^0 . The finite-index inclusion gives $NG^0N^{-1} = G^0$.

Suppose first that G^0 consists of scalar matrices. Then the projective image of G is finite. The scalar subgroup of the linear image of Γ is bounded. Indeed, an affine element with linear part aI , $a \neq 1$, has a fixed point and hence finite order, so a is a root of unity. Since there are only finitely many scalar cosets, the full linear image is bounded.

Otherwise, let $\mathcal{S}$ be the common invariant lines of G^0 . Lie–Kolchin gives at least one such line, and a nonscalar element has at most two, so $1 \leq |\mathcal{S}| \leq 2$. By normality, Γ permutes $\mathcal{S}$, and N permutes it as well. If there is just one line, it is the required common direction.

Suppose there are two lines. We will show that the linear image is finite. Since the ambient affine group is a connected Lie group, the discrete-solvable subgroup theorem [51, Main Theorem] makes a solvable finite-index subgroup of Γ finitely generated. Hence Γ is finitely generated as well. Selberg’s lemma [52] gives a torsion-free finite-index normal subgroup $\mathcal{T}$, which we intersect with the inverse image of G^0 .

We put $m = [\Gamma : \mathcal{T}]$ and intersect the kernels of all homomorphisms from Γ to groups of order at most m . Finite generation makes this a finite intersection. The resulting subgroup Δ is torsion-free, has finite index, is contained in $\mathcal{T}$, and is fully invariant. In particular, $N\Delta N^{-1} \subset \Delta$ with finite index, and its linear Zariski closure is G^0 . Thus Δ fixes both lines and N^2 fixes them individually.

Every nonidentity element of Δ is fixed-point-free, so its diagonal linear part has eigenvalue one. One of the diagonal characters must be identically one. Otherwise, multiplying elements on which different characters are nontrivial contradicts this condition. In suitable coordinates, we write

$$\delta(x, y) = (a_\delta x + b_\delta, y + c_\delta), \quad M = N^2 = (\lambda x, \mu y), \quad |\lambda|, |\mu| > 1.$$

Consider the subgroup Δ_x of pure x -translations, which is a discrete additive subgroup of $\mathbb{C}$. If it is nonzero, multiplication by each a_δ is a scalar lattice automorphism. A rank-one lattice permits only ± 1 . For a rank-two lattice, these scalars preserve covolume and form a finite subgroup of the circle. Thus the scalar image is finite in either case.

Now suppose $\Delta_x = 0$. Every commutator is then trivial. If some $a_\delta \neq 1$, the commuting first-coordinate affine maps have a common fixed point x_0 . Restriction to $x = x_0$ identifies Δ with a discrete transverse translation lattice $C_0 \subset \mathbb{C}$. Indeed, the restriction is injective by fixed-point-freeness, and its image is discrete by properness on that line. Normalization forces $\lambda x_0 = x_0$, so $x_0 = 0$, and we obtain

$$\mu C_0 \subset C_0, \quad a(\mu c) = a(c).$$

The endomorphism $\mu - 1$ of the real span of C_0 is invertible, so $(\mu - 1)C_0$ has finite index. The character a therefore factors through the finite group $C_0/(\mu - 1)C_0$, and has finite image. If every $a_\delta = 1$, the linear image is already trivial. Thus in every case the linear image of Δ , and hence that of Γ , is finite, as required. $\square$

Proposition 3.17 (Bounded holonomy gives full quotient descent). *In Lemma 3.15, suppose the linear image of Γ is bounded. Then $E \cap J = \emptyset$ and $\bar{\Phi} : Q \rightarrow \mathbb{P}^2 \setminus E$ is a biholomorphism.*

Proof. The closure of the linear image is compact, since boundedness also applies to inverses. We average a Hermitian norm over this compact group, so that Γ acts by affine isometries. We then choose r with $c = \|N^{-r}\| < 1$, together with right-coset representatives T_i of $N^r\Gamma N^{-r}$ in Γ . The maps $S_i = N^{-r}T_i$ are c -contractions and enumerate $F^{-r}([z])$.

If $M = \max_i \|S_i(0)\|$, a closed source ball of radius at least $M/(1 - c)$ is invariant under all S_i . We enlarge this ball to contain representatives of the finitely many remaining depths modulo r . Hence the full backward orbit of $[0]$ has compact representatives. By density of the backward orbit of p , their compact closure maps onto J , and every value avoids E . Thus $E \cap J = \emptyset$.

The fibers over J are singletons, so $K = \bar{\Phi}^{-1}(J)$ is compact: finitely many finite local source pieces give compact representatives. A finite local piece of $\bar{\Phi}$ at a point of K cannot have degree greater than one, since a nearby regular backward center would then have multiple quotient preimages. Hence $\bar{\Phi}$ is locally biholomorphic along K .

We claim that there is one neighborhood V of K on which the map is injective. Otherwise, we could find collisions in shrinking neighborhoods. Compactness would make them converge to a collision on K , contradicting injectivity there and local injectivity near its points.

We equip the quotient with the metric induced by the invariant source norm. Every preimage of $[z]$ under F^n has a representative $N^{-n}\gamma(z)$, while $N^{-n}\gamma(0)$ represents a point of K . Consequently,

$$\sup_{q \in F^{-n}([z])} \text{dist}(q, K) \leq \|N^{-n}\| \|z\| \rightarrow 0.$$

The estimate is uniform for $[z]$ in relatively compact quotient pieces, which admit compact source representatives through finitely many local finite-quotient charts.

Suppose that two quotient points have the same image. We take disjoint finite local pieces about them with a common target ball. For large n , all inverse fibers of both pieces under F^n lie in V . We choose a target point outside the two local discriminants and the critical values of f^n . By the chain rule and local invertibility of $\bar{\Phi}$, each of its two chosen quotient preimages has d^{2n} distinct preimages. Injectivity of $\bar{\Phi}$ on V would then place $2d^{2n}$ distinct points in a fiber of f^n of size d^{2n} , a contradiction. Thus $\bar{\Phi}$ is injective. Since it has finite local degree one and the target is normal, it is a biholomorphism. $\square$

When the exceptional set is finite, the quotient is in fact compact. We prove this using a plurisubharmonic function provided by the Euclidean geometry of the quotient; compactness then gives the abelian-surface presentation.

Proposition 3.18 (Finite exceptional set in the bounded case). *Under Proposition 3.17, if E is finite then it is empty and f has a polarized abelian-surface Lattès presentation.*

Proof. By the Bieberbach theorem for complete flat orbifolds [53, Theorem 3.18], there is a Γ -invariant real affine subspace O on which the action is cocompact. If the full quotient were noncompact, O would be proper. Squared distance to O would then be a nonconstant smooth convex quadratic function, hence plurisubharmonic on $\mathbb{C}^2$, and would be Γ -invariant. It descends continuously to the quotient. The removable-singularity theorem for locally bounded

plurisubharmonic functions shows that the descended function is plurisubharmonic through the finite quotient branch loci.

However, every continuous plurisubharmonic function on $\mathbb{P}^2 \setminus E$ is constant when E is finite. Indeed, its restriction to a projective line avoiding E is constant by compactness. Given two points, we choose a third outside the finitely many lines joining either point to E . The two connecting projective lines avoid E , so the function has the same value at the two points. This contradicts the descended squared-distance function. Hence Q is compact, and its biholomorphism with $\mathbb{P}^2 \setminus E$ forces $E = \emptyset$.

The compact Bieberbach theorem now makes the full translation subgroup $\Lambda \subset \Gamma$ a real rank-four lattice of finite index. The torus $A = \mathbb{C}^2/\Lambda$ maps finitely onto $Q \simeq \mathbb{P}^2$. To prove that it is projective, we average the pullback of ω_{FS} over torus translations. The resulting form represents the integral class of $L = \pi_A^* \mathcal{O}(1)$ and is positive definite, since the original pullback is positive definite on a nonempty open set. The polarization criterion for complex tori therefore makes A an abelian surface and L ample. Since $N\Lambda \subset \Lambda$, it induces an isogeny φ satisfying

$$f\pi_A = \pi_A\varphi, \quad \varphi^*L \simeq \pi_A^*f^*\mathcal{O}(1) \simeq L^{\otimes d}.$$

All maps are algebraic by projectivity. $\square$

The quotient obtained here is the geometry appearing in the characterization of Lattès maps by Berteloot–Loeb [54]. Complex crystallographic plane quotients were classified in [13], and Rong developed their dynamical classification in [12]. We have obtained this quotient from the affine return identities by proving properness, descent, and compactness.

Theorem 3.19 (Conditional affine endpoint). *Under the affine actual-return and good-center hypotheses, either the actual returns preserve a common parallel foliation, or the bounded-holonomy quotient construction applies. In the latter case f is Lattès when E is finite, and a fixed further iterate is regular polynomial when E has a curve. The former case is covered by Proposition 3.3 and Corollary 3.14.*

Proof. We apply Lemma 3.16 to the proper group constructed in Lemma 3.15. Every actual return satisfies $h = N^{-m}T$, where $T = N^mh$ is its global affine deck transformation. Thus a common invariant direction is preserved by all actual returns and by N . Since all regular base-fiber lifts are deck translates of zero, they push this direction to the same line at p . We may therefore apply local descent and majority continuation.

In the remaining case, the linear image is bounded, and Propositions 3.17 and 3.18 apply. If the exceptional set has a curve, we use the at-most-three-lines theorem and pass to the sixth iterate, as in Corollary 3.14. $\square$

Remark 3.20. Theorems 3.12, 3.13, and 3.19 begin with the stated holomorphic return identities. To apply them to a stable family, we need the full return alternatives, including these identities, and then the classification of the polynomial and pencil cases. Constant multipliers alone do not supply the return hypotheses. Likewise, the potential-good-reduction arguments for the polynomial case require their own non-archimedean input; a divisorial atom alone does not give that conclusion. We keep these hypotheses separate when assembling the classification.

4 Stable families of regular polynomial maps

Throughout this section, we work over smooth connected complex algebraic curves. We say that a family is *everywhere stable* if its Lyapunov sum is harmonic on the whole parameter curve. A regular polynomial map of degree d is a polynomial map $F : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ whose leading

homogeneous pair Φ has no common projective zero; equivalently, its extension to $\mathbb{P}^2$ is a morphism. We write $g = [\Phi_1 : \Phi_2]$ for its restriction to infinity.

Theorem 4.1 (Regular polynomial families). *Let $(F_s)_{s \in S}$ be an everywhere stable algebraic family of regular polynomial maps of degree $d \geq 2$. After finite algebraic base change, either the family is affinely isotrivial, or*

$$F_s(X) = c_s + \Phi_s(X - c_s),$$

where $g_s = [\Phi_{1,s} : \Phi_{2,s}]$ is a nonisotrivial flexible Lattès family. The center is algebraic and is regular wherever the original family is regular. Coordinate presentations and trivializations may be taken on a nonempty algebraic open subset.

We first recall the rigidity results that enter the proof. By McMullen’s algebraic-stability theorem, an everywhere stable algebraic family of rational maps is isotrivial or consists of flexible Lattès maps [4]. An algebraic stable polynomial family therefore is isotrivial. We will also use Gauthier–Vigny’s theorem that, for a nonisotrivial polarized endomorphism over a *fixed* complex function field, all rational points of canonical height zero are mapped by one fixed iterate into one proper closed subset [45, Theorem A and Corollary 1.5]. Finally, Xie proves that a regular polynomial plane map over an algebraically closed characteristic-zero field, with no exceptional point at infinity and infinitely many periodic curves, is homogeneous after a translation [28, Theorem 6.2]. The condition on the map at infinity is Xie’s *NP-type* condition, and we will verify it before applying the theorem.

To apply McMullen’s spectral formulation, we explain why analytic stability on an algebraic curve makes the periodic multipliers constant. A periodic multiplier is a meromorphic function on the compactification of a finite periodic-point marking curve. A pole would give a repelling sheet near that puncture. Stability continues that sheet as repelling along every path; it cannot collide, since its multiplier cannot equal one. Its reciprocal is then bounded, extends to the complete marking curve, and is constant, contradicting the pole. All multiplier functions are consequently pole-free and constant. The compactification of the parameter curve is essential here; stability on a small analytic disk alone does not give this conclusion.

Our proof uses critical heights to pass from stability to algebraic rigidity. Bedford–Jonsson developed the relation between the dynamics at infinity, critical escape, Lyapunov exponents, and external geometry for regular polynomial maps [26]. Ingram proved rigidity and height estimates for important restricted classes, including maps with a power-map restriction at infinity and minimally critical regular maps [55, 56]. We will prove a coercivity estimate for the general regular polynomial family considered here. The interaction between critical heights and bifurcation in higher dimension is central to Gauthier–Taflin–Vigny’s work on the sparsity of postcritically finite maps [57].

4.1 Additive divisor escape and its specialization

We begin by defining an escape rate for divisors that keeps track of cycle multiplicities. For an effective affine divisor $E = \operatorname{div} P$, write P_r for its highest homogeneous part, $r = \deg P$, and set

$$\lambda(E) = \log \|P\| - \log \|P_r\|.$$

Here and below, we use coefficient maximum norms and include cycle multiplicities in every pushforward of divisors. Over a complete non-Archimedean field of residue characteristic zero, we put

$$\Delta_F(E) = \lim_{n \rightarrow \infty} d^{-2n} \lambda(F_*^n E).$$

The same definition over $\mathbb{C}$ uses the usual coefficient norms.

Lemma 4.2 (Escape, normalization, and slope). *The limit exists, is nonnegative and additive, and*

$$\Delta_F(F_*E) = d^2 \Delta_F(E), \quad \left| \Delta_F(E) - d^{-2n} \lambda(F_*^n E) \right| \leq C_F r d^{-n}.$$

The escape is invariant under affine conjugacy, with the divisor transported by that conjugacy. For a meromorphic complex degeneration at $t = 0$, the constant can be chosen $C_{F_t} = O(1 + \log |t|^{-1})$, and, with $|t|_v = e^{-1}$,

$$\lim_{t \rightarrow 0} \frac{\Delta_{F_t}(E_t)}{\log |t|^{-1}} = \Delta_{F_v}(E_v).$$

For the affine critical divisor $C_F = \operatorname{div} \det DF$,

$$\Delta_F(C_F) = L(F) - L(g) - \log d \quad \text{over } \mathbb{C}. \quad (4.1)$$

Proof. We first express the escape rate in terms of norms of homogeneous forms, keeping track of the multiplicities introduced by homogenization. Put $\mathcal{F}(X, Z) = (F_1^h, F_2^h, Z^d)$, and homogenize P to $\tilde{P}$. For a homogeneous finite map A in q variables let $A_*Q = \operatorname{Norm}_A(Q)$. Ingram's form escape theorem [27, Lemmas 8–9] supplies

$$G_A(Q) = \lim_{n \rightarrow \infty} d^{-qn} m(A_*^n Q), \quad |G_A(Q) - m(Q)| \leq B_A \deg Q.$$

Here m is logarithmic Gauss norm non-Archimedeanly and logarithmic Mahler measure over $\mathbb{C}$; the latter differs from logarithmic coefficient norm by $O(\deg Q)$. The constants B_A are bounded by fixed linear combinations of coefficient and resultant logarithms, hence have the asserted meromorphic growth. The same theorem gives additivity and $G_A(cQ) = G_A(Q) + \log |c|$.

If $R_n = \mathcal{F}_*^n \tilde{P}$, then

$$\operatorname{div}_{\mathbb{P}^2}(R_n) = d^n F_*^n E, \quad R_n(X, 0) = (\Phi_*^n P_r)^{d^n}.$$

In the first identity, we identify the affine divisor with its projective closure. To prove the second identity, we restrict the finite free source algebra of the norm to the target $Z = 0$. Its filtration by powers of the source Z has d^n successive quotients, each acted on by Φ^n . Taking determinants along that filtration gives the displayed power. Consequently, we obtain

$$\Delta_F(E) = G_{\mathcal{F}}(\tilde{P}) - G_{\Phi}(P_r).$$

This proves existence and additivity. Nonnegativity follows from the coefficient-norm approximants, and we obtain the error estimate by applying the form estimate to $F_*^n E$, whose degree is $d^n r$.

We next check affine invariance. For a fixed affine coordinate change and its inverse, substitution changes the logarithmic coefficient norm of a degree- r polynomial by at most a constant times r . The same holds for its top part, on which only the linear part of the coordinate change acts. Thus it changes $\lambda(E)$ by $O(r)$. Under n pushforwards this is $O(d^n r)$, which disappears after division by d^{2n} and proves affine invariance.

To prove the specialization formula, fix n . Every coefficient of R_n is meromorphic, and its coefficient logarithm divided by $\log |t|^{-1}$ tends to the non-Archimedean coefficient logarithm. We may therefore apply the uniform error estimate, let $t \rightarrow 0$ at fixed n , and then let $n \rightarrow \infty$. This gives the asserted slope.

It remains to identify the critical escape over the complex numbers. We have $J_{\mathcal{F}} = dZ^{d-1} \tilde{J}_F$ and $G_{\mathcal{F}}(Z) = 0$. The homogeneous Jacobian formula [27, Remark 4] gives

$$L(F) = G_{\mathcal{F}}(J_{\mathcal{F}}) - \log d = G_{\mathcal{F}}(\tilde{J}_F), \quad L(g) = G_{\Phi}(J_{\Phi}) - \log d.$$

Subtracting these identities proves (4.1), which is also the relative formula of Bedford–Jonsson [26]. $\square$

The norm calculation above gives the slope directly for regular polynomial maps. Favre's hybrid-space approach treats degenerating endomorphisms and Lyapunov asymptotics [58]. Variation of canonical heights for subvarieties is treated more generally by Gauthier–Vigny [59]. We use the additive normalization on cycles throughout, so the degree of a divisor remains part of its escape rate.

Lemma 4.3 (All-place consequence of stability). *For an everywhere stable algebraic regular polynomial family, $L(F_s)$ and $L(g_s)$ are constant, g_s is stable, and $\Delta_{F,v}(C_F) = 0$ at every place of its function field. These conclusions remain valid after finite base change and algebraic affine conjugacy on an algebraic open.*

Proof. We first show that the Lyapunov sums are constant. The negative of the bounded-below harmonic function $L(F_s)$ is subharmonic and bounded above. It extends across the finitely many punctures of the smooth projective compactification, hence is constant. By (4.1) and nonnegativity of Δ , the subharmonic function $L(g_s)$ is bounded above. The same compactification argument makes it constant. The difference in (4.1) is constant, so its logarithmic slope is zero at every puncture. Lemma 4.2 gives the asserted non-Archimedean vanishing. At ordinary parameters, we apply the same argument to the holomorphic germ. Finally, conjugacy preserves Lyapunov sums, and deleting finitely many parameters introduces only more meromorphic germs to which the identical argument applies. $\square$

4.2 Good infinity and translation centering

We next consider the case in which the map at infinity has good reduction. The main point is to choose a translation for which a nonintegral lower coefficient forces positive critical escape. We begin with an algebraic observation about conical branch divisors.

Lemma 4.4 (Conical branch rigidity). *Let $R : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be regular of degree d over an algebraically closed characteristic-zero field, with leading pair Ψ . If its reduced affine branch divisor is a cone about zero, then $R(X) = \Psi(X - c)$ for some c .*

Proof. We first show that a regular vector field tangent to the branch divisor of a finite morphism between smooth varieties has a regular lift. Indeed its rational derivation extends uniquely through the separable function-field extension. At a generic ramification divisor, tame local coordinates give $y_1 = z_1^e$, $y_2 = z_2$ after a faithfully flat local extension. A field $y_1 a_1 \partial_{y_1} + a_2 \partial_{y_2}$ lifts to

$$\frac{z_1}{e}(a_1 \circ R)\partial_{z_1} + (a_2 \circ R)\partial_{z_2},$$

which has no pole. Regularity in codimension one implies regularity everywhere, since the tangent sheaf on a smooth variety is reflexive.

We apply this observation to the Euler field on the target $\mathbb{P}^2$. This field is tangent to the conical affine branch divisor and to infinity, and vanishes on infinity. Its lift V is tangent to source infinity; the generically étale restriction of R to that line forces V to vanish there. A global projective vector field vanishing pointwise on infinity has affine form $V = (aX + b) \cdot \partial_X$, with scalar a . Hence $DR(X)(aX + b) = R(X)$. Comparing leading terms gives $a = 1/d$; with $c = -db$,

$$DR(X)(X - c) = dR(X).$$

By Euler's identity, $R(X + c)$ is homogeneous, with leading part Ψ , as desired. $\square$

Lemma 4.5 (Local centered coercivity). *Suppose F is defined over a completed algebraic closure of a finite extension of $\mathbb{C}((t))$, and Φ has integral coefficients and unit resultant. There is a*

translation conjugate G with

$$\Delta_F(C_F) = \Delta_G(C_G) \geq \frac{d-1}{d^2} B(G), \quad B(G) = \max_{i, |I| < d} \frac{\log^+ |g_{i,I}|}{d - |I|}.$$

In particular zero critical escape makes G a good-reduction map.

Proof. We first choose the translation. The reduction of $c \mapsto D\Phi(X)c$ is injective: otherwise, after a linear change, both leading forms would be independent of one variable and would have a common projective zero. We may therefore choose two coefficient rows giving a unit 2×2 minor and hence an integral left inverse ℓ to that map. Translation conjugacy changes the degree- $(d-1)$ layer by $D\Phi(X)c$, so we choose the unique translation for which $\ell(G_{d-1}) = 0$.

To estimate escape in these coordinates, let $E = \operatorname{div} P$ have degree r . We normalize $\|P_r\| = 1$ and define

$$M_E(s) = \max_{k \leq r} \{\log \|P_k\| - (r - k)s\}.$$

For $s \geq B(G)$, we choose $|a| = e^s$. The map $G_s(X) = a^{-d}G(aX)$ then has good reduction. Reduction of a finite-map norm is the norm of the reduction; since the latter is a finite dominant map,

$$\|\operatorname{Norm}_{G_s}(A)\| = \|A\|^{d^2}.$$

We can check this identity over a finite discretely valued extension and extend it to all radii by continuity. If $Q = \operatorname{Norm}_G(P)$, then

$$Q(a^d Y) = a^{rd^2} \operatorname{Norm}_{G_s}(a^{-r} P(aX))(Y),$$

so $M_{G_*E}(ds) = d^2 M_E(s)$. Iteration and $M_E(0) \geq M_E(s)$ give $\Delta_G(E) \geq M_E(B(G))$.

We now suppose $B = B(G) > 0$ and put $R(X) = a^{-d}G(aX)$ for $|a| = e^B$. Its reduction is regular and has a nonzero lower term. We claim that its branch divisor is not conical. Indeed, Lemma 4.4 would give $\bar{R} = \bar{\Phi}(X - c)$, whose degree- $(d-1)$ layer and the reduced left inverse force $c = 0$, a contradiction. Thus the residual norm $\operatorname{Norm}_{\bar{R}}(\det D\bar{R})$ has a nonzero lower homogeneous term of some degree $k < M = 2d(d-1)$. Both the top norm and this lower coefficient norm are one. Writing $B_G = \operatorname{Norm}_G(\det DG)$, derivative scaling gives

$$B_G(a^d Y) = a^{dM} B_R(Y), \quad \log \|(B_G)_k\| = d(M - k)B.$$

Therefore

$$d^2 \Delta_G(C_G) = \Delta_G(G_* C_G) \geq (d-1)(M - k)B \geq (d-1)B.$$

Finally, we check that the translation has not changed the escape rate. Translation changes $\lambda(E)$ by at most $\deg(E) \log \max(1, \|c\|)$. Under n pushforwards the degree is $d^n \deg(E)$, so its normalized effect tends to zero. This proves translation invariance of Δ and finishes the proof. $\square$

Proposition 4.6 (Isotrivial infinity). *An everywhere stable algebraic regular polynomial family whose infinity family is isotrivial is affinely isotrivial.*

Proof. After a finite base change, we may use a linear conjugacy to make the leading pair a scalar multiple of one constant pair. By extracting a $(d-1)$ st root and applying a scalar conjugacy, we may make it exactly constant. We choose the coefficient left inverse ℓ in Lemma 4.5 over $\mathbb{C}$ and use it to center the family globally. The chosen minor is a unit at every function-field place. By Lemma 4.3 and local coercivity every centered coefficient is integral at every place. It is therefore a regular function on the complete parameter curve, hence constant. $\square$

The passage from potential good reduction at every place to isotriviality is also the content of the general criterion of Petsche–Szpiro–Tepper [46]. Here our common choice of center gives affine isotriviality. By Lemma 4.3 and McMullen’s theorem, it remains to consider nonisotrivial flexible Lattès infinity. In this case, we will use vanishing critical escape to control divisor profiles on the Tate skeleton; we do not require it to determine the full non-Archimedean Green measure.

4.3 Tate profiles and finite annuli

In this subsection, all non-Archimedean absolute values have residue characteristic zero. We carry out the local analytic constructions over completed algebraic closures and then descend the formal coefficients to a single finite function-field extension. We write the homogeneous Green gauge as

$$g_\Phi(X) = \lim_{n \rightarrow \infty} d^{-n} \log \|\Phi^n(X)\| \quad \text{on } \mathbb{A}^2 \setminus \{0\};$$

its truncation is the usual escape function $G_\Phi = \max(0, g_\Phi)$. For a nonhomogeneous regular map the usual escape function is $G_F = \lim_{n \rightarrow \infty} d^{-n} \log^+ \|F^n\|$.

We will use Tate uniformization [60] and the interval model for non-Archimedean Lattès maps developed by Favre–Rivera-Letelier [61, §5.1]. Explicit interval measures and quadratic logarithmic potentials for Legendre Lattès maps appear in DeMarco–Krieger–Ye [62, §3.1, Propositions 3.2–3.4]. We first compute the homogeneous normalization that will enter our divisor profiles.

Lemma 4.7 (Homogeneous Tate geometry). *Suppose the infinity map of Φ is the involution quotient of a Tate elliptic map $[m] + b$, $b \in E[2]$, and $d = m^2$. After a linear change there are $a, b_0 \in \mathbb{R}$ and $l > 0$ such that, with logarithmic radii,*

$$\gamma(u) = (a, b_0) + l(-u^2/2, u - u^2/2), \quad 0 \leq u \leq 1,$$

the homogeneous filled set is the union of the centered polydiscs D_u with these radii, and the homogeneous Green-zero Gauss points form this arc. The map on the arc is an m -branch tent map T , possibly endpoint flipped. Off finitely many vertices, each coordinate has one dominant degree- d monomial, and the determinant of the two exponent rows has absolute value dm .

Proof. By Tate uniformization [60], we may identify the elliptic skeleton with a circle, on which $[m] + b$ is multiplication by m followed possibly by a half-period translation. Quotienting by inversion gives the interval tent map. At an interior skeleton point the local degree is m and there are m inverse branches. Normalized length is therefore a balanced measure of degree m^2 . Its continuous potential makes it the canonical projective measure; compare [61, §5.1].

We choose a projective coordinate so that the interval consists of Gauss points with logarithmic coordinate radius $r \in [r_0, r_0 + l]$, and write the homogeneous gauge as $\log |x| + \varphi(\log |y/x|)$. The potential of uniform length has derivative zero before the interval, derivative $(r - r_0)/l$ inside it, and derivative one after it. Thus inside the interval $\varphi(r) = c + (r - r_0)^2/(2l)$. The uniqueness of a continuous potential up to a constant, and the homogeneity functional equation, identify this with the actual gauge. Its zero section gives γ above; the constant and coordinate scaling are absorbed into a, b_0 . The zero sublevel is exactly the union of the indicated polydiscs.

To describe the monomials along the arc, we note that the image of a Green-zero point is Green-zero, and its projective skeleton coordinate is $T(u)$. Hence its image is the corresponding Gauss point. Along an open tent branch the homogeneous coordinate norms are maxima of finitely many affine functions in the projective logarithmic radius. Their nonidentical intersections are finite, so the dominant monomials are unique away from finitely many parameters. Their row sums are d ; their difference acts on the projective radius with slope $\pm m$. Their determinant is therefore $\pm dm$. $\square$

We now allow lower terms in the map. For $F = \Phi +$ lower terms, we define a finite number $\tau \geq 0$ by requiring every lower monomial on D_u to be bounded by the corresponding output radius of $D_{T(u)}$, after weighting its coefficient by $e^{-(d-k)\tau}$ for its degree $k < d$. Explicitly,

$$\tau = \max \left(0, \max_{\substack{i, I, |I| < d \\ u \in [0, 1]}} \frac{\log |c_{i, I}| + \langle I, \gamma(u) \rangle - \gamma_i(Tu)}{d - |I|} \right).$$

For $E = \text{div } P$, $r = \deg P$, we set

$$M_E(s) = \int_0^1 \left(\log |P|_{\zeta_{\gamma(u)+s(1,1)}} - \log |P_r|_{\zeta_{\gamma(u)+s(1,1)}} \right) du.$$

Here ζ_ρ denotes the Gauss point of the centered polydisc with logarithmic polyradius ρ .

Lemma 4.8 (Averaged profile bound). *For $s > \tau$, $M_{F_*E}(ds) = d^2 M_E(s)$, and $\Delta_F(E) \geq M_E(\tau)$. If $\Delta_F(C_F) = 0$, every lower homogeneous part of every equation for $F_*^j C_F$ is bounded by its top part at every point $\zeta_{\gamma(u)+\tau(1,1)}$.*

Proof. Let ν_s be the pushforward of du to the shifted arc. We first establish the transport formula for this measure. When $s > \tau$, strict lower-term dominance leaves the initial monomial maps of Lemma 4.7 unchanged. A strict invertible monomial initial map has analytic local degree equal to the lattice index, the absolute exponent determinant: this follows by Hensel lifting the initial torus isogeny, or by the corresponding graded residue-field computation. Residue characteristic zero precludes a defect factor. A generic target arc point has m such preimages of degree dm . They exhaust the total degree d^2 , and changing variables on the m branches of slope m gives

$$F^* \nu_{ds} = d^2 \nu_s.$$

The excluded vertices have measure zero.

We now apply this transport formula to the polynomial norm $Q = \text{Norm}_F(P)$. It defines F_*E , has degree dr , and has top part $Q_{dr} = \text{Norm}_\Phi(P_r)$. The last identity follows by homogenizing the finite flat family $z^d F(X/z)$ and specializing to $z = 0$. At an analytic point, the logarithm of the norm is the weighted sum of the source logarithms. We integrate this identity against ν_{ds} and subtract the corresponding identity for Φ, P_r . This gives the asserted transport formula for the profiles.

To deduce the lower bound for escape, we observe that the integrand of M_E is the maximum of zero and finitely many affine functions of s of nonpositive slope. It is therefore nonnegative, continuous, and nonincreasing. If $C = \max_{u,i} |\gamma_i(u)|$, comparison with the ordinary Gauss norm gives

$$0 \leq M_E(s) \leq M_E(0) \leq \lambda(E) + 2Cr.$$

Iterating transport and using $\deg F_*^n E = d^n r$ yields

$$M_E(s) \leq d^{-2n} \lambda(F_*^n E) + 2Crd^{-n}.$$

Letting $n \rightarrow \infty$ and then $s \downarrow \tau$ gives the desired bound. For the last assertion, escape functoriality gives zero escape for every $F_*^j C_F$. The corresponding integrand is nonnegative and continuous with zero integral, so it vanishes at every u . This proves the asserted domination. $\square$

We wish to turn this control of critical divisors into inverse branches near infinity. For this purpose, we choose periodic directions whose reductions avoid the branch directions at every place.

Lemma 4.9 (Safe directions and finite inverse annuli). *Let $K = \mathbb{C}(S)$ and suppose F has flexible Lattès infinity and zero critical escape at every place. There are infinitely many periodic directions ξ with the following property. For each one choose a finite extension L/K , a period q , and $a \in L^2 \setminus \{0\}$ such that $[a] = \xi$ and $\Phi^q(a) = a$. Put $H = F^q$, $D = d^q$. At each place v of L there is a finite $\tau_v \geq 0$ and, for all $n \geq 1$, a parametrized analytic annulus*

$$H^n(h_{n,v}(w)) = aw^{D^n}, \quad h_{n,v}(w) = aw + O(1), \quad |w|_v > e^{\tau_v/D^n}.$$

It satisfies

$$G_{F,v}(h_{n,v}(w)) = \log |w|_v, \quad Hh_{n,v}(w) = h_{n-1,v}(w^D), \quad h_{0,v}(w) = aw.$$

Proof. After a finite base change, we may present the map at infinity by $[m] + b$ with $b \in E[2]$. If m is odd, its square removes b ; if m is even, translation by b conjugates it to $[m]$. The latter descends to a projective coordinate change on infinity. Since our aim is to construct periodic curves, we may make this fixed replacement. We choose an odd prime $N \nmid m$ and a nonzero point $P \in E[N]$. Its involution quotient is periodic and avoids the orbifold branch set. After marking the direction, we scale a representative by a $(d^q - 1)$ st root so that $\Phi^q(a) = a$. Both choices can be made over one finite extension. There are infinitely many different such directions as N varies.

We make a semistable base change for the elliptic family and examine the chosen torsion point at each place. At a Tate place, we can write

$$P = \zeta_N^r q_T^{s/N}, \quad 0 \leq r, s < N.$$

If $s \neq 0$, its quotient skeleton point is interior, since N is odd. If $s = 0$, then $r \neq 0$, and on the endpoint component the involution quotient is $u \mapsto u + u^{-1}$. The value of ζ_N^r avoids the two branch values ± 2 and the node at infinity, which contains the directions of the other two two-torsion branch points. Thus the selected direction avoids every reduced branch direction, even at an endpoint. A retained two-torsion translation only exchanges endpoints or signs and changes nothing in this assertion.

Let Q_j define $F_*^j C_F$, $j \geq 1$, and let $r_j = \deg Q_j$. Its top homogeneous part defines $\Phi_*^j C_\Phi$, whose directions are postcritical branch directions. Since the chosen direction avoids these branch directions, evaluation at a attains the boundary Gauss norm of the top homogeneous part. Also $g_\Phi(a) = 0$, by $\Phi^q(a) = a$. Lemma 4.8 gives, for $|z| > e^{\tau_v}$ and $k < r_j$,

$$\frac{|(Q_j)_k(az)|}{|(Q_j)_{r_j}(az)|} \leq e^{-(r_j-k)(\log |z| - \tau_v)} < 1.$$

It follows that the target radial annulus az avoids every forward critical image.

We also have $G_F = G_\Phi$ when $G_\Phi > \tau_v$. To see this, let p lie on the boundary of $e^s K_\Phi$, $s > \tau_v$, and choose $e^s D_u$ containing it. Its leading image is on the boundary of $e^{ds} K_\Phi$ inside $e^{ds} D_{T(u)}$. Lower terms are strictly smaller than both coordinate bounds of this latter polydisc. At an interior parameter, the boundary requires both bounds to be attained; at an endpoint it requires the active coordinate bound. These attained norms cannot change under the strict perturbation. Hence $G_\Phi(Fp) = dG_\Phi(p)$, and iteration plus bounded comparison with coordinate norm gives the assertion. On the target ray it gives $G_F(az) = \log |z|$.

We next consider a place of good elliptic reduction. Here odd prime-to-characteristic torsion specializes injectively and avoids the reduced branch locus. To normalize the leading pair, we first choose a projective coordinate in which the elliptic quotient has good reduction. We then adjust the scalar of its homogeneous lift by a scalar linear conjugacy, extracting a $(d-1)$ st root if necessary. Affine invariance preserves zero escape, so Lemma 4.5 allows us to make

the whole map good by a translation. For an escaping critical point, the reduced direction is critical for the reduced leading map. Indeed, we may scale the point by its norm and reduce the leading Jacobian. Forward escaping critical values therefore have reduced postcritical directions. Bounded critical points stay bounded. The chosen ray avoids those directions. In the original coordinates an affine change translates the ray, but increasing the finite threshold dominates that fixed translation. Its direction and Green value are then unchanged. We have therefore proved both assertions about the target annulus at good places as well.

We can now construct the inverse annuli. The pullback of the target annulus by H^n is finite étale of degree D^{2n} . Its projective closure is a curve of degree D^n . It meets infinity at the D^n distinct inverse directions of ξ ; they are simple because ξ is outside the global postcritical set. The curve is smooth and transverse there, and the radial ramification index at every such end is D^n .

Over the completed algebraic closure, every connected finite étale cover of an annulus in residue characteristic zero is Kummer, including open annuli and punctured disks. This is the tame annulus theorem of Berkovich [63, Theorem 6.3.5]; the extension from compact subannuli is recalled in the proof of [64, Proposition 2.21]. Residue characteristic zero ensures tameness. Each component has one end over target infinity. Thus there are exactly D^n components, each of degree D^n . On the component with end ξ , we choose the Kummer coordinate so that the target radial coordinate is w^{D^n} and the leading vector is a . The normalization $h_n(w) = aw + O(1)$ removes the residual root-of-unity ambiguity and makes this choice unique. The Green identity follows from $G_F H^n = D^n G_F$; it determines the lower radius e^{τ_v/D^n} . The same normalization at infinity proves the compatibility with H . $\square$

The finite inverse covers we have constructed are closely related to non-Archimedean Böttcher coordinates. In one variable, the Kummer interpretation and behavior of these coordinates in families are studied by Ingram [65] and DeMarco–Ghioca–Krieger–Nguyen–Tucker–Ye [66, §6]. For the higher-dimensional complex external geometry, see Bedford–Jonsson [26, §§4,7]. We will use the radii of our inverse annuli to bound all coefficients of the resulting formal parametrization by a single pole divisor.

4.4 Uniform coefficients, heights, and periodic curves

The next two lemmas pass from the local annuli to a global algebraic curve. We first show that the coefficients of the formal branch belong to one Riemann–Roch space. We then use the resulting bound on heights to prove algebraicity unless the map is isotrivial.

Lemma 4.10 (One pole divisor for all formal coefficients). *For the data of Lemma 4.9 there is a unique formal series*

$$h(w) = aw + b_0 + \sum_{k \geq 1} b_{-k} w^{-k} \in L((w^{-1}))^2, \quad Hh(w) = h(w^D),$$

whose coefficient coordinates all have poles bounded by one effective divisor on the complete curve of L .

Proof. We first construct the formal solution. The derivative of the homogeneous leading part $\Psi = \Phi^q$ at a is invertible: its radial eigenvalue is D and its transverse derivative is the nonzero multiplier of the noncritical periodic direction. Formal implicit inversion of $H^n(h_n) = aw^{D^n}$ therefore determines every coefficient of $h_n = aw + O(1)$ in the same field L . At every place the analytic annulus of Lemma 4.9 has precisely this formal expansion. Thus the coefficients descend to the chosen function field, although we used a completed algebraic closure to construct the local annuli.

To compare successive expansions, let $e_n \leq 0$ be the highest exponent of a nonzero difference $h_n - h_{n-1}$. Comparing the leading terms in

$$Hh_n - Hh_{n-1} = (h_{n-1} - h_{n-2})(w^D)$$

gives

$$e_n + D - 1 = De_{n-1}.$$

The leading coefficient on the left is multiplied by the invertible matrix $D\Psi(a)$; all higher Taylor terms have smaller exponents. Since $e_1 \leq 0$, we obtain $e_n \leq 1 - D^{n-1}$. A zero difference remains zero by formal uniqueness. Thus every coefficient stabilizes exactly, giving h and its functional equation. If there were two distinct solutions, the same comparison would give $e + D - 1 = De$ with $e \leq 0$, which is impossible. This proves uniqueness.

It remains to bound the poles of the coefficients. By regularity, there are constants $C_v \geq 1$, equal to one outside finitely many places in the original coordinates, such that

$$\|z\|_v \leq C_v e^{G_{F,v}(z)}$$

over all complete valued extensions. This follows from the usual resultant lower bound for the leading pair and the escaping-radius estimate. On a Gauss circle of radius $r > e^{\tau_v/D^n}$, the annulus Green identity gives $\|h_{n,v}\| \leq C_v r$. Hence

$$|b_{-k,n}|_v \leq C_v r^{k+1}.$$

We fix k and take n beyond its stabilization index. We may then decrease r to the annulus threshold and let $n \rightarrow \infty$. We obtain $|b_{-k}|_v \leq C_v$, independently of k ; the same calculation includes the constant coefficient. The leading vector satisfies the same bound. Rounding the finitely many local bounds gives the required effective pole divisor. The bounds hold coefficient by coefficient; they require neither a limiting analytic leaf nor convergence at $|w| = 1$. $\square$

Lemma 4.11 (Finite coefficient span). *Let H be regular over a fixed complex function field L . Suppose a nonconstant formal solution $Hh(w) = h(w^D)$ has all coefficient coordinates in one finite-dimensional complex subspace of L . Then either H is projectively isotrivial, or the formal image of h lies in an irreducible H -invariant algebraic curve over L .*

Proof. We choose a complex basis $f_1, \dots, f_s$ of the coefficient span and write $h_j = \sum_i f_i H_{ij}$, with $H_{ij} \in \mathbb{C}((w^{-1}))$. Let $A = \mathbb{C}[H_{ij}]$, which we view as a finitely generated subalgebra of the Laurent-series field. We claim that substitution $\sigma : w \mapsto w^D$ preserves A . Indeed, in the identity

$$\sum_i f_i \sigma(H_{ij}) = H_j \left(\sum_i f_i H_{i1}, \sum_i f_i H_{i2} \right),$$

we apply complex-linear dual functionals of the f_i , extended to L . The right side belongs to $L \otimes_{\mathbb{C}} A$, and coefficient extraction gives polynomials in the generators with complex coefficients. The map

$$L \otimes_{\mathbb{C}} A \longrightarrow L((w^{-1}))$$

is injective. To see this, we expand a finite relation in complex-linearly independent elements of L ; every Laurent coefficient must then vanish separately.

Thus σ is an injective endomorphism of A , giving a dominant morphism $R : X = \operatorname{Spec} A \rightarrow X$ over $\mathbb{C}$. There is an L -morphism $\pi : X_L \rightarrow \mathbb{A}_L^2$ with coordinates $\sum_i f_i H_{ij}$ and $H\pi = \pi R$. For every $x \in X(\mathbb{C})$, all iterates of $\pi(x)$ have coordinates in the same Riemann–Roch space. Their pole divisors, and hence their projective Weil heights, are uniformly bounded. It follows that their canonical heights vanish.

If π were dominant, these rational height-zero points would be Zariski dense over L . Indeed, complex points remain dense after base extension. To check this, we expand a proposed equation in finitely many complex-linearly independent coefficients of L . Each resulting equation over $\mathbb{C}$ vanishes on $X(\mathbb{C})$ and is zero. By Gauthier–Vigny [45, Theorem A and Corollary 1.5], this density is impossible for a nonisotrivial polarized map: the inverse image under the fixed finite iterate in their theorem of the corresponding proper closed set is still proper. Therefore the image closure has dimension at most one. It has positive dimension because substitution recovers the nonconstant formal series. It is irreducible, and dominance of R makes it H -invariant. $\square$

Proof of Theorem 4.1. By Proposition 4.6, we may assume that the infinity family is nonisotrivial and flexible Lattès. We apply Lemmas 4.9–4.11 separately to infinitely many safe torsion directions. We first exclude the isotrivial outcome for $H = F^q$. Indeed, after conjugacy to a constant plane map, its infinity line becomes a totally invariant line of that map. There are at most three such lines, since each occurs in the critical divisor with multiplicity $D - 1$ and the critical divisor has degree $3(D - 1)$. The chosen line therefore becomes constant after a finite marking. Its restriction would be isotrivial, whereas the elliptic modulus of the flexible Lattès infinity family remains nonconstant under iteration.

Thus each selected direction belongs to an invariant curve for an iterate of F . A curve has only finitely many points at infinity, so there are infinitely many distinct periodic curves. Although their fields of definition and periods may vary, we may apply Xie’s theorem over the algebraic closure. Flexible Lattès maps have no exceptional point, so [28, Theorem 6.2] gives $F(X) = c + \Phi(X - c)$. If a fixed preliminary iterate was used to remove a two-torsion translation, its periodic curves are also periodic for the original F , so the theorem still applies to the original map. Xie’s construction of invariant analytic germs at infinity [28, §5] is also relevant background to the passage from formal branches to periodic curves here.

It remains to check the assertion about the center. Two centers would give $D\Phi(X)(c - c') = 0$ from the degree- $(d - 1)$ layer, contradicting regularity unless $c = c'$. Thus the center is unique and descends to the original function field. At any ordinary parameter the same layer and an invertible two-row minor of $c \mapsto D\Phi(X)c$ recover it holomorphically. It has no pole there. $\square$

5 Families preserving a pencil of lines

We now consider a stable family preserving a pencil of lines. Restriction to the lines gives polynomial dynamics on the fibers, while the pencil itself carries a rational map. We will use the stability of periodic fiber returns to recover the coefficients of the plane map.

Theorem 5.1 (Families preserving a marked pencil). *Let (f_s) be an everywhere stable algebraic family of degree- d endomorphisms of $\mathbb{P}^2$ preserving a pencil of lines, marked algebraically after finite base change. Then, after finite base change and projective conjugacy on a nonempty algebraic open, either the family is constant or*

$$f_s[x : y : z] = [P_s(x, y) : Q_s(x, y) : z^d],$$

with nonisotrivial flexible Lattès base $g_s = [P_s : Q_s]$.

The relation between the base and fiber dynamics is well understood for regular polynomial skew products. Jonsson constructed the equilibrium disintegration and studied the fiber dynamics in this setting [67]. Dupont–Taflin extend the disintegration and the relative Lyapunov formula to fibered endomorphisms of projective space [68]. Astorg–Bianchi studied stability and bifurcation in families of polynomial skew products, including approximation through periodic fibers [69]. We will apply the linear-fibration results of [68] to the pencil and then use algebraic rigidity for the return polynomials.

5.1 Normal form, support, and periodic returns

We begin by choosing coordinates with pencil center $o = [0 : 0 : 1]$. Since the map preserves the pencil, it has the form

$$f = [P(x, y) : Q(x, y) : R(x, y, z)], \quad g = [P : Q].$$

The binary forms P, Q have no common projective zero: otherwise their common source line would meet the degree- d divisor $R = 0$, creating a base point. The coefficient of z^d is also nonzero, so o is totally invariant and superattracting. We scale z by a $(d - 1)$ st root to make that coefficient one, and then use a shear by a linear form to remove the coefficient of z^{d-1} . The output correction of the shear changes only the constant coefficient in z . Thus, over one function field, we obtain the normalization

$$R(x, y, z) = z^d + \sum_{j=0}^{d-2} a_j(x, y)z^j, \quad \deg a_j = d - j. \quad (5.1)$$

We also check that the pencil center can be marked algebraically. For a center c , let W_c be the two-dimensional space of linear forms vanishing at c . The condition $f^*W_c \subset \text{Sym}^d W_c$ is closed in (f, c) . Its projection is proper because the center space is projective. Every center is fixed. A degree- d polarized map has no positive-dimensional fixed locus: on a fixed curve, the ample line bundle would have both degrees $d \deg L$ and $\deg L$. Thus the incidence has finite fibers. A generic center is therefore defined over a finite function-field extension.

Lemma 5.2 (Base and periodic-fiber stability). *The rational base g_s is stable. If a repelling base cycle is marked, its polynomial return on the corresponding line is stable.*

Proof. We first prove stability of the base. Dupont–Taflin show that the difference of the plane bifurcation current and the rational-base bifurcation current is positive [68, Theorem 1.8]. Thus the two positive currents $T_{\text{Bif}}(g)$ and $T_{\text{Bif}}(f) - T_{\text{Bif}}(g)$ have sum $T_{\text{Bif}}(f) = 0$; both vanish. In particular the base current vanishes.

To study a periodic fiber, fix one map and write L_a for the pencil line of direction a , and $\mu_a = T_f \wedge [L_a]$. We claim that $a \mapsto \mu_a$ is weakly continuous. We may choose smooth local parametrizations of the lines and pull back a continuous local potential of the Green current. Uniform convergence of these potentials, followed by integration by parts, gives the claim. Dupont–Taflin’s disintegration formula [68, Theorem 1.1 and Corollary 1.2] gives

$$\mu_f = \int \mu_a d\mu_g(a).$$

Therefore

$$a \in J_g \implies \text{supp } \mu_a \subset J_f.$$

Indeed, a test function of positive μ_a -integral has positive integral on a neighborhood of a , and that neighborhood has positive μ_g -mass. If a is periodic, restricting the Green-current defining sequence to L_a makes μ_a exactly the equilibrium measure of its polynomial return. Thus weak continuity allows us to use the disintegration at each repelling periodic direction, although the integral formula initially gives a statement only almost everywhere.

Every repelling periodic point of that return over a repelling base cycle is consequently an ambient repelling point in the small Julia set: the two return eigenvalues are the vertical and base multipliers. The Berteloot–Bianchi–Dupont theorem moves every ambient repelling Julia cycle holomorphically in a stable family [9, Theorem 1.1]; see also the repelling-cycle persistence statement in [11]. In dimension two, this theorem applies over every simply connected parameter

manifold, as explained immediately after its statement in [9]. Its projection remains the selected base cycle by uniqueness of repelling continuation and the identity theorem. It never reaches o , which is superattracting. The vertical multiplier therefore remains repelling. We have therefore obtained holomorphic motion of every repelling periodic point of the return. By the one-variable stability criterion, the return is stable.

We now pass to the finite algebraic marking curve of the base cycle and algebraically trivialize its line and infinity. The return then becomes an algebraic polynomial family. The repelling base sheet remains repelling throughout its component by base stability. Thus the local argument applies everywhere on that algebraic open, and McMullen's polynomial consequence [4] makes the return isotrivial. If the return is already monic and centered, it is constant, since a fixed polynomial conjugacy class has only finitely many monic centered representatives. A projective conjugacy first has only finitely many choices for the totally invariant polynomial infinity and then is affine. $\square$

Lemma 5.3 (Recovery of centered factors). *If A, B are monic polynomials and A is centered of degree m , then*

$$B = [(A \circ B)^{1/m}]_{\geq 0},$$

where the right side is the polynomial part of the unique monic Laurent root at infinity. A composition of prescribed degrees of monic centered polynomials therefore determines each factor uniquely.

Proof. Since $A(u) = u^m + O(u^{m-2})$, expansion of the monic root gives $(A(B))^{1/m} = B + O(B^{-1})$. This recovers the inner factor. We then recover the outer composite by comparing coefficients in the polynomial identity and repeat the argument for the remaining factors. $\square$

We can now settle the case in which the rational base is isotrivial. After conjugacy, we fix a homogeneous lift $\Phi = (P, Q)$ over $\mathbb{C}$. We choose a repelling cycle of length $n > d$ and constant representatives v_i of its directions, with $\Phi(v_i) = \lambda_i v_{i+1}$. We also choose constant scales u_i satisfying $u_{i+1} = u_i^d / \lambda_i$. Such a cyclic choice exists because the consistency condition is a nonzero root equation of degree $d^n - 1$. The resulting fiber polynomials are monic centered. Their return is constant by Lemma 5.2, so all factors are constant by Lemma 5.3. Thus every a_j has constant values at the $n > d$ distinct base directions. Subtracting the coefficient form at a fixed parameter gives a binary form of degree at most d with more than d zeros. This form is zero, so the plane family is constant. By McMullen's theorem, it remains to treat a nonisotrivial flexible Lattès base.

5.2 Constant observations and a constant factor

We work over $K = \mathbb{C}(S)$ in normalization (5.1). Our aim is to show that a nonzero lower coefficient would give a constant rational factor of the base map. Let a periodic direction of g have period n . Over a finite extension, we choose v such that $\Phi^n(v) = v$, and put $v_i = \Phi^i(v)$. In the fiber coordinates $[v_i : z]$, the one-step maps are exactly

$$q_i(z) = z^d + \sum_{j=0}^{d-2} a_j(v_i) z^j.$$

Every Lattès periodic point is repelling. Lemmas 5.2 and 5.3 show that all these polynomials are constant. Hence

$$a_j(\Phi^i(v)) \in \mathbb{C} \tag{5.2}$$

for every periodic homogeneous lift and every i . Different cycles may be marked on different finite covers; their constants lie in the same algebraically closed field $\mathbb{C}$.

Suppose that some $a = a_j$ is nonzero, of degree $k = d - j > 0$. We define

$$r_i = \frac{a \circ \Phi^i}{a^{d^i}} \in K(\mathbb{P}^1), \quad i \geq 1.$$

We first observe that r_1 is nonconstant in the phase variable. Otherwise, the nonempty zero divisor of a would have totally invariant support under g , which is impossible for a Lattès map. At every periodic direction outside the finitely many zeros of a , equation (5.2) gives $r_i(u) \in \mathbb{C}$. There are infinitely many distinct observation tuples, since the nonconstant rational map r_1 has finite fibers.

The fields $K(r_1, \dots, r_N)$ form an ascending chain inside the finite extension $K(\mathbb{P}^1)/K(r_1)$, so the chain stabilizes. We choose a stabilization index. The identities

$$r_i \circ g = \frac{r_{i+1}}{r_1^{d^i}}$$

show that this field is invariant under g^* ; they also place every later r_i in it. Let Y_K be the image curve of the observation map to $(\mathbb{P}^1)^N$. We claim that it descends to $\mathbb{C}$. Let a polynomial over K vanish on its infinitely many constant points. We expand its coefficients in a finite complex-linearly independent set in K . Each constant coefficient polynomial in that expansion vanishes on those points. Their complex Zariski closure therefore base changes into Y_K ; it has positive dimension, and irreducibility of the curve forces equality. Applying the same argument to pairs of observation tuples at u and $g(u)$, we also descend the graph of the induced self-map.

The normalization of this constant curve has genus zero, since it is dominated by $\mathbb{P}_K^1$. Over $\mathbb{C}$ it is $\mathbb{P}^1$. We have obtained

$$\pi \circ g = h \circ \pi, \quad h \in \mathbb{C}(z), \quad \pi \in K(z) \setminus K, \quad \deg h = d. \quad (5.3)$$

After shrinking the parameter curve, we may assume that the degree e of π_s is fixed.

Lemma 5.4 (Constant-target obstruction). *A flexible Lattès family admitting (5.3) is isotrivial.*

Proof. Let $q_s : E_s \rightarrow \mathbb{P}^1$ be the degree-two elliptic quotient and $A_s : E_s \rightarrow E_s$ the inducing affine unramified map. Then $\varphi_s = \pi_s q_s$ has degree $2e$ and $h\varphi_s = \varphi_s A_s$. We claim that the branch values of every such φ_s are exactly the postcritical set P_h . The argument is the one in Milnor's local-degree lemma [70, Lemma 3.4]. Since A_s is unramified, we have

$$\deg_{A_s u} \varphi_s = \deg_u \varphi_s \deg_{\varphi_s(u)} h.$$

Thus all critical values of h are branch values of φ_s , and the latter finite set is forward invariant; hence it contains P_h . Conversely, if a critical point u of φ_s lay over a point outside P_h , all d^n distinct points in $A_s^{-n}(u)$ would still be critical for φ_s , since h^n is unramified at their images. This contradicts finiteness of its critical set.

There are only finitely many connected degree- $2e$ covers of $\mathbb{P}^1$ branched over a fixed finite set. Indeed, over the punctured sphere, these covers correspond to transitive homomorphisms from its finitely generated fundamental group to the finite group S_{2e} , and each cover extends uniquely across the punctures. Consequently the elliptic sources have only finitely many isomorphism classes. Their algebraic j -invariant is constant. On a fixed elliptic curve a fixed-degree flexible Lattès map has only finitely many conjugacy classes: its linear part is $\pm m$, $m^2 = d$, and the translation allowed by the involution is two-torsion; see also [70]. The moduli image of a connected algebraic family is therefore one point. The algebraic conjugacy scheme has a point over a finite function-field extension, giving isotriviality in the stated sense. $\square$

Proof of Theorem 5.1. We have already treated the case of an isotrivial base. If the base is nonisotrivial and flexible Lattès, any nonzero lower coefficient in (5.1) gives (5.3). This contradicts Lemma 5.4, so every lower coefficient vanishes. The resulting map has the totally invariant line $z = 0$ and gives the homogeneous regular polynomial family in the statement. $\square$

Remark 5.5. The arguments above apply to families with polynomial structure or a preserved pencil. To classify a general stable plane family, one must separately reduce to these cases or to a plane Lattès family. Such a reduction may require control of local return characters, monodromy, or the extension of a foliation from a Julia neighborhood. Those questions lie outside the arguments given here.

Proposition 5.6 (Lyapunov sum for homogeneous maps). *For a homogeneous regular polynomial plane map $F = \Phi$ whose infinity restriction is rational Lattès of degree d , one has*

$$L(F) = \frac{3}{2} \log d.$$

If an everywhere stable connected family has this presentation after conjugacy and finite base change on a nonempty algebraic open, its Lyapunov sum has this value at every original parameter.

Proof. We first observe that the affine critical equation of a homogeneous map, as well as every pushforward of its critical divisor, is homogeneous. The relative coefficient norms are therefore zero, so $\Delta_\Phi(C_\Phi) = 0$. Equation (4.1) gives $L(\Phi) = \log d + L(g)$. The elliptic presentation of the rational Lattès map is an affine map whose linear multiplier has modulus $\sqrt{d}$. Its invariant Haar measure descends to the rational equilibrium measure; the finite quotient changes the logarithmic derivative by an integrable coboundary. Thus $L(g) = \frac{1}{2} \log d$, in accordance with the standard description in [70]. Finally, conjugacy preserves Lyapunov sums. The Lyapunov sum of the original connected family is harmonic and equals the displayed constant on a nonempty open set. By uniqueness for harmonic functions, it has this value everywhere. $\square$

6 Algebraic selection and removal of iterates

The preceding arguments produce geometric structures for individual maps and their iterates. We now explain how to choose these structures algebraically in a family and then pass from an iterate back to the original map.

Throughout this section, S is a smooth irreducible complex algebraic curve and $K = \mathbb{C}(S)$. By a finite base change, we mean normalization in a finite extension of K , followed, if necessary, by removing finitely many points from the original curve and their inverse images. Recall that a family is projectively isotrivial if its generic fiber becomes conjugate to a map defined over $\mathbb{C}$ after such an extension. We make all coordinate choices on the resulting nonempty open curve.

We first record a selection argument that will allow us to pass from uncountably many pointwise constructions to a single algebraic family.

Lemma 6.1 (A countable selection principle). *Let $W_i \rightarrow S$, $i \geq 1$, be schemes of finite type. If the union of their images contains an uncountable subset of $S(\mathbb{C})$, some irreducible component W dominates S . After finite base change and shrinking, there is a morphism $S' \rightarrow W$ over S .*

Proof. Suppose first that no component dominates the base. Then the closure of the image of each component is a proper closed subset of S and is therefore finite. Since there are only countably many components, the union of their images is countable, contrary to the hypothesis.

We may therefore choose a dominating component. Its generic fiber is a nonempty scheme of finite type over K . By Zariski's lemma, a closed point of this fiber has residue field K' finite

over K . The corresponding map from the generic point of the normalization S' extends over a nonempty open subset. We obtain the desired morphism after removing from the original curve the images of the finitely many points where this extension is not defined. $\square$

Lemma 6.2 (Marking pencils and exceptional lines). *For a fixed algebraic family $g : S \times \mathbb{P}^2 \rightarrow S \times \mathbb{P}^2$ of degree $D > 1$, the loci of parameters at which g_s preserves a pencil of lines, or has a totally invariant line, are closed algebraic subsets of S . A dominating incidence permits an algebraic marking after finite base change. The pencil-center incidence has finite fibers.*

Proof. Let $V = \mathbb{C}^3$. For $c \in \mathbb{P}(V)$, let $W_c \subset V^*$ be the space of linear forms vanishing at c . We first consider the pencil with center at this point. The condition that the map preserves this pencil is

$$g_s^* W_c \subset \text{Sym}^D W_c.$$

This is a closed condition on $S \times \mathbb{P}(V)$, since it is an inclusion of the corresponding vector bundles. Choosing coordinates in which $c = [0 : 0 : 1]$, we can write it as

$$g_s[X : Y : Z] = [P_s(X, Y) : Q_s(X, Y) : R_s(X, Y, Z)].$$

We claim that the binary forms P_s, Q_s have no common projective zero. Indeed, a common zero would give a source line on which both forms vanish. This line would meet the zero divisor of R_s , producing a base point of the endomorphism. It follows that $g_s^{-1}(c) = \{c\}$, and hence c is fixed.

We also recall that a polarized endomorphism of degree greater than one has finitely many fixed points. Otherwise its fixed locus would contain a projective curve on which the pullback of $\mathcal{O}(1)$ has both its original degree and D times that degree, which is impossible. Thus the pencil incidence has finite fibers over the parameter curve. Its projection is proper because the space of centers is projective, so its image is closed.

We next consider a line with equation $\ell = 0$. The line is totally invariant precisely when the nonzero homogeneous forms $\ell \circ G_s$ and ℓ^D are proportional, where G_s is a local homogeneous lift. The vanishing of the coefficient minors expresses this as a closed condition on $S \times (\mathbb{P}^2)^*$. Since this projection is also proper, its image is closed.

For either incidence, a closed point of a dominating generic fiber gives a marking after finite base change, as in Lemma 6.1. We may then choose projective coordinates over the resulting function field that send the marked center or line to standard position. These coordinates extend over a nonempty open subset of the base. $\square$

Endomorphisms preserving pencils were classified by Dabija–Jonsson [21]. Their work on invariant algebraic webs [22] treats a broader geometric structure. For our purposes, the preceding incidence construction gives the algebraic marking that we need.

To apply this construction to exceptional curves, we use the fact that the curve part of the maximal proper algebraic totally invariant set is a union of at most three lines, permuted by inverse image [40, Proposition 1.1]. Each of these lines is therefore totally invariant under the sixth iterate. Consequently, if for one fixed return $g = f^q$ every parameter of an analytic open has either a preserved pencil or an exceptional curve, then

$$U \subset P(g) \cup T(g^6),$$

where both sets on the right are closed algebraic incidence images. At least one of them must equal S , since two proper closed subsets of a curve are finite and cannot cover an analytic open. We can therefore choose one algebraic alternative even when the pointwise alternatives vary with the parameter.

It remains to explain how the exceptional set and isotriviality behave under iteration.

Lemma 6.3 (Exceptional sets and roots). *For a projective endomorphism f of degree at least two, its maximal proper algebraic totally invariant set satisfies $E_{f^r} = E_f$ for every $r \geq 1$.*

Proof. Total invariance gives the inclusion from the exceptional set of the map into that of its iterate. For the converse, consider

$$Z = \bigcup_{j=0}^{r-1} f^{-j}(E_{f^r})$$

This is a proper algebraic subset: each term is proper, and a finite union of proper closed subsets cannot cover irreducible projective space. The identity $f^{-r}(E_{f^r}) = E_{f^r}$ implies that $f^{-1}Z = Z$. The required inclusion now follows from the maximality of E_f . $\square$

Proposition 6.4 (A unique polynomial chart for a homogeneous Lattès lift). *Suppose $F = f^r$ has the presentation*

$$F[X : Y : Z] = [P(X, Y) : Q(X, Y) : Z^D],$$

where $[P : Q]$ is a rational Lattès map. Then the line $H = \{Z = 0\}$ is the unique curve component of E_F , and $f^{-1}(H) = H$. Thus f itself is regular polynomial on $\mathbb{P}^2 \setminus H$.

Proof. We first show that there is no second exceptional line. Suppose that such a line M exists. Then some iterate F^a preserves it totally, so its intersection $b = M \cap H$ satisfies $(F^a|_H)^{-1}(b) = \{b\}$. This contradicts the fact that a rational Lattès map has no exceptional point [70].

For completeness, we can see the latter fact from the elliptic presentation. The map lifts to an unramified affine isogeny of an elliptic curve of degree greater than one. The iterated inverse images of the finite nonempty fiber over any point grow exponentially, whereas the fibers of the quotient to $\mathbb{P}^1$ have bounded cardinality. These inverse images cannot all lie above a finite backward orbit. We conclude that H is the unique exceptional line.

By Lemma 6.3, H is also the only curve component of E_f . The nonempty effective divisor $f^{-1}H$ is supported in E_f , so its support is H . This proves the proposition. $\square$

Lemma 6.5 (Isotrivial iterates). *If an iterate of an algebraic family of projective endomorphisms is projectively isotrivial, the original family is projectively isotrivial.*

Proof. After finite base change and conjugacy on a connected open subset, we may write $g_s^r = F$ with F constant. It follows that $g_s F = F g_s$. For each $x \in \text{Fix}(F^n)$, the map $s \mapsto g_s(x)$ takes values in the finite set $\text{Fix}(F^n)$, and is therefore constant. Here the finiteness follows from the polarization argument in Lemma 6.2.

Periodic points of a projective endomorphism are Zariski dense [36, 33]. Since all the morphisms g_s agree on this dense set, they are identical. This proves isotriviality of the original family. For the corresponding statement over general function fields, see [46, Corollary 16]. $\square$

7 Families of Lattès maps

We now consider the Lattès case. We will first show that, in a stable connected family, the presence of one Lattès map forces every member to be Lattès. We will then choose the abelian covers algebraically after a finite base change.

The first assertion follows from the characterization of Lattès maps by their Lyapunov exponents. Briend–Duval [33] show that, for a degree- d endomorphism of $\mathbb{P}^k$, every equilibrium Lyapunov exponent is at least $\frac{1}{2} \log d$. Equality for all exponents characterizes the Lattès

maps by [71, Corollary 1]. This uses Dupont's implication from minimal exponents to absolute continuity [72], the characterization by the Green measure in [71], and the earlier geometric characterization of Berteloot–Loeb [54]. In particular, we have

$$L(f) \geq \frac{k}{2} \log d, \quad L(f) = \frac{k}{2} \log d \iff f \text{ is Lattès.} \quad (7.1)$$

The equality case is also equivalent to absolute continuity of the equilibrium measure. Since each of the k exponents satisfies the same lower bound, their sum attains its lower bound precisely when every exponent does.

Proposition 7.1 (Propagation from one parameter). *Let (f_s) be a connected holomorphic family of degree- d endomorphisms of $\mathbb{P}^k$, with pluriharmonic equilibrium Lyapunov sum. If $f_{s_0}^r$ is Lattès for one interior parameter and some $r \geq 1$, every f_s is Lattès.*

Proof. We first remove the iterate from the hypothesis. The defining Green-current limits give $T_{f^r} = T_f$, and hence $\mu_{f^r} = \mu_f$. The characterization by absolute continuity therefore shows that f^r is Lattès if and only if f is Lattès. We can also obtain this from $\chi_i(f^r) = r\chi_i(f)$ and (7.1).

We now consider the pluriharmonic function

$$u(s) = L(f_s) - \frac{k}{2} \log d$$

By (7.1), this function is nonnegative and vanishes at s_0 . The strong minimum principle shows that it vanishes identically on the connected parameter manifold. Applying (7.1) once more, we conclude that every member of the family is Lattès. $\square$

This proposition is [9, Theorem 6.3]. The proof shows that algebraicity and a simultaneous choice of tori are not needed for this conclusion. Conversely, a holomorphic family consisting entirely of Lattès maps has constant minimal Lyapunov sum and is therefore equilibrium-stable.

The classification and deformation theory of Lattès maps have been studied by Milnor [70] in dimension one, Rong [12] in dimension two, and Gauthier–Hutz–Kaschner [14] for symmetric products and families. To obtain a relative abelian presentation, we will parametrize the covering diagrams themselves by countably many algebraic parameter spaces. This allows us to apply Lemma 6.1 without a dimension formula for a Lattès locus. In particular, the construction also applies to the nonscalar degree-two lift in Section 8.

Theorem 7.2 (Algebraization of pointwise abelian covers). *Let f be an algebraic degree- d family on $\mathbb{P}^2$ over S . Suppose an uncountable set of parameters admits finite surjective maps $\pi_s : A_s \rightarrow \mathbb{P}^2$ from abelian surfaces and isogenies α_s with $f_s \pi_s = \pi_s \alpha_s$. After finite base change and shrinking, there exist an abelian scheme $\mathcal{A} \rightarrow S'$ of relative dimension two, an isogeny $\alpha : \mathcal{A} \rightarrow \mathcal{A}$, and a finite surjective map $\pi : \mathcal{A} \rightarrow \mathbb{P}^2 \times S'$ satisfying*

$$f\pi = \pi\alpha, \quad \alpha^* \pi^* \mathcal{O}(1) \simeq (\pi^* \mathcal{O}(1))^{\otimes d}. \quad (7.2)$$

If the pointwise presentations are finite-group quotients, a fixed finite abstract quotient group can also be retained.

Proof. We first construct algebraic parameter spaces for the covering diagrams. Embed each abelian surface in some $\mathbb{P}^N$. There are countably many possible choices of N and of its Hilbert polynomial. For each such choice, we restrict the corresponding Hilbert scheme to the locus of smooth geometrically connected surfaces and let X be the universal projective surface. We also

record the identity point and the graphs of the addition map, inverse map, quotient map, and isogeny:

$$e \in X, \quad m : X \times X \rightarrow X, \quad i : X \rightarrow X, \quad \pi : X \rightarrow \mathbb{P}^2, \quad \alpha : X \rightarrow X.$$

We fix a Hilbert polynomial for each graph. The locus of graphs of morphisms is open in the relative Hilbert scheme [73, 74], so each choice gives a parameter space of finite type. There are again only countably many choices.

On these parameter spaces, we impose the group identities, commutativity, preservation of the identity by α , the homomorphism identity, and $f\pi = \pi\alpha$. These are closed equalizer conditions for morphisms from projective flat families to separated targets. Equivalently, each condition is the inverse image of the diagonal in a separated relative morphism functor. Thus these are identities of morphisms on the whole fibers.

We next restrict to the open locus where π has zero-dimensional fibers. To see that this locus is open, note that the points of X at which π is not quasi-finite form a closed set. Its image in the parameter space is closed by properness. On the complementary open subset, π is proper and quasi-finite, and hence finite. On each smooth connected surface fiber, dimension and irreducibility then imply that it is surjective onto $\mathbb{P}^2$.

The group scheme we have constructed is smooth, proper and connected, so it is an abelian scheme. Moreover, $\pi\alpha = f\pi$ is finite. It follows that α has finite fibers; properness then makes it finite, and dimension makes it surjective. Thus it is an isogeny on every fiber and, in characteristic zero, an isogeny of abelian schemes.

We have therefore constructed countably many parameter spaces of finite type over S , each carrying a universal diagram, and every pointwise presentation in the hypothesis occurs in one of them. Applying Lemma 6.1, we obtain a diagram over a finite extension of K that extends over a nonempty open curve. This gives the first identity in (7.2).

It remains to check the polarization identity. The bundle $\mathcal{L} = \pi^*\mathcal{O}(1)$ is relatively ample. For a family on the trivial projective bundle, the pullback may have the form

$$f^*\mathcal{O}(1) \simeq \mathcal{O}(d) \otimes \mathrm{pr}_S^* \mathcal{M}$$

for a line bundle $\mathcal{M}$ on the base. We choose a nonzero rational section of this line bundle, which trivializes $\mathcal{M}$ away from finitely many points. After deleting those points, the semiconjugacy identity gives the second identity in (7.2).

Finally, suppose that the pointwise presentations are finite-group quotients. We add to our parameter spaces a finite multiplication table G and graphs of automorphisms u_g satisfying $u_g u_h = u_{gh}$ and $\pi u_g = \pi$. We impose faithfulness of the action and $\deg \pi = |G|$ on constructible strata of finite type. Since there are only countably many multiplication tables, we may apply the same selection argument.

Over the selected function field, the induced finite map $\mathcal{A}/G \rightarrow \mathbb{P}^2$ is birational by the equality of degrees. It is therefore an isomorphism, since the target is normal. After shrinking the base, this isomorphism extends to the family. The automorphisms in the group action may include translations. $\square$

Remark 7.3 (Affine lifts and origins). The same theorem applies when the pointwise lift is affine. Indeed, suppose that $\beta(x) = \alpha(x) + b$. Translation acts trivially on the numerical class of $\pi^*\mathcal{O}(1)$, so the polarization gives a positive definite Hermitian form H satisfying $H(\alpha v, \alpha w) = dH(v, w)$. Every analytic eigenvalue of α consequently has modulus $\sqrt{d} > 1$, and hence $1 - \alpha$ is an isogeny.

We may therefore choose a with $(1 - \alpha)a = b$. Then $T_{-a}\beta T_a = \alpha$ and $f(\pi T_a) = (\pi T_a)\alpha$, which gives the pointwise presentation required in Theorem 7.2. These choices only need to be made for the individual fibers, since the Hilbert-scheme argument then selects an algebraic family of presentations. As in the theorem, the resulting presentation is defined over a nonempty open subset of the new base; we do not assert an extension across the deleted parameters.

8 A quadratic plane Lattès family

We give a degree-two family to explain why the plane Lattès alternative in Theorem 1.2 must allow nonsquare degrees. The construction belongs to the signed-permutation quotient class in Rong's classification [12, Proposition 3.3]. In particular, the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ occurs there for a varying elliptic lattice, and Rong's equation (15) is the $t = 0$ specialization of the map below. We write this known construction as an explicit one-parameter family and verify that it is nonisotrivial.

Let

$$S = \{t \in \mathbb{A}^1 : 64 - 27t^2 \neq 0\}, \quad E_t : y^2 = 4x^3 - 4x - t,$$

and define

$$F_t[X : Y : Z] = [2XY - 2XZ - tZ^2 : (Y + Z)^2 + tXZ : X^2 - 4YZ]. \quad (8.1)$$

Proposition 8.1. *Equation (8.1) is an algebraic, nonisotrivial, equilibrium-stable family of degree-two endomorphisms of $\mathbb{P}^2$.*

Proof. We first check that the three defining forms have no common zero. At $Z = 0$, they are $2XY, Y^2, X^2$, which do not vanish simultaneously at any projective point. At $Z = 1$, the vanishing of the first and third gives $Y = X^2/4$ and $t = X^3/2 - 2X$. Substituting into the second gives $(3X^2/4 - 1)^2$. Thus a common zero would imply $3X^2 = 4$ and $t = -X^3$, contrary to $27t^2 \neq 64$.

We next construct the abelian presentation. Write $x(u) = \wp(u)$ on E_t . The quotient by signed permutations is given by

$$q_t : E_t^2 \longrightarrow \mathbb{P}^2, \quad q_t(u, v) = [x(u) + x(v) : x(u)x(v) : 1].$$

This map is the composition of the two elliptic involution quotients with $\text{Sym}^2 \mathbb{P}^1 \simeq \mathbb{P}^2$, and hence extends to a finite morphism everywhere. Consider the isogeny

$$H(u, v) = (u + v, u - v)$$

Its matrix normalizes the signed permutation group: conjugating either a sign-change generator or the swap generator again gives a signed permutation matrix. To compute the induced map on the quotient, we use the elliptic addition formulas

$$\begin{aligned} x(u + v) + x(u - v) &= \frac{(2x(u)x(v) - 2)(x(u) + x(v)) - t}{(x(u) - x(v))^2}, \\ x(u + v)x(u - v) &= \frac{(x(u)x(v) + 1)^2 + t(x(u) + x(v))}{(x(u) - x(v))^2}. \end{aligned}$$

They give $q_t H = F_t q_t$ on a dense open subset, and therefore everywhere. The topological degree of H is $|\det H|^2 = 4$, which agrees with the topological degree of F_t . The latter has algebraic degree two. We conclude that F_t is Lattès, so the family is stable by (7.1).

It remains to prove that the family is nonisotrivial. Since $H^2 = 2I$, we have $F_t^2 = \text{Sym}^2(g_t)$, where g_t is the degree-four rational Lattès map induced by multiplication by two on E_t . Its postcritical set is the four-point set $B_t = x(E_t[2])$.

We claim that the postcritical divisor of $\text{Sym}^2(g_t)$ consists of the discriminant conic C and the four lines L_b , $b \in B_t$, parametrizing unordered pairs that contain b . Indeed, away from the diagonal the symmetric quotient is locally unramified. Criticality in either coordinate produces the lines, while pairs of distinct points with the same image produce the conic. At a generic point of the diagonal, the ramification of the source and target quotients cancels, so no

additional divisor appears. Since the conic maps onto itself and $g_t(B_t) \subset B_t$, these components give exactly the postcritical closure. We refer also to [14] for the critical geometry of symmetric products.

Any projective conjugacy between two members must preserve the unique conic in this divisor and its four tangent lines. We can therefore recover the unordered branch configuration B_t from the map, and hence the elliptic invariant

$$j(E_t) = 1728 \frac{64}{64 - 27t^2}.$$

Since this invariant varies with t , the family F_t^2 is nonisotrivial. The same is then true of F_t . $\square$

The degree of the lift. In this example, $\deg_{\text{alg}} F_t = 2$ and $\deg_{\text{top}} F_t = 4$. The analytic eigenvalues of H are $\sqrt{2}$ and $-\sqrt{2}$, so the lift is nonscalar even though its square is scalar. This explains how the original map can have nonsquare algebraic degree while an iterate is the symmetric square of a flexible rational Lattès map. In comparing this example with the general square-degree and dimension criteria of [14, Section 3, especially Corollary 3.7] (arXiv version 3), it is necessary to retain this distinction and to allow the explicit nonscalar lift. We do not invoke those criteria for the full class of plane Lattès families, and the arguments here do not require a correction or extension of them.

9 Proof of the main theorem

We are now ready to combine the local return theorem with the polynomial, pencil, and Lattès cases. Before doing so, we prove two lemmas. The first lets us choose the normalization data of the Poincaré family over a finite extension of the parameter function field. The second allows us to choose one iterate and one geometric alternative algebraically, even when the alternatives produced by the local argument depend on the reference parameter.

Lemma 9.1 (Algebraic normalization data). *After a finite algebraic base change, the marked return in Theorem 2.14 admits a fixed normal form and a formal Poincaré conjugacy whose coefficients belong to the parameter function field. Its analytic realization is the entire family of Lemma 3.5. Replacing a locally chosen Poincaré family by this one preserves the full rigid return conclusion.*

Proof. We first make the formal normalization over the function field. The periodic point is algebraic, so after a finite extension we may mark it and its constant eigenvalues. We can then choose eigenvectors or a Jordan chain over this field. At each degree, the homogeneous conjugacy equations form a finite linear system over the same field, and we solve for the nonresonant coefficients using the invertible homological operators.

In dimension two, the only possible nonlinear obstruction is the coefficient of x^q in the second coordinate. After the preceding finite normalization, this coefficient is a rational function. If it is nonzero, we scale the second coordinate to make it one; otherwise the term is absent. We set the free resonant conjugacy coefficients to zero and continue the recursion. This constructs all formal coefficients over the same function field, with any finite algebraic choices absorbed into the initial extension.

If the rigid gauge requires a finite local parameter cover, we can also realize this algebraically. We choose an algebraic local uniformizer at the reference point and adjoin a root of the required ramification order. The resulting finite algebraic cover agrees with the analytic cover in a neighborhood of the reference point.

By Lemma 3.5, the formal construction has a jointly holomorphic entire realization over a nonempty algebraic open subset. We must check that passing to this realization preserves the rigid return conclusion. Two locally invertible realizations at the marked repeller differ by a germ commuting with the fixed normal form, and Lemma 3.4 shows that this germ is a polynomial automorphism of the source.

Write $\sigma_t^{\text{local}} = \sigma_t^{\text{standard}} C_t$ for the change of realization. If Θ_t rigidly conjugates the locally chosen return system to its reference system, then $C_t \Theta_t C_{t_0}^{-1}$ gives the corresponding gauge for the standard family. It commutes with N and is the identity at t_0 , so the rigid return conclusion is preserved. $\square$

Lemma 9.2 (Countable incidence suffices). *Suppose an uncountable set of parameters $s \in S$ has the following property: for some positive integer $q = q(s)$, the map f_s^q preserves a pencil or has an exceptional curve. Then one fixed iterate of the family has, after finite algebraic base change, either an algebraically marked pencil or an algebraically marked totally invariant line.*

Proof. For each positive integer q , let P_q be the pencil incidence image for f^q , and let T_{6q} be the totally invariant line incidence image for f^{6q} . Lemma 6.2 shows that these are closed algebraic subsets of S . Every parameter in the given uncountable set belongs to

$$\bigcup_{q \geq 1} (P_q \cup T_{6q}).$$

If all of these images were proper, they would be finite and their union would be countable. Thus one image is all of S . A closed point of the generic fiber of its incidence scheme is defined over a finite extension of $\mathbb{C}(S)$, and gives the required marking after extending over a nonempty open curve. This argument allows both the geometric alternative and the return period to vary with the reference parameter. $\square$

9.1 Completion of the proof

Proof of Theorem 1.2. We may assume that the family is not projectively isotrivial. We first consider the case where some fiber, or some positive iterate of a fiber, is Lattès. By Proposition 7.1, every fiber of the original family is then Lattès. Theorem 7.2 gives a simultaneous polarized abelian-surface presentation after finite base change and shrinking, and the Lyapunov sum is $\log d$. This proves the theorem in this case.

We may now assume that no fiber is Lattès. We apply the local return theorem at the eligible reference parameters, using Lemma 2.2 for the good-center hypothesis, and consider its three alternatives.

Suppose first that we obtain a source foliation. A regular lift z of the marked repeller is obtained from zero by a germ $N^n h$, where h is an actual return. Both factors preserve the source foliation. Consequently, the pushed-forward directions agree on the regular base fiber, and we may apply Proposition 3.3 and Theorem 3.13. Corollary 3.14 then gives either a preserved pencil or an exceptional curve for the selected return.

In the affine case, Theorem 3.19 either produces a foliation, which we treat as above, or gives a bounded-holonomy quotient. In the latter case, a finite exceptional set would make the selected return Lattès, contrary to our assumption. Thus this case also produces an exceptional curve.

It remains to consider the rigid case. By Lemma 9.1, we may use the algebraically normalized Poincaré family in the global continuation argument. The full conjugacy of actual returns gives the fixed base-fiber identities (3.1), so Theorem 3.12 applies. It either makes the selected iterate isotrivial or gives a foliation for the reference return system. The first possibility would make

the original family isotrivial by Lemma 6.5, again contrary to our assumption. In the second possibility, we apply the preceding foliation argument at the reference parameter. Since this step is pointwise, we do not need these foliations to form a holomorphic family.

By Proposition 2.15, these alternatives hold at an uncountable set of eligible reference parameters. We can therefore apply Lemma 9.2 to obtain one fixed iterate $g = f^q$ with either an algebraically marked pencil or a totally invariant line. This iterate is stable because $L(f^q) = qL(f)$. After an algebraic coordinate change and restriction to a nonempty open subset, we may apply Theorem 5.1 or Theorem 4.1.

If the resulting family g is isotrivial, Lemma 6.5 makes the original family isotrivial, which is a contradiction. Hence g is a homogeneous lift of a nonisotrivial flexible rational Lattès family.

We finally return to the original map. By Proposition 6.4, the line at infinity of the homogeneous lift is its unique exceptional curve and is totally invariant for the original map as well. The original family is therefore regular polynomial in this marked affine chart. Applying Theorem 4.1 in degree d , we obtain (1.1). The rational base family is nonisotrivial, since otherwise the isotrivial-infinity case of that theorem would make the plane family isotrivial.

Proposition 5.6 gives $L = \frac{3}{2} \log d$ on the resulting open subset and then on the whole original connected curve. We have thus obtained the asserted alternatives for the original family, rather than only for an iterate. $\square$

Remark 9.3. Both nonisotrivial classes in the theorem have constant Lyapunov sum, and hence satisfy the stability hypothesis. We emphasize that the projective coordinates and the abelian presentation are chosen only over a nonempty open subset after finite base change. The theorem does not assert that these choices extend across every deleted parameter. The conclusions on the whole original curve concern the intrinsic alternatives and the Lyapunov values.

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