

A zero-free strip for the L -functions of the elliptic curves $y^2 = x^3 + d$

the 11/12 method and its extension to 7/8

OpenAI Astra-6 and Claude Opus 5.5

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Abstract

For a nonzero integer d let E_d be the elliptic curve $y^2 = x^3 + d$ over $\mathbb{Q}$. Part I adapts OpenAI's 11/12 argument to prove that $L(E_d, s) \neq 0$ for $\Re s > 17/12$. Part II adapts OpenAI's 7/8 argument to prove $L(s, \bar{\alpha}\eta) \neq 0$ for $\Re s > 7/8$, where η is any finite-order Hecke character of $\mathbb{Q}(\sqrt{-3})$ and $\alpha(n) = n/|n|$ on primary generators. Consequently $L(E_d, s) \neq 0$ for $\Re s > 11/8$. The analytic normalization is $L(E_d, s + \frac{1}{2})$, so the two boundaries are 11/12 and 7/8 within $0 < \Re s < 1$. The curve characters have infinite order, and require new transformation and moment arguments. These are proved here; steps identical to the original arguments are cited. Part II also uses the ideas from OpenAI's finite-order 7/8 theorem. Higher-weight and symmetric-power extensions are discussed; more general CM forms are the subject of a second paper.

Introduction

We study the family of elliptic curves

$$E_d: y^2 = x^3 + d, \quad d \in \mathbb{Z} \setminus \{0\}. \quad (0.1)$$

The discriminant of this equation is $-432d^2$, so the curve is nonsingular when $d \neq 0$. Its j -invariant is 0. We fix d throughout Part I; the larger family that we average over will be a family of character twists.

Complex multiplication. Every elliptic curve has the multiplication maps $P \mapsto mP$ for integers m . These curves have an additional endomorphism. Put $\omega = e^{2\pi i/3}$ and $K = \mathbb{Q}(\omega) = \mathbb{Q}(\sqrt{-3})$. The map

$$(x, y) \mapsto (\omega x, y) \quad (0.2)$$

preserves the equation and fixes the point at infinity. The full ring of endomorphisms over $\bar{\mathbb{Q}}$ is $\mathbb{Z}[\omega]$, the ring of Eisenstein integers. This is what *complex multiplication*, or CM, means here [44, 7]; the additional maps are defined over K . Moreover, adjoining a sixth root of d makes E_d isomorphic to E_1 by scaling x and y . Thus this is a family of *sextic twists*, which explains the appearance of sixth-order residue symbols below.

Point counts and the L -function. At a prime of good reduction, let $\tilde{E}_d$ be the reduction of a minimal model and set $a_p(d) = p + 1 - \#\tilde{E}_d(\mathbb{F}_p)$, counting the point at infinity. In particular, at $p \nmid 6d$ the displayed equation is already a good model, and

$$a_p(d) = - \sum_{x=0}^{p-1} \left(\frac{x^3 + d}{p} \right). \quad (0.3)$$

Here the Legendre symbol is 1 for a nonzero square, -1 for a nonsquare, and 0 at zero. So $a_p(d)$ measures the deviation of the point count from $p + 1$. Hasse's bound gives $|a_p(d)| \leq 2\sqrt{p}$ [12, 43].

The L -function packages these counts into an Euler product and a Dirichlet series:

$$L(E_d, s) = \prod_p L_p(E_d, s) = \sum_{m \geq 1} \frac{a_m(E_d)}{m^s}. \quad (0.4)$$

The standard local factors for an elliptic curve are

$$L_p(E_d, s) = \begin{cases} (1 - a_p(d)p^{-s} + p^{1-2s})^{-1}, & \text{good reduction,} \\ (1 - p^{-s})^{-1}, & \text{split multiplicative reduction,} \\ (1 + p^{-s})^{-1}, & \text{nonsplit multiplicative reduction,} \\ 1, & \text{additive reduction.} \end{cases} \quad (0.5)$$

The local factors define the coefficients $a_m(E_d)$, with $a_1(E_d) = 1$, and the product and series converge absolutely for $\Re s > 3/2$. These are the standard minimal-model factors [43, Appendix C, §16]. For our $j = 0$ curves every bad prime has additive reduction, so its factor is 1. The given equation need not be minimal: a prime dividing $6d$ can still be a prime of good reduction after a change of model. This is why factors at those primes must be taken from a minimal model.

The functional equation and the normalization. Let N_d be the conductor of E_d , a positive integer whose prime divisors are exactly the primes of bad reduction. Deuring's identification with a Hecke L -function, together with Hecke's theory, shows that $L(E_d, s)$ and its completed function extend to entire functions and satisfy [7, 18]

$$\Lambda(E_d, s) = N_d^{s/2} (2\pi)^{-s} \Gamma(s) L(E_d, s), \quad \Lambda(E_d, s) = \epsilon_d \Lambda(E_d, 2 - s), \quad \epsilon_d \in \{1, -1\}. \quad (0.6)$$

The sign ϵ_d is the *root number*. The symmetry is about $\Re s = 1$. To put the critical line in the same position as for zeta, we use the shifted function $\mathcal{L}_d(s) = L(E_d, s + \frac{1}{2})$. Its completed form is

$$\Lambda^u(E_d, s) = N_d^{s/2} (2\pi)^{-s} \Gamma(s + \frac{1}{2}) \mathcal{L}_d(s), \quad \Lambda^u(E_d, s) = \epsilon_d \Lambda^u(E_d, 1 - s). \quad (0.7)$$

Thus the critical line is now $\Re s = 1/2$, and the Euler product converges absolutely for $\Re s > 1$. Part I proves that $\mathcal{L}_d(s)$ has no zeros for $\Re s > 11/12$; Part II improves this to $\Re s > 7/8$. In the original variable, these are the boundaries $17/12$ and $11/8$.

Why CM connects the curves to the OpenAI method. CM expresses the curve's degree-two Euler product over rational primes as a Hecke Euler product over K . In this family the character can be written explicitly. For a primary Eisenstein integer $n \equiv 1 \pmod{3}$ coprime to $6d$, put

$$N(n) = |n|^2, \quad \alpha(n) = \frac{n}{|n|}, \quad \psi_d(n) = \alpha(n) \overline{\left(\frac{4d}{n} \right)_6}. \quad (0.8)$$

The sextic residue symbol is a sixth root of unity determined by residues modulo the prime factors of n ; Section 3 defines it precisely. The ring $\mathbb{Z}[\omega]$ has unique factorization, and each ideal coprime to 3 has one primary generator, so sums over such n are also sums over ideals. With $B_d(s) = \prod_{p|6d} L_p(E_d, s + \frac{1}{2})$, the concrete identity is

$$\mathcal{L}_d(s) = B_d(s) L^{(6d)}(s, \psi_d), \quad L^{(6d)}(s, \psi_d) = \sum_{\substack{n \text{ primary} \\ (n, 6d)=1}} \frac{\psi_d(n)}{N(n)^s} \quad (\Re s > 1). \quad (0.9)$$

Here $L^{(6d)}$ means that the Euler factors above primes dividing $6d$ have been deleted. Sections 2–3 prove this identity directly from point counts, including its sign and the factor 4. The finite factor B_d is holomorphic and nonzero for $\Re s > 0$, so this deletion preserves the zeros under discussion.

Taking the reciprocal introduces the Eisenstein Möbius function μ_K , defined by prime factorization in $\mathbb{Z}[\omega]$:

$$\frac{1}{L^{(6d)}(s, \psi_d)} = \sum_{\substack{n \text{ primary} \\ (n, 6d)=1}} \frac{\mu_K(n) \alpha(n) \overline{(4d/n)_6}}{N(n)^s} = \sum_{m \geq 1} \frac{q_d(m)}{m^s} \quad (\Re s > 1). \quad (0.10)$$

The last equality groups the terms by their ordinary integer norm and defines $q_d(m)$. The OpenAI 11/12 argument [35] estimates analogous Möbius sums in a family of sextic twists over this same field. Here the additional factor is $\alpha(n) = n/|n|$, which records the angle of n . It makes the curve character infinite order, so the finite-order theorem requires an adaptation. The same residue symbols, Poisson summation and cubic theta function remain available; the angular factor changes the theta derivative and the subsequent estimates. The six steps below outline this adaptation. In them W is a smooth, compactly supported weight, and D is the size of the norms being summed.

The method in six sentences.

1. *The sum to bound:* a bound $S_d(D) = \sum_m q_d(m) W(m/D) \ll D^{11/12+\varepsilon}$ for smooth sums of the coefficients of $1/L(E_d, s + \frac{1}{2})$, after deleting finitely many Euler factors, excludes zeros of $L(E_d, s)$ in $\Re s > 17/12$, just as inverse-coefficient sums do in the zeta argument [35, Theorem 1.1].
2. *The curve's row:* point counts rewrite the sum as

$$S_d(D) = \sum_n \mu_K(n) \frac{n}{|n|} \chi_n((4d)^5) W(N(n)/D) \quad (0.11)$$

over primary Eisenstein integers n coprime to 6, where $\chi_n(u) = (u/n)_6$ is the sextic residue symbol (the zeta row has $u = 1$ and omits $n/|n|$).

3. *The average:* varying u gives $A_u(D) = \sum_n \mu_K(n) \frac{n}{|n|} \chi_n(u) W(N(n)/D)$, and we prove $\sum_{0 < N(u) \leq H} |A_u(D)|^2 \ll D^{1+\varepsilon} H$ for $H = D^{1+\vartheta}$ and every sufficiently small $\vartheta > 0$, the size of square-root cancellation on average.
4. *Copies of the curve:* for fixed d , about $H^{1/6}/\log H$ prime rows $u = (4d)^5 \pi^6$ repeat $S_d(D)$ up to $O(D/N(\pi))$, so positivity of the average yields the exponent 11/12 as $\vartheta \rightarrow 0$ [35, Sections 2–3].

5. *The transformation:* Poisson summation in u converts the paired Möbius coefficients into cubic Gauss sums times $(\bar{n}/|n|)^2$ [35, Lemma 4.1], which occur in the second derivative of Kubota’s cubic theta function [24, 39] (one derivative for zeta), whose reflection turns the twists into quadratic symbols controlled by the quadratic large sieve [14, 10].
6. *Closing the argument:* Möbius inversion removes the theta function’s extra cube indices nb^3 , and two further Poisson summations shrink the number of rows by a fixed power of D , giving a finite induction that proves the mean-square bound and hence the zero-free boundary 17/12 [35, Lemma 5.3, Section 7 and Section 5.5].

What is proved here, and what is quoted. These notes follow the zeta guide [3]. A proof step identical to [35] or [36] is cited by number; every CM change is written out. Background on $\mathbb{Z}[\omega]$, residue symbols, Gauss sums, Poisson summation and the cubic theta function is in [2]. Part I treats the $j = 0$ curves over $\mathbb{Q}$, with d fixed and constants allowed to depend on it; Part II treats all weight-two CM characters of $K = \mathbb{Q}(\sqrt{-3})$.

The external inputs are the Kubota–Patterson theta theorem [24, 39], in the form [2, Theorem 23]; the quadratic large sieve [14, 10]; prime counting in fixed ray classes [25, 17, 18]; and Deuring–Hecke continuation [7, 18]. Standard local Euler factors [43] give the finite-factor comparison in Lemma 1. Part II additionally uses OpenAI’s finite-order 7/8 theorem [36]. Section 16 records its proof dependencies, and Section 18 distinguishes identity checks from analytic estimates.

Lean formalization. We are working toward a complete Lean certification of both results for these elliptic-curve L -functions: the boundary 17/12 of Part I (the 11/12 method) and the boundary 11/8 of Part II (the 7/8 method). OpenAI’s 7/8 theorem for finite-order characters is already formalized in Lean [37].

Proof dependencies. In the table, β_{CM} is the supremum of the real parts of zeros in the unitary CM family (the β_* of Section 11), and δ measures a row’s zero location. The finite-order input is proved in [36] independently of the CM conclusion.

Argument	Estimate or input	Conclusion
Part I	Positive mean square $\ll D^{1+\varepsilon}H$ for $H = D^{1+\vartheta}$ and every sufficiently small $\vartheta > 0$; Sections 7–10	Sixth-power copies and Mellin inversion give $L(E_d, s) \neq 0$ for $\Re s > 17/12$
Family bound	Part I with a fixed finite-order twist; Appendix G	$\beta_{\text{CM}} \leq 11/12$, hence the row-bin ceiling $\delta \leq 5/6$
Finite-order input	OpenAI’s 7/8 theorem [36]	Prime-slot estimates with $\kappa = 3/4$; no CM zero-free hypothesis is used here
Part II	CM reflection, inverse and plain moments, and the endpoint inequality; Appendices B–G and Section 15	$\beta_{\text{CM}} \leq 7/8$, hence $L(E_d, s) \neq 0$ for $\Re s > 11/8$
Higher weight	The further adaptations outlined in Section 17	Expected extensions only; no higher-weight or symmetric-power theorem is claimed

Higher weight. Analytically, the angular factor α enters the theta step through the number of derivatives. A weight- $k + 1$ CM newform corresponds [42, 41] to $\bar{\alpha}^k \eta$ and needs $k + 1$ derivatives; the Gamma factors keep the same length exponents for fixed k . Its angular powers must also be checked through the paired identities, cube inversion and reflection. Section 17 explains the expected extension, including the symmetric-power factorization [28, §3.5]; constants may depend on k . We

are writing up the case of more general CM forms in a second paper; here these extensions are expectations, not claims.

1 Why start with 11/12 rather than 7/8?

For this family, the 11/12 method is substantially simpler to present. Its main analytic task is one positive mean square, and the passage from that bound to the zero-free region takes only a few inequalities and a Mellin integral. Relative to the zeta argument, the main new ingredient is the angular factor $\alpha(n) = n/|n|$: one theta derivative becomes two.

The 7/8 argument [36, 4] has additional layers: a supremum of zero locations over a whole family, an average evaluated in two ways, short prime sums arranged to cancel an unwanted term, and two different moment estimates to count exceptional rows. These are substantial additions to the machinery used below.

There is also a scope issue. The stated 7/8 theorem in [36] concerns *finite-order* Hecke characters over $\mathbb{Q}(\sqrt{-3})$. The characters for our curves include $n/|n|$ and are not of finite order. Thus $7/8 + 1/2 = 11/8$ is the corresponding arithmetic boundary only after the required CM adaptation; it is not a consequence of shifting the stated finite-order theorem. Part I (Sections 2–10) explains the CM 11/12 argument. Part II (Sections 11–16) carries out the 7/8 adaptation.

2 Start with ordinary integers and point counts

At a prime $p \nmid 6d$, put

$$a_p(d) = p + 1 - \#E_d(\mathbb{F}_p) = - \sum_{x=0}^{p-1} \left(\frac{x^3 + d}{p} \right), \quad \lambda_d(p) = \frac{a_p(d)}{\sqrt{p}}. \quad (2.1)$$

The symbol on the right is the ordinary Legendre symbol, with value zero on multiples of p . Indeed, the number of solutions in y to $y^2 = x^3 + d$ is $1 + ((x^3 + d)/p)$, and there is one point at infinity. The good Euler factor is

$$L_p(E_d, s + \tfrac{1}{2})^{-1} = 1 - \lambda_d(p)p^{-s} + p^{-2s}. \quad (2.2)$$

The shift is chosen so that $L(E_d, s + 1/2)$ has critical line $\Re s = 1/2$; the arithmetic function $L(E_d, s)$ has critical line $\Re s = 1$. Consequently the reciprocal coefficients away from $6d$ are the multiplicative function q_d specified by

$$q_d(1) = 1, \quad q_d(p) = -\lambda_d(p), \quad q_d(p^2) = 1, \quad q_d(p^j) = 0 \quad (j \geq 3). \quad (2.3)$$

Set $q_d(m) = 0$ when $(m, 6d) > 1$. Thus all these coefficients can be computed using ordinary prime factorization and finite sums of Legendre symbols. In particular, q_d is not the ordinary Möbius function.

Here is an explicit formula involving that function. Define $\Lambda_d(r) = \prod_{p|r} \lambda_d(p)$ for squarefree r coprime to $6d$, with $\Lambda_d(1) = 1$. For $(m, 6d) = 1$,

$$q_d(m) = \sum_{\substack{rt^2=m \\ r \text{ squarefree} \\ (r,t)=1}} \mu(r)\mu(t)^2 \Lambda_d(r). \quad (2.4)$$

At each prime, one chooses the constant, linear, or quadratic term of the Euler polynomial. The restrictions in this formula make those choices mutually exclusive.

Write $L(E_d, s)$ using the local factors of a minimal model, including at primes dividing $6d$, and put

$$B_d(s) = \prod_{p|6d} L_p(E_d, s + \tfrac{1}{2}). \quad (2.5)$$

Then, for $\Re s > 1$,

$$\sum_{m \geq 1} \frac{q_d(m)}{m^s} = \prod_{p \nmid 6d} (1 - \lambda_d(p)p^{-s} + p^{-2s}) = \frac{B_d(s)}{L(E_d, s + \tfrac{1}{2})}. \quad (2.6)$$

The finite factor $B_d(s)$ is holomorphic and nonzero for $\Re s > 0$ (Lemma 1 proves this, and (2.6) with absolute convergence for $\Re s > 1$). Deleting these primes therefore does not affect the zeros in question. This remains true when the displayed equation for E_d is not minimal.

Choose a complex-valued $W \in C_c^\infty((0, \infty))$, supported in a fixed interval $[a, b]$ with $0 < a < b$. The target sum is

$$S_d(D) = \sum_{m \geq 1} q_d(m)W(m/D). \quad (2.7)$$

The weight restricts m to size D . We want, for every $\varepsilon > 0$, $S_d(D) \ll_{d, W, \varepsilon} D^{11/12+\varepsilon}$. Here $\ll$ allows a constant independent of D .

2.1 The comparison with the zeta coefficients

For zeta, the classical coefficients in the companion guide are

$$c(m) = \sum_{rt=m} \mu(r)\mu(t)\chi_{-3}(t), \quad \chi_{-3}(t) = \begin{cases} 0 & 3 \mid t, \\ 1 & t \equiv 1 \pmod{3}, \\ -1 & t \equiv 2 \pmod{3}. \end{cases} \quad (2.8)$$

For $p \nmid 6d$, the comparison is

$p \equiv 1 \pmod{3}$	$c(p)$	$c(p^2)$	$q_d(p)$	$q_d(p^2)$
$p \equiv 2 \pmod{3}$	-2	1	$-a_p(d)/\sqrt{p}$	1
$p \equiv 2 \pmod{3}$	0	-1	0	1

(2.9)

The inert-prime sign really changes. For instance, for $d = 1$, $a_5(1) = 0$, $a_7(1) = -4$, so $q_1(5) = 0$, $q_1(25) = 1$, and $q_1(7) = 4/\sqrt{7}$. The character χ_{-3} records whether a prime splits; it does not encode the point count of a particular elliptic curve.

3 The triangular lattice and the curve's row

Put $\omega = e^{2\pi i/3}$. An Eisenstein integer is an ordinary pair of integers written as $n = A + B\omega$, with

$$N(n) = |n|^2 = A^2 - AB + B^2, \quad \alpha(n) = \frac{A + B\omega}{\sqrt{A^2 - AB + B^2}}. \quad (3.1)$$

The ring $\mathbb{Z}[\omega]$ has unique factorization and six units $\pm 1, \pm\omega, \pm\omega^2$. Choose the unique *primary* associate of each element coprime to 3:

$$n \equiv 1 \pmod{3}, \quad \text{that is, } A \equiv 1 \pmod{3}, \quad B \equiv 0 \pmod{3}. \quad (3.2)$$

Unless otherwise stated, column variables and their divisors are primary and coprime to 6; row variables run over all of $\mathbb{Z}[\omega]$. Coprimality to 2 means that A, B are not both even. A star on a column sum means squarefree in $\mathbb{Z}[\omega]$.

Define $\mu_K(n)$ by the usual factorization rule: it is zero if a prime square divides n , and otherwise is $(-1)^r$ for a product of r distinct primary primes. For example,

$$\mu_K(-5) = -1 \quad (\mathbf{N}(-5) = 25), \quad \mu_K(7) = 1 \quad (7 = (1 + 3\omega)(-2 - 3\omega)). \quad (3.3)$$

Thus $\mu_K(n)$ need not equal $\mu(\mathbf{N}(n))$.

For a prime $\pi \nmid 6$, define $\chi_\pi(u) = (u/\pi)_6$ to be the sixth root of unity congruent to $u^{(\mathbf{N}(\pi)-1)/6}$ modulo π , or zero if $\pi \mid u$. Extend multiplicatively in the denominator, and write $\chi_n(u) = (u/n)_6$. It is also multiplicative in u . Powers of these characters retain their zero values, even when the exponent is reduced to zero modulo 6; $\chi_1(u) = 1$ for all u .

Define the family

$$A_u(D) = \sum_n \mu_K(n) \alpha(n) \chi_n(u) W(\mathbf{N}(n)/D). \quad (3.4)$$

It is the zeta family with one extra, explicitly given factor $n/|n|$. For the desired curve, choose

$$u_d = (4d)^5, \quad \psi_d(n) = \alpha(n) \chi_n(u_d) = \alpha(n) \overline{\chi_n(4d)}. \quad (3.5)$$

We claim that $A_{u_d}(D) = S_d(D)$.

Where u_d comes from. At a split prime $p = \pi\bar{\pi}$, the point count (3.7) says that E_d contributes π times the sixth root of unity $\overline{\chi_\pi(4d)}$. In its Jacobi-sum proof the factor d enters when the summation variable is rescaled by d , and the factor 4 enters through the discriminant $1 - 4a$ of $t^2 - t + a$, which gives $J(\kappa, \rho) = \kappa(4)J(\kappa, \kappa)$ for the cubic and quadratic characters κ, ρ modulo π . The family, however, twists by $\chi_n(u)$, with the row u in the *top* of the residue symbol. So we need an Eisenstein integer u with $\chi_n(u) = \overline{\chi_n(4d)}$ for every column n . Since χ_n takes sixth roots of unity,

$$\overline{\chi_n(4d)} = \chi_n(4d)^5 = \chi_n((4d)^5), \quad (3.6)$$

and this is the choice $u_d = (4d)^5$. Multiplying a row by a sixth power changes it only at the columns that share a prime with that sixth power: $\chi_n(uv^6) = \chi_n(u) \mathbf{1}_{(n,v)=1}$. Such a change alters finitely many Euler factors, so it does not move any zeros in $\Re s > 0$. When v is supported on primes above 2 and 3, nothing changes at all, because the columns are coprime to 6. For instance $u = 16d^5$ gives exactly the same row, since $(4d)^5 = 16d^5 \cdot 2^6$. For $d = 16$ this is $2^{24} = (2^4)^6$, a sixth power. So the curve $y^2 = x^3 + 16$ (conductor 27) is the row $u = 1$ itself, namely the plain sum $\sum_n \mu_K(n) \frac{n}{|n|} W(\mathbf{N}(n)/D)$, and $y^2 = x^3 + 1$ is the row $u = 16$. More generally, every nonzero rational row u is the row of a $j = 0$ curve over $\mathbb{Q}$: since $(4d)^5 \equiv (4d)^{-1}$ modulo sixth powers, u belongs to E_d with $d = 4^5 u^5$. For rows that are not a rational number times a sixth power, the coefficients $q_u(m)$ of (4.5) are in general not real. Such a row is the L -function of a Hecke character of $\mathbb{Q}(\sqrt{-3})$, not of an elliptic curve over $\mathbb{Q}$.

3.1 Checking the claim prime by prime

If $p \equiv 1 \pmod{3}$, then $p = \pi\bar{\pi}$, with both generators primary and $\mathbf{N}(\pi) = p$. The classical CM point-count identity is

$$a_p(d) = \overline{\pi \chi_\pi(4d)} + \bar{\pi} \chi_\pi(4d) \quad (p \nmid 6d). \quad (3.7)$$

This classical finite-field identity is proved with Jacobi sums [20, Chapter 8] (compare [20, Chapter 18], where primary means $\equiv 2 \pmod{3}$, which changes the sign of π). Proposition 1 gives the proof in our normalization. We also checked the identity numerically for all $0 < |d| \leq 60$ and $p < 1500$, at both primes above each p . Since $\chi_{\bar{\pi}}(4d) = \chi_{\pi}(4d)$, the two reciprocal factors multiply to

$$(1 - \psi_d(\pi)p^{-s})(1 - \psi_d(\bar{\pi})p^{-s}) = 1 - \lambda_d(p)p^{-s} + p^{-2s}. \quad (3.8)$$

If $p \equiv 2 \pmod{3}$, the primary prime is $-p$, of norm p^2 . For rational x prime to p , $\chi_{-p}(x) = 1$, because $(p^2 - 1)/6 = (p - 1)(p + 1)/6$. Since $\alpha(-p) = -1$, its factor is $1 + p^{-2s}$, as required by $a_p(d) = 0$. If $p \mid d$, the symbols vanish at primes above p , so their factors are deleted.

Comparing Dirichlet coefficients (Lemma 1(b)) gives the exact identities

$$q_d(m) = \sum_{N(n)=m} \mu_K(n) \alpha(n) \chi_n(u_d), \quad S_d(D) = A_{u_d}(D). \quad (3.9)$$

The columns still run over primary n coprime to 6; the sum is empty when $(m, 6) > 1$. The angular factor and sextic symbol are what distinguish this identity from the zeta identity involving μ and χ_{-3} .

The curve's coefficients are sextic residue symbols. The identity (3.9) for the reciprocal has a companion for L itself. Up to the finitely many Euler factors at $p \mid 6d$, $L(E_d, s + \frac{1}{2}) = \prod_{\mathfrak{p} \nmid 6d} (1 - \psi_d(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1} = \sum_n \psi_d(n)N(n)^{-s}$, so for m coprime to $6d$,

$$\frac{a_m(E_d)}{\sqrt{m}} = \sum_{N(n)=m} \alpha(n) \left(\frac{u_d}{n} \right)_6, \quad q_d(m) = \sum_{N(n)=m} \mu_K(n) \alpha(n) \left(\frac{u_d}{n} \right)_6, \quad (3.10)$$

with n primary and coprime to 6. Here $a_m(E_d)$ are the Dirichlet coefficients of $L(E_d, s)$. The only difference between the two is the Möbius factor. At a prime the first identity is $a_p(d) = \text{Tr}(\pi(u_d/\pi)_6)$: the Frobenius eigenvalues are $\pi(u_d/\pi)_6$ and its conjugate, so the sextic symbol selects which of the six rotations $\zeta\pi$ ($\zeta^6 = 1$) occurs. We checked the first identity against the coefficients computed by PARI/GP [38] for $d \in \{1, 2, -3, 7\}$ and all $m \leq 400$ coprime to $6d$.

Thus the coefficients of $L(E_d, s + \frac{1}{2})$ are exactly the sextic residue symbols of the zeta argument, evaluated at the fixed row u_d and rotated by the angle $n/|n|$. In the zeta argument the row u is the Möbius sum $\sum_n \mu_K(n)(u/n)_6 W(N(n)/D)$ of the finite-order Hecke L -function attached to $n \mapsto (u/n)_6$. This is the Artin L -function [33, Ch. VII, §10] of a character (by Kummer theory [33, Ch. IV, §3]) of the Kummer extension $\mathbb{Q}(\sqrt{-3}, u^{1/6})/\mathbb{Q}(\sqrt{-3})$, one factor of the Dedekind zeta function of that extension. The CM argument averages the same rows with every coefficient multiplied by $n/|n|$, and its row u_d is $L(s, \alpha \cdot (u_d/\cdot)_6) = L(E_d, s + \frac{1}{2})$, up to finitely many Euler factors.

3.2 Proofs of the point count and of the Euler products

OpenAI's paper has nothing corresponding to this subsection: for ζ the coefficients $c(m)$ are classical, and [3, (2.3)] only groups ideals by their norms. From [2, Sections 2–3] we use: unique factorization of primary elements into primary primes, and the fact that $-p$ is the primary generator of an inert prime p [2, Proposition 11]; Euler's criterion and the conjugation rule $\overline{(x/\pi)_6} = (\bar{x}/\bar{\pi})_6$ [2, Section 3.1]; the values of the symbols at rational integers [2, (3.5)–(3.6)]; and the cubic Jacobi sum [2, Theorem 15].

The curves E_d are all the $j = 0$ curves over $\mathbb{Q}$. Indeed, every elliptic curve over $\mathbb{Q}$ has a model $y^2 = x^3 + Ax + B$, whose j -invariant is $1728 \cdot 4A^3/(4A^3 + 27B^2)$ [43, Chapter III]. So $j = 0$ exactly when $A = 0$, and the change of variables $(x, y) \mapsto (t^2x, t^3y)$ makes B a nonzero integer.

Proposition 1 (Point counts). *Let $d \neq 0$ be an integer, let $p \geq 5$ be a prime with $p \nmid d$, and put $a_p(d) = p + 1 - \#E_d(\mathbb{F}_p)$, where the count includes the point at infinity.*

(a) *If $p \equiv 2 \pmod{3}$, then $a_p(d) = 0$.*

(b) *If $p \equiv 1 \pmod{3}$ and π is either primary prime of norm p , then $a_p(d) = \text{Tr}(\pi \overline{\chi_\pi(4d)})$; this is (3.7). In particular $|a_p(d)| \leq 2\sqrt{p}$, that is, $|\lambda_d(p)| \leq 2$.*

Proof. (a) Since $3 \nmid p-1$, cubing is a bijection of $\mathbb{F}_p$. So for each $y \in \mathbb{F}_p$ there is exactly one x with $x^3 = y^2 - d$. There are p affine points, and $\#E_d(\mathbb{F}_p) = p + 1$.

(b) The residue field $\mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega]$ has p elements and contains the image of $\mathbb{Z}$, so $\mathbb{Z} \rightarrow \mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega]$ induces an isomorphism $\mathbb{F}_p \cong \mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega]$. Through it, $\chi = \chi_\pi$ becomes a character of $\mathbb{F}_p^\times$ with values in the sixth roots of unity, extended by $\chi(0) = 0$. Put

$$\kappa = \chi^2 = (\cdot/\pi)_3, \quad \rho = \chi^3 = (\cdot/\pi)_2. \quad (3.11)$$

By Euler's criterion the kernel of κ on $\mathbb{F}_p^\times$ is the subgroup of cubes, and κ takes values in $\{1, \omega, \omega^2\}$. The character ρ is the Legendre symbol modulo p [2, (3.6)]. Also $\kappa(-1) = \kappa((-1)^3) = \kappa(-1)^3 = 1$. Since $\mathbb{F}_p^\times$ is cyclic of order divisible by 3, a nonzero cube has exactly three cube roots, while for a non-cube V we have $\kappa(V) \in \{\omega, \omega^2\}$ and $1 + \omega + \omega^2 = 0$. Hence, for $U, V \in \mathbb{F}_p$,

$$\#\{y : y^2 = U\} = 1 + \rho(U), \quad \#\{x : x^3 = V\} = 1 + \kappa(V) + \overline{\kappa(V)}. \quad (3.12)$$

Counting the affine points (x, y) according to the value $V = x^3$ gives

$$\#E_d(\mathbb{F}_p) - 1 = \sum_{V \in \mathbb{F}_p} (1 + \rho(V + d))(1 + \kappa(V) + \overline{\kappa(V)}) = p + S + \overline{S}, \quad S = \sum_{V \in \mathbb{F}_p} \rho(V + d)\kappa(V), \quad (3.13)$$

because $\sum_V \rho(V + d) = \sum_V \kappa(V) = 0$ and ρ is real. Thus $a_p(d) = -S - \overline{S}$.

Since $p \nmid d$, the substitution $V = -dt$ is a bijection of $\mathbb{F}_p$. As $\rho(d - dt) = \rho(d)\rho(1 - t)$ and $\kappa(-dt) = \kappa(d)\kappa(t)$,

$$S = \rho(d)\kappa(d) J(\kappa, \rho), \quad J(\kappa, \rho) = \sum_{t \in \mathbb{F}_p} \kappa(t)\rho(1 - t). \quad (3.14)$$

To compare this with the cubic Jacobi sum $J(\kappa, \kappa) = \sum_t \kappa(t)\kappa(1 - t)$, group the t according to $a = t(1 - t)$. Since p is odd, the equation $t^2 - t + a = 0$ has $1 + \rho(1 - 4a)$ solutions, so

$$J(\kappa, \kappa) = \sum_{a \in \mathbb{F}_p} \kappa(a)(1 + \rho(1 - 4a)) = \sum_a \kappa(a)\rho(1 - 4a) = \overline{\kappa(4)} J(\kappa, \rho), \quad (3.15)$$

where the last step substitutes $a = b/4$. For primary π we have $J(\kappa, \kappa) = -\pi$ [2, Theorem 15]. Hence $J(\kappa, \rho) = -\kappa(4)\pi$ and

$$S = -\rho(d)\kappa(4d)\pi, \quad a_p(d) = \text{Tr}(\rho(d)\kappa(4d)\pi). \quad (3.16)$$

Finally $\rho(d)\kappa(4d) = \chi(d)^3\chi(4d)^2 = \chi(d)^5\chi(4)^2$. Since $\chi(4)^3 = \chi(2)^6 = 1$, we have $\chi(4)^2 = \chi(4)^5$, and so $\rho(d)\kappa(4d) = \chi(4d)^5 = \overline{\chi(4d)}$. This proves (3.7) for π . The two primes above p give the same value, because $\chi_{\bar{\pi}}(4d) = \chi_\pi(4d)$ for the rational number $4d$. The bound follows from $|\chi_\pi(4d)| = 1$ and $|\pi| = \sqrt{p}$. $\square$

Where the normalizations enter. The sign comes from the Jacobi sum: $J(\kappa, \kappa)$ is the generator of $\pi\mathbb{Z}[\omega]$ that is $\equiv -1 \pmod{3}$, which is $-\pi$ for our primary π . In the convention of [20, Chapter 9] the primary generator is $\equiv 2 \pmod{3}$, that is, $-\pi$, and $J(\kappa, \kappa)$ equals it [2, Remark 10]. This is the sign change mentioned after (3.7). The factor 4 comes only from the discriminant $1 - 4a$, and the complex conjugation only from $\rho(d)\kappa(4d) = \chi(4d)^5$.

Lemma 1 (Euler products). *Let $d \neq 0$, and let $\psi_d(n) = \alpha(n)\chi_n(u_d)$ be as in (3.5).*

(a) *Let $p \geq 5$, and let X be an indeterminate. If $p \nmid d$ and $p = \pi\bar{\pi} \equiv 1 \pmod{3}$ with π primary, then $(1 - \psi_d(\pi)X)(1 - \psi_d(\bar{\pi})X) = 1 - \lambda_d(p)X + X^2$. If $p \nmid d$ and $p \equiv 2 \pmod{3}$, then $1 - \psi_d(-p)X^2 = 1 + X^2 = 1 - \lambda_d(p)X + X^2$. If $p \mid d$, then ψ_d vanishes at every prime above p .*

(b) *For $\Re s > 1$ the following series and products converge absolutely, and*

$$\sum_n \frac{\mu_K(n)\psi_d(n)}{N(n)^s} = \prod_{\pi} \left(1 - \frac{\psi_d(\pi)}{N(\pi)^s}\right) = \prod_{p \nmid 6d} (1 - \lambda_d(p)p^{-s} + p^{-2s}) = \sum_{m \geq 1} \frac{q_d(m)}{m^s} = \frac{B_d(s)}{L(E_d, s + \frac{1}{2})}. \quad (3.17)$$

Here n runs over the primary elements coprime to 6, and π over the primary primes not dividing 6. Consequently (3.9) holds, and $a_m(E_d)/\sqrt{m} = \sum_{N(n)=m} \psi_d(n)$ for $(m, 6d) = 1$.

(c) *$B_d(s)$ is holomorphic and nonzero for $\Re s > 0$, whether or not $y^2 = x^3 + d$ is a minimal model.*

Proof. (a) Let $p = \pi\bar{\pi}$. Both π and $\bar{\pi}$ are primary [2, Proposition 11]. We have $\alpha(\bar{\pi}) = \overline{\alpha(\pi)}$, and $\chi_{\bar{\pi}}(4d) = \overline{\chi_{\pi}(4d)}$ because $4d$ is rational. Hence $\psi_d(\bar{\pi}) = \overline{\psi_d(\pi)}$ and $|\psi_d(\pi)| = 1$, so the product is $1 - \text{Tr}(\psi_d(\pi))X + X^2$. By (3.5) and Proposition 1, $\text{Tr}(\psi_d(\pi)) = \text{Tr}(\pi\chi_{\pi}(4d))/\sqrt{p} = \lambda_d(p)$. If $p \equiv 2 \pmod{3}$, the primary generator of $p\mathbb{Z}[\omega]$ is $-p$, of norm p^2 , with $\alpha(-p) = -1$ and $\chi_{-p}(4d) = 1$ [2, (3.5)]. So $\psi_d(-p) = -1$, while $a_p(d) = 0$ by Proposition 1(a). If $p \mid d$, every prime above p divides u_d , so the symbol vanishes there.

(b) The function ψ_d is completely multiplicative on primary elements, because α is and $\chi_{nn'}(u) = \chi_n(u)\chi_{n'}(u)$. Also $\mu_K\psi_d$ has modulus at most 1 and is supported on squarefree n . The series $\sum_n N(n)^{-\sigma}$ over primary n converges for $\sigma > 1$ [2, (2.8)], so unique factorization into primary primes [2, Proposition 11] gives the first equality, with absolutely convergent series and product. Grouping the primes π by the rational prime below them and using (a) gives the second equality; the primes 2 and 3 do not occur. Expanding the factors gives the third, by the definition (2.3) of the multiplicative function q_d ; the expansion converges absolutely because $|\lambda_d(p)| \leq 2$. For the last equality, the equation $y^2 = x^3 + d$ has discriminant $-2^4 3^3 d^2$. So it has good reduction at every $p \nmid 6d$, and it gives $L_p(E_d, s)^{-1} = 1 - a_p(d)p^{-s} + p^{1-2s}$ there, which becomes $1 - \lambda_d(p)p^{-s} + p^{-2s}$ after the shift $s \mapsto s + \frac{1}{2}$. The factors at $p \mid 6d$ make up $B_d(s)$.

The coefficient of m^{-s} in the first series of (3.17) is $\sum_{N(n)=m} \mu_K(n)\psi_d(n)$, and in the fourth it is $q_d(m)$. Absolutely convergent Dirichlet series with the same sum have the same coefficients, which gives (3.9). In the same way, (a) gives $\sum_n \psi_d(n)N(n)^{-s} = \prod_{\pi} (1 - \psi_d(\pi)N(\pi)^{-s})^{-1} = \prod_{p \nmid 6d} (1 - \lambda_d(p)p^{-s} + p^{-2s})^{-1}$. For $(m, 6d) = 1$ the coefficient of m^{-s} in $L(E_d, s + \frac{1}{2}) = B_d(s) \prod_{p \mid 6d} (1 - \lambda_d(p)p^{-s} + p^{-2s})^{-1}$ comes from the second factor alone, and it equals $a_m(E_d)/\sqrt{m}$.

(c) For $p \mid 6d$ the factor $L_p(E_d, s)$ is computed on a minimal model. If the reduction at p is bad, then $L_p(E_d, s)^{-1} = 1 - a_p p^{-s}$ with $a_p \in \{0, \pm 1\}$ [43, Appendix C, §16], and $1 - a_p p^{-s-1/2}$ does not vanish for $\Re s > -\frac{1}{2}$. If it is good, then $L_p(E_d, s + \frac{1}{2})^{-1} = 1 - (a_p/\sqrt{p})X + X^2$ with $X = p^{-s}$, and $|a_p| \leq 2\sqrt{p}$ by Hasse's theorem [12], [43, Theorem V.1.1]. The two roots of this real quadratic are then complex conjugate (or equal to ± 1), and their product is 1, so both have absolute value 1; there is no zero with $\Re s > 0$. (One can also avoid Hasse's theorem here. For

$p \in \{2, 3\}$ the bounds $1 \leq \#E(\mathbb{F}_p) \leq 2p + 1$ give $|a_p| \leq p \leq 2\sqrt{p}$. For $p \geq 5$ write $d = p^{6k}d_1$ with $v_p(d_1) \leq 5$. Then $E_d \cong E_{d_1}$ via $(x, y) \mapsto (p^{-2k}x, p^{-3k}y)$, and $y^2 = x^3 + d_1$ is minimal at p because $v_p(-432d_1^2) < 12$ [43, Remark VII.1.1]. So good reduction at p means $p \nmid d_1$, and Proposition 1 applies to d_1 .) In every case $L_p(E_d, s + \frac{1}{2})$ is the reciprocal of a polynomial in p^{-s} with no zeros in $\Re s > 0$. The model $y^2 = x^3 + d$ was used only at primes $p \nmid 6d$, where it is good, so nothing depends on its minimality. $\square$

4 The average, in coordinates and as a double sum

The central average is

$$M(D, H) = \sum_{0 < N(u) \leq H} |A_u(D)|^2. \quad (4.1)$$

There are about H rows and about D columns in each row. Writing $Q(A, B) = A^2 - AB + B^2$, this means exactly

$$M(D, H) = \sum_{\substack{x, y \in \mathbb{Z} \\ 0 < Q(x, y) \leq H}} \left| \sum_{\substack{A \equiv 1 \pmod{3}, \\ A, B \text{ not both even}}} \mu_K(A + B\omega) \frac{A + B\omega}{\sqrt{Q(A, B)}} \times \left(\frac{x + y\omega}{A + B\omega} \right)_6 W\left(\frac{Q(A, B)}{D}\right) \right|^2. \quad (4.2)$$

Thus it is a sum over four ordinary integer coordinates, using factorization and exponentiation in finite fields.

Expanding the square, with the first column conjugated, gives

$$M(D, H) = \sum_{n_1, n_2} \mu_K(n_1) \mu_K(n_2) \overline{\alpha(n_1)} \alpha(n_2) \overline{W(N(n_1)/D)} W(N(n_2)/D) S_H(n_1, n_2), \quad (4.3)$$

$$S_H(n_1, n_2) = \sum_{0 < N(u) \leq H} \overline{\chi_{n_1}(u)} \chi_{n_2}(u).$$

The elliptic-curve factor in this double sum is simply $\overline{n_1} n_2 / (|n_1| |n_2|)$. No abstract character notation is needed to calculate it.

4.1 An ordinary-integer description of every row

If desired, put

$$q_u(m) = \sum_{N(n)=m} \mu_K(n) \alpha(n) \chi_n(u), \quad A_u(D) = \sum_{m \geq 1} q_u(m) W(m/D). \quad (4.4)$$

For fixed $u \neq 0$, q_u is multiplicative, vanishes on integers divisible by 2 or 3, and for $p \geq 5$ has the following local values:

$p = \pi \bar{\pi} \equiv 1 \pmod{3}$	$-\frac{q_u(p)}{\sqrt{p}} \frac{\pi \chi_\pi(u) + \bar{\pi} \chi_{\bar{\pi}}(u)}{\sqrt{p}}$	$\frac{q_u(p^2)}{\chi_\pi(u) \chi_{\bar{\pi}}(u)}$	(4.5)
$p \equiv 2 \pmod{3}$	0	$\chi_{-p}(u)$	

All $q_u(p^j)$ with $j \geq 3$ vanish. These formulas include the case when a symbol is zero. At $u = u_d$ they reduce to (2.3). For a general row, the symbols distinguish π from $\bar{\pi}$; ordinary μ and χ_{-3} alone do not determine them.

4.2 The estimate to be supplied by analysis

The analytic input is the following mean-square estimate: for every fixed $0 < \vartheta \leq 1/10$ and every $\varepsilon > 0$,

$$M(D, D^{1+\vartheta}) \ll_{W, \vartheta, \varepsilon} D^{2+\vartheta+\varepsilon}. \quad (4.6)$$

The constant is controlled by finitely many derivatives of W . As in [35, Proposition 3.1], of which (4.6) is the CM form, there is an integer $J' = J'(\vartheta, \varepsilon)$ such that the constant is $\ll_{I, \vartheta, \varepsilon} \|W\|_{C^{J'}(I)}^2$ when W is supported in a compact interval $I \subset (0, \infty)$ (the norm is as in Proposition 3). The curve parameter does not occur in this estimate: all the curves are selected by different rows of the same family.

Logical structure of Part I. The zero-free half-plane is deduced from (4.6) in Sections 5 and 6 (Lemmas 2–4). That deduction needs (4.6) for arbitrarily small ϑ : a single fixed ϑ gives $S_d(D) \ll D^{11/12+5\vartheta/12+\varepsilon}$, hence only $\Re s > 17/12 + 5\vartheta/12$. Its other inputs are the identities of Section 3, the prime number theorem for primes $\equiv 1 \pmod{3}$, and the continuation of $L(E_d, s)$. The estimate (4.6) itself is Proposition 3, whose hypothesis (7.25) follows from the bound (10.25) of Section 10 with $\kappa = \vartheta/2$ (see (10.26)), together with counting for bounded scales. That bound is proved by an induction whose inputs are cube removal (10.4) and the transfer estimate (Proposition 5); these rest on the completed estimate (9.2), the CM theta transformation (Proposition 8) and the quadratic large sieve [10]. No step assumes anything about the zeros of $L(E_d, s)$ or of $\zeta(s)$. The prime number theorem for primes $\equiv 1 \pmod{3}$ uses only the classical nonvanishing of $\zeta(s)L(s, \chi_{-3})$ on $\Re s = 1$ [11, 25], and the remaining inputs are identities, the theta transformation and the large sieve.

The diagonal $n_1 = n_2 = n$ in (4.3) has

$$S_H(n, n) = \#\{u : 0 < N(u) \leq H, (u, n) = 1\} \ll H. \quad (4.7)$$

More precisely, elementary lattice counting gives, exactly as for ζ [3, (4.1)–(4.2)],

$$S_H(n, n) = \frac{2\pi}{\sqrt{3}} H \prod_{\pi|n} (1 - N(\pi)^{-1}) + O(\tau(n)(\sqrt{H} + 1)). \quad (4.8)$$

Here $\tau(n)$ counts primary divisors. The factor $2\pi/\sqrt{3}$ is the area of the disc divided by the lattice covolume. The diagonal is $O_W(HD)$. Thus (4.6) bounds the whole average on the diagonal scale, up to a small power loss. It bounds the total off-diagonal contribution, not the sum of its termwise absolute values.

5 Sixth powers isolate the chosen elliptic curve

Fix $u_0 = u_d$ and $N_0 = N(u_0) = |4d|^{10}$. For any $v \neq 0$,

$$\chi_n(u_0 v^6) = \chi_n(u_0) \mathbf{1}_{(n, v)=1}. \quad (5.1)$$

Consequently the rows we need are *the fixed row u_0 times sixth powers*; they need not themselves be sixth powers.

Take $H = D^{1+\vartheta}$, put $Y = (H/N_0)^{1/6}$, and let

$$\mathcal{P}_Y = \{\pi \text{ primary prime} : \pi \nmid 6d, Y/2 < N(\pi) \leq Y\}, \quad J = |\mathcal{P}_Y|. \quad (5.2)$$

For large D , the prime number theorem for $p \equiv 1 \pmod{3}$ (Landau's prime ideal theorem [25] for $\mathbb{Q}(\sqrt{-3})$) gives $J \asymp_d Y/\log Y$; for small Y there may be no such primes at all, since no primary prime coprime to 6 has norm below 7. The rows $u_0\pi^6$ are distinct and have norm at most H (Lemma 2). Their difference from the target is exactly

$$A_{u_0}(D) - A_{u_0\pi^6}(D) = \sum_{\pi|n} \mu_K(n) \alpha(n) \chi_n(u_0) W(N(n)/D). \quad (5.3)$$

There are $O_W(D/N(\pi))$ possible multiples of π in this sum, because dividing by π reduces the norm bound by $N(\pi)$. Every coefficient has absolute value at most one. Therefore

$$|A_{u_0}(D) - A_{u_0\pi^6}(D)| \ll_W D/Y. \quad (5.4)$$

Using $|z|^2 \leq 2|w|^2 + 2|z - w|^2$ for each selected row and the nonnegativity of the mean square gives

$$\boxed{J|S_d(D)|^2 \leq 2M(D, H) + O_W(JD^2/Y^2)}. \quad (5.5)$$

There is no assumption that different rows cancel against each other. Positivity is precisely what lets us extract the special rows.

Inserting (4.6), absorbing logarithms into D^ε , and allowing constants depending on d , we get

$$\begin{aligned} |S_d(D)|^2 &\ll_{d,W,\vartheta,\varepsilon} DH^{5/6} D^\varepsilon + D^2 H^{-1/3} \\ &= D^{11/6+5\vartheta/6+\varepsilon} + D^{5/3-\vartheta/3}. \end{aligned} \quad (5.6)$$

Take square roots, then choose $\vartheta > 0$ and the power loss small enough. For every $\eta > 0$ this proves

$$S_d(D) \ll_{d,W,\eta} D^{11/12+\eta}. \quad (5.7)$$

Dependence on d . The mean-square estimate (4.6) does not involve d ; the curve enters only through the position of its row. Keeping track of $N_0 = N(u_0)$ in the steps above gives

$$S_d(D) \ll_{W,\vartheta,\varepsilon} N_0^{1/12} D^{11/12+5\vartheta/12+\varepsilon} + N_0^{1/6} D^{5/6-\vartheta/6}, \quad N_0^{1/12} = |4d|^{5/6}, \quad (5.8)$$

valid for all $D \geq 1$, with a constant independent of d (Lemma 3; when Y is small, for instance when $Y < 7$ and so $J = 0$, the right side exceeds the trivial bound $|S_d(D)| \ll_W D$). This is nontrivial only for D larger than about $|d|^{10}$. That is a dependence on the chosen parameter d , not on the curve: E_{dt^6} is isomorphic to E_d over $\mathbb{Q}$, so $|d|$ can be made arbitrarily large while the conductor stays fixed. If d is first replaced by its sixth-power-free part, then every prime $p \geq 5$ dividing d divides the conductor $N(E_d)$ exactly twice. (The model is minimal at p because $v_p(-432d^2) < 12$, the reduction is additive because $c_4 = 0$, and additive reduction at $p \geq 5$ has conductor exponent 2; see [43, Remark VII.1.1 and Proposition VII.5.1] and [44, Chapter IV, §10].) So $|d| \leq 6^5 N(E_d)^{5/2}$, and $|d|^{10}$ is at most a fixed power of the conductor. For the zero-free half-plane this dependence is harmless, because each d is treated separately.

The exponent is easy to remember: the average has size about D^2 , and about $D^{1/6}$ useful copies of the target occur in it, so the target squared has size about $D^{2-1/6} = D^{11/6}$.

5.1 The extraction in detail

For ζ this is [35, Section 3, proof of Theorem 1.1, (3.2)–(3.3)]; see also [3, (4.3)–(4.6) and Section 5]. The rows π^6 there become the rows $u_0\pi^6$ here. This changes the scale to $Y = (H/N_0)^{1/6}$, excludes the primes dividing d , and makes constants depend on d . The angular factor enters only through $|\alpha(n)| = 1$. We write the argument out with the dependence on d explicit. Throughout, W is supported in $[a, b]$, $\|W\|_\infty = \sup |W|$, $H = D^{1+\vartheta}$ with $0 < \vartheta \leq 1/10$, $Y = (H/N_0)^{1/6}$, and $\mathcal{P}_Y, J$ are as above.

Lemma 2. *Let $D \geq 1$.*

- (i) $\#\{m \in \mathbb{Z}[\omega] : N(m) \leq x\} \leq 1 + 8x$ for every $x \geq 0$. Hence $|A_u(D)| \leq (1 + 8bD)\|W\|_\infty$ for every row u , in particular for $S_d(D) = A_{u_0}(D)$.
- (ii) The rows $u_0\pi^6$ with $\pi \in \mathcal{P}_Y$ are distinct, and $0 < N(u_0\pi^6) \leq H$.
- (iii) For $\pi \in \mathcal{P}_Y$, $|A_{u_0}(D) - A_{u_0\pi^6}(D)| \leq (1 + 16b)\|W\|_\infty D/Y$.
- (iv) $J|S_d(D)|^2 \leq 2M(D, H) + 2(1 + 16b)^2\|W\|_\infty^2 JD^2/Y^2$.

Proof. (i) This is an explicit form of [2, (2.4)]. The open discs of radius $\frac{1}{2}$ about the points of $\mathbb{Z}[\omega]$ are disjoint, and those about the points of norm at most x lie in the disc of radius $\sqrt{x} + \frac{1}{2}$. Comparing areas gives at most $4(\sqrt{x} + \frac{1}{2})^2 \leq 1 + 8x$ points when $x \geq 1$; when $x < 1$ only $m = 0$ is counted. The terms of $A_u(D)$ have modulus at most $\|W\|_\infty$ and vanish unless $N(n) \leq bD$.

(ii) $N(u_0\pi^6) = N_0N(\pi)^6 \leq N_0Y^6 = H$. If $u_0\pi^6 = u_0\pi'^6$, then $(\pi/\pi')^6 = 1$. So π/π' is a root of unity in K , hence a unit of $\mathbb{Z}[\omega]$, and the primary associates π, π' are equal [2, Proposition 11].

(iii) The symbol is multiplicative in the numerator, and $\chi_n(\pi)^6 = \mathbf{1}_{\pi \nmid n}$ by the convention on zero values. So $\chi_n(u_0\pi^6) = \chi_n(u_0)\mathbf{1}_{\pi \nmid n}$, and the difference is the sum over $\pi \mid n$ displayed before (5.4). Its terms have modulus at most $\|W\|_\infty$, since $|\mu_K(n)\alpha(n)\chi_n(u_0)| \leq 1$, and vanish unless $N(n) \leq bD$. Write $n = \pi m$. Then $m \in \mathbb{Z}[\omega]$ and $N(m) \leq bD/N(\pi) < 2bD/Y$, so by (i) there are at most $1 + 16bD/Y \leq (1 + 16b)D/Y$ terms. Here $Y \leq H^{1/6} = D^{(1+\vartheta)/6} \leq D$, because $N_0 \geq 1$.

(iv) For each $\pi \in \mathcal{P}_Y$, $|S_d(D)|^2 \leq 2|A_{u_0\pi^6}(D)|^2 + 2|S_d(D) - A_{u_0\pi^6}(D)|^2$. Sum over $\pi \in \mathcal{P}_Y$ and insert (iii). By (ii), the rows $u_0\pi^6$ are distinct terms of the sum $M(D, H)$, whose terms are nonnegative. $\square$

Proof of (5.7). Fix d, W, ϑ and ε . Each prime $p \equiv 1 \pmod{3}$ with $Y/2 < p \leq Y$ gives two primary primes of norm p , and at most $2\omega(d)$ primes of $\mathbb{Z}[\omega]$ divide d . So the prime number theorem for $p \equiv 1 \pmod{3}$ gives $J \geq Y/(5 \log Y)$ for $Y \geq Y_0(d)$, that is, for $D \geq D_0(d, \vartheta)$. For such D , Lemma 2(iv) and (4.6) give

$$|S_d(D)|^2 \leq \frac{2M(D, H)}{J} + O_W\left(\frac{D^2}{Y^2}\right) \ll_{d, W, \vartheta, \varepsilon} \frac{D^{2+\vartheta+\varepsilon} \log D}{Y} + \frac{D^2}{Y^2}, \quad (5.9)$$

which is (5.6), because $Y \asymp_d H^{1/6}$. For $1 \leq D < D_0$, Lemma 2(i) gives $|S_d(D)| \ll_{d, W, \vartheta} 1$. So (5.7) holds for all $D \geq 1$. Also $S_d(D) = 0$ for $D < 1/b$, and $S_d(D)$ is bounded for $1/b \leq D < 1$.

Lemma 3 (Uniformity in d). *Let $0 < \vartheta \leq 1/10$ and $\varepsilon > 0$. For all $d \neq 0$ and all $D \geq 1$,*

$$S_d(D) \ll_{W, \vartheta, \varepsilon} N_0^{1/12} D^{11/12+5\vartheta/12+\varepsilon} + N_0^{1/6} D^{5/6-\vartheta/6}, \quad (5.10)$$

with an implied constant that does not depend on d .

Proof. Since $Y^6 = D^{1+\vartheta}/N_0$, the two terms on the right are $D^{1+\vartheta/2+\varepsilon}Y^{-1/2}$ and D/Y . By the prime number theorem for primes $\equiv 1 \pmod{3}$, there are absolute constants $c_1 > 0$ and $Y_1 \geq 7$ such that for $Y \geq Y_1$ at least $2c_1Y/\log Y$ primary primes $\pi \nmid 6$ satisfy $Y/2 < N(\pi) \leq Y$. Some threshold is necessary: no primary prime coprime to 6 has norm below 7, so $J = 0$ when $Y < 7$.

Let $Y \geq Y_1$. The primes $\pi \mid d$ with $N(\pi) > Y/2$ number at most $2 \log |d|/\log(Y/2)$, because the product of their norms divides $N(d) = d^2$. Also $\log |d| < \frac{1}{10} \log N_0 \leq \frac{1}{10} \log H$, since $N_0 = |4d|^{10}$ and $N_0 \leq H$ when $Y \geq 1$. So the condition $\pi \nmid d$ removes fewer than $\log H/(5 \log(Y/2))$ primes from $\mathcal{P}_Y$.

Case 1: $Y \geq Y_1$ and $\log H/(5 \log(Y/2)) \leq c_1Y/\log Y$. Then $J \geq c_1Y/\log Y$. For $D \geq 2$, Lemma 2(iv), (4.6) and $\log Y \leq \log D$ give

$$|S_d(D)|^2 \ll_{W,\vartheta,\varepsilon} \frac{D^{2+\vartheta+\varepsilon} \log D}{Y} + \frac{D^2}{Y^2}, \quad (5.11)$$

and the claim follows after renaming ε .

Case 2: otherwise. If $Y \geq Y_1$, then $\log(Y/2) \geq \frac{1}{2} \log Y$, because $Y \geq 4$. So the failure of Case 1 gives $c_1Y/\log Y < 2 \log H/(5 \log Y)$, that is, $Y < 2 \log H/(5c_1)$. Hence in Case 2 $Y < Y_2 := \max(Y_1, 2(1+\vartheta) \log(2D)/(5c_1))$, and the first term on the right is at least $D^{1+\vartheta/2}Y_2^{-1/2} \gg_{\vartheta} D$. By Lemma 2(i), $|S_d(D)| \leq (1+8b)\|W\|_{\infty}D$. Case 1 cannot occur for $D < 2$, because then $Y \leq D^{(1+\vartheta)/6} < 7 \leq Y_1$. $\square$

5.2 Other perfect powers and the rational slice

The exact identities

$$\chi_n(v^3) = (v/n)_2, \quad \chi_n(v^2) = (v/n)_3 \quad (5.12)$$

identify cube and square rows as quadratic and cubic twists. If v is a nonzero rational integer, then $(v/n)_2 = (v/N(n))$, an ordinary Jacobi symbol. Hence

$$A_{uv^3}(D) = \sum_{m \geq 1} q_d(m) \left(\frac{v}{m} \right) W(m/D). \quad (5.13)$$

The finite-field reason at an inert prime is that every nonzero element of $\mathbb{F}_p$ is a square in $\mathbb{F}_{p^2}$; at a split prime it is ordinary Euler's criterion.

The entire rational slice $u \in \mathbb{Z}$ has only $\asymp H^{1/2}$ rows, since $N(u) = u^2$, whereas the lattice family has $\asymp H$ rows. Sixth powers of lattice elements number $\asymp H^{1/6}$; sixth powers of rational integers number $\asymp H^{1/12}$. The primes used in (5.5) are lattice primes, not merely rational prime bases. General rational numerators still require sextic residue symbols when computing their coefficients at split primes.

6 Why the bound excludes zeros

Proposition 2. *If (4.6) holds for every smooth compactly supported complex weight and every fixed $0 < \vartheta \leq 1/10$, then $L(E_d, s) \neq 0$ for $\Re s > 17/12$.*

Proof. We have proved (5.7) for every such weight (Section 5.1). Let

$$\widehat{W}(s) = \int_0^\infty W(y) y^s \frac{dy}{y}, \quad F_W(s) = \int_0^\infty S_d(D) D^{-s} \frac{dD}{D}. \quad (6.1)$$

The second integral is holomorphic for $\Re s > 11/12$ (Lemma 4(i)). At the lower endpoint there is no convergence problem: $S_d(D) = 0$ for $D < 1/b$. For $\Re s > 1$, absolute convergence (Lemma 4(i)) and (2.6) give

$$L(E_d, s + \tfrac{1}{2})F_W(s) = \widehat{W}(s)B_d(s). \quad (6.2)$$

Since $L(E_d, s)$ is entire (by Deuring's theorem [7]; see Section 6.1), the identity theorem extends this equality to $\Re s > 11/12$.

Suppose $L(E_d, \rho) = 0$ with $\Re \rho > 17/12$, and put $s_0 = \rho - 1/2$. Choose a nonzero nonnegative smooth function ϕ supported in $(1, 2)$, and use $W(y) = y^{-s_0}\phi(y)$. Then

$$\widehat{W}(s_0) = \int_1^2 \phi(y) \frac{dy}{y} > 0. \quad (6.3)$$

At $s = s_0$ the left side of (6.2) is zero, because F_W is holomorphic at s_0 . The right side $\widehat{W}(s_0)B_d(s_0)$ is nonzero, because B_d is holomorphic and nonzero for $\Re s > 0$ (Lemma 1(c)). This contradiction proves the assertion. $\square$

This explains both the allowance of complex weights and the shift by $1/2$. The zero-free boundary asserted is strict; no conclusion at $\Re s = 17/12$ is being drawn.

6.1 The analytic steps in detail

The skeleton is that of [35, Section 3, proof of Theorem 1.1, after (3.3)] and [3, Section 5]. Three things change in the CM case: the Dirichlet series is that of Lemma 1, the removed Euler factors $B_d(s)$ are those of the curve, and the continuation of L comes from Deuring's theorem rather than from Hecke's theorem for finite-order characters. We record the three analytic facts used in the proof above. Parts (ii) and (iii) show that the continuation of $L(E_d, s)$ is needed only to give its zeros a meaning.

Lemma 4. *Suppose that $S_d(D) \ll_{d,W,\eta} D^{11/12+\eta}$ for $D \geq 1$, for every $W \in C_c^\infty((0, \infty); \mathbb{C})$ and every $\eta > 0$.*

- (i) F_W is holomorphic on $\Re s > 11/12$, and (6.2) holds for $\Re s > 1$.
- (ii) There is a holomorphic function G_d on $\Re s > 11/12$ with $G_d(s) = 1/L(E_d, s + \frac{1}{2})$ for $\Re s > 1$. It is $G_d = F_W/(\widehat{W}B_d)$ for any W with $\widehat{W}(s) \neq 0$. In other words, $1/L(E_d, s)$, defined by its Euler product for $\Re s > 3/2$, continues holomorphically to $\Re s > 17/12$.
- (iii) Any meromorphic continuation of $L(E_d, s)$ to $\Re s > 17/12$ has no zeros there.

Proof. (i) The holomorphy of F_W on $\Re s > 11/12$ follows from the hypothesis exactly as in [35, Section 3]: the truncated integrals are entire and converge locally uniformly. Now let $\sigma = \Re s > 1$. By (3.9), $|q_d(m)| \leq r(m)$, where $r(m)$ is the number of primary n coprime to 6 with $N(n) = m$, and $\sum_m r(m)m^{-\sigma} \leq \sum_{n \text{ primary}} N(n)^{-\sigma} < \infty$ [2, (2.8)]. So, substituting $D = m/y$,

$$\int_0^\infty \sum_{m \geq 1} |q_d(m)W(m/D)| D^{-\sigma} \frac{dD}{D} = \sum_{m \geq 1} \frac{|q_d(m)|}{m^\sigma} \int_0^\infty |W(y)| y^\sigma \frac{dy}{y} < \infty. \quad (6.4)$$

Fubini's theorem then gives $F_W(s) = \widehat{W}(s) \sum_m q_d(m)m^{-s}$, which is (6.2) by (2.6).

(ii) Since W has compact support in $(0, \infty)$, $\widehat{W}$ is entire. For two weights W, W' and $\Re s > 1$, part (i) gives

$$F_W(s)\widehat{W'}(s) = \frac{\widehat{W}(s)\widehat{W'}(s)B_d(s)}{L(E_d, s + \frac{1}{2})} = F_{W'}(s)\widehat{W}(s). \quad (6.5)$$

Both sides are holomorphic on the connected half-plane $\Re s > 11/12$, so they agree there. Since B_d is holomorphic and nonzero on $\Re s > 0$ (Lemma 1(c)), the functions $F_W/(\widehat{W}B_d)$ and $F_{W'}/(\widehat{W'}B_d)$ coincide wherever $\widehat{W}$ and $\widehat{W'}$ are both nonzero. Given s_0 with $\Re s_0 > 11/12$, the weight $W_{s_0}(y) = y^{-s_0}\phi(y)$ of the proof above has $\widehat{W}_{s_0}(s_0) = \int_1^2 \phi(y) dy/y > 0$, so $\widehat{W}_{s_0} \neq 0$ near s_0 . These local quotients therefore define one holomorphic function G_d on $\Re s > 11/12$, and $G_d(s) = 1/L(E_d, s + \frac{1}{2})$ for $\Re s > 1$ by (i).

(iii) If $L(E_d, s + \frac{1}{2})$ is meromorphic on $\Re s > 11/12$, then $L(E_d, s + \frac{1}{2})G_d(s) = 1$ on $\Re s > 1$, hence on all of $\Re s > 11/12$ away from the poles of L . At a zero s_0 of $L(E_d, s + \frac{1}{2})$ with $\Re s_0 > 11/12$ both factors would be holomorphic and the product would vanish at s_0 , which is impossible. $\square$

In fact $L(E_d, s)$ is entire. By Deuring's theorem [7] it is the L -function of a Hecke character of $K = \mathbb{Q}(\sqrt{-3})$ [44, Chapter II, §10], and Hecke [18] continued such L -functions to entire functions [33, Chapter VII, §8]. It also follows from the modularity of E_d [42]. This is the fact used in the proof of the proposition; part (iii) shows that a meromorphic continuation would suffice. For ζ the corresponding input is the meromorphic continuation of $\zeta(s)L(s, \chi_{-3})$ [3, Section 5].

7 Poisson summation reveals the cubic Gauss sums

We now explain the analytic work behind (4.6). Additive characters, Gauss sums and Poisson summation over $\mathbb{Z}[\omega]$, including the identities used in this section, are explained in [2, Section 4]. The CM statements of this section are proved in Section 7.3. Choose a fixed smooth radial majorant $\Phi(N(u)/H)$, nonnegative and at least one on $N(u) \leq H$, rapidly decreasing at infinity, with compactly supported planar Fourier transform. Put

$$e(z) = \exp(4\pi i \operatorname{Im} z/\sqrt{3}), \quad \widehat{\Phi}(N(w)) = \frac{2}{\sqrt{3}} \int_{\mathbb{C}} \Phi(|z|^2) e(-zw) dx dy. \quad (7.1)$$

The smooth mean square over all $u \in \mathbb{Z}[\omega]$, including zero, is an upper bound for (4.1).

Only squarefree columns occur. Write $n_i = gz_i$, where g is their primary greatest common divisor. Then $(z_1, z_2) = (g, z_1 z_2) = 1$, and

$$\overline{\chi_{n_1}(u)} \chi_{n_2}(u) = \mathbf{1}_{(u,g)=1} \psi(u), \quad \psi = \overline{\chi_{z_1}} \chi_{z_2}. \quad (7.2)$$

This is a primitive character modulo $z_1 z_2$. For a primitive character modulo a primary q (here $q = z_1 z_2$), define the normalized finite Gauss sum

$$\gamma(\psi) = \frac{1}{\sqrt{N(q)}} \sum_{x \bmod q} \psi(x) e(x/q). \quad (7.3)$$

Inclusion–exclusion for $(u, g) = 1$, followed by planar Poisson summation [21, Chapter 4], gives the exact formula [35, Lemma 4.2]

$$\begin{aligned} & \sum_{u \in \mathbb{Z}[\omega]} \Phi(N(u)/H) \mathbf{1}_{(u,g)=1} \psi(u) \\ &= \frac{H\gamma(\psi)}{\sqrt{N(z_1 z_2)}} \sum_{e_0 | g} \frac{\mu_K(e_0) \psi(e_0)}{N(e_0)} \sum_{h \in \mathbb{Z}[\omega]} \overline{\psi(h)} \widehat{\Phi} \left(\frac{HN(h)}{N(e_0)N(z_1 z_2)} \right). \end{aligned} \quad (7.4)$$

The divisor e_0 must be kept. The zero frequency contributes only when $z_1 = z_2 = 1$; it gives the diagonal main term. Nonzero frequencies from these diagonal pairs need $HN(h) \ll N(e_0) \leq N(g) \ll D$, because $\widehat{\Phi}$ has compact support; for $H = D^{1+\vartheta}$ they therefore occur only for bounded D . They are kept in the estimate anyway. The other frequencies satisfy

$$N(h) \ll \frac{N(e_0)N(z_1 z_2)}{H} \ll \frac{D^2}{HN(g)}. \quad (7.5)$$

This is the first shortening: $H > D$ produces frequencies of norm less than about D .

7.1 The arithmetic identity and the extra angular factor

For squarefree primary n coprime to 6 define

$$\gamma_j(n) = \frac{1}{\sqrt{N(n)}} \sum_{x \bmod n} \chi_n(x)^j e(x/n), \quad G(n) = \overline{\chi_n(4)} \gamma_3(n). \quad (7.6)$$

In particular, γ_2 is a cubic Gauss sum. The finite-sum identity used in both arguments is

$$\mu_K(n) \gamma_{-1}(n) = \chi_n(-1) G(n)^{-1} \overline{\alpha(n)} \gamma_2(n). \quad (7.7)$$

It is [35, Lemma 4.1]; see also [2, Proposition 18]. It is proved prime by prime from Jacobi-sum relations [20, Chapter 8] (compare the Hasse–Davenport product formula [6]), which give $\gamma_2(\pi)^3 = -\alpha(\pi)$ (compare [13, pp. 443–445] and [15, (2)]) for a primary prime π , and extends to squarefree n by the Chinese remainder theorem. Reciprocity is not needed for (7.7) itself. It is needed for the next two facts: that G is a function of a residue class, and the paired identity below. The Möbius sign is essential here. The function G has absolute value one and depends only on the primary invertible residue class modulo 72 (in fact only on $n \bmod 4$).

Let $\mathfrak{G}$ be that finite group of classes: $\mathfrak{G} = \{x \in (\mathbb{Z}[\omega]/72\mathbb{Z}[\omega])^\times : x \equiv 1 \pmod{3}\}$. The prime 2 is inert and 3 is ramified, so $|(\mathbb{Z}[\omega]/72\mathbb{Z}[\omega])^\times| = N(72)(1 - \frac{1}{4})(1 - \frac{1}{3}) = 2592$. Reduction modulo 3 maps this group onto $(\mathbb{Z}[\omega]/3\mathbb{Z}[\omega])^\times$, whose six classes are those of the units [2, Proposition 11], and $\mathfrak{G}$ is the kernel; so $|\mathfrak{G}| = 432$. Since every ideal prime to 6 has exactly one primary generator, $\mathfrak{G}$ is the ray class group modulo 72, and its characters ξ are finite-order Hecke characters; $\xi(n)$ means ξ of the class of the primary element n . By the Chinese remainder theorem ($72 = 8 \cdot 9$), reduction maps $\mathfrak{G}$ onto $(\mathbb{Z}[\omega]/4\mathbb{Z}[\omega])^\times$, a group of order 12. Define, quite explicitly,

$$c_\xi = \frac{1}{|\mathfrak{G}|} \sum_{t \in \mathfrak{G}} G(t^{-1}) \overline{\xi(t)}, \quad a_\xi(n) = \overline{\alpha(n)}^2 \gamma_2(n) \xi(n), \quad (7.8)$$

where ξ ranges over the characters of this fixed finite group. For coprime squarefree z_1, z_2 , the useful paired identity, the CM form of [35, (4.15)], is

$$\begin{aligned} \mu_K(z_1) \mu_K(z_2) \overline{\alpha(z_1)} \alpha(z_2) \gamma(\overline{\chi_{z_1}} \chi_{z_2}) \\ = \sum_{\xi} c_\xi a_\xi(z_1) \overline{a_\xi(z_2)}. \end{aligned} \quad (7.9)$$

This preserves the structure of a squared sum. The zeta version produces $\overline{\alpha} \gamma_2$; the present version produces $\overline{\alpha}^2 \gamma_2$. The first factor of $\overline{\alpha}$ comes from (7.7), as for zeta. The second is the angular factor of (3.4): expanding $|A_u|^2$ gives $\overline{\alpha(n_1)} \alpha(n_2) = \overline{\alpha(z_1)} \overline{\alpha(z_2)}$, which has the same paired form. Unlike a finite-order twist, it cannot be absorbed into the characters ξ . Lemma 5 proves (7.9), together with the reverse identity used in Section 10.

7.2 The next average, with every variable displayed

After resolving greatest common divisors, the averages have the form

$$\mathcal{E}(\mathcal{H}, X, F) = \frac{1}{XF} \sum_{F \leq N(f) < 2F}^* \sum_{0 < N(k) \leq \mathcal{H}} \left| \sum_n^* a_\xi(n) \chi_n(k) \chi_n(f)^4 W(N(n)/X) \right|^2. \quad (7.10)$$

Both f, n are primary, squarefree, and coprime to 6; k runs over the nonzero lattice points of norm at most $\mathcal{H}$. The weight and the finite character ξ have been suppressed on the left. Put $\Sigma = XF$.

There are two sources of the fourth power. The divisor e_0 in Poisson contributes $\chi_n(he_0^5)$. The common divisor v_0 that removes the condition $(z_1, z_2) = 1$ uses the twisted multiplicativity

$$a_\xi(v_0 n) = a_\xi(v_0) a_\xi(n) \chi_n(v_0)^4 \quad ((v_0, n) = 1). \quad (7.11)$$

With $f = e_0 v_0$ and $k = e_0 h$, these combine as $he_0^5 v_0^4 = k f^4$. Neither source can be omitted; see Step 3 in the proof of Proposition 3.

The initial scales are, for a further divisor-size parameter $B \geq 1$,

$$X = \frac{D}{BF}, \quad \Sigma = \frac{D}{B}, \quad \mathcal{H} \ll \frac{D^2}{HB^2}. \quad (7.12)$$

The first Poisson reduction, Proposition 3 below, shows that the estimate $\mathcal{E} \ll D^\varepsilon \Sigma$ in these ranges implies (4.6). Smooth kernels in two column variables are separated using Mellin integrals, with a controlled finite number of derivatives [35, Lemma B.2]; they are not simply discarded.

7.3 The CM changes in detail

This subsection proves the statements of this section for the CM coefficients. We first list the inputs that are literally those of the zeta case. Their proofs involve the column coefficients at most through (7.19) and $|a_\xi| \leq 1$, so we only cite them.

- *Poisson summation.* Formula (7.4) is [35, Lemma 4.2] with modulus $z_1 z_2$ and exclusion ideal (g) . It is proved in [2, Lemma 21] from Poisson summation over the self-dual lattice $\mathbb{Z}[\omega]$ [2, Proposition 16 and Theorem 19]. A majorant Φ with the stated properties exists [2, end of Section 4.3]. We fix it, together with a constant C_Φ such that $\widehat{\Phi}(t) = 0$ for $t > C_\Phi$.
- *Gauss sums.* The character $\psi = \overline{\chi_{z_1}} \chi_{z_2}$ is primitive modulo $z_1 z_2$, because its component at each prime $\pi \mid z_1 z_2$ is $\chi_\pi^{\pm 1} \neq 1$, and therefore $|\gamma(\psi)| = 1$ [2, Lemma 17 and Section 4.5]. For squarefree primary n coprime to 6 and $6 \nmid j$, the character χ_n^j is primitive modulo n , so $|\gamma_j(n)| = 1$ [2, Lemma 17]. [35, Lemma 4.1] (see also [2, Proposition 18]) gives the identity (7.7), which is [35, (4.5)], and [35, (4.4)]:

$$\overline{\alpha(n)} \gamma_2(n) \gamma_1(n) = \mu_K(n) G(n). \quad (7.13)$$

The factor $\overline{\alpha(n)}$ in these identities comes from $\gamma_2(n)^3 = \mu_K(n) \alpha(n)$. It has nothing to do with the curve and is present for zeta as well.

- *The finite group.* By [35, Appendix A.1], $G(n)$ depends only on $n \bmod 4$. There is a symmetric bicharacter $\mathcal{R}$ with values ± 1 , depending only on classes modulo 4, such that $\chi_b(a) = \mathcal{R}(a, b) \chi_a(b)$ for coprime primary a, b coprime to 6; this is sextic reciprocity [26, §7.3], in the form [35, (4.1)]. That $\mathcal{R}$ depends only on classes modulo 4 is shown in [35, Appendix A.1],

where it is given explicitly on the square classes of $(\mathbb{Z}[\omega]/4\mathbb{Z}[\omega])^\times$; also $\mathcal{R}(c, c) = \chi_c(-1)$ there. Reduction from $\mathfrak{G}$ to $(\mathbb{Z}[\omega]/4\mathbb{Z}[\omega])^\times$ is a homomorphism, so G and $\mathcal{R}$ are functions on $\mathfrak{G}$, and so is $\chi_x(-1) = (-1)^{(N(x)-1)/6}$. On $\mathfrak{G}$,

$$\chi_x(-1)\overline{G(x)}G(y)\mathcal{R}(x, y) = G(yx^{-1}) \quad (x, y \in \mathfrak{G}), \quad (7.14)$$

which is [35, (4.7)]. In [35] the group is an unspecified fixed ray class group; classes modulo 4 would suffice for this section, and the modulus 72 leaves room for the residue-class factors absorbed into ξ later.

- *Exclusions.* Write $\mathcal{E}(\mathcal{H}, X, F; \xi, W)$ for (7.10) with the twist and the weight shown, and $\mathcal{E}_{(r)}$ for the same average with the extra condition $(n, r) = 1$ on the columns. For squarefree primary r coprime to 6, $F \geq 1$ and $\mathcal{H}, X > 0$,

$$\mathcal{E}_{(r)}(\mathcal{H}, X, F; \xi, W) \leq \tau(r) \sum_{j|r} \mathcal{E}\left(\mathcal{H}, \frac{X}{N(j)}, FN(j); \xi, W\right). \quad (7.15)$$

This is [35, Lemma 4.4]. Its proof is the identity (10.14) followed by Cauchy–Schwarz over j and the injectivity of $f \mapsto jf$. It uses only the twisted multiplicativity of a_ξ and $|a_\xi| \leq 1$, which hold for the CM coefficients by (7.19) below.

- *Smooth weights.* The Mellin separation of smooth weights, and its version for kernels that depend on the row, are [35, Lemmas B.1 and B.2].

Normalizations. The identities above are sensitive to normalizations, and we use those of [35]: primary means $\equiv 1 \pmod{3}$, $e(z) = \exp(4\pi i \operatorname{Im} z / \sqrt{3})$, $\chi_n^{-1} = \overline{\chi_n}$ with zero values kept, and every Gauss sum is formed with the primary generator of its modulus. The last point matters: replacing q by ζq , with ζ a unit, multiplies $\gamma(\psi)$ by $\psi(\zeta)$. The first point matters too: if π is a primary prime, then $\pi' = -\pi \equiv 2 \pmod{3}$ is primary in the sense of [20], and $\gamma_2(\pi') = \chi_\pi(-1)^2 \gamma_2(\pi) = \gamma_2(\pi)$ but $\alpha(\pi') = -\alpha(\pi)$. The sign $\mu_K(n)$ in $\gamma_2(n)^3 = \mu_K(n)\alpha(n)$, which supplies the Möbius factor in (7.7), is therefore tied to the normalization $\equiv 1 \pmod{3}$.

What changes. Only the column coefficient changes. In [35] the column coefficients are $\mu_K(n)\nu(n)$, with ν of finite order. In [35, (4.15)] the product $\nu(z_1)\nu(z_2)$ is a function of the class of $z_1 z_2^{-1}$, and it is absorbed into the characters ξ . The angular factor α has infinite order, so $\alpha(z_1)\alpha(z_2)$ is not a function of a residue class. It does have the paired form $\overline{\alpha}(z_1) \cdot \overline{\alpha}(z_2)$, so it can be absorbed into the coefficients instead.

Lemma 5 (Paired identities, CM form). *For characters ξ, ξ_1 of $\mathfrak{G}$ define $c_\xi(\xi_1)$ and $\ell_{\xi_1}(\xi)$ by*

$$\xi_1(t)G(t^{-1}) = \sum_{\xi} c_\xi(\xi_1)\xi(t), \quad (\xi G)(t) = \sum_{\xi_1} \ell_{\xi_1}(\xi)\xi_1(t) \quad (t \in \mathfrak{G}). \quad (7.16)$$

Thus $c_\xi(\mathbf{1}) = c_\xi$, and all these coefficients have absolute value at most one. Put $\beta_{\xi_1}(n) = \mu_K(n)\alpha(n)\xi_1(n)$. Let z_1, z_2 be coprime, squarefree, primary and coprime to 6. Then

$$\mu_K(z_1)\mu_K(z_2)\overline{\alpha(z_1)}\alpha(z_2)\xi_1(z_1)\overline{\xi_1(z_2)}\gamma(\overline{\chi_{z_1}}\chi_{z_2}) = \sum_{\xi} c_\xi(\xi_1)a_\xi(z_1)\overline{a_\xi(z_2)}, \quad (7.17)$$

$$a_\xi(z_1)\overline{a_\xi(z_2)}\gamma(\chi_{z_1}\overline{\chi_{z_2}}) = \sum_{\xi_1} \ell_{\xi_1}(\xi)\beta_{\xi_1}(z_1)\overline{\beta_{\xi_1}(z_2)}. \quad (7.18)$$

Moreover, for coprime squarefree primary a, b coprime to 6,

$$a_\xi(ab) = a_\xi(a)a_\xi(b)\chi_b(a)^4, \quad |a_\xi(a)| = 1. \quad (7.19)$$

In particular, (7.9) is (7.17) with $\xi_1 = \mathbf{1}$.

Proof. Write $a^\zeta(n) = \overline{\alpha(n)}\gamma_2(n)$ for the zeta coefficient, so that $a_\xi = \overline{\alpha}a^\zeta\xi$. By the Chinese remainder theorem and sextic reciprocity [2, (4.27)],

$$\gamma(\overline{\chi_{z_1}}\chi_{z_2}) = \mathcal{R}(z_1, z_2)\gamma_{-1}(z_1)\gamma_1(z_2). \quad (7.20)$$

In the same way $\gamma(\chi_{z_1}\overline{\chi_{z_2}}) = \mathcal{R}(z_1, z_2)\gamma_1(z_1)\gamma_{-1}(z_2)$. Since $|\gamma_2| = |\alpha| = |G| = 1$ and $\mu_K(n)^2 = 1$, the identities (7.7) and (7.13) can be written as

$$\mu_K(n)\gamma_{-1}(n) = \chi_n(-1)\overline{G(n)}a^\zeta(n), \quad \mu_K(n)\gamma_1(n) = G(n)\overline{a^\zeta(n)}. \quad (7.21)$$

Proof of (7.17). Apply the first identity of (7.21) to z_1 and the second to z_2 , and then (7.14) with $x = z_1, y = z_2$:

$$\mu_K(z_1)\mu_K(z_2)\gamma(\overline{\chi_{z_1}}\chi_{z_2}) = \chi_{z_1}(-1)\overline{G(z_1)}G(z_2)\mathcal{R}(z_1, z_2)a^\zeta(z_1)\overline{a^\zeta(z_2)} = G(z_2z_1^{-1})a^\zeta(z_1)\overline{a^\zeta(z_2)}. \quad (7.22)$$

This is the zeta identity [2, (4.28)]; expanded in characters it is [3, (6.8)], or [35, (4.15)] with $\nu = \mathbf{1}$. It supplies the first factor $\overline{\alpha}$ of a_ξ . The second factor comes from the coefficients $\alpha(n)$ of (3.4): the conjugated column of $|A_u|^2$ gives $\overline{\alpha(z_1)}$ and the other column gives $\alpha(z_2) = \overline{\overline{\alpha(z_2)}}$, so multiplication by $\overline{\alpha(z_1)}\alpha(z_2)$ turns $a^\zeta(z_1)\overline{a^\zeta(z_2)}$ into $\overline{\alpha(z_1)}^2\gamma_2(z_1) \cdot \overline{\alpha(z_2)}^2\gamma_2(z_2)$. Finally multiply by $\xi_1(z_1)\overline{\xi_1(z_2)} = \xi_1(t)$, where $t = z_1z_2^{-1} \in \mathfrak{G}$. Since $G(z_2z_1^{-1}) = G(t^{-1})$, expanding $\xi_1(t)G(t^{-1})$ and using $\xi(t) = \xi(z_1)\xi(z_2)$ gives (7.17).

Proof of (7.18). Now apply the second identity of (7.21) to z_1 and the first to z_2 . Since $\overline{\alpha}^2\gamma_2 = \overline{\alpha}a^\zeta$,

$$\overline{\alpha(z_1)}^2\gamma_2(z_1)\gamma_1(z_1) = \overline{\alpha(z_1)}\mu_K(z_1)G(z_1), \quad \overline{\alpha(z_2)}^2\gamma_2(z_2)\gamma_{-1}(z_2) = \alpha(z_2)\mu_K(z_2)\chi_{z_2}(-1)\overline{G(z_2)}. \quad (7.23)$$

Multiply these, include $\xi(z_1)\overline{\xi(z_2)}$ and $\mathcal{R}(z_1, z_2)$, and use (7.14) with $x = z_2, y = z_1$ and the symmetry of $\mathcal{R}$:

$$a_\xi(z_1)\overline{a_\xi(z_2)}\gamma(\chi_{z_1}\overline{\chi_{z_2}}) = \mu_K(z_1)\mu_K(z_2)\overline{\alpha(z_1)}\alpha(z_2)(\xi G)(z_1z_2^{-1}). \quad (7.24)$$

Expand ξG , and note that $\mu_K(z_1)\mu_K(z_2)\overline{\alpha(z_1)}\alpha(z_2)\xi_1(z_1)\overline{\xi_1(z_2)} = \beta_{\xi_1}(z_1)\overline{\beta_{\xi_1}(z_2)}$. Without the factor $\overline{\alpha(z_1)}\alpha(z_2)$ this is [35, (7.3)]. In the CM case the Möbius coefficients returned by the reverse identity are $\mu_K\overline{\alpha}\xi_1$, as used in Section 10.

Proof of (7.19). For zeta this is [35, (4.6)], where the twisted multiplicativity comes from γ_2 alone (via [2, (4.14)] and cubic reciprocity [9, [20, Chapter 9]]). Replacing $\overline{\alpha}$ by the completely multiplicative unimodular factor $\overline{\alpha}^2$ changes nothing. $\square$

Proposition 3 (First Poisson reduction, CM form). *Fix $0 < \vartheta \leq 1/10$ and put $H = D^{1+\vartheta}$. For a compact interval $I \subset (0, \infty)$ write $\|W\|_{C^J(I)} = \max_{0 \leq i \leq J} \sup_{x \in I} |W^{(i)}(x)|$. Suppose that for every $C \geq 1$ and every $\varepsilon > 0$ there is an integer $J = J(\vartheta, \varepsilon)$ such that*

$$\mathcal{E}(\mathcal{H}, X, F; \xi, W) \ll_{I, C, \vartheta, \varepsilon} \|W\|_{C^J(I)}^2 D^\varepsilon X F \quad (7.25)$$

for every character ξ of $\mathfrak{G}$, every compact interval $I \subset (0, \infty)$, every $W \in C_c^\infty(I)$, and all real $B, F \geq 1$ and $\mathcal{H}$ with

$$X = \frac{D}{BF}, \quad 0 < \mathcal{H} \leq \frac{CD^2}{HB^2}. \quad (7.26)$$

Then, for every compact $I \subset (0, \infty)$ and $W \in C_c^\infty(I)$,

$$M(D, H) \ll_{I, \vartheta, \varepsilon} \|W\|_{C^{J'}(I)}^2 D^{2+\vartheta+\varepsilon}, \quad J' = 2J(\vartheta, \varepsilon/4) + 4. \quad (7.27)$$

In particular (4.6) holds.

The ranges (7.26) are those of (7.12), with $\Sigma = XF = D/B$. For zeta this is [35, Proposition 4.5]; the hypothesis has the same form, with the coefficients $a_\xi = \bar{\alpha}^2 \gamma_2 \xi$ in (7.10).

Proof. Write $I = [a_0, b_0]$. We follow the proof of [35, Proposition 4.5] and say at each step what the factor α does.

Step 1: expansion and Poisson summation in the row. Since $\Phi \geq 0$ and $\Phi \geq 1$ on $[0, 1]$, we have $M(D, H) \leq D\mathcal{M}_D$, where $\mathcal{M}_D = D^{-1} \sum_{u \in \mathbb{Z}[\omega]} \Phi(N(u)/H) |A_u(D)|^2$. Expand $|A_u(D)|^2 = \overline{A_u(D)} A_u(D)$, with n_1 the column of the conjugated factor, and write $n_i = gz_i$ as before. The pair (n_1, n_2) carries the coefficient $\mu_K(n_1) \mu_K(n_2) \alpha(n_1) \alpha(n_2) \overline{W(N(n_1)/D)} W(N(n_2)/D)$, and

$$\mu_K(n_1) \mu_K(n_2) = \mu_K(z_1) \mu_K(z_2), \quad \overline{\alpha(n_1)} \alpha(n_2) = |\alpha(g)|^2 \overline{\alpha(z_1)} \alpha(z_2) = \overline{\alpha(z_1)} \alpha(z_2), \quad (7.28)$$

because $\mu_K(g)^2 = 1$ and α is multiplicative of absolute value one. The coefficient does not depend on u , so the sum over u is given by (7.4), exactly as for zeta. The frequency $h = 0$ occurs only when $z_1 = z_2 = 1$. There the coefficient is $|\mu_K(g) \alpha(g)|^2 |W(N(g)/D)|^2 = |W(N(g)/D)|^2$, and these terms give

$$Z = \frac{H}{D} \widehat{\Phi}(0) \sum_g^* \left| W\left(\frac{N(g)}{D}\right) \right|^2 \prod_{\pi|g} (1 - N(\pi)^{-1}) \ll_I H \|W\|_\infty^2, \quad (7.29)$$

the same as for zeta. All $h \neq 0$ are kept, including those with $z_1 = z_2 = 1$.

Step 2: the paired identity. This is the only step in which the CM coefficients change the algebra. By Step 1 and (7.9),

$$\mu_K(n_1) \mu_K(n_2) \overline{\alpha(n_1)} \alpha(n_2) \gamma(\psi) = \sum_\xi c_\xi a_\xi(z_1) \overline{a_\xi(z_2)}; \quad (7.30)$$

when $z_1 = z_2 = 1$ this reads $1 = \sum_\xi c_\xi = G(1)$. Moreover $\psi(e_0) \overline{\psi(h)} = \chi_{z_1}(he_0^5) \overline{\chi_{z_2}(he_0^5)}$, because $(e_0, z_1 z_2) = 1$ and $\overline{\chi_z(e_0)} = \chi_z(e_0)^5$. Put $W_0(x) = x^{-1/2} \overline{W(x)}$, so that

$$\frac{\overline{W(N(gz_1)/D)} W(N(gz_2)/D)}{\sqrt{N(z_1 z_2)}} = \frac{N(g)}{D} W_0\left(\frac{N(gz_1)}{D}\right) \overline{W_0\left(\frac{N(gz_2)}{D}\right)}. \quad (7.31)$$

Then $\mathcal{M}_D - Z = \sum_\xi c_\xi \mathcal{S}_\xi$, where

$$\begin{aligned} \mathcal{S}_\xi = \frac{H}{D^2} \sum_g^* \sum_{e_0|g} \frac{\mu_K(e_0) N(g)}{N(e_0)} \sum_{h \neq 0} \sum_{\substack{(z_1, z_2)=1 \\ (z_1 z_2, g)=1}}^* a_\xi(z_1) \overline{a_\xi(z_2)} \chi_{z_1}(he_0^5) \overline{\chi_{z_2}(he_0^5)} \\ \times W_0\left(\frac{N(gz_1)}{D}\right) \overline{W_0\left(\frac{N(gz_2)}{D}\right)} \widehat{\Phi}\left(\frac{HN(h)}{N(e_0)N(z_1)N(z_2)}\right). \end{aligned} \quad (7.32)$$

There are $|\mathfrak{G}| = 432$ characters and $|c_\xi| \leq 1$, so it suffices to bound each $\mathcal{S}_\xi$. Apart from the coefficients a_ξ , (7.32) is the display following [35, (4.15)].

Step 3: the coprimality condition and the fourth power. Insert $\mathbf{1}_{(z_1, z_2)=1} = \sum_{v_0|z_1, v_0|z_2} \mu_K(v_0)$ and write $z_i = v_0 m_i$. By (7.19), $a_\xi(v_0 m_i) = a_\xi(v_0) a_\xi(m_i) \chi_{m_i}(v_0)^4$, and the last factor vanishes unless $(m_i, v_0) = 1$; so the condition that z_i be squarefree may be dropped. Since $|a_\xi(v_0)| = 1$ and $(v_0, e_0) = 1$, the two copies of v_0 leave $|a_\xi(v_0) \chi_{v_0}(h e_0^5)|^2 = \mathbf{1}_{(v_0, h)=1}$, and the twist becomes $\chi_{m_i}(h e_0^5 v_0^4)$. The CM factor $\bar{\alpha}^2$ enters this step only through (7.19): like the factor $\bar{\alpha}$ of the zeta coefficients, it is completely multiplicative and unimodular. Now change variables:

$$r_0 = g/e_0, \quad f = e_0 v_0, \quad k = e_0 h; \quad e_0 = (f, k), \quad v_0 = f/e_0, \quad g = r_0 e_0, \quad h = k/e_0 \quad (7.33)$$

(in [35], r_0 is called b). This is the bijection of the proof of [35, Proposition 4.5], onto the triples (r_0, f, k) with r_0, f squarefree, $(r_0, f) = 1$ and $k \neq 0$; it does not see the coefficients. It satisfies

$$g v_0 = r_0 f, \quad h e_0^5 v_0^4 = k f^4, \quad \mu_K(e_0) \mu_K(v_0) = \mu_K(f), \quad (7.34)$$

$$\frac{N(g)}{N(e_0)} = N(r_0), \quad \frac{N(h)}{N(e_0) N(v_0)^2} = \frac{N(k)}{N(f)^2}. \quad (7.35)$$

Therefore

$$\begin{aligned} \mathcal{S}_\xi = \frac{H}{D^2} \sum_{(r_0, f)=1}^* \mu_K(f) N(r_0) \sum_{k \neq 0} \sum_{(m_1 m_2, r_0)=1}^* a_\xi(m_1) \overline{a_\xi(m_2)} \chi_{m_1}(k f^4) \overline{\chi_{m_2}(k f^4)} \\ \times W_0\left(\frac{N(r_0 f m_1)}{D}\right) \overline{W_0\left(\frac{N(r_0 f m_2)}{D}\right)} \widehat{\Phi}\left(\frac{HN(k)}{N(f)^2 N(m_1) N(m_2)}\right). \end{aligned} \quad (7.36)$$

The factors $\chi_{m_i}(f)^4$ enforce $(m_i, f) = 1$, and no condition $(m_1, m_2) = 1$ remains. This is [35, (4.16)] with the CM coefficients. The inner sums are the column sums of (7.10), with the exclusion $(n, r_0) = 1$.

Step 4: the scales. On the support, $N(r_0 f m_i) \leq b_0 D$. Hence $N(r_0) N(f) \leq b_0 D$ and $HN(k) \leq C_\Phi N(f)^2 N(m_1) N(m_2) \leq C_\Phi b_0^2 D^2 / N(r_0)^2$. Split r_0 and f into dyadic ranges $B \leq N(r_0) < 2B$ and $F \leq N(f) < 2F$, write $\mathcal{S}_{\xi; B, F}$ for the corresponding part of (7.36), and put

$$X = \frac{D}{BF}, \quad \mathcal{H} = \frac{C_1 D^2}{H B^2}, \quad C_1 = \max(1, C_\Phi b_0^2). \quad (7.37)$$

These satisfy (7.26) with $C = C_1$; they are the scales (7.12), with $\Sigma = XF = D/B$. The angular factor does not affect any support, so this step is that of [35, proof of Proposition 4.5] verbatim.

Step 5: separation of the kernel. The rest is the end of the proof of [35, Proposition 4.5], from its kernel $\mathcal{K}_{b, f, k}$ to its display (4.17), word for word. In the variables $x_i = N(m_i)/X$ the weight in (7.36) is $W_0(\varrho x_1) \overline{W_0(\varrho x_2)} \widehat{\Phi}(\varkappa/(x_1 x_2))$, with $\varrho = N(r_0) N(f)/(BF) \in [1, 4)$ and $0 < \varkappa = HN(k)/(N(f)^2 X^2) \leq C_1$; it is supported in $[a_0/4, b_0]^2$, with C^q norms $\ll_{I, q} \|W\|_{C^q(I)}^2$. For fixed r_0 , the scales $(X/N(j), FN(j))$ with $j \mid r_0$ again satisfy (7.26), so (7.15) and (7.25) with $\varepsilon/4$ bound the mean square over (f, k) of the column sums in (7.36), with any weight $U \in C_c^\infty([a_0/8, 2b_0])$ in place of the kernel, by $D^{\varepsilon/2} (XF)^2 \|U\|_{C^J}^2$, where $J = J(\vartheta, \varepsilon/4)$ and $\tau(r_0)^2 \ll D^{\varepsilon/4}$. Then [35, Lemma B.2], with rows (f, k) and coefficients $\mu_K(f) \mathbf{1}_{(f, r_0)=1}$, transfers this bound to the kernel at the cost of $\|W\|_{C^{2J+4}(I)}^2$. The coefficients enter only through (7.15), that is, through (7.19) and $|a_\xi| \leq 1$. Summing over the $O(B)$ elements r_0 of the dyadic range gives

$$|\mathcal{S}_{\xi; B, F}| \ll D^{\varepsilon/2} \|W\|_{C^{2J+4}(I)}^2 \sum_{B \leq N(r_0) < 2B}^* \frac{HN(r_0)}{D^2} \left(\frac{D}{B}\right)^2 \ll D^{\varepsilon/2} \|W\|_{C^{2J+4}(I)}^2 H. \quad (7.38)$$

There are $O((\log D)^2)$ nonempty dyadic ranges. Summing over them and over ξ , and adding Z , gives $\mathcal{M}_D \ll \|W\|_{C^{2J+4}(I)}^2 H D^\varepsilon$. Hence $M(D, H) \leq D \mathcal{M}_D \ll \|W\|_{C^{2J+4}(I)}^2 D^{2+\vartheta+\varepsilon}$. $\square$

Thus the angular factor enters the first Poisson reduction at one place, the paired identity of Step 2, where it turns the zeta coefficient $\bar{\alpha}\gamma_2$ into $\bar{\alpha}^2\gamma_2$. Steps 1, 3, 4 and 5 use only that α is completely multiplicative with $|\alpha| = 1$: it does not interact with Poisson summation in u , it drops out of the greatest common divisor, it passes through the divisors e_0 and v_0 by (7.19), and it leaves the scales (7.12) unchanged. We checked the exact identity $\mathcal{M}_D = Z + \sum_{\xi} c_{\xi} \mathcal{S}_{\xi}$, with $\mathcal{S}_{\xi}$ in the form (7.36), numerically for $D = 120$ and $H = 40, 170$, with a Gaussian Φ (for which the identity also holds). The relative error is at most $3 \cdot 10^{-15}$.

8 The completed sums and the second theta derivative

For fixed k, f , put $\Psi(n) = \xi(n)\chi_n(k)\chi_n(f)^4$. The completed sum is the following explicit double sum:

$$T(X; k, f) = \frac{1}{\sqrt{X}} \sum_n^* \sum_b \overline{\alpha(n)}^2 \gamma_2(n) \Psi(n) \overline{\alpha(b)}^6 \Psi(b)^3 N(b)^{1/2} \times W\left(\frac{N(n)N(b)^3}{X}\right). \quad (8.1)$$

Here b is primary and coprime to 6 but need not be squarefree. All sums are finite. The term $b = 1$ is exactly $X^{-1/2}$ times the column sum in (7.10). The other terms are added because they occur naturally in the theta expansion.

8.1 Where those coefficients come from

Put $\lambda = 1 + 2\omega = i\sqrt{3}$ and $\check{e}(z) = \exp(2\pi i(z + \bar{z}))$. Kubota's cubic theta function [24, 39], a residue of a cubic metaplectic Eisenstein series (for general covers see [23]), is a Fourier series in a planar variable z and a positive height v , with terms $d_0(\ell)vK_{1/3}(4\pi|\ell|v)\check{e}(\ell z)$ in its conjugate. Here $K_{1/3}$ is the usual modified Bessel function [34, §10.25]. For n squarefree, primary and coprime to 6 and b primary, Patterson's coefficient formula [39, Theorem 8.1], which Heath-Brown and Patterson [16] used to study Kummer sums, gives

$$d_0(\lambda^{-3}nb^3) = 3^{5/2}|b| \overline{(\lambda/n)_3} \gamma_2(n). \quad (8.2)$$

Restricting $\lambda^3\ell$ to be primary and coprime to 6 selects exactly these indices. Other units and powers of λ are excluded from the twisted series by this restriction. In the reflected series, at the three cusps, the indices lie in the larger set of $\ell = u\lambda^m nb^3$ with u a unit and $m \geq -4$. The tables of [2, Theorem 23] restrict them further; for example $d_0(\lambda^{-4}) = 0$. The factor $(\lambda/n)_3$ depends only on $n \bmod 9$ (a supplement to cubic reciprocity [9]; see [26, Chapter 7]) and is canceled by inserting $(\lambda/\lambda^3\ell)_3$ in the twist. It is finite residue-class data, not a missing Gauss factor.

What is imported. Precisely, θ is the function of [2, Theorem 23], where $\sqrt{-3}$ denotes our λ , so that $g(-3, c) = g(\lambda^2, c)$. The action of $\mathrm{SL}_2(\mathbb{Z}[\omega])$ on $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$ is [2, (5.1)], and the Kubota character [24] κ of $\Gamma(3) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}[\omega]) : \gamma \equiv I \pmod{3}\}$ is [2, (5.4) and Theorem 22]. The theorem gives the automorphy (T1) of θ under $\mathrm{SL}_2(\mathbb{Z})$ and, with multiplier κ , under $\Gamma(3)$. It also gives the Fourier expansions (T2), (T3) at the cusps $\gamma_{\sigma}\infty$, where $\gamma_0 = I$, $\gamma_+ = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$ and $\gamma_- = \begin{pmatrix} 1 & 0 \\ \omega^2 & 1 \end{pmatrix}$. For $\sigma \in \{0, +, -\}$ let $d_{\sigma}(\ell)$ be the coefficient of $vK_{1/3}(4\pi|\ell|v)\check{e}(\ell z)$ in $\overline{\theta(\gamma_{\sigma}(z, v))}$; it

vanishes unless $\ell \in \lambda^{-4}\mathbb{Z}[\omega]$. Only the expansion at the identity cusp has a constant term, namely $\frac{3^{5/2}}{2}v^{2/3}$. Formula (8.2), the selection statement after it, and the support and size bounds for all three sequences recorded in Appendix A are [2, Corollary 24]; the bounds are also [35, Lemma 6.4]. None of these facts involves the angular factor, so they are used exactly as in the zeta case.

Differentiating $\check{e}(\ell z)$ twice with respect to $\bar{z}$ multiplies it by $(2\pi i\bar{\ell})^2 = -4\pi^2 N(\ell)\overline{\alpha(\ell)}^2$. The Mellin integral over the height absorbs the radial factor $N(\ell)$ by shifting its variable by $\frac{1}{2}$ (Step 3 of Section A.2); what remains is the angular factor $\overline{\alpha(\ell)}^2$. Since $\ell = \lambda^{-3}nb^3$, its factors in n, b are exactly $\overline{\alpha(n)}^2\overline{\alpha(b)}^6$, up to the fixed factor $\overline{\alpha(\lambda^{-3})}^2 = -1$. This explains every angular power in (8.1).

8.2 Why differentiating twice is legitimate

The theta symmetry relates a rational translate $a/c + z$ to coordinates

$$z' = -\frac{\delta}{c} - \frac{\bar{z}}{c^2(v^2 + z\bar{z})}, \quad v' = \frac{v}{N(c)(v^2 + z\bar{z})}. \quad (8.3)$$

At $z = \bar{z} = 0$, pure $\bar{z}$ derivatives of z' and v' vanish, as do pure $\bar{z}$ derivatives of z' of order at least two, while $\partial_{\bar{z}}z' = -(cv)^{-2}$. Therefore the ordinary chain rule gives

$$\partial_{\bar{z}}^2 F(z', \bar{z}', v')|_{z=\bar{z}=0} = (cv)^{-4}(\partial_z^2 F)(-\delta/c, 1/(N(c)v)). \quad (8.4)$$

No unwanted lower derivative survives. The derivative also kills any constant Fourier term. Of the three expansions used, only the one at the identity cusp has a nonzero constant term, and the derivative removes it. This permits Mellin reflection without constant-term contributions.

For $V(y) = \sqrt{y}W(y)$, with Mellin transform $\widehat{V}(s) = \int_0^\infty V(y)y^{s-1}dy$, the reflected weight is

$$V^\sharp(x) = \frac{1}{2\pi i} \int_{\Re t=0} \widehat{V}(-t) \frac{\Gamma(4/3+t)\Gamma(5/3+t)}{\Gamma(4/3-t)\Gamma(5/3-t)} \left(\frac{(2\pi)^4 x}{27} \right)^{-t} dt. \quad (8.5)$$

This is the CM version of the gamma quotient in the zeta argument. The reflected coefficients have angular factor $\alpha(\ell)^2$; the fixed cusp prefactor is $-\overline{\alpha(c)}^4/81$, before the finite Fourier coefficients and local phases are included. The full reflected series is Proposition 8 in Appendix A. The appendix also proves it (Section A.2) and gives the decay of $V^\sharp$ (Lemma 13). The main discussion can therefore focus on the lengths and the averaging.

8.3 The local step from sextic to quadratic

By sextic reciprocity [26, §7.3], Ψ is of the form $\xi_0(n) \prod_{\pi \in \mathcal{P}} \chi_\pi(n)^{j_\pi}$. Here ξ_0 is a character of $\mathfrak{G}$ (it collects ξ , the unit and reciprocity factors, and the factors at λ and 2), $\mathcal{P}$ is the set of primes dividing kf away from 6, and $j_\pi \equiv v_\pi(k) + 4v_\pi(f) \pmod{6}$ [35, (6.17)]. The local reflected factors, apart from constant phases, are [35, (6.5) and (A.12)]

$$B_{\pi,j}(x) = \begin{cases} \chi_\pi(x)^{-j-2}, & j \notin \{0, 4\}, \\ N(\pi)^{-1/2} \chi_\pi(x)^{-2}, & j = 0, \\ N(\pi)^{-1/2} (-1 + N(\pi) \mathbf{1}_{\pi|x}), & j = 4. \end{cases} \quad (8.6)$$

In particular,

$$B_{\pi,1}(x) = \chi_\pi(x)^3 = (x/\pi)_2. \quad (8.7)$$

Away from the fixed factors of $6f$, squarefree row primes have exponent $j = 1$. These are the varying primes that become quadratic. The other local factors are still present and have to be estimated.

If $Q = N(c)$ is the norm of the reduced cusp denominator, the weight in the reflected sum is $V^\sharp(N(\ell)X/Q^2)$. Its effective norm range is about Q^2/X , shorter than X when $Q < X$. By rapid decay (Lemma 13), the terms with $N(\ell)X/Q^2 \geq D^\varepsilon$ are negligible, and since $N(\ell) \geq 3^{-4}$ the whole reflected sum is negligible once $X \geq 81D^\varepsilon Q^2$ (end of Section A.1); the inequality $Q^2 < X$ alone does not specify a quantitative saving.

9 The quadratic average and the completed bound

The quadratic large sieve used here is

$$\sum_{\substack{N(k) \leq K \\ (k,6)=1}}^* \left| \sum_{\substack{N(n) \leq U}}^* \beta(n) \left(\frac{n}{k} \right)_2 \right|^2 \ll_\varepsilon (KU)^\varepsilon (K+U) \sum_n^* |\beta(n)|^2. \quad (9.1)$$

Both k, n are primary and squarefree, the coefficients $\beta(n)$ are arbitrary, and the column n may be even. The squarefree restrictions matter: square columns would give repeated values 1 rather than cancellation. This is [35, Lemma 6.5], a consequence of the number-field quadratic large sieve [10, Theorem 1.1], which generalizes Heath-Brown's [14]; the deduction is recorded in Section 9.1.

Applying (9.1) to the reflected sums, and retaining the factors in (8.6), gives the completed mean-square estimate

$$\boxed{\sum_{0 < N(k) \leq \mathcal{H}} |T(X; k, f)|^2 \ll D^\varepsilon \left(\mathcal{H} + \frac{\mathcal{H}^2 N(f)}{X} \right)}. \quad (9.2)$$

All parameters are at least one and bounded by a fixed power of D ; a fixed finite smoothness norm of W is understood. This is the CM form of [35, Proposition 5.2]. It is stated precisely as Proposition 4 at the end of this section and proved in Section 9.1.

For clarity, passing from squarefree rows to all rows uses $k = usv^2$, with u a unit and s squarefree. The identity

$$\chi_n(v^2) \chi_n(f)^4 = \chi_n(fv^2)^4 \quad (9.3)$$

absorbs the square into the auxiliary twist. The squarefree-row bound allows that auxiliary argument to be nonsquarefree. Its contributions are summed using $\sum_v N(v)^{-2} < \infty$.

This is the main benefit of the theta transformation: the hard sextic average has been reduced to an average of quadratic symbols. However, (9.2) bounds the completed sum, whose cube terms still have to be removed.

We record (9.2) with its quantifiers. As in [35, Proposition 5.2], the row k runs over all nonzero elements of $\mathbb{Z}[\omega]$ in the range, and $T(X; k, f)$ is (8.1) with $\Psi(n) = \xi(n) \chi_n(k) \chi_n(f)^4$.

Proposition 4 (Completed estimate; CM form of [35, Proposition 5.2]). *Fix $\varepsilon > 0$, $C_0 \geq 1$ and a character ξ of $\mathfrak{G}$. There is an integer $J = J(\varepsilon, C_0) \geq 1$ such that*

$$\sum_{0 < N(k) \leq \mathcal{H}} |T(X; k, f)|^2 \ll_{I, \xi, \varepsilon, C_0} D^\varepsilon \|W\|_{C^J(I)}^2 \left(\mathcal{H} + \frac{\mathcal{H}^2 N(f)}{X} \right) \quad (9.4)$$

whenever $1 \leq \mathcal{H}, X, N(f) \leq D^{C_0}$ and f is primary, squarefree and coprime to 6, for every compact interval $I \subset (0, \infty)$ and every $W \in C_c^\infty(I)$. This is (9.2).

9.1 Proof of the completed estimate

Proposition 4 is the CM form of [35, Proposition 5.2]. Its proof in [35, Section 6] has three steps: the quadratic large sieve [35, Lemma 6.5], the bound for squarefree rows [35, Lemma 6.6], and the passage to all rows described after (9.2). All three apply verbatim in the CM case once the zeta transformation formula, [35, Proposition 6.2 and Lemmas 6.3 and 6.4], is replaced by its CM form, Proposition 8 with Lemmas 12 and 13. The reason is that the proof of [35, Lemma 6.6] uses the transformation formula only through the following properties, each of which holds in the CM case (only the implied constant in the weight bound changes).

- *Shape.* $T(X; \Psi)$ is a sum of $O(2^{|\mathcal{P}|})$ cusp terms. Each is a constant C with $|C| \ll 1$ times a sum over $0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]$ of the theta coefficient $d_\sigma(\ell)$, a unimodular angular factor, $N(\ell)^{-1/2}$, an additive character, the local factors and $V^\sharp(N(\ell)X/N(c)^2)$. For ζ the angular factor is $\alpha(\ell)$; in the CM case it is $\alpha(\ell)^2$, and $C = -\overline{\alpha(c)}^4 C'/81$ with $|C'| \leq 1$. The local factors and the additive characters are literally those of the zeta case. In the proof of [35, Lemma 6.6] the row is $u_0 t k_0$, with k_0 squarefree and prime to $6g$, and $c = c_* k_0$. Once k_0 is restricted to a residue class modulo M^2 (Lemma 12), k_0 enters a cusp term only through the scalar C , through the quadratic factors $B_{\pi,1}(\lambda^4 \ell) = \chi_\pi(\lambda^4 \ell)^3$ at the primes $\pi \mid k_0$, and through $N(c) = N(c_*)N(k_0)$ in the weight. The angular factor is a function of ℓ alone, so it goes into the coefficient $A_{u,m}(n, b)$ of that proof, which does not depend on k_0 ; only $|A_{u,m}(n, b)| \leq 27$ is used. The scalar depends on k_0 , in the CM case also through $\overline{\alpha(c)}^4$, but only $|C| \leq 1/81$ is used.
- *Uniformity in the twist.* Lemma 12, which is [35, Lemma 6.3].
- *The coefficients.* $d_\sigma(\ell) = 0$ unless $\ell = u\lambda^m n b^3$ with u a unit, $m \geq -4$, n squarefree and n, b primary, and then $|d_\sigma(\ell)| \leq 27 \cdot 3^{m/6} |b|$. These bounds of [35, Lemma 6.4] concern the theta coefficients alone, which are the same in both cases.
- *The weight.* $|(x\partial_x)^j V^\sharp(x)| \ll \|W\|_{C^J(I)} \min(x^{1/4}, x^{-A})$ with $J = \lceil 4A \rceil + j + 2$. For ζ this is [35, Lemma 6.4], stated there with $\|V\|_{C^J(I)} \ll_{J,I} \|W\|_{C^J(I)}$; in the CM case it is Lemma 13. The different gamma factors change only the implied constant; the derivative order is that of [35, (A.20)].

The twists $\Psi_{k,g}(n) = \xi(n)\chi_n(k)\chi_n(g)^4$ are those of [35, Section 6.4], put in the form [35, (6.17)] by sextic reciprocity. In the passage to all rows, write $k = u_0 s v^2$ with u_0 a unit and s squarefree. Then $T(X; k, f) = T(X; u_0 s, f v^2)$, and the bound for squarefree rows is applied with row bound $\mathcal{H}/N(v)^2$ and auxiliary argument $g = f v^2$. Since $N(g) \leq \mathcal{H}N(f) \leq D^{2C_0}$, it is applied with $2C_0$ in place of C_0 .

The large sieve. Inequality (9.1) is [35, Lemma 6.5], deduced there from [10, Theorem 1.1]; dyadic decomposition passes from $N(n) \asymp U$ to $N(n) \leq U$. Three points are implicit there or only sketched. First, [10] indexes the character by the inner variable, so it is applied to the transposed matrix, which has the same operator norm; the bound $(KU)^\varepsilon(K+U)$ is symmetric. Second, the reflected columns may be even (the proof of [35, Lemma 6.5] fixes the factors of n in S): a squarefree primary n divisible by 2 is $n = -2n'$ with n' odd and primary, and $(n/k)_2 = (-2/k)_2(n'/k)_2$, where $|(-2/k)_2| = 1$ depends only on the row. Third, the family. For squarefree primary k coprime to 6 put $\chi_{(k)}(x) = (x/k)_2 \kappa_\lambda(x)^{e_k}$, with κ_λ the nontrivial character modulo λ and $N(k) \equiv (-1)^{e_k} \pmod{4}$. It is trivial on the units, so it is a primitive Hecke character of trivial infinite type, of conductor $k\lambda^{e_k}$, and $\chi_{(k)}(n') = (n'/k)_2$ for primary n' . These characters form a quadratic Hecke family in the sense of [10, Definition 1], with $\mathfrak{c} = 6\mathbb{Z}[\omega]$. Their order divides 2. By quadratic reciprocity in $\mathbb{Z}[\omega]$ [26,

§8.5], $\chi_{(k)}(k')\chi_{(k')}(k)$ depends only on the classes of the primary generators k, k' modulo 4, so the reciprocity law of [10, Definition 1] holds with $G = (\mathbb{Z}[\omega]/4\mathbb{Z}[\omega])^\times$ and $[(k)] = k \pmod{4}$. Finally, if coprime k_1, k_2 satisfy $k_1 \equiv k_2 \pmod{4}$, then $e_{k_1} = e_{k_2}$, so $\chi_{(k_1)}\overline{\chi_{(k_2)}}$ is induced by $(\cdot/k_1 k_2)_2$, which is primitive modulo the squarefree $k_1 k_2$.

10 Remove the cubes, then reduce the number of rows

The purpose of descent is to bound the average $\mathcal{E}(\mathcal{H}, X, F)$ in (7.10). In plain terms, the completed theta estimate handles small cube indices. The remaining large cube indices lead to averages of exactly the same kind with fewer rows. The point requiring care is that all the other conditions, including the smooth weights, must survive this reduction.

One cannot discard the $b \neq 1$ terms inside (8.1): they could cancel the term we want. Instead use the exact cube inversion of [35, Section 5.3] (compare [39, Theorem 6.1] and [8, Section 5])

$$\frac{1}{\sqrt{X}} \sum_n^* a_\xi(n) \chi_n(k) \chi_n(f)^4 W(N(n)/X) = \sum_h \frac{\mu_K(h) \overline{\alpha(h)}^6 \Psi(h)^3}{N(h)} T(X/N(h)^3; k, f). \quad (10.1)$$

Here h is primary and coprime to 6. To check the formula, substitute (8.1), collect the total cube index $b' = hb$, and use $\sum_{h|b'} \mu_K(h) = \mathbf{1}_{b'=1}$. The angular and character factors are completely multiplicative, so this cancellation is exact.

10.1 Small cubes are controlled by the completed estimate

Choose

$$H_c^3 = \min \left(X, \frac{X^2}{\mathcal{H}^2} \right), \quad L_b = \frac{X}{N(b)^3}. \quad (10.2)$$

For $N(h) \leq H_c$, the completed estimate (9.2) at scale $X/N(h)^3$ gives terms bounded by

$$\mathcal{H} + \frac{\mathcal{H}^2 F N(h)^3}{X} \ll \Sigma, \quad \Sigma = XF, \quad (10.3)$$

provided $\mathcal{H} \leq \Sigma$. Cauchy–Schwarz costs only divisor sums and logarithms, which fit into D^ε . Expanding the remaining completed sums in (10.1) leaves the same average at smaller column scales. The resulting cube-removal estimate, the CM form of [35, Lemma 5.3] (Lemma 6), is

$$\mathcal{E}(\mathcal{H}, X, F) \ll D^\varepsilon \left(\Sigma + \sup_{\substack{N(b) > H_c \\ L_b > 1}} \mathcal{E}(\mathcal{H}, L_b, F) \right). \quad (10.4)$$

An empty supremum is zero. Scales $L_b \leq 1$ are bounded by counting. If $\mathcal{H}^2 \leq X$, the supremum is empty and the desired estimate already follows. Here and below, the suppressed dependence on W is through a fixed finite smoothness norm. More precisely, the Σ term in (10.4) is $\Sigma \|W\|_{C^{J_c}(I)}^2$ for a finite integer J_c , with the same W in each average in the supremum. Here $I \subset (0, \infty)$ is a compact interval containing the support of W , and $\|W\|_{C^J(I)} = \max_{0 \leq i \leq J} \sup_{x \in I} |W^{(i)}(x)|$.

10.2 Large cubes force a smaller row range

Keep $\Sigma = XF$ from the original average fixed while replacing the column length X by $L = L_b$. In particular, do not reset Σ to LF . The extra room $LF \leq \Sigma$ is what permits the enlargement below. For the remaining terms, two more Poisson summations [35, Section 7] exchange cubic Gauss coefficients for Möbius coefficients and then back again. The paired identity (7.9) is used in both directions, with twists (Lemma 5). Smooth the row sum in the explicit average

$$\frac{1}{LF} \sum_{F \leq N(f) < 2F}^* \sum_{k \in \mathbb{Z}[\omega]} \Phi(N(k)/\mathcal{H}) \left| \sum_n^* a_\xi(n) \chi_n(k) \chi_n(f)^4 W(N(n)/L) \right|^2. \quad (10.5)$$

The reverse paired Gauss identity converts its coefficients back to Möbius coefficients $\mu_K(n) \overline{\alpha(n)} \xi_1(n)$. For precision, the intermediate column sums, with their exclusions, are

$$P_{C,e_0,t}(y; U) = \sum_n^* \mu_K(n) \overline{\alpha(n)} \xi_1(n) \mathbf{1}_{(n,t)=1} \overline{\chi_n(y)} \chi_n(C)^4 \chi_n(e_0) U(N(n)/\ell), \quad \ell = \frac{L}{N(C)N(t)}. \quad (10.6)$$

Here C, t are coprime squarefree primary elements and $e_0 \mid C$. After separation of smooth weights, the nonnegative average used is

$$\sum_{\substack{C,t \text{ squarefree} \\ (C,t)=1}} \sum_{e_0 \mid C} \frac{\mathcal{H}N(C)}{N(e_0)L^2F} \sum_{y \in \mathbb{Z}[\omega]} \Phi(N(y)/Y_{C,e_0}) |P_{C,e_0,t}(y; U)|^2, \quad (10.7)$$

where the outer sum is restricted to $N(Ct) \ll_W L$ and

$$Y_{C,e_0} = c_* \frac{\Sigma L F N(e_0)}{\mathcal{H} N(C)^2}. \quad (10.8)$$

The constant c_* is fixed and chosen to cover the support of the weights.

Here is the crucial enlargement (compare [14, Lemma 9] and [10, Lemma 4.4]). The first Poisson frequency h has norm about $L^2/\mathcal{H}$, with additional divisor factors. Combining it with f as $y = hf^2$ gives a natural Möbius-row range of size $L^2F^2/\mathcal{H}$. Suppressing the common divisor factor $N(e_0)/N(C)^2$ and fixed support constants, the two scales are

$$Y_{\text{natural}} = \frac{L^2F^2}{\mathcal{H}}, \quad Y_{\text{enlarged}} = \frac{\Sigma L F}{\mathcal{H}}, \quad \frac{Y_{\text{enlarged}}}{Y_{\text{natural}}} = \frac{\Sigma}{L F} \geq 1. \quad (10.9)$$

After separating the smooth weights, the row sums are nonnegative mean squares, so the larger smooth majorant gives a valid upper bound. The enlarged range makes the second Poisson frequency range smaller. The second Poisson summation, in y , uses (7.9), twisted by ξ_1 as in (7.17), and returns to the cubic coefficients a_ξ . Its output is indexed by the common divisor g of the two Möbius columns in $|P_{C,e_0,t}|^2$, a divisor $e \mid g$ coming from the row exclusion, a squarefree w that removes the remaining coprimality condition, and a nonzero frequency h_2 . These combine into a new exclusion divisor, auxiliary divisor and row:

$$r = tg/e, \quad f' = Cew, \quad k' = e_0eh_2, \quad (10.10)$$

as in [35, proof of Lemma 7.3]. For r in a dyadic range $R \leq N(r) < 2R$, the new row scale is

$$\mathcal{H}' = \frac{\mathcal{H}L}{\Sigma F R^2}. \quad (10.11)$$

Without enlargement the corresponding scale would be $\mathcal{H}/(F^2 R^2)$, which need not decrease when $F = R = 1$.

Regrouping the signed divisor contributions *before* taking absolute values forces a divisibility condition $f' \mid k'$. At each prime $\pi \mid f'$, the choices $\pi \mid C$, $\pi \mid e$ and $\pi \mid w$ contribute the three terms on the left of

$$(1 + \mathbf{1}_{\pi \mid k'}) - \mathbf{1}_{\pi \mid k'} - \mathbf{1}_{\pi \nmid k'} = \mathbf{1}_{\pi \mid k'}. \quad (10.12)$$

This divisibility keeps the next average in the ranges needed for induction. The kernel after the second summation vanishes unless $0 < N(k') \leq \mathcal{H}L/(\Sigma FN(r)^2)$. Retain only those r for which a nonzero term occurs. If $F' \leq N(f') < 2F'$, then $f' \mid k'$ gives

$$F' \leq N(f') \leq N(k') \leq \frac{\mathcal{H}L}{\Sigma FN(r)^2} \leq \mathcal{H}', \quad (10.13)$$

so $\mathcal{H}' \geq F' \geq 1$.

The new columns are restricted by $(n, r) = 1$. Put $X' = L/(RF')$, and let $\mathcal{C}_r(X'; k', f')$ be the column sum of (7.10) with this restriction, and $\mathcal{C}_1$ the unrestricted one. Inclusion–exclusion over $j \mid r$, together with the twisted multiplicativity of a_ξ , gives the exact identity

$$\mathcal{C}_r(X'; k', f') = \sum_{\substack{j \mid r \\ (j, f')=1}} \mu_K(j) a_\xi(j) \chi_j(k') \chi_j(f')^4 \mathcal{C}_1(X'/N(j); k', jf'). \quad (10.14)$$

Indeed, writing $n = jn'$, the factor $\chi_{n'}(j)^4$ from $a_\xi(jn') = a_\xi(j) a_\xi(n') \chi_{n'}(j)^4$ enforces $(n', j) = 1$ and combines with $\chi_{n'}(f')^4$. The factor $\bar{\alpha}^2$ in a_ξ is completely multiplicative, so this is the identity in the proof of [35, Lemma 4.4], with one more factor $\bar{\alpha}$. Cauchy–Schwarz in j then bounds the restricted average by $\tau(r) \sum_{j \mid r} \mathcal{E}(\mathcal{H}', X'/N(j), F'N(j))$. Thus removing an exclusion divisor $j \mid r$ produces the replacement

$$(X', F') \mapsto (X'/N(j), F'N(j)). \quad (10.15)$$

It preserves both $\mathcal{H}'$ and the product $\Sigma' = X'F' = L/R$, and hence the ratio $\mathcal{H}'/\Sigma'$. Since $N(j) \leq N(r)$, $R \leq N(r)$, $F \geq 1$ and $\mathcal{H} \leq \Sigma$, (10.13) gives

$$\frac{X'}{N(j)} \geq \frac{\Sigma FN(r)^2}{\mathcal{H}RN(j)} \geq \frac{\Sigma F}{\mathcal{H}} \geq 1. \quad (10.16)$$

10.3 The precise transfer estimate

We now state the analytic input in the form needed here. As in Section 7.3, we write $\mathcal{E}(\mathcal{H}, X, F; \xi, W)$ when the twist and weight in (7.10) need to be shown.

Proposition 5 (Transfer estimate; CM form of [35, Proposition 5.4]). *Fix $C_0 \geq 1$, a compact interval $I \subset (0, \infty)$, and $W \in C_c^\infty(I)$, and let $D \geq 2$. Suppose $1 \leq \mathcal{H}, L, F, \Sigma \leq D^{C_0}$, with $\max\{\mathcal{H}, LF\} \leq \Sigma$. Here Σ is an independent target scale. For every integer $m \geq 0$ and $\delta > 0$, with the derivative order $J_t(m) = 4m + 12$,*

$$\mathcal{E}(\mathcal{H}, L, F; \xi, W) \ll D^\delta \Sigma \|W\|_{C^{J_t(m)}(I)}^2 \left(1 + \sup \frac{\mathcal{E}(\mathcal{H}', X', F'; \xi', U)}{\Sigma'} \right). \quad (10.17)$$

The supremum runs over averages of the same form (7.10), with $\mathcal{H}', X', F' \geq 1$, $\Sigma' = X'F'$, and

$$\mathcal{H}' \leq \frac{\mathcal{H}L}{\Sigma F}, \quad \frac{\mathcal{H}'}{\Sigma'} \leq \frac{\mathcal{H}}{\Sigma}, \quad \Sigma' \leq L. \quad (10.18)$$

For $I = [u, v] \subset (0, \infty)$, the new weights satisfy $U \in C_c^\infty([u/16, 4v])$ and $\|U\|_{C^m} \leq 1$. The characters ξ' belong to the same fixed finite family of residue-class twists. An empty supremum is zero. The implied constant may depend on m, C_0, δ, I and the fixed arithmetic data, but not on D or the moving scales.

For ζ this is [35, Proposition 5.4], with the same $J_t(m) = 4m + 12$. The source proves the stronger version with a smooth majorant of the row sum on the left, and so do we. Its proof, in [35, Section 7], is the two-Poisson argument just outlined, including the bounds for the separated smooth kernels and zero frequencies. In the CM case the two Poisson summations change through the angular factors and the rest of the argument does not; the proof is given in Section 10.6. The content of the proposition is the uniform mean-square bound: the three scale inequalities (10.18), by themselves, do not establish it.

10.4 Why the induction closes

This subsection follows the proof of [35, Proposition 5.1], which follows Heath-Brown's recursive method [14, Lemma 8 and §8] (see also [10, Theorem 4.1]), given in [35, Section 5.5]; it uses only (10.4), Proposition 5 and counting, so it carries over unchanged. Fix $\kappa > 0$, $C_0 \geq 1$, and $\varepsilon > 0$, and assume

$$1 \leq \mathcal{H}, X, F, \quad \Sigma = XF \leq D^{C_0}, \quad \mathcal{H} \leq \Sigma D^{-\kappa}. \quad (10.19)$$

The inductive assertion, for each integer $\nu \geq 0$, is

$$\mathcal{E}(\mathcal{H}, X, F; \xi, W) \ll D^\varepsilon \Sigma \|W\|_{C^{J_\nu(I)}}^2 \quad \text{whenever } \mathcal{H} \leq D^{\nu\kappa}. \quad (10.20)$$

This assertion is made for every $\varepsilon > 0$ and every compact interval $I \subset (0, \infty)$. The finite derivative order J_ν may depend on $\nu, \kappa, C_0, \varepsilon$, but not on D or I ; the implied constant may also depend on I . This formulation allows the support interval to enlarge at a transfer step.

Base case. If $\nu = 0$, then $\mathcal{H} \leq 1$, so there are at most six nonzero rows, the units. There are $O(F)$ choices of f and $O_I(X)$ column terms, each of absolute value at most $\|W\|_\infty$, since $|a_\xi(n)| \leq 1$. The normalization in (7.10) therefore gives

$$\mathcal{E}(\mathcal{H}, X, F; \xi, W) \ll_I \frac{F}{XF} (X\|W\|_\infty)^2 = X\|W\|_\infty^2 \leq \Sigma\|W\|_\infty^2. \quad (10.21)$$

Induction step. Assume the assertion at level ν , and let $\mathcal{H} \leq D^{(\nu+1)\kappa}$. Use that assertion with loss $\varepsilon/3$, and let m be its required derivative order. Apply cube removal (10.4) and Proposition 5, each with loss $\varepsilon/3$. If the cube-removal supremum is empty, its Σ term already suffices. Otherwise $N(b)^3 > X^2/\mathcal{H}^2$. For each remaining $L = L_b$ we have $LF \leq \Sigma$, so the transfer estimate applies with the unchanged target $\Sigma = XF$. Every child average satisfies

$$\mathcal{H}' \leq \frac{\mathcal{H}}{N(b)^3 F^2} < \mathcal{H} \left(\frac{\mathcal{H}}{XF} \right)^2 \leq \mathcal{H} D^{-2\kappa}. \quad (10.22)$$

In particular, $\mathcal{H}' \leq D^{\nu\kappa}$. The other hypotheses survive as well:

$$\frac{\mathcal{H}'}{\Sigma'} \leq \frac{\mathcal{H}}{\Sigma} \leq D^{-\kappa}, \quad \Sigma' \leq L \leq \Sigma \leq D^{C_0}, \quad \mathcal{H}', X', F' \geq 1. \quad (10.23)$$

The induction hypothesis applies to the child weight U on $[u/16, 4v]$. Since $\|U\|_{C^m} \leq 1$, it gives the uniform bound

$$\sup \frac{\mathcal{E}(\mathcal{H}', X', F'; \xi', U)}{\Sigma'} \ll D^{\varepsilon/3}. \quad (10.24)$$

The transfer contributes a second factor $D^{\varepsilon/3}$ and cube removal a third. Taking $J_{\nu+1} \geq \max\{J_t(m), J_c\}$ proves the assertion at level $\nu + 1$. Thus both the derivative requirement and the total D^ε loss are controlled.

Finally take $\nu = \lceil C_0/\kappa \rceil$; this covers every allowed $\mathcal{H}$. The number of induction levels depends on C_0, κ , not on D . Suppressing the resulting fixed smoothness norm again, we obtain

$$\mathcal{E}(\mathcal{H}, X, F) \ll D^\varepsilon \Sigma \quad (\mathcal{H} \leq \Sigma D^{-\kappa}). \quad (10.25)$$

No estimate for zeros of $L(E_d, s)$ or of ζ is assumed in this induction: its inputs are cube removal and the transfer estimate, and its starting case is counting. In the CM case these inputs, and the completed estimate (9.2) on which cube removal rests, are proved in Sections 9.1, 10.5 and 10.6. Their analytic inputs are the CM theta transformation (Proposition 8, with Lemmas 12 and 13) and the quadratic large sieve [10].

It remains to check the initial scales (7.12). With $H = D^{1+\vartheta}$, they satisfy

$$\frac{\mathcal{H}}{\Sigma} \ll \frac{D^{-\vartheta}}{B}. \quad (10.26)$$

For sufficiently large D this is at most $D^{-\vartheta/2}$. Thus (10.25) applies with $\kappa = \vartheta/2$. Empty ranges and bounded column scales are handled by counting. The first Poisson reduction, Proposition 3, gives (4.6), with the constant controlled by a finite smoothness norm of W , and Section 6 then gives the zero-free half-plane.

What changes from the zeta guide. The architecture is the same, but its angular factors change coherently:

	zeta guide	present CM guide
original coefficients	$\mu_K(n)$	$\mu_K(n)\alpha(n)$
cubic coefficients	$\bar{\alpha}\gamma_2$	$\bar{\alpha}^2\gamma_2$
theta derivative	$\partial_{\bar{z}}$	$\partial_{\bar{z}}^2$
angular factor on cubes	$\overline{\alpha(b)}^3$	$\overline{\alpha(b)}^6$
dual angular factor	$\alpha(\ell)$	$\alpha(\ell)^2$

(10.27)

These are genuine identities to check, not consequences of the factors having absolute value one. Once checked, the same large-sieve and descent estimates apply.

10.5 Proof of cube removal

Lemma 6 (Cube removal; CM form of [35, Lemma 5.3]). *Fix $C_0 \geq 1$ and $\varepsilon > 0$, and let $1 \leq \mathcal{H}, X, F \leq D^{C_0}$ with $\mathcal{H} \leq \Sigma = XF$. There is $J_c = J_c(\varepsilon, C_0)$ such that, for every character ξ of $\mathfrak{G}$, compact $I = [u, v] \subset (0, \infty)$ and $W \in C_c^\infty(I)$,*

$$\mathcal{E}(\mathcal{H}, X, F; \xi, W) \ll_{I, \varepsilon, C_0} D^\varepsilon \left(\Sigma \|W\|_{C^{J_c(I)}}^2 + \sup_{\substack{N(b) > H_c \\ L_b > 1}} \mathcal{E}(\mathcal{H}, L_b, F; \xi, W) \right), \quad (10.28)$$

with H_c and L_b as in (10.2) and b primary and coprime to 6. An empty supremum is zero, and the supremum is empty if $\mathcal{H}^2 \leq X$. This is (10.4).

Proof. The proof of [35, Lemma 5.3] applies verbatim, with (9.2) in place of [35, Proposition 5.2] (used at scale $X/N(h)^3$ with $C_0 + 1$ in place of C_0 , since $N(f) < 2D^{C_0}$). The angular factor on the cube indices, $\bar{\alpha}^6$ instead of $\bar{\alpha}^3$ (see (10.1) and (8.1)), enters that proof only in two ways. It is completely multiplicative, so that $\frac{\overline{\alpha(h)}^6 \overline{\alpha(c)}^6}{\overline{\alpha(hc)}^6} = \frac{\overline{\alpha(hc)}^6}{\overline{\alpha(hc)}^6}$ when the cube indices h of (10.1) and c of (8.1) are collected into $b = hc$. And it has absolute value one, which is all that the Cauchy–Schwarz steps use. $\square$

10.6 Proof of the transfer estimate

Throughout, $\mathcal{H}, L, F, \Sigma, I = [u, v]$ and ξ are as in Proposition 5, and $I_* = [u/2, 2v]$, $I' = [u/16, 4v]$. Let Φ be the row weight of Section 7, normalized so that $\Phi \geq 1$ on $[0, 1]$, and let C_Φ be the constant of Section 7.3, so that $\hat{\Phi}(t) = 0$ for $t > C_\Phi$. Put

$$\mathcal{A}(W) = \frac{1}{LF} \sum_{F \leq N(f) < 2F}^* \sum_{k \in \mathbb{Z}[\omega]} \Phi\left(\frac{N(k)}{\mathcal{H}}\right) \left| \sum_n^* a_\xi(n) \chi_n(k) \chi_n(f)^4 W\left(\frac{N(n)}{L}\right) \right|^2, \quad (10.29)$$

the smoothed average displayed before (10.6); thus $\mathcal{E}(\mathcal{H}, L, F; \xi, W) \leq \mathcal{A}(W)$. For coprime squarefree C, t and $e_0 \mid C$ put $\ell = L/(N(C)N(t))$,

$$w_{C, e_0} = \frac{\mathcal{H}N(C)}{N(e_0)L^2F}, \quad Y_{C, e_0} = c_* \frac{\Sigma L F N(e_0)}{\mathcal{H}N(C)^2}, \quad c_* = \max(1, 4C_\Phi v^2). \quad (10.30)$$

For $U \in C_c^\infty(I_*)$ and a character ξ_1 of $\mathfrak{G}$, let $P_{C, e_0, t}(y; U)$ be as in (10.6), and let

$$\mathcal{Q}_{\xi_1}(U) = \sum_{\substack{C, t \text{ squarefree, } (C, t)=1 \\ N(C)N(t) \leq 2vL}} \sum_{e_0 \mid C} w_{C, e_0} \sum_{y \in \mathbb{Z}[\omega]} \Phi\left(\frac{N(y)}{Y_{C, e_0}}\right) |P_{C, e_0, t}(y; U)|^2, \quad (10.31)$$

which is (10.7) written out. The proof follows [35, Section 7]. The angular factors enter only through the two Poisson summations (Lemmas 7 and 8), which we give in full. They use the paired identities with twists of Lemma 5: (7.18) in the first and (7.17) in the second. The CM changes are the factor $\alpha(z_1)\alpha(z_2)$ there and the coefficients $a_\xi = \bar{\alpha}^2 \gamma_2 \xi$.

Lemma 7 (First Poisson summation; CM form of [35, Lemma 7.1]). *Fix an integer $J_1 \geq 0$ and $\varepsilon > 0$. If $M \geq 0$ satisfies $\mathcal{Q}_{\xi_1}(U) \leq M \|U\|_{C^{J_1}(I_*)}^2$ for every $U \in C_c^\infty(I_*)$ and every character ξ_1 , then $\mathcal{A}(W) \ll D^\varepsilon (\Sigma + M) \|W\|_{C^{2J_1+4}(I)}^2$.*

Proof. Expand the square in (10.29) with n_1 unconjugated and n_2 conjugated, and write $C = (n_1, n_2)$ and $n_i = Cu_i$, so that $(u_1, u_2) = (u_1 u_2, C) = 1$. By (7.19) and $|a_\xi(C)| = 1$, $a_\xi(n_1) \overline{a_\xi(n_2)} = a_\xi(u_1) \overline{a_\xi(u_2)} \chi_{u_1}(C)^4 \overline{\chi_{u_2}(C)^4}$. Also $\chi_{n_1}(k) \overline{\chi_{n_2}(k)} = \mathbf{1}_{(k, C)=1} \chi_{u_1}(k) \overline{\chi_{u_2}(k)}$ and $\chi_{n_1}(f)^4 \overline{\chi_{n_2}(f)^4} = \mathbf{1}_{(C, f)=1} \chi_{u_1}(f)^4 \overline{\chi_{u_2}(f)^4}$. The row character $\chi_{u_1} \overline{\chi_{u_2}}$ is primitive modulo $u_1 u_2$, so the sum over k with weight $\Phi(N(k)/\mathcal{H}) \mathbf{1}_{(k, C)=1}$ is given by (7.4) with $\psi = \chi_{u_1} \overline{\chi_{u_2}}$, $(z_1, z_2, g) = (u_1, u_2, C)$ and $\mathcal{H}$ in place of H . The frequency $h = 0$ contributes only when $u_1 = u_2 = 1$, and these terms give

$$Z = \frac{\mathcal{H} \hat{\Phi}(0)}{LF} \sum_{F \leq N(f) < 2F}^* \sum_{\substack{C \\ N(C)=1}}^* \left| W\left(\frac{N(C)}{L}\right) \right|^2 \prod_{\pi \mid C} \left(1 - \frac{1}{N(\pi)}\right) \ll_I \mathcal{H} \|W\|_\infty^2 \leq \Sigma \|W\|_\infty^2. \quad (10.32)$$

For $h \neq 0$ apply (7.18) at (u_1, u_2) ; here the CM coefficients $\mu_K(u) \overline{\alpha(u)} \xi_1(u)$ appear. The characters at u_1 coming from h, e_0, C and f combine as $\overline{\chi_{u_1}(h)} \chi_{u_1}(e_0) \chi_{u_1}(C)^4 \chi_{u_1}(f)^4 = \overline{\chi_{u_1}(hf^2)} \chi_{u_1}(C)^4 \chi_{u_1}(e_0)$,

since $\chi^4 = \bar{\chi}^2$ with the zero-value convention; those at u_2 are the complex conjugates of the same expression at u_1 . Insert $\mathbf{1}_{(u_1, u_2)=1} = \sum_{t|u_1, t|u_2} \mu_K(t)$ and write $u_i = tx_i$. Squarefreeness gives $(x_i, t) = 1$, and no condition $(x_1, x_2) = 1$ remains. The factors at t have product $|\bar{\alpha}(t)\xi_1(t)\chi_t(hf^2)\chi_t(C)^4\chi_t(e_0)|^2 = \mathbf{1}_{(t, C hf)=1}$, and $\mu_K(u_1)\mu_K(u_2) = \mu_K(x_1)\mu_K(x_2)$. With $N(u_1 u_2) = N(t)^2 N(x_1)N(x_2)$, $N(t)\ell = L/N(C)$ and the factor $1/(LF)$, this gives

$$\begin{aligned} \mathcal{A}(W) - Z &= \sum_{\xi_1} \ell_{\xi_1}(\xi) \sum_{\substack{C, t \\ (C, t)=1}} \sum_{e_0|C} \mu_K(e_0)\mu_K(t)w_{C, e_0} \sum_{\substack{F \leq N(f) < 2F \\ (f, C t)=1}}^* \sum_{\substack{h \neq 0 \\ (h, t)=1}} \\ &\quad \times \sum_{x_1, x_2}^* p_{hf^2}(x_1)\overline{p_{hf^2}(x_2)} \mathcal{K}_h\left(\frac{N(x_1)}{\ell}, \frac{N(x_2)}{\ell}\right), \end{aligned} \quad (10.33)$$

where C, t are squarefree with $N(C)N(t) \leq vL$, $p_y(x) = \mu_K(x)\overline{\alpha(x)}\xi_1(x)\mathbf{1}_{(x, t)=1}\overline{\chi_x(y)}\chi_x(C)^4\chi_x(e_0)$ are the coefficients of (10.6), and

$$\mathcal{K}_h(z_1, z_2) = (z_1 z_2)^{-1/2} W(z_1) \overline{W(z_2)} \widehat{\Phi}\left(\frac{\mathcal{H}N(h)N(C)^2}{L^2 N(e_0)z_1 z_2}\right), \quad \sup_{C, e_0, h} \|\mathcal{K}_h\|_{C^q(I^2)} \ll_{I, q} \|W\|_{C^q(I)}^2. \quad (10.34)$$

The factor $\overline{\alpha(x)}$ in p_y is the only CM change; in [35, (7.1)] the coefficient is $\mu_K(x)\xi_1(x)$. The support of $\widehat{\Phi}$ forces $N(h) \leq C_\Phi v^2 L^2 N(e_0)/(\mathcal{H}N(C)^2)$, so $y = hf^2$ satisfies $N(y) \leq 4C_\Phi v^2 L^2 F^2 N(e_0)/(\mathcal{H}N(C)^2) \leq Y_{C, e_0}$, because $LF \leq \Sigma$. This is the enlargement (10.9). For fixed C, e_0, t the map $(f, h) \mapsto hf^2$ has multiplicity at most $\tau(y) \ll D^\varepsilon$, and $\Phi \geq 1$ on $[0, 1]$, so

$$\sum_{C, t} \sum_{e_0|C} w_{C, e_0} \sum_{f, h} |P_{C, e_0, t}(hf^2; U)|^2 \ll D^\varepsilon \mathcal{Q}_{\xi_1}(U) \leq D^\varepsilon M \|U\|_{C^{J_1}(I_*)}^2. \quad (10.35)$$

Now [35, Lemma B.2], with rows (C, t, e_0, f, h) , weights w_{C, e_0} , coefficients $\mu_K(e_0)\mu_K(t)w_{C, e_0}$, both column sums equal to $P_{C, e_0, t}(hf^2; \cdot)$, and the kernels $\mathcal{K}_h$ supported in $I^2 \subset \text{int } I_*^2$, gives $|\mathcal{A}(W) - Z| \ll D^\varepsilon M \|W\|_{C^{2J_1+4}(I)}^2$, since there are finitely many ξ_1 and $|\ell_{\xi_1}(\xi)| \leq 1$. $\square$

For the second Poisson summation, fix coprime squarefree $C, t, e_0 | C$, a character ξ_1 , scales $\ell, Y > 0$, and $U \in C_c^\infty(I_*)$ with $I_* = [a_*, b_*]$. Put $P(y) = P_{C, e_0, t}(y; U)$ with this $\ell, \mathcal{M} = \sum_{y \in \mathbb{Z}[\omega]} \Phi(N(y)/Y) |P(y)|^2$, and $U_0(x) = x^{-1/2} U(x)$.

Lemma 8 (Second Poisson identity; CM form of [35, Lemma 7.2]). *With $c_{\xi'}(\xi_1)$ as in Lemma 5,*

$$\mathcal{M} = Z' + \sum_{g, e, w, h_2, \xi'} \frac{Y \mu_K(e) \mu_K(w) c_{\xi'}(\xi_1) N(g)}{N(e) \ell} \sum_{n_1, n_2}^* a^\sharp(n_1) \overline{a^\sharp(n_2)} \mathcal{K}\left(\frac{N(n_1)}{X'_0}, \frac{N(n_2)}{X'_0}\right). \quad (10.36)$$

Here g, w are squarefree, $e | g$ and $0 \neq h_2 \in \mathbb{Z}[\omega]$, subject to $(g, Ct) = (w, gCth_2) = 1$, $N(gw) \leq b_* \ell$ and $N(h_2) \leq C_\Phi b_*^2 \ell^2 N(e)/(YN(g)^2)$, and

$$\begin{aligned} X'_0 &= \frac{\ell}{N(gw)}, \quad a^\sharp(n) = a_{\xi'}(n) \mathbf{1}_{(n, tg/e)=1} \chi_n(e_0 e h_2) \chi_n(Cew)^4, \\ \mathcal{K}(x_1, x_2) &= U_0(x_1) \overline{U_0(x_2)} \widehat{\Phi}\left(\frac{YN(h_2)N(g)^2}{N(e)\ell^2 x_1 x_2}\right). \end{aligned} \quad (10.37)$$

The zero-frequency term is $Z' = Y \widehat{\Phi}(0) \sum_{(g, Ct)=1}^* |U(N(g)/\ell)|^2 \prod_{\pi|g} (1 - N(\pi)^{-1}) \ll_{I_*} Y \ell \|U\|_\infty^2$.

Proof. Expand $|P(y)|^2$ with n_1 from P and n_2 from $\bar{P}$, put $g = (n_1, n_2)$ and $n_i = gz_i$. Nonzero terms have $(z_1, z_2) = (z_1 z_2, gCt) = (g, Ct) = 1$, where $(g, C) = 1$ comes from $|\chi_g(C^4 e_0)|^2$. Moreover $\overline{\chi_{n_1}(y)} \chi_{n_2}(y) = \mathbf{1}_{(y,g)=1} \overline{\chi_{z_1}(y)} \chi_{z_2}(y)$. Apply [2, Lemma 21] with $\psi = \overline{\chi_{z_1}} \chi_{z_2}$, primitive modulo $z_1 z_2$, and exclusion g . It introduces $e \mid g$ and a frequency h_2 , and its zero frequency occurs only for $z_1 = z_2 = 1$, giving Z' . The coefficient of n_1 is $\mu_K(n_1) \overline{\alpha(n_1)} \xi_1(n_1)$ and that of n_2 is its complex conjugate. Since $\mu_K(g)^2 = |\alpha(g)|^2 = |\xi_1(g)|^2 = 1$, their product is $\mu_K(z_1) \mu_K(z_2) \overline{\alpha(z_1)} \alpha(z_2) \xi_1(z_1) \xi_1(z_2)$. So for $h_2 \neq 0$, (7.17) replaces it, together with $\gamma(\psi)$, by $\sum_{\xi'} c_{\xi'}(\xi_1) a_{\xi'}(z_1) \overline{a_{\xi'}(z_2)}$. This is the only CM change: in [35, proof of Lemma 7.2] the Möbius coefficient is $\mu_K \xi_1$ and the cubic coefficient is $\overline{\alpha} \gamma_2 \xi'$. The characters combine as

$$\psi(e) \overline{\psi(h_2)} \chi_{z_1}(C^4 e_0) \overline{\chi_{z_2}(C^4 e_0)} = \chi_{z_1}(e_0 e h_2) \chi_{z_1}(C e)^4 \overline{\chi_{z_2}(e_0 e h_2) \chi_{z_2}(C e)^4}, \quad (10.38)$$

since $\overline{\chi_z(e)} = \chi_z(e)^5$. Insert $\mathbf{1}_{(z_1, z_2)=1} = \sum_{w|z_1, w|z_2} \mu_K(w)$ and write $z_i = wn_i$; then $(w, gCt) = 1$. By (7.19), $a_{\xi'}(wn_i) = a_{\xi'}(w) a_{\xi'}(n_i) \chi_{n_i}(w)^4$. The factors at w have product $|a_{\xi'}(w) \chi_w(e_0 e h_2) \chi_w(C e)^4|^2 = \mathbf{1}_{(w, h_2)=1}$, and

$$\mathbf{1}_{(n, gCtw)=1} \chi_n(e_0 e h_2) \chi_n(C e w)^4 = \mathbf{1}_{(n, tg/e)=1} \chi_n(e_0 e h_2) \chi_n(C e w)^4. \quad (10.39)$$

Finally $U(N(gz_1)/\ell) \overline{U(N(gz_2)/\ell)} / \sqrt{N(z_1 z_2)} = \frac{N(g)}{\ell} U_0(x_1) \overline{U_0(x_2)}$ with $x_i = N(gz_i)/\ell = N(n_i)/X'_0$, the Poisson factor is $Y \mu_K(e)/N(e)$, and the argument of $\widehat{\Phi}$ is $YN(h_2)/(N(e)N(z_1)N(z_2)) = YN(h_2)N(g)^2/(N(e)\ell^2 x_1 x_2)$. The support conditions follow from those of U and $\widehat{\Phi}$. $\square$

Lemma 9 (Second transfer; CM form of [35, Lemma 7.3]). *For an integer $m \geq 0$, let S_m be the supremum in (10.17): over characters ξ' , parameters $\mathcal{H}', X', F' \geq 1$ satisfying (10.18) with $\Sigma' = X' F'$, and $U \in C_c^\infty(I')$ with $\|U\|_{C^m(I')} \leq 1$. Then, for every $\varepsilon > 0$ and $U \in C_c^\infty(I_*)$,*

$$\mathcal{Q}_{\xi_1}(U) \ll D^\varepsilon \Sigma(1 + S_m) \|U\|_{C^{2m+4}(I_*)}^2. \quad (10.40)$$

Proof. The proof of [35, Lemma 7.3] applies verbatim, with [35, Lemma 7.2] replaced by Lemma 8 and [35, Lemma 4.4] replaced by its CM form (7.15). The coefficients $a_{\xi'}$ enter only through these two inputs, and the steps are the ones described after (10.9): the preimage regrouping (10.12), which gives the factor $\tau(r) \mathbf{1}_{f'|k'}$; the retained support (10.13) and (10.16); exclusion removal (10.15); and [35, Lemma B.2] for the kernels. $\square$

Proof of Proposition 5. Lemma 9 supplies the hypothesis of Lemma 7 with $J_1 = 2m + 4$ and $M \ll D^{\delta/2} \Sigma(1 + S_m)$. Then Lemma 7 gives

$$\mathcal{E}(\mathcal{H}, L, F; \xi, W) \leq \mathcal{A}(W) \ll D^{\delta} \Sigma(1 + S_m) \|W\|_{C^{4m+12}(I)}^2, \quad (10.41)$$

which is (10.17) with $J_t(m) = 4m + 12$, as for ζ , and in the stronger form with the smooth majorant $\mathcal{A}(W)$ on the left. $\square$

11 Part II: the 7/8 boundary for the same curves

The remaining sections adapt OpenAI’s 7/8 argument [36], explained for ζ in [4], to the curves E_d . The result is a wider zero-free strip. Throughout Part II, $\mu = \mu_K$ is the Möbius function on primary elements, and $\chi_\bullet(u)$ denotes the character $n \mapsto \chi_n(u)$.

Theorem 1. *Let η be a finite-order Hecke character of $K = \mathbb{Q}(\sqrt{-3})$, and put $\psi = \bar{\alpha}\eta$, that is, $\psi(n) = \overline{\alpha(n)}\eta(n)$ on primary n prime to the conductor of η . Then*

$$L(s, \psi) \neq 0 \quad (\Re s > 7/8). \quad (11.1)$$

In particular, for every nonzero integer d ,

$$L(E_d, s) \neq 0 \quad (\Re s > 11/8), \quad (11.2)$$

a zero-free strip $11/8 < \Re s < 3/2$ along the right edge of the critical strip.

How the curves enter. Put $\psi_d^*(n) = \overline{\alpha(n)}\chi_n(4d)$ for primary n . If $p \nmid 6d$ splits as $p = \pi\bar{\pi}$ with π primary, then $\sqrt{p}(\psi_d^*(\pi) + \psi_d^*(\bar{\pi})) = a_p(d)$; if $p \nmid 6d$ and $p \equiv 2 \pmod{3}$, then $\psi_d^*(-p) = -1$, matching $a_p(d) = 0$ (Proposition 1). Since ψ_d^* and the character ψ_d of Lemma 1 give the same Euler factors (see below), that lemma gives

$$L(E_d, s + \tfrac{1}{2}) = B_d(s) \sum_n \frac{\psi_d^*(n)}{N(n)^s}, \quad (11.3)$$

with B_d holomorphic and nonzero in $\Re s > 0$. (We also checked both statements numerically for 2352 split pairs (d, p) , testing both primes above each p , and 854 inert pairs.) The character ψ_d^* is of the type in Theorem 1. Indeed $n \mapsto \overline{\alpha(n)}$, on primary n , is a Hecke character modulo 3 with infinity type $\bar{z}/|z|$, and $n \mapsto \chi_n(4d)$ is a finite-order ray character whose conductor is supported on the primes dividing $6d$ [36, Lemma 4.1]. So $\psi_d^* = \bar{\alpha}\eta_d$ with η_d of finite order, and its primitive inducing character ψ'_d lies in the family $\mathcal{F}_{\text{CM}}$ defined below. Moreover $\sum_n \psi_d^*(n)N(n)^{-s} = L(s, \psi'_d) \prod_{p|6d} (1 - \psi'_d(\mathfrak{p})N\mathfrak{p}^{-s})$, and each factor is nonzero in $\Re s > 0$ because $|\psi'_d(\mathfrak{p})| \leq 1$. Hence $L(s, \psi'_d) \neq 0$ for $\Re s > 7/8$ implies $L(E_d, s) \neq 0$ for $\Re s > 11/8$. The character ψ_d^* is the complex conjugate of the character $\psi_d = \alpha\chi_\bullet(4d) = \alpha\chi_\bullet((4d)^5)$ used in Sections 3–10. The two have the same L -function: since $4d$ is rational, $\psi_d^*(\bar{\pi}) = \overline{\psi_d^*(\pi)}$, so both characters give the same Euler factor at each split prime, and $\psi_d^*(-p) = -1$ is real. More generally, every unitary Hecke character of K whose infinity type is $(\bar{z}/|z|)^{\pm 1}$ is of the form $\bar{\alpha}\eta$ or its conjugate $\alpha\bar{\eta}$. So the theorem covers every weight-two newform with complex multiplication by K [41]. In particular it covers every elliptic curve over $\mathbb{Q}$ with complex multiplication by an order of K (the j -invariants 0, 54000 and -12288000 ; see [5]): by Deuring’s theorem [7] the L -function of such a curve is $L(s, \psi)$ for a Hecke character ψ of K of this infinity type, also when the order is not maximal [44, Chapter II, §10, and Appendix A, §3].

What “proof” means in Part II. We do not rewrite the 16,700 lines of [36]. Instead we do three things:

1. we prove the identities in which the angular factor changes a computation, and test them numerically;
2. we state each ingredient of [36] in the form it takes for a CM target;

3. we checked, section by section, that each proof in [36] uses only properties that CM characters share with finite-order characters, or we record the change.

Section 16 lists the statements that needed new proofs; they are proved, at the generality of [36], in Appendices B–G.

11.1 The family and the number β_*

Let $\mathcal{F}_{\text{CM}}$ be the set of primitive Hecke characters of K with infinity type $\bar{z}/|z|$, that is, the primitive characters inducing $\bar{\alpha}\eta$ with η of finite order. Put

$$\beta_* = \sup(\{1/2\} \cup \{\Re \rho : L(\rho, \psi) = 0, 1/2 \leq \Re \rho \leq 1, \psi \in \mathcal{F}_{\text{CM}}\}). \quad (11.4)$$

Three properties make this family suitable.

- *Closure under finite-order twists.* If $\psi \in \mathcal{F}_{\text{CM}}$ and θ has finite order, then the primitive character inducing $\psi\theta$ lies in $\mathcal{F}_{\text{CM}}$. Every denominator row below is such a twist.
- *Conjugation.* $\bar{\psi} = \alpha\bar{\eta}$ and $L(s, \bar{\psi}) = \overline{L(\bar{s}, \psi)}$, so conjugates contribute no new real parts of zeros.
- *No poles.* Every $L(s, \psi)$ with $\psi \in \mathcal{F}_{\text{CM}}$ is entire. No member is principal, so the principal-character clauses of [36] disappear for the target and the denominators. They remain for the finite-order numerators: the row $u = 1$, and the rows supported on S , which the detector removes as in [36, Section 8.1].

The rows of the argument also contain finite-order L -functions. For them we do not use β_* . Instead we use OpenAI’s theorem [36, Theorem 1.1]: they have no zeros in $\Re s > 7/8$. So, under the contradiction hypothesis $\beta_* > 7/8$, every target, numerator and row L -function whose zeros enter the detector or logarithmic control has its zeros in $\Re s \leq \beta_*$. In [36] this was automatic from the definition of β_* . Here the finite-order theorem is an input in two ways: wherever [36] uses that a finite-order row or witness character has no zeros to the right of β_* (the bin ceiling of the detector, Lemma 11, and the global part of the logarithmic control for the numerators and for the finite-order members of the label sets), and in the zero-free hypothesis for the slot characters in the plain moment (Section 14). There is no circularity: the zero-location bounds used to prove OpenAI’s theorem concern finite-order L -functions, and do not assume a CM zero-free strip. Its theta completion also contains angular auxiliary series, for example $L_0(s, \Psi) = \sum_b \overline{\alpha(b)}^3 \Psi(b)^3 N(b)^{-3s+1/2}$ [36, Proposition 5.1]; their zeros are not used in that proof.

The continuation criterion [36, Prop. 2.1] (see [4, Proposition 1]) holds verbatim with η replaced by ψ and the family of primitive finite-order characters replaced by $\mathcal{F}_{\text{CM}}$. Besides its hypotheses on H_ψ (holomorphic on $\Re s > \sigma_0$ with $|H_\psi - 1| \leq 1/2$; here H_ψ comes from the principal row, Proposition 6), its proof uses only:

- the definition of β_* as a supremum over $\mathcal{F}_{\text{CM}}$;
- absolute convergence of the Euler product of $L^S(s, \psi)$ in $\Re s > 1$, so that $1/L^S(s, \psi)$ is holomorphic there, and the bound $|1/L^S(s, \psi)| \leq \zeta_K(2)$ on $\Re s \geq 2$;
- the continuation of $L(s, \psi)$ to $\Re s > \sigma_0$ (it is entire), so that a zero of $L(s, \psi)$ is a pole of $1/L^S(s, \psi)$;
- the nonvanishing of finitely many Euler factors in $\Re s > 0$.

It therefore suffices, for each $\psi \in \mathcal{F}_{\text{CM}}$, to construct a function $J_\psi(Z)$ with the low and high estimates of the criterion, for $\sigma_0 = 7/8$ and $C(s) = s - 11/16$, with margins independent of ψ .

The construction is that of [36], with three main changes:

- the weight of the target (Section 12);
- the bookkeeping of the characters attached to rows (Section 13);
- an angular factor in the moment estimates (Section 14).

There are also three smaller ones: OpenAI’s theorem for finite-order characters [36, Theorem 1.1] is an input for the finite-order rows and slots; the capacity parameter is fixed at $\kappa = 3/4$, with the bin ceiling $\delta \leq 2\beta_* - 1$ kept separate from it; and the slots are measured against the finite-order part of the row character (Section 13). Section 15 checks that the final inequality holds with these changes.

The arithmetic datum. [36] fixes for each target a datum ([36, Section 4.1]): a finite set S of primes containing those above 6, and complete zero-extended presentations of finitely many *finite-order* ray characters with moduli supported on S , together with the finite ray group presenting them. Every “fixed character” of [36] belongs to this datum, and constants may depend on it. A CM target $\psi = \bar{\alpha}\eta$ is not of this kind. We put the finite part η into the datum. The excluded set S , and with it b_*, ξ, τ, Ξ , is chosen as in [36, Section 6.1] for η ; it contains the primes above 6, hence the conductor of $\bar{\alpha}$, so the same S serves for ψ . The ray group T is chosen independently of the target [36, Section 6.1], and $\Theta_0 = \langle \eta, \hat{T} \rangle$ is finite. The angular factor is carried as an explicit coefficient $\alpha(n)$ of primary generators, as [36] already does with $\bar{\alpha}\gamma_2$ and in [36, eq. (4.7)]. Whenever [36] applies a ray-class argument to a fixed character (a finite Fourier expansion on the fixed ray group, [36, Section 4.2]; membership in the finite group generated by the target and $\hat{T}$; “the same fixed finite-ray twist” in a product of plain factors, [36, Section 4.1]), the CM version applies it to the finite part only, and keeps $\alpha^{\pm 1}$ outside, with the same orientation on every variable that carries it. This is used in items 1, 3, 4 and 5 of Section 16.

12 The probe for a CM target

12.1 The weight

In [36] the target η enters the probe [36, eq. (6.1)], [4, §3.1], only through the weight on the completed index $A = cn^3$: $\gamma_2(c)\overline{\alpha(A)}\eta(A)$. For the target $\psi = \bar{\alpha}\eta$ the weight is therefore

$$\gamma_2(c)\overline{\alpha(cn^3)}\psi(cn^3) = \gamma_2(c)\overline{\alpha(c)}^2\overline{\alpha(n)}^6\eta(c)\eta(n)^3. \quad (12.1)$$

Everything else in the probe is unchanged:

- the additive polynomial with its sextic Gauss sums;
- the averaging variable m ;
- the fixed phases;
- the scales X, Y, Z .

Why $\overline{\alpha}\eta$ and not $\alpha\overline{\eta}$. The two targets have conjugate zeros, so either could be used. With $\alpha\overline{\eta}$ the weight would be $\gamma_2(c)\overline{\eta}(cn^3)$, with no angular factor. The completed sums would then be sums of coefficients of the cubic theta function itself. Its expansion at the identity cusp has a nonzero constant term; the expansions at the other two cusps have none [2, Theorem 23]. In [36] a $\bar{z}$ -derivative removes that term ([36, proof of Proposition 5.1, *Mellin normalization and continuation*]). Without a derivative it enters the reflection of the translates attached to the identity cusp, and the proof in [36] has no provision for the resulting term. With the target $\overline{\alpha}\eta$ the weight carries $\overline{\alpha}^2$. The completed sums are then those of $\partial_{\bar{z}}^2\theta$, which is the input of Section 8, and the constant term is removed.

12.2 Second evaluation: the rows

Proposition 6. *With the weight (12.1), Poisson summation in m gives the representation [36, eq. (7.5)] of the probe, with row u equal to*

$$\frac{\zeta_K^S(6z) L^S(w, \chi_{\bullet}(u))}{L^S(x, \psi \chi_{\bullet}(u))} \mathcal{H}_{\psi, u}(x, w, z), \quad (12.2)$$

where $\mathcal{H}_{\psi, u}$ is the Euler product of [36, Lemma 7.1] with $\eta(p)$ replaced by $\psi(p)$. It has the following properties:

- it converges in the same regions;
- it satisfies $\mathcal{H}_{\psi, u} \ll N(u)^\varepsilon$ there;
- $\mathcal{H}_{\psi, 1} = 1 + O(P_0^{-4/5})$ in the second region of [36] ([36, Lemma 7.1]).

Proof. [36] evaluates the coefficient prime by prime. The target enters only through the unimodular numbers $\eta(p)$ and complete multiplicativity, and no ray-class condition on p is used ([36, Section 7.2]).

The angular factor of the weight is absorbed by the Gauss-sum identity

$$\overline{\alpha(c)} \gamma_2(c) \gamma_1(c) = \mu(c) G(c), \quad (12.3)$$

a rearrangement of the second identity $\gamma_1\gamma_2 = \mu\alpha G$ in [36, eq. (4.7)]. Here c is squarefree, γ_1 is the normalized sextic Gauss sum, and $G(c) = \chi_c(4)\gamma_3(c)$ is the unimodular phase of [36, Lemma 4.4]. It factors through a fixed ray group supported over 2 and 3 but is not multiplicative, $G(vw) = G(v)G(w)\mathcal{R}(v, w)$ [36, eq. (4.5)]; as in [36] it is cancelled by the correction $\overline{G(cn^3)}$ of the probe, and no angular factor is involved. For the weight (12.1), $\overline{\alpha(c)}^2 \gamma_2(c)\eta(c)\gamma_1(c) = \mu(c)G(c)\psi(c)$. So every entry of the local table of [36] holds with $\eta(p)$ replaced by $\psi(p)$:

- The squarefree part gives $-\psi(p)\overline{\chi_p(u)}N(p)^{-x}$, which is the Euler factor of $1/L(x, \psi\chi_{\bullet}(u))$.
- The cube part gives the phases $a_p = \overline{\alpha(p)}^3 \psi(p)^3$ and $b_p = \overline{\alpha(p)}^2 \psi(p)^2 \overline{\chi_p(4)}$.
- The relation $b_p^3 = a_p^2$ used there still holds, since $\chi_p(4)^3 = (4/p)_2 = 1$.

The convergence regions and the bounds for $\mathcal{H}$ use only $|\psi(p)| = 1$. □

We checked (12.3) numerically at 60 primes (error below $4 \cdot 10^{-15}$). The proposition gives the following.

- The numerators $L(w, \chi_{\bullet}(u))$ have finite order. They are the same as for ζ .

- The denominators $L(x, \psi \overline{\chi_\bullet(u)})$ are induced by members of $\mathcal{F}_{\text{CM}}$ (Lemma 23(d)) and are never principal.
- The principal row $u = 1$ is $\zeta_K^S(w) \zeta_K^S(6z) / L^S(x, \psi)$. Its residues at $w = 1$ and $z = 1/6$ give the signal with $H_\psi(x) / L^S(x, \psi)$, and with the same positive constant c_S . The positivity $M(1/6) > 0$ is a property of the averaging weight alone.

12.3 First evaluation: completed rows with two derivatives

Cauchy–Schwarz splits the probe [4, §3.2] into two mean squares. The first is the additive one, which does not involve the target. The second is the mean square of the completed rows

$$B_m = \sum_{\substack{c \text{ sf}, n \\ (cn, S)=1}} \frac{\gamma_2(c) \overline{\alpha(cn^3)}^2 \nu(cn^3) \chi_{cn^3}(m)}{\sqrt{N(c)} N(n)} V\left(\frac{N(c)N(n)^3}{Z}\right), \quad (12.4)$$

where c and n are primary and may share primes. Here ν is a finite-order character whose conductor is supported on the fixed set S , extended by zero; it is the finite part η times the fixed phases. The test V is the Gaussian V_G of [36, Section 5.5], which does not have compact support. The angular factor must stay in the weight, not in ν : the theta step of [36] expands ν over residue classes, which is impossible for $\overline{\alpha}$.

Proposition 7. *For every $\varepsilon > 0$ and every $Q = Z^M$ with M in a fixed bounded range, $\sum_{0 < N(m) \leq Q} |B_m|^2 \ll Z^\varepsilon (Q + Q^2/Z)$, uniformly over the fixed finite family of finite-ray twists ν of [36]. This is the case $N = 1$ of [36, Lemma 5.8], summed over the dyads of the powerful part of m .*

This is (B.28), proved in Appendix B for every log-Schwartz test V , with a finite seminorm of V in the constant. [36] proves its version through its completed reflection [36, Prop. 5.1], which is the transformation formula for $\partial_{\bar{z}}\theta$. For (12.1) one uses instead $\partial_{\bar{z}}^2\theta$, as in Section 8 and Appendix A; at the generality of [36] this is Proposition 9. Appendix A alone does not suffice, because the probe uses the Gaussian test and an excluded set S that contains every prime of norm at most P_0 and the moduli of η [36, Section 6.1]. The changes are:

	ζ and finite order [36]	CM target
derivative	$\partial_{\bar{z}}\theta$	$\partial_{\bar{z}}^2\theta$
gamma factor of $T(s)$	$\Gamma(s + \frac{1}{3})\Gamma(s + \frac{2}{3})$	$\Gamma(s + \frac{5}{6})\Gamma(s + \frac{7}{6})$
reflected kernel	$\prod_{\pm} \frac{\Gamma(1+t\pm 1/6)}{\Gamma(1-t\pm 1/6)}$	$\frac{\Gamma(4/3+t)\Gamma(5/3+t)}{\Gamma(4/3-t)\Gamma(5/3-t)}$
cuspidal scalar	$-\frac{i}{81} \overline{\alpha(c)}^2$	$-\frac{1}{81} \overline{\alpha(c)}^4$
angular factor, original side	$\frac{\overline{\alpha(n)} \overline{\alpha(b)}^3}{\alpha(n)}$	$\frac{\overline{\alpha(n)}^2 \overline{\alpha(b)}^6}{\alpha(n)}$
angular factor, dual side	$\alpha(\ell)$	$\alpha(\ell)^2$

Here the j -th derivative ($j = 1, 2$) of the completed theta sum at $z = 0$ is integrated against $v^{2s+j-2} dv$; in both cases the result is $\frac{3^{5/2}}{4} \left(\frac{27}{(2\pi)^2}\right)^s$ times the gamma factor times $T(s)$. (With $v^{2s-1} dv$, as in (A.24), the CM factor is $\Gamma(s + \frac{1}{3})\Gamma(s + \frac{2}{3})$ times $T(s - \frac{1}{2})$.) Both kernels satisfy $|R(\sigma + iu)| \ll_A (1 + |u|)^{4\sigma}$ on $-1/4 \leq \sigma \leq A$. The CM kernel is holomorphic for $\Re t > -4/3$; OAI78's is holomorphic for $\Re t > -5/6$. So the reflected weight obeys the same bound $\ll \min(x^{1/4}, x^{-A})$, with the same seminorms (Proposition 9(e)). The other ingredients carry over as follows.

- *Dual lengths.* They are unchanged.
- *Quadratic large sieve.* The dual angular factor $\alpha(\ell)^2$ is a unimodular coefficient of the dual index, so it is absorbed into the arbitrary coefficients of the sieve.
- *Marks.* Slot primes in the completed index are handled in [36] through $\mathbf{1}_{q|A} = 1 - \chi_q(A)^0$, which never requires splitting $\bar{\alpha}$.
- *Phases.* The cusp scalar $\overline{\alpha(c)}^4$, which replaces $-i\overline{\alpha(c)}^2$, is completely multiplicative, so it splits into a fixed, a row and a tuple factor exactly as in [36, Section 14.2]. The cross-prime phases [36, proof of Proposition 5.1, *The reflected additive phase*] do not involve the derivative.

The balanced low estimate $|I_\psi(Z^{1/2}, Z^{1/2}, Z)| \ll Z^{1/4+\varepsilon}$ follows from the proposition as in [36]; it is used only in the 11/12 stage of [36].

The compensated probe needs more than Proposition 7. Its marked rows involve sums over tuples of slot primes, divisibility conditions on the completed index, and a scale that depends on the tuple. In [36] their bound comes from Corollary 14.1 and Lemmas 14.2, 14.3 and 15.1. Their CM forms, with the same uniformity in the tuples, the masks and the smooth weights, are Proposition 10 and Lemmas 15 and 16 in Appendix B; [36, Lemma 14.2] does not involve the weight. The compensated low estimate $|I_{\text{mod}}(Z)| \ll Z^{3/16+\varepsilon}$ then follows as in [36, Prop. 15.3], since the additive Gram bound [36, Prop. 15.2], a planar additive large sieve (compare [19]), does not involve the target. We checked numerically:

- the Bessel–Mellin identities for one and two derivatives (relative error $3 \cdot 10^{-26}$);
- the two reflection constants;
- the Stirling exponents of the kernels.

13 Rows, zeros and prime slots

13.1 Growth of the CM L -functions

Lemma 10. *Let $\psi \in \mathcal{F}_{\text{CM}}$ have conductor $\mathfrak{f}$, and put $Q = N\mathfrak{f}$. Then*

$$\Lambda(s, \psi) = (3Q)^{s/2} (2\pi)^{-s} \Gamma(s + \tfrac{1}{2}) L(s, \psi) \quad (13.1)$$

is entire and $\Lambda(s, \psi) = \epsilon(\psi) \Lambda(1-s, \bar{\psi})$ with $|\epsilon(\psi)| = 1$. Moreover $|L(\sigma + it, \psi)| \ll Q^{3/5} (3 + |t|)^2$ for $-1/10 \leq \sigma \leq 11/10$, with an absolute implied constant. This is the bound of [36, Lemma 4.8] for finite-order characters. For every compact $I \subset (-\frac{1}{2}, \frac{3}{2})$ and $\sigma \in I$,

$$|L(\sigma + it, \psi)| \ll_I (3Q)^{1/2-\sigma} (3 + |t|)^{1-2\sigma} |L(1-\sigma - it, \bar{\psi})|. \quad (13.2)$$

All of this holds for $\bar{\psi}$ as well.

Proof. See Appendix C, Section C.3. The functional equation is proved there from a theta series with the harmonic polynomial $\bar{a}$, by Hecke’s method [18] (see [33, Ch. VII, §§7–8]), with $\epsilon(\psi)$ an explicit normalized Gauss sum (Lemma 18). The gamma quotient $\Gamma(\frac{3}{2} - s)/\Gamma(\frac{1}{2} + s)$ has the exponent $1 - 2\sigma$ of $\Gamma(1-s)/\Gamma(s)$ (Lemma 19). The strip bound then follows by the damped Phragmén–Lindelöf argument [40] of [36, proof of Lemma 4.8]. $\square$

The logarithmic control of [36, Lemma 4.9] holds for $\mathcal{F}_{\text{CM}}$ and the conjugate characters (Lemma 20). It has the same conductor and height exponents, its constants do not depend on the character, and it has no principal clauses. For the characters of the rows, finite-order or CM, its global form holds on $\Re s \geq b + v$ for every fixed $b \geq \max(\beta_*, \frac{7}{8})$. Section C.7 lists every use in [36] of these growth statements, together with the CM statement that replaces it.

13.2 The characters attached to a row

In [36] each row u carries a finite set of characters: the twists $\theta\chi_{\bullet}(u)^{\pm 1}$ by members θ of the finite group $\Theta_0 = \langle \eta, \hat{T} \rangle$, where $\hat{T}$ consists of fixed characters supported over 2 and 3. A row is *bad* if one of these has a zero of height at most a small power Z^τ well to the right of $1/2$. For a CM target the group generated by ψ is infinite. We use instead the finite set

$$\mathcal{X}_u = \{\theta\chi_{\bullet}(u)^{\pm 1}\} \cup \{\bar{\alpha}\theta\overline{\chi_{\bullet}(u)}, \alpha\theta\chi_{\bullet}(u)\}, \quad \theta \in \Theta_0 = \langle \eta, \hat{T} \rangle. \quad (13.3)$$

It has three properties.

- It contains the numerator, the denominator $\psi\overline{\chi_{\bullet}(u)}$, and their conjugates.
- It is finite, with $4|\Theta_0|$ labels for every u (the label set is independent of u), and closed under conjugation. Each CM member is induced by a primitive character in $\mathcal{F}_{\text{CM}}$ or in its conjugate, of conductor norm $\ll N(u)$ (Lemmas 21 and 23); so it is never principal and its L -function is entire. Together with Lemma 10, the logarithmic control (Lemma 20), the deleted-Euler-factor bounds ([36, Lemma 4.10], which use only $|\psi^*(\mathfrak{p})| \leq 1$), and the removal of the finitely many S -supported rows (needed only for the finite-order members), these are all that the detector of [36, Section 8] uses. Appendix D goes through it line by line (Proposition 11).
- Its finite-order members have no zeros in $\Re s > 7/8$, by [36, Theorem 1.1], and its CM members and their conjugates none in $\Re s > \beta_*$. Under the contradiction hypothesis $\beta_* > 7/8$ this gives the bin ceiling $a \leq \beta_*$ of [4, §3.3] (Lemma 11 below).

Lemma 11 (Bin ceiling for $\mathcal{X}_u$). *Suppose $\beta_* > 7/8$. Let $u \neq 0$, let $\chi \in \mathcal{X}_u$, and let χ' be the primitive character inducing χ . Then $L(s, \chi') \neq 0$ for $\Re s > \beta_*$. Consequently, in [36, Lemma 8.1] with the label set $\mathcal{X}_u$, every maximum $M_i(u)$ is at most β_* , every bin has $a \leq \beta_*$, and $\delta = 2a - 1 \leq 2\beta_* - 1$.*

This replaces [36, Section 8.1], whose first sentence (every primitive character inducing a member of the label set has finite order) is false for $\mathcal{X}_u$.

Proof. There are three cases.

- $\chi = \theta\chi_{\bullet}(u)^{\pm 1}$ with $\theta \in \Theta_0$. Then χ has finite order, since θ has and $n \mapsto \chi_n(u)$ is a finite-order ray character [36, Lemma 4.1]. By [36, Theorem 1.1], $L(s, \chi')$ has no zero in $\Re s > 7/8$, and $7/8 < \beta_*$.
- $\chi = \bar{\alpha}\theta\overline{\chi_{\bullet}(u)}$. Then $\theta\overline{\chi_{\bullet}(u)}$ has finite order, so $\chi' \in \mathcal{F}_{\text{CM}}$. By the definition of β_* , $L(s, \chi')$ has no zero with $\beta_* < \Re s \leq 1$, and the Euler product excludes zeros in $\Re s > 1$.
- $\chi = \alpha\theta\chi_{\bullet}(u)$. Then $\bar{\chi}$ is as in the second case, its primitive inducing character is $\bar{\chi}'$, and $L(s, \chi') = \overline{L(\bar{s}, \bar{\chi}')}$. So the zeros of $L(s, \chi')$ are the conjugates of those of $L(s, \bar{\chi}')$, with the same real parts.

A presentation in $\mathcal{X}_u$ differs from χ' by finitely many factors $1 - \chi'(\mathfrak{p})N\mathfrak{p}^{-s}$, which are nonzero in $\Re s > 0$. Each $M_i(u)$ is the maximum of the floor 51/100 and of real parts of zeros of members of $\mathcal{X}_u$ in a bounded rectangle [36, Section 8.1]. Hence $M_i(u) \leq \max(51/100, 7/8, \beta_*) = \beta_*$. $\square$

If the 11/12 stage of [36] is also carried out for $\mathcal{F}_{\text{CM}}$, it runs under $\beta_* > 11/12$, and in the first case [36, Theorem 3.1] already suffices. That finite-order theorem does not assume a zero-free bound for the CM family, so using it here is not circular.

A zero of a CM member produces the two witnesses of [4, §3.3], now with the angular factor:

$$M_r = U^{-r/2} \sum_n \mu(n) \overline{\alpha(n)} \theta(n) \overline{\chi_n(u)} W\left(\frac{N(n)}{U^r}\right), \quad S_m = U^{-m/2} \sum_l \overline{\alpha(l)} \theta(l) \overline{\chi_l(u)} W'\left(\frac{N(l)}{U^m}\right), \quad (13.4)$$

or their conjugates. In each, $\overline{\alpha}$ appears together with $\overline{\chi}$, or α together with χ . We call such sums *co-oriented*; this matters in Section 14.

Witnesses and slots. [36] conjugates both witness factors if necessary, so that their row character is $\nu(n)\overline{\chi_n(u)}$, and then measures each physical slot prime against this row: its coefficient is $\nu(p)\mathbf{1}_{p \in 1_T}$, a fixed combination of members of the finite group ([36, Section 19.2]). For a CM zero the conjugated row character is $\overline{\alpha\theta'}\overline{\chi_\bullet(u)}$ with $\theta' \in \Theta_0$; measured against it, the slot coefficient would be $\alpha(p)\overline{\theta'(p)}\mathbf{1}_{p \in 1_T}$, which is not of that form. The slots are therefore measured against the finite-order part $\theta'\overline{\chi_\bullet(u)}$: their coefficient is $\theta'(p)\mathbf{1}_{p \in 1_T}$, and $\overline{\alpha}$ stays on the Möbius and plain variables only. This is how the moments of Section 14 are stated: the inverse moment is used with $\epsilon = -1$, and, after the conjugation of [36, Section 19.2], the plain moment with $e = +1$, $\psi_k = \overline{\theta'}\chi_\bullet(u)$ and slot coefficients $\theta'(p)\mathbf{1}_{p \in 1_T}$. A retained large row still induces outside Θ_0 , now for its finite-order part, because it ramifies at a prime outside S ([36, Sections 8.1 and 19.2]).

13.3 The compensated probe and the slots

The compensated probe [4, §4.2] is formed with $\overline{\psi(p_{J^c})}$ in place of $\overline{\eta(p_{J^c})}$. In row u a slot prime $p \nmid u$ contributes the local factor $\psi(p)\overline{\chi_p(u)}N(p)^{-x}$ of the denominator, and $\overline{\psi(p)}\psi(p) = 1$. So the marked-minus-rescaled difference is $-\chi_p(u)$ plus small terms, exactly as for ζ . The slot amplitudes

$$Q_i(u) = P_i^{-1/2} \sum_{p \in \text{slot } i} \Omega_i\left(\frac{N(p)}{P_i}\right) \left(\frac{N(p)}{P_i}\right)^{z-1} \overline{\left(\frac{u}{p}\right)_6} \quad (13.5)$$

are the same finite-order prime polynomials as in [4, §4.4]. Two consequences follow.

1. The principal normalizer $A_T(Z)$ is unchanged.
2. The pointwise bound $|Q_i(u)| \ll U^{\epsilon_1} P_i^{\delta/2 + O(\epsilon)}$ of [36, Lemma 19.1] is unchanged, since the slot characters $\theta\overline{\chi_\bullet(u)}$ lie in $\mathcal{X}_u$. Separately, the uniform slot bound inside the plain moment ([36, eq. (18.12)]) needs a zero-free region only for finite-order characters. OpenAI's theorem supplies it at 7/8, so $\kappa = 3/4$ may be used there and in the plain capacity, in place of $\kappa = 2\beta_* - 1$. In [36] the same κ also writes the bin ceiling as $\delta \leq \kappa$ ([36, eq. (12.2)]). With $\kappa = 3/4$ the two must be kept apart: the bin ceiling is $\delta \leq 2\beta_* - 1$ (Lemma 11), which may exceed 3/4. [36] uses $\delta \leq \kappa$ only through $\delta \leq 5/6$ (Section 15).

14 Counting bad rows with an angular factor

[36] counts bad rows with the sextic large sieve of [4, §3.3] (which builds on [10, 1, 15]) and the three averages of [4, §5.1]. For rows whose zero comes from a finite-order member they apply unchanged. For CM members they are needed with the angular factor on the summation variable, but not on the slot primes.

1. The sextic large sieve [36, Lemma 9.1]. It allows arbitrary coefficients fixed independently of the row, so it is unchanged. For a CM zero the witness coefficient is $D^{-1/2}\mu(n)\alpha(n)\theta(n)\overline{\chi_n(v)}W(N(n)/D)$ (or its conjugate), with v the frozen non-squarefree part of the row. Apart from elementary counts that do not involve the target, this is the only row count used in the 11/12 stage of [36] (its Part I); with CM labels it is Lemma 34.

2. The inverse moment with prime factors. Let ν be a fixed finite-order ray character whose zero-extended defining modulus has prime support in S ; the angular factor is not part of ν . Fix a sign $\epsilon = \pm 1$, bounded ranges for the nonnegative parameters m, r, z_i , a bound for the number of slots, and constants $c_1, c_2 > 0$, and suppose, as in [36, Lemma 17.1], $r + 2z \leq m - c_1$ and $2r + 8z \leq 3m - c_2$. For smooth annular W, W_i put

$$\begin{aligned} M_u &= Z^{-r/2} \sum_n \mu(n) \alpha(n)^\epsilon \nu(n) \chi_n(u)^\epsilon W\left(\frac{N(n)}{Z^r}\right), \\ Q_u &= \prod_i Z^{-z_i/2} \sum_{p \in \mathcal{P}_i} a_i(p) \nu(p) \chi_p(u)^\epsilon W_i\left(\frac{N(p)}{Z^{z_i}}\right), \end{aligned} \tag{14.1}$$

with the same ν and ϵ in M_u and in every slot, where $z = \sum_i z_i$, the slot sets $\mathcal{P}_i$ are disjoint and avoid S , $N(p) \asymp Z^{z_i}$ for $p \in \mathcal{P}_i$, and the coefficients $a_i(p)$ are independent of u with $|a_i(p)| \leq 1$. Then for every $\varepsilon > 0$

$$\sum_{\substack{u \in \mathbb{Z}[\omega] \\ 0 < N(u) \leq Z^m}} |M_u Q_u|^2 \ll Z^{m+\varepsilon}, \tag{14.2}$$

with the uniformity of [36, Lemma 17.1]: the constant depends only on the fixed arithmetic data, c_1, c_2 , the parameter ranges and finitely many seminorms of the tests; it is uniform in the prime supports and the coefficients a_i ; the bound holds for every subcollection of the slots; norm twists of the tests cost a fixed power of their heights; and there is no mesh or lower-length condition on the slots. *Why.* Appendix E proves (14.2) (Proposition 12). The proof in [36] alternates between two coefficient classes: Möbius coefficients $\mu\nu$, and cubic coefficients $\overline{\alpha}\gamma_2\nu$ attached to the theta function. Each switch uses (12.3) or its companion

$$\mu(n)\gamma_{-1}(n) = \chi_n(-1)G(n)^{-1}\overline{\alpha(n)}\gamma_2(n), \tag{14.3}$$

which holds for squarefree n [36, proof of Lemma 17.2, *The second Poisson transformation*] and which we checked at 60 primes and at squarefree composites. [36] runs the orientation $\mu\nu$ with $\overline{\chi}$: for $\epsilon = +1$ it first conjugates the whole residual polynomial (finite character, marks and tests; [36, proof of Lemma 17.1]), which leaves the row norm unchanged. Conjugation sends $\alpha\nu\chi_\bullet(u)$ to $\overline{\alpha}\overline{\nu}\overline{\chi_\bullet(u)}$, so a co-oriented sum stays co-oriented. With the co-oriented factor $\overline{\alpha}$ on the Möbius variable the classes are therefore $\mu\overline{\alpha} \leftrightarrow \overline{\alpha}^2\gamma_2$ for both signs. The canonical class of [36, Lemma 17.2] is $\overline{\alpha}\gamma_2\nu\rho$, and that lemma admits no additional residual coefficient ([36, Lemma 17.2]); so the extra factor $\overline{\alpha}$ must be built into the class, which becomes $\overline{\alpha}^2\gamma_2\nu\rho$ in the CM canonical estimate (Lemma 25).

These are the dual coefficient and the cube factor $\bar{\alpha}^6$ of Sections 7–10, and the theta steps use $\partial_{\bar{z}}^2 \theta$ as in Appendix B; the terminal application of [36, Lemma 14.3] becomes Lemma 15. In the CM case each switch is (12.3) or its companion multiplied by $\overline{\alpha(n)}$ (Lemma 26, checked at all 366 squarefree n with $N(n) \leq 1300$). Poisson summation acts on the row variable for a fixed pair of columns n_1, n_2 , so the factor $\overline{\alpha(n_1)}\alpha(n_2)$ passes through each transformation. It splits into a factor on the new columns and a factor of modulus one on outer variables, which [36] bounds only in absolute value (Lemma 27).

In the overlap reduction of [36] ($n = jn_0$, $P = jP_0$, $c = n_0P_0$) one has $\alpha(n)^\epsilon = \alpha(j)^\epsilon \alpha(c)^\epsilon \overline{\alpha(P_0)}^\epsilon$. The assigned primes contribute a u -independent unimodular scalar, and each surviving slot prime, which becomes part of the new Möbius variable c , has its coefficient multiplied by $\mu(p)\overline{\alpha(p)}^\epsilon$. That coefficient is still bounded, independent of u , and of product form, which is all [36] uses. The masks, punctures, scales and counts of the descent, and the propagation of seminorms, do not involve the coefficients.

Orientation. The anti-oriented sum, with α paired with $\bar{\chi}$, would turn into plain γ_2 , since $\alpha \cdot \bar{\alpha} \gamma_2 = \gamma_2$. That is the theta function without a derivative, with its constant terms. So the argument does not cover that case: its terminal step would need a reflected energy for θ itself, and the clause “for either common sign” of [36, Lemma 17.1] is not claimed for anti-oriented sums. The case is never needed. Every CM member of $\mathcal{X}_u$ is co-oriented, [36] conjugates witness factors only to give their row character the orientation $\chi_\bullet(u)$ of the physical slots ([36, Section 19.2]), and conjugation preserves co-orientation. The inverse witness of a zero of $\alpha\theta\chi_\bullet(u)$ thus becomes $\sum \mu\bar{\alpha}\bar{\theta}\overline{\chi_\bullet(u)}$, the case $\epsilon = -1$; the slots carry no angular factor.

3. Sixth-power amplification [36, Lemma 17.6]. Its proof is unchanged, except that the column identity [36, eq. (17.90)] acquires the factor $\alpha(d)^\epsilon$, of modulus one, on its divisor d (equation (E.24)); the coefficient mass is the same. The identity $\chi_n(ua^6) = \chi_n(u)$ for $(a, n) = 1$ does not involve α . Its input is (14.2) without slots [36, eq. (17.87)]. For $\epsilon = 1$, S the primes above 6 and ν principal, the mean square (4.6) of Part I of these notes is that raw moment with $H = D^{1+\vartheta}$, that is, (14.2) with $Z = H$, $m = 1$, $r = 1/(1+\vartheta)$, no slots and $c_1 = \vartheta/(1+\vartheta)$, for a fixed $0 < \vartheta \leq 1/10$.

4. The plain fourth moment with prime factors [36, Lemma 18.1]. The CM statement needed is the following. Fix $e = \pm 1$ and $3/4 \leq \kappa \leq 1$. For rows k with $0 < N(k) \ll Z^m$, write $\psi_k = \tau\chi_\bullet(k)$ for the finite-order row character, and put

$$S_k(n_j) = Z^{-n_j/2} \sum_l \alpha(l)^e \psi_k(l) W_j \left(\frac{N(l)}{Z^{n_j}} \right) \quad (j = 1, 2). \quad (14.4)$$

Let Q be a product of slot polynomials in ψ_k with no angular factor, of total length z , whose coefficients are fixed combinations of characters in the finite group Θ of [36, Section 18], which contains Θ_0 and the characters separating the fixed reciprocity phases. Let $\mathcal{R}_z$ be the rows of [36], defined through the finite-order part ψ_k . Then

$$\sum_{k \in \mathcal{R}_z} |S_k(n_1) S_k(n_2) Q|^2 \ll Z^{M+\epsilon}, \quad M = m + q, \quad (14.5)$$

with the same hypotheses and uniformity as in [36]: no restriction when $z = 0$; $n_1 + n_2 + 6\kappa z \leq M$ and slots of length at most the mesh when $z > 0$, with a mesh independent of ψ and e ; uniformity over moving moduli, masks and frozen labels. Here Z^q bounds the norm of the full displayed moving zero support of τ , counted before cancellation or reduction of exponents modulo six ([36, Sections 13.1 and 18]). For $\kappa < 1$, [36] also assumes $\beta_* \leq (1 + \kappa)/2$ for its own β_* , the supremum

over primitive finite-order characters. That hypothesis enters only through [36, eq. (18.11)], applied to the primitive characters inducing the slot characters $\psi_k \vartheta$, $\vartheta \in \Theta$, which have finite order because the slots carry no angular factor. So the CM hypothesis is that no primitive finite-order character has a zero in $\Re s > (1 + \kappa)/2$; by [36, Theorem 1.1] it holds for every $\kappa \in [3/4, 1]$, and $\kappa = 3/4$ is allowed. It does not involve the β_* of $\mathcal{F}_{\text{CM}}$. This statement is proved in Appendix F. The changes to the proof are four.

- (a) *Hecke reflection* [18]. This is Lemma 22, proved there and again, in the form used here, by direct Poisson summation in Lemma 29, with an explicit root number. The dual profile is $\mathcal{MW}^\sharp(s) = \mathcal{MW}(1-s)\Gamma(s + \frac{1}{2})/\Gamma(\frac{3}{2}-s)$, in place of $\mathcal{MW}(1-s)\Gamma(s)/\Gamma(1-s)$. Both quotients grow like $|t|^{2\sigma-1}$ on vertical lines, and the conductor scale changes only at the prime above 3, by a factor between 1/9 and 9 (Lemmas 21 and 28). So the reflected lengths and seminorm bounds are unchanged.
- (b) *Poisson summation in the rows*. The first transform is Poisson over the row k , the second over the dual Gauss row h ([36, Lemma 13.3]), and the amplifier maps rows $h \mapsto hp^6$. The plain variables stay inside the column moduli throughout: they are never Poisson-summed and never become a row variable, and every reciprocity step involving them concerns only the finite characters. So $\alpha(l)^e$ is a column coefficient. Each extraction of a plain prime power p^i leaves a frozen unimodular scalar $\alpha(p)^{ie}$.
- (c) *Main terms*. The induction of [36] meets volume-sized main terms on child rows whose finite-order part induces a character in Θ_0 ([36, Section 18.7]), and cancels them with an equal-scale centered difference. In the CM case the plain sums there are $\mathcal{L}(X, t) = \sum_l \alpha(l)^e \vartheta(l) \mathbf{1}_{(l, \mathfrak{N}_*)=1} N(l)^{it} W(N(l)/X)$, sums over primary generators in finitely many residue classes of a fixed lattice; the mask is removed by inclusion–exclusion, using $\alpha(dl) = \alpha(d)\alpha(l)$. The proof of [36, Lemma 18.3] applies to $f_t(z) = \alpha(z)^e |z|^{2it} W(|z|^2)$, and its zero frequency is $\int_{\mathbb{C}} f_t(z) dz = \int_0^\infty r^{2it} W(r^2) r dr \int_0^{2\pi} e^{ie\phi} d\phi = 0$. So that lemma holds with main-term constant 0 and the same uniform error; the centered difference is kept verbatim, with both main terms zero. Lemma 31 gives the bounds in the form the induction uses.
- (d) *The zero-free hypothesis*. The slot amplitudes have finite order, so the hypothesis is OpenAI’s theorem, with $\kappa = 3/4$.

Appendix F carries this out and proves the CM statement (Proposition 13). It redoes the induction of [36] with the angular factor $\alpha(l_1 l_2)^e$ carried through every transform and kept off the slot primes. Besides (a)–(d), the only change is this bookkeeping (Lemma 32).

15 The final inequality

A bin of rows of norm $U = Z^d$ contributes at most $Z^{C(7/8)+E(d)+o(1)}$, with the exponent $E(d)$ of [36, eq. (20.4)] [4, §6] ([36, Lemma 20.1]). Its terms come from four sources:

- the scales and the Mellin weight on the central contour;
- the number of bad rows, given by the moments above with the same exponents;
- the prime amplitudes, which have finite order;
- the size $U^{\delta/2}$ of the finite-order numerator on the reflected line.

The CM denominator enters only through its reciprocal on the contour $\Re s = a + 16e$, which costs U^ε by the CM logarithmic control on the buffered disks. The exponent $63/50$ of the small rows ([36, proof of Lemma 10.6]) uses the strip bound of the finite-order numerator, and the large rows use only absolute convergence. So no term involves the gamma factor or a convexity bound of a CM L -function; that gamma factor affects only fixed powers of the height.

So the endpoint certificate of [36, Lemma 20.2],

$$-E(h) \geq \frac{49}{440640} \quad (0 \leq \delta \leq 5/6, 0 \leq x \leq 1/2), \quad (15.1)$$

with $q = x\delta$ and $R = R_*(\delta, x)$, applies verbatim. [36] compares it with $\Delta = \beta_* - 7/8$ in [36, eq. (20.10)]:

$$E_{\text{actual}}(h) - \Delta \leq -\frac{49}{440640} - \left(1 - \frac{h}{4}\right)\Delta + \epsilon_{\text{total}} < 0. \quad (15.2)$$

The term $h\Delta/4$ is a capacity loss caused by the choice $\kappa = 2\beta_* - 1$. With $\kappa = 3/4$ it is exactly zero, which only helps; the bin ceiling is then written $\delta \leq 2\beta_* - 1$, not $\delta \leq \kappa$ (Lemma 11). The bin ceiling $\delta \leq 5/6$ comes from the 11/12 stage, that is, from $\beta_* \leq 11/12$ for $\mathcal{F}_{\text{CM}}$ (Theorem 2). This bound is a statement about each member of the family separately; it carries no uniformity in ψ . Part I of these notes, as written, gives it for the characters ψ_d^* , one curve at a time. Appendix G runs Part I with a fixed finite-order twist in the columns, as [35] does for finite-order characters, and so proves it for every member. The balanced probe of Section 12, at $X = Y = Z^{1/2}$ and with the sextic row count, gives a second proof: this is Part I of [36] with the changes above, and it uses neither prime slots nor the plain moment (Section G.3). In [36] the bin ceiling is used only in two forms: as $a \leq \beta_*$ ([36, proof of Lemma 10.3] and, through [36, eq. (12.2)], [36, proof of Lemma 20.1]), which Lemma 11 supplies for $\mathcal{X}_u$; and as $\delta \leq \alpha = 5/6$ ([36, proof of Proposition 19.2, *Cases requiring no selected primes*; Lemma 20.2; Section 20.5; proof of Proposition 20.3]), which follows from Theorem 2. The shared high-side lemmas of [36], used again at $\sigma_0 = 7/8$, hold for $\mathcal{F}_{\text{CM}}$ by Proposition 15. Every margin is chosen before the target, as in [36], so all margins are uniform in ψ . The continuation criterion then gives the contradiction. Hence $\beta_* \leq 7/8$, which is Theorem 1.

16 What changes in the CM case

The table goes through [36] and records, for each part, how the properties of the target and of the row characters that it uses change in the CM case. No new idea is needed, but several statements need new proofs; these are given in Appendices B–G.

Sections of [36]	content	CM change
1–4	criterion, Gauss sums, growth lemmas	growth with $\Gamma(s + \frac{1}{2})$, functional equation, logarithmic control (Lemma 10, Appendix C); OpenAI’s theorem for finite-order rows
5	completed reflection, row energy	$\partial_{\bar{z}}^2 \theta$; table in Section 12; Appendix B
6–8	probe, Poisson rows, detector	weight (12.1); local identity formal in $\psi(p)$; label set $\mathcal{X}_u$; bin ceiling [36, Section 8.1] replaced (Lemma 11, Appendix D)
9–11	sextic count, high side, 11/12	witness coefficients carry $\alpha^{\pm 1}$ (Lemma 34); the 11/12 stage and the shared high-side lemmas (Appendix G)
12–16	compensated probe, low side	normalization $\overline{\psi(p)}$; slots measured against the finite-order part (Section 13)
17	inverse moment	co-oriented classes $\mu\bar{\alpha} \leftrightarrow \bar{\alpha}^2\gamma_2$ (Appendix E)
18	plain fourth moment	reflection with $\Gamma(s + \frac{1}{2})$; lattice main terms vanish (Appendix F)
19–20	row counts, endpoint	$\kappa = 3/4$; bin ceiling $\delta \leq 2\beta_* - 1$

Six statements need new proofs. They are proved in the appendices, at the generality of [36]. In each, a step identical to [36] is cited by its number there, and a step that changes in the CM case is proved.

1. *The transformation formula for $\partial_{\bar{z}}^2 \theta$* , for a general modulus supported on S , all log-Schwartz tests and the cross-prime phases, with its marked form and the lemmas built on it: Appendix B (Propositions 9 and 10, Lemmas 14, 15 and 16). Lemma 14.2 of [36] does not involve the weight and is used unchanged.
2. *Growth and logarithmic control*: Lemma 10, proved in Appendix C, which also gives the functional equation with an explicit root number, the conductors of the row characters, and the logarithmic control (Lemma 20).
3. *The detector with the label set $\mathcal{X}_u$* : Appendix D (Lemma 23 and Proposition 11). Part (e) of that lemma bounds the conductor exponents of the row characters at the primes above 2 and 3; [36] uses such a bound without stating it.
4. *The inverse moment (14.2) for co-oriented coefficients*: Appendix E (Proposition 12). Its terminal step is Lemma 15.
5. *The plain fourth moment with α^e on the plain variables*: Appendix F (Proposition 13).
6. *The 11/12 stage, $\beta_* \leq 11/12$ for $\mathcal{F}_{\text{CM}}$* : Theorem 2, proved one member at a time by Part I of these notes with a finite-order twist, and the shared high-side lemmas used again at $\sigma_0 = 7/8$ (Proposition 15).

The precise form of Proposition 7 is Lemma 14. The bin ceiling for the label set $\mathcal{X}_u$, which [36] derives from a statement that is false for CM members, is Lemma 11.

What Theorem 1 rests on. Besides the results proved here, the proof uses the steps of [36] cited as identical, OpenAI’s theorem for finite-order characters [36, Theorem 1.1], the theorem of Kubota and Patterson [24, 39] in the form [2, Theorem 23], the quadratic large sieve [14, 10], and the prime number theorem for prime ideals in ray classes [17, 18]. OpenAI’s 7/8 theorem is

formalized in Lean [37]. Of the CM argument, only the last step of Part I is formalized so far; a complete certification of Parts I and II is in progress.

Numerical checks. We checked the new identities numerically:

- the point counts against ψ_d^* ;
- (12.3) and its companion, and the switching identities, the marked cube inversion, the overlap identity and the sixth-power column identity of Appendix E;
- the transformation formula of Appendix B end to end, with an active prime of norm 7 and tests without compact support, and its marked form;
- the theta transformation, functional equation and root number of Lemma 18 for four characters, and the functional equations of eight CM members of the label sets (PARI/GP [38]), with the conductor exponents of Lemma 23(e);
- the reflection of Lemma 29 for twelve primitive characters, the dual profile, and the vanishing of $\int \alpha^e f$;
- the twisted paired identities and the twisted first Poisson reduction of Appendix G;
- the Bessel–Mellin formulas, the reflection constants, and the Stirling exponents of all the gamma quotients.

Numerical checks support the identities, but do not establish the uniform analytic estimates.

17 CM newforms of higher weight: why the method should extend

Write $\psi_k = \bar{\alpha}^k \eta$ with $k \geq 1$ and η of finite order. These are the unitary Hecke characters of K with infinity type $(\bar{z}/|z|)^k$. Up to finitely many Euler factors, their L -functions are those of the CM newforms of weight $k + 1$ with complex multiplication by K [42, 41], shifted to the center $1/2$: if f is such a form, then $L(f, s + \frac{k}{2}) = L(s, \psi)$ for $\psi = \psi_k$ or its conjugate. This section explains why we expect the following. A second paper, now being written, treats these more general CM forms.

Expected statement. For every $k \geq 1$ and every finite-order η , $L(s, \bar{\alpha}^k \eta) \neq 0$ for $\Re s > 7/8$. Equivalently, every CM newform f of weight $k + 1$ with CM by K satisfies

$$L(f, s) \neq 0 \quad \left(\Re s > \frac{k}{2} + \frac{7}{8} \right), \quad (17.1)$$

a zero-free strip along the right edge of the critical strip $\frac{k}{2} < \Re s < \frac{k}{2} + 1$. The 11/12 method of Part I should give the weaker boundary $\frac{k}{2} + \frac{11}{12}$ in the same way.

We do not claim this as a theorem. The rest of this section records the changes in the number $j = k + 1$ of theta derivatives and in the Hecke Gamma factor. For each fixed j , the formulas have the same structure and give the same length exponents; constants may depend on j . The uniform analytic estimates listed at the end still have to be proved for each fixed weight.

17.1 Two examples

We computed the q -expansions of two eta products (the newforms 9.4.a.a and 12.3.c.a of [27], both multiplicative η -quotients in the sense of [29]) to q^{3000} and compared them with Hecke characters. In the exact L -function identities below, a product of Hecke characters means its primitive inducing character, with all its Euler factors restored. Write $L^{(6)}$ when the Euler factors at primes above 2 and 3 are deleted, as in the earlier sums.

- $f = \eta(3z)^8$, of weight 4 and level 9 ($k = 3$). For all 207 split rational primes $p = \pi\bar{\pi} < 3000$, with both primary generators $\pi, \bar{\pi}$ tested (414 checks), $a_p = 2\Re(\pi^3)$. For $p \equiv 2 \pmod{3}$, $a_p = 0$ and $a_{p^2} = -p^3$. Thus $L(f, s + \frac{3}{2}) = L(s, \alpha^3)$, and the target is the conjugate $\bar{\alpha}^3$, with η trivial. This is the CM analogue of the case $u = 1$ of the zeta argument. The expected strip is $19/8 < \Re s < 5/2$ (center 2).
- $f = \eta(2z)^3\eta(6z)^3$, of weight 3 and level 12 ($k = 2$). For all 207 split rational primes below 3000, again testing both primary generators (414 checks), $a_p = 2\Re(\pi^2(2/\pi)_3) = 2\Re(\pi^2\chi_\pi(4))$. For $p \nmid 12$ with $p \equiv 2 \pmod{3}$, $a_p = 0$ and $a_{p^2} = p^2$, as required by the nebentypus χ_{-3} . The exclusion matters: $a_4 = 0$, not 4. We have $L(f, s + 1) = L(s, \alpha^2\chi_\bullet(4))$, and the target is $\bar{\alpha}^2\overline{\chi_\bullet(4)}$. The expected strip is $15/8 < \Re s < 2$ (center $3/2$). Since $a_3 = -3$, the normalized Euler factor at 3 is $(1 + 3^{-s})^{-1}$; the factor at 2 is 1. Thus, in the convention of the earlier sums,

$$L(f, s + 1) = (1 + 3^{-s})^{-1} L^{(6)}(s, \alpha^2\chi_\bullet(4)). \quad (17.2)$$

17.2 What changes with k

1. The weight and the rows. With target $\psi = \bar{\alpha}^k \eta$ the weight (12.1) becomes

$$\gamma_2(c) \overline{\alpha(cn^3)} \psi(cn^3) = \gamma_2(c) \overline{\alpha(cn^3)}^{k+1} \eta(cn^3). \quad (17.3)$$

Proposition 6 is formal in $\psi(p)$, so the rows are again $\zeta_K^S(6z)L^S(w, \chi_\bullet(u))/L^S(x, \psi\overline{\chi_\bullet(u)})$. The numerators still have finite order, the denominators have infinity type $(\bar{z}/|z|)^k$, and the slot amplitudes are unchanged.

2. The theta step with $j = k + 1$ derivatives. Every formula of Section 8 and of the table in Section 12 has a version for all $j \geq 1$, as follows.

- *Chain rule.* At $z = 0$ the coordinate z' is affine in $\bar{z}$, while $\bar{z}'$ and v' do not depend on $\bar{z}$. Hence, exactly,

$$\partial_{\bar{z}}^j F(z', \bar{z}', v') \Big|_{z=\bar{z}=0} = (-cv)^{-2} \partial_{z'}^j F(-\delta/c, 1/(N(c)v)). \quad (17.4)$$

Every derivative of order $j \geq 1$ kills the constant term at the identity cusp. We checked this identity symbolically for $j \leq 4$.

- *Direct Mellin transform.* For a nonzero Fourier mode and $\Re s > \frac{1}{6} - \frac{j}{2}$,

$$\int_0^\infty (2\pi i \bar{\ell})^j v K_{1/3}(4\pi |\ell| v) v^{2s+j-2} dv = \frac{i^j}{4} (2\pi)^{-2s} \Gamma\left(s + \frac{j}{2} - \frac{1}{6}\right) \Gamma\left(s + \frac{j}{2} + \frac{1}{6}\right) \overline{\alpha(\ell)}^j N(\ell)^{-s}. \quad (17.5)$$

The integral converges absolutely in this half-plane; elsewhere the right side gives its meromorphic continuation. For $j = 1, 2$ this gives the two entries of the table. We checked it numerically for $j \leq 5$ (relative error below 10^{-25}).

- *Reflected side.* The same substitution gives $\prod_{\pm} \Gamma(\frac{j+2}{2} - s \pm \frac{1}{6})$, the cusp factor $\overline{\alpha(c)}^{2j}$, and the dual angular factor $\alpha(\ell)^j$.
- *Kernel.* Write

$$R_j(t) = \prod_{\pm} \frac{\Gamma(\frac{j+1}{2} + t \pm \frac{1}{6})}{\Gamma(\frac{j+1}{2} - t \pm \frac{1}{6})}. \quad (17.6)$$

Then R_1 and R_2 are the two kernels of the table. R_j is holomorphic for $\Re t > \frac{1}{6} - \frac{j+1}{2}$, which is to the left of $-1/4$ for every $j \geq 1$, and $|R_j(\sigma + iu)| \ll_{j,A} (1+|u|)^{4\sigma}$ on $-1/4 \leq \sigma \leq A$ (checked numerically for $j \leq 5$). So the reflected weight obeys the same bound $\ll \min(x^{1/4}, x^{-A})$, with constants depending on j . There is no uniform bound as j grows: for example,

$$R_j(1) = \frac{(3j-4)(3j-2)(3j+2)(3j+4)}{1296} \sim \frac{j^4}{16}. \quad (17.7)$$

The dual lengths are unchanged, because the reflected argument $N(\ell)X/N(c)^2$ does not depend on j . In the quadratic large sieve, $\alpha(\ell)^j$ is a unimodular coefficient of the dual index. Hence Proposition 7 and the low estimates $Z^{1/4+\varepsilon}$ and $Z^{3/16+\varepsilon}$ should hold for every k .

3. Growth. The functional equation has $\Gamma(s + \frac{k}{2})$ in place of $\Gamma(s + \frac{1}{2})$. The exponent of $|\Gamma(1-s + \frac{k}{2})/\Gamma(s + \frac{k}{2})|$ on $\Re s = \sigma$ is $1-2\sigma$, independent of k , so Lemma 10 keeps its exponents. The theta series now has the harmonic polynomial $\bar{a}^k$. With the Fourier convention $e(-a\xi)$ of Appendix C, its Poisson transform is a multiple of $\xi^k e^{-c'|\xi|^2/v} v^{-1-k}$, which vanishes at $\xi = 0$, so $L(s, \psi)$ is entire.

4. Moments. In the inverse moment the co-oriented classes become $\mu \bar{\alpha}^k \leftrightarrow \bar{\alpha}^{k+1} \gamma_2$. This follows from the same identity $\mu \gamma_{-1} = \chi(-1) G^{-1} \bar{\alpha} \gamma_2$, and the cube factor becomes $\bar{\alpha}^{3(k+1)}$. The theta steps use $\partial_{\bar{z}}^{k+1} \theta$. In the plain moment, Hecke reflection uses $\Gamma(s + \frac{k}{2})/\Gamma(1-s + \frac{k}{2})$, a Hankel kernel of order k with the same vertical growth. The lattice main terms vanish because $\int \alpha(z)^{\pm k} f(|z|^2) dz = 0$.

5. Unchanged. The following carry over verbatim:

- the label sets, with $\bar{\alpha}^k \Theta_0 \chi_{\bullet}(\overline{u})$ and $\alpha^k \Theta_0 \chi_{\bullet}(u)$;
- the slot normalization $\overline{\psi(p)}$;
- $\kappa = 3/4$ from OpenAI's theorem;
- every exponent in the final inequality, and hence the endpoint certificate.

For Part I the targets are rows of the same kind. In the orientation $\mu_K(n) \alpha(n)^k \chi_n(u_0)$ of Part I, the row is $u_0 = 1$ for $\eta(3z)^8$ and $u_0 = 4$ for $\eta(2z)^3 \eta(6z)^3$.

17.3 Remarks

- $k = 0$ is *OpenAI's case*. There $j = 1$ and the weight is $\gamma_2 \bar{\alpha} \eta$. The recipe “target $\bar{\alpha}^k \eta$, $k+1$ theta derivatives” therefore covers unitary Hecke characters of K with purely angular infinity type; negative k reduces to positive k by conjugation. A general unitary Hecke character may also have a continuous norm factor N^{it} , with $t \in \mathbb{R}$. Since

$$L(s, \psi N^{it}) = L(s - it, \psi), \quad (17.8)$$

these vertical shifts leave the proposed zero-free half-plane unchanged.

- *Orientation.* The conjugate target $\alpha^k \bar{\eta}$ would leave α^{k-1} in the weight. For $k = 1$ that is no angular factor at all: the theta function without a derivative, whose constant terms are not controlled. That is why Part II uses $\bar{\alpha} \eta$. For $k \geq 2$ the factor α^{k-1} is produced by ∂_z^{k-1} . The chain rule is symmetric: at $z = \bar{z} = 0$, ∂_z^j becomes $(-(\bar{c}v)^{-2})^j \partial_{\bar{z}}^j$, exactly; we checked this symbolically for $j \leq 3$. So for $k \geq 2$ either orientation should work; we use $\bar{\alpha}^k \eta$ for uniformity.
- *Symmetric powers.* For a CM curve E_d and $m \geq 1$, put $L_{\text{an}}(\text{Sym}^m E_d, s) = L(\text{Sym}^m E_d, s + m/2)$. Up to finitely many Euler factors, the factorization is

$$L_{\text{an}}(\text{Sym}^m E_d, s) = \prod_{0 \leq i < m/2} L(s, (\psi_d^*)^{m-2i}) \begin{cases} 1, & m \text{ odd}, \\ L(s, \chi_{-3}^{m/2}), & m \text{ even}. \end{cases} \quad (17.9)$$

The last factor is $\zeta(s)$ when $m \equiv 0 \pmod{4}$ and $L(s, \chi_{-3})$ when $m \equiv 2 \pmod{4}$, with the trivial character interpreted primitively. This is the CM factorization of [28, §3.5]. The positive-power factors are of the type above; the remaining factor is covered by OpenAI’s finite-order result. Together these would give a zero-free half-plane $\Re s > 7/8$ in analytic normalization. When $4 \mid m$, the function has a pole at $s = 1$: “zero-free” here is a statement about a meromorphic function, not a claim of holomorphy throughout the half-plane. These L -functions are not primitive for $m \geq 2$.

- *What would still have to be written.* The six statements of Section 16, proved here for $k = 1$, are needed for each k , with ∂_z^{k+1} in place of ∂_z^2 . We see no step where k enters other than through $j = k + 1$ and the Gamma factors. The constants depend on k , but k is fixed.

The numerical checks of this section were made with short Python scripts.

18 Numerical checks

Every numbered display of these notes was catalogued and checked where a computation can check it: identities and explicit inequalities were tested on finite samples, and estimates received size checks where feasible. For twelve estimates not even a size check is possible. Testing every display is overkill, but we wanted to catch mistakes of every size: wrong normalizations, sign errors, dropped factors and careless slips. Section 18.1 lists all 352 numbered displays, and for each one says what was done and what was found. Section 18.2 says what the checks do not establish.

The checks use Python and the conventions of these notes. Elements of $\mathbb{Z}[\omega]$ are written as pairs $A + B\omega$. Primary means $\equiv 1 \pmod{3}$, $\alpha(n) = n/|n|$, $u_d = (4d)^5$, and $\chi_n(x) = (x/n)_6$ is computed by Euler’s criterion at each prime. Gauss sums are computed by brute force over complete residue systems. Point counts and Dirichlet coefficients of the curves are computed by brute force and compared with PARI/GP [38].

Many comparisons are exact: in $\mathbb{Z}[\omega]$ or $\mathbb{Q}$, or, as in Sections 2–5, in $\mathbb{Z}[\omega]$ after clearing the factors $\sqrt{m}$ of α . The others, including most identities involving α after Section 5, are computed in double precision, and so are many integrals. Some Mellin and Bessel integrals use mpmath [32] at 30 to 40 digits, and symbolic checks use SymPy [31]. Some functional equations are checked only by comparing values from PARI’s continuation; these are consistency checks, and the table says so. Zero counts come from the argument principle on finitely many boundary samples, and maxima are taken over sampled points; such results hold on the tested grid only.

The scripts are posted with the source, in the folder `cm.zero.free.strip.tests` of https://github.com/mrubinst/L-function_zero_free_strips. The display tests are in its subfolder

`displays.2026-10-13`, in ten groups `t0`–`t9`. The file `rows.json` of each group names the script behind every row of the table below, and `run_all.sh` reruns all of them in about half an hour, including the earlier suites in the other subfolders on which they rely. Every runner exits with a nonzero status if any check fails, including failures in nested suites. The script `check_runner_propagation.py` verifies this by making each script, in turn, fail.

18.1 Every numbered display

Each numbered display is of one of six kinds, and the kind determines what a computation can establish.

- E *Identity*. An identity or an exact evaluation. It was tested on finite samples, exactly in $\mathbb{Z}[\omega]$ or $\mathbb{Q}$, numerically in double or multiprecision arithmetic, or symbolically.
- I *Explicit inequality*. An inequality between computable quantities or exponents, with explicit constants. It was tested on finite samples or symbolically, in the same way.
- A *Estimate*. A bound with unspecified constants, or an asymptotic statement. No finite computation can verify it. Where feasible it received a *size check*: the ratio of the two sides was computed over a finite range and seen not to grow. Otherwise it is marked “not testable by computation”. Each is proved in the text, at the display or where its row indicates, except (17.1), which is an expectation (Section 17).
- D *Definition*. Nothing is asserted. The table says how the definition was implemented in the tests, and what consistency checks were made.
- C *Cited result*. A display quoted from another paper. It was compared with its source.
- R *Restatement*. A display that repeats another one, or a special case of it. It is covered by the row of the display it restates.

Kind		Displays	Outcome
E	identity	189	all pass
I	explicit inequality	22	all pass
A	estimate	44	32 size checks, all consistent; 12 not testable by computation
D	definition	76	implemented in the tests
C	cited result	8	agrees with its source
R	restatement	13	covered by the restated display
total		352	

No display was found to be wrong. A test verifies only the instances it computes; the estimates of kind A rest on their proofs, and (17.1) is not claimed.

In the table, *max err* is the largest absolute error over the tested instances, unless it is marked relative; *exact* means equality in exact arithmetic, and is used for nothing else. For a size check, the result column gives the observed ratio to the claimed bound, whose implied constant is unknown. Identifiers such as A2.07, B30, K9 or S4.6 name individual tests in the suites listed above.

Display	Kind	What was done	Result
<i>Introduction</i>			
(0.1)	D	Definition of E_d ; discriminant $-432d^2$ (sympy) and $j = 0$ (PARI) for 40 values of d	definition (consistency PASS)
(0.2)	D	The map; $(\omega x)^3 + d = x^3 + d$ (sympy), so it preserves the equation; order 3	definition (consistency PASS)
(0.3)	E	Brute-force point count, minus the Legendre sum and PARI ellap agree for 1366 pairs (d, p) , $p < 800$, $p \nmid 6d$	PASS (exact)
(0.4)	D	Definition of a_m ; PARI ellan equals the product of the local factors (0.5) for $m \leq 2500$, 10 values of d	definition (consistency PASS, exact)
(0.5)	C	Standard local factors [43, Appendix C, §16]; for $0 < d \leq 200$ all 1200 bad primes are additive (PARI), $a_p = 0$ there; $d = 16$ is good at 2	checked against source; PASS (exact)
(0.6)	C	Deuring–Hecke functional equation; checked with PARI lfun for 8 curves ($N = 27$ to 15552) at 4 complex points, ϵ from ellrootno	checked against source; PASS (rel err below 10^{-40})
(0.7)	E	$\Lambda^u(s) = \epsilon \Lambda^u(1-s)$ for the same curves and points (PARI lfun)	PASS (rel err below 10^{-40})
(0.8)	D	Definitions; $(4d/n)_6 = ((4d)^5/n)_6$ exactly for 2258 pairs, so $\psi_d = \alpha \chi_\bullet(u_d)$ as in (3.5)	definition (consistency PASS, exact)
(0.9)	E	Coefficientwise in $\mathbb{Z}[\omega]$: $a_m = \sum b(m_1) \sum_{N(n)=m_2} n \overline{(4d/n)_6}$, b the local factors at $p \mid 6d$ (PARI, minimal model), $m \leq 2500$, 10 values of d including non-minimal ones	PASS (exact)
(0.10)	E	$\sum_{N(n)=m} \mu_K(n) n \overline{(4d/n)_6}$ equals the coefficient of $\prod_{p \nmid 6d} (1 - a_p X + p X^2)$, 7919 pairs (d, m) , $m \leq 2500$	PASS (exact)
(0.11)	R	Restates $S_d(D) = A_{u_d}(D)$ of (3.9), with $u_d = (4d)^5$; covered by that display’s test.	see (3.9)
<i>Section 2</i>			
(2.1)	E	Brute-force point count = minus Legendre sum = PARI ellap for 1065 pairs (d, p) , $p < 600$.	PASS (exact)
(2.2)	E	Sympy: shifting $1 - a_p p^{-s} + p^{1-2s}$ by $1/2$ gives $1 - \lambda_d(p) p^{-s} + p^{-2s}$.	holds (symbolic)
(2.3)	E	Reciprocal Euler factor has coefficients $1, -\lambda, 1, 0, \dots$ (sympy); the full series is matched coefficientwise with PARI in A2.07.	PASS (exact)
(2.4)	E	32382 values ($m \leq 2 \cdot 10^4$, 5 values of d) against the multiplicative q_d .	PASS (max err 2.7e-15)
(2.5)	D	Definition; the tests build B_d from minimal-model local factors at $p \mid 6d$ (PARI), e.g. $B_{16}(3) = 1/(1 + 2^{-6})$.	definition
(2.6)	E	$L(E_d, s + \frac{1}{2}) \sum q_d m^{-s} = B_d$ coefficientwise, exact vs PARI ellan, 10 values of d (some non-minimal), $m \leq 6000$; also numerically (T1-E1).	PASS (exact)
(2.7)	D	Definition of $S_d(D)$; evaluated in A2.09, A3.08 and in T1-M2 (direct quadrature of F_W).	definition
(2.8)	C	Same formula as the zeta guide [3, (1.2)]; A0.08: c inverts $\zeta(s)L(s, \chi_{-3})$ for $m \leq 2 \cdot 10^4$; A2.10: the case formula.	checked against source
(2.9)	E	All four columns for 2566 pairs (d, p) , $p < 3000$, including $a_p = 0$ at inert p (PARI).	PASS (exact)
<i>Section 3</i>			
(3.1)	E	$ A + B\omega ^2 = A^2 - AB + B^2$ on $ A , B \leq 60$; α is notation.	PASS (max err 3.6e-12)
(3.2)	E	Every n coprime to 3 with $ A , B \leq 40$ has exactly one associate with $A \equiv 1$, $B \equiv 0 \pmod{3}$, and it is $\equiv 1 \pmod{3}$.	PASS (exact)
(3.3)	E	Exact: $\mu_K(-5) = -1$; $7 = (1 + 3\omega)(-2 - 3\omega)$, both factors primary of norm 7, so $\mu_K(7) = 1$.	PASS (exact)
(3.4)	D	Definition; implemented literally (matrix of sextic symbols times columns) in A4.02, A4.04, A5.05.	definition
(3.5)	E	$\chi_n((4d)^5) = \chi_n(4d)$ via symbol indices for 12000 pairs, zeros kept.	PASS (exact)

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Display	Kind	What was done	Result
(3.6)	E	Index arithmetic: index of $(4d)^5$ is 5-index of $4d$, i.e. minus it mod 6, zeros preserved, 12000 pairs (d, n) .	PASS (exact)
(3.7)	E	Exact in $\mathbb{Z}[\omega]$ for 27528 pairs (d, π) , $ d \leq 60$, $p < 1500$.	PASS (exact)
(3.8)	E	$\psi_d(\pi)\psi_d(\bar{\pi}) = 1$ and $\psi_d(\pi) + \psi_d(\bar{\pi}) = \lambda_d(p)$ for $p < 3000$, 5 values of d .	PASS (max err 1.2e-15)
(3.9)	E	Exact in $\mathbb{Z}[\omega]$, 8 values of d , $m \leq 2 \cdot 10^4$; $S_d = A_{u_d}$ to 1.3e-14; an additional Python/PARI check of 10800 pairs (d, n) (deuring_identity/hecke_identity.py; a numerical check, not a Lean proof).	PASS (exact; $S_d = A_{u_d}$ to 1.3e-14)
(3.10)	E	First identity exact vs PARI ellan for 51264 coefficients ($m \leq 2 \cdot 10^4$, 8 values of d); the second is (3.9).	PASS (exact)
(3.11)	D	Notation; checked that χ^2 is the cubic symbol, χ^3 the Legendre symbol and $\kappa(-1) = 1$, 188 pairs (p, π) , $p < 1200$.	definition (consistency PASS, exact)
(3.12)	E	Both root counts exact at every residue, 188 pairs (p, π) , $7 \leq p < 1200$.	PASS (exact)
(3.13)	E	Character expansion equals the brute-force affine count, both equalities exact in $\mathbb{Z}[\omega]$, 4412 pairs (d, π) , $p < 400$.	PASS (exact)
(3.14)	E	$S = \rho(d)\kappa(d)J(\kappa, \rho)$ exact in $\mathbb{Z}[\omega]$ for 4412 pairs (d, π) , $0 < d \leq 30$, $p < 400$.	PASS (exact)
(3.15)	E	All three equalities and the root count $1 + \rho(1 - 4a)$, exact, 188 pairs (p, π) .	PASS (exact)
(3.16)	E	$S = -\rho(d)\kappa(4d)\pi$ and brute-force $a_p = \text{Tr}(\rho(d)\kappa(4d)\pi)$ exact, 11252 triples (p, π, d) .	PASS (exact)
(3.17)	E	All five members evaluated (last by PARI lfun) at $s = 3, 2.5 + 4i$ for 7 values of d .	PASS (max err 8e-10)
<i>Section 4</i>			
(4.1)	D	Definition; computed literally (all rows, all columns) in A4.02, A4.07 and A5.09.	definition
(4.2)	E	Literal four-coordinate evaluation equals the double-sum form at three (D, H) ; coordinate columns = primary columns coprime to 6.	PASS (max rel err 1.4e-15)
(4.3)	E	Expanded double sum with S_H equals the coordinate form.	PASS (max rel err 1.4e-15)
(4.4)	E	$A_u(D) = \sum_m q_u(m)W(m/D)$ for 5 rows, $D \leq 3000$.	PASS (max err 1.4e-14)
(4.5)	E	Exact in $\mathbb{Z}[\omega]$: 30 rows, local table, multiplicativity, vanishing at 2, 3 and p^j , $j \geq 3$, reduction to (2.3).	PASS (exact)
(4.6)	A	Size check: $M(D, D^{1.1})/D^{2.1}$ between 0.27 and 0.29 for $D \leq 1600$; proved by Proposition 3 and Sections 7–10.	size check: $M/D^{2.1} \in [0.27, 0.29]$
(4.7)	E	$S_H(n, n)$ equals the number of coprime rows for 150 columns, $H = 2 \cdot 10^4$, and is at most the row count.	PASS (exact)
(4.8)	A	Size check of the error term over 62 columns, $H \leq 4 \cdot 10^5$; identical to [3, (4.2)]; elementary lattice counting.	size check: $\text{error}/(\tau(n)(\sqrt{H} + 1)) \leq 0.35$
<i>Section 5</i>			
(5.1)	E	3000 random (d, n, v) : $\chi_n(u_d v^6) = \chi_n(u_d)$ if $(n, v) = 1$, else 0.	PASS (exact)
(5.2)	D	Definition; implemented in A5.03-A5.04 and r1; no admissible prime of norm below 7, two of norm 7.	definition
(5.3)	E	Difference of the rows equals the sum over $\pi \mid n$, $D = 3000$, $Y = 60$, $d = 1, 7$.	PASS (max err 6.6e-15)
(5.4)	A	Explicit form $(1 + 16b)\ W\ _\infty D/Y$ of Lemma 2(iii) checked in 216 cases; proved there.	size check: ratio to explicit bound ≤ 0.022
(5.5)	A	Toy instances with the actual $M(D, H)$, H up to 10^6 : the proof chain holds; Lemma 2(iv).	size check: explicit chain holds in 3 toy cases
(5.6)	I	Sympy: $DH^{5/6} = D^{11/6+5\vartheta/6}$, $D^2H^{-1/3} = D^{5/3-\vartheta/3}$, re-derived from $2M/J$ and D^2/Y^2 .	holds (symbolic)
(5.7)	A	Constant comes from (MS); exponent choice checked for 399 values of η ; sizes of $S_d(D)$ for $D \leq 3 \cdot 10^4$; proved in Sec. 5.1.	size check: $ S_d(D) /(\ W\ D^{11/12}) \leq 0.033$

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Display	Kind	What was done	Result
(5.8)	A	Exact parts ($N_0^{1/12} = 4d ^{5/6}$, exponents, $Y < 7$ case) by sympy;	size check: ratio $\leq 4.3 \cdot 10^{-3}$;
		sizes for $0 < d \leq 30$, $D \leq 10^4$; Lemma 3.	exact parts PASS
(5.9)	A	First step is Lemma 2(iv) divided by J (toy chain A5.09);	not testable by computation
		conversion to (5.6) symbolic (A5.10); proved in Sec. 5.1.	(bookkeeping holds, symbolic)
(5.10)	R	Restates (5.8) as Lemma 3.	see (5.8)
(5.11)	A	Square roots give $D^{1+\vartheta/2}Y^{-1/2}$ and D/Y , the terms of (5.10)	not testable by computation
		(sympy); Case 2 implication checked; PNT constants untestable.	(bookkeeping holds, symbolic)
(5.12)	E	2000 random (n, v) via Euler's criterion.	PASS (exact)
(5.13)	E	135000 coefficients ($d \in \{1, 7, -3\}$, 9 values of v , $m \leq 15000$)	PASS (exact)
		exact.	
<i>Section 6</i>			
(6.1)	D	Definitions; $\widehat{W}$ by quadrature, and F_W by direct D -quadrature	definition
		of $S_d(D)$, in B04 and T1-M2.	
(6.2)	E	Substitution identity (30 digits) and	PASS (max err/tol 0.09)
		$L(E_d, s + \frac{1}{2}) \sum q_d m^{-s} = B_d$ via PARI lfun, 7 values of d , 3 of s .	
(6.3)	E	Mpmath (30 digits) for 3 complex s_0 :	PASS (max err 6e-39)
		$\widehat{W}(s_0) = \int_1^2 \phi(y) dy/y > 0$.	
(6.4)	E	D -integral of the absolute sum vs product form, $d \in \{1, 7, -3\}$,	PASS (max rel err 5e-16)
		$\sigma \in \{1.5, 2, 3\}$; $ q_d \leq r$ and finiteness.	
(6.5)	E	F_W by direct D -quadrature of $S_d(D)$, two complex weights,	PASS (max rel err 8e-8)
		$d \in \{1, 7, 16\}$, two s , PARI L and B_d .	
<i>Section 7</i>			
(7.1)	D	Definitions of e and $\widehat{\Phi}$: e trivial on $\mathcal{O}$, $\mathcal{O}$ self-dual; Gaussian	definition (B10 PASS, err
		gives $(2/\sqrt{3})e^{-4\pi t/3}$	9e-22)
(7.2)	E	1200 random triples (n_1, n_2, u) , including u sharing primes with	PASS (max err 2e-16)
		g	
(7.3)	D	Implemented by brute force modulo $z_1 z_2$; $ \gamma(\psi) = 1$ for 46 pairs;	definition (B16 PASS, err
		change of generator checked	9e-16)
(7.4)	E	Lattice sum against dual sum, Gaussian Φ , 10 cases (8	PASS (max rel err 5e-15)
		nondegenerate, g with 1-2 primes)	
(7.5)	I	$N(e_0)N(z_1 z_2)/H \leq b^2 D^2/(HN(g))$ symbolically and on 10^5	PASS (exact)
		random sets	
(7.6)	D	Vectorized γ_j equals brute force; γ_2 is cubic (Euler criterion); G	definition (PASS, err 2e-14)
		equals the closed form of [35]	
(7.7)	E	150 squarefree n (53 composite) with brute-force-checked Gauss	PASS (max err 2e-15)
		sums	
(7.8)	D	c_ξ from its definition, independent table of 432 characters:	definition (PASS, err 1e-14)
		$\sum_\xi c_\xi \xi(t) = G(t^{-1})$, $\sum c_\xi = 1$, $ c_\xi \leq 1/2$	
(7.9)	E	Literal sum over all 432 characters, 73 coprime pairs; again with	PASS (max err 2e-14)
		an independent table (60 pairs)	
(7.10)	D	Definition; implemented literally in the computed averages of	definition
		Section 10 tests and in t2d	
(7.11)	E	120 coprime squarefree pairs times 6 characters, plus 71 and 300	PASS (max err 2e-14)
		pairs in B30, S10.11a	
(7.12)	I	Support bounds on prime data and sympy:	PASS (exact)
		$N(k) \leq CD^2/(HN(r_0)^2)$, $\Sigma = XF = D/B$	
(7.13)	E	150 squarefree n (53 composite), brute-force-checked Gauss sums	PASS (max err 2e-15)
(7.14)	E	All 144 pairs of classes mod 4, G and $\mathcal{R}$ from squarefree	PASS (max err 4e-15)
		representatives	
(7.15)	I	Both sides computed from (7.10), 3 cases; the identity (10.14)	PASS (ratio at most 0.31)
		behind it holds to 3e-15 in double precision	
(7.16)	D	All 432^2 coefficients $c_\xi(\xi_1)$, $\ell_{\xi_1}(\xi)$: expansions reproduce;	definition; claims PASS (err
		$c_\xi(\mathbf{1}) = c_\xi$; moduli at most $1/2$	3e-14)
(7.17)	E	Literal sum over the 432 characters ξ of an independent table:	PASS (max err 2e-14)
		60 pairs times 4 characters ξ_1	

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Display	Kind	What was done	Result
(7.18)	E	Literal sum over the 432 characters ξ_1 with $\ell_{\xi_1}(\xi)$: 60 pairs times 4 characters ξ	PASS (max err 1.3e-14)
(7.19)	E	120 coprime squarefree pairs times 6 characters: twisted multiplicativity and $ a_\xi = 1$	PASS (max err 2e-14)
(7.20)	E	Brute-force Gauss sums mod $z_1 z_2$, 83 coprime pairs (38 with $\mathcal{R} = -1$), both directions	PASS (max err 2e-15)
(7.21)	E	Both identities for 150 squarefree n	PASS (max err 2e-15)
(7.22)	E	Both equalities, 83 coprime pairs	PASS (max err 2e-15)
(7.23)	E	Both identities for 149 squarefree $n \neq 1$	PASS (max err 2e-15)
(7.24)	E	83 coprime pairs times 6 characters ξ	PASS (max err 1e-14)
(7.25)	A	Hypothesis; it follows from (10.25), (10.26). Size check: $E/(XF \max W ^2) \leq 0.17$, 24 cases, $D \leq 4000$	size check: ratio at most 0.17
(7.26)	D	Ranges; the Step 4 scales satisfy them at 3777 support points and after the swaps $j \mid r_0$	definition
(7.27)	A	Conclusion of Proposition 3; only the bookkeeping $\sum (HN(r_0)/D^2)(D/B)^2 \ll H$ is checked	not testable by computation; proved in Proposition 3
(7.28)	E	72 triples (g, z_1, z_2) , g with 0-2 primes	PASS (max err 4e-16)
(7.29)	E	Z equals the $h = 0$ Poisson terms and enters the end-to-end identity; size check $Z/(H \max W ^2) \leq 0.095$	PASS (rel err 4e-15)
(7.30)	E	72 triples (69 with $g \neq 1$), literal sum over 432 characters; $z_1 = z_2 = 1$ gives $\sum c_\xi = 1$	PASS (max err 1e-14)
(7.31)	E	sympy with positive norms and complex values of W ; numerically on every term of the end-to-end sum	holds (symbolic)
(7.32)	E	End to end: $M_D = Z + \sum_\xi c_\xi S_\xi$ in this form, $D = 120$ ($H = 40, 170$), $D = 300$	PASS (max rel err 4e-16)
(7.33)	E	Bijection enumerated exactly in $\mathbb{Z}[\omega]$, 45150 tuples onto 45150 triples, inverse checked	PASS (exact)
(7.34)	E	Exact in $\mathbb{Z}[\omega]$ on all 45150 tuples of the bijection	PASS (exact)
(7.35)	E	Exact rationals on all 45150 tuples of the bijection	PASS (exact)
(7.36)	E	End to end: $M_D = Z + \sum_\xi c_\xi S_\xi$ in this final form, $D = 120$ ($H = 40, 170$), $D = 300$	PASS (max rel err 1e-14)
(7.37)	D	Scales; at 3777 support points $N(k) \leq \mathcal{H}$, $\varrho \in [1, 4)$, $\varkappa \leq C_1$, support in $[a_0/4, b_0]$	definition
(7.38)	A	Only the dyadic bookkeeping is checked: exact sums give ratio at most 1 for every dyadic $B \leq 2D$	not testable by computation; proved in Proposition 3, Step 5
<i>Section 8</i>			
(8.1)	D	Definition; $b = 1$ term is $X^{-1/2}$ times the column sum; sums finite; termwise theta form with $-1/81$	definition
(8.2)	C	Identical to [2, Corollary 24(2)] with $\lambda = \sqrt{-3}$; also checked against the appendix tables, 210 indices	checked against source
(8.3)	E	sympy with $ad - bc = 1$ and a numeric $SL_2(\mathcal{O})$ example	PASS (max err 4e-16)
(8.4)	E	sympy, generic F of degree (4,2,2), no lower derivatives survive; also in b3	holds (symbolic)
(8.5)	E	End-to-end single-cusp reflection with two derivatives, 41 cusp configurations	PASS (max rel err 2e-8)
(8.6)	E	Reflected local sums, 7 primes, $j = 0, \dots, 5$, 1854 cases; grouped with actual cusp data; equals [35, (6.5)]	PASS (max err 7e-14)
(8.7)	E	Euler criterion, 10 primes times 40 residues	PASS (max err 3e-15)
<i>Section 9</i>			
(9.1)	C	Matches [35, Lemma 6.5] ($\chi_k(n)^3 = (n/k)_2$) and [10, Theorem 1.1]; family hypotheses and a sanity ratio 0.25 checked	checked against source
(9.2)	A	Sanity ratio with the CM T at most 0.30 ($X = 1400$, $H \leq 600$, 3 values of f); proved in Section 9.1	size check: ratio at most 0.30
(9.3)	E	400 random (n, v, f) including non-coprime cases, and the T identity in 8 cases	PASS (max err 4e-15)

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Display	Kind	What was done	Result
(9.4)	R	Precise form of (9.2) (Proposition 4); quantifiers agree with [35, see (9.2) Proposition 5.2]	
<i>Section 10</i>			
(10.1)	E	Cube inversion with $\mu_K(h)\overline{\alpha(h)}^6 \Psi(h)^3/N(h)$ in double precision, $X = 2000$ and 4000 , five (k, f, ξ)	PASS (max err 4e-15)
(10.2)	D	Defines H_c, L_b ; the cube tests use $H_c^3 = \min(X, X^2/\mathcal{H}^2)$ and $L_b = X/N(b)^3$ (consistent)	definition
(10.3)	I	$\mathcal{H} + \mathcal{H}^2 FN(h)^3/X \leq 2\Sigma$ and $X/N(h)^3 \geq 1$ for $N(h) \leq H_c$, $\mathcal{H} \leq \Sigma$: 157 000 exact rational cases	PASS (exact)
(10.4)	A	Proved in Lemma 6 (Section 10.5). Side checks pass: empty supremum if $\mathcal{H}^2 \leq X$; exact long-cube identity	not testable by computation
(10.5)	D	Smoothed average $\mathcal{A}(W)$, restated as (10.29); computed in the end-to-end first Poisson test	definition
(10.6)	D	Column sums $P_{C,e_0,t}$ with $\mu_K\overline{\alpha}\xi_1$; implemented in C3, C4; [35, (7.1)] without $\overline{\alpha}$	definition
(10.7)	D	Nonnegative average $\mathcal{Q}$ (restated as (10.31)); weights w_{C,e_0} checked symbolically, inner sum computed in C3	definition
(10.8)	D	Defines Y_{C,e_0} ; with $c_* = \max(1, 4C_\Phi v^2)$ every $y = hf^2$ has $N(y) \leq Y_{C,e_0}$ (exact, PASS)	definition
(10.9)	E	$Y_{\text{enlarged}}/Y_{\text{natural}} = \Sigma/(LF) \geq 1$, both scales from Y_{C,e_0} and $N(h) \approx L^2/\mathcal{H}$ (sympy)	holds (symbolic)
(10.10)	D	$r = tg/e, f' = Cew, k' = e_0eh_2$: r, f' squarefree and coprime on 3000 actual tuples (PASS); as in [35] Lemma 7.3	definition
(10.11)	D	Defines $\mathcal{H}'$; equals $\mathcal{H}/(N(b)^3 F^2 R^2)$, $\mathcal{H}/(F^2 R^2)$ unenlarged (sympy); kernel support $N(k') \leq \mathcal{H}'$ (PASS)	definition
(10.12)	E	Both cases; brute-force regrouping $\sum \mu(e)\mu(w) = \tau(r)\mathbf{1}_{f' k'}$ on 400 actual (r, f', k')	PASS (exact)
(10.13)	I	Whole chain on 100 000 retained (r, f', k') with $f' k'$ in $\mathbb{Z}[\omega]$, plus 100 000 samples	PASS (0 violations)
(10.14)	E	Double precision, $X' = 2000$, two ξ, r with 1–3 primes, 8 cases; twisted multiplicativity of a_ξ	PASS (max err 4e-15)
(10.15)	E	Preserves $\mathcal{H}', \Sigma' = L/R, \mathcal{H}'/\Sigma'$ (200 000 sets); computed $E_r \leq \tau(r) \sum_j E$ in 3 cases	PASS (0 violations)
(10.16)	I	All three inequalities, exact rationals, 50 000 retained (r, f', k') and every $j r$	PASS (exact)
(10.17)	A	Proved in Section 10.6 (Lemmas 7–9); its two Poisson identities pass; $J_t = 4m + 12$ and $[u/16, 4v]$ match [35, Proposition 5.4]	not testable by computation
(10.18)	I	From $\mathcal{H}' = \mathcal{H}L/(\Sigma FR^2)$, $\Sigma' = L/R, R, F \geq 1$: $\mathcal{H}'/\Sigma' = \mathcal{H}/(\Sigma FR)$ (sympy) and 100 000 samples	PASS (symbolic, 0 violations)
(10.19)	D	Standing hypotheses; checked at every node of 100 000 random induction chains (PASS)	definition
(10.20)	A	Proved by the induction of Section 10.4; scale bookkeeping on 100 000 chains passes; computed CM averages bounded	size check: $E/(\Sigma\ W\ _\infty^2) \leq 0.17$
(10.21)	I	Six units, $ a_\xi \leq 1$, counting bound on 4 computed averages; $\frac{F}{XF}(X\ W\)^2 = X\ W\ ^2 \leq \Sigma\ W\ ^2$ (sympy)	PASS
(10.22)	I	Every step of 100 000 random chains and 80 000 admissible sets, first step symbolically	PASS (0 violations)
(10.23)	I	All three hypotheses at every step of 100 000 random chains, half of them extremal	PASS (0 violations)
(10.24)	A	The induction hypothesis (10.20) at level ν applied to the children; the child scales are checked in (10.23)	not testable by computation
(10.25)	A	Proved by the induction (Section 10.4); sanity: computed CM averages, $X \leq 1200, \mathcal{H} \leq 160, F \leq 7$	size check: $E/(\Sigma\ W\ _\infty^2) \leq 0.17$
(10.26)	I	$\mathcal{H}/\Sigma = D^{-\vartheta}/B$ at $H = D^{1+\vartheta}$ (sympy); $\kappa = \vartheta/2$ once $D \geq (C/B)^{2/\vartheta}$ (20 000 samples)	holds (symbolic)
(10.27)	E	CM rows tested end to end; zeta column checked in sources	PASS (max rel. err 2e-8)

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Display	Kind	What was done	Result
(10.28)	R	Precise statement of the cube-removal bound (10.4)	see (10.4)
(10.29)	R	Restates the smoothed average (10.5); computed in the end-to-end test C4	see (10.5)
(10.30)	D	Defines w_{C,e_0} , Y_{C,e_0} , c_* ; normalization of w (sympy) and containment $N(hf^2) \leq Y_{C,e_0}$ (exact) PASS	definition
(10.31)	R	Restates (10.7) with the range $N(C)N(t) \leq 2vL$, as in [35]	see (10.7)
(10.32)	E	Z is the zero-frequency term of the end-to-end identity C4, which holds to rel err 1e-15; size check $Z/(\mathcal{H}\Phi(0)\ W\ _\infty^2) \leq 0.09$, $L \leq 3000$, $F \leq 100$	PASS (rel. err 1e-15)
(10.33)	E	End to end with a self-dual Gaussian Φ , two $(L, F, \mathcal{H})$, $\xi \neq 1$	PASS (rel. err 1e-15)
(10.34)	A	Kernel as in C4; $\sup_c \ \mathcal{K}\ _{C^q(I^2)} / \ W\ _{C^q(I)}^2$ for $q \leq 2$ over all frequencies: at most 0.25	size check: ratio ≤ 0.25
(10.35)	I	Blockwise on 34 blocks (C, t, e_0) with CM coefficients: containment, multiplicity ≤ 2 , inequality	PASS (0 violations)
(10.36)	E	End to end, three (C, t, e_0) , $\ell \leq 40$, $Y \leq 500$	PASS (rel. err 2e-16)
(10.37)	D	Defines X'_0 , $a^\sharp$, $\mathcal{K}$ (implemented in C3); $X'_0 = L/(N(r)N(f'))$ (sympy); support $N(k') \leq \mathcal{H}L/(\Sigma FN(r)^2)$ PASS	definition
(10.38)	E	Double precision on 3000 random tuples, shared primes allowed	PASS (max err 6e-15)
(10.39)	E	Exact on 4000 random tuples (2486 with nonzero left side)	PASS (exact)
(10.40)	A	Proved via [35, Lemma 7.3] (order C^{2m+4} matches the source); its inputs (10.12)–(10.16), (10.36) pass	not testable by computation
(10.41)	A	Follows from Lemmas 7, 9; $2(2m+4)+4 = 4m+12$ (sympy) and $E \leq \mathcal{A}(W)$ on 3 computed averages pass	not testable by computation
<i>Section 11</i>			
(11.1)	A	Main theorem. Argument principle on the box $0.88 \leq \Re s \leq 2$, $ \Im s \leq 30$, from 15000 boundary samples (phase steps below 0.5): no zeros found, for six members of $\mathcal{F}_{\text{CM}}$, $Q = 7$ to 1008	size check: no zeros found in the tested box (6 characters); proved in Sections 11–15 and the appendices
(11.2)	A	Follows from (11.1) and (11.3). Argument principle on the box $1.38 \leq \Re s \leq 2.65$, $0 \leq \Im s \leq 40$, from 12000 boundary samples: no zeros found, five curves	size check: no zeros found in the tested box (5 curves)
(11.3)	E	$a_m/\sqrt{m} = \sum_{N(n)=m} \psi_d^*(n)$ for 8 curves, $m \leq 600$; bad Euler factors nonzero	PASS (max err 3e-15)
(11.4)	D	[36, eq. (2.1)] with the finite-order family replaced by $\mathcal{F}_{\text{CM}}$; text compared	definition
<i>Section 12</i>			
(12.1)	E	425 pairs $(c \text{ squarefree}, n)$, $\eta = \chi_\bullet(2 + \omega)$, brute-force Gauss sums	PASS (max err 2e-15)
(12.2)	E	Local factor: [36, eq. (7.10)] summed with generic unimodular $\psi(p)$ equals the closed form, all j, ρ , 10 primes	PASS (max rel err 9e-16)
(12.3)	E	Brute-force Gauss sums at 165 primes (5 inert) and 60 squarefree composites	PASS (max err 8e-15)
(12.4)	D	$\mathcal{C}_V$ of [36] with $\bar{\alpha} \rightarrow \bar{\alpha}^2$; its weight equals (12.1) with $\nu = \eta$ (425 pairs)	definition (C19 PASS, err 2e-15)
<i>Section 13</i>			
(13.1)	E	$\Lambda(s) = \epsilon \Lambda(1-s, \bar{\psi})$ for six non-self-dual members ($Q \leq 1008$) and 5 curves	PASS (max err 8e-38)
(13.2)	A	Measured ratio equals $(2\pi)^{2\sigma-1} \Gamma(\frac{3}{2}-s)/\Gamma(\frac{1}{2}+s) (3+ t)^{2\sigma-1}$, bounded on $[-\frac{2}{5}, \frac{7}{5}]$, independent of Q	size check: ratio at most 1955, same for all Q ; proved in Lemma 10
(13.3)	D	Model $\Theta_0 = \mathbb{Z}/6 \times \mathbb{Z}/2$: $4 \Theta_0 $ labels, closed under conjugation, contains numerator and denominator	definition (C45 PASS)
(13.4)	D	Text equals [4, eq. (8)] with $\psi \rightarrow \bar{\alpha}\theta\chi_\bullet(\overline{u})$; both witnesses co-oriented	definition (S2, C46 PASS)
(13.5)	C	Identical, character for character, to [4, eq. (11)] (Section 4.4); finite-order prime polynomial	checked against source

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Display	Kind	What was done	Result
<i>Section 14</i>			
(14.1)	D	[36, Lemma 17.1] set-up with $\alpha(n)^\epsilon$ on the Möbius variable only; definition text compared; implemented in M1	
(14.2)	A	With one slot, $m = 1$, $r = 0.3$, $z_1 = 0.25$: $\sum M_u Q_u ^2 / Z^m \approx 0.13$ for $Z = 10^4$ to $1.6 \cdot 10^5$, no growth	size check: ratio 0.13, growth exponent at most 0.05; proved in Proposition 12
(14.3)	E	Brute-force Gauss sums at 165 primes and 60 squarefree composites	PASS (max err 7e-15)
(14.4)	D	[36, eq. (4.1)] with $\psi \rightarrow \alpha^e \psi_k$, $\psi_k = \tau \chi_\bullet(k)$; text compared; implemented in M2	definition
(14.5)	A	$z = 0$, $Z = 4000$: ratio 0.29 to 0.34 for $e = \pm 1$; control $e = 0$ with principal rows gives 252	size check: ratio at most 0.34; proved in Proposition 13
<i>Section 15</i>			
(15.1)	I	The certificate of [36] re-derived symbolically, exact rational and float grids; minimum $2.28 \cdot 10^{-4}$	PASS (exact)
(15.2)	I	$1 - h/4 = 51/64$ at $h = 13/16$; exact for $0 < \Delta \leq 1/24$, $\epsilon_{\text{total}} < 49/440640$	PASS (exact)
<i>Section 17</i>			
(17.1)	A	Expected, not claimed. Argument principle from 15000 boundary samples: no zeros found in $\Re s > k/2 + 0.88$, $ \Im s \leq 30$, for two eta products and four characters with $k = 2, 3$	size check: no zeros found in the tested box (6 L-functions)
(17.2)	E	Dirichlet coefficients of $\eta(2z)^3 \eta(6z)^3$ against the right side, $n \leq 3000$	PASS (max err 4e-15)
(17.3)	E	60 random (c, n, k) ; also $k = 0$	PASS (max err 4e-16)
(17.4)	E	sympy on a complete monomial basis, $j \leq 6$	holds (symbolic)
(17.5)	E	40-digit quadrature, $j = 1, \dots, 6$, three (ℓ, s) each	PASS (max rel err 2e-40)
(17.6)	D	R_1, R_2 equal the two table kernels; rightmost pole $\frac{1}{6} - \frac{i+1}{2}$ for $j \leq 40$	definition (D04.1 PASS)
(17.7)	E	Exact (sympy) for $j \leq 60$; ratio to $j^4/16$ tends to 1	PASS (exact)
(17.8)	E	Truncated Euler products, $\psi = \alpha^2 \chi_\bullet(4)$, $s = 2.5 + 0.3i$, $t = 1.7$	PASS (err 2e-24)
(17.9)	E	Local Euler polynomials at good $p \leq 600$, $m \leq 8$, 7 curves (5960 cases)	PASS (max err 1e-12)
<i>Appendix A</i>			
(A.1)	D	Definition of Ψ ; checked complete multiplicativity and vanishing at even n : 5 characters, 1500 random pairs	definition; consistency PASS (exact)
(A.2)	E	End to end: $T(X; \Psi)$ from (8.1) vs the grouped sum, 72 groups, $P = \{\pi\}$, $j = 0, \dots, 5$; also P empty	PASS (max rel err 2.5e-12)
(A.3)	I	Factors of C' in (A.41): $ \hat{\phi}(h_0) \leq 1$ for all h_0 (5 twists), $ \varpi = 1$, $ \kappa_0 = 1$	PASS (max $ \hat{\phi} = 0.083$; $ \varpi = 1$ to 3e-14)
(A.4)	D	Same as [2, Theorem 23 (T3)]; these coefficients reproduce $\bar{\theta}$ at the three cusps numerically	definition; checked against source (rel err 4e-15)
(A.5)	I	Bound checked for every nonzero $d_\sigma(\ell)$ with $N(9\ell) \leq 30000$; sharp; same as [2, Corollary 24]	PASS (0 violations; equality attained)
(A.6)	D	Definition: (A.1) with $j_\pi = 1$ for $\pi \mid k_0$; row primes have $j = 1$ (B59), hence are active	definition
(A.7)	A	Size check with compact $W \in C^{17}$: ratio to $\min(x^{1/4}, x^{-A})$ bounded, $A \leq 3$, $j \leq 2$; slope 4/3 at 0	size check: bounded on $[10^{-8}, 10^8]$; proved in Lemma 13
(A.8)	A	Fitted growth exponents of $ R(\eta + iu) $ equal 4η for five η ; the ratio stays bounded on the sampled grid	size check: exponents 4η within 2e-5; proved by Stirling
(A.9)	I	Inequality at 33 values $x \in [10^{-8}, 10^8]$, $j \leq 2$, $\eta = -1/4, 1, 2$; left side from the line $\Re t = 0$	PASS (0 violations)
(A.10)	E	mpmath, 30 digits, 8 cases including $s = -0.3$ and complex s ; same as [2, (5.21)]	PASS (max rel err 1e-29)
(A.11)	D	Definition of Θ_Ψ ; implemented in the pointwise tests of (A.14) and (A.31)	definition

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Display	Kind	What was done	Result
(A.12)	E	3424 indices: d_0 from the table of [2] times the twist vs the right side with brute-force γ_2	PASS (max abs err 2e-13)
(A.13)	D	Definitions of $\hat{\phi}$ and $C_{\pi,j}$; implemented literally with $L = 72$	definition
(A.14)	E	8 twists, $L = 72$: coefficient identity for all $N(x) \leq 1500$ and pointwise with the actual θ ; $\sum c_F = 0$	PASS (max err 9e-14)
(A.15)	E	$g_1 \in \Gamma(3)$ in exact arithmetic (528 translates); $\theta(HP) = \theta(\gamma_\sigma P)$ numerically, 12 matrices H	PASS (exact for g_1 ; θ to rel err 4e-15)
(A.16)	E	Pointwise at 61 cusps of all types, θ from [2, Theorem 23], own cusp data	PASS (max rel err 4e-14)
(A.17)	E	Exact in $\mathbb{Z}[\omega]$ for 46 groups (528 translates), $L = 72$, $M = \lambda^{12} 7^2 2^4$	PASS (exact)
(A.18)	E	Literal $C_{\pi,j}$: 4 primes (one inert), all j , all x ; also with actual cusp data	PASS (max err 6e-14)
(A.19)	D	Definition of $\mathcal{T}(w)$; used as the product of the two series in the test of (A.20)	definition
(A.20)	E	$T(X; \Psi)$ from (8.1) vs the w -integral on $\Re w = 1.5, 2$, $X = 60, 300$	PASS (max rel err 8e-13)
(A.21)	D	Definitions of f and $\mathcal{J}$; implemented in the tests of (A.24) and (A.35)	definition
(A.22)	E	mpmath, 40 digits, 9 pairs (ℓ, s)	PASS (max rel err 1e-40)
(A.23)	E	720 pairs (n, b) , three values of s ; $\alpha(\lambda^{-3}) = i$	PASS (max rel err 4e-15)
(A.24)	E	Quadrature of $\int f(v) v^{2s-1} dv$ from the twisted theta series vs the Dirichlet series, $s = 2.5, 3 + 1.2i$	PASS (max rel err 2e-14)
(A.25)	E	Formula vs the action of g^{-1} at 61 cusps; sympy with $ad - bc = 1$	PASS (max err 3e-15)
(A.26)	E	sympy: the six first and second derivatives	PASS (exact)
(A.27)	E	sympy, generic polynomial F , general z ; at $z = 0$ only $Y = Y' = z'$ survives	PASS (exact)
(A.28)	E	Pointwise at 61 cusps: second $\bar{z}$ -derivative from the expansion at infinity vs the cusp side	PASS (max rel err 2e-14)
(A.29)	E	Expansions from d_σ equal $\overline{\theta(\gamma_\sigma P)}$; constant term only for $\sigma = 0$	PASS (rel err 4e-15)
(A.30)	E	Used in the pointwise test of (A.28)	PASS (max rel err 2e-14)
(A.31)	E	$f(v)$ from the twisted series vs the sum over 36 to 252 translates, 8 twists	PASS (max rel err 5e-14)
(A.32)	E	Single mode (mpmath) and truncated series at three cusps by quadrature	PASS (max rel err 2e-14)
(A.33)	I	Termwise bound for all $d_\sigma(\ell)$, $N(9\ell) \leq 30000$; partial sums for $s' = 1.6, 2, 3$	PASS (0 violations)
(A.34)	E	Substitution by mpmath quadrature at two s ; $c^{-4} N(c)^2 = \overline{\alpha(c)}^4$	PASS (max rel err 1e-41)
(A.35)	E	$\mathcal{J}(-3 + 0.5i)$ by quadrature in v vs the right side, 8 twists	PASS (max rel err 2e-10)
(A.36)	E	sympy: (A.24) at $s = w + 1/2$	holds (symbolic)
(A.37)	A	Fitted exponents 6.0, 4.0, 2.4 at $\sigma_2 = -1, -0.5, -0.1$ equal $2 - 4\sigma_2$	size check: exponents to 1e-6; proved by Stirling
(A.38)	E	sympy, and numerically at three values of t	holds (symbolic)
(A.39)	E	End to end, P empty, three twists, all cusp types: (8.1) vs this formula at $\Re t = 0$	PASS (max rel err 1.1e-12)
(A.40)	E	$C_{\pi,0}(0) = 1 - 1/N(\pi)$ and $C_{\pi,j}(0) = 0$ for $j \neq 0$, 4 primes	PASS (max err 4e-16)
(A.41)	E	Grouped end to end with this C , $j = 0, \dots, 5$; κ_0 agrees with [35, (A.10)]	PASS (max rel err 2.5e-12)
<i>Appendix B</i>			
(B.1)	D	R_1 is the kernel R of [36] verbatim, R_2 the quotient of (8.5); both as implemented, Stirling and poles checked	definition (consistent; rerun PASS)
(B.2)	D	ϕ from sextic symbols for three Ψ_0 : 18-periodic and equal to the closed forms used in the tests; equals [36]	definition (K9 PASS, 324 cases)
(B.3)	A	Identity $(-t)^N \widehat{V} = \text{Mellin of } (x\partial_x)^N V$ and the bound with constant 2^N , two tests; equals [36, eq. (4.11)]	size check: ratio ≤ 0.37 (identity err 6e-31)

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Display	Kind	What was done	Result
(B.4)	D	Equals [36] verbatim; closed form $e^{A^2} \int (1+ u)^B e^{-u^2} du$ for V_G checked on an η -grid	definition (K2 PASS)
(B.5)	D	Equals D, L_0 of [36] with $\bar{\alpha}^2, \bar{\alpha}^6$; implemented as two separate Dirichlet series in the test of (B.8)	definition (used in B.8 test, PASS)
(B.6)	D	Equals the completed sum of [36] with $\bar{\alpha}(nb^3)^2$; implemented on the whole index nb^3 with zeros	definition
(B.7)	D	As implemented (trapezoid on a line); same value on $\Re t = -1, -\frac{1}{4}, 0, \frac{1}{2}, 2$ for two tests; $\sigma_0 = 0$ gives (8.5)	definition (line independence 3.5e-8)
(B.8)	E	T_V from (B.6) vs line integral of $\widehat{V} X^{s-1/2} \mathcal{D}_2 \mathcal{L}_2$, 2 tests, 2 X , $j = 0..5, a = 1.5, 2$	PASS (max rel err 6e-13)
(B.9)	E	Definition vs grouped formula, active prime of norm 7 (both), $j = 0..5$, log-Gaussian and $e^{-y-1/y}$	PASS (max rel err 1.7e-12 / 5.2e-8)
(B.10)	E	Scalar C inside the grouped end-to-end test; $ C \leq 1/81$ and $C = -i\bar{\alpha}(c)^2 \zeta$ with ζ as in [36]	PASS (1.7e-12; K8 3e-19)
(B.11)	A	Ratio of $ (x\partial_x)^j V^\sharp $ to the right side on $[10^{-8}, 10^6]$, $j \leq 2$, $A = 1, 2, 3$, two tests; proved in Step 5	size check: ratio $\leq 3.1 \cdot 10^{-3}$
(B.12)	E	$\sum_{\mathbf{h}} C(\mathbf{h}) \tilde{e}(\ell z_{\mathbf{h}}) = \phi \chi_p^j$ at all 2268 classes mod $18p$, $j = 0..5$, both p ; constant term 0	PASS (max err 1.3e-13)
(B.13)	E	$\bar{\theta}(a/c + z, v)$ from the identity expansion vs $\bar{\kappa}(G) \times$ cusp expansion at $g^{-1}(a/c + z, v)$: 252 translates, 8 cusps $\lambda \mid c$	PASS (max rel err 2.3e-13)
(B.14)	E	$\bar{\theta}(H(z', v'))$ from the identity expansion vs $C_H + \sum d_\sigma(\ell) v' K_{1/3} \tilde{e}(\ell z')$, 11 matrices H , 33 points	PASS (max rel err 1.8e-15)
(B.15)	E	Mellin integral of $f(v)$ by quadrature vs the right side with $\mathcal{D}_2 \mathcal{L}_2$, $s = 4, 3.5 + 2i$, $j = 0, 1, 4$; constant by sympy	PASS (max rel err 2.0e-12)
(B.16)	E	∂_z^2 of the identity expansion at a/c vs $\bar{\kappa}(G)(cv)^{-4} \mathcal{G}_{\mathbf{h}}(1/(N(c)v))$, 260 cusps	PASS (max rel err 3.6e-13)
(B.17)	E	The series $\mathcal{G}_{\mathbf{h}}$ in the same pointwise test	PASS (max rel err 3.6e-13)
(B.18)	E	Closed form used for the cusp side of (B.20) vs quadrature; termwise mode Mellin with two z -derivatives	PASS (1.9e-13; mode 4e-31)
(B.19)	I	$ d_\sigma(\ell) \leq 27 \cdot 3^{k/6} b $ for all 60770 table entries, at most 6 units, exponents by sympy, partial sums below the right side	PASS (0 violations; max ratio 1)
(B.20)	E	$\mathcal{J}_2(s)$ by quadrature of f (both expansions) vs the right side, $s = -2, -2.5 + 1.5i$, $j = 0, 1, 4$	PASS (max rel err 1.9e-13)
(B.21)	E	Equivalent to (B.15) at $s = w + \frac{1}{2}$ (tested there); Gamma shifts and growth exponent $2 - 4\sigma_2$ by sympy and numerically	PASS (symbolically; numerically to 2e-12)
(B.22)	I	$\max y V(y) $ on a grid of y is finite, and the truncated left side is at most $C_V X \sum N(n)^{-3/2} \sum N(b)^{-4}$, two tests, $X = 10, 1000$; exponents by sympy	PASS
(B.23)	E	Constant identity by sympy (ratio 1) and mpmath at three t	PASS (exact; 4e-31)
(B.24)	E	Ungrouped right side over all 252 translates vs definition, both primes, $j = 0..5$, two tests	PASS (max rel err 1.7e-12 / 5.2e-8)
(B.25)	A	$\max_{ u \leq 10^4} R_2(\eta + iu) (1+ u)^{-4\eta}$ on a grid of u is finite for $\eta \in [-1.3, 6]$; fitted exponent 4η ; proved by Stirling in Step 5	size check: largest sampled value 25 at $\eta = -1.3$, 1 at $\eta = 0$
(B.26)	D	Equals [36, eq. (5.30)] with $\bar{\alpha}^2$; its Mellin form (R6) with $\Psi = \nu \chi_\bullet(m)$ tested for 3 rows m	definition (Mellin form PASS, 1.3e-12)
(B.27)	A	Equals [36, eq. (5.36)] verbatim; O -dyad sums for $Z = 100, 300$, all rows $N(m) \leq 8Z$; proved by Lemma 14	size check: ratio ≤ 0.02
(B.28)	A	Exponent inequality on a grid and $\sum \mathcal{C}^{(2)} ^2 / (Q + Q^2/Z)$ for $Z = 100, 300$, $Q \leq 8Z$	holds (grid); size check: ratio ≤ 0.021
(B.29)	E	Marked sum, $\mathcal{Q} = \{p\}$, $\mathcal{P}_0 = \emptyset$, vs the inactive and active branches, two tests, both p	PASS (max rel err 8e-13 / 1.8e-8)
(B.30)	E	$C_{\mathcal{Q}}$ in the same marked test; $ C_{\mathcal{Q}} \leq 1/81$, $C_{\mathcal{Q}} = -i\bar{\alpha}(c)^2 \zeta_{\mathcal{Q}}$; local $-\varpi_{q,0}$ checked	PASS (8e-13; local 7e-15)
(B.31)	E	Exact in $\mathbb{Z}[\omega]$ for 300 random $(c_F, F_{\text{act}}, R, P)$	PASS (exact)

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Display	Kind	What was done	Result
(B.32)	E	Exact in $\mathbb{Z}[\omega]$ for random $\ell = u\lambda^h N_{Fn}(B_F b)^3$, $h \geq -4$	PASS (exact)
(B.33)	D	Equals [36, eq. (14.2)] with $\bar{\alpha}^2, \bar{\alpha}^6$; implemented (with Ψ_0 , two tests) as the direct marked sum of the end-to-end test	definition
(B.34)	A	Equals [36, Lemma 14.3] display with the same E_{ref} ; proved in the proof of Lemma 15	not testable by computation
(B.35)	A	Equals [36, eq. (14.18)], Q, V renamed Q_ρ, V_f ; proved by Lemma 15	not testable by computation
(B.36)	D	Equals B^J of [36] with $a(p) = \alpha(p)\bar{\eta}(p)$; the inner integral equals the marked sum with V_G ($D = p$)	definition (inner integral PASS, 9e-12)
(B.37)	D	Equals T_D of [36] with $\bar{\alpha}(cn^3)^2$; checked in the same Mellin test	definition (PASS, 9e-12)
(B.38)	A	Equals [36, eq. (15.2)]; $((d_J - \frac{1}{6})/4)_+ = 0$ for $0 \leq d_J \leq \frac{1}{6}$ checked; bound proved by Lemma 16	holds (exponent); not testable by computation
<i>Appendix C</i>			
(C.1)	E	For 8 primitive ϕ : ψ is generator-independent, multiplicative, of type $\bar{z}/ z $; $\bar{\alpha}$ case	PASS (max err 1.4e-15)
(C.2)	D	Definitions, implemented directly in the tests; example $\gamma(\bar{\alpha}) = -1$ exactly; W independent of the generator f	definition; example PASS (exact)
(C.3)	E	Exact check in $\mathbb{Z}[\zeta_M]$ for all $h \bmod f$, 8 characters; matches [2, Lemma 17]	PASS (exact)
(C.4)	E	Both sides from direct theta sums at 4–5 values of v , 4 + 8 characters	PASS (max rel err 3e-15)
(C.5)	E	Quadrature of the direct theta series vs $6(2\pi/\sqrt{3})^{-s-1/2}\Gamma(s + \frac{1}{2})L(s, \psi)$ at 3 points, ψ and $\bar{\psi}$, 8 characters	PASS (max rel err 1.6e-12)
(C.6)	E	FE from direct theta quadratures in the strip; PARI lfuncheckfeq with explicit $W = -\gamma\sqrt{Q}/f$	PASS (max rel err 7e-15; PARI -253 bits)
(C.7)	E	Symbolic evaluation (sympy) and quadrature at 5 points; agrees with [2, Example 20]	holds (symbolic); PASS (quad err 2e-25)
(C.8)	E	Wirtinger derivative symbolically; transform of $\bar{w}e^{-2\pi v w ^2/\sqrt{3}}$ by quadrature at 5 points	holds (symbolic); PASS (quad err 2e-25)
(C.9)	E	Poisson step for every residue $t \bmod f$; both lines vs direct theta sums, 8 characters	PASS (max rel err 1.5e-13)
(C.10)	E	Quadrature of the direct theta series vs the termwise element sum, at 3 points with $\text{Re } s > 1$, 8 characters	PASS (max rel err 1.6e-12)
(C.11)	D	Definitions of B, B^- ; quadrature agrees with the incomplete-gamma series used in the s2 tests	definition; consistency PASS (err 4e-14)
(C.12)	E	Three expressions by separate quadratures and series, 8 characters, 2 points	PASS (max rel err 6.5e-15)
(C.13)	E	Both sides by quadrature at 4 points (2 with $\text{Re } s < 1$), also conjugate identity; constants symbolically	PASS (max rel err 1.1e-11)
(C.14)	E	PARI values of both sides at 6 points, 8 characters (a consistency check of PARI's continuation); sympy equivalence with (C.6)	PASS (max rel err 9e-16)
(C.15)	E	Equality from PARI values at 6 heights, 8 characters (a consistency check of PARI's continuation); ratio to $Q^{3/5}(3 + t)^{6/5}$ at most 0.30 on the tested heights; gamma quotient on a grid	PASS (max rel err 4e-16)
(C.16)	A	For 8 characters, $a = 0.6$, $t = 0, 10, 40$: the winding number of $\mathcal{L}$ estimated from 256 samples on a circle is 0 (no zero found in the disk), and the largest of $ \mathcal{L} + \mathcal{L}^{-1} $ over 64 samples on a smaller circle is about 8.4; proved in Lemma 20	size check on the sampled circles: log ratio at most 0.30
(C.17)	E	Deletion identity at $s = 2.5 + 1.3i$ for 8 CM members; conductor bounds and exponents by brute force and random sampling	PASS (max err 1.2e-9)
(C.18)	E	Both sides computed independently (Mellin–Barnes $W^\sharp$), 3 characters	PASS (max rel err 6e-15)

Appendix D

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Display	Kind	What was done	Result
(D.1)	D	Definition, as in [36]; the tests use a model Θ_0 of order 216 containing η	definition (checked against source)
(D.2)	D	Definition of $\beta^\sharp$; nothing to compute	definition
(D.3)	D	Label set as implemented in the s3 tests: $4 \Theta_0 = 864$ distinct members, closed under conjugation	definition; consistency PASS
(D.4)	D	Members with the zero convention of [36], implemented by s3lib.Member; multiplicativity checked	definition; consistency PASS
(D.5)	D	Set of members; numerator, denominator, conjugates, slot characters found in it	definition; consistency PASS
(D.6)	E	Direct sum vs Euler product and vs reciprocal of the Möbius series, 4 members, 2 points	PASS (max err 4e-10)
(D.7)	E	Deletion identity at $s = 2.5 + 1.3i$, 8 CM members and a control; $\varphi^*(p) = 0$ at the good primes of u	PASS (max err 1.2e-9)
(D.8)	I	PARI values for 4 CM members on a grid of (a, e, t) : ratio at most 0.96, so $C_0 = 1$ works; gamma quotient grid	PASS (ratio at most 0.96)
(D.9)	E	Both expressions for 4 members and 400 random a : generator-independent, type $(z/ z)^\varsigma$, equal to the CM member	PASS (max err 7e-15)
(D.10)	E	PARI FE check (lfunccheckfeq, $ \epsilon = 1$; a consistency check of PARI's continuation) and the identity at 4 points for 4 CM members, both ς	PASS (max rel err 6e-16)
(D.11)	R	Lemma 24 is Lemma 20 applied to φ^* or its conjugate	see (C.16)
(D.12)	D	Same as [36, eqs. (4.2), (4.1)] at base U with a CM label; coefficients $\mu(n)\varphi(n)$ as in S3-T1d	definition (checked against source)
<i>Appendix E</i>			
(E.1)	D	Definition $\varphi = \overline{\alpha}$; complete multiplicativity on 3721 pairs of primary elements and $ \varphi = 1$ checked	definition; PASS (max err 5e-16)
(E.2)	R	First line of (14.1); equals [36] with $\alpha(n)^\epsilon$ inserted (compared with the source); implemented in s4 S4.6, S4.7	see (14.1)
(E.3)	R	Second line of (14.1); equals [36] verbatim (compared with the source); implemented in s4 S4.6	see (14.1)
(E.4)	C	Identical to the hypotheses of [36, Lemma 17.1]; exact check that the MS application $m = 1$, $r = 1/(1 + \vartheta)$ satisfies them	checked against source; application holds (exact)
(E.5)	R	Restates (14.2) (= [36, eq. (17.1)] with α^ϵ); size check: $\sum M_u Q_u ^2 / Z^m \leq 0.29$ for $Z^m \leq 6 \cdot 10^4$, both signs, one slot	see (14.2); size check: ratio 0.03-0.29
(E.6)	D	Definition; equals φ times [36] coefficient (compared with the source); both forms and $ a = 1$ checked on 400 squarefree n	definition; PASS (max err 9e-16)
(E.7)	D	Definition; equals [36, eq. (17.3)] with $a(n)$ of (E.6) (compared with the source); without w , mark, puncture it is (7.10)	definition
(E.8)	R	Identical to (B.33); [36, eq. (14.2)] with $\overline{\alpha}^2, \overline{\alpha}^6$ (compared with the source); evaluated in s4 S4.5	see (B.33)
(E.9)	E	Pointwise at all 366 squarefree primary u with $N(u) \leq 1300$ (157 composite)	PASS (max err 2.5e-15)
(E.10)	E	Pointwise at the same 366 u	PASS (max err 1.6e-15)
(E.11)	E	Pointwise at the same 366 u	PASS (max err 1.8e-15)
(E.12)	E	Pointwise at the same 366 u	PASS (max err 2.5e-15)
(E.13)	E	Pointwise at the same 366 u	PASS (max err 1.8e-15)
(E.14)	E	20 random fibrewise identities (25 fibres) survive the twist $\varphi(n_1)\overline{\varphi(n_2)}$	PASS (max err 9e-15)
(E.15)	E	Complete finite sums in double precision, $X = 900, 3000$, with ν , puncture, two-slot mark	PASS (rel err 7.1e-16)
(E.16)	E	Reopened completion in (h, h') dyads equals the block with $\beta^{\text{CM}} = \overline{\alpha(b)}^3 \beta$, one β for 3 rows; $X = 3000, 2 \cdot 10^5$	PASS (rel err 2.7e-15)
(E.17)	E	3000 random $n_a = \mathfrak{A}_a C t x_a$	PASS (max err 8e-16)
(E.18)	D	Equals [36, eq. (17.30)] with $\overline{\alpha(x)}$ inserted (compared with the source); its class $\mu\overline{\alpha}$ is that of (E.9)	definition

continued on the next page

Display	Kind	What was done	Result
(E.19)	E	Switch $\mu\bar{\alpha}\nu \rightarrow a_0\nu\bar{\vartheta}$ with ray factor expanded over $(\mathbb{Z}[\omega]/4)^\times$, 3 twists, 52 pairs; equals [36] with $\bar{\alpha}^2$	PASS (max err 3.3e-15)
(E.20)	D	Equals [36, eq. (17.55)] with $\overline{\alpha(n)}^2$ (compared with the source); same coefficient class as (E.6)	definition
(E.21)	E	Summed over assignments, $\epsilon = \pm 1$, 3 rows (S4.6); pointwise in 4000 cases (T8.12)	PASS (max err 9e-15)
(E.22)	D	[36, eq. (17.68)] times $\varphi(c)$ (compared with the source); orientation step checked numerically for both ϵ	definition; orientation step PASS (max err 2.5e-16)
(E.23)	E	Initialization switch with $\nu_* = \nu_0\bar{\vartheta}$ over 12 ray characters, 3 twists; extraction identity (14580 cases)	PASS (max err 7.8e-15)
(E.24)	E	$D = 400$, $\epsilon = \pm 1$, three (u, a) , one sharing a prime	PASS (rel err 6.5e-16)
<i>Appendix F</i>			
(F.1)	E	On 14161 pairs of primary ideals: products primary (exact, in $\mathbb{Z}[\omega]$); α multiplicative and unimodular (double precision).	PASS (exact; α to 4e-16)
(F.2)	D	Definition; equals [36, eq. (4.1)] with $\alpha(l)^e$ inserted; implemented in d6 and s5/t5.	definition
(F.3)	A	Case (1) on 451 rows, X_1X_2 from 10^2 to $9 \cdot 10^8$, $e = \pm 1$: the fitted exponent of X_1X_2 in the mean fourth moment is -0.11 , and the largest mean at the upper four scales is 0.69 times that at the lower four; without α it grows (exponent 0.73).	size check: no growth on the tested scales; proved in Section F.6
(F.4)	E	Pointwise on all 1510 ideals of norm at most 2500, 8 rows, $e = \pm 1$, λ moving both ways.	PASS (max err 5e-15)
(F.5)	E	All a prime to $3\mathfrak{g}_k$ in a box, 8 rows; also φ_α a character mod 3 and $\varphi_k(u) = u^{-e}$.	PASS (max err 5e-15)
(F.6)	D	Definition; implemented in s5 hecke.gauss; $ \gamma(\varphi) = 1$ for 12 characters and $\epsilon(\alpha) = 1$ (s5/t3 rerun).	definition
(F.7)	D	Definition; implemented in s5/t1, s5/t3 and d4; closed form $x^{1/2}e^{-x} \mapsto y^{1/2}e^{-y}$ rechecked (s5/t1 1b).	definition
(F.8)	E	Rerun of s5/t3: 12 primitive characters, $e = \pm 1$, 3 profiles, 3 values of X ; s5/t5 on rows.	PASS (max err 3.3e-11)
(F.9)	E	Sum over ideals (one generator each) equals one sixth of the element sum, 6 characters, 3 scales; $c(ua) = c(a)$.	PASS (max err 3e-15)
(F.10)	E	Shifted Poisson for every $t \bmod f$, 5 characters, 3 values of X ; transform computed from its definition.	PASS (max err 7e-15)
(F.11)	E	Both sides numerically, 5 characters, 3 values of X ; [2, Lemma 17] for every $w \bmod f$.	PASS (max err 5e-13)
(F.12)	E	Both equalities by quadrature at 60 random (r, ρ, β) , $e = \pm 1$.	PASS (max err 4e-15)
(F.13)	E	2-d quadrature of the definition vs the Hankel side, 3 profiles, $X = 0.37, 1, 4.2$, $e = \pm 1$; value 0 at $w = 0$.	PASS (max err 5e-13)
(F.14)	E	sympy: first and third identities symbolically; second on character values.	holds (symbolic)
(F.15)	E	Both sides with closed-form $W^\sharp$, 6 characters (conductor norm up to 1197), 2 profiles, 4 values of X .	PASS (max err 8e-15)
(F.16)	A	Explicit small- y bound of the proof at 81 points; decay of y^A times the derivatives seen up to $y = 10^6$.	size check: holds for $j \leq 2$, $A \leq 2$; proved in Lemma 30(c)
(F.17)	E	Direct y -integral (mpmath, 30 digits) at $x = 0.3, 2.5$ and 4 values of s near both ends of the strip.	PASS (max rel err 3e-31)
(F.18)	D	Definition; implemented ideal by ideal in d5 (F20) and in s5/t4.	definition
(F.19)	A	Rerun of s5/t4: unmasked sums bounded and decaying to $X = 3 \cdot 10^5$, masked sums at most 3.3; $e = 0$ grows.	size check: at most 2.5; proved in Lemma 31
(F.20)	E	Ideal-by-ideal sum equals the sum with w tabulated mod 12, 3 characters ϑ , $e = \pm 1$, two values of t .	PASS (rel err 6e-13)
(F.21)	E	Poisson with $W = x^{11/2}e^{-x}$, 3 characters, $e = \pm 1$, $t = 0, 1.3$, $X = 20, 50$; transform checked against its definition.	PASS (rel err 2e-13)
(F.22)	A	Lattice sum with $J_0 = 3$ at most 523 for $1 \leq X \leq 10^6$, diverges for $J_0 = 2$; transform bound checked for $t \leq 64$.	size check: holds; proved in Lemma 31

continued on the next page

Display	Kind	What was done	Result
(F.23)	D	Same as [36, eq. (18.18)] with $\alpha(l_1 l_2)^e$; allocation scalar checked in d1.	definition
(F.24)	E	Moebius identity on 8 rows, bump and $x^{1/2}e^{-x}$ profiles, 5 scales.	PASS (max err 7e-16)
(F.25)	E	Restored factors with $W^\sharp = y^{1/2}e^{-y}$, 8 rows, 3 scales.	PASS (max err 6e-16)
(F.26)	E	Expanded difference vs product form with the row character of [36] on the product, 3 rows, $e = \pm 1$.	PASS (max err 6e-17)
<i>Appendix G</i>			
(G.1)	C	Both bounds verbatim in [36], $\omega = \Delta_1/2$, C_1 ; $C_1(11/12) = 1/4$	checked against source exactly
(G.2)	D	Definition of $\psi \in \mathcal{F}_{\text{CM}}$ as in Theorem 1; implemented as $\bar{\alpha}\bar{\nu}$ in s6-T5 and test T8.9	definition
(G.3)	D	Definition; s6 and t8 tests use ν of order 12 mod 4 + 3 ω , zero off primary elements prime to S	definition
(G.4)	E	Checked at 300 primes (holds by construction); infinity type $\bar{z}/ z $ checked	PASS (exact)
(G.5)	D	Definition; equals [35] with $\alpha(n)$ inserted (compared with the source); implemented in s6-T1 and t8 SZ1	definition
(G.6)	A	Size check $D = 100\text{--}6400$, $\vartheta = 1/10$: $\text{sum}/D^{2.1} \in [0.25, 0.29]$, $\text{sum}/\text{diagonal} \in [0.93, 0.99]$; equals [35, eq. (3.1)]; proved after Proposition 14	size check: ratio at most 0.29
(G.7)	E	Literal sum over the 144 characters of $\mathfrak{G}_S$, ν of order 12, 60 coprime pairs	PASS (max err 1.2e-14)
(G.8)	A	Size check: $ A_1^\nu(D) \leq 0.57D^{1/2}$ at 120 $D \in [10^2, 10^6]$; extraction exponents exact; proved in proof of Theorem 2	size check: $ A_1 /D^{11/12} \leq 0.07$
(G.9)	E	Euler-product coefficients agree for $N(n) \leq 4000$; values at two s ; Mellin side by quadrature	PASS (max err 3e-15)
(G.10)	R	(E.2) with $\nu = \theta$, $Z^r = D$ (compared with the source); $\epsilon = -1$ is M_r of (13.4)	see (E.2)
(G.11)	R	(B.28) at $M = 1$: $\max(M, 2M - 1) = 1$ (exact); stated as Proposition 7 with $Q = Z$	see (B.28)

18.2 What the checks do not establish

No computation verifies an estimate with unspecified constants. This applies to the 44 displays of kind A, among them the mean square (4.6), the completed estimate (9.2), cube removal (10.4), the transfer estimate (10.17), and their Part II counterparts; 12 of them are not testable by computation even for size. The Part I estimates are proved in Sections 7, 9 and 10 and Appendix A, using the large sieve (9.1), which is cited from [10]. The Part II statements are proved in Appendices B–G. In both parts the exact identities used in the proofs were tested, as the table shows. In particular, the random descent chains behind (10.22) and (10.23) check scale inequalities under the displayed transition formulas, not the transfer estimate itself. The one display of kind A that is not proved is (17.1), the expectation of Section 17; its size check, which finds no zeros of six such L -functions in a box, is evidence only. A few remarks on presentation are recorded with the scripts.

A The CM transformation formula

This appendix states the CM transformation formula precisely and proves it for the twists of Part I. Appendix B extends it to the generality used in Part II. The coefficient data are tabulated in [2, Theorem 23], so every series in the theta step is given in finite-sum terms. It can be skipped on a first reading. The imported theta function is that of Kubota and Patterson [24, 39] in the form [2, Theorem 23], which follows [8, Section 5.1 and Appendix] and the conventions of [35];

see the paragraph “What is imported” in Section 8. Only these theta facts are used from [8]; its GRH-conditional prime Gauss-sum asymptotic is not an input here.

Let ξ_0 be a character of $\mathfrak{G}$, extended by zero to elements that are not primary or not coprime to 6. Let $\mathcal{P}$ be a finite set of primary primes coprime to 6, and let $0 \leq j_\pi \leq 5$ for $\pi \in \mathcal{P}$. Put

$$\Psi(n) = \xi_0(n) \prod_{\pi \in \mathcal{P}} \chi_\pi(n)^{j_\pi}, \quad (\text{A.1})$$

with the convention $\chi_\pi(n)^0 = \mathbf{1}_{\pi \nmid n}$. Then Ψ is completely multiplicative on primary elements and vanishes unless $(n, 6) = 1$. Write $T(X; \Psi)$ for the right side of (8.1) with this Ψ . By Section 8, $T(X; k, f)$ has this form. Fix a compact interval $I \subset (0, \infty)$, and let $W \in C_c^\infty(I)$ and $V(y) = \sqrt{y} W(y)$.

Proposition 8 (CM transformation formula). *For every $X > 0$, $T(X; \Psi)$ is a sum of at most $5184 \cdot 2^{|\mathcal{P}|}$ terms of the form*

$$C \sum_{0 \neq \ell \in \lambda^{-4} \mathbb{Z}[\omega]} \frac{d_\sigma(\ell) \alpha(\ell)^2}{\sqrt{N(\ell)}} \psi(\lambda^4 \ell) \prod_{\pi \in \mathcal{A}} B_{\pi, j_\pi}(\lambda^4 \ell) V^\# \left(\frac{N(\ell) X}{N(c)^2} \right). \quad (\text{A.2})$$

In each term, $\sigma \in \{0, +, -\}$ and d_σ is the coefficient sequence of $\bar{\theta}$ at the cusp $\gamma_\sigma \infty$; $\mathcal{A}$ is a set with $\{\pi \in \mathcal{P} : j_\pi \neq 0\} \subseteq \mathcal{A} \subseteq \mathcal{P}$; $c = c_0 \prod_{\pi \in \mathcal{A}} \pi$, where c_0 divides $8\lambda^2$ up to a unit; $\psi(x) = e(\beta x / (\lambda^3 c_0))$ for some $\beta \in \mathbb{Z}[\omega]$; and

$$C = -\frac{\overline{\alpha(c)}^4}{81} C', \quad |C'| \leq 1. \quad (\text{A.3})$$

Each series converges absolutely. The weight $V^\#$ and the local factors $B_{\pi, j}$ are those of (8.5) and (8.6).

The scalar C' collects a finite Fourier coefficient of the residue-class twist, the conjugated Kubota character and the local phases; it is given explicitly in (A.41). This is the CM form of [35, Proposition 6.2], a variant of the theta transformation of [8, Section 5], which extends work of Patterson [39] and Yoshimoto [46]. Relative to the zeta case, $\alpha(\ell)$ becomes $\alpha(\ell)^2$, the prefactor $-\frac{i}{81} \overline{\alpha(c)}^2$ becomes $-\frac{1}{81} \overline{\alpha(c)}^4$, and the gamma factors of the reflected weight are shifted by $\frac{1}{2}$. The proof is in Section A.2. Lemma 12 shows how the terms vary with the twist.

The coefficient sequences are explicit. The coefficients τ of θ at ∞ and τ_1, τ_2 at the other two cusps (the notation of [2]; elsewhere τ is the divisor function) are tabulated in [2, Theorem 23, (5.7), (5.9), (5.10)] in terms of the cubic Gauss sums $g(r, c)$ of [2, (5.5)]; there $\sqrt{-3}$ is our λ , so $g(-3, c) = g(\lambda^2, c)$, and c is primary, squarefree and possibly even. With them,

$$t_0(\ell) = \tau(\ell), \quad t_+(\ell) = \omega \tau_2(\omega \ell) \check{e}(\ell), \quad t_-(\ell) = \omega^2 \tau_1(\omega^2 \ell) \check{e}(\ell), \quad d_\sigma(\ell) = \overline{t_\sigma(-\ell)}. \quad (\text{A.4})$$

These are precisely the coefficients of the conjugated theta functions at the three cusps. Their support is contained in the set of indices $\ell = u \lambda^m n b^3$, with u a unit, $m \geq -4$, n squarefree, and n, b primary (the tables of [2] impose further restrictions); this representation is unique, and, by [2, Corollary 24] and [35, Lemma 6.4],

$$|d_\sigma(u \lambda^m n b^3)| \leq 27 \cdot 3^{m/6} |b|. \quad (\text{A.5})$$

In the varying quadratic symbol in (A.2), the factor b^3 contributes $(b/k)_2$, since $(b^3/k)_2 = (b/k)_2$. Fixing u, m, b leaves squarefree columns n , exactly as required for (9.1). This is why the explicit support of the theta coefficients matters to the averaging argument.

A.1 Uniformity in the twist and decay of the weight

The sieve of Section 9 averages Proposition 8 over squarefree rows k_0 . For this the cusp data must be chosen consistently as k_0 varies, and the reflected weight must decay.

Lemma 12 (Uniformity in the twist). *Fix ξ_0 , a finite set $\mathcal{P}_{\text{fix}}$ of primary primes coprime to 6, and exponents $0 \leq j_\pi \leq 5$ for $\pi \in \mathcal{P}_{\text{fix}}$. For squarefree primary k_0 coprime to $6 \prod_{\pi \in \mathcal{P}_{\text{fix}}} \pi$ put*

$$\Psi_{k_0}(n) = \xi_0(n) \prod_{\pi \in \mathcal{P}_{\text{fix}}} \chi_\pi(n)^{j_\pi} \prod_{\pi | k_0} \chi_\pi(n). \quad (\text{A.6})$$

The terms of Proposition 8 for Ψ_{k_0} can be indexed by pairs $(h_0, \mathcal{A}_{\text{fix}})$, with $h_0 \in \mathbb{Z}[\omega]/72\mathbb{Z}[\omega]$ and $\{\pi \in \mathcal{P}_{\text{fix}} : j_\pi \neq 0\} \subseteq \mathcal{A}_{\text{fix}} \subseteq \mathcal{P}_{\text{fix}}$, so that $\mathcal{A} = \mathcal{A}_{\text{fix}} \cup \{\pi : \pi \mid k_0\}$ and $c = c_0 k_0 \prod_{\pi \in \mathcal{A}_{\text{fix}}} \pi$. For each index, the data σ , ψ and c_0 depend on k_0 only through its residue class modulo M^2 , where $M = \lambda^{12} 72^4$.

Proof. This is [35, Lemma 6.3] for the present family, and its proof at the end of [35, App. A.2] applies verbatim. Every prime dividing k_0 has exponent 1, so it is active, and $r = k_0 \prod_{\pi \in \mathcal{A}_{\text{fix}}} \pi$ in Step 2 of Section A.2. The data c_0 , σ , κ_0 and ψ of that step depend only on h_0 , $\mathcal{A}_{\text{fix}}$ and $r \bmod M^2$. They are fixed before any derivative is taken, so the number of derivatives plays no role. The scalar C depends on k_0 , also through $\overline{\alpha(c)}^4$, but only the bound $|C| \leq 1/81$ is used, as in the zeta case. $\square$

Lemma 13 (Decay of the reflected weight). *Let $I \subset (0, \infty)$ be a compact interval, $W \in C_c^\infty(I)$ and $V(y) = \sqrt{y} W(y)$. For every $A > 0$ and every integer $j \geq 0$ there is $J = J(A, j)$ such that*

$$|(x\partial_x)^j V^\sharp(x)| \ll_{A,j,I} \|W\|_{C^J(I)} \min(x^{1/4}, x^{-A}) \quad (x > 0), \quad (\text{A.7})$$

where $\|W\|_{C^J(I)} = \max_{0 \leq i \leq J} \sup_I |W^{(i)}|$.

Proof. Let $R(t)$ be the gamma quotient in (8.5). Its numerator has poles only at $t = -\frac{4}{3} - k$ and $t = -\frac{5}{3} - k$ ($k \geq 0$), and $1/\Gamma$ is entire, so R is holomorphic in $\Re t > -\frac{4}{3}$. Stirling's formula [34, (5.11.9)] $|\Gamma(x + iy)| = \sqrt{2\pi} |y|^{x-1/2} e^{-\pi|y|/2} (1 + O(1/|y|))$, uniform for x in compact sets and $|y| \geq 1$, gives $|\Gamma(a + \eta + iu)/\Gamma(a - \eta - iu)| \asymp |u|^{2\eta}$ for $a \in \{\frac{4}{3}, \frac{5}{3}\}$ and $|u| \geq 1$. With continuity for $|u| \leq 1$,

$$|R(\eta + iu)| \ll_A (1 + |u|)^{4\eta} \quad (-\frac{1}{4} \leq \eta \leq A, u \in \mathbb{R}). \quad (\text{A.8})$$

The zeta kernel $\Gamma(\frac{7}{6} + t)\Gamma(\frac{5}{6} + t)/(\Gamma(\frac{7}{6} - t)\Gamma(\frac{5}{6} - t))$ has the same exponent and is holomorphic in $\Re t > -\frac{5}{6}$; both regions contain $\Re t \geq -\frac{1}{4}$, which is all that is used. Next, $\widehat{V}(-t) = \int_I V(y) y^{-t-1} dy$ is entire. For $|\Im t| \geq 1$, k integrations by parts give $|\widehat{V}(-t)| \ll_{k,A,I} \|V\|_{C^k(I)} (1 + |t|)^{-k}$ uniformly for $-\frac{1}{4} \leq \Re t \leq A$, and the same bound is trivial for $|\Im t| \leq 1$. Also $\|V\|_{C^k(I)} \ll_{k,I} \|W\|_{C^k(I)}$, since $\sqrt{y}$ and its derivatives are bounded on I .

The integrand of (8.5) therefore decays faster than any power of $|\Im t|$ in the strip $-\frac{1}{4} \leq \Re t \leq A$. We move the contour to $\Re t = -\frac{1}{4}$ if $x \leq 1$ and to $\Re t = A$ if $x \geq 1$, crossing no poles. Each application of $x\partial_x$ multiplies the integrand by $-t$. With $Y = (2\pi)^4 x/27$ and $\eta = -\frac{1}{4}$ or A ,

$$|(x\partial_x)^j V^\sharp(x)| \leq \frac{Y^{-\eta}}{2\pi} \int_{\mathbb{R}} |\eta + iu|^j |\widehat{V}(-\eta - iu)| |R(\eta + iu)| du. \quad (\text{A.9})$$

By (A.8) and the bound for $\widehat{V}$ with $k = J := \lceil 4A \rceil + j + 2$, the integral is $\ll_{A,j,I} \|W\|_{C^J(I)}$. Since $Y \asymp x$, this proves the lemma. The argument is that of [35, (A.20)], with the CM kernel. $\square$

In particular, suppose $1 \leq X, N(c) \leq D^{C_0}$ and $\|W\|_{C^J(I)} \ll 1$ for J large. Then, in (A.2), the terms with $N(\ell)X/N(c)^2 \geq D^\varepsilon$ contribute $O(D^{-B})$ for every $B > 0$. This follows from Lemma 13 with A large, the coefficient bound for d_σ and $\prod_{\pi \in \mathcal{A}} |B_{\pi, j_\pi}| \leq N(c)^{1/2}$, as at the end of Section A.2. Since $N(\ell) \geq 3^{-4}$, the whole series is negligible when $X \geq 81D^\varepsilon N(c)^2$.

A.2 Proof of Proposition 8

We follow the proof of [35, Proposition 6.2] given in [35, App. A.2]. Steps 1 and 2 construct the twisted theta series and choose the cusp data. They do not involve the derivative, so we quote them and record what they produce. Steps 3–6 are the Mellin transform, the chain rule at a cusp, the functional equation and the assembly. This is where the second derivative enters, and we write these steps out.

Throughout, $\partial_{\bar{z}}$ and ∂_z are the Wirtinger derivatives $\frac{1}{2}(\partial_{\Re z} \pm i\partial_{\Im z})$, so that $\partial_{\bar{z}}\check{e}(\ell z) = 2\pi i \bar{\ell} \check{e}(\ell z)$ and $\partial_z\check{e}(\ell z) = 2\pi i \ell \check{e}(\ell z)$. The Mellin integral of the Bessel function is [2, (5.21)] (see [34, (10.43.19)])

$$\int_0^\infty v^{2s} K_{1/3}(4\pi|\ell|v) dv = 2^{2s-1} \Gamma(s + \frac{1}{3}) \Gamma(s + \frac{2}{3}) (4\pi|\ell|)^{-2s-1} \quad (\Re s > -\frac{1}{3}). \quad (\text{A.10})$$

Step 1: the twisted series and its translates. Let $L = 72$. In [35, Appendix A.2] L is any modulus divisible by sufficiently high powers of 2 and 3; $L = 72$ suffices here. In that proof L enters only through the periodicity of ϕ below, which holds modulo 72, and through $M = \lambda^{12}L^4$, which for $L = 72$ is the rational integer $729 \cdot 72^4$, with $v_\lambda(M) = 28$ and $v_2(M) = 12$. The congruences of Step 2 are imposed modulo the powers of λ and 2 dividing $M c_0$, with c_0 as in Step 2(a), and modulo the active primes. We checked Step 2(a)–(e) with $L = 72$ and this M in exact arithmetic, for groups covering every cusp type (Section 18). Define $\phi : \mathbb{Z}[\omega] \rightarrow \mathbb{C}$ by $\phi(x) = (\lambda/x)_3 \xi_0(x)$ if x is primary and coprime to 6, and $\phi(x) = 0$ otherwise. Since $(\lambda/x)_3$ depends only on $x \bmod 9$, ϕ is periodic modulo L , and $|\phi| \leq 1$. Put

$$\Theta_\Psi(z, v) = \sum_{0 \neq \ell \in \lambda^{-3}\mathbb{Z}[\omega]} d_0(\ell) \phi(\lambda^3 \ell) \prod_{\pi \in \mathcal{P}} \chi_\pi(\lambda^3 \ell)^{j_\pi} v K_{1/3}(4\pi|\ell|v) \check{e}(\ell z). \quad (\text{A.11})$$

By (8.2) and the selection statement after it, the coefficient of Θ_Ψ at ℓ vanishes unless $\lambda^3 \ell = nb^3$ with n squarefree and n, b primary and coprime to 6. This representation is unique, and the coefficient is then

$$3^{5/2} |b| \overline{(\lambda/n)_3} \gamma_2(n) \cdot (\lambda/nb^3)_3 \Psi(nb^3) = 3^{5/2} |b| \gamma_2(n) \Psi(n) \Psi(b)^3, \quad (\text{A.12})$$

because $(\lambda/b^3)_3 = 1$ and Ψ is completely multiplicative. For $h_0 \in \mathbb{Z}[\omega]/L\mathbb{Z}[\omega]$ and $h \in \mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega]$ put

$$\widehat{\phi}(h_0) = \frac{1}{N(L)} \sum_{x \bmod L} \phi(x) e\left(-\frac{h_0 x}{L}\right), \quad C_{\pi, j}(h) = \frac{1}{N(\pi)} \sum_{y \bmod \pi} \chi_\pi(y)^j e\left(-\frac{hy}{\pi}\right). \quad (\text{A.13})$$

Finite Fourier expansion [2, (4.5)] of ϕ modulo L and of $\chi_\pi^{j_\pi}$ modulo π gives, as in [35, Lemma 6.1 and (A.2)–(A.5)],

$$\Theta_\Psi(z, v) = \sum_{\mathbf{h}} c_F(\mathbf{h}) \overline{\theta(z + z_{\mathbf{h}}, v)}, \quad z_{\mathbf{h}} = \lambda^2 \left(\frac{h_0}{L} + \sum_{\pi \in \mathcal{P}} \frac{h_\pi}{\pi} \right), \quad c_F(\mathbf{h}) = \widehat{\phi}(h_0) \prod_{\pi \in \mathcal{P}} C_{\pi, j_\pi}(h_\pi), \quad (\text{A.14})$$

where $\mathbf{h} = (h_0, (h_\pi)_{\pi \in \mathcal{P}})$ runs over $\mathbb{Z}[\omega]/L\mathbb{Z}[\omega] \times \prod_\pi \mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega]$. Indeed, multiplying the mode $\check{e}(\ell z)$ by $e(\lambda^3 \ell \eta) = \check{e}(\lambda^2 \ell \eta)$ (recall $e(x) = \check{e}(x/\lambda)$) translates z by $\lambda^2 \eta$. The constant terms $\frac{3^{5/2}}{2} v^{2/3}$ of the translates cancel, because $\sum_{\mathbf{h}} c_F(\mathbf{h}) = \phi(0) \prod_\pi \chi_\pi(0)^{j_\pi} = 0$.

We have $|\hat{\phi}(h_0)| \leq 1$, $C_{\pi,j}(0) = 0$ for $j \neq 0$, and $C_{\pi,0}(0) = 1 - N(\pi)^{-1}$ [35, (A.5)]. So $c_F(\mathbf{h}) \neq 0$ forces the active set $\mathcal{A}(\mathbf{h}) = \{\pi : h_\pi \neq 0\}$ to contain every π with $j_\pi \neq 0$. Group the $\mathbf{h}$ according to $(h_0, \mathcal{A}(\mathbf{h}))$. There are at most $N(L) 2^{|\mathcal{P}|} = 5184 \cdot 2^{|\mathcal{P}|}$ groups. Changing h_π modulo π moves $z_{\mathbf{h}}$ by an element of $\lambda^2 \mathbb{Z}[\omega] = 3\mathbb{Z}[\omega]$, under which θ is invariant by (T1). So we may take representatives with $h_\pi \equiv 0 \pmod{M^2}$, where $M = \lambda^{12} L^4$.

Step 2: the cusp data. Fix a group $(h_0, \mathcal{A})$ and put $r = \prod_{\pi \in \mathcal{A}} \pi$. The following facts are proved in [35, App. A.2], in the paragraphs on reduced denominators and on the local character transformation (see also [2, (5.18)]). They concern only the translates $\theta(z + z_{\mathbf{h}}, v)$, not their derivatives, so they apply without change. For each $\mathbf{h}$ in the group:

- (a) $z_{\mathbf{h}} = a/c$ in lowest terms, normalized by a common unit so that $a \equiv 1 \pmod{3}$ if $\lambda \mid c$ and $c \equiv 1 \pmod{3}$ otherwise (this fixes the unit in $\overline{\alpha(c)}^4$ in (A.41)), where $c = c_0 r$ is the same for the whole group and c_0 is an associate of the reduced denominator of $\lambda^2 h_0/L = h_0/(8\lambda^2)$. In particular c_0 divides $8\lambda^2$ up to a unit. For $\pi \in \mathcal{A}$, $a \equiv \sigma_\pi h_\pi \pmod{\pi}$ with $\sigma_\pi = \lambda^2 c/\pi$.
- (b) There are $\delta \in \mathbb{Z}[\omega]$ with $a\delta \equiv 1 \pmod{c}$, a matrix $H \in \mathrm{SL}_2(\mathbb{Z}[\omega])$ and $\sigma \in \{0, +, -\}$ such that

$$g = \begin{pmatrix} a & (a\delta - 1)/c \\ c & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[\omega]), \quad g_1 = gH^{-1} \in \Gamma(3), \quad \theta(HP) = \theta(\gamma_\sigma P) \quad (P \in \mathbb{H}^3). \quad (\text{A.15})$$

The label σ and the residue of δ modulo $\lambda^3 c_0$ are the same for the whole group. Since $\theta(gP) = \kappa(g_1)\theta(HP)$ by (T1),

$$\overline{\theta(a/c + z, v)} = \overline{\kappa(g_1)} F_\sigma(g^{-1}(a/c + z, v)), \quad F_\sigma(P) = \overline{\theta(\gamma_\sigma P)}. \quad (\text{A.16})$$

- (c) $\kappa(g_1) = \kappa_0 \prod_{\pi \in \mathcal{A}} \chi_\pi(a)^2$, where $|\kappa_0| = 1$ and κ_0 is the same for the whole group [35, (A.10)].
- (d) For $\ell \in \lambda^{-4} \mathbb{Z}[\omega]$,

$$\check{e}\left(-\frac{\delta \ell}{c}\right) = \psi(\lambda^4 \ell) \prod_{\pi \in \mathcal{A}} e\left(\frac{\epsilon_\pi h_\pi^{-1} \lambda^4 \ell}{\pi}\right), \quad (\text{A.17})$$

where $\psi(x) = e(\beta x/(\lambda^3 c_0))$ with $\beta \in \mathbb{Z}[\omega]$ the same for the whole group, and $\epsilon_\pi \in (\mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega])^\times$ depends only on c and π [35, (A.11)].

- (e) For $\pi \in \mathcal{A}$, $0 \leq j \leq 5$ and $x \in \mathbb{Z}[\omega]$,

$$\sum_{h \neq 0 \pmod{\pi}} C_{\pi,j}(h) \chi_\pi(\sigma_\pi h)^{-2} e\left(\frac{\epsilon_\pi h^{-1} x}{\pi}\right) = \chi_\pi(\sigma_\pi)^{-2} \varpi_{\pi,j} B_{\pi,j}(x), \quad |\varpi_{\pi,j}| = 1, \quad (\text{A.18})$$

with $B_{\pi,j}$ as in (8.6). This is [35, (A.12)], where $\varpi_{\pi,j}$ is called $\omega_{p,j}$.

Step 3: the Mellin transform at the identity cusp. For $\Re w > 1$ put

$$\mathcal{T}(w) = \left(\sum_n^* \frac{\overline{\alpha(n)}^2 \gamma_2(n) \Psi(n)}{N(n)^w} \right) \left(\sum_b \frac{\overline{\alpha(b)}^6 \Psi(b)^3}{N(b)^{3w-1/2}} \right), \quad (\text{A.19})$$

a product of absolutely convergent series. The summand of (8.1) is the coefficient times $V(N(n)N(b)^3/X)/(\sqrt{N(n)}N(b))$. Inserting $V(y) = \frac{1}{2\pi i} \int_{(\sigma_1)} \widehat{V}(w - \frac{1}{2}) y^{1/2-w} dw$, where $\int_{(\sigma)}$ runs upward along $\Re w = \sigma$, gives

$$T(X; \Psi) = \frac{1}{2\pi i} \int_{(\sigma_1)} \widehat{V}(w - \frac{1}{2}) \mathcal{T}(w) X^{w-1/2} dw \quad (\sigma_1 > 1). \quad (\text{A.20})$$

The series for Θ_Ψ has polynomially bounded coefficients, and $K_{1/3}(x) \ll e^{-x}$ for $x \geq 1$ [34, (10.25.3)], so it may be differentiated termwise. Put

$$f(v) = \partial_{\bar{z}}^2 \Theta_\Psi(z, v)|_{z=0}, \quad \mathcal{J}(s) = \int_0^\infty f(v) v^{2s-1} dv. \quad (\text{A.21})$$

Since $(2\pi i \bar{\ell})^2 = -4\pi^2 N(\ell) \overline{\alpha(\ell)}^2$, (A.10) shows that a mode ℓ contributes its coefficient times

$$-\frac{\pi}{2} (2\pi)^{-2s} \Gamma(s + \frac{1}{3}) \Gamma(s + \frac{2}{3}) \overline{\alpha(\ell)}^2 N(\ell)^{1/2-s}. \quad (\text{A.22})$$

For $\ell = \lambda^{-3} n b^3$ we have $\alpha(\lambda^{-3}) = i$, so

$$\overline{\alpha(\ell)}^2 = -\overline{\alpha(n)}^2 \overline{\alpha(b)}^6, \quad N(\ell)^{1/2-s} = 27^{s-1/2} N(n)^{1/2-s} N(b)^{3/2-3s}. \quad (\text{A.23})$$

Inserting (A.12) and using $3^{5/2} \cdot 27^{-1/2} = 3$ and $|b| N(b)^{3/2-3s} = N(b)^{2-3s}$, we get

$$\mathcal{J}(s) = \frac{3\pi}{2} \left(\frac{27}{(2\pi)^2} \right)^s \Gamma(s + \frac{1}{3}) \Gamma(s + \frac{2}{3}) \mathcal{T}(s - \frac{1}{2}) \quad (\Re s > \frac{3}{2}). \quad (\text{A.24})$$

Termwise integration is justified. Since $K_{1/3} > 0$, the sum over the modes of the absolute values of the integrals is a constant times $\sum_n N(n)^{1/2-\Re s} \sum_b N(b)^{2-3\Re s}$, which is finite for $\Re s > \frac{3}{2}$. With one derivative the same computation gives $\mathcal{T}$ at s itself [35, (A.16)]. The shift by $\frac{1}{2}$, caused by the extra factor $|\ell|$, is the source of the CM gamma factors.

Step 4: the chain rule at a cusp. Fix $\mathbf{h}$, with the data of Step 2, and fix $v > 0$. By [2, (5.1), (5.19)] and $\det g = 1$,

$$(z', v') := g^{-1}(a/c + z, v) = \left(-\frac{\delta}{c} - \frac{\bar{z}}{c^2(v^2 + z\bar{z})}, \frac{v}{N(c)(v^2 + z\bar{z})} \right), \quad (\text{A.25})$$

and $\bar{z}' = \overline{z'}$. Regard z' , $\bar{z}'$ and v' as smooth functions of $(z, \bar{z})$ near 0. Differentiation gives

$$\begin{aligned} \partial_{\bar{z}} z' &= -\frac{v^2}{c^2(v^2 + z\bar{z})^2}, & \partial_{\bar{z}}^2 z' &= \frac{2zv^2}{c^2(v^2 + z\bar{z})^3}, \\ \partial_{\bar{z}} \bar{z}' &= \frac{z^2}{\bar{c}^2(v^2 + z\bar{z})^2}, & \partial_{\bar{z}}^2 \bar{z}' &= -\frac{2z^3}{\bar{c}^2(v^2 + z\bar{z})^3}, \\ \partial_{\bar{z}} v' &= -\frac{zv}{N(c)(v^2 + z\bar{z})^2}, & \partial_{\bar{z}}^2 v' &= \frac{2z^2v}{N(c)(v^2 + z\bar{z})^3}. \end{aligned} \quad (\text{A.26})$$

At $z = 0$ all six vanish except $\partial_{\bar{z}} z' = -(cv)^{-2}$. For a smooth function F of (z', v') , the chain rule in the variables $Y \in \{z', \bar{z}', v'\}$ gives

$$\partial_{\bar{z}}^2 [F(z', v')] = \sum_{Y, Y'} (\partial_Y \partial_{Y'} F) (\partial_{\bar{z}} Y) (\partial_{\bar{z}} Y') + \sum_Y (\partial_Y F) \partial_{\bar{z}}^2 Y, \quad (\text{A.27})$$

and at $z = 0$ only the term with $Y = Y' = z'$ survives. This proves (8.4). Taking $F = F_\sigma$ and using (A.16),

$$\partial_{\bar{z}}^2 \overline{\theta(a/c + z, v)}|_{z=0} = \overline{\kappa(g_1)} (cv)^{-4} G\left(\frac{1}{N(c)v}\right), \quad G(v') := (\partial_{z'}^2 F_\sigma)(-\delta/c, v'). \quad (\text{A.28})$$

In the expansion

$$F_\sigma(z', v') = \frac{3^{5/2}}{2} \mathbf{1}_{\sigma=0} v'^{2/3} + \sum_{0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]} d_\sigma(\ell) v' K_{1/3}(4\pi|\ell|v') \check{e}(\ell z'), \quad (\text{A.29})$$

the operator $\partial_{z'}^2$ removes the constant term and multiplies the mode ℓ by $(2\pi i\ell)^2 = -4\pi^2 N(\ell)\alpha(\ell)^2$. Hence

$$G(v') = -4\pi^2 \sum_{0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]} d_\sigma(\ell) N(\ell) \alpha(\ell)^2 v' K_{1/3}(4\pi|\ell|v') \check{e}\left(-\frac{\delta\ell}{c}\right). \quad (\text{A.30})$$

On this side the derivative is in z' rather than $\bar{z}'$, which is why $\alpha(\ell)^2$ appears instead of $\overline{\alpha(\ell)}^2$. In the zeta case the single factor $-(cv)^{-2}$ produces the sign and the factor $\overline{\alpha(c)}^2$ of [35, (A.18)]. Here its square $(cv)^{-4}$ produces no sign.

Step 5: continuation and the functional equation. On $\lambda^{-3}\mathbb{Z}[\omega] \setminus \{0\}$ we have $|\ell| \geq 3^{-3/2}$, so the defining series shows that $f(v)$ decays exponentially as $v \rightarrow \infty$. As $v \rightarrow 0$, by (A.14) and (A.28), $f(v)$ is the finite sum of the terms

$$c_F(\mathbf{h}) \overline{\kappa(g_{1,\mathbf{h}})} (c_{\mathbf{h}}v)^{-4} G_{\mathbf{h}}(1/(N(c_{\mathbf{h}})v)), \quad (\text{A.31})$$

where the subscript $\mathbf{h}$ marks the data of $\mathbf{h}$. By (A.30), and since $|\ell| \geq 1/9$ on $\lambda^{-4}\mathbb{Z}[\omega] \setminus \{0\}$, each $G_{\mathbf{h}}(v')$ decays exponentially as $v' \rightarrow \infty$. Hence $\mathcal{J}(s)$ converges absolutely for every s , is entire, and is bounded on vertical strips of bounded width.

For $\Re s' > \frac{3}{2}$ the computation of Step 3, with ℓ in place of $\bar{\ell}$, gives

$$\mathcal{J}_{\mathbf{h}}^\vee(s') := \int_0^\infty G_{\mathbf{h}}(v') v'^{2s'-1} dv' = -\frac{\pi}{2} (2\pi)^{-2s'} \Gamma(s' + \frac{1}{3}) \Gamma(s' + \frac{2}{3}) \sum_{0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]} \frac{d_\sigma(\ell) \alpha(\ell)^2}{N(\ell)^{s'-1/2}} \check{e}\left(-\frac{\delta\ell}{c}\right). \quad (\text{A.32})$$

The series and the termwise integration converge absolutely. Indeed, write $\ell = u\lambda^m n b^3$ uniquely as in Appendix A. The bound $|d_\sigma(\ell)| \leq 27 \cdot 3^{m/6} |b|$ gives

$$\sum_{\ell} |d_\sigma(\ell)| N(\ell)^{1/2-\Re s'} \leq 6 \cdot 27 \sum_{m \geq -4} 3^{m(2/3-\Re s')} \sum_n N(n)^{1/2-\Re s'} \sum_b N(b)^{2-3\Re s'} < \infty. \quad (\text{A.33})$$

Now let $\Re s < \frac{1}{2}$. In each term of f substitute $v = 1/(N(c)v')$:

$$\int_0^\infty (cv)^{-4} G\left(\frac{1}{N(c)v}\right) v^{2s-1} dv = c^{-4} N(c)^{4-2s} \int_0^\infty G(v') v'^{3-2s} dv' = \overline{\alpha(c)}^4 N(c)^{2-2s} \mathcal{J}^\vee(2-s). \quad (\text{A.34})$$

This integral converges absolutely, because $\Re(2-s) > \frac{3}{2}$; here $c^{-4} N(c)^2 = (\bar{c}/c)^2 = \overline{\alpha(c)}^4$. Summing over the finitely many $\mathbf{h}$,

$$\mathcal{J}(s) = \sum_{\mathbf{h}} c_F(\mathbf{h}) \overline{\kappa(g_{1,\mathbf{h}})} \overline{\alpha(c_{\mathbf{h}})}^4 N(c_{\mathbf{h}})^{2-2s} \mathcal{J}_{\mathbf{h}}^\vee(2-s) \quad (\Re s < \frac{1}{2}). \quad (\text{A.35})$$

By (A.24), for $\Re w > 1$,

$$\mathcal{T}(w) = \mathcal{J}(w + \tfrac{1}{2}) / \left(\frac{3\pi}{2} \left(\frac{27}{(2\pi)^2} \right)^{w+1/2} \Gamma(w + \tfrac{5}{6}) \Gamma(w + \tfrac{7}{6}) \right). \quad (\text{A.36})$$

The right side is entire, so $\mathcal{T}$ continues to an entire function. Fix $\sigma_1 \in (1, 2]$ and $\sigma_2 \in [-1, 0)$. On $\Re w = \sigma_1$, $\mathcal{T}$ is bounded. On $\Re w = \sigma_2$, apply (A.35) and (A.32) at $s' = \frac{3}{2} - w$. Stirling's formula then gives

$$|\mathcal{T}(w)| \ll \left| \frac{\Gamma(\frac{11}{6} - w) \Gamma(\frac{13}{6} - w)}{\Gamma(w + \frac{5}{6}) \Gamma(w + \frac{7}{6})} \right| \ll (1 + |\Im w|)^{2-4\sigma_2}. \quad (\text{A.37})$$

In the strip $\sigma_2 \leq \Re w \leq \sigma_1$, $\mathcal{J}(w + \frac{1}{2})$ is bounded and $1/\Gamma$ is entire of order one, so $\mathcal{T}$ has finite order there. The Phragmén–Lindelöf principle [40], [45, §5.65] gives $|\mathcal{T}(w)| \ll (1 + |\Im w|)^{2-4\sigma_2}$ throughout the strip. Since $\hat{V}(w - \frac{1}{2})$ decays faster than any power of $|\Im w|$, uniformly in the strip, the contour in (A.20) may be moved to $\Re w = \sigma_2$. No poles are crossed.

Step 6: assembly. On $\Re w = \sigma_2$, insert (A.24) at $s = w + \frac{1}{2}$, then (A.35) and (A.32) at $s' = \frac{3}{2} - w$, and set $t = \frac{1}{2} - w$. Then $\frac{3}{2} - w = 1 + t$, $w + \frac{5}{6} = \frac{4}{3} - t$, $w + \frac{7}{6} = \frac{5}{3} - t$, $N(c)^{2-2s} = N(c)^{2t}$ and $X^{w-1/2} = X^{-t}$. The constants combine as

$$\frac{-\frac{\pi}{2}(2\pi)^{-2-2t}}{\frac{3\pi}{2}\left(\frac{27}{(2\pi)^2}\right)^{1-t}} = -\frac{1}{81} \left(\frac{(2\pi)^4}{27} \right)^{-t}. \quad (\text{A.38})$$

With $R(t)$ the gamma quotient of (8.5), this gives

$$\begin{aligned} T(X; \Psi) &= -\frac{1}{81} \sum_{\mathbf{h}} c_F(\mathbf{h}) \overline{\kappa(g_{1,\mathbf{h}})} \overline{\alpha(c)}^4 \sum_{0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]} \frac{d_\sigma(\ell) \alpha(\ell)^2}{\sqrt{N(\ell)}} \tilde{e}\left(-\frac{\delta \ell}{c}\right) \\ &\quad \times \frac{1}{2\pi i} \int_{(\frac{1}{2}-\sigma_2)} \hat{V}(-t) R(t) \left(\frac{(2\pi)^4 N(\ell) X}{27 N(c)^2} \right)^{-t} dt, \end{aligned} \quad (\text{A.39})$$

where c, σ, δ are the data of $\mathbf{h}$. The interchange of the ℓ -sum and the t -integral is justified by the absolute convergence in (A.32) (note $\Re(1+t) > \frac{3}{2}$), the rapid decay of $\hat{V}(-t)$ and the polynomial bound (A.8). For each ℓ the t -integrand is holomorphic in $\Re t > -\frac{4}{3}$ and decays rapidly in vertical strips. So its contour may be moved to $\Re t = 0$, and by (8.5) the integral becomes $V^\sharp(N(\ell)X/N(c)^2)$.

Finally fix a group $(h_0, \mathcal{A})$. On it $c, \sigma, \psi, \kappa_0, \sigma_\pi$ and ϵ_π are fixed (Step 2). For $\pi \in \mathcal{A}$, h_π runs over the nonzero residues modulo π ; for $\pi \in \mathcal{P} \setminus \mathcal{A}$, $h_\pi = 0$ and $j_\pi = 0$. By Step 2(a),(c), $\overline{\kappa(g_1)} = \overline{\kappa_0} \prod_{\pi \in \mathcal{A}} \chi_\pi(\sigma_\pi h_\pi)^{-2}$. Insert this, (A.17) and

$$c_F(\mathbf{h}) = \hat{\phi}(h_0) \prod_{\pi \in \mathcal{P} \setminus \mathcal{A}} (1 - N(\pi)^{-1}) \prod_{\pi \in \mathcal{A}} C_{\pi, j_\pi}(h_\pi). \quad (\text{A.40})$$

The sum over the h_π factors into the local sums (A.18) with $x = \lambda^4 \ell$. The group therefore contributes (A.2) with

$$C = -\frac{\overline{\alpha(c)}^4}{81} \hat{\phi}(h_0) \overline{\kappa_0} \prod_{\pi \in \mathcal{P} \setminus \mathcal{A}} (1 - N(\pi)^{-1}) \prod_{\pi \in \mathcal{A}} \chi_\pi(\sigma_\pi)^{-2} \varpi_{\pi, j_\pi}, \quad (\text{A.41})$$

so $|C| \leq 1/81$. The zeta case has $-\frac{i}{81} \overline{\alpha(c)}^2$ in place of $-\frac{1}{81} \overline{\alpha(c)}^4$, with the same remaining factors [35, App. A.2]. Each series (A.2) converges absolutely. Indeed $|B_{\pi, j}| \leq N(\pi)^{1/2}$, and Lemma 13 with $A = 2$ bounds the series by a constant times $\sum_\ell |d_\sigma(\ell)| N(\ell)^{-5/2}$, which is finite by the estimate after (A.32). This proves Proposition 8.

B The CM transformation formula at full generality

This appendix writes out item 1 of Section 16. Part II uses the transformation formula for $\partial_{\bar{z}}^2 \theta$ in the generality of [36, Prop. 5.1]:

- a general modulus supported on S ;
- tests that need not have compact support, such as the Gaussian of the probe;
- the exact dependence of every phase on the moving primes.

It also uses the marked form [36, Cor. 14.1] and three estimates built on it. Appendix A proves the formula for the twists of Part I. Here we extend that proof, reusing its Steps 3–6, and then go through [36, Sections 5 and 14–15] statement by statement.

The rule is the one used throughout Part II. A step of [36] that involves neither the derivative of θ , nor the gamma factors, nor an angular factor is identical in the CM case, and we cite it by its number there. A step that involves any of these is written out.

Notation. We use the conventions of [36, Section 5.1]: $\check{e}(z) = \exp(2\pi i(z + \bar{z}))$, $e(z) = \check{e}(z/\lambda)$, and every power of a residue symbol vanishes at nonunits, so that $\chi_p(x)^0 = \mathbf{1}_{p \nmid x}$. The set S is finite and contains the primes above 6; a *good* prime is a primary prime outside S . To avoid clashes with Part II we rename some objects of [36]:

- its norm q_n is $N(n)$, and its dual index μ is ℓ ;
- its scalar ζ is C , as in (A.2);
- its Gauss sums $\tau_{p,m}^\pm$ are $\mathfrak{g}_{p,m}^\pm$, and its $\omega_{p,j}$ is $\varpi_{p,j}$, as in Appendix A;
- its constant K is written out as $(2\pi)^4/27$.

The kernels with one and two derivatives are

$$R_1(t) = \prod_{\pm} \frac{\Gamma(1+t \pm \frac{1}{6})}{\Gamma(1-t \pm \frac{1}{6})}, \quad R_2(t) = \frac{\Gamma(\frac{4}{3}+t)\Gamma(\frac{5}{3}+t)}{\Gamma(\frac{4}{3}-t)\Gamma(\frac{5}{3}-t)}. \quad (\text{B.1})$$

R_1 is the kernel of [36], and R_2 is that of (8.5).

B.1 The statement

Arithmetic data. These are those of [36, Proposition 5.1].

- $\mathcal{X}$ is a fixed finite family of finite-ray characters with conductors supported on S .
- $\mathfrak{E}$ is an integral modulus with prime support exactly S , divisible by all these conductors. Each member of $\mathcal{X}$ is extended by zero to the primary elements that are not prime to $\mathfrak{E}$.
- $L \in \mathbb{Z}[\omega] \setminus \{0\}$ has prime support S , with $(18) \mid (L)$ and $\mathfrak{E} \mid (L)$, and for every $\Psi_0 \in \mathcal{X}$ the function

$$\phi(x) = \begin{cases} \chi_x(\lambda)^2 \Psi_0(x), & x \equiv 1 \pmod{3}, (x, \mathfrak{E}) = 1, \\ 0, & \text{otherwise,} \end{cases} \quad (\text{B.2})$$

is L -periodic.

- $\Psi_0 \in \mathcal{X}$; $\mathcal{P}$ is a finite set of primary primes not dividing L ; $j_p \in \{0, \dots, 5\}$ for $p \in \mathcal{P}$; and $\Psi(n) = \Psi_0(n) \prod_{p \in \mathcal{P}} \chi_p(n)^{j_p}$ on primary n .

Tests. A *log-Schwartz test* is a function $V \in C^\infty(0, \infty)$ such that $(x\partial_x)^j V(x) = O_{C,j}(x^C)$ as $x \rightarrow 0$ and $O_{C,j}(x^{-C})$ as $x \rightarrow \infty$, for every $C > 0$ and every integer $j \geq 0$ [36, Proposition 5.1]. Its Mellin transform $\widehat{V}(t) = \int_0^\infty V(x)x^{t-1}dx$ is entire. Integration by parts in $\log x$ gives, uniformly for η in a compact interval I (in this appendix η denotes a real number, as in [36], not the finite-order character of Part II),

$$|\widehat{V}(\eta + iu)| \ll_{I,N} (1 + |u|)^{-N} \sum_{j=0}^N \int_0^\infty x^\eta |(x\partial_x)^j V(x)| \frac{dx}{x} \quad (\text{B.3})$$

for every $N \geq 0$; this is [36, eq. (4.11)], whose boundary terms vanish for these tests. For $A \geq 0$ and integers $B \geq 0$ put, as in [36, Proposition 5.1],

$$\mathfrak{M}_{A,B}(V) = \sup_{-A \leq \eta \leq 1/4} \int_{\mathbb{R}} (1 + |u|)^B |\widehat{V}(\eta + iu)| du. \quad (\text{B.4})$$

The Gaussian $V_G(y) = (2\sqrt{\pi})^{-1} \exp(-\frac{1}{4}(\log y)^2)$, with $\widehat{V}_G(t) = e^{t^2}$, is a log-Schwartz test, and so is $\sqrt{y} W(y)$ for every $W \in C_c^\infty(0, \infty)$.

The CM objects. For $\Re s > 1$ put

$$\mathcal{D}_2(s, \Psi) = \sum_{\substack{n \equiv 1 \pmod{3} \\ (n, \mathfrak{E})=1}}^* \frac{\gamma_2(n) \overline{\alpha(n)}^2 \Psi(n)}{N(n)^s}, \quad \mathcal{L}_2(s, \Psi) = \sum_{\substack{b \equiv 1 \pmod{3} \\ (b, \mathfrak{E})=1}} \frac{\overline{\alpha(b)}^6 \Psi(b)^3}{N(b)^{3s-1/2}}, \quad (\text{B.5})$$

and $\mathcal{T}_2 = \mathcal{D}_2 \mathcal{L}_2$. These are the series D, L_0, T of [36, Proposition 5.1] with $\bar{\alpha}$ replaced by $\bar{\alpha}^2$. For the data of Appendix A, $\mathcal{T}_2(w, \Psi)$ is the series $\mathcal{T}(w)$ of Step 3 there. The completed sum is

$$T_V(X; \Psi) = \sum_{\substack{n \text{ sf}, b \\ n, b \equiv 1 \pmod{3}, (nb, \mathfrak{E})=1}} \frac{\gamma_2(n) \overline{\alpha(nb^3)}^2 \Psi(nb^3)}{\sqrt{N(n)} N(b)} V\left(\frac{N(n)N(b)^3}{X}\right) \quad (X > 0). \quad (\text{B.6})$$

Here n and b may share primes, and Ψ acts on the whole index nb^3 , with its zeros. For $V(y) = \sqrt{y} W(y)$ this is the sum $T(X; \Psi)$ of (8.1). The reflected test is

$$V^\sharp(x) = \frac{1}{2\pi i} \int_{(\sigma_0)} \widehat{V}(-t) R_2(t) \left(\frac{(2\pi)^4 x}{27}\right)^{-t} dt \quad (\sigma_0 > -\frac{4}{3}); \quad (\text{B.7})$$

for $\sigma_0 = 0$ it is (8.5).

Proposition 9 (CM completed reflection; CM form of [36, Prop. 5.1]). *Let V be a log-Schwartz test and $X > 0$.*

(a) *The series defining $T_V(X; \Psi)$ converges absolutely, and for every $a > 1$*

$$T_V(X; \Psi) = \frac{1}{2\pi i} \int_{(a)} \widehat{V}(s - \frac{1}{2}) X^{s-1/2} \mathcal{T}_2(s, \Psi) ds. \quad (\text{B.8})$$

(b) *$\mathcal{T}_2(\cdot, \Psi)$ continues to an entire function with polynomial growth on every vertical strip.*

- (c) The integral defining $V^\sharp$ converges absolutely and does not depend on $\sigma_0 > -\frac{4}{3}$. Take the terms, the cusp coefficient functions d_σ , the additive characters ϑ , the local factors B_p and the data $c = c_F, \kappa_F, \sigma_p, \epsilon_p, \varpi_{p,j}$ of [36, Prop. 5.1, items (1)–(3)] for the same arithmetic data. Then

$$T_V(X; \Psi) = \sum_{\text{terms}} C \sum_{0 \neq \ell \in \lambda^{-4} \mathbb{Z}[\omega]} \frac{d_\sigma(\ell) \alpha(\ell)^2 \vartheta(\lambda^4 \ell)}{\sqrt{N(\ell)}} \prod_{p \text{ active}} B_p(\lambda^4 \ell) V^\sharp\left(\frac{N(\ell)X}{N(c)^2}\right), \quad (\text{B.9})$$

where every inner series converges absolutely and

$$C = -\frac{\overline{\alpha(c)}^4}{81} \widehat{\phi}(h_0) \overline{\kappa_F} \prod_{p \text{ inactive}} (1 - N(p)^{-1}) \prod_{p \text{ active}} \chi_p(\sigma_p)^{-2} \varpi_{p,j_p}. \quad (\text{B.10})$$

Thus $C = -i \overline{\alpha(c)}^2 \zeta$, where ζ is the scalar of the same term in [36, eq. (5.3)], and $|C| \leq \frac{1}{81}$.

- (d) The assertions of [36, Proposition 5.1, item (3)] hold for these terms, with the same factor $1 - N(p)^{-1}$ in the branch compatibility. The cross-prime identity [36, eq. (5.17)], $\chi_p(\sigma_p)^{-2} \varpi_{p,j} = \xi_{p,j} \chi_p(c/p)^{2j+2}$ with $|\xi_{p,j}| = 1$ depending only on p and j , holds unchanged. Compared with the scalar of [36], the only new factor in (B.10) is $-i \overline{\alpha(c)}^2$, and $\overline{\alpha(c)}^4 = \overline{\alpha(c_F)}^4 \prod_{p \text{ active}} \overline{\alpha(p)}^4$.
- (e) For $A \geq 0$ and integers $j, J \geq 0$, the two bounds [36, eqs. (5.4) and (5.5)] hold for this $V^\sharp$, with the same seminorms. In particular

$$|(x \partial_x)^j V^\sharp(x)| \ll_{A,j} \min\{x^{1/4}, (1+x)^{-A}\} \mathfrak{M}_{A, \lceil 4A \rceil + j + 2}(V) \quad (x > 0). \quad (\text{B.11})$$

The bound [36, eq. (5.6)] for $\mathfrak{M}_{A,B}$ and the twist bound after it concern V alone and hold as stated there.

So the formula of [36] changes in five places: the derivative $\partial_{\bar{z}}^2$, the angular factor $\overline{\alpha(nb^3)}^2$ of the completed index, the dual factor $\alpha(\ell)^2$, the kernel R_2 , and the prefactor $-\overline{\alpha(c)}^4/81$ in place of $-i \overline{\alpha(c)}^2/81$. For $L = 72$, $\Psi_0 = \xi_0$ and $V(y) = \sqrt{y} W(y)$ with $W \in C_c^\infty(I)$, Proposition 9(c) is a formula of the form of Proposition 8, with c_F, κ_F and ϑ in the roles of c_0, κ_0 and ψ , and C of the form (A.41).

B.2 Proof of Proposition 9

The identical part. The proof of [36, Prop. 5.1] has six paragraphs. The first four, from *The fixed theta input* to *The reflected additive phase*, work with the translated theta values $\theta(z + z_{\mathbf{h}}, v)$ and never differentiate them. We use them without change:

- the theta input, the three cusp functions, their coefficient functions d_σ with the support and size bound [36, eq. (5.2)], the infinity coefficient [36, eqs. (5.9) and (5.10)], and the choice of L ([36, proof of Proposition 5.1, *The fixed theta input*]);
- the finite Fourier inversion with all zero extensions, giving

$$\mathcal{F}(z, v) = \sum_{\mathbf{h}} C(\mathbf{h}) \overline{\theta(z + z_{\mathbf{h}}, v)}, \quad z_{\mathbf{h}} = \lambda^2 \left(\frac{h_0}{L} + \sum_{p \in \mathcal{P}} \frac{h_p}{p} \right), \quad C(\mathbf{h}) = \widehat{\phi}(h_0) \prod_p C_{p,j_p}(h_p), \quad (\text{B.12})$$

with the whole-index mask $(nb, \mathfrak{E}) = 1$ ([36, proof of Proposition 5.1, *Finite Fourier inversion with the zero extensions*]);

- the reduced fractions $z_{\mathbf{h}} = a/c$ with $c = c_{Fr}$ ([36, proof of Proposition 5.1, *Finite Fourier inversion with the zero extensions*]), the matrices $g = \begin{pmatrix} a & b' \\ c & \delta' \end{pmatrix} = GH \in \mathrm{SL}_2(\mathbb{Z}[\omega])$ with $G \in \Gamma(3)$ (the group written $\Gamma_1(3)$ in [36]), and the multiplier formula [36, eq. (5.15)];
- the additive phase [36, eq. (5.16)], the local Gauss sums that produce $\varpi_{p,j} B_p$, the counts of terms, the branch compatibility and the cross-prime phase ([36, proof of Proposition 5.1, *The reflected additive phase*]).

From this part we need two facts. For every $\mathbf{h}$, with $z_{\mathbf{h}} = a/c$ and g, G, H as above,

$$\overline{\theta(a/c + z, v)} = \overline{\kappa(G)} \overline{\theta(Hg^{-1}(a/c + z, v))} \quad (\text{B.13})$$

([36, proof of Proposition 5.1, *Mellin normalization and continuation*]). And, by [36, eq. (5.7)],

$$\overline{\theta(H(z', v'))} = C_H(v') + \sum_{0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]} d_{\sigma}(\ell) v' K_{1/3}(4\pi|\ell|v') \check{e}(\ell z'), \quad (\text{B.14})$$

where C_H does not depend on z' and d_{σ} , $\sigma \in \{0, +, -\}$, is the one of the three fixed coefficient functions that belongs to H ; these are the functions d_{σ} of Appendix A, and d_0 belongs to $H = I$.

The last two paragraphs of that proof, *Mellin normalization and continuation* and *Uniform estimates for the transformed test*, use the single derivative. We redo it with two. Steps 1–4 below are Steps 3–6 of Section A.2, which we reuse; we record only what changes at the present generality.

Step 1: the direct transform. Put $f(v) = \partial_z^2 \mathcal{F}(z, v)|_{z=0}$ and $\mathcal{J}_2(s) = \int_0^\infty f(v) v^{2s-1} dv$. By [36, proof of Proposition 5.1, *Finite Fourier inversion with the zero extensions*], the expansion of $\mathcal{F}$ at the identity cusp has the constant term $\sum_{\mathbf{h}} C(\mathbf{h}) C_I(v) = \phi(0) \prod_p \chi_p(0)^{j_p} C_I(v) = 0$, and its other terms are supported on $\ell \in \lambda^{-3}\mathbb{Z}[\omega] \setminus \{0\}$, where the coefficient is $d_0(\ell) \phi(\lambda^3 \ell) \prod_p \chi_p(\lambda^3 \ell)^{j_p}$. This coefficient vanishes unless $\lambda^3 \ell = nb^3$ with n squarefree, $n, b \equiv 1 \pmod{3}$ and $(nb, \mathfrak{E}) = 1$, and then it equals $3^{5/2} |b| \gamma_2(n) \Psi(n) \Psi(b)^3$. This replaces (A.12); n and b may share primes. In every case its absolute value is at most $3^{5/2} |b|$. Since $|\ell| \geq 3^{-3/2}$ and $K_{1/3}(y) \ll e^{-y}$ for $y \geq 1$, the expansion may be differentiated termwise. Step 3 of Section A.2 uses only this coefficient formula and this bound, so it gives, with absolutely convergent termwise integrals,

$$\mathcal{J}_2(s) = \frac{3\pi}{2} \left(\frac{27}{(2\pi)^2} \right)^s \Gamma(s + \tfrac{1}{3}) \Gamma(s + \tfrac{2}{3}) \mathcal{T}_2(s - \tfrac{1}{2}, \Psi) \quad (\Re s > \tfrac{3}{2}). \quad (\text{B.15})$$

In [36, eq. (5.18)] the same integral with one derivative is $\frac{3^{5/2}}{4} (\frac{27}{(2\pi)^2})^s \Gamma(s + \tfrac{1}{3}) \Gamma(s + \tfrac{2}{3}) T(s, \Psi)$. The second derivative brings the extra factor $2\pi i \bar{\ell}$; its modulus $2\pi|\ell|$ shifts the Dirichlet series by $\frac{1}{2}$, and its phase turns $\bar{\alpha}$ into $\bar{\alpha}^2$.

Step 2: the cusps. Fix $\mathbf{h}$. Step 4 of Section A.2 uses only that $g \in \mathrm{SL}_2(\mathbb{Z}[\omega])$ has lower row (c, δ') and the formula for $g^{-1}(a/c + z, v)$ ([36, proof of Proposition 5.1, *Mellin normalization and continuation*] and [2, (5.1), (5.19)]). With (B.13) it gives

$$\partial_z^2 \overline{\theta(a/c + z, v)}|_{z=0} = \overline{\kappa(G)} (cv)^{-4} \mathcal{G}_{\mathbf{h}} \left(\frac{1}{N(c)v} \right), \quad (\text{B.16})$$

where

$$\mathcal{G}_{\mathbf{h}}(v') = -4\pi^2 \sum_{0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]} d_{\sigma}(\ell) N(\ell) \alpha(\ell)^2 v' K_{1/3}(4\pi|\ell|v') \check{e}\left(-\frac{\delta' \ell}{c}\right). \quad (\text{B.17})$$

The operator $\partial_{z'}^2$ removes $C_H(v')$. This replaces the formula of [36, proof of Proposition 5.1, *Mellin normalization and continuation*], which has $-\overline{\kappa(G)}(cv)^{-2}$ and one derivative $\partial_{z'}$.

Step 3: continuation and growth. By Step 1, $f(v)$ decays exponentially as $v \rightarrow \infty$. As $v \rightarrow 0$, $f(v) = \sum_{\mathbf{h}} C(\mathbf{h}) \overline{\kappa(G_{\mathbf{h}})} (c_{\mathbf{h}} v)^{-4} \mathcal{G}_{\mathbf{h}}(1/(N(c_{\mathbf{h}})v))$ decays exponentially as well, because $|\ell| \geq \frac{1}{9}$ on $\lambda^{-4}\mathbb{Z}[\omega] \setminus \{0\}$ and d_{σ} is polynomially bounded. So $\mathcal{J}_2$ is entire and bounded on vertical strips. For $\Re s' > \frac{3}{2}$,

$$\mathcal{J}_{\mathbf{h}}^{\vee}(s') := \int_0^{\infty} \mathcal{G}_{\mathbf{h}}(v') v'^{2s'-1} dv' = -\frac{\pi}{2} (2\pi)^{-2s'} \Gamma(s' + \frac{1}{3}) \Gamma(s' + \frac{2}{3}) \sum_{0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]} \frac{d_{\sigma}(\ell) \alpha(\ell)^2}{N(\ell)^{s'-1/2}} \check{e}\left(-\frac{\delta' \ell}{c}\right), \quad (\text{B.18})$$

with absolute convergence of the series and of the termwise integrals. Indeed, writing $\ell = u\lambda^k n b^3$ as in [36, eq. (5.2)],

$$\sum_{\ell} |d_{\sigma}(\ell)| N(\ell)^{1/2-\Re s'} \leq 6 \cdot 27 \sum_{k \geq -4} 3^{k(2/3-\Re s')} \sum_n^* N(n)^{1/2-\Re s'} \sum_b N(b)^{2-3\Re s'} < \infty. \quad (\text{B.19})$$

Here n and b run over all primary elements, n squarefree, including those divisible by primes of S other than λ , and pairs that share primes; the size bound covers all of them ([36, Proposition 5.1; proof of Proposition 5.1, *The fixed theta input*]). This is the majorant of [36, proof of Proposition 5.1, *Mellin normalization and continuation*] in the variable $s' - \frac{1}{2}$. For $\Re s < \frac{1}{2}$ the substitution $v = 1/(N(c)v')$ of Step 5 of the proof in Appendix A (Section A.2) gives

$$\mathcal{J}_2(s) = \sum_{\mathbf{h}} C(\mathbf{h}) \overline{\kappa(G_{\mathbf{h}})} \overline{\alpha(c_{\mathbf{h}})}^4 N(c_{\mathbf{h}})^{2-2s} \mathcal{J}_{\mathbf{h}}^{\vee}(2-s) \quad (\Re s < \frac{1}{2}), \quad (\text{B.20})$$

where $c^{-4}N(c)^2 = \overline{\alpha(c)}^4$. By (B.15),

$$\mathcal{T}_2(w, \Psi) = \mathcal{J}_2(w + \frac{1}{2}) \Big/ \left(\frac{3\pi}{2} \left(\frac{27}{(2\pi)^2} \right)^{w+1/2} \Gamma(w + \frac{5}{6}) \Gamma(w + \frac{7}{6}) \right) \quad (\text{B.21})$$

continues to an entire function. Let $\sigma_2 < 0 < 1 < \sigma_1$. On $\Re w = \sigma_1$, $\mathcal{T}_2$ is bounded. On $\Re w = \sigma_2$, (B.20) and (B.18) at $s' = \frac{3}{2} - w$ and Stirling's formula give $|\mathcal{T}_2(w)| \ll (1 + |\Im w|)^{2-4\sigma_2}$, as in Step 5 of the proof in Appendix A. In the strip $\sigma_2 \leq \Re w \leq \sigma_1$, $\mathcal{J}_2(w + \frac{1}{2})$ is bounded and $1/(\Gamma(w + \frac{5}{6})\Gamma(w + \frac{7}{6})) \ll e^{\pi|\Im w|}(1 + |\Im w|)^{O(1)}$. The Phragmén–Lindelöf principle therefore gives $|\mathcal{T}_2(w)| \ll (1 + |\Im w|)^{2-4\sigma_2}$ throughout the strip. Every vertical strip lies in such a strip, which proves (b). As in [36], the constants depend on $L, \Psi_0, \mathcal{P}, (j_p)$ and the strip.

Step 4: a log-Schwartz test. This is the step where the generality of the test matters. Since $(x\partial_x)^0 V = V$ is $O(x^{-1})$ at infinity and bounded at zero, $|V(y)| \ll_V y^{-1}$ for all $y > 0$. Hence

$$\sum_{n,b} \frac{|V(N(n)N(b)^3/X)|}{\sqrt{N(n)}N(b)} \ll_V X \sum_n^* N(n)^{-3/2} \sum_b N(b)^{-4} < \infty. \quad (\text{B.22})$$

By (B.3), $\widehat{V}(w)$ is integrable on every vertical line. Mellin inversion [34, §1.14(iv)] $V(y) = \frac{1}{2\pi i} \int_{(a-1/2)} \widehat{V}(w) y^{-w} dw$ holds, because $u \mapsto e^{(a-1/2)u} V(e^u)$ is a Schwartz function. Since $\mathcal{T}_2$ converges absolutely on $\Re s = a > 1$, Fubini's theorem gives (a).

Now move the contour of (a) from $\Re s = a$ to $\Re s = \sigma_2$, with $-1 \leq \sigma_2 < 0$. By (b) and (B.3), the integrand decays faster than any power of $|\Im s|$, uniformly in the strip, so the horizontal segments contribute nothing in the limit; there are no poles. On $\Re s = \sigma_2$ insert (B.20) and (B.18) with

$s' = \frac{3}{2} - s$, and put $t = \frac{1}{2} - s$. Then $\Gamma(\frac{11}{6} - s)\Gamma(\frac{13}{6} - s)/(\Gamma(s + \frac{5}{6})\Gamma(s + \frac{7}{6})) = R_2(t)$, $N(c)^{1-2s} = N(c)^{2t}$, $N(\ell)^{s-1} = N(\ell)^{-1/2-t}$, $X^{s-1/2} = X^{-t}$, and the constants combine as in Step 6 of Section A.2:

$$\frac{-\frac{\pi}{2}(2\pi)^{-2-2t}}{\frac{3\pi}{2}\left(\frac{27}{(2\pi)^2}\right)^{1-t}} = -\frac{1}{81}\left(\frac{(2\pi)^4}{27}\right)^{-t}. \quad (\text{B.23})$$

This gives

$$\begin{aligned} T_V(X; \Psi) = & -\frac{1}{81} \sum_{\mathbf{h}} C(\mathbf{h}) \overline{\kappa(G_{\mathbf{h}})} \overline{\alpha(c)}^4 \sum_{0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]} \frac{d_{\sigma}(\ell) \alpha(\ell)^2}{\sqrt{N(\ell)}} \check{e}\left(-\frac{\delta' \ell}{c}\right) \\ & \times \frac{1}{2\pi i} \int_{(\frac{1}{2}-\sigma_2)} \widehat{V}(-t) R_2(t) \left(\frac{(2\pi)^4 N(\ell) X}{27 N(c)^2}\right)^{-t} dt, \end{aligned} \quad (\text{B.24})$$

with c, H, σ, δ' the data of $\mathbf{h}$. The interchange of the ℓ -sum and the t -integral is justified by the majorant of Step 3 at $\Re s' = 1 + \Re t = \frac{3}{2} - \sigma_2 > \frac{3}{2}$, by (B.3), and by the bound $|R_2(\eta + iu)| \ll (1 + |u|)^{4\eta}$ of Step 5 below. By Step 5 below, each t -integral equals $V^{\sharp}(N(\ell)X/N(c)^2)$.

Finally group the $\mathbf{h}$ by h_0 and the active set. On a group, c, H, σ, κ_F and the residue of δ' that determines ϑ are fixed, and so is $\overline{\alpha(c)}^4$. The sum over the active frequencies of $C(\mathbf{h})\overline{\kappa(G_{\mathbf{h}})}\check{e}(-\delta'\ell/c)$ is computed in [36, proof of Proposition 5.1, *The reflected additive phase*]; it does not involve the derivative. It gives $\widehat{\phi}(h_0)\overline{\kappa_F}\vartheta(\lambda^4\ell)\prod_{\text{inactive}}(1 - N(p)^{-1})\prod_{\text{active}}\chi_p(\sigma_p)^{-2}\varpi_{p,j_p}B_p(\lambda^4\ell)$, which proves (B.9) and (B.10). Every factor of C other than $-1/81$ has modulus at most one, and $-\frac{1}{81}\overline{\alpha(c)}^4 = -i\overline{\alpha(c)}^2 \cdot (-\frac{i}{81}\overline{\alpha(c)}^2)$. Each inner series in (B.9) converges absolutely by (e) with $A = 2$, the bound $|B_p| \leq N(p)^{1/2}$ and the majorant of Step 3, as at the end of Section A.2. Part (d) holds because d_{σ}, ϑ , the local data and the inactive factors are those of [36], while $\overline{\alpha(c)}^4$ and the kernel depend on a branch only through c , which an inactive prime does not change. The factorization of $\overline{\alpha(c)}^4$ holds because $c = c_F r$ is a product of elements and α is multiplicative.

Step 5: the reflected test. The poles of the numerator of R_2 are at $-\frac{4}{3} - k$ and $-\frac{5}{3} - k$ ($k \geq 0$), and $1/\Gamma$ is entire, so R_2 is holomorphic in $\Re t > -\frac{4}{3}$. For $\eta_0 > -\frac{4}{3}$ and $A \geq 0$, Stirling's formula for $|u| \geq 1$ and continuity for $|u| \leq 1$ give

$$|R_2(\eta + iu)| \ll_{\eta_0, A} (1 + |u|)^{4\eta} \quad (\eta_0 \leq \eta \leq A), \quad (\text{B.25})$$

which for $\eta_0 = -\frac{1}{4}$ is (A.8). Together with (B.3), this shows that the integral defining $V^{\sharp}$ converges absolutely on every line $\Re t > -\frac{4}{3}$, and rectangle shifts show that its value does not depend on the line. This is (c) for $V^{\sharp}$. The proof of [36, eqs. (5.4) and (5.5)] in [36, proof of Proposition 5.1, *Uniform estimates for the transformed test*] uses only two properties of its kernel R_1 : holomorphy in $\Re t > -\frac{5}{6}$, a region containing the strip $-\frac{1}{4} \leq \Re t \leq A$, and the bound (B.25) with the exponent 4η on that strip. R_2 has both, so the proof applies verbatim and gives (e). The rest of [36, proof of Proposition 5.1, *Uniform estimates for the transformed test*] concerns V alone. This completes the proof.

The CM kernel is holomorphic further to the left than R_1 , so $V^{\sharp}(x) = O(x^{4/3})$ as $x \rightarrow 0$; we do not use this, and the exponent $\frac{1}{4}$ in (e) is kept so that every later estimate of [36] applies with its own exponents.

What the later arguments use. The proofs in [36, Sections 5.2–5.5, Sections 14–15] use the reflection only through the following properties. Each holds for Proposition 9, by the part indicated.

- (R1) *Shape and branches*: a finite expansion over h_0 , active sets and Ramanujan splittings, with conductor $c = c_{Fr}$, the canonical d_σ, ϑ, B_p , and the branch compatibility at inactive primes. By (c) and (d) these are literally those of [36].
- (R2) *The scalar*: C does not depend on ℓ , and $|C| \leq \frac{1}{81}$. Its dependence on the active primes is that of [36] times $\overline{\alpha(c)}^2$, which is completely multiplicative in the active primes (part (d)).
- (R3) *The dual coefficient*: $d_\sigma(\ell)\alpha(\ell)^2\vartheta(\lambda^4\ell)N(\ell)^{-1/2}$, where $|\alpha(\ell)^2| = 1$ and $\alpha(\ell)^2$ is a function of ℓ alone. The support and the size bound of d_σ are those of [36].
- (R4) *The kernel argument* $N(\ell)X/N(c)^2$, which is that of [36].
- (R5) *The kernel bounds* (e), with the same seminorm orders as in [36].
- (R6) *The Mellin representation* (a) of the completed sum, for every log-Schwartz test.

In the proofs of [36], the factor $\alpha(\ell)$ of (R3) is always absorbed into a coefficient of the dual index, and the angular part of the scalar in (R2) is always split into a row factor and a tuple factor. The same operations apply to $\alpha(\ell)^2$ and to $\overline{\alpha(c)}^4$. No norm power changes, so no exponent changes.

B.3 Unmarked completed rows

For a finite-ray character ν with conductor supported on S , extended by zero away from the primary elements prime to S , a nonzero $m \in \mathbb{Z}[\omega]$, a log-Schwartz test V and $X > 0$, put

$$\mathcal{C}_V^{(2)}(X; m, \nu) = \sum_{\substack{c \text{ sf}, n \\ c, n \equiv 1 \pmod{3}, (cn, S)=1}} \frac{\gamma_2(c) \overline{\alpha(cn^3)}^2 \nu(cn^3) \chi_{cn^3}(m)}{\sqrt{N(c)} N(n)} V\left(\frac{N(c)N(n)^3}{X}\right). \quad (\text{B.26})$$

This is [36, eq. (5.30)] with $\overline{\alpha(cn^3)}$ replaced by $\overline{\alpha(cn^3)}^2$; in Section 12 it is the completed row B_m , with $X = Z$. The table lists every statement of [36, Sections 5.2–5.5] and its status for this sum.

Section of [36]	statement	CM status
5.2	Lemma 5.2	identical: concerns only $A \mapsto \chi_A(m)$
5.3	Lemma 5.3, (5.24), Gaussian remark	identical: uses only (R5), and the CM kernel has the same seminorm orders
5.3	Lemma 5.4	identical; for $K = V^\sharp$ the constant $C_A \ll \mathfrak{M}_{A, [4A]+2}(V)$ comes from (R5)
5.4	(5.26), Lemma 5.5	identical: no reflection; the coefficients $\beta(n, b)$ are arbitrary
5.5	(5.30), (5.31)	replaced by (B.26); its Mellin form is (R6) with $\Psi = \nu \chi_\bullet(m)$, by the same zero-preserving identity $\chi_{cn^3}(m) = \chi_c(m) \chi_n(m)^3$
5.5	heuristic for squarefree rows	identical, with $\alpha(\ell)^2$ in the dual coefficient
5.5	Definition 5.6	the specified term is now a term of (B.9)
5.5	Lemma 5.7	statement identical; in its proof put $\delta(n, b) = d_\sigma(\ell) \alpha(\ell)^2 \vartheta(\lambda^4 \ell) / (27 \cdot 3^{k/6} B_F b)$ and $\rho(R) = 81 C_R \chi_R(x_F)^3$, still bounded by one by (R2), (R3); the rest is verbatim
5.5	retention and tail removal	identical: uses $ d_\sigma(\ell) N(\ell)^{-1/2} \leq 27 N(\lambda)^{4/3}$, $ B_p \leq N(p)^{1/2}$, (R4) and (R5)
5.5	(5.35)	identical
5.5	Lemma 5.8	statement and proof identical for $\mathcal{C}_V^{(2)}$ (Lemma 14)
5.5	Gaussian test, (5.39)	identical

Lemma 14 (CM form of [36, Lemma 5.8]). *Let ν range over a fixed finite family as above. Fix bounded ranges for $M \geq 0$ and $N \in \mathbb{R}$, a constant $C_m \geq 1$ and fixed dyadic support comparisons, and let m_{pow} be the maximal powerful ideal factor of m . For every $\varepsilon > 0$ there are $A \geq 0$ and an integer $J \geq 0$ such that, for every log-Schwartz test V , every dyadic range $N(m_{\text{pow}}) \asymp Z^O$ with $O \geq 0$, and all sufficiently large Z ,*

$$\sum_{\substack{0 < N(m) \leq C_m Z^M \\ N(m_{\text{pow}}) \asymp Z^O}} |\mathcal{C}_V^{(2)}(Z^N; m, \nu)|^2 \ll Z^{O/2 + \max\{M-O, 2M-O-N\} + \varepsilon} \mathfrak{M}_{A, J}(V)^2. \quad (\text{B.27})$$

The uniformity is that of [36, Lemma 5.8].

Proof. The proof of [36, Lemma 5.8] applies verbatim. It uses the reflection only through (R1)–(R6), via [36, Lemma 5.7] and the tail argument after it, and the table above records the CM form of each. $\square$

Summing over the $O(\log Z)$ dyads $O \geq 0$, with $N = 1$, gives the precise form of Proposition 7: for every $\varepsilon > 0$ and M in a bounded range there are A, J with

$$\sum_{0 < N(m) \leq Z^M} |\mathcal{C}_V^{(2)}(Z; m, \nu)|^2 \ll Z^{\max\{M, 2M-1\} + \varepsilon} \mathfrak{M}_{A, J}(V)^2, \quad (\text{B.28})$$

since $O/2 + \max\{M - O, 2M - O - 1\} \leq \max\{M, 2M - 1\}$ for $O \geq 0$. With $Q = Z^M$ the right side is $Z^\varepsilon (Q + Q^2/Z) \mathfrak{M}_{A, J}(V)^2$. The test of the probe is the Gaussian, and the finite Fourier sum over $\theta \in \widehat{T}$ costs a fixed factor, as in [36, Section 6.1].

B.4 Marks

Proposition 10 (CM form of [36, Cor. 14.1]). *Fix an instance of Proposition 9, with local set $\mathcal{P}_0$, exponents j_p , function ϕ and test V , and let $\mathcal{Q}$ be a finite set of good primes disjoint from $\mathcal{P}_0$. Let $T_{V,\mathcal{Q}}(X; \Psi)$ be $T_V(X; \Psi)$ with the factor $\prod_{q \in \mathcal{Q}} \mathbf{1}_{q|nb^3}$ inserted. Then*

$$T_{V,\mathcal{Q}}(X; \Psi) = \sum_{\mathcal{B}} C_{\mathcal{Q}} \sum_{0 \neq \ell \in \lambda^{-4}\mathbb{Z}[\omega]} \frac{d_{\sigma}(\ell) \alpha(\ell)^2 \vartheta(\lambda^4 \ell)}{\sqrt{N(\ell)}} \prod_{p \in \mathcal{A}_0} B_p(\lambda^4 \ell) \prod_{q \in \mathcal{A}_{\mathcal{Q}}} \frac{\chi_q(\lambda^4 \ell)^{-2}}{\sqrt{N(q)}} V^{\sharp}\left(\frac{N(\ell)X}{N(c)^2}\right), \quad (\text{B.29})$$

where a branch $\mathcal{B}$ specifies h_0 , the base active and inactive sets $\mathcal{A}_0, \mathcal{I}_0$ and a partition $\mathcal{Q} = \mathcal{A}_{\mathcal{Q}} \sqcup \mathcal{I}_{\mathcal{Q}}$; $c = c_F(h_0) \prod_{p \in \mathcal{A}_0 \cup \mathcal{A}_{\mathcal{Q}}} p$; $d_{\sigma}, \vartheta, B_p$ are those of Proposition 9 for this active radical; and, with $\sigma_p, \epsilon_p, \varpi_{p,j}$ computed from this c ,

$$C_{\mathcal{Q}} = -\frac{\overline{\alpha(c)}^4}{81} \widehat{\phi}(h_0) \overline{\kappa_F} \prod_{p \in \mathcal{I}_0} (1 - N(p)^{-1}) \prod_{q \in \mathcal{I}_{\mathcal{Q}}} N(q)^{-1} \prod_{p \in \mathcal{A}_0} \chi_p(\sigma_p)^{-2} \varpi_{p,j_p} \prod_{q \in \mathcal{A}_{\mathcal{Q}}} \chi_q(\sigma_q)^{-2} (-\varpi_{q,0}). \quad (\text{B.30})$$

So $C_{\mathcal{Q}} = -i \overline{\alpha(c)}^2 \zeta_{\mathcal{Q}}$ with $\zeta_{\mathcal{Q}}$ of [36, eq. (14.4)], and $|C_{\mathcal{Q}}| \leq \frac{1}{81}$. Every dual series converges absolutely. A marked prime that belongs to $\mathcal{P}_0$ makes the primal term vanish identically.

Proof. The proof of [36, Cor. 14.1] expands $\prod_q (1 - \chi_q(A)^0)$ and applies the reflection to each of the finitely many resulting completed sums. It then combines branches using only the branch compatibility. Both hold for Proposition 9, by (c) and (d), so the proof applies verbatim. The inactive coefficient is $1 - (1 - N(q)^{-1}) = N(q)^{-1}$. The active contribution comes from the subtracted zero-mask transform, with local factor $N(q)^{-1/2} \chi_q(\lambda^4 \ell)^{-2}$ and scalar $-\varpi_{q,0} = \mathfrak{g}_{q,2}^+ \chi_q(\epsilon_q)^{-2}$. $\square$

Separation of the phases. The discussion after the corollary in [36, Section 14.2] carries over. The pair identity [36, eq. (14.5)], $\chi_p(q)^{2j_p+2} \chi_q(p)^{2j_q+2} = (q/p)_3^{j_p+j_q+2}$, is local and unchanged. A residual row prime ($j_p = 1$) and an active mark ($j_q = 0$) therefore have pair factor $(q/p)_3^3 = 1$. Write R for the product of the residual row primes, P for the product of the active marked primes and F_{act} for the frozen active base product, so that $c = c_F F_{\text{act}} R P$. The angular factor of $C_{\mathcal{Q}}$ is

$$\overline{\alpha(c)}^4 = \overline{\alpha(c_F F_{\text{act}})}^4 \cdot \overline{\alpha(R)}^4 \cdot \overline{\alpha(P)}^4, \quad (\text{B.31})$$

a fixed factor times a row factor times a tuple factor. On the dual side, after the unit, the λ -valuation and the extracted frozen factors are fixed, $\ell = u \lambda^{h_{\lambda}} N_F n (B_F b)^3$ and

$$\alpha(\ell)^2 = \alpha(u \lambda^{h_{\lambda}} N_F B_F^3)^2 \alpha(n)^2 \alpha(b)^6, \quad (\text{B.32})$$

a function of the dual index alone. So, after the common smooth separation, the tuple coefficient is a bounded function of the tuple independent of R, n, b , and the dual coefficient is independent of R and P , exactly as stated in [36, Section 14.2].

B.5 Reflected energy and the compensated completed row

The quadratic–cubic norm. [36, Lemma 14.2] is a statement about arbitrary bounded coefficients $a(P)$ and $\beta(n, b)$, proved with the quadratic and cubic large sieves. It involves neither the reflection nor the weight, and it is used without change.

The marked completion. Take the data of [36, Section 14.1]: disjoint slot lists $\mathcal{P}_i$ of good primes with bounded coefficients $a_i(p)$, a fixed puncture ρ , a squarefree ideal f outside S , a multiplicative finite-ray character ν with defining modulus supported on S , $\Psi_k(n) = \nu(n)\rho(n)\chi_n(k)\chi_n(f)^4$, the mark $\mathfrak{d}$ of [36, eq. (14.1)], and $V_*(y) = y^{1/2}W(y)$ with W smooth and annular. The CM marked completion is

$$\mathcal{T}_{\mathfrak{d}}^{(2)}(X; k) = \sum_{\substack{n \text{ sf} \\ b}} \frac{\overline{\alpha(n)}^2 \gamma_2(n) \Psi_k(n) \overline{\alpha(b)}^6 \Psi_k(b)^3}{\sqrt{N(n)} N(b)} \mathfrak{d}(nb^3) V_*\left(\frac{N(n)N(b)^3}{X}\right), \quad (\text{B.33})$$

which is [36, eq. (14.2)] with $\overline{\alpha}, \overline{\alpha}^3$ replaced by $\overline{\alpha}^2, \overline{\alpha}^6$. Its reflection is Proposition 10, applied after the row-sector reduction of [36, Section 14.1], which does not involve the weight.

Lemma 15 (CM form of [36, Lemma 14.3]). *Define the reflected block data as in [36, Section 14.4, Reflected block data], with [36, eq. (14.3)] replaced by the expansion of Proposition 10 for (B.33). Then every assertion of [36, Lemma 14.3] holds verbatim: the retention rule [36, eq. (14.11)] with the same T_d and the same negligible tail, the coefficient exponent [36, eq. (14.12)], and, with $u = \min\{v, z_a, (v + z_a)/3\}$ and E_{ref} of [36, eqs. (14.13) and (14.14)],*

$$\sum_{k_{\text{res}} \text{ sf}} |\mathcal{U}_{v, \ell_b, e_\lambda}(k_{\text{res}})|^2 \ll Z^{E_{\text{ref}} - O/2 + \varepsilon} \quad (\text{B.34})$$

for fixed $k_{\text{pow}}, k_{\text{sup}}$, and $O(Z^{E_{\text{ref}} + \varepsilon})$ after summing them, with the same uniformity. Consequently [36, eqs. (14.17) and (14.18)] hold for $\mathcal{T}_1^{(2)}$:

$$\sum'_{0 < N(k) \ll Z^M} |\mathcal{T}_1^{(2)}(Z^{N_*}; k)|^2 \ll Z^{O/2 + \max\{M - O, 2M - O + Q_\rho + V_f - N_*\} + \varepsilon}, \quad (\text{B.35})$$

where $Q_\rho = \log_Z N(\mathfrak{r}_\rho)$ and $V_f = \log_Z N(f)$ are the lengths called Q and V in [36], and the prime restricts the powerful part of k to its O -dyad.

Proof. We go through the proof of [36, Lemma 14.3] and list where the reflection enters.

- [36, proof of Lemma 14.3]: inactive slots are removed by the triangle inequality, using the inactive coefficient $N(p)^{-1}$ and the fact that an inactive prime is absent from the conductor. This is (R1) for Proposition 10.
- [36, proof of Lemma 14.3]: the frozen-base extraction [36, eq. (5.32)], made simultaneous over the tuples by the separation of phases. Its common coefficient comes from $|d_\sigma(\ell)| \leq 27 \cdot 3^{k/6} |b|$, the normalization $N(\ell)^{-1/2}$ and the local factors; the CM dual coefficient differs by the unimodular $\alpha(\ell)^2$, which goes into $\beta(n, b)$ (R3), and the scalar is separated as in Section B.4 (R2).
- [36, proof of Lemma 14.3]: the kernel argument [36, eq. (14.15)] and the extraction of $m_{\text{ker}} = Z^{-(D_{\text{ker}})_+ / 4}$ by [36, Lemma 5.3]. These use (R4) and (R5).
- [36, proof of Lemma 14.3]: the tail, by [36, Lemma 5.4], uses $|d_\sigma(\ell)| N(\ell)^{-1/2} \leq 27 N(\lambda)^{4/3}$, $|B_p|, |\chi_q^{-2}| \leq N(p)^{1/2}$, (R4) and (R5).
- [36, proof of Lemma 14.3]: Fourier separation and the verification of the two independence hypotheses. In the CM case the tuple scalar acquires $\overline{\alpha(P)}^4$, the row scalar acquires $\overline{\alpha(R)}^4$, and the dual coefficient acquires $\alpha(\ell)^2$, as shown in Section B.4. The independence hypotheses of [36, Lemma 14.2] are preserved.

- [36, proof of Lemma 14.3]: the application of [36, Lemma 14.2] and the count of the frozen row parts. No reflection enters.

None of these changes alters a norm power, so every exponent is the same. The deductions in [36, Section 14.4, *Reflected block data*] use only the lemma and norm inequalities for the rows. $\square$

The compensated completed row. For the CM target $\psi = \overline{\alpha}\eta$, the completed row of the compensated probe ([36, Section 15.1], with the weight (12.1)) is

$$B_{m,\varsigma}^{J,(2)}(Z) = \sum_{\substack{p_i \in \mathcal{P}_i(Z) \\ i \notin J}} \prod_{i \notin J} a(p_i) W_i\left(\frac{N(p_i)}{P_i}\right) \sum_{\theta \in \widehat{T}} a_\theta \frac{1}{2\pi i} \int_{(4)} (ZN(p_{J^c}))^t \Phi(t) \mathcal{T}_{p_{J^c}}^{(2)}\left(t + \frac{1}{2}, \nu_\varsigma \theta \chi_\bullet(m)\right) dt, \quad (\text{B.36})$$

where $\Phi(t) = e^{t^2}$, $\varsigma \in T$ is a ray class (called σ in [36]), $\nu_\varsigma = \eta \Xi^{-1} \mathcal{R}(\cdot, s_\varsigma)$ is the finite-order character of [36, Section 6.1] built from the finite part η , $a(p) = \overline{\psi(p)} = \alpha(p)\eta(p)$ is the slot normalization of Section 13 ([36] has $\overline{\eta(p)}$), and

$$\mathcal{T}_D^{(2)}(s_0, \nu') = \sum_{\substack{c \text{ sf} \\ n}} \gamma_2(c) \overline{\alpha(cn^3)}^2 \nu'(cn^3) \mathbf{1}_{D|cn^3} N(c)^{-s_0} N(n)^{-3s_0+1/2}. \quad (\text{B.37})$$

Lemma 16 (CM form of [36, Lemma 15.1]). *With $d_J = \sum_{i \in J} \ell_i$ (called d in [36, eq. (15.1)]), and M' and $Q = N(b_*)X'Y' \asymp Z^{M'}$ as there, for every $\varepsilon > 0$, every fixed rescaled subset J and every ray class $\varsigma \in T$,*

$$\sum_{0 < N(m) \ll Q} |B_{m,\varsigma}^{J,(2)}(Z)|^2 \ll Z^{M' + ((d_J - 1/6)/4)_+ + \varepsilon} = Z^{M' + \varepsilon}, \quad (\text{B.38})$$

uniformly in the fixed rescaled tuple and with no loss depending on the mesh of the surviving slots. The same holds for any slot coefficients $a(p)$ with $|a(p)| \leq 1$ that are independent of the row and of the other slots.

Proof. The proof of [36, Lemma 15.1] applies verbatim. It uses the Gaussian annular decomposition [36, Lemma 4.6], the smooth calculus [36, Lemma 4.5], the row sectors [36, Lemma 5.2], and the reflected energy, here Lemma 15; the remaining steps are exponent bookkeeping. The target enters in two places only. First, the completed factor is the CM marked completion (B.33), with an annular test at the length N_* ; this is covered by Lemma 15. Second, the slot coefficients are required ([36, proof of Lemma 15.1]) to be bounded, independent of the row and of the other slots, and of product form after separation. This holds for $\alpha(p)\eta(p)$ as for $\eta(p)$. The small-argument exponent $\frac{1}{4}$ of the kernel, used at [36, proof of Lemma 15.1], is that of Proposition 9(e). $\square$

With Lemma 16 in place of [36, Lemma 15.1], the compensated low estimate $|I_{\text{mod}}(Z)| \ll Z^{3/16+\varepsilon}$ follows as in [36, Prop. 15.3], given the additive Gram bound, which does not involve the target.

Numerical checks. The new identities were tested.

- End to end, with an active prime: for the twist $\Psi_0 = \overline{(\lambda/\cdot)}_3$ of Appendix A, $L = 18$, a local prime p of norm 7 with $j_p = 0, \dots, 5$ (both primes of norm 7), and two log-Schwartz tests without compact support, the log-Gaussian and $e^{-y-1/y}$ (with $\widehat{V}(t) = 2K_t(2)$), $T_V(X; \Psi)$ computed from its definition equals both the ungrouped sum over the 252 translates and the grouped formula (B.9). The relative error is at most $1.7 \cdot 10^{-12}$ and $5.2 \cdot 10^{-8}$ for the two tests; the cusps cover $\sigma \in \{0, +, -\}$ and $N(c) \in \{1, 4, 7, 28\}$. The combination $\overline{\kappa_F} \vartheta(\lambda^4 \ell)$ extracted

from each active translate is the same for all $h_p \neq 0$ to $2.4 \cdot 10^{-12}$; this checks the multiplier and additive-phase bookkeeping. The marked formula of Proposition 10 with $\mathcal{Q} = \{p\}$ holds to $8 \cdot 10^{-13}$.

- The local identities: the Fourier coefficients $C_{p,j}$, the local sums giving $\varpi_{p,j}B_p$ for every unit c/p , the cross-prime identity and the pair identity, at 15 good primes (split and inert) and all j , to 10^{-13} .
- The constant $-\frac{1}{81}((2\pi)^4/27)^{-t}$, the Stirling exponent $4\Re t$ of R_2 , the line independence of $V^\sharp$ down to $\Re t = -1$, the small- x expansion of $V^\sharp$ from the poles at $-\frac{4}{3}, -\frac{5}{3}$, and the bound (e) for both tests on $10^{-8} \leq x \leq 10^6$.

C Hecke L -functions of infinity type $\bar{z}/|z|$

This appendix proves Lemma 10. It also proves the CM form of [36, Lemma 4.9], the logarithmic control. Section C.7 lists every place where [36] uses its growth lemmas, and the CM statement that replaces each. The only change is the gamma factor: $\Gamma(s + \frac{1}{2})$ in place of $\Gamma(s)$. Every exponent of the conductor and of the height used in [36] stays the same, and no constant depends on the character.

We write $\zeta_K(\sigma) = \zeta(\sigma)L(\sigma, \chi_{-3})$ for the Dedekind zeta function of K [2, Theorem 3], $e(z) = \exp(4\pi i \operatorname{Im} z / \sqrt{3})$ as in Section 7, and $\alpha(a) = a/|a|$ for every $a \neq 0$. Since $|\psi(\mathfrak{p})| \leq 1$ for every character below, we have $|L(s, \psi)| \leq \zeta_K(\sigma)$ and $|L(s, \psi)^{-1}| \leq \zeta_K(\sigma)$ for $\sigma = \Re s > 1$.

C.1 The characters

Let $\mathfrak{m}$ be a nonzero ideal. Following Hecke [18] (see [33, Ch. VII, §6]), a *character of modulus $\mathfrak{m}$ and infinity type $\bar{z}/|z|$* is a homomorphism ψ from the group of fractional ideals prime to $\mathfrak{m}$ to the unit circle such that $\psi((a)) = \overline{\alpha(a)}$ whenever $a = b/c$ with $b, c \in \mathbb{Z}[\omega]$ prime to $\mathfrak{m}$ and $b \equiv c \pmod{\mathfrak{m}}$. Its conductor is the least modulus, for divisibility, from which it is induced, and it is primitive if its modulus is its conductor. Infinity type $z/|z|$ is defined in the same way with $\alpha(a)$ in place of $\overline{\alpha(a)}$; ψ has infinity type $\bar{z}/|z|$ exactly when $\bar{\psi}$ has infinity type $z/|z|$. We write $\overline{\mathcal{F}}_{\text{CM}} = \{\bar{\psi} : \psi \in \mathcal{F}_{\text{CM}}\}$.

Lemma 17. (i) *Let ψ have modulus $\mathfrak{m}$ and infinity type $\bar{z}/|z|$. For $a \in \mathbb{Z}[\omega]$ prime to $\mathfrak{m}$ put $\phi(a) = \psi((a))\alpha(a)$, and put $\phi(a) = 0$ for the other $a \in \mathbb{Z}[\omega]$. Then ϕ is a character of $(\mathbb{Z}[\omega]/\mathfrak{m})^\times$, extended by zero, with $\phi(\zeta) = \zeta$ for the six units ζ , and*

$$\psi((a)) = \phi(a)\overline{\alpha(a)} \quad (a \in \mathbb{Z}[\omega] \text{ prime to } \mathfrak{m}). \quad (\text{C.1})$$

Conversely, every character ϕ of $(\mathbb{Z}[\omega]/\mathfrak{m})^\times$ with $\phi(\zeta) = \zeta$ for all units defines by (C.1) a character of modulus $\mathfrak{m}$ and infinity type $\bar{z}/|z|$.

(ii) *ψ is primitive modulo $\mathfrak{m}$ if and only if ϕ is primitive modulo $\mathfrak{m}$ in the sense of [2, Section 4.2]. The conductor of ψ is never (1).*

(iii) *A character has infinity type $\bar{z}/|z|$ if and only if it agrees on primary elements with $\bar{\alpha}\eta$ for a finite-order Hecke character η . Thus $\mathcal{F}_{\text{CM}}$ (Section 11) is the set of primitive characters of infinity type $\bar{z}/|z|$. The character $\bar{\alpha}$, that is $(n) \mapsto \bar{\alpha}(n)$ for primary n , has $\phi(a) =$ the unit congruent to a modulo 3, and conductor (3).*

Proof. (i) Let $a, b \in \mathbb{Z}[\omega]$ be prime to $\mathfrak{m}$ with $a \equiv b \pmod{\mathfrak{m}}$. Then $\psi((a))/\psi((b)) = \psi((a/b)) = \overline{\alpha(a)}/\overline{\alpha(b)}$, that is, $\phi(a) = \phi(b)$. So ϕ is a function on $(\mathbb{Z}[\omega]/\mathfrak{m})^\times$. It is multiplicative because ψ and α are, and $\phi(\zeta) = \psi((1))\alpha(\zeta) = \zeta$. Conversely, the right side of (C.1) does not change when a is replaced by ζa , since $\phi(\zeta a)\overline{\alpha(\zeta a)} = \zeta\bar{\zeta}\phi(a)\overline{\alpha(a)}$. Every ideal of $\mathbb{Z}[\omega]$ is principal, so (C.1) defines a character of the integral ideals prime to $\mathfrak{m}$, and hence of the fractional ones. If $b \equiv c \pmod{\mathfrak{m}}$, then $\phi(b) = \phi(c)$ and $\psi((b/c)) = \overline{\alpha(b)}/\overline{\alpha(c)} = \overline{\alpha(b/c)}$.

(ii) For a divisor $\mathfrak{m}'$ of $\mathfrak{m}$, ψ is induced from modulus $\mathfrak{m}'$ exactly when $\psi((a)) = \overline{\alpha(a)}$ for all $a \in \mathbb{Z}[\omega]$ prime to $\mathfrak{m}$ with $a \equiv 1 \pmod{\mathfrak{m}'}$, that is, when ϕ is trivial on these classes. This is the definition of [2, Section 4.2]. If the conductor were (1), ϕ would be a character of the trivial group, but $\phi(-1) = -1$.

(iii) Every ideal $\mathfrak{a}$ prime to 3 has a unique primary generator $n_{\mathfrak{a}}$, and products of primary elements are primary [2, Section 2.7]. If $a = b/c$ with $b, c \in \mathbb{Z}[\omega]$ prime to $3\mathfrak{m}$ and $b \equiv c \pmod{3\mathfrak{m}}$, write $b = \zeta_b n_{(b)}$ and $c = \zeta_c n_{(c)}$ with units ζ_b, ζ_c . Since the six units are distinct modulo 3, $b \equiv c \pmod{3}$ forces $\zeta_b = \zeta_c$, and then $n_{(b)} \equiv n_{(c)} \pmod{\mathfrak{m}}$.

Let ψ have modulus $\mathfrak{m}$ and infinity type $\bar{z}/|z|$, and put $\eta(\mathfrak{a}) = \psi(\mathfrak{a})\alpha(n_{\mathfrak{a}})$ for $\mathfrak{a}$ prime to $3\mathfrak{m}$. By (C.1), $\eta(\mathfrak{a}) = \phi(n_{\mathfrak{a}})$. So η is multiplicative, takes values in roots of unity, and is trivial on (b/c) when $b \equiv c \pmod{3\mathfrak{m}}$, by the previous paragraph. Hence η is a finite-order ray class character modulo $3\mathfrak{m}$, and $\psi((n)) = \overline{\alpha(n)}\eta((n))$ for primary n . Conversely, let η be a ray class character modulo $\mathfrak{m}_0$, and put $\psi(\mathfrak{a}) = \overline{\alpha(n_{\mathfrak{a}})}\eta(\mathfrak{a})$ for $\mathfrak{a}$ prime to $3\mathfrak{m}_0$. If $b \equiv c \pmod{3\mathfrak{m}_0}$, then $\zeta_b = \zeta_c$, so $\overline{\alpha(n_{(b)})}/\overline{\alpha(n_{(c)})} = \overline{\alpha(b/c)}$, and $\eta((b/c)) = 1$. So ψ has modulus $3\mathfrak{m}_0$ and infinity type $\bar{z}/|z|$.

For $\bar{\alpha}$: if $\zeta \equiv a \pmod{3}$ is a unit, then $n_{(a)} = \zeta^{-1}a$ and $\overline{\alpha(n_{(a)})} = \zeta \overline{\alpha(a)}$, so $\phi(a) = \zeta$, a character modulo 3. It is not trivial on the classes $a \equiv 1 \pmod{\lambda}$, because $\omega \equiv 1 \pmod{\lambda}$ and $\phi(\omega) = \omega$. The only proper divisors of $(3) = (\lambda)^2$ are (λ) and (1), so the conductor is (3). $\square$

C.2 Theta series and the functional equation

We follow Hecke's method [18] (see [33, Ch. VII, §§7–8]). For $\psi \in \mathcal{F}_{\text{CM}}$ with conductor $\mathfrak{f} = (f)$, $Q = \text{N}\mathfrak{f} = |f|^2$, and ϕ as in Lemma 17, put

$$\Theta_\phi(v) = \sum_{a \in \mathbb{Z}[\omega]} \phi(a) \bar{a} e^{-2\pi v \text{N}(a)/\sqrt{3}} \quad (v > 0), \quad \gamma(\phi) = \frac{1}{\sqrt{Q}} \sum_{x \bmod f} \phi(x) e\left(\frac{x}{f}\right). \quad (\text{C.2})$$

The term $a = 0$ vanishes. Since ϕ is primitive modulo f (Lemma 17(ii)), [2, Lemma 17] gives $|\gamma(\phi)| = 1$ and, for every $h \in \mathbb{Z}[\omega]$,

$$\sum_{x \bmod f} \phi(x) e\left(\frac{hx}{f}\right) = \overline{\phi(h)} \sqrt{Q} \gamma(\phi). \quad (\text{C.3})$$

Lemma 18. *Let $\psi \in \mathcal{F}_{\text{CM}}$, with f , Q and ϕ as above.*

(i) *For $v > 0$,*

$$\Theta_\phi(v) = -\frac{\gamma(\phi)}{f\sqrt{Q}} v^{-2} \overline{\Theta_\phi(1/(Qv))}. \quad (\text{C.4})$$

(ii) *For $\Re s > 1$,*

$$\int_0^\infty \Theta_\phi(v) v^{s+1/2} \frac{dv}{v} = 6 \left(\frac{2\pi}{\sqrt{3}}\right)^{-s-1/2} \Gamma(s + \frac{1}{2}) L(s, \psi), \quad (\text{C.5})$$

and the same holds with $\overline{\Theta_\phi}$ and $\bar{\psi}$ in place of Θ_ϕ and ψ .

(iii) The function $\Lambda(s, \psi) = (3Q)^{s/2} (2\pi)^{-s} \Gamma(s + \frac{1}{2}) L(s, \psi)$ extends to an entire function that is bounded on every vertical strip of finite width, and

$$\Lambda(s, \psi) = W(\psi) \Lambda(1-s, \bar{\psi}), \quad W(\psi) = -\frac{\gamma(\phi)\sqrt{Q}}{f}, \quad |W(\psi)| = 1. \quad (\text{C.6})$$

The same holds for $\bar{\psi}$, with $W(\bar{\psi}) = \overline{W(\psi)}$.

(iv) $L(s, \psi)$ and $L(s, \bar{\psi})$ are entire and have no zeros in $\Re s > 1$. For each fixed ψ and each bounded interval J , $|L(\sigma + it, \psi)| \ll_{\psi, J} e^{2|t|}$ for $\sigma \in J$, and likewise for $\bar{\psi}$.

Proof. (i) Fix $v > 0$ and put $G(w) = \bar{w} e^{-2\pi v|w|^2/\sqrt{3}}$ and $\hat{G}(\eta) = \int_{\mathbb{C}} G(w) e(-w\eta) dx dy$. We first compute $\hat{G}$. By [2, Example 20],

$$\int_{\mathbb{C}} e^{-2\pi v|w|^2/\sqrt{3}} e(-w\eta) dx dy = \frac{\sqrt{3}}{2v} e^{-2\pi|\eta|^2/(\sqrt{3}v)}. \quad (\text{C.7})$$

Since $\text{Im } z = (z - \bar{z})/(2i)$, we have $e(-w\eta) = \exp(-\frac{2\pi}{\sqrt{3}}(w\eta - \bar{w}\bar{\eta}))$, so $\partial_{\bar{\eta}} e(-w\eta) = \frac{2\pi}{\sqrt{3}} \bar{w} e(-w\eta)$, where $\partial_{\bar{\eta}}$ is the Wirtinger derivative. Differentiating under the integral sign, which the Gaussian decay permits,

$$\hat{G}(\eta) = \frac{\sqrt{3}}{2\pi} \partial_{\bar{\eta}} \left(\frac{\sqrt{3}}{2v} e^{-2\pi\eta\bar{\eta}/(\sqrt{3}v)} \right) = -\frac{\sqrt{3}}{2v^2} \eta e^{-2\pi|\eta|^2/(\sqrt{3}v)}. \quad (\text{C.8})$$

Now split $a = t + fz$, with t running over a set of representatives modulo f and $z \in \mathbb{Z}[\omega]$, so that $\Theta_{\phi}(v) = \sum_t \phi(t) \sum_{z \in \mathbb{Z}[\omega]} G(t + fz)$. Poisson summation over $\mathbb{Z}[\omega]$ [2, Theorem 19] gives $\sum_z G(t + fz) = \frac{2}{\sqrt{3}} \sum_{y \in \mathbb{Z}[\omega]} \hat{F}_t(y)$ with $F_t(z) = G(t + fz)$. The substitution $w = t + fz$ has Jacobian $|f|^2 = Q$, and $e(-zy) = e(ty/f) e(-wy/f)$, so $\hat{F}_t(y) = Q^{-1} e(ty/f) \hat{G}(y/f)$. Summing over t and using (C.3) with $h = y$,

$$\begin{aligned} \Theta_{\phi}(v) &= \frac{2}{\sqrt{3}Q} \sum_{y \in \mathbb{Z}[\omega]} \overline{\phi(y)} \sqrt{Q} \gamma(\phi) \left(-\frac{\sqrt{3}}{2v^2} \right) \frac{y}{f} e^{-2\pi N(y)/(\sqrt{3}Qv)} \\ &= -\frac{\gamma(\phi)}{f\sqrt{Q}} v^{-2} \sum_{y \in \mathbb{Z}[\omega]} \overline{\phi(y)} y e^{-2\pi N(y)/(\sqrt{3}Qv)}. \end{aligned} \quad (\text{C.9})$$

The last sum is $\overline{\Theta_{\phi}(1/(Qv))}$.

(ii) For $\sigma > 1$, $\sum_{a \neq 0} |a| \int_0^{\infty} e^{-2\pi v N(a)/\sqrt{3}} v^{\sigma+1/2} dv/v = 6(2\pi/\sqrt{3})^{-\sigma-1/2} \Gamma(\sigma + \frac{1}{2}) \zeta_K(\sigma) < \infty$, because every nonzero ideal has six generators. So we may integrate termwise:

$$\int_0^{\infty} \Theta_{\phi}(v) v^{s+1/2} \frac{dv}{v} = \left(\frac{2\pi}{\sqrt{3}} \right)^{-s-1/2} \Gamma(s + \frac{1}{2}) \sum_{a \neq 0} \phi(a) \bar{a} N(a)^{-s-1/2}. \quad (\text{C.10})$$

Since $\bar{a} N(a)^{-1/2} = \overline{\alpha(a)}$, the summand is $\psi((a)) N(a)^{-s}$ by (C.1), and it vanishes unless a is prime to $\mathfrak{f}$. Each ideal prime to $\mathfrak{f}$ has six generators, so the sum is $6L(s, \psi)$. For $\bar{\psi}$ use $\overline{\Theta_{\phi}(v)} = \sum_a \overline{\phi(a)} a e^{-2\pi v N(a)/\sqrt{3}}$ and $\bar{\psi}((a)) = \overline{\phi(a)} \alpha(a)$.

(iii) Let $I_{\psi}(s)$ denote the left side of (ii), and $I_{\bar{\psi}}(s)$ the same integral with $\overline{\Theta_{\phi}}$. Put $v_0 = Q^{-1/2}$ and

$$B(w) = \int_{v_0}^{\infty} \Theta_{\phi}(v) v^{w+1/2} \frac{dv}{v}, \quad B^-(w) = \int_{v_0}^{\infty} \overline{\Theta_{\phi}(v)} v^{w+1/2} \frac{dv}{v}. \quad (\text{C.11})$$

Since $N(a) \geq 1$ for $a \neq 0$, for $v \geq v_0$ we have $|\Theta_\phi(v)| \leq e^{-\pi v/\sqrt{3}} \sum_{a \neq 0} |a| e^{-\pi v_0 N(a)/\sqrt{3}}$. So B and B^- are entire and bounded on every vertical strip of finite width. For $\Re s > 1$ split $I_\psi(s)$ at v_0 , substitute $v = 1/(Qv)$ in the integral over $(0, v_0)$, and use (i):

$$\int_0^{v_0} \Theta_\phi(v) v^{s+1/2} \frac{dv}{v} = -\frac{\gamma(\phi)}{f\sqrt{Q}} \int_0^{v_0} \overline{\Theta_\phi(1/(Qv))} v^{s-3/2} \frac{dv}{v} = -\frac{\gamma(\phi)}{f\sqrt{Q}} Q^{3/2-s} B^-(1-s). \quad (\text{C.12})$$

With $W = -\gamma(\phi)\sqrt{Q}/f$ this gives

$$Q^{s/2} I_\psi(s) = Q^{s/2} B(s) + W Q^{(1-s)/2} B^-(1-s). \quad (\text{C.13})$$

Conjugating (i) gives $\overline{\Theta_\phi(v)} = -\frac{\overline{\gamma(\phi)}}{f\sqrt{Q}} v^{-2} \Theta_\phi(1/(Qv))$, and the same computation gives $Q^{s/2} I_{\overline{\psi}}(s) = Q^{s/2} B^-(s) + \overline{W} Q^{(1-s)/2} B(1-s)$. Here $|W| = |\gamma(\phi)| = 1$. By (ii), $Q^{s/2} I_\psi(s) = \kappa \Lambda(s, \psi)$ and $Q^{s/2} I_{\overline{\psi}}(s) = \kappa \Lambda(s, \overline{\psi})$, with $\kappa = 6 \cdot 3^{1/4} (2\pi)^{-1/2}$, because $(2\pi/\sqrt{3})^{-s-1/2} = 3^{1/4} (2\pi)^{-1/2} 3^{s/2} (2\pi)^{-s}$. The right sides of these two identities are entire and bounded on vertical strips of finite width, which gives the continuation. Replacing s by $1-s$ in the second identity and multiplying by W gives the first one, since $|W|^2 = 1$; that is, $\Lambda(s, \psi) = W \Lambda(1-s, \overline{\psi})$. In the same way $\Lambda(s, \overline{\psi}) = \overline{W} \Lambda(1-s, \psi)$.

(iv) $1/\Gamma$ and $(3Q)^{-s/2} (2\pi)^s$ are entire, so (iii) shows that $L(s, \psi)$ is entire. The Euler product converges absolutely in $\Re s > 1$, so there are no zeros there. On $\sigma \in J$, Stirling's formula gives $|1/\Gamma(s + \frac{1}{2})| \ll_J (1 + |t|)^{c_J} e^{\pi|t|/2} \ll_J e^{2|t|}$, and Λ is bounded there. The same argument applies to $\overline{\psi}$. $\square$

The root number $W(\psi)$ is the number $\epsilon(\psi)$ of Lemma 10. It does not depend on the choice of the generator f : replacing f by ζf multiplies $\gamma(\phi)$ by $\phi(\zeta) = \zeta$ (substitute $x = \zeta x'$). For $\overline{\alpha}$ ($f = 3$) the six units represent $(\mathbb{Z}[\omega]/3)^\times$, and $\sum_\zeta \zeta e(\zeta/3) = -3$, so $\gamma(\phi) = -1$ and $W(\overline{\alpha}) = 1$. This agrees with the root number $+1$ of $y^2 = x^3 + 16$, the curve E_{16} of conductor 27: since $4 \cdot 16 = 2^6$, its character ψ_{16}^* of Section 11 is $\overline{\alpha}$ on primary n prime to 2.

C.3 The gamma quotient and the strip bound

Put $G(\sigma, t) = |\Gamma(\frac{3}{2} - \sigma - it)/\Gamma(\frac{1}{2} + \sigma + it)|$. This is the gamma quotient in the functional equation: by Lemma 18(iii), for $\psi \in \mathcal{F}_{\text{CM}} \cup \overline{\mathcal{F}}_{\text{CM}}$ with conductor norm Q ,

$$L(s, \psi) = W(\psi) \left(\frac{3Q}{(2\pi)^2} \right)^{1/2-s} \frac{\Gamma(\frac{3}{2} - s)}{\Gamma(\frac{1}{2} + s)} L(1-s, \overline{\psi}). \quad (\text{C.14})$$

Lemma 19. (a) For every compact $I \subset (-\frac{1}{2}, \frac{3}{2})$, $G(\sigma, t) \asymp_I (3 + |t|)^{1-2\sigma}$ for $\sigma \in I$ and $t \in \mathbb{R}$.

(b) For every compact $I \subset (-\infty, \frac{3}{2})$, $G(\sigma, t) \ll_I (3 + |t|)^{1-2\sigma}$ for $\sigma \in I$. Equivalently, since $G(1-\sigma, -t) = |\Gamma(\frac{1}{2} + \sigma + it)/\Gamma(\frac{3}{2} - \sigma - it)|$, for every compact $I \subset (-\frac{1}{2}, \infty)$, $|\Gamma(\frac{1}{2} + \sigma + it)/\Gamma(\frac{3}{2} - \sigma - it)| \ll_I (3 + |t|)^{2\sigma-1}$ for $\sigma \in I$.

The exponent $1 - 2\sigma$ is that of $|\Gamma(1-\sigma-it)/\Gamma(\sigma+it)|$, the quotient of [36] for finite-order characters.

Proof. Stirling's formula gives $|\Gamma(x+iy)| = \sqrt{2\pi} |y|^{x-1/2} e^{-\pi|y|/2} (1 + O_K(|y|^{-1}))$ for $|y| \geq 1$, uniformly for x in a compact set $K \subset \mathbb{R}$. For $|t|$ larger than a constant depending on I , it gives $G(\sigma, t) = |t|^{(1-\sigma)-\sigma} (1 + O_I(|t|^{-1}))$, since the factors $e^{-\pi|t|/2}$ cancel. In (a) the real parts $\frac{3}{2} - \sigma$ and $\frac{1}{2} + \sigma$ lie in a compact subset of $(0, 2)$, so G is continuous and positive on $I \times [-T, T]$ for every T . In (b) the real part $\frac{3}{2} - \sigma$ is bounded below by a positive constant, so $\Gamma(\frac{3}{2} - \sigma - it)$ is continuous on $I \times [-T, T]$, and $1/\Gamma$ is entire. $\square$

Proof of Lemma 10. The first assertion is Lemma 18(iii), with $\epsilon(\psi) = W(\psi)$. The last assertion follows from (C.14) and Lemma 19(a). For the strip bound, let $\psi \in \mathcal{F}_{\text{CM}} \cup \overline{\mathcal{F}}_{\text{CM}}$. On $\sigma = \frac{11}{10}$, $|L(s, \psi)| \leq \zeta_K(\frac{11}{10})$. On $\sigma = -\frac{1}{10}$, (C.14) and Lemma 19(a) give

$$|L(-\frac{1}{10} + it, \psi)| = (3Q)^{3/5} (2\pi)^{-6/5} G(-\frac{1}{10}, t) |L(\frac{11}{10} - it, \overline{\psi})| \ll Q^{3/5} (3 + |t|)^{6/5}. \quad (\text{C.15})$$

This is the left-line bound of [36, proof of Lemma 4.8], with the same exponents. The rest is the damped maximum principle of [36, proof of Lemma 4.8], which applies verbatim. Namely, $H(s) = Q^{-3/5} L(s, \psi) (s+2)^{-2}$ is holomorphic on the closed strip $-\frac{1}{10} \leq \sigma \leq \frac{11}{10}$ (Lemma 18(iv)). It is bounded on both boundary lines by an absolute constant C_0 , because $|s+2|^2 \geq (\frac{19}{10})^2 + t^2$. Inside the strip, $|H(s)| \ll_{\psi} e^{2|t|}$ by Lemma 18(iv). For $\delta > 0$, $|e^{\delta(s-1/2)^2}| = e^{\delta((\sigma-1/2)^2 - t^2)}$ is at most $e^{9\delta/25}$ in the strip and at most $e^{\delta(9/25 - T^2)}$ at height $\pm T$, which beats e^{2T} . The maximum principle on the rectangles $|t| \leq T$, with $T \rightarrow \infty$, gives $|H(s)e^{\delta(s-1/2)^2}| \leq C_0 e^{9\delta/25}$ in the strip. Letting $\delta \rightarrow 0$ gives $|H| \leq C_0$. Hence $|L(s, \psi)| \leq C_0 Q^{3/5} |s+2|^2 \leq (\frac{31}{30})^2 C_0 Q^{3/5} (3 + |t|)^2$. $\square$

Numerically, $\zeta_K(\frac{11}{10}) \approx 6.6298$, and sampling the left-boundary gamma quotient gives a contribution about 0.446 to C_0 , compared with about 0.690 on the right boundary. These are numerical observations, not certified supremum bounds; the proof uses only the existence of an absolute C_0 established above.

Zeros. [36] never counts zeros. Its detector [36, Lemma 8.1] uses only that the zeros of each $L(s, \psi^*)$ are discrete and lie in $\Re s \leq 1$ [36, proof of Lemma 8.1], which for CM characters is Lemma 18(iv). If a count is wanted, Jensen's formula [22], [45, §3.61] on the disks $|s - (2 + it)| \leq 2.07$ and 2.1, with $|L(2 + it, \psi)| \geq \zeta_K(4)/\zeta_K(2)$ and the strip bound, shows that the number of zeros with $0 \leq \Re \rho \leq 1$ and $|\Im \rho - t| \leq \frac{1}{2}$ is $\ll \log(Q(3 + |t|))$. This is the same bound as for finite order.

C.4 Logarithmic control

Lemma 20. Let $\psi \in \mathcal{F}_{\text{CM}} \cup \overline{\mathcal{F}}_{\text{CM}}$ have conductor norm Q , and put $\mathcal{L}(s) = L(s, \psi)$. Fix $a \in [\frac{1}{2}, 1]$ and $0 < e < 10^{-3}$, and suppose that $\mathcal{L}$ has no zero in the open disk of radius $2 - a - 2e$ centred at $2 + it$. Then, for every $\epsilon > 0$, uniformly on the closed concentric disk of radius $2 - a - 6e$,

$$|\mathcal{L}(s)| + |\mathcal{L}(s)^{-1}| \ll_{e, \epsilon} \{2Q(3 + |t|)^2\}^{\epsilon}, \quad (\text{C.16})$$

and on the closed disk of radius $2 - a - 8e$, $|\mathcal{L}'(s)/\mathcal{L}(s)| \ll_{\epsilon} \log\{2Q(3 + |t|)^2\}$. The implied constants depend only on e and ϵ . In particular, for every fixed $b \geq \beta_*$ and $v > 0$ these bounds hold on $\Re s \geq b + v$, with the height of s in place of t and constants depending only on v and ϵ .

Proof. This is [36, Lemma 4.9] without its principal clauses, and its proof [36, proof of Lemma 4.9] applies word for word. That proof uses exactly the following four properties of $\mathcal{L}$. Everything else in it is complex analysis that does not involve the character: Borel–Carathéodory [45, §5.5] between the radii R_3 and R_4 , Hadamard's three circles [45, §5.3] with inner radius $r_0 = 49/100$ and exponent $\theta = \log(R_6/r_0)/\log(R_4/r_0) < 1$, uniformly in $a \in [\frac{1}{2}, 1]$, and Cauchy's estimate between R_6 and R_8 , where $R_j = 2 - a - je$.

1. $\mathcal{L}$ is holomorphic on the disks used; they lie in $\Re s \geq a + 2e > \frac{1}{2}$. This holds by Lemma 18(iv).
2. $\log |\mathcal{L}(s)| \ll \log\{2Q(3 + |t|)^2\}$ on the disk of radius R_3 . That disk lies in $a + 3e \leq \sigma \leq \frac{7}{2}$, at heights within $\frac{3}{2}$ of t . Lemma 10 gives $|\mathcal{L}(s)| \ll Q^{3/5} (3 + |t| + \frac{3}{2})^2$ for $\sigma \leq \frac{11}{10}$, and $|\mathcal{L}(s)| \leq \zeta_K(\frac{11}{10})$ for $\sigma \geq \frac{11}{10}$. The constants are absolute, as in [36, proof of Lemma 4.9].

3. The Euler logarithm $\log \mathcal{L}(s) = \sum_{\mathfrak{p}} \sum_{k \geq 1} \psi(\mathfrak{p})^k / (k N \mathfrak{p}^{ks})$ converges absolutely for $\Re s > 1$, and $|\log \mathcal{L}(s)| \leq \log \zeta_K(\sigma)$ because $|\psi(\mathfrak{p})| = 1$. It is therefore at most $\log \zeta_K(2)$ at the centre and at most $\log \zeta_K(\frac{151}{100})$ on the disk of radius r_0 .
4. For the global assertion with $\frac{1}{2} \leq b \leq 1$, take $a = b$ and $e < \min(10^{-3}, v/8)$. The disk of radius R_2 about $2 + it$ lies in $\Re s > b + 2e > \beta_*$, so it contains no zero of $\mathcal{L}$, by the definition of β_* in Section 11. For $\bar{\psi}$ note that $L(s, \bar{\psi}) = \overline{L(\bar{s}, \psi)}$. For $b > 1$ the Euler product suffices.

In particular the constants depend only on e, ϵ , the absolute constant of Lemma 10, and $\zeta_K(\frac{151}{100})$, and not on ψ . $\square$

The whole row family. For a finite-order character, [36, Lemma 4.9] applies as it stands. Its global assertion needs b at least the supremum [36, eq. (2.1)] over finite-order characters. That supremum is at most $7/8$ by [36, Theorem 1.1]. Hence, for the label sets $\mathcal{X}_u$ of Section 13, which contain finite-order and CM members, the global assertion holds for every fixed $b \geq \max(\beta_*, \frac{7}{8})$. This is $b \geq \beta_*$ under the contradiction hypothesis $\beta_* > \frac{7}{8}$ of Section 15, and under the hypothesis $\beta_* > \frac{11}{12}$ of statement 6 of Section 16. So it is the same range as in [36]. The deleted Euler factors of [36, Lemma 4.10] use only $|a_p| \leq 1$, so they are unchanged.

C.5 Conductors of the row characters

Lemma 21. *Let ψ_0 be a primitive finite-order Hecke character with conductor $\mathfrak{f}_0$, and let ψ^* be the primitive character inducing $\bar{\alpha}\psi_0$ (on primary n prime to $3\mathfrak{f}_0$). Then $\psi^* \in \mathcal{F}_{\text{CM}}$, and its conductor divides $\text{lcm}((3), \mathfrak{f}_0)$. In particular $Q_{\psi^*} \leq 9N\mathfrak{f}_0$. Moreover ψ^* and ψ_0 have the same conductor exponent at every prime other than λ . The same holds for $\alpha\psi_0$, with $\overline{\mathcal{F}}_{\text{CM}}$ in place of $\mathcal{F}_{\text{CM}}$.*

Consequently, let $\theta \in \Theta_0$ and let u be a sixth-power-free row that is not supported on S . Then the CM members $n \mapsto \bar{\alpha}(n)\theta(n)\chi_n(u)$ and $n \mapsto \alpha(n)\theta(n)\chi_n(u)$ of $\mathcal{X}_u$ satisfy the conclusions of [36, Section 8.1]. That is, the primitive character ψ^ inducing such a member satisfies*

$$Q_{\psi^*} \ll_{\mathcal{A}} N(u), \quad L_{\text{orig}}(s) = L(s, \psi^*) \prod_{p \in E_u} (1 - \psi^*(p)N(p)^{-s}), \quad E_u \subset S \cup \{p : p \mid u\}, \quad (\text{C.17})$$

where L_{orig} is the Dirichlet series of the zero-extended member. No CM member is principal. (In fact $E_u \subset S$, since every good prime dividing u divides the conductor; see Lemma 23.)

Proof. By Lemma 17(iii), $\bar{\alpha}$ has conductor (3) . Put $\mathfrak{m} = \text{lcm}((3), \mathfrak{f}_0)$. On ideals prime to $\mathfrak{m}$, the product $\bar{\alpha}\psi_0$ is a character of modulus $\mathfrak{m}$ and infinity type $\bar{z}/|z|$: if $b \equiv c \pmod{\mathfrak{m}}$, then $\bar{\alpha}((b/c)) = \bar{\alpha}(b/\bar{c})$ and $\psi_0((b/c)) = 1$. So $\psi^* \in \mathcal{F}_{\text{CM}}$ and its conductor divides $\mathfrak{m}$, and $N\mathfrak{m} \leq 9N\mathfrak{f}_0$. Conversely ψ_0 is induced by $\alpha\psi^*$, and α has conductor (3) , so $\mathfrak{f}_0$ divides $\text{lcm}((3), \mathfrak{f}(\psi^*))$. At a prime $\mathfrak{p} \neq (\lambda)$ the two divisibilities give $v_{\mathfrak{p}}(\mathfrak{f}(\psi^*)) \leq v_{\mathfrak{p}}(\mathfrak{f}_0) \leq v_{\mathfrak{p}}(\mathfrak{f}(\psi^*))$. For $\alpha\psi_0$, conjugate: $\alpha\psi_0 = \overline{\bar{\alpha}\psi_0}$.

For the rows, let ψ_0 be the primitive character inducing $\theta\chi_{\bullet}(u)$. By Lemma 23(e) (which proves the bound that [36, Section 8.1] uses without stating it), $N\mathfrak{f}_0 \ll_{\mathcal{A}} N(u)$, and $\theta(n)\chi_n(u) = \psi_0((n))$ for primary n prime to S and u . At such n the CM member equals $\bar{\alpha}(n)\psi_0((n)) = \psi^*((n))$, because $(\lambda) \in S$ and the primes of $\mathfrak{f}_0$ lie in $S \cup \{p : p \mid u\}$. So $L_{\text{orig}}(s)$ is the Euler product of ψ^* over the primes outside $S \cup \{p : p \mid u\}$. This gives the factorization, with E_u the primes of $S \cup \{p : p \mid u\}$ that do not divide $\mathfrak{f}(\psi^*)$. A principal character has trivial infinity type, so no CM member is principal. The second member is the conjugate of a first one, since Θ_0 is a group. $\square$

C.6 Reflection of a CM plain sum

For a character ψ and a smooth function W with compact support in $(0, \infty)$, put $T_\psi(X; W) = X^{-1/2} \sum_{\mathfrak{a}} \psi(\mathfrak{a}) W(N\mathfrak{a}/X)$. The sum is over nonzero integral ideals, with $\psi(\mathfrak{a}) = 0$ unless $\mathfrak{a}$ is prime to the conductor. We use the Mellin convention $\mathcal{M}W(s) = \int_0^\infty W(y) y^s dy/y$ of [36, Lemma 4.5]. The next lemma replaces the reflection step [36, Section 18.1] of the plain fourth moment when the row character has infinite order.

Lemma 22. *Let $\psi \in \mathcal{F}_{\text{CM}} \cup \overline{\mathcal{F}_{\text{CM}}}$ have conductor norm Q and root number $W(\psi)$, and put $C_\psi = 3Q/(2\pi)^2$. For $X > 0$,*

$$T_\psi(X; W) = W(\psi) T_{\overline{\psi}}(C_\psi/X; W^\sharp), \quad W^\sharp(y) = \frac{1}{2\pi i} \int_{(c)} \mathcal{M}W(1-u) \frac{\Gamma(u + \frac{1}{2})}{\Gamma(\frac{3}{2} - u)} y^{-u} du, \quad (\text{C.18})$$

for any $c > -\frac{1}{2}$. Thus $\mathcal{M}W^\sharp(u) = \mathcal{M}W(1-u)\Gamma(u + \frac{1}{2})/\Gamma(\frac{3}{2} - u)$ on $\Re u > -\frac{1}{2}$, in place of $\mathcal{M}W(1-u)\Gamma(u)/\Gamma(1-u)$ in [36]. Moreover:

- (a) $W^\sharp$ is smooth on $(0, \infty)$. For all integers $j \geq 0$ and all $A \geq 0$, $(y\partial_y)^j W^\sharp(y) \ll \min(y^{1/2}, y^{-A})$. The implied constant is at most a constant depending on j and A times finitely many integrals $\int_0^\infty y^\sigma |(y\partial_y)^k W(y)| dy/y$, with σ in a compact set depending only on j and A .
- (b) For $W_\omega(y) = W(y)y^{i\omega}$, the bounds in (a) hold for $W_\omega^\sharp$ with an extra factor $(1 + |\omega|)^h$, where h depends only on j and A .

The new kernel is holomorphic on $\Re u > -\frac{1}{2}$, so $W^\sharp(y) = \mathcal{M}W(\frac{3}{2})y^{1/2} + O(y^{3/2})$ has no constant term at 0. In [36], $W^\sharp(0) = \mathcal{M}W(1)$. On $\Re u = c$ both kernels are $\ll_c (3 + |t|)^{2c-1}$, by Lemma 19(b) and its finite-order analogue. So the reflected lengths, the annular decomposition, the seminorm bounds and the height polynomials of [36, Section 18.1] are unchanged. The only difference is that the annuli below the main scale now carry coefficients $O(2^{-j})$ in place of $O(2^{-j/2})$, which only helps.

Proof. Use the direct Poisson proof in Lemma 29, which establishes the reflection for both infinity types, and Lemma 30 for its profile. Those proofs use the finite Gauss identity and the planar Fourier transform; they do not use the present lemma. For infinity type $\bar{z}/|z|$, the root number there is $-\gamma(\phi)\bar{\alpha}(f) = W(\psi)$ of Lemma 18; the other type follows by conjugation. The conductor scale is $C_\psi = 3N(f)/(4\pi^2)$ in both formulas.

Lemma 30(a) gives the Mellin identity initially on $-\frac{1}{2} < \Re u < \frac{1}{4}$. Its decay bounds in (c) make the Mellin transform holomorphic throughout $\Re u > -\frac{1}{2}$, so the identity extends there. The rapid vertical decay of $\mathcal{M}W(1-u)$ permits inversion and contour shifts within this half-plane. The seminorm and norm-twist bounds in (a)–(b) are exactly those proved in Lemma 30(c).

For clarity, the expansion at zero comes directly from $W^\sharp(y) = \int W(x) J_1(2\sqrt{xy}) dx$ and $J_1(t) = t/2 + O(t^3)$: compact support of W gives $W^\sharp(y) = \mathcal{M}W(\frac{3}{2})y^{1/2} + O(y^{3/2})$. $\square$

C.7 Where these statements are used

Below we list the uses in [36] of its Lemma 4.8 (equation (4.14) there), of Lemmas 4.9 and 4.10, and of the primitive functional equation, each with the CM statement that replaces it; [36] uses no count of zeros. The characters that occur are of three kinds: the target; the denominators $\psi_{\chi_\bullet}(u)$ and the other CM members of $\mathcal{X}_u$; and finite-order characters (numerators, slot characters, and the finite-order members of $\mathcal{X}_u$). For the third kind, [36] applies as it stands, with zero-free input $b \geq \frac{7}{8}$ from [36, Theorem 1.1].

1. *Buffered bins* [36, Lemma 8.1].
 - The maxima $M_j(u)$ exist ([36, proof of Lemma 8.1]): Lemma 18(iv). No count of zeros is used.
 - Disk control ([36, proof of Lemma 8.1]): Lemma 20, with $Q_{\psi^*} \ll_{\mathcal{A}} N(u)$ from Lemma 21 and $(3 + |t|)^2 \ll U^{1/50}$ as there.
 - Reflected estimate ([36, proof of Lemma 8.1]): (C.14) and Lemma 19(a) on the line $\sigma = 1 - a - 6e \in [-6e, \frac{1}{2} - 6e]$. The conductor and the gamma factor contribute $(3Q_{\psi^*})^{a-1/2+6e}(3 + |t|)^{2a-1+12e}$, exactly as for finite order. The conjugate member lies in $\mathcal{X}_u$.
 - Bin ceiling ([36, Section 8.1]): $M_i(u) \leq \max(\beta_*, \frac{7}{8})$, because the CM members have no zeros in $\Re s > \beta_*$ and the finite-order members none in $\Re s > \frac{7}{8}$.
2. *Pointwise dyadic estimates* [36, Lemma 8.2 and its proof]. The plain witness of a CM member is shifted to $1 - a - 6e$: $L(s, \psi^*)$ is entire (Lemma 18(iv)), and the joins use Lemma 10.
3. *Witness construction* [36, Prop. 8.3 and its proof]. On $\Re z = -\frac{1}{4}$ the argument $\rho + z$ has real part in $[0.26, 0.75]$, and Lemma 10 gives the same bound $Q_{\psi^*}^{3/5}(3 + |\gamma + v|)^2$ as [36, eq. (4.14)].
4. *Contour moves of the high expansion* [36, Lemmas 10.3, 10.5 and 10.6], [36, proof of Lemma 10.3], [36, proof of Lemma 10.5], [36, proof of Lemma 10.6]. These need the reciprocal of a CM denominator, or of the target, on $\Re s = \beta_* + 20e$ or $\beta_* + e$. That is the global part of Lemma 20, with Lemma 21 and the deleted factors. The principal specialization ([36, proof of Lemma 10.6]) is not needed, since no CM denominator is principal. The numerator bounds there, the exponent 3/5 at [36, proof of Lemma 10.6] included, concern finite-order numerators and are unchanged.
5. *Compensated probe*, [36, Section 16.2] and [36, proof of Proposition 16.1]. These use the buffered estimates of item 1 and the functional equation of the finite-order numerator. They are unchanged.
6. *Plain fourth moment*. The reflection [36, Section 18.1] becomes Lemma 22 for rows $\alpha^e \tau \chi_{\bullet}(k)$. The conductor scale [36, eq. (18.7)] keeps its proof, because by Lemma 21 the CM conductor differs from that of $\tau \chi_{\bullet}(k)$ only at $\lambda \in S$. The prime estimate [36, eq. (18.11)] concerns finite-order slot characters. It needs $s_{\kappa} \geq \frac{7}{8}$, which allows $\kappa = \frac{3}{4}$.
7. *Prime bound in a bin* [36, Lemma 19.1 and its proof]. Here $\mathcal{L}'/\mathcal{L}$ is taken for finite-order slot characters, inside disks that the CM bins of item 1 make zero-free for all of $\mathcal{X}_u$. [36, Lemma 4.9] applies as it stands.

In every item the CM statement has the same conductor exponent and the same height exponent as the finite-order one, and its constants do not depend on the character. So the uniformity in the row and in the conductor required by [36] holds as stated there.

D The zero detector for a CM target

This appendix writes out item 3 of Section 16: the zero detector (a truncated-inverse detector; compare [30, Appendix C]) of [36, Section 8] for a target $\psi \in \mathcal{F}_{\text{CM}}$, with the label set $\mathcal{X}_u$ of Section 13. In [36] the detector is used by the sextic row count [36, Prop. 9.2], by the contour

moves of the high side [36, Lemma 10.3], by the dynamic local errors of the compensated probe [36, Section 16.2], and by the Part II row counts [36, Section 19]. We go through [36, Section 8] in order. Every step that involves a row character is restated and proved below; every other step is identical and is cited by line. Apart from [36], the inputs are Lemma 10 and the logarithmic control of Lemma 20 (item 2 of Section 16, proved in Appendix C).

D.1 Setting and the label set

Hecke characters. As in [33, Ch. VII, §6], a Hecke character modulo an integral ideal $\mathfrak{m}$ is a character φ of the group of fractional ideals prime to $\mathfrak{m}$ for which there are characters φ_f of $(\mathbb{Z}[\omega]/\mathfrak{m})^\times$ and φ_∞ of $\mathbb{C}^\times$ with $\varphi((a)) = \varphi_f(a)\varphi_\infty(a)$ for every $a \in \mathbb{Z}[\omega]$ prime to $\mathfrak{m}$. Here φ_∞ is the infinity type. Such a φ is induced by a unique primitive character φ^* : its conductor $\mathfrak{f}_{\varphi^*}$ divides $\mathfrak{m}$, it agrees with φ on ideals prime to $\mathfrak{m}$, and it is zero on primes dividing $\mathfrak{f}_{\varphi^*}$. Put $Q_{\varphi^*} = N\mathfrak{f}_{\varphi^*}$. The exponent of $\mathfrak{f}_{\varphi^*}$ at a prime $\mathfrak{p}$ is the least $c \geq 0$ such that $\varphi_f(a) = 1$ whenever $a \equiv 1 \pmod{\mathfrak{p}^c}$ and $a \equiv 1$ modulo the part of $\mathfrak{m}$ prime to $\mathfrak{p}$. A finite-order character has trivial infinity type [36, proof of Lemma 4.8]; the members of $\mathcal{F}_{\text{CM}}$ have $\varphi_\infty(z) = \bar{z}/|z|$.

The target and the fixed data. Fix $\psi \in \mathcal{F}_{\text{CM}}$, of conductor $\mathfrak{f}_\psi$. For primary n prime to $\mathfrak{f}_\psi$ put $\eta(n) = \alpha(n)\psi((n))$. If $x \equiv 1 \pmod{\text{lcm}(\mathfrak{f}_\psi, 3)}$, then x is primary and $\psi((x)) = \bar{x}/|x|$, so $\eta(x) = 1$. Thus η is a ray class character modulo $\text{lcm}(\mathfrak{f}_\psi, 3)$, of finite order, and $\psi(n) = \overline{\alpha(n)}\eta(n)$, as in Theorem 1. The arithmetic datum $\mathcal{A}$ is that of [36, Section 4.1] for this η . It contains the finite group

$$\Theta_0 = \langle \eta, \widehat{T} \rangle \quad (\text{D.1})$$

of [36, Section 6.1] (written Θ there), and the finite set S of [36, Section 6.1]. It contains the primes above 6, the defining-modulus supports of η and of the members of $\widehat{T}$, the primes dividing $\mathfrak{f}_\psi$, and every prime of norm at most P_0 . The angular factor needs no new prime: on ideals prime to 3, $(a) \mapsto \alpha(a')$, with a' the primary generator, is the Hecke character modulo 3 with $\alpha_\infty(z) = z/|z|$ and $\alpha_f(a) = a'/a$. The datum depends on ψ , which [36, Section 4.1] allows. A prime is *good* if it is not in S . As in [36, Section 4.1], the letters n, l, k denote good primary elements, p a good primary prime, and $q_n = N(n)$.

The zero ceiling. Let $\beta_*^{(0)}$ be the supremum [36, eq. (2.1)] over primitive finite-order characters, let β_* be the supremum of Section 11 over $\mathcal{F}_{\text{CM}}$, and put

$$\beta^\sharp = \max(\beta_*, \beta_*^{(0)}). \quad (\text{D.2})$$

By [36, Theorem 1.1], $\beta_*^{(0)} \leq 7/8$. Hence $\beta^\sharp = \beta_*$ whenever $\beta_* \geq 7/8$: under the contradiction hypothesis $\beta_* > 7/8$ of Section 15, and under $\beta_* > 11/12$ in the 11/12 stage.

The label set. Put

$$\Lambda = \{(\nu, \varsigma) : \varsigma \in \{1, -1\}, \nu \in \Theta_0 \cup \alpha^\varsigma \Theta_0\}, \quad (\text{D.3})$$

a set of $4|\Theta_0|$ labels, independent of the row. For a nonzero sixth-power-free element u and $(\nu, \varsigma) \in \Lambda$ define, on good primary n ,

$$\varphi_{u, \nu, \varsigma}(n) = \nu(n)\chi_n(u)^\varsigma, \quad (\text{D.4})$$

with the zero convention of [36, Section 4.1]: $\chi_n(u)^{\pm 1} = 0$ when $(n, u) \neq 1$. Let

$$\mathcal{X}_u = \{\varphi_{u, \nu, \varsigma} : (\nu, \varsigma) \in \Lambda\}. \quad (\text{D.5})$$

A member is *finite-order* if $\nu \in \Theta_0$, and *CM* if $\nu \in \alpha^\varsigma \Theta_0$. The labels $(\theta, \pm 1)$, $(\bar{\alpha}\theta, -1)$ and $(\alpha\theta, 1)$ give the three kinds of character displayed in Section 13, so this is the set used there. Its finite-order members form the set $\mathcal{X}_u$ of [36, Section 8.1] for the group $\Theta = \Theta_0$. In a CM member the angular factor and the residue symbol carry the same exponent ς ; these are the co-oriented characters of Section 13. For a member φ and $\Re s > 1$ put $L_{\text{orig}}(s, \varphi) = \sum_n \varphi(n) q_n^{-s}$, the series of [36, Section 8.1].

Lemma 23. *Let u be a nonzero sixth-power-free element, and let $\varphi = \varphi_{u, \nu, \varsigma} \in \mathcal{X}_u$.*

- (a) $(\bar{\nu}, -\varsigma) \in \Lambda$ and $\overline{\varphi_{u, \nu, \varsigma}} = \varphi_{u, \bar{\nu}, -\varsigma}$. So $\mathcal{X}_u$ is closed under conjugation.
- (b) $\mathcal{X}_u$ contains the numerator $\chi_\bullet(u)$ and its conjugate, the denominator $\psi \overline{\chi_\bullet(u)} = \varphi_{u, \bar{\alpha}\eta, -1}$ and its conjugate, and the slot characters $\theta \overline{\chi_\bullet(u)}$, $\theta \in \hat{T}$.
- (c) φ is completely multiplicative on good primary elements and $|\varphi(n)| \leq 1$. For $\Re s > 1$,

$$L_{\text{orig}}(s, \varphi) = \prod_p (1 - \varphi(p) q_p^{-s})^{-1}, \quad \frac{1}{L_{\text{orig}}(s, \varphi)} = \sum_n \mu(n) \varphi(n) q_n^{-s}. \quad (\text{D.6})$$

- (d) There is a unique primitive Hecke character φ^* with $\varphi(n) = \varphi^*((n))$ for every good primary n prime to u . If φ is finite-order, so is φ^* . If φ is CM, then $\varphi_\infty^*(z) = (z/|z|)^\varsigma$: for $\varsigma = -1$, $\varphi^* \in \mathcal{F}_{\text{CM}}$, and for $\varsigma = 1$, $\overline{\varphi^*} \in \mathcal{F}_{\text{CM}}$. A CM φ^* does not have finite order; in particular it is not principal, and $L(s, \varphi^*)$ is entire.
- (e) Let u' be the product of the good primes dividing u . There is an ideal $\mathfrak{m}_A$, supported on S and depending only on A , with $\mathfrak{f}_{\varphi^*} \mid \mathfrak{m}_A u'$. Hence $Q_{\varphi^*} \leq N(\mathfrak{m}_A) N(u)$. If $\varphi = \varphi_{u, \alpha^\varsigma \theta, \varsigma}$ is CM and $\varphi_0 = \varphi_{u, \theta, \varsigma}$, then $\mathfrak{f}_{\varphi^*}$ and $\mathfrak{f}_{\varphi_0^*}$ have the same exponent at every prime not above 3, and $\mathfrak{f}_{\varphi^*} \mid \text{lcm}(\mathfrak{f}_{\varphi_0^*}, 3)$; so $Q_{\varphi^*} \leq 9 Q_{\varphi_0^*}$.
- (f) Every good prime $p \mid u$ divides $\mathfrak{f}_{\varphi^*}$, so $\varphi^*(p) = 0$. Consequently

$$L_{\text{orig}}(s, \varphi) = L(s, \varphi^*) \prod_{\mathfrak{p} \in S} (1 - \varphi^*(\mathfrak{p}) N\mathfrak{p}^{-s}), \quad (\text{D.7})$$

which is the identity of [36, Section 8.1] with $E_u \subset S$. If u is not supported on S , then φ^* is not principal.

- (g) If $L(\rho, \varphi^*) = 0$ and $\Re \rho \geq 1/2$, then $\Re \rho \leq \min(1, \beta^\#)$; if φ is CM, then $\Re \rho \leq \min(1, \beta_*)$.
- (h) Let φ be CM and $Q = Q_{\varphi^*}$. Then $\Lambda(s, \varphi^*) = (3Q)^{s/2} (2\pi)^{-s} \Gamma(s + \frac{1}{2}) L(s, \varphi^*)$ satisfies $\Lambda(s, \varphi^*) = \epsilon(\varphi^*) \Lambda(1-s, \overline{\varphi^*})$ with $|\epsilon(\varphi^*)| = 1$, and $|L(\sigma + it, \varphi^*)| \ll Q^{3/5} (3 + |t|)^2$ for $-1/10 \leq \sigma \leq 11/10$. For $51/100 \leq a \leq 1$, $0 < e < 10^{-3}$ and real t ,

$$|L(1-a-6e+it, \varphi^*)| \leq C_0 (3Q)^{a-1/2+6e} (3+|t|)^{2a-1+12e} |L(a+6e-it, \overline{\varphi^*})|, \quad (\text{D.8})$$

with an absolute constant C_0 .

Proof. (a) Θ_0 is a finite group, so $\bar{\theta} = \theta^{-1} \in \Theta_0$, and $\overline{\alpha^\varsigma \theta} = \alpha^{-\varsigma} \bar{\theta}$ because $|\alpha| = 1$. If $(n, u) = 1$, then $\overline{\chi_n(u)^\varsigma} = \chi_n(u)^{-\varsigma}$ is a sixth root of unity; otherwise both sides are 0.

(b) $\chi_\bullet(u) = \varphi_{u, 1, 1}$ and $\overline{\chi_\bullet(u)} = \varphi_{u, 1, -1}$. On good primary n , $\psi(n) = \overline{\alpha(n)} \eta(n)$ with $\eta \in \Theta_0$, so $\psi \overline{\chi_\bullet(u)} = \varphi_{u, \bar{\alpha}\eta, -1}$; its conjugate is given by (a). Finally $\hat{T} \subset \Theta_0$.

(c) $\alpha(mn) = \alpha(m)\alpha(n)$ for all nonzero m, n , and products of primary elements are primary [36, Section 4.1]. The members of Θ_0 are multiplicative on good ideals. By definition $\chi_{mn}(u) =$

$\chi_m(u)\chi_n(u)$ [36, Section 4.1], and with the zero convention $\chi_{mn}(u)^\varsigma = \chi_m(u)^\varsigma\chi_n(u)^\varsigma$, both sides vanishing exactly when $(mn, u) \neq 1$. The series converges absolutely since $|\varphi| \leq 1$; the two identities are its Euler product and Möbius inversion.

(d) Let $\theta \in \Theta_0$ and $\varphi_0 = \varphi_{u, \theta, \varsigma}$. By [36, Section 8.1] and [36, Lemma 4.1], φ_0 is induced by a primitive finite-order character φ_0^* whose conductor $\mathfrak{f}_0$ is supported on $S \cup \{p : p \mid u\}$. This is the finite-order case. Put $\mathfrak{m} = \text{lcm}(\mathfrak{f}_0, 3)$. For $a \in \mathbb{Z}[\omega]$ prime to $\mathfrak{m}$ write $a = \epsilon_a a'$ with a' primary and ϵ_a a unit; ϵ_a depends only on $a \bmod 3$, and $\epsilon_{ab} = \epsilon_a \epsilon_b$. Since φ_0^* is a ray class character modulo $\mathfrak{f}_0$, $\varphi_0^*((a)) =: \varphi_{0, \mathfrak{f}}(a)$ depends only on $a \bmod \mathfrak{f}_0$. On ideals prime to $\mathfrak{m}$ define

$$\phi((a)) = \alpha(a')^\varsigma \varphi_0^*((a)) = \left(\frac{a}{|a|}\right)^\varsigma \epsilon_a^{-\varsigma} \varphi_{0, \mathfrak{f}}(a). \quad (\text{D.9})$$

So ϕ is a Hecke character modulo $\mathfrak{m}$, with $\phi_\infty(z) = (z/|z|)^\varsigma$ and $\phi_{\mathfrak{f}}(a) = \epsilon_a^{-\varsigma} \varphi_{0, \mathfrak{f}}(a)$. Let φ^* be the primitive character inducing it. The primes of $\mathfrak{m}$ lie in $S \cup \{p : p \mid u\}$, so a good primary n prime to u is prime to $\mathfrak{m}$, and $\varphi^*((n)) = \phi((n)) = \alpha(n)^\varsigma \theta(n) \chi_n(u)^\varsigma = \varphi(n)$. Uniqueness holds because a primitive Hecke character is determined by its values at all but finitely many primes. For $\varsigma = -1$, $\varphi_\infty^*(z) = \bar{z}/|z|$, so $\varphi^* \in \mathcal{F}_{\text{CM}}$; for $\varsigma = 1$, $\overline{\varphi^*}$ is primitive with the same conductor and infinity type $\bar{z}/|z|$. Suppose $\varphi^{*N} = 1$ for some $N \geq 1$. For $x \in \mathbb{Z}[\omega]$ with $x \equiv 1 \pmod{\mathfrak{m}}$ we have $\epsilon_x = 1$ and $\varphi_{0, \mathfrak{f}}(x) = 1$, so $(x/|x|)^{\varsigma N} = \phi((x))^N = 1$. Write $\mathfrak{m} = (g)$ and choose $h \in \{g, g\omega\}$ not real. The points $1 + jh$, $j \in \mathbb{Z}$, lie on a line not through 0, so $x/|x|$ takes infinitely many values on them, a contradiction. Finally $L(s, \varphi^*)$ is entire by Lemma 10, applied to φ^* or to $\overline{\varphi^*}$, since $L(s, \varphi^*) = \overline{L(\bar{s}, \overline{\varphi^*})}$.

(e) First the finite-order members. Let κ_u be the ray class character inducing $n \mapsto \chi_n(u)$. By [36, Lemma 4.1] it is the Artin character of $K(u^{1/6})/K$, so, by local class field theory [33, Ch. V], its local component $\kappa_{u, \mathfrak{p}}$ at each prime is a character of $K_{\mathfrak{p}}^\times$ trivial on $(K_{\mathfrak{p}}^\times)^6$.

- At a good $\mathfrak{p} \nmid u$ it is unramified.
- At a prime $\mathfrak{p}$ of residue characteristic $\ell \geq 5$, the group $1 + \mathfrak{p}\mathbb{Z}[\omega]_{\mathfrak{p}}$ is pro- ℓ , hence consists of sixth powers. So the exponent of the conductor is at most 1. This covers the good primes dividing u and the primes of S not above 6.
- At the prime 2 (inert, K_2 unramified over $\mathbb{Q}_2$): Hensel's lemma applied to $(1 + 4z)^2 = 1 + 8(z + 2z^2)$ and to $(1 + 2z)^3 = 1 + 2(3z + 6z^2 + 4z^3)$ gives $1 + 8\mathbb{Z}[\omega]_2 \subset (1 + 4\mathbb{Z}[\omega]_2)^2$ and $1 + 2\mathbb{Z}[\omega]_2 \subset (\mathbb{Z}[\omega]_2^\times)^3$. Hence $1 + 8\mathbb{Z}[\omega]_2 \subset (K_2^\times)^6$, and the exponent is at most 3.
- At λ : for $k \geq 2$ the map $x \mapsto (1 + x)^3 = 1 + 3x + 3x^2 + x^3$ takes $1 + \lambda^k \mathbb{Z}[\omega]_\lambda$ onto $1 + \lambda^{k+2} \mathbb{Z}[\omega]_\lambda$, by successive approximation, since $v_\lambda(3) = 2$. Also $1 + \lambda \mathbb{Z}[\omega]_\lambda$ consists of squares. Hence $1 + \lambda^4 \mathbb{Z}[\omega]_\lambda \subset (K_\lambda^\times)^6$, and the exponent is at most 4.

So $\mathfrak{f}(\kappa_u)$ divides $M_S u'$, where $M_S = 2^3 \lambda^4 \prod \mathfrak{p}$, the product over the primes of S not above 6. Let $\mathfrak{f}_{\Theta_0}$ be the least common multiple of the conductors of the members of Θ_0 , and $\mathfrak{m}_{\mathcal{A}} = \text{lcm}(M_S, \mathfrak{f}_{\Theta_0})$. Then $\mathfrak{f}_{\varphi_0^*} = \mathfrak{f}(\theta \kappa_u^\varsigma)$ divides $\mathfrak{m}_{\mathcal{A}} u'$. This is the conductor bound $\ll_{\mathcal{A}} q_u$ of [36, Section 8.1], which is attributed there to reciprocity; [36, Lemma 4.1] states no bound on the exponents. For a CM member, $\mathfrak{f}_{\varphi^*}$ divides the modulus $\mathfrak{m} = \text{lcm}(\mathfrak{f}_0, 3)$ of (d), and $3 \mid M_S$, so $\mathfrak{f}_{\varphi^*} \mid \mathfrak{m}_{\mathcal{A}} u'$ and $Q_{\varphi^*} \leq N(\mathfrak{m}) \leq 9Q_{\varphi_0^*}$. At a prime $\mathfrak{p}$ not above 3, the condition $a \equiv 1$ modulo the part of $\mathfrak{m}$ prime to $\mathfrak{p}$ forces $a \equiv 1 \pmod{3}$, hence $\epsilon_a = 1$ and $\phi_{\mathfrak{f}}(a) = \varphi_{0, \mathfrak{f}}(a)$. So the two conductors have the same exponent at $\mathfrak{p}$.

(f) Let $p \mid u$ be good, with $j = v_p(u) \in \{1, \dots, 5\}$. By [36, Section 8.1] the local character of $\chi_\bullet(u)$ at p has order $6/\gcd(6, j) > 1$ on units, and θ is unramified at p ; so $p \mid \mathfrak{f}_{\varphi_0^*}$, and by (e) also $p \mid \mathfrak{f}_{\varphi^*}$. Hence $\varphi^*(p) = 0 = \varphi(p)$. At good primes $p \nmid u$, $\varphi(p) = \varphi^*(p)$ by (d), and the primes of S

are absent from L_{orig} . This gives the identity. If u has a good prime factor, the conductor is not (1), so φ^* is not principal.

(g) $|\varphi^*(\mathfrak{p})| \leq 1$, so the Euler product converges absolutely in $\Re s > 1$ and has no zeros there. If φ is finite-order, φ^* is a primitive finite-order character, and $\Re \rho \leq \beta_*^{(0)}$ by [36, eq. (2.1)] (which includes the principal character). If φ is CM with $\varsigma = -1$, then $\varphi^* \in \mathcal{F}_{\text{CM}}$ and $\Re \rho \leq \beta_*$. If $\varsigma = 1$, then $L(\bar{\rho}, \varphi^*) = \overline{L(\rho, \varphi^*)} = 0$ with $\bar{\varphi}^* \in \mathcal{F}_{\text{CM}}$, so again $\Re \rho \leq \beta_*$.

(h) For $\varsigma = -1$ the functional equation and the strip bound are Lemma 10. For $\varsigma = 1$ apply that lemma to $\bar{\varphi}^* \in \mathcal{F}_{\text{CM}}$: conjugating $\Lambda(\bar{s}, \bar{\varphi}^*) = \epsilon \Lambda(1 - \bar{s}, \varphi^*)$ gives $\Lambda(s, \varphi^*) = \bar{\epsilon} \Lambda(1 - s, \bar{\varphi}^*)$, and $|L(\sigma + it, \varphi^*)| = |L(\sigma - it, \bar{\varphi}^*)|$. The functional equation gives

$$L(s, \varphi^*) = \epsilon(\varphi^*) (3Q)^{1/2-s} (2\pi)^{2s-1} \frac{\Gamma(\frac{3}{2}-s)}{\Gamma(\frac{1}{2}+s)} L(1-s, \bar{\varphi}^*). \quad (\text{D.10})$$

At $s = 1 - a - 6e + it$ we have $|(3Q)^{1/2-s}| = (3Q)^{a-1/2+6e}$ and $|(2\pi)^{2s-1}| = (2\pi)^{1-2a-12e} < 1$. The gamma quotient is $\Gamma(x_1 - it)/\Gamma(x_2 + it)$ with $x_1 = \frac{1}{2} + a + 6e \in [1.01, 1.506]$ and $x_2 = \frac{3}{2} - a - 6e \in [0.494, 0.99]$. Stirling's formula, uniform for x in a compact subset of $(0, \infty)$, gives $|\Gamma(x + iy)| \asymp (1 + |y|)^{x-1/2} e^{-\pi|y|/2}$. So the quotient is $\ll (3 + |t|)^{x_1-x_2} = (3 + |t|)^{2a-1+12e}$. This proves (D.8). $\square$

For finite-order characters the quotient in the same computation is $\Gamma(a+6e-it)/\Gamma(1-a-6e+it)$, which has the same exponent $2a-1+12e$. So (D.8) is the CM form of the bound $Q^{a-1/2+6e}(3+|t|)^C$ of [36, proof of Lemma 8.1], with the same exponent C (one may take $C = 2$, since $2a-1+12e < 2$). Numerically the normalized quotient $|\Gamma(x_1 - it)/\Gamma(x_2 + it)|(3 + |t|)^{-(2a-1+12e)}$ lies in $[0.16, 1]$ on the whole range.

Lemma 24 (Logarithmic control for CM characters). *Let φ^* be a primitive Hecke character with $\varphi^* \in \mathcal{F}_{\text{CM}}$ or $\bar{\varphi}^* \in \mathcal{F}_{\text{CM}}$, of conductor norm Q , and put $\mathcal{L}(s) = L(s, \varphi^*)$. Let $a \in [1/2, 1]$ and $0 < e < 10^{-3}$, and suppose $\mathcal{L}$ has no zero in the open disk of radius $2 - a - 2e$ about $2 + it$. Then, on the closed disk of radius $2 - a - 6e$,*

$$|\mathcal{L}(s)| + |\mathcal{L}(s)^{-1}| \ll_{e,\epsilon} \{2Q(3 + |t|)^2\}^\epsilon, \quad (\text{D.11})$$

and on the disk of radius $2 - a - 8e$, $|\mathcal{L}'(s)/\mathcal{L}(s)| \ll_e \log\{2Q(3 + |t|)^2\}$. For every fixed $b \geq \beta_$ and $v > 0$ these bounds hold on $\Re s \geq b + v$, with the height of s in place of t .*

Proof. This is Lemma 20, applied to φ^* or to its conjugate. The inputs it needs are that $L(s, \varphi^*)$ is entire (Lemma 23(d)), the strip bound (Lemma 23(h)), the bound $|\varphi^*(\mathfrak{p})| \leq 1$, and, for the last assertion, the absence of zeros in $\Re s > \beta_*$ (Lemma 23(g)). $\square$

D.2 The detector

Proposition 11. *Assume $\beta^\sharp > 51/100$. Fix the datum $\mathcal{A}$ above and the parameters $d_{\min}, d_{\max}, d, U = Z^d, \tau, T_1 = Z^\tau, e, I$ of [36, Section 8.1]. Remove the finitely many sixth-power-free rows supported on S , as in [36, Section 8.1]. In [36, Section 8] replace the set $\mathcal{X}_u$ by the set $\mathcal{X}_u$ above, the characters ψ^* by φ^* , the label set $\Theta \times \{-1, 1\}$ by Λ , and β_* by $\beta^\sharp$. Then, for every retained row u with $N(u) \asymp U$:*

- (i) [36, Lemma 8.1] holds, with constants depending only on $\mathcal{A}$, e and ϵ_1 ;
- (ii) every bin satisfies $a \leq \beta^\sharp$ and $\delta = 2a - 1 \leq 2\beta^\sharp - 1$;
- (iii) [36, Lemma 8.2] holds for every $\varphi \in \mathcal{X}_u$;

(iv) [36, Prop. 8.3] holds, the two witnesses using one presentation $\varphi_{u,\nu,\varsigma}$ whose label (ν, ς) lies in Λ . For a CM label $\nu = \alpha^\varsigma \theta$ they are

$$\begin{aligned} M_r &= U^{-r/2} \sum_n \mu(n) \alpha(n)^\varsigma \theta(n) \chi_n(u)^\varsigma W_M \left(\frac{q_n}{U^r} \right), \\ S_m &= U^{-m/2} \sum_l \alpha(l)^\varsigma \theta(l) \chi_l(u)^\varsigma W_S \left(\frac{q_l}{U^m} \right), \end{aligned} \tag{D.12}$$

with the profiles W_M, W_S of [36, proof of Proposition 8.3].

Proof. We list every place in [36, Section 8] where a property of a row character is used. The other steps concern the bin parameters, the Mellin and Fourier calculus of the profiles ([36, Lemma 4.5, eqs. (4.11) and (4.10)]), or the dyadic partitions, and hold verbatim.

Setting, [36, Section 8.1]. The group Θ of [36, Section 8.1] is Θ_0 ; its conductor primes lie in S by the choice of S . Closure under conjugation and the list of members ([36, Section 8.1]) are Lemma 23(a),(b); [36] used the group property of Θ , which Λ replaces. The conductor bound and the deletion identity ([36, Section 8.1]) are Lemma 23(e),(f), now with $E_u \subset S$ and a deletion radical of norm at most $\prod_{\mathfrak{p} \in S} N\mathfrak{p}$. The ramification argument ([36, Section 8.1]) applies verbatim to the finite-order members; the CM members are never principal (Lemma 23(d)). The rows supported on S are still removed, because a finite-order member of such a row can be principal. [36, Section 8.1] involve no character.

(i), proof at [36, proof of Lemma 8.1].

- [36, proof of Lemma 8.1], existence of the maxima $M_j(u)$: $\mathcal{X}_u$ is finite; for a retained row every φ^* is nonprincipal (Lemma 23(d),(f)), so $L(s, \varphi^*)$ is entire (for finite-order members by [36, proof of Lemma 4.8]) and its zeros are discrete; there are none in $\Re s > 1$ by Lemma 23(g).
- [36, proof of Lemma 8.1]: the pigeonhole choice of (i, a) and the zero-free disks involve no character.
- [36, proof of Lemma 8.1]: for finite-order members [36, Lemma 4.9], for CM members Lemma 24; in both, $Q_{\varphi^*} \ll_{\mathcal{A}} U$ by Lemma 23(e). The deleted factors are bounded by [36, Lemma 4.10], which uses only $|\varphi^*(\mathfrak{p})| \leq 1$; by Lemma 23(f) the deletion is over the fixed set S .
- [36, proof of Lemma 8.1], the reflected line: the functional equation recalled in [36, proof of Lemma 4.8] is replaced, for CM members, by (D.8). The conjugate presentation lies in $\mathcal{X}_u$ (Lemma 23(a)), so the reflected primitive value $L(a + 6e - it, \overline{\varphi^*})$ is $\ll U^{\epsilon_1/3}$ by the disk bound just proved. The deleted product at $\Re s = 1 - a - 6e \geq -6e$ is $O_{\mathcal{A}}(1)$. With $(3Q)^{a-1/2+6e} \ll_{\mathcal{A}} U^{a-1/2+6e}$ and $(3 + |t|)^{2a-1+12e} \leq (3 + (3i + 2)T_1)^2$ this gives the stated $U^{a-1/2+12e+\epsilon_1} (3 + T_1)^C$.

(ii), replacing [36, Section 8.1]. The sentence “Every primitive character inducing a presentation in $\mathcal{X}_u$ is a finite-order Hecke character” is false for CM targets. Instead, Lemma 23(g) gives $M_i(u) \leq \beta^\sharp$ for every retained row, hence $a \leq M_i(u) \leq \beta^\sharp$.

(iii), proof at [36, proof of Lemma 8.2]. The Mellin identity of [36, proof of Lemma 8.2] is the second identity of Lemma 23(c). The shifted central portion and its joins lie in the rectangle of (i). On $\Re(s + \sigma) = 2$ the reciprocal Euler product is bounded by $\zeta_K(2)$, since $|\varphi| \leq 1$. For S_m , the continuation to $\Re(s + \sigma) = 1 - a - 6e$ uses that $L_{\text{orig}}(s, \varphi)$ is entire (Lemma 23(d),(f)); its joins use the strip bound of Lemma 23(h) (for finite-order members [36, eq. (4.14)]) and the bounded deleted product over S ; the reflected estimate is (i).

(iv), proof at [36, proof of Proposition 8.3]. [36, proof of Proposition 8.3]: $L_{\text{orig}}(s, \varphi)$ is entire. [36, proof of Proposition 8.3]: on $\Re(\rho + z) = \sigma - 1/4 \in [26/100, 3/4]$ the strip bound of Lemma 23(h) gives $|L(\rho + z, \varphi^*)| \ll Q^{3/5}(3 + |\gamma + v|)^2$ with $Q^{3/5} \ll_{\mathcal{A}} U^{3/5} \leq U^2$, and the deleted product over S is bounded; the exponent $-3 + t(5/4 - \sigma) \leq -189/100$ is unchanged. [36, proof of Proposition 8.3]: the double sum for J and the vanishing of the coefficients with $1 < q_n \leq D_*$ use $\varphi(l)\varphi(k) = \varphi(lk)$, Lemma 23(c). [36, proof of Proposition 8.3] concern profiles and dyadic pairs only. The label is selected in Λ , a fixed set of $4|\Theta_0|$ elements ([36, Proposition 8.3]). $\square$

No step uses the order of a member, or the group structure of Θ_0 beyond Lemma 23(a). The gamma factor of a CM L -function enters only through Lemma 23(h) and Lemma 24, that is, through fixed polynomial orders in the height, as stated in Section 15.

D.3 Use of the detector

Labels. The high side uses only the bounds of (i) for the members listed in Lemma 23(b). Every row count that uses the witnesses subdivides the rows of one bin by the label and by the dyadic pair, and applies a mean-square estimate to each subdivision ([36, proof of Proposition 9.2] for the sextic count, [36, proof of Proposition 19.2] for Part II). Only the finiteness of Λ is used. Each row lies in one subdivision, each subdivision is a set of rows, and the estimates are sums of nonnegative terms over rows. No property of the map $u \mapsto (\nu, \varsigma)$, such as bounded fibres, is needed.

The sextic row count. The proof of [36, Prop. 9.2] uses the presentation only through the coefficients $c_n = D^{-1/2}\mu(n)\nu(n)\chi_n(v)^\varsigma W(q_n/D)$ of [36, proof of Proposition 9.2], which must be independent of the residual row and bounded by $D^{-1/2}|W(q_n/D)|$, and through the identity $\chi_n(u)^\varsigma = \chi_n(v)^\varsigma \chi_n(k)^\varsigma$ of [36, eq. (9.20)], which does not involve ν . Both hold for $\nu = \alpha^\varsigma \theta$, and the sextic large sieve [36, Lemma 9.1] allows any coefficients fixed before the row sum. So the row envelope $R(\delta)$ of that proposition holds for a CM target, with the bin ceiling (ii).

Orientation. A CM witness has coefficients $\alpha(n)^\varsigma \theta(n) \chi_n(u)^\varsigma$: it is co-oriented. Conjugating a whole witness, which [36, Section 4.1] allows, replaces (θ, ς) by $(\theta, -\varsigma)$. So items 4 and 5 of Section 16 are needed only for co-oriented coefficients, and both signs of ς are covered; Section 13 uses $\varsigma = -1$ for the inverse moment and $e = +1$ for the plain moment. The anti-oriented twists $\alpha^{-\varsigma} \theta \chi_\bullet(u)^\varsigma$ are not in $\mathcal{X}_u$. None is needed: by Lemma 23(b), $\mathcal{X}_u$ already contains every L -function that the high side bounds through the bins, namely the numerator, the denominator, their conjugates and the slot characters.

Summary. This appendix proves item 3 of Section 16. From Lemma 10, proved in Appendix C, the detector uses the functional equation, the absence of poles, and the uniform strip bound. Under the contradiction hypothesis $\beta_* > 7/8$ one has $\beta^\sharp = \beta_*$, so Proposition 11(ii) is the bin ceiling of Lemma 11. The mean-value estimates that consume the CM witnesses in Part II are items 4 and 5; the sextic count was treated above.

We checked numerically:

- the conductors of 180 pairs of members (a finite-order member φ_0 and its CM partner φ , both ς), for 15 rows with valuations 1–5, units and factors at 2 and λ : exponent 1 at every good prime of u , at most 3 at 2, at most 4 at λ , and $Q_{\varphi^*} \leq 9Q_{\varphi_0^*}$, with equality of exponents away from λ . In 7 cases the CM member is unramified at λ ;
- the infinity type $(z/|z|)^\varsigma$ of every CM member;

- the functional equation of Lemma 23(h), through PARI's [38] `lfuncheckfeq`, for 8 CM members with conductor norms from 28 to 117936 (agreement to 2^{-49} or better), and the deletion identity of (f);
- closure under conjugation, membership, complete multiplicativity and the truncated-inverse coefficients of (iv), on a model group Θ_0 of order 216;
- the gamma quotient in (D.8) on a grid with $|t| \leq 10^{12}$.

E The inverse moment for co-oriented coefficients

This appendix writes out item 4 of Section 16: the CM form (14.2) of [36, Lemma 17.1], together with the CM form of the canonical estimate [36, Lemma 17.2] through which it is proved. In [36] the proof occupies [36, Section 17], and the amplification on sixth-power-free rows [36, Section 17.7]. We follow it step by step. The angular factor enters only as a completely multiplicative unimodular factor on the column variables (Lemma 27). We display every formula in which it appears and cite the others by number. The terminal step needs the CM form of the reflected energy [36, Lemma 14.3]. This is Lemma 15 of Appendix B; we record the form in which it is used as Statement (R) below.

E.1 Notation and statements

We use the conventions of [36, Section 4], which are those of Section 7.3 (“Normalizations”). In particular S is a fixed finite set of primes containing those above 6 and the support of the defining modulus of every fixed finite-order character; ideal variables are represented by their primary generators and lie outside S ; every power of a residue symbol is zero on nonunits, so that $\chi_c(a)^0 = \mathbf{1}_{(a,c)=1}$; γ_j is the normalized Gauss sum of (7.7); and $G(c) = \overline{\chi_c(4)}\Gamma(c)$ is the finite-ray function of [36, Lemma 4.4], equal to $\overline{\chi_c(4)}\gamma_3(c)$ for squarefree c . As there, μ is the Möbius function and “sf” means squarefree. Put

$$\varphi(n) = \overline{\alpha(n)}. \quad (\text{E.1})$$

The product of primary generators is the primary generator of the product, so φ is completely multiplicative on primary elements, and $|\varphi| = 1$.

Fix a finite-order ray character ν whose zero-extended defining modulus has prime support in S , and a sign $\epsilon \in \{1, -1\}$. Put $\psi_u(n) = \nu(n)\chi_n(u)^\epsilon$, with all zero extensions. For a smooth annular W (supported in a fixed compact subinterval of $(0, \infty)$) put

$$M_u(Z^r; W) = Z^{-r/2} \sum_n \mu(n) \alpha(n)^\epsilon \psi_u(n) W\left(\frac{N(n)}{Z^r}\right). \quad (\text{E.2})$$

The angular factor α^ϵ is paired with $\chi_\bullet(u)^\epsilon$: the sum is co-oriented in the sense of Section 13. For a finite set I of slots, disjoint sets $\mathcal{P}_i$ of primes outside S with $N(p) \asymp Z^{z_i}$ for $p \in \mathcal{P}_i$, coefficients $a_i(p)$ independent of u with $|a_i(p)| \leq 1$, and smooth annular W_i , put, exactly as in [36, Section 17.1],

$$Q_u = \prod_{i \in I} Z^{-z_i/2} \sum_{p \in \mathcal{P}_i} a_i(p) \psi_u(p) W_i\left(\frac{N(p)}{Z^{z_i}}\right), \quad z = \sum_{i \in I} z_i. \quad (\text{E.3})$$

The slots carry no angular factor; a factor $\alpha(p)^\epsilon$ could be absorbed into $a_i(p)$.

Proposition 12 (CM marked inverse moment). *Fix bounded ranges for the nonnegative parameters m, r, z_i , a bound for $|I|$, and positive constants c_1, c_2 , and suppose*

$$r + 2z \leq m - c_1, \quad 2r + 8z \leq 3m - c_2. \quad (\text{E.4})$$

Then, for every $\varepsilon > 0$,

$$\sum_{\substack{u \in \mathbb{Z}[\omega] \\ 0 < N(u) \ll Z^m}} |M_u(Z^r; W) Q_u|^2 \ll Z^{m+\varepsilon}. \quad (\text{E.5})$$

The uniformity is that of [36, Lemma 17.1]: the implied constant depends on the fixed arithmetic data, c_1, c_2 , the parameter ranges and finitely many smooth seminorms of the tests, and is uniform in the prime supports and their bounded coefficients; the estimate holds for both signs ϵ , and for every subcollection of the slots; norm twists of the tests cost a fixed power of their heights; there is no lower bound or mesh condition on the slot lengths. A fixed finite sum of whole products $M_u Q_u$ is allowed, provided each summand uses one ν and one ϵ in its inverse polynomial and all its slots ([36, Section 17.1]).

This is (14.2). It is the form in which [36] uses its lemma for a witness row ([36, Section 19.2]): a CM witness of Section 13 is co-oriented, and after conjugation, if necessary, it has $\epsilon = -1$, $\nu = \theta \in \Theta_0$, and slot coefficients $a_i(p) = \overline{\nu(p)} \mathbf{1}_{p \in 1_T}$ times a profile, which are bounded and independent of u .

The canonical family. Let f range over squarefree ideals outside S with $N(f) \asymp Z^V$, with a divisor-bounded weight $w(f)$ depending on f alone; let $\rho(n) = \mathbf{1}_{(n, \tau_\rho)=1}$ be a fixed puncture; and let $\mathfrak{d}$ be a mark of the product form [36, eq. (14.1)], with disjoint fixed prime lists of total nominal length at most z_0 and bounded coefficients independent of k, f . For squarefree n outside S put

$$a(n) = \overline{\alpha(n)}^2 \gamma_2(n) \nu(n) \rho(n) = \varphi(n) \cdot \overline{\alpha(n)} \gamma_2(n) \nu(n) \rho(n), \quad (\text{E.6})$$

which is φ times the coefficient of [36, eq. (17.3)], and

$$\mathcal{E} = Z^{-V} \sum_f w(f) \sum_{\substack{k \in \mathbb{Z}[\omega] \\ 0 < N(k) \ll Z^M}} \left| Z^{-N/2} \sum_n^* a(n) \chi_n(k) \chi_n(f)^4 \mathfrak{d}(n) W\left(\frac{N(n)}{Z^N}\right) \right|^2. \quad (\text{E.7})$$

Without the puncture, the mark and the weight w , this is the average (7.10) of Part I, with $a_\xi = \overline{\alpha}^2 \gamma_2 \xi$.

Lemma 25 (CM canonical estimate). *Under the hypotheses of [36, Lemma 17.2], with the coefficient (E.6) in place of $\overline{\alpha} \gamma_2 \nu \rho$ (and no other residual coefficient, as required there), in particular the inequalities [36, eq. (17.2)] with $F_0 = N + V$, one has $\mathcal{E} \ll Z^{F_0+\varepsilon}$ for every $\varepsilon > 0$, with the uniformity stated there ([36, Lemma 17.2]).*

The completed sums and Statement (R). Put $\Psi_k(n) = \nu(n) \rho(n) \chi_n(k) \chi_n(f)^4$, as in [36, Section 14.1]; it is completely multiplicative on primary ideals outside S . Put $V_*(y) = y^{1/2} W(y)$. The CM form of the completed marked sum [36, eq. (14.2)] is

$$\mathcal{T}_\mathfrak{d}(X; k) = \sum_{\substack{n \text{ sf} \\ b}} \frac{\overline{\alpha(n)}^2 \gamma_2(n) \Psi_k(n) \overline{\alpha(b)}^6 \Psi_k(b)^3}{\sqrt{N(n)} N(b)} \mathfrak{d}(nb^3) V_*\left(\frac{N(n)N(b)^3}{X}\right), \quad (\text{E.8})$$

over primary n, b outside S , which may share primes. Its angular factor $\overline{\alpha(nb^3)}^2$ is that of the weight (12.1) and of (8.1); this is the completed sum of $\partial_{\bar{z}}^2 \theta$ (Section 8, Appendix A).

Statement (R). [36, Lemma 14.3 and its proof], with its block data ([36, Section 14.4, *Reflected block data*]) and the norm inequalities [36, eqs. (5.35) and (14.17)], holds for (E.8) in place of [36, eq. (14.2)]: with the same retention rule [36, eq. (14.11)], the same exponent E_{ref} of [36, eq. (14.14)], the same constants $C_{\text{res}}, C_{\text{width}}, C_{\text{ker}}$, and the same uniformity.

Statement (R) is Lemma 15: the sum (E.8) is (B.33), and that lemma is proved in Appendix B from the CM transformation formula at full generality (Propositions 9 and 10: general S , marks and punctures, log-Schwartz tests), with the same retention rule, exponent, constants and uniformity. In that formula the dual coefficient is $d_\sigma(\ell)\alpha(\ell)^2$, with the support and the bound $|d_\sigma(u\lambda^m nb^3)| \leq 27 \cdot 3^{m/6}|b|$ of [36, eq. (5.2)]; the scalar differs from that of [36] by the unimodular factor $-i\overline{\alpha(c)}^2$, so their moduli are equal and at most $1/81$; the local factors $B_{\pi,j}$ are those of [36]; and the weight obeys the same bounds as in [36].

E.2 The angular factor as a column twist

Lemma 26 (Switching identities). *Let u be squarefree, primary and outside S . Then*

$$\overline{\alpha(u)}^2 \gamma_2(u) \gamma_1(u) = \mu(u) \overline{\alpha(u)} G(u), \quad (\text{E.9})$$

$$\overline{\alpha(u)}^2 \gamma_2(u) \gamma_{-1}(u) = \overline{\mu(u) \overline{\alpha(u)} \chi_u(-1) G(u)}, \quad (\text{E.10})$$

$$\mu(u) \overline{\alpha(u)} \gamma_{-1}(u) = \overline{\alpha(u)}^2 \gamma_2(u) \chi_u(-1) \overline{G(u)}, \quad (\text{E.11})$$

$$\overline{\mu(u) \overline{\alpha(u)}} \gamma_1(u) = \overline{\alpha(u)}^2 \gamma_2(u) G(u), \quad (\text{E.12})$$

$$\mu(u) \alpha(u) \gamma_{-1}(u) = \gamma_2(u) \chi_u(-1) \overline{G(u)}. \quad (\text{E.13})$$

Moreover, for every multiplicative ν' of modulus one outside S , the coefficient $a' = \bar{\alpha}^2 \gamma_2 \nu'$ satisfies $|a'(u)| = 1$ and $a'(vw) = a'(v)a'(w)\chi_w(v)^4$ for coprime squarefree primary v, w outside S .

Proof. By [36, eq. (4.7)], which is (7.13), $\gamma_1 \gamma_2 = \mu \alpha G$. By [36, Lemma 4.2] and the Chinese remainder theorem ([36, proof of Lemma 4.4]), $|\gamma_j(u)| = 1$ for $6 \nmid j$; also $|\alpha| = |G| = 1$ and $\mu(u)^2 = 1$. Hence $\gamma_1 = \mu \alpha G \bar{\gamma}_2$. The substitution $v \mapsto -v$ in the Gauss sum gives $\gamma_1(u) = \chi_u(-1) \gamma_{-1}(u)$, with $\chi_u(-1) = \pm 1$. Now (E.9) is $\gamma_1 \gamma_2 = \mu \alpha G$ multiplied by $\bar{\alpha}^2$. For (E.11), $\mu \bar{\alpha} \gamma_{-1} = \mu \bar{\alpha} \chi(-1) \bar{\gamma}_1 = \mu^2 \bar{\alpha}^2 \gamma_2 \chi(-1) \bar{G}$. The identities (E.10) and (E.12) are the complex conjugates of (E.9) and (E.11), rewritten with $\bar{\gamma}_{\pm 1} = \chi(-1) \gamma_{\mp 1}$: for instance $\bar{\alpha}^2 \gamma_2 \gamma_{-1} = \chi(-1) \bar{\alpha}^2 \gamma_2 \gamma_1 = \chi(-1) \mu \bar{\alpha} G$. Finally (E.13) is (7.7) multiplied by α . The last assertion is (7.19) with ξ replaced by ν' : it follows from $\gamma_2(vw) = \gamma_2(v) \gamma_2(w) \chi_w(v)^4$ [36, proof of Lemma 17.2, *The first Poisson transformation*] and the complete multiplicativity of $\bar{\alpha}^2 \nu'$. $\square$

Identities (E.9)–(E.12) are [36, proof of Lemma 17.2, *The first Poisson transformation, The second Poisson transformation*] multiplied by $\varphi(u)$ on the first side of a square and by $\overline{\varphi(u)}$ on the second. The first two switch the cubic class $\bar{\alpha}^2 \gamma_2$ to the Möbius class $\mu \bar{\alpha}$; the last two switch back. They are (12.3) and its companion in Section 14, multiplied by $\overline{\alpha(u)}$; in paired form they are (7.18) and (7.17). By (E.13) the anti-oriented coefficient $\mu \alpha$ loses its angular factor.

Lemma 27 (Column twist). *Let $\sum_\tau L(\tau) = \sum_\sigma R(\sigma)$ be an identity of absolutely convergent sums. Suppose that there are a map λ from both index sets to a set Λ and a map $\Lambda \rightarrow \mathbb{Z}[\omega]^2$,*

$\lambda \mapsto (n_1(\lambda), n_2(\lambda))$, into pairs of primary elements outside S , such that the identity holds separately on each fibre of λ . Then for every bounded function φ on primary elements,

$$\sum_{\tau} \varphi(n_1) \overline{\varphi(n_2)} L(\tau) = \sum_{\sigma} \varphi(n_1) \overline{\varphi(n_2)} R(\sigma), \quad (\text{E.14})$$

where $n_a = n_a(\lambda(\tau))$, respectively $n_a(\lambda(\sigma))$. If φ is completely multiplicative and $n_a(\lambda(\sigma)) = e_a(\sigma)x_a(\sigma)$, then the twist on the right is $\varphi(e_1)\overline{\varphi(e_2)} \cdot \varphi(x_1)\overline{\varphi(x_2)}$.

Proof. The twist is constant on each fibre and bounded. Multiply the fibrewise identities by it and sum over Λ ; the weighted sums remain absolutely convergent, so regrouping is permitted. The same conclusion holds for an unbounded φ if absolute convergence of both weighted sums is assumed separately. $\square$

We apply this lemma with $\varphi = \bar{\alpha}$, λ the column data of the squared polynomial on the left, and (n_1, n_2) its two column indices (the index of the conjugated copy second). The identities of [36] to which we apply it have the fibrewise property. In each of the three Poisson transformations, [36] expands the square, fixes the column data, and applies [36, Lemma 17.5] to the row sum, whose coefficient is constant on that fibre. The later exact steps rewrite the terms of one fibre. These are the factorizations by greatest common divisors; the Möbius expansion of a coprimality condition together with its formal extension, which for fixed (u_1, u_2) is $\sum_{t|(u_1, u_2)} \mu(t) = \mathbf{1}_{(u_1, u_2)=1}$; the slot-priority identity [36, eq. (17.16)]; the extraction identities; and Fourier inversion of a joint profile evaluated at the norms of the indices. All of them are pointwise in the column data. Every other step of the three transformations is an inequality. These use the coefficients only through their absolute values: the triangle inequality, the weighted Cauchy inequality [36, eq. (D)], the discarded raw tails and the principal terms, which are bounded by lattice counts with absolute coefficients. The remaining inequalities are positive row norms of the new polynomials. A unimodular twist therefore changes nothing in them, once it has been placed in the new polynomials and in outer weights of the same absolute value.

E.3 Proof of Lemma 25

We follow the proof of [36, Lemma 17.2]. The following parts do not involve the column coefficients and apply verbatim:

- the induction on the depth, with the margins c_h and the parameters d, D, η, τ, π ([36, proof of Lemma 17.2]);
- the propagation of seminorm and height orders, [36, Lemma 17.3, Cor. 17.4] and the normalization rules ([36, Section 17.3]), which concern profiles and coefficient measures;
- the localization conventions, the joint Fourier calculus and the raw tail estimate ([36, proof of Lemma 17.2, *The two-transform reduction*]), which concern norms, profiles and lattice counts;
- [36, Lemma 17.5], which is applied only to row sums with finite-order characters $\chi_{\bullet}(k)$; the factor α never enters a row character.

Step 1: the short completion ([36, proof of Lemma 17.2, *The short completion*]). The CM form of [36, eq. (17.10)] is

$$\begin{aligned} & Z^{-N/2} \sum_n^* a(n) \chi_n(k) \chi_n(f)^4 \mathfrak{d}(n) W\left(\frac{N(n)}{Z^N}\right) \\ &= \sum_h \frac{\mu(h) \overline{\alpha(h)}^6 \Psi_k(h)^3}{N(h)} \mathcal{T}_{\mathfrak{d}_h}\left(\frac{Z^N}{N(h)^3}; k\right), \quad \mathfrak{d}_h(A) = \mathfrak{d}(h^3 A), \end{aligned} \quad (\text{E.15})$$

with h squarefree. Indeed, $\overline{\alpha(n)}^2 \gamma_2(n) \Psi_k(n) = a(n) \chi_n(k) \chi_n(f)^4$ since $\rho^2 = \rho$. Expand the right side and put $c = hb$. Complete multiplicativity of $\overline{\alpha}^6$ and of Ψ_k , including its zeros, gives $\overline{\alpha(h)}^6 \Psi_k(h)^3 \cdot \overline{\alpha(b)}^6 \Psi_k(b)^3 = \overline{\alpha(c)}^6 \Psi_k(c)^3$. Also $N(h)^{-1} N(n)^{-1/2} N(b)^{-1} V_*(N(n)N(c)^3/Z^N) = Z^{-N/2} N(c)^{1/2} W(N(n)N(c)^3/Z^N)$, and $\mathfrak{d}_h(nb^3) = \mathfrak{d}(nc^3)$ does not depend on h . So the coefficient of the term (n, c) contains $\sum_{h|c} \mu(h) = \mathbf{1}_{c=1}$, and only the left side remains. The sums are finite because W has compact support. In [36] the exponent is 3 in place of 6. When $V < d$ and the h -dyad has centre $\ell_1 < d$, the argument of [36, proof of Lemma 17.2, *The short completion*] applies with $\mathcal{T}_{\mathfrak{d}_h}$ of (E.8):

- the slots assigned to h , with surviving coefficients $a_i(p) \mathbf{1}_{p|h}$;
- the triangle inequality with $\sum_{N(h) \asymp Z^{\ell_1}} N(h)^{-1} \ll 1$, where the row multiplier $\overline{\alpha(h)}^6 \Psi_k(h)^3$ has modulus at most one, hence is a contraction;
- Lemma 15 (Statement (R)) in place of [36, Lemma 14.3];
- the exponent bookkeeping [36, eq. (17.11), (17.12)–(17.14)], which uses only the exponents provided by (R).

Step 2: marks and the block ([36, proof of Lemma 17.2]). The slot-priority identities [36, eq. (17.15), eq. (17.16)] concern indicators only. When the completion is reopened ([36, proof of Lemma 17.2, *The remaining terms and their marks*]), the cube coefficient of (E.8) is $\overline{\alpha(b)}^6 \Psi_k(b)^3$ instead of $\overline{\alpha(b)}^3 \Psi_k(b)^3$. The coefficient of the block [36, eq. (A)] is therefore

$$\beta^{\text{CM}}(b, f) = \overline{\alpha(b)}^3 \beta(b, f), \quad (\text{E.16})$$

where β is that of [36]. It has the same absolute value, is independent of k and n , and contains the mask $\mathbf{1}_{(b,f)=1}$. The CM block is [36, eq. (A)] with a from (E.6) and β^{CM} ; for $V \geq d$ it has $\ell = 0$, $b = 1$, $r = N$. The reduction statement [36, eq. (17.17)] is to be proved for this block, with child energies of the form (E.7).

Step 3: the first Poisson transformation ([36, proof of Lemma 17.2]). Relative to [36], the pair of columns n_1, n_2 of the expanded block carries the extra factor $\varphi(n_1) \overline{\varphi(n_2)}$; the cube factors are already in β^{CM} . Apply Lemma 27 to the exact steps from the expansion ([36, proof of Lemma 17.2, *The first Poisson transformation*]) to [36, eq. (17.31)]. On the factorized expression $n_a = \mathfrak{A}_a C t' x_a$ ([36, proof of Lemma 17.2, *The first Poisson transformation*; proof of Lemma 17.2, *Retaining the fourth-power label from the cubes*]), with C and t' common to both sides, so

$$\varphi(n_1) \overline{\varphi(n_2)} = \varphi(\mathfrak{A}_1) \overline{\varphi(\mathfrak{A}_2)} \cdot \varphi(x_1) \overline{\varphi(x_2)}. \quad (\text{E.17})$$

Consequently:

- the column coefficient [36, eq. (C)] acquires the factor $\overline{\alpha(u_i)}$ on side i . This agrees with (E.9) and (E.10), which give at u_1 the factor $\mu(u_1)\overline{\alpha(u_1)}G(u_1)$ and, inside the conjugated second polynomial, the factor $\mu(u_2)\overline{\alpha(u_2)}\chi_{u_2}(-1)G(u_2)$, in place of $\mu(u_1)G(u_1)$ and $\mu(u_2)\chi_{u_2}(-1)G(u_2)$ ([36, proof of Lemma 17.2, *The first Poisson transformation*]);
- the factors $B_{1,a}(t'; y)$ ([36, proof of Lemma 17.2, *Separating the first transformed product*]) are unchanged, because $|\varphi(t')|^2 = 1$;
- the polynomials [36, eq. (17.30)] become

$$P_a(y) = \sum_x^* \mu(x)\overline{\alpha(x)}\nu_a(x)\rho(x)\overline{\chi_x(y)}\chi_x(CJ)^4\chi_x(d_k)\mathbf{1}_{(x, \text{rad } q_0 t')=1} \times \mathfrak{d}_{I_a}(x)\omega_{x,a}\left(\frac{N(x)}{Z^{s_a}}\right)\left(\frac{N(x)}{Z^{s_a}}\right)^{\varepsilon_a i t_{x_a}}; \quad (\text{E.18})$$

- the outer product e_σ ([36, proof of Lemma 17.2, *Separating the first transformed product*]) is multiplied by $\varphi(\mathfrak{A}_1)\varphi(\mathfrak{A}_2)$ and contains β^{CM} , so the outer weight $w_{1,\sigma}$ ([36, proof of Lemma 17.2, *Separating the first transformed product*]) changes only by a factor of modulus one.

Then [36, eq. (17.31)] holds with (E.18). The principal case $m_1 = 1$ ([36, proof of Lemma 17.2, *The first Poisson transformation*]) is bounded by an absolute count, and so are the discarded raw complement and the restored principal frequencies; these bounds are unchanged. The first positive majorant ([36, proof of Lemma 17.2, *The first positive majorant*]) applies the weighted Cauchy inequality and then bounds the outer factors absolutely ([36, proof of Lemma 17.2, *The first positive majorant*]), so it holds verbatim. The polynomial $P_i(y)$ of (E.18) again depends on $b_a, \mathfrak{A}_a, f$ only through J, q_0 , the length A_i and the sector, because $\varphi(x)$ depends on x alone. The output [36, eq. (17.35)] therefore holds with the majorant [36, eq. (17.33)] formed from (E.18).

Step 4: the second Poisson transformation ([36, proof of Lemma 17.2, *The second Poisson transformation*]). Now $P_i(y) = \sum_x c_i(x)\overline{\chi_x(y)}$ with $c_i = \varphi c_i^{\text{OAI}}$, where c_i^{OAI} is the coefficient of [36, proof of Lemma 17.2, *The second Poisson transformation*]. Apply Lemma 27 with $\lambda = (x_1, x_2)$. Here $x_a = g'z_a = g'v'n_a$ ([36, proof of Lemma 17.2, *The second Poisson transformation*]), so $\varphi(x_1)\varphi(x_2) = \varphi(n_1)\varphi(n_2)$ and the outer twist is 1. By (E.11) and (E.12), the signal calculation of [36, proof of Lemma 17.2, *The second Poisson transformation*] holds with

$$a_0(z) = \overline{\alpha(z)}^2 \gamma_2(z) \quad (\text{E.19})$$

in place of $a_0 = \overline{\alpha}\gamma_2$ ([36, proof of Lemma 17.2, *The second Poisson transformation*]), and with the same ray factor $G([z_2][z_1]^{-1})$. Hence:

- [36, eq. (17.41)] holds with $L_{o,d_2,k''}$ formed from (E.19). The factor $|B_o(g')|^2$ is unchanged, since the common factor $\varphi(g')$ of the two sides has modulus one;
- the extraction [36, eq. (17.42)] holds for (E.19) by the twisted multiplicativity in Lemma 26. Its $D_o(v'; d_2, k'')$ acquires $\varphi(v')$, and $|D_o|^2$ is unchanged;
- the moving characters [36, eq. (17.43)] and the expansion of G over the ray group ([36, proof of Lemma 17.2, *The second Poisson transformation*]) are unchanged.

The principal case $x_1 = x_2$ ([36, proof of Lemma 17.2, *The second Poisson transformation*]) and the discarded raw complement are bounded by absolute counts, which are unchanged. The new canonical data and their admissibility ([36, proof of Lemma 17.2, *The new canonical data and their admissibility*]) involve no coefficient: the child data, lengths and clipping, the source triples, the transition identities, the new puncture, the child margins [36, eq. (17.51)] and the row decrease [36, eq. (17.52)]. In the common density ([36, proof of Lemma 17.2, *A common density for the second transformed sum*]) the child polynomial [36, eq. (17.55)] becomes

$$Q_{a,\gamma,\zeta}(k_{\text{new}}, f_{\text{new}}) = \sum_n^* \overline{\alpha(n)}^2 \gamma_2(n) \nu'_a(n) \rho'(n) \chi_n(k_{\text{new}}) \chi_n(f_{\text{new}})^4 \mathfrak{d}'_a(n) W'_a\left(\frac{N(n)}{Z^{N_{\text{ch}}}}\right), \quad (\text{E.20})$$

of the form (E.7), while the outer weight [36, eq. (17.56)] is literally unchanged, since it contains only $|B_o|^2$ and $|D_o|^2$. Thus [36, eq. (17.57)] holds with (E.20). The second Cauchy inequality, the fibre and source counts, the positive children and the energy exponent ([36, proof of Lemma 17.2]) use only the absolute weights and the positive child sums. The latter are CM canonical energies (E.7), with label weight $D_{K_{\text{slot}}}(f)$, puncture ρ_γ and mark $\mathfrak{d}'_a$; by [36, eqs. (17.51) and (17.52)] they satisfy the hypotheses of Lemma 25 with shorter rows.

Step 5: order of choices and termination ([36, proof of Lemma 17.2, *Order of choices and termination*]). The parameter choices [36, eq. (17.66)], the window family $\mathcal{I}_D$, the crude exponent B_{crude} and the Fourier orders are unchanged. The exponent B_{ref} ([36, proof of Lemma 17.2, *Order of choices and termination*]) uses [36, eq. (5.2)] for the dual coefficients. The CM dual coefficients $d_\sigma(\ell)\alpha(\ell)^2$ satisfy the same bound (Proposition 9 and the coefficient bounds after Proposition 8), and their local factors are the same, so B_{ref} is unchanged. The thresholds for $C_{\text{res}}, C_{\text{width}}, C_{\text{ker}}$ are those of (R). The backward induction of [36, proof of Lemma 17.2, *Order of choices and termination*] then gives $\mathcal{E} \ll Z^{F_0+\varepsilon}$. $\square$

The family (E.7) is closed under the recursion for the following reason. In [36] the classes alternate between $\bar{\alpha}\gamma_2\nu$ (cubic) and $\mu\nu$ (Möbius). The twist multiplies every column by $\varphi = \bar{\alpha}$, so the CM classes alternate between $\bar{\alpha}^2\gamma_2\nu$ and $\mu\bar{\alpha}\nu$, and the child is again of the form (E.6).

E.4 Proof of Proposition 12

We follow the proof of [36, Lemma 17.1].

Overlap with the slots ([36, proof of Lemma 17.1]). Let P be the product of the slot primes, $j = (n, P)$, $n = jn_0$, $P = jP_0$ and $c = n_0P_0$. Then $\alpha(n)^\epsilon = \alpha(j)^\epsilon \alpha(c)^\epsilon \overline{\alpha(P_0)^\epsilon}$. Together with $\mu(n) = \mu(j)\mu(c)\mu(P_0)$ and $\psi_u(n)\psi_u(P) = \psi_u(j)^2\psi_u(c)$ ([36, proof of Lemma 17.1]), this gives, for a fixed assigned tuple with surviving subset I_0 ,

$$\begin{aligned} & \mu(n)\alpha(n)^\epsilon \psi_u(n) \prod_{i \in I} a_i(p_i) \psi_u(p_i) \\ &= \left[\mu(j)\alpha(j)^\epsilon \psi_u(j)^2 \prod_{i \notin I_0} a_i(p_i) \right] \cdot \mu(c)\alpha(c)^\epsilon \psi_u(c) \cdot \prod_{i \in I_0} \mu(p_i) \overline{\alpha(p_i)^\epsilon} a_i(p_i). \end{aligned} \quad (\text{E.21})$$

So [36, eq. (17.67)] holds with three changes:

- the assigned scalar is multiplied by $\alpha(j)^\epsilon$, which has modulus one and does not depend on u ;
- the residual coefficient is $\mu(c)\alpha(c)^\epsilon \psi_u(c)$;

- the surviving slot coefficients are $\mu(p)\overline{\alpha(p)}^\epsilon a_i(p)$, which are bounded by one, independent of u , and of product form.

The Fourier separation of the overlap profile ([36, proof of Lemma 17.1]) then gives marks $\mathfrak{d}_v$ with these individual coefficients times $W_i(y_i)y_i^{iv_i}$.

Orientation ([36, proof of Lemma 17.1]). For $\epsilon = -1$ the residual polynomial is unchanged; for $\epsilon = +1$ conjugate it, with its character, mark and test. Since $\overline{\mu(c)\alpha(c)\nu(c)\chi_c(u)} = \mu(c)\overline{\alpha(c)}\overline{\nu(c)}\overline{\chi_c(u)}$, in both cases it suffices to bound

$$\mathcal{R}_0(u) = Z^{-D'/2} \sum_c^* \mu(c)\overline{\alpha(c)}\nu_0(c)\mathbf{1}_{(c,j)=1}\overline{\chi_c(u)}\mathfrak{d}_0(c)W_0\left(\frac{N(c)}{Z^{D'}}\right), \quad (\text{E.22})$$

which is [36, eq. (17.68)] times $\varphi(c)$. Co-orientation is what makes the twist $\varphi = \bar{\alpha}$ for both signs; an anti-oriented sum would give the twist α .

The Poisson transformation ([36, proof of Lemma 17.1]). Apply Lemma 27 with $\lambda = (c_1, c_2)$. Here $c_a = Cz_a = Csn_a$ ([36, proof of Lemma 17.1]), so the outer twist is 1 and the columns n_a carry $\varphi(n_a)$. By (E.11) and (E.12), the signal step ([36, proof of Lemma 17.1]) gives

$$A_*(z) = \overline{\alpha(z)}^2 \gamma_2(z)\nu_*(z), \quad \nu_* = \nu_0\bar{\vartheta}, \quad (\text{E.23})$$

in place of $\bar{\alpha}\gamma_2\nu_*$. Then:

- [36, eq. (17.71)] holds with (E.23), the common factor being $|\mu(C)\overline{\alpha(C)}\nu_0(C)|^2 = 1$ ([36, proof of Lemma 17.1]);
- the extraction [36, eq. (17.72)] holds for (E.23) by the twisted multiplicativity in Lemma 26, and $|A_*(s)| = 1$, so the outer arithmetic weight [36, eq. (17.73)] is unchanged;
- the child polynomial [36, eq. (17.77)] has the coefficient (E.23), so it is of the form (E.7), and the outer weight [36, eq. (17.78)] and the separated identity [36, eq. (17.79)] hold unchanged.

The rest ([36, proof of Lemma 17.1]) involves no coefficient beyond absolute weights. This covers the weighted Cauchy inequality, the fibre bound [36, eq. (17.80)], the quotient measure, the puncture bound [36, eq. (17.84)], the margins [36, eq. (17.85)], the energy [36, eq. (17.86)], and the raw-tail orders. It applies with Lemma 25 in place of [36, Lemma 17.2]. The principal columns have the direct bound $O(Z^{m+\epsilon})$ ([36, proof of Lemma 17.1]), with absolute coefficients. This proves (E.5); the remaining clauses follow as in [36, proof of Lemma 17.1]. $\square$

E.5 Amplification on sixth-power-free rows

[36, Lemma 17.6] holds for the polynomials (E.2) without slots. Its input [36, eq. (17.87)] is Proposition 12 with $Z = H$, $m = 1$, $r = \log_H D$ and no slots ([36, Section 17.7]). In the proof ([36, proof of Lemma 17.6]) only the column identity [36, eq. (17.90)] changes. It becomes

$$M_u(D; W) = \sum_{d|\text{rad } a} \mu(d)\alpha(d)^\epsilon \psi_u(d)N(d)^{-1/2} M_{ua^6}(D/N(d); W), \quad (\text{E.24})$$

because $n = dm$ with $d \mid \text{rad } a$, $(m, a) = 1$, gives $\alpha(n)^\epsilon = \alpha(d)^\epsilon \alpha(m)^\epsilon$, and $\psi_{ua^6}(m) = \psi_u(m)\mathbf{1}_{(m,a)=1}$ ([36, proof of Lemma 17.6]). The coefficient mass is still at most $d_{\mathbb{Z}[\omega]}(a)$, and the rest is verbatim.

E.6 Orientation, and what this appendix proves

The descent itself would also run for the anti-oriented sum $\sum \mu \alpha \nu \overline{\chi_{\bullet}(u)}$, with twist α in Lemma 27. Its canonical class would be $\gamma_2 \nu$, with no angular factor, by (E.13). That is the completed sum of θ itself, and Statement (R) would have to be replaced by a reflected energy for θ , whose expansion at the identity cusp has a constant term (Section 12). This case is not covered here, and it is not needed: every CM witness is co-oriented (Section 13), and conjugation preserves co-orientation.

In summary:

- *Proved here:* Lemmas 26 and 27, and every display of [36] that changes. These are the definitions (E.2), (E.6), (E.7), (E.8) and the identities (E.15), (E.16), (E.18), (E.19), (E.20), (E.21), (E.22), (E.23) and (E.24). Hence Lemma 25, Proposition 12 (that is, (14.2)) and the CM form of [36, Lemma 17.6] hold.
- *Cited verbatim:* all other steps of [36, Section 17], at the lines given above.
- *From Appendix B:* Statement (R), the CM reflected energy (Lemma 15).

We checked numerically:

- (E.9)–(E.13) at all 366 squarefree u with $N(u) \leq 1300$, 157 of them composite;
- the twisted multiplicativity for $\nu' = \chi_{\bullet}(\lambda)$;
- the ray-quotient forms of the two switches;
- (E.15) with a character, a puncture and a two-slot mark;
- (E.21) for both signs, including a row sharing a slot prime;
- (E.24) for both signs.

The errors are at most $3.3 \cdot 10^{-15}$.

F The plain fourth moment with an angular factor

This appendix writes out item 5 of Section 16: the CM form of the fourth moment with short prime factors [36, Lemma 18.1], stated in paragraph 4 of Section 14. Steps that are identical to [36] are cited by number, with the reason they carry over. Every step that changes is proved here. There are four such steps:

- the reflection of a plain sum, now with a Bessel kernel of order one (Lemmas 28–30);
- the vanishing of the lattice main terms (Lemma 31);
- the bookkeeping of the angular factor through the transforms (Lemma 32);
- the zero-free input for the prime factors, which becomes a theorem (Step 5 of the proof).

F.1 Setting and statement

We use the notation of [36, §§18 and 13]. In particular:

- $Z > 1$ is the base, and $\mathcal{S}$ is the fixed finite set of primes; it contains the primes above 6, so $\lambda \in \mathcal{S}$. Column sums run over ideals outside $\mathcal{S}$, through their primary generators; [36] writes q_l for $N(l)$.
- The rows are $k \in \mathbb{Z}[\omega]$ with $0 < N(k) \ll Z^m$, and $\psi_k(n) = \tau(n)\chi_n(k)$ is the finite-order row character, with τ fixed in the row sum. The moving support has norm at most Z^q , and $M = m + q$.
- Masks are as in [36, Section 18] and [36, Section 13.1].
- Θ is the fixed finite group of finite-order ray characters with conductor primes in $\mathcal{S}$ ([36, Section 18]). The row sets $\mathcal{R}_0$ and $\mathcal{R}_z$ ([36, Section 18]) are defined through ψ_k .
- $A = n_1 + n_2 + z$.

The prime slots $Q_{\psi_k, i}$ are those of [36, eq. (13.1)]. They carry ψ_k and never the angular factor.

For an ideal l outside $\mathcal{S}$ put $\alpha(l) = \alpha(l_0)$, where l_0 is its primary generator. Products of primary elements are primary ([36, Section 4.1]), so

$$\alpha(ll') = \alpha(l)\alpha(l'), \quad |\alpha(l)| = 1 \quad (l, l' \text{ outside } \mathcal{S}). \quad (\text{F.1})$$

For $e \in \{1, -1\}$ the *CM plain polynomial* of log-length n is

$$S_{\psi_k}^{(e)}(n; W) = Z^{-n/2} \sum_l \alpha(l)^e \psi_k(l) W\left(\frac{N(l)}{Z^n}\right), \quad (\text{F.2})$$

the analogue of [36, eq. (4.1)], with every mask of ψ_k retained.

Proposition 13 (Fourth moment with an angular factor). *Let $3/4 \leq \kappa \leq 1$ and $e \in \{1, -1\}$. Let n_1, n_2 , the slot product $Q = \prod_i Q_{\psi_k, i}$, its coefficients ν_i , the masks and the profiles satisfy the hypotheses of [36, Lemma 18.1]. For every $\varepsilon > 0$ there is a slot mesh $\eta > 0$ such that, if $z_i \leq \eta$ for every i , then*

$$\sum_{k \in \mathcal{R}_z} \left| S_{\psi_k}^{(e)}(n_1; W_1) S_{\psi_k}^{(e)}(n_2; W_2) \right|^2 \ll Z^{M+\varepsilon} \quad (\text{F.3})$$

in either of the following cases:

1. $z = 0$, with no restriction on the lengths $n_1, n_2 \geq 0$ and no lower bound on the conductor;
2. $z > 0$ and $n_1 + n_2 + 6\kappa z \leq M$.

No zero-free hypothesis is assumed. The remaining clauses of [36, Lemma 18.1] hold verbatim, uniformly in $\kappa \in [3/4, 1]$ and e . These clauses cover zero-length slots, bounded length sets, a mesh depending only on those sets and ε , finitely many seminorms and a fixed polynomial in the separated heights, and uniformity over moving moduli, admissible masks and frozen outer labels.

For $\kappa < 1$, [36, Lemma 18.1] assumes $\beta_* \leq (1 + \kappa)/2$. Here β_* is the supremum [36, eq. (2.1), Section 2] over *finite-order* characters, not the CM quantity of Section 11. OpenAI's theorem [36, Theorem 1.1] gives $\beta_* \leq 7/8 \leq (1 + \kappa)/2$ for every $\kappa \geq 3/4$. That is why the hypothesis disappears, and why $\kappa = 3/4$ is admissible (Step 5 below).

F.2 The primitive CM row character

Lemma 28. *Fix a row k and $e \in \{1, -1\}$. Write:*

- ψ_k^0 for the natural character of [36] ([36, Section 18.1]);
 - ψ_k^* for the primitive finite-order character inducing ψ_k^0 , and $\mathfrak{g}_k$ for its conductor;
 - $R_{0,k}$ for the redundant natural radical ([36, Section 18.1]): the product of the primes not dividing $\mathfrak{g}_k$ at which ψ_k^0 , extended by zero on $\mathcal{S}$, vanishes.
- (a) *There is a unique primitive Hecke character Ξ_k of K , of infinity type $(a/|a|)^e$, with $\Xi_k(l) = \alpha(l)^e \psi_k^*(l)$ for every ideal l prime to $3\mathfrak{g}_k$. Explicitly, $\Xi_k((a)) = \varphi_k(a)(a/|a|)^e$ for a prime to its conductor $\mathfrak{f}_k$. Here φ_k is a primitive character of $(\mathbb{Z}[\omega]/\mathfrak{f}_k)^\times$ with $\varphi_k(u) = u^{-e}$ for the six units u .*
- (b) *$v_p(\mathfrak{f}_k) = v_p(\mathfrak{g}_k)$ for every prime $p \neq \lambda$. Moreover $v_\lambda(\mathfrak{f}_k) \leq \max(2, v_\lambda(\mathfrak{g}_k))$ and $v_\lambda(\mathfrak{g}_k) \leq \max(2, v_\lambda(\mathfrak{f}_k))$. In particular $1/9 \leq N(\mathfrak{f}_k)/N(\mathfrak{g}_k) \leq 9$.*
- (c) *Let $\mathfrak{r}_k$ be the product of the primes not dividing $\mathfrak{f}_k$ that lie in $\mathcal{S}$ or at which ψ_k^0 vanishes. Write $[\alpha^e \psi_k^0](l)$ for $\alpha(l)^e \psi_k^0(l)$ when l is outside $\mathcal{S}$, and for 0 otherwise. Then, for every nonzero ideal l ,*

$$[\alpha^e \psi_k^0](l) = \Xi_k(l) \mathbf{1}_{(l, \mathfrak{r}_k)=1}. \quad (\text{F.4})$$

The ideal $\mathfrak{r}_k$ is squarefree, prime to $\mathfrak{f}_k$, and differs from $R_{0,k}$ at most at λ .

- (d) *Put $C_k^{\text{CM}} = 3N(\mathfrak{f}_k)/(4\pi^2)$, and let $C_k = 3N(\mathfrak{g}_k)/(4\pi^2)$ be the conductor scale of [36] ([36, Section 18.1]). Then $C_k^{\text{CM}} N(\mathfrak{r}_k) \leq 27 C_k N(R_{0,k})$. Hence [36, eq. (18.7)] holds for the CM data, with C_A replaced by $27C_A$.*

Proof. (a) Reduction modulo 3 maps the six units bijectively onto $(\mathbb{Z}[\omega]/3)^\times$ ([36, Section 4.1]). So each a prime to 3 is uniquely $a = u(a)a_0$, with $u(a)$ a unit and a_0 primary. Since $u(ab) = u(a)u(b)$, the function $\varphi_\alpha(a) = u(a)^{-1}$ is a character of $(\mathbb{Z}[\omega]/3)^\times$ with $\varphi_\alpha(u) = u^{-1}$. Hence $\varphi_\alpha(a) a/|a| = a_0/|a_0|$ depends only on the ideal (a) , and on an ideal outside $\mathcal{S}$ it is the $\alpha(l)$ of (F.2). So α is a Hecke character modulo 3 of infinity type $a/|a|$.

Since K has class number one, $\psi_k^*((a)) = \varphi_0(a)$, where φ_0 is a primitive character of $(\mathbb{Z}[\omega]/\mathfrak{g}_k)^\times$ that is trivial on units. Thus, for a prime to $3\mathfrak{g}_k$,

$$\alpha^e \psi_k^*((a)) = \varphi_\alpha(a)^e \varphi_0(a) (a/|a|)^e. \quad (\text{F.5})$$

Let φ_k be the primitive character inducing $\varphi_\alpha^e \varphi_0$, and let $\mathfrak{f}_k$ be its conductor. On units $\varphi_k(u) = \varphi_\alpha(u)^e \varphi_0(u) = u^{-e}$, so $\Xi_k((a)) = \varphi_k(a)(a/|a|)^e$ is well defined on ideals prime to $\mathfrak{f}_k$. It agrees with $\alpha^e \psi_k^*$ on ideals prime to $3\mathfrak{g}_k$. It is primitive because φ_k is: a character modulo a proper divisor of $\mathfrak{f}_k$ would induce φ_k . Two primitive characters that agree outside a finite set of primes coincide, which gives uniqueness.

(b) By the Chinese remainder theorem, a character of $(\mathbb{Z}[\omega]/m)^\times$ is a product of components on the groups $(\mathbb{Z}[\omega]/p^{v_p(m)})^\times$. Its conductor exponent at p is that of the p -component. Since $3 = -\lambda^2$, the character φ_α lives on $(\mathbb{Z}[\omega]/\lambda^2)^\times$. So the p -components of $\varphi_\alpha^e \varphi_0$ and φ_0 agree for $p \neq \lambda$. At λ , a product of characters trivial on $1 + \lambda^a \mathbb{Z}[\omega]$ and on $1 + \lambda^b \mathbb{Z}[\omega]$ is trivial on $1 + \lambda^{\max(a,b)} \mathbb{Z}[\omega]$. This gives the first inequality; the second follows from $\varphi_0 = \varphi_\alpha^{-e}(\varphi_\alpha^e \varphi_0)$. If $v_\lambda(\mathfrak{g}_k) \geq 3$, the two inequalities force equality. Otherwise both exponents are at most 2, and $N(\lambda) = 3$.

(c) Let l be a nonzero ideal. Suppose first that l is prime to $\mathfrak{f}_k \mathfrak{r}_k$, and let $p \mid l$. Then $p \notin \mathcal{S}$ and $\psi_k^0(p) \neq 0$, so $p \nmid \mathfrak{g}_k$, because ψ_k^0 vanishes at the conductor primes of ψ_k^* . Hence $\psi_k^0(p) = \psi_k^*(p)$, and $\alpha(p)^e \psi_k^*(p) = \Xi_k(p)$ by (a). Multiplicativity then gives (F.4).

If $p \mid l$ and $p \mid \mathfrak{f}_k$, both sides vanish. Indeed $\Xi_k(p) = 0$, and $p \mid 3\mathfrak{g}_k$, so either $p \in \mathcal{S}$ or $\psi_k^0(p) = \psi_k^*(p) = 0$. If $p \mid l$ and $p \mid \mathfrak{r}_k$, both sides vanish by definition.

The ideals $\mathfrak{r}_k$ and $R_{0,k}$ are built from the same zero set, since α never vanishes. By (b), $\mathfrak{f}_k$ and $\mathfrak{g}_k$ have the same prime divisors other than λ .

(d) Away from λ , $\mathfrak{f}_k \mathfrak{r}_k$ and $\mathfrak{g}_k R_{0,k}$ coincide by (b) and (c). At λ , write $v = v_\lambda(\mathfrak{g}_k)$. The λ -part of $N(\mathfrak{f}_k \mathfrak{r}_k)$ is at most $3^{\max(2,v)+1} \leq 27 \cdot 3^v$. So $N(\mathfrak{f}_k)N(\mathfrak{r}_k) \leq 27N(\mathfrak{g}_k)N(R_{0,k})$. The last claim is [36, eq. (18.7), Section 18.1]. $\square$

The λ -part of the conductor can move in both directions: for $\psi = \chi_\pi^5$ with $N(\pi) = 7$ and $e = 1$, the finite part has conductor norm 63 but Ξ has conductor norm 7. So λ can pass from the conductor into $\mathfrak{r}_k$, and the reverse also occurs. Only the bound in (d) is used.

F.3 Reflection of a plain sum with an angular factor

Lemma 29 (CM reflection). *Let Ξ be a primitive Hecke character of K with conductor $\mathfrak{f} = (f)$ and infinity type $(a/|a|)^e$, $e \in \{1, -1\}$. Thus $\Xi((a)) = \varphi(a)(a/|a|)^e$ for a prime to $\mathfrak{f}$, with φ primitive modulo f and $\varphi(u) = u^{-e}$ on units; in particular $\mathfrak{f} \neq (1)$, since $\varphi(-1) = -1$. Put $C_\Xi = 3N(\mathfrak{f})/(4\pi^2)$, and*

$$\gamma(\varphi) = \frac{1}{\sqrt{N(f)}} \sum_{x \bmod f} \varphi(x) e\left(\frac{x}{f}\right), \quad \epsilon(\Xi) = e \gamma(\varphi) \left(\frac{f}{|f|}\right)^e. \quad (\text{F.6})$$

Let W be smooth on $(0, \infty)$, such that $g(z) = (z/|z|)^e W(|z|^2)$, with $g(0) = 0$, is a Schwartz function on $\mathbb{C}$. This holds, for example, for annular W . Define the Hankel transform [34, §10.22(v)]

$$W^\sharp(y) = \int_0^\infty W(x) J_1(2\sqrt{xy}) dx. \quad (\text{F.7})$$

Then $|\epsilon(\Xi)| = 1$, and for every $X > 0$

$$X^{-1/2} \sum_{\mathfrak{a}} \Xi(\mathfrak{a}) W\left(\frac{N\mathfrak{a}}{X}\right) = \epsilon(\Xi) Y^{-1/2} \sum_{\mathfrak{b}} \overline{\Xi(\mathfrak{b})} W^\sharp\left(\frac{N\mathfrak{b}}{Y}\right), \quad Y = \frac{C_\Xi}{X}. \quad (\text{F.8})$$

Both sums run over nonzero ideals and converge absolutely.

Proof. For $a \in \mathbb{Z}[\omega]$ put $c(a) = \varphi(a)(a/|a|)^e$, which is 0 unless $(a, f) = 1$. Then $c(ua) = c(a)$ for units u , and every ideal prime to $\mathfrak{f}$ has six generators. So

$$\sum_{\mathfrak{a}} \Xi(\mathfrak{a}) W(N\mathfrak{a}/X) = \frac{1}{6} \sum_{a \in \mathbb{Z}[\omega]} \varphi(a) g_X(a), \quad g_X(z) = (z/|z|)^e W(|z|^2/X). \quad (\text{F.9})$$

Write $\widehat{h}(w) = \int_{\mathbb{C}} h(z) e(-zw) d^2z$. Write $a = t + fy$ with $t \bmod f$ and $y \in \mathbb{Z}[\omega]$. Poisson summation over $\mathbb{Z}[\omega]$ [2, Theorem 19], applied to $y \mapsto g_X(t + fy)$, and the substitution $z = t + fy$ (as in [2, (4.20)]) give

$$\sum_{y \in \mathbb{Z}[\omega]} g_X(t + fy) = \frac{2}{\sqrt{3}} \frac{1}{N(f)} \sum_{w \in \mathbb{Z}[\omega]} e\left(\frac{tw}{f}\right) \widehat{g_X}\left(\frac{w}{f}\right). \quad (\text{F.10})$$

By [2, Lemma 17], $\sum_{t \bmod f} \varphi(t) e(tw/f) = \overline{\varphi(w)} \sqrt{N(f)} \gamma(\varphi)$ for every $w \in \mathbb{Z}[\omega]$; both sides vanish when $(w, f) \neq 1$. The same lemma gives $|\gamma(\varphi)| = 1$. Hence

$$\sum_{a \in \mathbb{Z}[\omega]} \varphi(a) g_X(a) = \frac{2}{\sqrt{3}} \frac{\gamma(\varphi)}{\sqrt{N(f)}} \sum_{w \in \mathbb{Z}[\omega]} \overline{\varphi(w)} \widehat{g}_X(w/f). \quad (\text{F.11})$$

We now compute $\widehat{g}_X$. Write $z = re^{i\theta}$ and $w = \rho e^{i\beta}$. Then $\text{Im}(zw) = r\rho \sin(\theta + \beta)$. The Jacobi–Anger expansion [34, (10.12.1)] $e^{-ix \sin \phi} = \sum_n J_n(x) e^{-in\phi}$ and $J_{-1} = -J_1$ give

$$\int_0^{2\pi} e^{ie\theta} \exp\left(-\frac{4\pi i r \rho}{\sqrt{3}} \sin(\theta + \beta)\right) d\theta = 2\pi J_e\left(\frac{4\pi r \rho}{\sqrt{3}}\right) e^{-ie\beta} = 2\pi e J_1\left(\frac{4\pi r \rho}{\sqrt{3}}\right) \left(\frac{\bar{w}}{|w|}\right)^e. \quad (\text{F.12})$$

Integrating against $W(r^2/X) r dr$ and substituting $x = r^2/X$ gives, for $w \neq 0$,

$$\widehat{g}_X(w) = \pi e X \left(\frac{\bar{w}}{|w|}\right)^e \int_0^\infty W(x) J_1\left(4\pi |w| \sqrt{Xx/3}\right) dx. \quad (\text{F.13})$$

At $w = 0$ we get $\widehat{g}_X(0) = \int g_X = 0$, because $\int_0^{2\pi} e^{ie\theta} d\theta = 0$.

Insert (F.13) at w/f into (F.11). Three identities simplify the result:

$$\left(\frac{\overline{w/f}}{|w/f|}\right)^e = \left(\frac{\bar{w}}{|w|}\right)^e \left(\frac{f}{|f|}\right)^e, \quad \overline{\varphi(w)} \left(\frac{\bar{w}}{|w|}\right)^e = \overline{c(w)}, \quad 4\pi \left|\frac{w}{f}\right| \sqrt{\frac{Xx}{3}} = 2\sqrt{\frac{xN(w)}{Y}}. \quad (\text{F.14})$$

The last uses $Y = C_\Xi/X$. Hence

$$\sum_a \varphi(a) g_X(a) = \frac{2\pi e X}{\sqrt{3}} \frac{\gamma(\varphi)}{\sqrt{N(f)}} \left(\frac{f}{|f|}\right)^e \sum_{w \neq 0} \overline{c(w)} W^\sharp\left(\frac{N(w)}{Y}\right). \quad (\text{F.15})$$

Group w into ideals (six generators each, and $\bar{c}$ is invariant under units) and divide by 6. Multiplying by $X^{-1/2}$ and using $X^{1/2}Y^{1/2} = C_\Xi^{1/2} = \sqrt{3N(f)}/(2\pi)$ gives the coefficient $e\gamma(\varphi)(f/|f|)^e = \epsilon(\Xi)$. Absolute convergence holds on the left because g is Schwartz. On the right it holds because $\widehat{g}_X$ is Schwartz, and the w -sum is the Poisson dual sum. $\square$

Remark (relation to Hecke’s functional equation). Take Mellin transforms in X of (F.8) and use Lemma 30(a). This gives $C_\Xi^{s/2} \Gamma(s + \frac{1}{2}) L(s, \Xi) = \epsilon(\Xi) C_\Xi^{(1-s)/2} \Gamma(\frac{3}{2} - s) L(1 - s, \bar{\Xi})$. This is the functional equation of Lemma 10, with the explicit root number (F.6). [36] goes the other way: it derives its reflection ([36, Section 18.1]) from the functional equation by Mellin inversion. For a CM character (F.8) is proved directly, so this step uses no analytic continuation. Take $\Xi = \alpha$, of conductor (3). Then $L(s, \alpha) = L(E_{16}, s + \frac{1}{2})$, where $E_{16}: y^2 = x^3 + 16$ has conductor $27 = 3N(3)$. We found $\epsilon(\alpha) = 1$ numerically, which is the root number of that curve.

Lemma 30 (The dual profile). *Let W be annular. Use the Mellin convention $\mathcal{M}W(s) = \int_0^\infty W(y) y^s dy/y$ of [36] ([36, Lemma 4.5]), and put $G(s) = \Gamma(s + \frac{1}{2})/\Gamma(\frac{3}{2} - s)$.*

- (a) $W^\sharp$ is smooth on $(0, \infty)$, and $\mathcal{M}W^\sharp(s) = \mathcal{M}W(1 - s) G(s)$ for $-\frac{1}{2} < \Re s < \frac{1}{4}$. In the notation of [36, Lemma 4.7], $W^\sharp = K_{W_1}$ with $W_1(y) = yW(y)$ and $m = G$.
- (b) G is holomorphic on $\Re s > -\frac{1}{2}$. Its poles are at $-\frac{1}{2} - n$ and its zeros at $\frac{3}{2} + n$ ($n \geq 0$). Uniformly for σ in compact subsets of $(-\frac{1}{2}, \infty)$, $|G(\sigma + it)| = |t|^{2\sigma-1} (1 + O(|t|^{-1}))$ as $|t| \rightarrow \infty$, the same exponent as for $\Gamma(s)/\Gamma(1 - s)$.

(c) For integers $j \geq 0$ and every $A \geq 0$,

$$|(y\partial_y)^j W^\sharp(y)| \ll_{j,A} \min(y^{1/2}, y^{-A}), \quad (\text{F.16})$$

with constants given by finitely many seminorms of W . Also $W^\sharp(y) = \mathcal{M}W(\frac{3}{2})y^{1/2} + O(y^{3/2})$ as $y \rightarrow 0$. For $W_\omega(y) = W(y)y^{i\omega}$ the same bounds hold with an extra factor $(1 + |\omega|)^h$, where $h = h(j, A)$ is fixed.

Proof. (a) Smoothness follows by differentiating under the integral, since W has compact support in $(0, \infty)$. For $-\frac{1}{2} < \sigma < \frac{1}{4}$ the double integral of $|W(x)J_1(2\sqrt{xy})|y^{\sigma-1}$ converges, by $|J_1(t)| \leq t/2$ and $|J_1(t)| \ll t^{-1/2}$ [34, (10.14.4), (10.7.8)]. The classical Mellin transform $\int_0^\infty t^{\mu-1}J_1(t)dt = 2^{\mu-1}\Gamma(\frac{1+\mu}{2})/\Gamma(\frac{3-\mu}{2})$, valid for $-1 < \Re\mu < \frac{3}{2}$ ([34, (10.22.43)], with $\mu - 1$ in place of μ), with $t = 2\sqrt{xy}$, gives

$$\int_0^\infty J_1(2\sqrt{xy})y^{s-1}dy = x^{-s}G(s) \quad (-\frac{1}{2} < \Re s < \frac{3}{4}). \quad (\text{F.17})$$

Fubini then gives $\mathcal{M}W^\sharp(s) = \mathcal{M}W(1-s)G(s)$. Since $\mathcal{M}W_1(-s) = \mathcal{M}W(1-s)$, Mellin inversion on a line in $(-\frac{1}{2}, \frac{1}{4})$ identifies $W^\sharp$ with K_{W_1} . The inversion is legitimate because $W^\sharp$ is continuous and $\mathcal{M}W(1-s)G(s)$ is integrable on the line, by [36, eq. (4.11)].

(b) $\Gamma(s + \frac{1}{2})$ has poles at $-\frac{1}{2} - n$, and $1/\Gamma(\frac{3}{2} - s)$ is entire with zeros at $\frac{3}{2} + n$. Stirling's formula $|\Gamma(\sigma + it)| = \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}(1 + O(|t|^{-1}))$ gives the exponent $\sigma - (1 - \sigma) = 2\sigma - 1$.

(c) Large y : G is holomorphic on the closed strip $-\frac{1}{4} \leq \Re s \leq A$, where $|G(s)| \ll (1 + |\Im s|)^{2A+1}$. So [36, Lemma 4.7] applies on the available line $\Re s = A$ and gives $|(y\partial_y)^j W^\sharp(y)| \ll y^{-A}$, with the stated seminorm and twist dependence. This is the argument of [36, Section 18.1] with G in place of $\Gamma(s)/\Gamma(1-s)$.

Small y : $y\partial_y[J_1(2\sqrt{xy})] = x\partial_x[J_1(2\sqrt{xy})]$. Integrating by parts j times, with no boundary terms, gives $(y\partial_y)^j W^\sharp(y) = \int_0^\infty W_j(x)J_1(2\sqrt{xy})dx$, where $W_j = (-\partial_x x)^j W$ is annular. Then $|J_1(t)| \leq t/2$ gives $|(y\partial_y)^j W^\sharp(y)| \leq y^{1/2} \int |W_j(x)|x^{1/2}dx$. The expansion $J_1(t) = t/2 + O(t^3)$ gives the asymptotic formula. Finally, W_ω is annular, with j -th seminorm $\ll (1 + |\omega|)^j$. $\square$

So the CM dual profile is better than that of [36] at the origin: it vanishes like $y^{1/2}$, while $\Gamma(s)/\Gamma(1-s)$ leaves a constant term. It is the same at infinity and on vertical lines. In particular, the annular coefficients below the main scale in [36, Section 18.1] are $O(2^{-j})$ instead of $O(2^{-j/2})$.

F.4 Lattice sums with an angular factor

Lemma 31 (CM form of [36, Lemma 18.3]). *Let:*

- ϑ range over a fixed finite set of finite-order ray characters with conductor primes in $\mathcal{S}$;
- $\mathfrak{r}_*$ be squarefree, with $N(\mathfrak{r}_*) \leq Z^{B_*}$;
- W_1, W_2 be smooth profiles on fixed annuli, and $e \in \{1, -1\}$.

For $X > 0$ and $t \in \mathbb{R}$ put

$$\mathcal{L}_i(X, t) = \sum_{l \text{ outside } \mathcal{S}} \alpha(l)^e \vartheta(l) \mathbf{1}_{(l, \mathfrak{r}_*)=1} N(l)^{it} W_i\left(\frac{N(l)}{X}\right). \quad (\text{F.18})$$

Then, with J fixed and finitely many seminorms, uniformly in $X > 0$, in $\mathfrak{r}_*$ and in the finite set,

$$|\mathcal{L}_i(X, t)| \ll Z^{\varepsilon_1} (1 + |t|)^J, \quad |\mathcal{L}_i(X, t)| \ll X. \quad (\text{F.19})$$

Consequently:

- [36, eq. (18.45)] holds with $c_{\vartheta, \mathfrak{r}_*} = 0$;
- [36, eq. (18.46)] holds;
- [36, eq. (18.47)] holds for the coefficient $\alpha(l_1)^e \alpha(l_2)^e \vartheta(l_1) \vartheta(l_2) \mathbf{1}_{(l_1 l_2, \mathfrak{r}_*)=1} N(l_1 l_2)^{it} D_{\mathbf{b}}(l_1, l_2)$, together with its fallback $O(T^{1/2})$ when there is no lower length.

Proof. First let $\mathfrak{r}_* = (1)$, and call the sum $\mathcal{L}_i^0$. The primary generators of ideals outside $\mathcal{S}$ are the $z \equiv 1 \pmod{3}$ that are prime to the primes of $\mathcal{S}$. The value $\vartheta(l)$ depends only on z modulo a fixed modulus. So there are a fixed nonzero $N_0 \in \mathbb{Z}[\omega]$ and a function w on $\mathbb{Z}[\omega]/N_0$ with $|w| \leq 1$ such that

$$\mathcal{L}_i^0(X, t) = \sum_{z \in \mathbb{Z}[\omega]} w(z) g_{X,t}(z), \quad g_{X,t}(z) = X^{it} h_t(z/\sqrt{X}), \quad h_t(z) = (z/|z|)^e |z|^{2it} W_i(|z|^2). \quad (\text{F.20})$$

Poisson summation in each class $t_0 + N_0 \mathbb{Z}[\omega]$, exactly as in the proof of Lemma 29, gives

$$\mathcal{L}_i^0(X, t) = \frac{2}{\sqrt{3}} \frac{1}{N(N_0)} \sum_{w \in \mathbb{Z}[\omega]} \widehat{w}(w) X^{1+it} \widehat{h}_t(\sqrt{X} w/N_0), \quad (\text{F.21})$$

where $\widehat{w}(w) = \sum_{t_0 \bmod N_0} w(t_0) e(t_0 w/N_0)$, so $|\widehat{w}| \leq N(N_0)$.

The zero frequency is $X^{1+it} \widehat{h}_t(0) = X^{1+it} \int_0^\infty r^{2it} W_i(r^2) r dr \int_0^{2\pi} e^{ie\theta} d\theta = 0$. This is the term $c_\vartheta X^{1+it} I_i(t)$ of [36]. With the angular factor it vanishes identically.

For the other frequencies, note that h_t is smooth and supported on a fixed annulus, with $\partial^\beta h_t \ll (1 + |t|)^{|\beta|}$. So $|\widehat{h}_t(\xi)| \ll (1 + |t|)^{J_0} (1 + |\xi|)^{-J_0}$ for any fixed $J_0 \geq 3$. For $X \geq 1$ this gives

$$|\mathcal{L}_i^0| \ll (1 + |t|)^{J_0} X \sum_{w \neq 0} (1 + \sqrt{X} |w|/|N_0|)^{-J_0} \ll (1 + |t|)^{J_0}. \quad (\text{F.22})$$

For $0 < X < 1$, absolute counting on the fixed annulus gives $O(1)$.

For a general mask, remove from $\mathfrak{r}_*$ its primes in $\mathcal{S}$. Then (F.1) and the multiplicativity of ϑ give $\mathcal{L}_i(X, t) = \sum_{d|\mathfrak{r}_*} \mu(d) \alpha(d)^e \vartheta(d) N(d)^{it} \mathcal{L}_i^0(X/N(d), t)$. The number of divisors is $\ll Z^{\varepsilon_1}$, which proves the first bound in (F.19). The second is [36, proof of Lemma 18.3] verbatim, since $|\alpha| = 1$.

As in [36, proof of Lemma 18.3], the masked double sum in [36, eq. (18.47)] is exactly $\mathcal{L}_1(U_1, t) \mathcal{L}_2(U_2, t) - \mathcal{L}_1(V_1, t) \mathcal{L}_2(V_2, t)$, where the angular factor is already split as $\alpha(l_1)^e \alpha(l_2)^e$. By (F.19) its modulus is $\ll Z^{2\varepsilon_1} (1 + |t|)^{2J}$. If all four scales are at least Z^r , then $T \geq Z^{2r}$, so $T^{-1/2} \leq T^{1/2} Z^{-2r} \leq T^{1/2} Z^{-r}$. This proves [36, eq. (18.47)]. The fallback follows from the second bound in (F.19). $\square$

In [36] the zero frequency is a volume term, and the equal-product centering exists to cancel it ([36, Section 18.3]). With the angular factor it is absent. So the centered estimate below holds with room to spare. We keep [36]'s argument unchanged rather than restructure the induction.

F.5 The coefficient class

Lemma 32. Replace the coefficient [36, eq. (18.18), Section 18.3] by

$$\left(\prod_{i \in \mathcal{I}'} \nu_i(p_i) W_i^{\text{slot}}(N(p_i)/P_i) \right) D_{\mathbf{b}}(l_1, l_2) \alpha(l_1 l_2)^e \tau_1(u) N(u)^{it} \mathbf{1}_{(u, \mathfrak{A})=1} \mathbf{1}_{s|u}, \quad u = \Pi l_1 l_2, \quad (\text{F.23})$$

with the angular factor on the two plain variables only. Every operation of [36] in [36, Sections 18.1–18.7] takes (F.23) to a coefficient of the same form. The angular factor of each extracted plain

prime power becomes a frozen scalar of modulus one, the same in both rectangles. In particular [36, Lemma 18.2] holds with $\alpha(l_1 l_2)^e$ included. In its exceptional conclusion the plain coefficient is $\alpha(l_1)^e \alpha(l_2)^e \vartheta(l_1) \vartheta(l_2) \mathbf{1}_{(l_1 l_2, \mathfrak{R}_*)=1} N(l_1 l_2)^{it} D_{\mathbf{b}}(l_1, l_2)$.

Proof. Each operation uses only (F.1).

- (i) *Mask deletion* [36, eq. (18.4)]. Writing $n = dl$ in a plain variable gives $\alpha(n)^e = \alpha(d)^e \alpha(l)^e$. The row-dependent scalar $\mu(d) \alpha(d)^e \psi_k^0(d)$ has modulus at most one, which is all [36, Section 18.1] use. The slot identity [36, eq. (18.5)] is unchanged.
- (ii) *Reflection*. By (F.4) and Lemma 29, a CM plain sum is $\epsilon(\Xi_k)$ times a combination of conjugates of CM plain sums with the same coefficient $\alpha^e \psi_k^0$ and profile $\overline{W}^\sharp$. The conjugation is removed inside the absolute value as in [36, Section 18.1] and [36, Section 4.1].
- (iii) *Complete common-support extraction* at the first and second transforms ([36, Sections 18.4 and 18.6]). A prime power p^{ij} allocated to plain variable j gives the scalar $\alpha(p)^{e_{ij}}$. The allocation is made once for the coefficient, so the same scalar multiplies both rectangles.
- (iv) *The amplifier* ([36, Section 18.5]). The powers p^{i_1}, p^{i_2} go only to the plain variables. The scalar $\alpha(p)^{e(i_1+i_2)}$ is absorbed in $c_{p,i}(h)$, whose modulus is unchanged.
- (v) *The $\mathfrak{t}$ -allocation* ([36, Section 18.6]). Writing $n_i = pl_i$ gives $\alpha(p)^e$, a scalar depending only on $\mathfrak{t}$ and the allocation. It is absorbed in $c_{\mathfrak{t},i}(h')$ ([36, Section 18.7]).
- (vi) *The second side* ([36, Section 18.6]). It enters as $\overline{B(v)}$, the complex conjugate of a convolution with coefficient α^e times a finite-order character. After conjugating the whole child, which is allowed inside an absolute value or a Cauchy–Schwarz norm, both children carry α^e with the same e . Their finite-order characters are those of [36].
- (vii) *Fourier separations* ([36, Sections 18.4 and 18.6]). The kernels and inverse roots depend on whole-product norms, and the fixed-ray phases are finite-ray characters of whole columns. They produce powers $N(u)^{it}$ and finite-ray characters, and never act on α .

For the exceptional conclusion, the proof in [36, proof of Lemma 18.2] factors the character, mask and norm power on $\Pi l_1 l_2$. The angular factor $\alpha(l_1)^e \alpha(l_2)^e$ already sits on the plain variables. $\square$

F.6 Proof of Proposition 13

We follow [36, Section 18] in order. *Verbatim* means that the cited text applies word for word when the plain coefficient $\psi_k(l)$ is read as $\alpha(l)^e \psi_k(l)$; the reason is given in each case.

Step 0: inputs ([36, Section 18]). The finite Gauss identities of [36, §13.3] concern the column Gauss sums $G(u, h)$ and correlations $F(u, v; j)$ only. [36, Lemma 4.5] concerns profiles. [36, Lemma 4.9] is used only in Step 5, for finite-order characters.

Step 1 ([36, Section 18]): zero-length slots, the induction on M by bands, and the threshold $A \leq 5M/6$. *Verbatim*.

Step 2: mask deletion ([36, Section 18.1]). This is Lemma 32(i); [36, eq. (18.6)] is verbatim. The finite-order part of the inducing character is unchanged, so the row family $\mathcal{R}_z$ is preserved.

Step 3: reflection ([36, Section 18.1]). This step changes. For a natural character, (F.4) gives, by Möbius inversion on $(l, \mathfrak{r}_k) = 1$ and multiplicativity of Ξ_k ,

$$T_k(X; W) = \sum_{d_0 | \mathfrak{r}_k} \frac{\mu(d_0) \Xi_k(d_0)}{\sqrt{N(d_0)}} T_{\Xi_k}(X/N(d_0); W). \quad (\text{F.24})$$

Here T_{Ξ_k} is the normalized sum of Ξ_k over all nonzero ideals. Reflect each term by (F.8). Then restore the deleted factors: every ideal is uniquely $h_0 \mathbf{b}'$ with $\text{rad}(h_0) \mid \mathfrak{r}_k$ and $(\mathbf{b}', \mathfrak{r}_k) = 1$, and (F.4) conjugated gives

$$T_{\Xi_k}(Y; W^\sharp) = \sum_{\text{rad}(h_0) \mid \mathfrak{r}_k} \frac{\overline{\Xi_k(h_0)}}{\sqrt{N(h_0)}} T_{\alpha^e \psi_k^0}(Y/N(h_0); W^\sharp). \quad (\text{F.25})$$

The result is the display [36, Section 18.1], with these substitutions:

- $\psi_k^* \rightarrow \Xi_k$, $R_{0,k} \rightarrow \mathfrak{r}_k$, $C_k \rightarrow C_k^{\text{CM}}$, $\varepsilon_k \rightarrow \epsilon(\Xi_k)$;
- $W^\sharp$ given by (F.7).

The coefficient mass is bounded as in [36, Section 18.1]. The scales [36, eq. (18.8)] satisfy $Y \leq 27C_{\mathcal{A}}Z^{M-\log_Z X}$, by Lemma 28(d). The profile bounds of [36, Section 18.1] are replaced by Lemma 30, which gives the same finite-seminorm and polynomial-height bounds.

Step 4: annuli and reflected lengths ([36, Section 18.1]). Verbatim, with C_{ref} enlarged by the factor 27. Below the main scale the annular coefficients are $O(2^{-j})$, by Lemma 30(c), which only improves [36, Section 18.1]. Equations [36, (18.9)] and [36, (18.10)] are unchanged.

Step 5: prime estimates ([36, Section 18.2]). The slots carry $\psi_k^0 \nu_i$, which has finite order and no angular factor. Expand ν_i in the $\vartheta_{ij} \in \Theta$. Each slot character $\psi_k^* \vartheta_{ij}$ is nonprincipal for $k \in \mathcal{R}_z$ ([36, Section 18.2], verbatim, because $\mathcal{R}_z$ is defined through ψ_k). For [36, eq. (18.11)] at $s_\kappa + e'$, with $e' > 0$ the small displacement of [36] (written e there), apply the global part of [36, Lemma 4.9] with $b = s_\kappa = (1 + \kappa)/2$. This is allowed because OpenAI's theorem [36, Theorem 1.1] gives $\beta_* \leq 7/8 \leq s_\kappa$ for $\kappa \geq 3/4$. Hence [36, eq. (18.12)] holds for every $\kappa \in [3/4, 1]$ without hypothesis. The rest is verbatim.

Step 6: induction order, terminal widths and comparisons ([36, Section 18.2]). Verbatim, with Step 3 for every reflection. The comparison has the same coefficient $\alpha^e \psi_k$ and the same profiles. The κ -dependent inequalities are those of [36] and hold on all of $[3/4, 1]$, endpoint included. They are [36, eq. (18.13)], the bound $6\kappa - 1 \geq 7/2$ at [36, Section 18.2], [36, eq. (18.15)] and [36, eq. (18.16)].

Step 7: the centered difference ([36, Section 18.2]). Now

$$\Delta_k = \frac{Q}{\sqrt{X_1 X_2}} \sum_{l_1, l_2} \alpha(l_1 l_2)^e \psi_k(l_1 l_2) \{\cdots\}, \quad (\text{F.26})$$

with the bracket of [36, Section 18.2]. The enlargement to the smooth row ball uses only nonnegativity. Verbatim.

Step 8: the centered coefficient ([36, Section 18.3]). [36, eq. (18.17)] is unchanged. The coefficient form is (F.23). The main term of [36, Section 18.3] has $c_{\vartheta, \mathfrak{R}_*} = 0$ (Lemma 31). The support convention concerns norms only.

Step 9: the first transform ([36, Section 18.4]). Poisson summation acts on the row k , on which α does not depend. [36, eq. (18.21)] holds with two changes:

- $A_C(a)$ and $A_D(b)$ include the residual angular factors (Lemma 32(iii));
- the frozen α -scalars of the allocation are part of the scalar of bounded modulus in the frozen labels ([36, Section 18.4]).

The zero frequency ([36, Section 18.4]) uses only that the coefficients have modulus at most one. [36, eqs. (18.23), (18.24)] are verbatim.

Step 10: the Gauss-row enlargement ([36, Section 18.5]). Verbatim, by Lemma 32(iv).

Step 11: the second transform and the children ([36, Section 18.6]). Verbatim, by Lemma 32(iii),(v)–(vii). A child is classified by its finite-order character $\tau_1 \rho \chi_\bullet(G_e V_{\text{id}} h')$ ([36, Section 18.6]). For a child outside Θ , the induction hypothesis is Proposition 13 at the smaller width M' , with the same e , for rows in $\mathcal{R}_{z'}$ (or $\mathcal{R}_0$ if $z' = 0$); both sets contain every such child row.

Step 12: exceptional rows ([36, Section 18.7]). The count [36, eq. (18.41)] involves only the finite-order child character, so it is verbatim. The volume bounds in [36, Section 18.7] use only $|\alpha| = 1$. For the centered exceptional terms, Lemma 32 supplies the hypotheses of Lemma 31. Then [36, Section 18.7] give [36, eq. (18.48)] verbatim, and [36, eq. (18.50)] follows.

Step 13: completion of the induction ([36, Section 18.8]). Verbatim. The “fixed gamma shape” of reflection ([36, Section 18.8]) is now $G(s) = \Gamma(s + \frac{1}{2})/\Gamma(\frac{3}{2} - s)$, which has the same vertical growth (Lemma 30(b)). This proves Proposition 13. $\square$

Remarks.

1. *Two strengthenings, not used.* In case (1), the proof uses $k \in \mathcal{R}_0$ only through the reflections of Steps 3 and 6. These are valid for every row in the CM case, because Ξ_k always exists and is never principal. So in the CM case, (1) holds with $\mathcal{R}_0$ replaced by all rows $0 < N(k) \ll Z^m$. Also, by Lemma 31 each exceptional plain sum is $O(Z^{\varepsilon_1}(1 + |t|)^J)$, far below the volume bound, so the centering is not needed. We have not used either simplification.
2. *Application.* The witnesses S_m of Section 13 have coefficient $\overline{\alpha} \theta \chi_\bullet(u)$. Conjugating the whole product, as in [36, Section 19.2] gives (F.2) with $e = 1$, $\tau = \bar{\theta}$ and $k = u$. The slot coefficients relative to $\psi_u = \bar{\theta} \chi_\bullet(u)$ are fixed combinations of members of Θ . Every retained row ramifies at a good prime, so it lies in $\mathcal{R}_z$. This is the instance of Proposition 13 used in Section 14.

Numerical checks. We checked the new identities numerically: six scripts and 106 checks, all of which pass.

- *The Bessel–Mellin formulas.* $\int_0^\infty J_1(2\sqrt{y}) y^{s-1} dy = G(s)$ agrees to 10^{-30} , and Mellin inversion reproduces $W^\sharp$ for a C_c^∞ bump on three lines to $4 \cdot 10^{-9}$.
- *The behaviour of $W^\sharp$.* At 0 it behaves like $y^{1/2}$, and the annular coefficients are $O(2^{-j})$.
- *Stirling exponents.* These were checked for both quotients.
- *The planar transform.* Formula (F.13) and $\hat{g}(0) = 0$ hold to 10^{-13} .
- *The reflection formula.* (F.8) was checked for 12 primitive characters: $\alpha^{\pm 1}$, and twists by sextic residue characters modulo primes of norm 7, 13 and 19 (one a product of two) and by the cubic character modulo 2. Both signs of e and three profiles were used, and the error was at most $4 \cdot 10^{-11}$.
- *The reflection on actual rows.* Rows with redundant zeros were checked, with deleted and restored factors and the bound of Lemma 28(d); the error was at most $2 \cdot 10^{-15}$.
- *The lattice sums.* $\mathcal{L}_i^0(X, t)$ stays bounded and tends to 0 for X up to $3 \cdot 10^5$, while the sum without the angular factor grows like $\frac{\pi}{6\sqrt{3}} X |I(t)|$. The inclusion–exclusion identity holds to $2 \cdot 10^{-13}$.

G The 11/12 stage for the family $\mathcal{F}_{\text{CM}}$

This appendix proves item 6 of Section 16. The family $\mathcal{F}_{\text{CM}}$ and the number β_* are those of Section 11.

Theorem 2. *For every $\psi \in \mathcal{F}_{\text{CM}}$, $L(s, \psi) \neq 0$ for $\Re s > 11/12$. Equivalently, $\beta_* \leq 11/12$.*

The two assertions are equivalent. The number β_* is the supremum of $1/2$ and of the real parts of the zeros ρ , with $1/2 \leq \Re \rho \leq 1$, of the members of $\mathcal{F}_{\text{CM}}$. No $L(s, \psi)$ vanishes in $\Re s > 1$, where its Euler product converges absolutely.

G.1 What [36] proves at this stage, and what Part II uses

The 11/12 stage of [36] is its Theorem 3.1: every finite-order Hecke L -function over K , and every Dirichlet L -function, has no zero in $\Re s > 11/12$. The proof ([36, Sections 10–11], with the probe, the Poisson representation and the detector of [36, Sections 6–9]) is by contradiction. Suppose that $\Delta_1 = \beta_* - 11/12 > 0$, where now β_* is the supremum for the finite-order family. For each target η the balanced probe $J_{1,\eta}(Z) = I_\eta(Z^{1/2}, Z^{1/2}, Z)/c_S$ satisfies two estimates:

$$|J_{1,\eta}(Z)| \ll_\eta Z^{1/4+\omega}, \quad |J_{1,\eta}(Z) - f_{1,\eta}(Z)| \ll_\eta Z^{C_1(\beta_*)-1/4800}, \quad (\text{G.1})$$

with $C_1(s) = s - 2/3$ and $\omega = \Delta_1/2$ ([36, Props. 6.3 and 11.2]). The high estimate uses five inputs:

- the Poisson representation, [36, Lemma 7.1];
- the detector, [36, Section 8];
- the sextic large sieve and the row count, [36, Lemma 9.1 and Prop. 9.2];
- the analytic lemmas of [36, Section 10];
- the late choice of the analysis height, [36, Lemma 11.1].

The continuation criterion [36, Prop. 2.1] then contradicts $\Delta_1 > 0$. A quadratic transfer, [36, Prop. 11.3], passes to Dirichlet L -functions.

Part II of [36] uses this stage in two ways.

- (U1) *The bound $\beta_* \leq 11/12$* ([36, Section 12]). It enters only through $0 < \Delta = \beta_* - 7/8 \leq 1/24$, $\kappa = 2\beta_* - 1 \in (3/4, 5/6]$ and the bin ceiling $\delta \leq \kappa$ ([36, eqs. (12.1) and (12.2)]; used again at [36, Section 19.2] and [36, proof of Proposition 20.3]). In Section 15 this is the range $0 \leq \delta \leq 5/6$ of the endpoint certificate.
- (U2) *The shared lemmas.* The 7/8 conclusion of [36] (its Section 20) applies the following results of its Part I again, with $\sigma_0 = 7/8$: Definition 10.1; eqs. (10.1) and (10.2); and Lemmas 10.2, 10.3 (inside the proof of the next one), 10.4, 10.5, 10.6 and 11.1.

The sextic large sieve and the sextic row count are used only in Part I of [36].

For $\mathcal{F}_{\text{CM}}$, (U1) is Theorem 2. Its conclusion is not uniform in anything: it is the conjunction of one statement for each member. In [36] the margins ω and σ of the continuation criterion must be independent of the target, because the criterion argues by contradiction about the supremum β_* . That requirement belongs to the proof, not to the conclusion. So we may prove Theorem 2 one character at a time. We do this in Section G.2, by running Part I of these notes with a fixed finite-order twist. This is how [35] proves the same theorem for finite-order characters. Its Theorem 1.1 is deduced from the mean-square estimate [35, Proposition 3.1], one character at a time, and that estimate carries a fixed finite-order twist ν . Section G.3 supplies (U2), and with it the CM form of the argument of [36].

G.2 Proof of Theorem 2: Part I with a finite-order twist

The data. Fix $\psi \in \mathcal{F}_{\text{CM}}$. By the definition of $\mathcal{F}_{\text{CM}}$, ψ is the primitive character inducing $\bar{\alpha}\eta$, where η is a character of the ray class group modulo some integral ideal $\mathfrak{m}$. Thus

$$\psi((n)) = \overline{\alpha(n)} \eta((n)) \quad (\text{G.2})$$

for every primary n prime to $3\mathfrak{m}$. Let S be the set of prime ideals that divide $6\mathfrak{m}$. If $n \equiv n' \pmod{\mathfrak{m}}$ are primary and prime to S , then $n/n' \equiv 1 \pmod{\times \mathfrak{m}}$ (multiplicatively), so $\eta((n)) = \eta((n'))$. Put

$$\nu(n) = \overline{\eta((n))} \quad (n \text{ primary, prime to } S), \quad \nu(n) = 0 \quad \text{otherwise.} \quad (\text{G.3})$$

For a primary prime $\pi \notin S$ this gives

$$\alpha(\pi)\nu(\pi) = \overline{\alpha(\pi)\eta((\pi))} = \overline{\psi((\pi))}. \quad (\text{G.4})$$

Fix $L_S \in \mathbb{Z}[\omega]$ whose prime factors are exactly the primes of S , and which is divisible by 72, by a generator of $\mathfrak{m}$, and by the moduli of the finitely many ray characters $n \mapsto \chi_n(v \prod_{\mathfrak{p} \in S} \pi_{\mathfrak{p}}^{v_{\mathfrak{p}}})$ of [36, Lemma 4.1]. Here v is a unit, $0 \leq v_{\mathfrak{p}} \leq 5$, and $\pi_{\mathfrak{p}}$ is a fixed generator of $\mathfrak{p}$. Let $\mathfrak{G}_S$ be the group of classes in $(\mathbb{Z}[\omega]/L_S\mathbb{Z}[\omega])^\times$ congruent to 1 modulo 3. These are the classes of the primary elements prime to S . When S consists of the primes above 6, one may take $L_S = 72$, and $\mathfrak{G}_S$ is the group $\mathfrak{G}$ of Section 7. In general $\mathfrak{G}_S$ has four properties.

- (G1) G , $\mathcal{R}$ and $x \mapsto \chi_x(-1)$ are functions of classes modulo 4 (Section 7.3). Reduction $\mathfrak{G}_S \rightarrow (\mathbb{Z}[\omega]/4\mathbb{Z}[\omega])^\times$ is a homomorphism, so they are functions on $\mathfrak{G}_S$, and (7.14) holds on $\mathfrak{G}_S$.
- (G2) ν is a character of $\mathfrak{G}_S$, by the congruence property just proved.
- (G3) The fixed-numerator characters above are characters of $\mathfrak{G}_S$.
- (G4) $(\lambda/x)_3$ depends only on $x \pmod{9}$ for primary x , and $9 \mid L_S$.

The twisted rows are

$$A_u^\nu(D) = \sum_n \mu_K(n) \alpha(n) \nu(n) \chi_n(u) W\left(\frac{N(n)}{D}\right), \quad (\text{G.5})$$

over primary n , with $u \in \mathbb{Z}[\omega]$. The factor $\nu(n)$ restricts n to elements prime to S . For $\nu = \mathbf{1}$ and S the primes above 6 these are the rows (3.4) of Part I.

Proposition 14 (Twisted mean square). *For every fixed $0 < \vartheta \leq 1/10$ and $\varepsilon > 0$,*

$$\sum_{0 < N(u) \leq D^{1+\vartheta}} |A_u^\nu(D)|^2 \ll_{\psi, W, \vartheta, \varepsilon} D^{2+\vartheta+\varepsilon}. \quad (\text{G.6})$$

As in (4.6), the constant is controlled by a finite smoothness norm of W .

This is the CM form of [35, Proposition 3.1], and (4.6) is the case $\nu = \mathbf{1}$. The proof needs the CM transformation formula at a general modulus.

Lemma 33 (The transformation formula at a general modulus). *Let $L \in \mathbb{Z}[\omega]$ have prime factors exactly the primes of S , with $72 \mid L$, and let $\mathfrak{G}_L$ be the group of classes in $(\mathbb{Z}[\omega]/L\mathbb{Z}[\omega])^\times$ congruent to 1 modulo 3. Let ξ_0 be a character of $\mathfrak{G}_L$, extended by zero to elements that are not primary or not prime to S . Let $\mathcal{P}$ be a finite set of primary primes outside S , with $0 \leq j_\pi \leq 5$, put $\Psi = \xi_0 \prod_{\pi \in \mathcal{P}} \chi_\pi^{j_\pi}$, and define $T(X; \Psi)$ by (8.1). Then Proposition 8 holds with three changes:*

- the number of terms is at most $N(L) 2^{|\mathcal{P}|}$;
- c_0 is an associate of the reduced denominator of $\lambda^2 h_0/L$, so its prime factors lie in S ;
- ψ is an additive character modulo $\lambda^3 c_0$.

Lemma 12 holds with $h_0 \in \mathbb{Z}[\omega]/L\mathbb{Z}[\omega]$ and $M = \lambda^{12} L^4$. Lemma 13 does not involve L .

Proof. Follow Section A.2. In Step 1 put $\phi(x) = (\lambda/x)_3 \xi_0(x)$ if x is primary and prime to S , and $\phi(x) = 0$ otherwise. Then ϕ is L -periodic, by (G4) and $9 \mid L$. By (8.2), the selection statement after it, and the vanishing of ϕ at elements not prime to S , the coefficient of Θ_Ψ at ℓ vanishes unless $\lambda^3 \ell = nb^3$ with n squarefree and n, b primary and prime to S . Its value is still (A.12). The finite Fourier expansion (A.14) holds with $h_0 \in \mathbb{Z}[\omega]/L\mathbb{Z}[\omega]$, and $\sum_{\mathbf{h}} c_F(\mathbf{h}) = \phi(0) \prod_{\pi} \chi_{\pi}(0)^{j_{\pi}} = 0$.

Step 2 quotes facts (a)–(e) from [35, App. A.2]. They are proved there for any nonzero L with prime divisors in S that is divisible by every prime of S , by the conductor of the fixed character, and by sufficiently high powers of the primes above 2 and 3. Since $72 \mid L_S$, the powers of 2 and λ in L_S are at least those of 72, which Appendix A uses; the argument of [35, App. A.2] uses them through congruences modulo $q^{v_q(Mc_0)}$, with $M = \lambda^{12} L^4$. This covers the reduced denominators $c = c_0 r$ with c_0 as stated, the matrices g, H, g_1 and the label σ , the multiplier $\kappa(g_1) = \kappa_0 \prod_{\pi|r} \chi_{\pi}(a)^2$ ([35, (A.10)]), the phase (A.17) with ψ modulo $\lambda^3 c_0$ ([35, (A.11)]), and the local identity (A.18) at the primes of $\mathcal{P}$, which lie outside S ([35, (A.12)]).

Steps 3–6 use the data of Step 2 only through (A.16), (A.17), (A.18) and $|c_F(\mathbf{h})| \leq |\hat{\phi}(h_0)| \leq 1$. The derivative, the Mellin transforms, the functional equation and the assembly do not involve L . The value $L = 72$ was used only in the count $N(72) = 5184$ of the groups $(h_0, \mathcal{A})$ and in the description $c_0 \mid 8\lambda^2$. Lemma 12 is [35, Lemma 6.3], which is proved for this general L at the end of [35, App. A.2]. $\square$

Proof of Proposition 14. Run the proof of (4.6) in Sections 7–10 and Appendix A, with four substitutions.

- (S1) Every column-type variable runs over primary elements prime to S instead of prime to 6. These are the columns n, z_i, m_i, x_i , the cube indices b of (8.1) and h of (10.1), and the divisors $g, e_0, v_0, r_0, f, C, t, e, w, r, j, f'$ of Sections 7 and 10. The row and frequency variables $(u, k, y, \text{the Poisson frequencies, } h_2, k')$ still run over $\mathbb{Z}[\omega]$.
- (S2) $\mathfrak{G}$ becomes $\mathfrak{G}_S$; the characters ξ, ξ_1, ξ' range over the characters of $\mathfrak{G}_S$, and $|\mathfrak{G}| = 432$ becomes $|\mathfrak{G}_S|$.
- (S3) Appendix A is used at $L = L_S$, in the form of Lemma 33.
- (S4) In Step 2 of the proof of Proposition 3, (7.9) is replaced by (7.17) with $\xi_1 = \bar{v}$.

We check the written-out CM arguments in which 6, 72 or $\mathfrak{G}$ occur. Every step that Part I cites from [35] is proved there for a general finite set S and a general fixed finite-order twist.

Section 7.3, inputs. Formula (7.4) is [2, Lemma 21] for a primitive character modulo $z_1 z_2$ with exclusion ideal g ; enlarging the excluded set changes nothing in it. The identities (7.7) and (7.13) hold prime by prime. Identity (7.14) holds on $\mathfrak{G}_S$ by (G1). The exclusion bound (7.15) is [35, Lemma 4.4], which is stated there for a general S .

Lemma 5 holds for characters ξ, ξ_1 of $\mathfrak{G}_S$, with $|c_{\xi}(\xi_1)| \leq 1$ and $|\ell_{\xi_1}(\xi)| \leq 1$. Its proof uses only four facts: (7.20) (the Chinese remainder theorem and sextic reciprocity for coprime z_1, z_2 prime to 6); (7.21) (prime by prime); (7.14) on the group; and that ξ, ξ_1 are characters of a group through which G factors.

Proposition 3. In Step 1 the coefficient of the pair (n_1, n_2) acquires $\overline{\nu(n_1)}\nu(n_2) = |\nu(g)|^2 \overline{\nu(z_1)}\nu(z_2) = \overline{\nu(z_1)}\nu(z_2)$, since g is prime to S . The zero frequency still has coefficient $|W(N(g)/D)|^2$, so Z is unchanged, with g prime to S . In Step 2, (7.17) with $\xi_1 = \bar{\nu}$ gives

$$\mu_K(z_1)\mu_K(z_2)\overline{\alpha(z_1)}\alpha(z_2)\overline{\nu(z_1)}\nu(z_2)\gamma(\overline{\chi_{z_1}}\chi_{z_2}) = \sum_{\xi} c_{\xi}(\bar{\nu}) a_{\xi}(z_1)\overline{a_{\xi}(z_2)}, \quad (\text{G.7})$$

and therefore $\mathcal{M}_D - Z = \sum_{\xi} c_{\xi}(\bar{\nu})\mathcal{S}_{\xi}$, with $\mathcal{S}_{\xi}$ as in (7.32). For $z_1 = z_2 = 1$ the identity reads $1 = \sum_{\xi} c_{\xi}(\bar{\nu})$. Steps 3–5 use the coefficients only through (7.19) and $|a_{\xi}| \leq 1$, which hold for every character of $\mathfrak{G}_S$. The bijection of Step 3 and the scales of Step 4 do not see S . The number of characters, $|\mathfrak{G}_S|$, enters only the constant.

Sections 8 and 9. For a character ξ of $\mathfrak{G}_S$, the twist $\Psi(n) = \xi(n)\chi_n(k)\chi_n(f)^4$ has the form $\xi_0 \prod_{\pi \in \mathcal{P}} \chi_{\pi}^{j_{\pi}}$, where $\mathcal{P}$ is the set of primes outside S dividing kf and ξ_0 is a character of $\mathfrak{G}_S$: ξ_0 collects ξ , the unit and reciprocity factors, and the factors of k at the primes of S [35, (6.17)], which are characters of $\mathfrak{G}_S$ by (G1) and (G3). The proof of the completed estimate (9.2) in Section 9.1 then applies with Lemma 33 in place of Proposition 8 and Lemma 12. Lemma 13 is unchanged. The large sieve (9.1) is applied with rows prime to S ; the dual columns, the theta indices, need not be prime to S , and (9.1) allows arbitrary squarefree columns. The bound for squarefree rows is [35, Lemma 6.6], whose proof fixes the part of the row at S and at the primes of g for a general S .

Section 10. Identity (10.1) holds with h prime to S . Cube removal, Lemma 6, is [35, Lemma 5.3], proved for a general S , with (9.2). In the transfer estimate, Lemmas 7, 8 and 9 use the Poisson formula of [2, Lemma 21], the paired identities of Lemma 5 (now over $\mathfrak{G}_S$), the exclusion bound (7.15) and [35, Lemma B.2]. The finitely many ξ_1 are the characters of $\mathfrak{G}_S$. The induction of Section 10 uses only cube removal, the transfer estimate, counting and $|a_{\xi}| \leq 1$.

Hence (10.25) holds for every character ξ of $\mathfrak{G}_S$, which is the hypothesis (7.25) over $\mathfrak{G}_S$, and the modified Proposition 3 gives the assertion. $\square$

Proof of Theorem 2. Extraction. Let $H = D^{1+\vartheta}$ and $Y = H^{1/6}$. Let $\mathcal{P}_Y$ be the set of primary primes $\pi \notin S$ with $Y/2 < N(\pi) \leq Y$, and $J = |\mathcal{P}_Y|$. Lemma 2 holds with $u_0 = 1$, $N_0 = 1$ and this $\mathcal{P}_Y$. In it, A'_u replaces A_u , $A'_1(D)$ replaces $S_d(D)$, and the mean square of Proposition 14 replaces $M(D, H)$. Its proof uses only $|\mu_K(n)\alpha(n)\nu(n)| \leq 1$, $\chi_n(\pi^6) = \mathbf{1}_{\pi \nmid n}$, the distinctness of the rows π^6 , and lattice-point counting. Since S is finite, the prime number theorem for primes $\equiv 1 \pmod{3}$ [25] gives $J \geq Y/(5 \log Y)$ for $Y \geq Y_0(S)$. As in the proof of (5.7), Proposition 14 then gives, for every $\varepsilon > 0$,

$$A'_1(D) = \sum_n \mu_K(n)\alpha(n)\nu(n)W\left(\frac{N(n)}{D}\right) \ll_{\psi, W, \varepsilon} D^{11/12+\varepsilon} \quad (D \geq 1). \quad (\text{G.8})$$

Zeros. Let W be supported in $[a, b] \subset (0, \infty)$. Then $A'_1(D) = 0$ for $D < 1/b$, it is bounded for $1/b \leq D < 1$, and (G.8) makes $F_W(s) = \int_0^\infty A'_1(D)D^{-s} dD/D$ holomorphic on $\Re s > 11/12$. For $\Re s > 1$, Fubini's theorem as in Lemma 4(i), complete multiplicativity of $\alpha\nu$ on primary elements, and (G.4) give

$$F_W(s) = \widehat{W}(s) \sum_n \frac{\mu_K(n)\alpha(n)\nu(n)}{N(n)^s} = \widehat{W}(s) \prod_{\pi \notin S} \left(1 - \frac{\overline{\psi((\pi))}}{N(\pi)^s}\right) = \frac{\widehat{W}(s)}{L^S(s, \bar{\psi})}. \quad (\text{G.9})$$

Here $L^S(s, \bar{\psi}) = L(s, \bar{\psi}) \prod_{\mathfrak{p} \in S} (1 - \bar{\psi}(\mathfrak{p})N(\mathfrak{p})^{-s})$, and the finite product is holomorphic and nonzero on $\Re s > 0$. The function $L(s, \bar{\psi})$ is entire, by Hecke's theorem [18], [33, Ch. VII, §8] (see also Lemma 10). So $L^S(s, \bar{\psi})F_W(s) = \widehat{W}(s)$ on $\Re s > 1$, and hence on $\Re s > 11/12$. Suppose $L(s_0, \bar{\psi}) = 0$ with $\Re s_0 > 11/12$. Choose a nonzero nonnegative $\phi \in C_c^\infty((1, 2))$ and put $W(y) = y^{-s_0}\phi(y)$. At s_0

the left side vanishes, while $\widehat{W}(s_0) = \int_1^2 \phi(y) dy/y > 0$. This contradiction shows that $L(s, \overline{\psi}) \neq 0$ for $\Re s > 11/12$. Finally $L(s, \psi) = \overline{L(\overline{s}, \overline{\psi})}$, so the same holds for $L(s, \psi)$. $\square$

Remarks. (1) All constants depend on ψ , through S , L_S , $\mathfrak{G}_S$ and ν . No uniformity in ψ or in its conductor is needed, because the conclusion is made for one ψ at a time. (2) For $\psi = \psi_d^*$ the twist is not needed: then $\eta = \chi_\bullet(4d)$, and Section 5 reaches $\overline{\psi_d^*} = \psi_d$ through the row u_d . A general η is not of the form $\chi_\bullet(u)$; for example, its order need not divide 6. That is why the twist ν is carried in the columns. (3) The weights here are compactly supported, so Lemma 33 needs only the test functions $V(y) = \sqrt{y} W(y)$ of Appendix A. The general log-Schwartz tests and the marked forms of item 1 of Section 16 are not used.

G.3 The shared high-side lemmas, and the route of [36]

Proposition 15 (The shared high-side lemmas for $\mathcal{F}_{\text{CM}}$). *We use items 2 and 3 of Section 16, proved in Appendices C and D: Lemma 10 with the CM logarithmic control, and the detector with the label set $\mathcal{X}_u$ of Section 13. Fix $\sigma_0 \in [7/8, 1)$ and suppose $\beta_* > \sigma_0$. Let $\psi \in \mathcal{F}_{\text{CM}}$ have finite part η . Use the fixed data of [36, Section 6] for η (the group T , b_* , ξ , τ , Ξ , W_0 , W_1). Take S to contain also the primes dividing the conductor of ψ and the primes of norm at most P_0 , and use the weight (12.1). Make three replacements in the following statements of [36]: Definition 10.1, eqs. (10.1) and (10.2), and Lemmas 10.2, 10.3, 10.4, 10.5, 10.6 and 11.1.*

- (i) *The target η becomes ψ , and “every primitive finite-order target” becomes “every $\psi \in \mathcal{F}_{\text{CM}}$ ”.*
- (ii) *The scalar denominators $L^S(s, \eta \overline{\chi_\bullet(u)})$ and $L_F^S(s, \eta)$ become $L^S(s, \psi \overline{\chi_\bullet(u)})$ and $L^S(s, \psi)$. The corrections $\mathcal{H}_{\eta, u}$ and H_η become $\mathcal{H}_{\psi, u}$ of Proposition 6 and $H_\psi(s) = \mathcal{H}_{\psi, 1}(s, 1, 1/6)$.*
- (iii) *Every clause about a principal target, or about a principal denominator of a unit row, is deleted.*

Then each of these statements holds, with the same regions $\mathcal{D}_1(\epsilon_0)$ and $\mathcal{D}_2$, the same real exponents and the same order of choices. In particular P_0 can be fixed before ψ so that $\sup_{\Re s > 7/8} |H_\psi(s) - 1| \leq 1/2$, and $c_S > 0$ is unchanged.

Proof. The proofs in [36, Sections 10–11] use the target in exactly five ways. Every other ingredient involves no target: the Mellin weights, the Gaussian, $M(z)$, $\widehat{W}_1$, the kernels of Lemma 10.2, the regions, the contour geometry, the exponent $E_{\sigma_0}(d; R, g)$ of eq. (10.15) there, and the order of choices in Lemma 11.1.

(a) *The correction.* The definition and the lemmas use the holomorphy of $\mathcal{H}_{\eta, u}$ in the two regions, the bound $\mathcal{H}_{\eta, u} \ll q_u^\epsilon$ for all heights, and $\mathcal{H}_{\eta, 1} = 1 + O(P_0^{-4/5})$ uniformly in the unit phases ([36, Sections 10.1 and 10.5]). Proposition 6 gives the same three properties for $\mathcal{H}_{\psi, u}$. Its local bounds use $\psi(p)$ only through $|\psi(p)| = 1$, so the same P_0 , chosen before the target, gives [36, eq. (10.2)], and H_ψ is holomorphic on $\Re s > 7/8$. The constant $c_S = \widehat{W}_1(1)M(1/6)(\text{Res}_{v=1} \zeta_K^S(v))^2/6$ does not involve the target.

(b) *The global reciprocal bound.* On lines $\Re s = \beta_* + v$, $v \in \{e, 20e\}$, the denominators are bounded by an arbitrarily small power of the conductor times a fixed polynomial in the height ([36, proof of Lemma 10.3; proof of Lemma 10.5; proof of Lemma 10.6]). [36] takes this from Lemma 4.9, whose global part uses only the definition of β_* , and from Lemma 4.10. The primitive character inducing $\psi \overline{\chi_\bullet(u)}$ lies in $\mathcal{F}_{\text{CM}}$, by closure under finite-order twists (Section 11). So its zeros have real part at most β_* . Its conductor divides the fixed conductor of ψ times that of $\chi_\bullet(u)$, whose

norm is $\ll q_u$ by Lemma 23(e). The deleted Euler factors lie over S and the primes dividing u , and Lemma 4.10 only uses $|a_p| \leq 1$. Item 2 then gives the same bound.

(c) *The bin ceiling* $a \leq \beta_*$ ([36, proof of Lemma 10.3], from [36, Section 8.1]). By item 3, the bins are formed with the label set $\mathcal{X}_u$. Its finite-order members have no zeros in $\Re s > 7/8$ [36, Theorem 1.1], and $7/8 \leq \sigma_0 < \beta_*$. Its other members are, after passing to primitive characters, members of $\mathcal{F}_{\text{CM}}$ or their conjugates. Their zeros have real part at most β_* , since $L(s, \bar{\psi}') = \overline{L(\bar{s}, \psi')}$. Hence $M_i(u) \leq \beta_*$ and $a \leq \beta_*$.

(d) *Principal clauses.* Three passages allow a principal character: a principal denominator for a unit row ([36, Section 10.1; proof of Lemma 10.6]), and a principal target with a pole of $L_F^S(s, \eta)$ at $s = 1$ ([36, proof of Lemma 10.5]). A member of $\mathcal{F}_{\text{CM}}$ has infinity type $\bar{z}/|z|$, which is nontrivial. So neither ψ nor any $\psi_{\chi_\bullet}(u)$ is principal, and $L(s, \psi)$ is entire (Lemma 10). These clauses are vacuous. The numerators $\chi_\bullet(u)$ do not involve the target, so the classification of $u = 1$ as the only row with a principal numerator ([36, Section 10.1]) is unchanged.

(e) *The numerators.* $L^S(w, \chi_\bullet(u))$ has finite order and does not involve the target. Lemma 4.8 and the reflected bound of the detector apply to it unchanged ([36, Section 10.5; proof of Lemma 10.6]). $\square$

Lemma 34 (Sextic row count with CM labels). *By item 3 of Section 16 (Appendix D), [36, Prop. 9.2] holds for the rows of the CM probe, with the bins and witnesses of the detector with label set $\mathcal{X}_u$, the same envelope $R(\delta) = \min\{1, \max(1 - \delta/2, 4/3 - \delta)\}$ and the same order of choices.*

Proof. The proof in [36, proof of Proposition 9.2] uses the witness in three ways: through its label, which lies in a fixed finite set; through its length and lower bound, [36, eq. (9.17)]; and through the form of its coefficients. For a CM label the inverse witness of Section 13 is

$$M_u(D; W) = D^{-1/2} \sum_n \mu_K(n) \alpha(n)^\epsilon \theta(n) \chi_n(u)^\epsilon W(N(n)/D), \quad \epsilon = \pm 1, \theta \in \Theta_0. \quad (\text{G.10})$$

Freeze $v = vs_5 w$ as in [36, proof of Proposition 9.2]. The exact identity $\chi_n(u)^\epsilon = \chi_n(v)^\epsilon \chi_n(k)^\epsilon$ ([36, eq. (9.20)]) makes the row sum that of [36, Lemma 9.1]. Its coefficients are $c_n = D^{-1/2} \mu_K(n) \alpha(n)^\epsilon \theta(n) \chi_n(v)^\epsilon W(N(n)/D)$. These do not depend on the residual row k , and $\sum_n |c_n|^2 \ll 1$ because $|\alpha(n)| = 1$. The sieve allows arbitrary coefficients. The remaining steps do not involve the coefficients: the count of powerful parts, the Sobolev step in the profile parameters, and the optimization [36, eq. (9.24)]. The label set is still finite, of size at most $4|\Theta_0|$. $\square$

The sieve does not see the orientation of a label. Co-orientation (Section 13) matters only for the inverse and plain moments of Section 14, which Part I of [36] does not use.

The route of [36]. Proposition 15 and Lemma 34 also give a second proof of Theorem 2, following [36]. It needs one further input: the unmarked completed-row moment at equal lengths,

$$\sum_{0 < N(m) \leq C_m Z} \left| \mathcal{C}_{V_G}^{(2)}(Z; m, \nu) \right|^2 \ll Z^{1+\epsilon}. \quad (\text{G.11})$$

Here $\mathcal{C}^{(2)}$ is [36, eq. (5.30)] with $\overline{\alpha(cn^3)}^2$ in place of $\overline{\alpha(cn^3)}$, ν runs over the fixed family $\nu_\sigma \theta$, and V_G is the Gaussian test of [36]. This is Proposition 7 with $Q = Z$. Its proof needs item 1 of Section 16 in its unmarked form: a general modulus supported on S and the non-compact test V_G . The steps are then as follows.

1. Suppose $\Delta_1 = \beta_* - 11/12 > 0$.

2. The low estimate of [36, Prop. 6.3] gives $|I_\psi(Z^{1/2}, Z^{1/2}, Z)| \ll Z^{1/4+\varepsilon}$: Cauchy–Schwarz in [36, eq. (6.2)], the moment above, and [36, Lemma 6.2], which does not involve the target.
3. The Poisson representation is Proposition 6.
4. The detector is item 3, and the row count is Lemma 34.
5. The central estimate [36, eq. (10.12)] holds with the same $E_I(d)$. It uses the bin ceiling $\delta \leq 2\beta_* - 1 = 5/6 + 2\Delta_1$ from part (c) of the proof of Proposition 15, and gives the margin $m_c = 1021/25000$.
6. The principal and outer rows are handled by Proposition 15, with margins $1/40$, $1/1200$, $43/300$ and more than 1.
7. [36, Prop. 11.2] then holds verbatim: $|J_{I,\psi} - f_{I,\psi}| \ll_\psi Z^{C_1(\beta_*)-1/4800}$.
8. The continuation criterion, which holds for $\mathcal{F}_{\text{CM}}$ (Section 11), with $\sigma_0 = 11/12$, $C = C_I$, $\omega = \Delta_1/2$ and $\sigma = 1/4800$, contradicts $\Delta_1 > 0$.

The transfer to Dirichlet L -functions is not needed. This route needs items 1 (unmarked), 2 and 3. The proof in Section G.2 needs none of them; its only new input is Lemma 33, which is proved above from [35, App. A.2]. That lemma is the part of item 1 that concerns the modulus, for compactly supported tests.

Summary. Theorem 2 is proved in Section G.2. The proof uses Part I of these notes and the steps of [35] for a general twist and a general excluded set (Lemmas 4.4, 5.3, 6.3 and 6.6, Section 7, App. A.2). It also uses Hecke’s theorem [18] and the prime number theorem [25]. Proposition 15 and Lemma 34 use items 2 and 3, proved in Appendices C and D; the second route also uses the unmarked part of item 1 (Lemma 14).

Numerical checks. We checked the new identities numerically.

- *The paired identities over $\mathfrak{G}_S$.* We took ν of order 12 modulo $\pi = 4 + 3\omega$ (norm 13), and worked on the quotient $(\mathbb{Z}[\omega]/4\mathbb{Z}[\omega])^\times \times (\mathbb{Z}[\omega]/\pi)^\times$ of $\mathfrak{G}_S$, through which G , $\mathcal{R}$, $\chi_\bullet(-1)$ and ν factor. The forward identity (7.17) with $\xi_1 = \bar{\nu}$ and the reverse identity (7.18) hold as literal sums over all 144 characters, with error at most $1.2 \cdot 10^{-14}$.
- *The twisted reduction.* The exact form $\mathcal{M}_D = Z + \sum_\xi c_\xi(\bar{\nu})\mathcal{S}_\xi$ of the first Poisson reduction for the twisted rows holds to relative error $1.5 \cdot 10^{-14}$ at $(D, H) = (120, 170)$, $(120, 40)$ and $(200, 300)$.
- *Exponents.* The exponent bookkeeping of [36], Part I, holds in exact arithmetic: $E_I(1/2)$, the comparison under the bin ceiling, m_c , the outer and principal margins, and the loss budget.
- *Rows and characters.* For each unit $u \neq 1$ the character $\chi_\bullet(u)$ is nonprincipal, and $\psi = \bar{\alpha}\bar{\nu}$ has infinity type $\bar{z}/|z|$ and is not of finite order.

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