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A seven-eighths zero-free half-plane for unitary Weil L-functions

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Abstract

We prove that for every number field F and every finite-dimensional continuous unitary representation r of the global Weil group W_F, the L-function $L_F(s,r)$ has no zeros in $\operatorname{Re} s > 7/8$, and that its only poles in this half-plane are at $s = 1 + it$, with order equal to the multiplicity of $\mathcal{N}_F^{it}$ in r. The analytic core is the rank-one case: $L_F(s,\psi) \neq 0$ for $\operatorname{Re} s > 7/8$ for every unitary Hecke character ψ of every number field, including characters of infinite order. This extends the cubic-theta method of the OpenAI quasi-Riemann preprint, which treats finite-order characters over $\mathbb{Q}(\sqrt{-3})$, by transferring its estimates to number fields containing μ_{12} and to families of fixed infinity type, followed by an abelian descent. Brauer induction and Frobenius reciprocity then give the result in every dimension. Consequences include the same half-plane for all Artin L-functions and their unitary Hecke twists, an effective Chebotarev theorem, cuspidal weight-one newforms, and CM motives after the appropriate weight shift.

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1 Introduction

Let F be a number field and let W_F be its classical global Weil group. We use the norm character $\mathcal{N}_F$ normalized by $\mathcal{N}_F(\mathrm{Frob}_{\mathfrak{p}}) = \mathrm{N}\,\mathfrak{p}$, and put $\nu_{F,t} = \mathcal{N}_F^{it}$ for $t \in \mathbb{R}$. Thus $L_F(s, \nu_{F,t}) = \zeta_F(s - it)$. For a continuous unitary representation $r : W_F \longrightarrow U(d)$ on V_r, write

$$L_F(s,r) = \prod_{\mathfrak{p}} \det\Bigl(1 - r(\mathrm{Frob}_{\mathfrak{p}})(\mathrm{N}\,\mathfrak{p})^{-s} \mid V_r^{I_{\mathfrak{p}}}\Bigr)^{-1} \qquad (\operatorname{Re} s > 1).$$

We use the meromorphic continuation of this finite-place Euler product. At a ramified prime the Frobenius acts on the inertia invariants, so the definition is independent of its lift. We use the classical Weil-group formalism of [14, Chapters 1, 2 and 20]; finite-dimensional unitary representations are automatically semisimple.

Theorem 1.1 (Unitary Weil form)**.** *For every number field F and every finite-dimensional continuous unitary representation r of W_F, the meromorphic function $L_F(s,r)$ has no zeros in $\operatorname{Re} s > 7/8$. Its only poles in this half-plane are prescribed by the norm-character constituents:*

$$\operatorname{ord}_{s=1+it} L_F(s,r) = -m_t(r), \qquad m_t(r) = \dim_{\mathbb{C}} \operatorname{Hom}_{W_F}(\nu_{F,t}, r). \tag{1.1}$$

Here positive order denotes a zero and negative order a pole. Consequently, if r contains no $\nu_{F,t}$, its L-function is holomorphic and nowhere zero in this half-plane. The statement and the pole orders are unchanged on deleting any fixed finite set of finite Euler factors. The boundary is independent of d, the field, the conductor, and the infinity type.

Sections 2–8 prove the following one-dimensional result. Section 9 then deduces Theorem 1.1 by classical induction theory.

Theorem 1.2 (Hecke analytic core)**.** *Let F be a number field and let $\psi : F^\times\backslash\mathbb{A}_F^\times \longrightarrow S^1$ be a continuous unitary Hecke character. The function $L_F(s,\psi)$ is holomorphic and nonzero in $\operatorname{Re} s > 7/8$ unless $\psi = \mathcal{N}_F^{it}$ for some $t \in \mathbb{R}$; in that case it is nonzero there and its only singularity is a simple pole at $s = 1 + it$. The same statement holds after deletion of any fixed finite set of Euler factors.*

In rank one the finite Euler product is $\prod_{\mathfrak{p}}(1-\psi(\mathfrak{p})(\mathrm{N}\,\mathfrak{p})^{-s})^{-1}$, with the usual convention at ramified primes. Its meromorphic continuation and pole classification are the standard Hecke theory [19].

The principal analytic sources are the 30 September 2026 preprint [16], denoted Q, and the 4 October 2026 preprint [17], denoted P. The theorem in [16, Theorem 1.1] treats finite-order Hecke characters over $\mathbb{Q}(\sqrt{-3})$ and Dirichlet characters, with boundary 7/8; [17, Theorem 1.2] treats finite-order Hecke characters over cyclotomic fields containing μ_{12}, with boundary $1-10^{-6}$.

Independently, Cureton and Thorner [3] followed the strategy of P over an arbitrary number field K and proved that $\zeta_K(s)$, and hence every finite-order Hecke L-function over K, has no zeros in $\operatorname{Re} s > 1-1/6000$; their proofs were generated and verified in Lean. Theorem 1.2 implies their nonvanishing statement, but the present argument has not been formally verified.

References to Q and P use the dated editions specified in the bibliography; all their numerical locators refer to those editions.

Earlier work. Patterson's cubic theta series [18] and the metaplectic theta construction of Kazhdan–Patterson [12, 13] underlie the reflection. The unconditional Eisenstein-field cusp formulas used in [16, Section 5] are recorded in [5, Section 5.1 and Appendix A]. For the general-field automorphic normalization we use [17, Sections 3, 5–7 and Appendix A], with the corrected Whittaker normalization explained below. The marked reflection and physical probe estimates are taken from [16, Sections 5–16]; the canonical inverse and paired fourth-moment arguments are those of [16, Lemmas 17.1–17.2 and 18.1]. The sixth-power amplification is [16, Lemma 17.6], and the witness comparison follows [16, Proposition 19.2]. We retain their finite-seminorm, polynomial-height, strict-width, and common-mask assertions, and use the signal-to-zero-free principle of [16, Proposition 2.1].

Field and infinity type. Sections 3 and 4 establish the ideal-norm and fixed-infinity-type versions of the source constructions. The unit quotient, complete marked expansion, and recursive norm identities are the three interfaces summarized in Appendix A. Sections 7 and 8 then make the exponent comparison and the common-family closure.

Weil representations and geometric applications. Brauer's induction method [2], in the Weil-group form of [14, Lemma 2.2], supplies a virtual decomposition into characters induced from finite extensions. Lemma 9.1 verifies that the inducing characters can be taken unitary. Frobenius reciprocity then accounts exactly for the apparent zeros that negative Brauer coefficients could produce at Hecke poles. This gives (1.1); it is the same classical pole-counting mechanism used on $\operatorname{Re} s \geq 1$ in [11, Proposition 11], now applied with the Hecke half-plane of Theorem 1.2.

The applications include Artin representations and their unitary Hecke twists, the effective Chebotarev theorem of Corollary 9.6, weight-one newforms through Deligne–Serre [4, Theorems 4.1 and 4.6], and potentially abelian geometric realizations through [11, Corollary 10 and Lemma 14]. For pure weight w, the change from unitary to arithmetic normalization moves the boundary to $w/2 + 7/8$; the CM realization used here is [6, Corollaries 7.6, 7.9 and Proposition 7.12].

1.1 Proof outline

The argument compares two estimates for one auxiliary function. Its principal term contains the reciprocal of the target L-function; controlling the difference extends that reciprocal to a larger half-plane.

Fix $K = F(\zeta_{12})$ and a unitary character ω of K. The family $\mathcal{F}_\omega = \{\theta\omega^e \mathrm{N}_K^{it} : e \in \{-1,0,1\},\ \theta \text{ finite order},\ t \in \mathbb{R}\}$ contains every character that appears in the estimates. Let B be the supremum of the real parts of their nontrivial zeros, with $B \geq 1/2$, as in (2.3). The Euler product gives $B \leq 1$. Assume $B > 7/8$ and fix a target ψ in this family.

The cubic theta series is summed against an additive test and finitely many smooth prime sums. This defines the auxiliary function $I_{\psi,\mathrm{mod}}$ in (6.9). The compensation at each prime is chosen so that its main prime factor contains only the finite-order row character; see (4.3) and (6.8). Thus the prime estimates used for an infinite-order target involve finite-order Hecke characters in the same family. Poisson summation and Mellin inversion express the auxiliary function as a sum over sixth-power-free rows. The principal row has poles at $w = 1$ and $z = 1/6$. After taking these residues and normalizing, its contribution is

$$f_\psi(Z) = \frac{1}{2\pi i}\int_{(2)} Z^{C(s)} e^{(s-5/6)^2} \frac{H_\psi^S(s)}{L_K^S(s,\psi)}\, ds, \qquad C(s) = s - \frac{11}{16}.$$

The explicit Euler factor in Lemma 6.1 makes H_ψ^S holomorphic and nonzero on $\operatorname{Re} s > 2/3$.

There are two ways to estimate the normalized auxiliary function J_ψ. On the theta side, the complete marked reflection and the additive Gram estimate give $|J_\psi(Z)| \ll_{\psi,\varepsilon} Z^{3/16+\varepsilon}$. On the Euler-product side, the zero detector associates exceptional rows with large values of inverse or ordinary character sums. The two canonical moments bound the number of these rows. With the actual fourth-moment parameter $\kappa = 2B - 1$, the comparison in Section 7 and its positive certificate leave a fixed saving. The resulting estimates, for the same J_ψ, are

$$|J_\psi(Z)| \ll_{\psi,\varepsilon} Z^{C(7/8)+\varepsilon},$$
$$|J_\psi(Z) - f_\psi(Z)| \ll_\psi Z^{C(B)-\eta_0},$$

where $\eta_0 > 0$ is common to the later finite-order twists of the fixed infinity type. Consequently the Mellin transform of f_ψ is holomorphic on $\operatorname{Re} s > \max\{7/8, B - \eta_0\}$. It equals $e^{(s-5/6)^2} H_\psi^S(s)/L_K^S(s,\psi)$ there. The nonvanishing numerator excludes all zeros to the right of a line strictly smaller than B, a contradiction. Abelian descent proves the Hecke theorem over F; unitary Brauer induction and Frobenius reciprocity give the Weil theorem and its pole formula.

The main geometric change occurs before these estimates. Over K, associates are organized on the compact quotient T_0/Γ, where $T_0 = K_S^1$ and $\Gamma = (\mathcal{O}_{K,S}^\times)^6$. Weighted integration on this quotient retains the continuous unit phases. Balancing the generators keeps the additive shapes in a fixed compact set, so the two Poisson transforms have the original norm scales; the additional kernel logarithms are absorbed through finite-depth propagation.

Relative to the source framework, the proof has five components:

(a) Ideal and unit-quotient formulas over every $K \supset \mu_{12}$, with complete local coefficients and both Poisson zero orbits.

(b) Fixed-infinity-type reflection and paired prime coefficients, which supply the moments for infinite-order targets.

(c) One prime-slot system for the entire moderate-row range and the alternative positive certificate (7.25).

(d) A common positive saving and the single Mellin closure of Lemma 8.1.

(e) The Weil deduction, its exact pole orders, and the arithmetic consequences of Section 9.

Appendix A matches each source step with the field-sensitive input, its replacement here, and the norm calculation that remains unchanged. The parameter choices are listed in order in Section 8.1.

1.2 Terminology

All lengths below are logarithmic: a variable of length a has norm of order Z^a. The following terms describe the sums and the finite recursions used in Sections 4–7.

Auxiliary function (physical probe).
The theta–additive sum $I_{\psi;D}(X,Y,Z)$ in (6.3), with its compensated and normalized form J_ψ in (6.19). Its principal Mellin contribution is f_ψ.

Prime slot and mark.
A slot is one factor $Q_{u,i}$: a smoothly weighted sum over primes in a prescribed annulus. In a completed theta sum its corresponding mark is the divisibility condition $\mathbf{1}_{p_i|A}$ on the whole index $A = cn^3$.

Ordinary sum (plain) and inverse sum.
The ordinary sum has coefficients $\lambda_u(\mathfrak{a})$; the inverse sum has coefficients $\mu_K(\mathfrak{a})\lambda_u(\mathfrak{a})$. Both are normalized by the square root of their length scale.

Capacity and strict width.
A capacity is an allowed total prime-slot length, specified by the moment inequalities. A strict width is the positive difference between the two sides of such an inequality. A width decrement is the fixed amount reserved when passing to a recursive child.

Terminal and canonical energy.
A terminal is a range estimated directly, without another recursive call. The canonical energy (4.27) is the row-and-label second moment whose coefficient form is recovered after two Poisson transforms.

Moving puncture and common mask.
A puncture is a radical $\mathfrak{r}$ imposing $\mathbf{1}_{(\mathfrak{a},\mathfrak{r})=1}$. It is fixed in the current inner sum but may vary with outer data; its logarithmic norm is charged to the capacity. A common mask is the same coprimality restriction in all factors of the current expression.

Frozen outer ideals.
These are ideals held fixed during an inner estimate and then summed with their displayed weights. Uniformity in them means that their allowed norm ranges, rather than their individual values, determine the estimate.

Coprime residual support.
This is the support left after the displayed greatest-common-divisor extractions and the original character zeros have been imposed. The full Möbius expansion is applied to this support before any unrestricted coefficient extension.

2 Field reduction

2.1 Abelian descent

Lemma 2.1. *Let K/F be a finite abelian extension and $\sigma_0 > 0$. If $L_K(s, \psi \circ \mathrm{N}_{K/F})$ is zero-free in $\operatorname{Re} s > \sigma_0$, then so is $L_F(s,\psi)$.*

Proof. Put $G = \mathrm{Gal}(K/F)$ and view $\widehat{G}$ as finite-order Hecke characters of F. After omitting

the ramified primes and the conductors from one fixed finite set S, class field theory gives

$$L_K^S(s, \psi \circ \mathrm{N}_{K/F}) = \prod_{\chi \in \widehat{G}} L_F^S(s, \psi\chi). \tag{2.1}$$

For completeness, at an unramified prime whose Frobenius has order f, the values $\chi(\mathrm{Frob}_{\mathfrak{p}})$ run through the fth roots of unity, each $|G|/f$ times. This proves the Euler identity on the absolute half-plane, and continuation proves (2.1) everywhere; see also [15].

If every factor is holomorphic at s_0, a zero of one factor would give a zero on the left. Otherwise the Hecke pole classification gives a unique χ_0 with $\psi\chi_0 = \mathrm{N}_F^{it}$ and $s_0 = 1 + it$. Its pullback is N_K^{it}. Both the left side and the χ_0 factor have a simple pole, so meromorphic orders give

$$-1 = -1 + \sum_{\chi \neq \chi_0} \mathrm{ord}_{s_0} L_F^S(s, \psi\chi).$$

Every summand is nonnegative and is therefore zero. Finally, a deleted Euler polynomial $1 - \xi(\mathfrak{p})(\mathrm{N}\,\mathfrak{p})^{-s}$ is nonzero for $\mathrm{Re}\, s > 0$ when ξ is unitary. This proves the lemma, including possible norm-character poles. □

The extension $K = F(\zeta_{12})$ is abelian over F, has degree at most four, is totally imaginary, and contains μ_{12}. We henceforth work over such a fixed K.

2.2 The fixed-type family

Fix a unitary Hecke character ω of K and set

$$\mathcal{F}_\omega = \{\theta\omega^e\, \mathrm{N}_K^{it} : e \in \{-1, 0, 1\},\ \theta \text{ finite order},\ t \in \mathbb{R}\}. \tag{2.2}$$

Ideal values are normalized by $\mathrm{N}_K^{it}(\mathfrak{a}) = (\mathrm{N}\,\mathfrak{a})^{it}$. The family contains all finite-order characters and is closed under finite twists, conjugation, and vertical translation. Define its common boundary by

$$B = \sup\big(\{1/2\} \cup \{\mathrm{Re}\,\rho : \eta \in \mathcal{F}_\omega,\ L_K(\rho, \eta) = 0,\ 1/2 \le \mathrm{Re}\,\rho \le 1\}\big). \tag{2.3}$$

The Euler product gives $B \le 1$. The parameter t does not affect this supremum. Under the eventual contradiction $B > 7/8$, put

$$\Delta = B - \frac{7}{8} \in (0, 1/8], \qquad \kappa = 2B - 1 = \frac{3}{4} + 2\Delta \in (3/4, 1], \qquad \alpha = \frac{5}{6}. \tag{2.4}$$

All arithmetic sums use ideal norms $q_{\mathfrak{a}} = \mathrm{N}\,\mathfrak{a}$. When a roman letter denotes a generator, its finite class and unit sector are specified by the allowed-generator convention below. Every power-residue character retains its zero extension on nonunits, including powers whose exponent is zero modulo six.

The real losses, strict-width decrements, and finite analytic slot system will be fixed for the three types $0, \omega_\infty, \bar{\omega}_\infty$ before a later finite-order twist is chosen. Constants, seminorm orders, height degrees, and lower thresholds may depend on that later character. Section 8 records the complete order of choices.

3 Arithmetic and reflection

Throughout this section $K \supset \mu_{12}$ is fixed and $[K : \mathbb{Q}] = 2r$. At a finite place we use $|\varpi_v|_v = (\mathrm{N}\, v)^{-1}$, and at a complex place we use $|z|_v = |z|_{\mathrm{ord}}^2$. Fix the standard trace additive character and self-dual additive measures. Choose S to contain the complex places, the primes over 6,

the conductors of the fixed characters, the ramification of the additive character, and the fixed exceptional places of the theta construction. Enlarge S by primes generating the ideal class group. Thus $R = \mathcal{O}_{K,S}$ is principal. Write

$$K_S = \prod_{v\in S} K_v, \qquad |y|_S = \prod_{v\in S} |y_v|_v, \qquad T_0 = K_S^1 = \{y \in K_S^\times : |y|_S = 1\}, \qquad \Gamma = (R^\times)^6.$$

An exterior ideal is an integral ideal of R. Its norm is $q_{\mathfrak{a}} = |R/\mathfrak{a}|$, extended multiplicatively to fractional ideals. The positive norm section ℓ_x has finite components 1 and complex components $x^{1/(2r)}$, so $|\ell_x|_S = x$ and $\ell_x\ell_y = \ell_{xy}$.

3.1 Generators, reciprocity and the Gauss corrections

Proposition 3.1 (Arithmetic calibration)**.** *The following choices and identities hold over K.*

(i) *The diagonal lattice $R \subset K_S$ is self-dual and has covolume 1. For $a \in K^\times$, $q_a = |a|_S = q_{aR}$ and $(aR)^\perp = a^{-1}R$.*

(ii) *Put $V = K_S^\times/K_S^{\times 6}$ and $H_S(x,y) = \prod_{v\in S}(x_v, y_v)_{6,v}$, with the reciprocity convention of [17, Section 3]. The image E of $R^\times/(R^\times)^6$ is Lagrangian, and $V = E \oplus B$ for an isotropic subgroup B. Each exterior fractional ideal has an allowed generator a with $[a] \in B$, unique modulo Γ. Such generators can be balanced so that a/ℓ_{q_a} lies in a fixed compact subset of T_0.*

(iii) *For an exterior prime $\mathfrak{p}$, define the sextic residue character $\chi_{\mathfrak{p}}$ by the power $(q_{\mathfrak{p}} - 1)/6$. Every character power is extended by zero on nonunits, including powers divisible by 6. On allowed coprime generators,*

$$\chi_{\mathfrak{a}}(b) = \chi_{\mathfrak{b}}(a). \tag{3.1}$$

For $u \in R\setminus\{0\}$, $\mathfrak{a} \mapsto \chi_{\mathfrak{a}}(u)$ is induced away from its displayed zeros by a finite-order Hecke character, whose primitive conductor satisfies

$$\mathrm{N}\,\mathfrak{f}_u \ll_{K,S} \mathrm{N}\,\mathrm{rad}(uR) \le q_u. \tag{3.2}$$

Two such inducing characters agree exactly when their parameters have quotient in $K^{\times 6}$. Characters of B extend to finite-order Hecke characters unramified outside S.

(iv) *If $\gamma_j(\mathfrak{c})$ denotes the normalized Gauss sum of $\chi_{\mathfrak{c}}^j$ for squarefree $\mathfrak{c}$, there are a unitary Hecke character Λ and a finite-order Hecke character G, unramified outside S, for which*

$$\gamma_2(\mathfrak{p})^3 = -\Lambda(\mathfrak{p}), \qquad G(\mathfrak{p}) = \overline{\chi_{\mathfrak{p}}(4)}\gamma_3(\mathfrak{p}), \qquad \gamma_1(\mathfrak{p})\gamma_2(\mathfrak{p}) = -\Lambda(\mathfrak{p})G(\mathfrak{p}). \tag{3.3}$$

The complex angular exponents of Λ are ± 1, and it has no radial infinity parameter.

Proof. Apply adelic Poisson summation to $\phi \otimes \prod_{v\notin S} \mathbf{1}_{\mathcal{O}_v}$. The exterior factors are self-dual by the choice of S, giving the lattice R with scalar factor one on both sides. Strong approximation and additive adelic duality identify K_S/R with the corresponding compact adelic quotient and identify the annihilator with R. Scaling and the product formula prove (i); see also [17, Lemma 3.1] and [19].

For $\ell = 2, 3$, local Kummer theory and the S-unit theorem give

$$\dim_{\mathbb{F}_\ell} K_S^\times/K_S^{\times\ell} = 2|S_f| + 2r, \qquad \dim_{\mathbb{F}_\ell} R^\times/(R^\times)^\ell = |S_f| + r.$$

Indeed a finite place contributes $2 + \mathbf{1}_{v|\ell}[K_v : \mathbb{Q}_\ell]$, and complex places contribute zero. An S-unit locally an ℓth power at S defines a Kummer extension unramified outside S and split at S. The trivial S-class group makes this extension trivial, proving injection of the unit quotient. Hilbert reciprocity makes its image isotropic. Since $-1 = \zeta_{12}^6$, the diagonal Hilbert symbols are

trivial, so the pairings are alternating. The dimension calculation gives a Lagrangian unit image; complementary isotropic subspaces for $\ell = 2, 3$ combine to give B. Multiplying any generator by an S-unit puts its class in B; uniqueness follows from $E \cap B = 1$. The norm-one idele class group is compact, whence $T_0/R^\times$, and therefore T_0/Γ, is compact. This proves the balancing assertion.

The exterior Hilbert symbols of two allowed generators and the trivial S-pairing give (3.1). If $m = \epsilon b$ with b allowed and $\epsilon \in R^\times$, the same calculation gives, with the original zeros retained,

$$\chi_{\mathfrak{a}}(m) = \chi_{\mathfrak{a}}(\epsilon)\chi_{\mathfrak{b}}(a), \qquad \chi_{\mathfrak{a}}(\epsilon) = H_S(\epsilon, a)^{-1}. \tag{3.4}$$

Kummer theory identifies the inducing character with the action on $u^{1/6}$. At an exterior prime it is tame, with conductor exponent at most one, and is unramified if u is a unit. At S only finitely many local power classes occur. This proves (3.2) and the equality criterion. A character of B extends trivially across E; the resulting character of $K_S^\times$, trivial on $R^\times$, defines the asserted Hecke character on exterior ideals. In particular, reducing exterior valuations modulo six gives distinct sixth-power-free row representatives modulo Γ; only the representative 1 induces the principal character.

For completeness, the Gauss calibration is independent of any Galois hypothesis on K. Put $K_0 = \mathbb{Q}(\sqrt{-3})$. If $\pi_+(\mathfrak{q})$ is the generator congruent to 1 modulo 3, the cubic Jacobi identity gives $J_q(\rho, \rho) = -\pi_+(\mathfrak{q})$ and $\gamma_2^3 = q^{-1/2}J_q(\rho, \rho)$. For residue degree f, the classical Davenport–Hasse lifting formula, in the normalization recorded in [10, Theorem 1.1], gives

$$J_{q^f}(\rho \circ N, \rho \circ N) = (-1)^{f-1}J_q(\rho, \rho)^f = -\pi_+(\mathfrak{q})^f.$$

Consequently $\Lambda = \Lambda_{K_0} \circ N_{K/K_0}$, where $\Lambda_{K_0}(\mathfrak{a}) = \pi_+(\mathfrak{a})/|\pi_+(\mathfrak{a})|$, has the first property in (3.3). The local norm at infinity is a product of coordinates or their conjugates, giving angular weights ± 1. For the quadratic sum we use Weil's local index and product formula [20, Chapter II], in the explicit normalization of [17, Lemma 3.4, equation (3.13)]: the normalized indices satisfy $w_v(a)w_v(b) = (a, b)_{2,v}w_v(ab)$ and $\prod_v w_v(a) = 1$. Thus $\gamma_3(\mathfrak{p}) = w_S(p^{-1})^{-1}$ is a square-class character on B, since its multiplicative defect is $H_S^3 = 1$ there. Extend it as in (iii) and multiply by $\overline{\chi_\bullet(4)}$ to obtain G. Finally the finite-field duplication identity $g_1g_4 = \overline{\chi(4)}g_2g_3$, together with $g_2g_4 = q$, gives the last identity. Chinese remaindering gives, for coprime squarefree ideals,

$$\gamma_j(\mathfrak{c}_1\mathfrak{c}_2) = \chi_{\mathfrak{c}_1}(c_2)^j\chi_{\mathfrak{c}_2}(c_1)^j\gamma_j(\mathfrak{c}_1)\gamma_j(\mathfrak{c}_2). \tag{3.5}$$

These are the calculations of [17, Lemmas 3.2–3.4]; all their hypotheses have now been checked over K. □

3.2 The exact theta period with fixed infinity type

We use the determinant-modified Kazhdan–Patterson cover with parameters $(d, n, c) = (2, 3, 2)$, divided by its split scalar subgroup. Its exceptional datum has normalized exponents $1/6, -1/6$. The construction of [12], with the normalization in [17, Proposition 5.1 and Appendix A], applies over K. Indeed the scalar cocycle has exponent $1 + 4c = 9$, so the scalar section is central and split; on a coroot cube the selected datum has value $|t|_v$, the exceptional condition. The rational splittings are compatible by reciprocity and the product formula. A rational ratio that is a cube at every completion is a cube in K, since otherwise its Kummer extension has a nonsplit unramified prime by Chebotarev. This verifies the global compatibility of the datum. The residual representation is the one in [12, Theorem II.2.1]; its Whittaker coefficients are given by [12, Theorem II.2.2], read with the correction [13]. Nonvanishing of the global functional and uniqueness of the local models use [12, Theorem II.2.5(a),(b)]. The correction inserts the finite-S normalizing factor $T(S)^{-1}$, with $T(S) = \prod_{v \in S_f}(1 + q_v^{-1})$ in rank two. We use the normalized-functional realization in [17, Appendix A.2], which incorporates this correction. At a complex

place the representation is the spherical principal series of $\mathrm{PGL}_2(\mathbb{C})$ with these exponents. Its constant term has radial complex inducing law, for every fixed right argument.

Use the rational Weyl representative $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of [17, Section 6.1]. The normalized exterior spherical Whittaker values, in the convention of [17, Proposition 5.1(ii)], are

$$W_{\mathfrak{p}}(a(\varpi^e u)) = \chi_{\mathfrak{p}}(u)^{2e} \begin{cases} q_{\mathfrak{p}}^{-j}, & e = 3j \geq 0, \\ q_{\mathfrak{p}}^{-j-1/2}\gamma_2(\mathfrak{p}), & e = 3j+1 \geq 1, \\ 0, & e < 0 \text{ or } e \equiv 2 \pmod 3. \end{cases} \tag{3.6}$$

The local identities and (3.5) show that the exterior coefficient at an allowed A is zero unless $\mathfrak{A} = \mathfrak{c}\mathfrak{n}^3$, with $\mathfrak{c}$ squarefree, and is then $\gamma_2(\mathfrak{c})/(q_{\mathfrak{c}}^{1/2} q_{\mathfrak{n}})$. The ideals $\mathfrak{c}$ and $\mathfrak{n}$ may share primes.

Fix a unitary target ψ and put $\vartheta = \overline{\Lambda}\,\overline{G}\psi$. Let $\mathcal{L}$ be its actual S-period character, so $\mathcal{L}(A) = \vartheta(\mathfrak{A})$ on allowed generators and $\mathcal{L}(R^\times) = 1$. At a complex place write

$$\mathcal{L}_v(z) = (z/|z|_{\mathrm{ord}})^{m_v} |z|_v^{i\tau_v}, \qquad m_v \in \mathbb{Z}, \quad \tau_v \in \mathbb{R}. \tag{3.7}$$

All relative radial parameters are retained. In ideal-character convention $m_v = k_v - b_v$, where k_v is the target angular weight and $b_v \in \{\pm 1\}$ is the weight of Λ. Replacing ψ by its conjugate if necessary ensures $m_v \neq 0$ at some place, since $k - b$ and $-k - b$ cannot both vanish. This orientation is fixed throughout its finite-order twists.

Proposition 3.2 (Fixed-type theta reflection)**.** *For the preceding target, the unmarked theta periods, including their whole-index coprimality masks and finite sectors, admit exact reflection for every smooth test supported in a fixed compact subinterval of* $(0, \infty)$*. For every fixed finite* S*-branch the reflected transform satisfies*

$$|(x\partial_x)^h \Phi_{\kappa,V}(x)| \ll_{K,S,\mathcal{L},A,h} p_J(V) \min\{x^{1/4}, x^{-A}\}, \qquad A \geq 1, \quad J > 4rA + h + 1. \tag{3.8}$$

The same identities give exact Gaussian smoothing. A moving norm phase $V(x)x^{iT}$ *has a fixed polynomial cost in* $1 + |T|$*.*

Proof. At a complex place normalize the holomorphic and antiholomorphic upper root operators $E, \bar{E}$ so that they multiply the Whittaker function by $z, \bar{z}$, respectively; this follows from Whittaker equivariance [17, (5.10)]. Starting from the spherical vector, take $\phi_m = E^m \phi^\circ$ for $m \geq 0$ and $\phi_m = \bar{E}^{-m}\phi^\circ$ for $m < 0$. Then

$$W_{\phi_m}(a(z)) = |z|_v^{a_v} (z/|z|_{\mathrm{ord}})^m K_{1/3}(c_v |z|_v^{1/2}), \qquad a_v = (|m|+1)/2, \quad c_v > 0. \tag{3.9}$$

Spherical invariance gives $(F - \bar{E})\phi^\circ = 0$, and F commutes with $\bar{E}$. Our Weyl representative satisfies $\mathrm{Ad}(w)E = F$ and fixes ϕ°, so $\pi(w)\phi_m = \phi_{-m}$. Thus the reflected angular factor is exactly the opposite one. The other customary Weyl representative introduces only the corresponding fixed sign.

At the finite places choose a common compact open product $U \subset K_{S_f}^\times$ inside the sixth powers and the kernels of the finitely many unit restrictions of $\mathcal{L}$ that occur. A vector with Whittaker function 1_U exists by [17, Lemma 5.3]: convolve a nonzero Whittaker vector by an upper-unipotent Schwartz function whose additive Fourier transform is 1_U, and normalize at 1. This construction uses the open kernels on units, irrespective of the orders of uniformizer values.

Let $X_l = \{y : |y|_S = l\}$ and choose measures so that $d^\times y = d_l^\times y\, dl/l$. The unreflected sheet profile is

$$V_0(l) = \int_{X_l} 1_U(y_f) \prod_{v|\infty} t_v^{a_v - i\tau_v} K_{1/3}(c_v\sqrt{t_v})\, d_l^\times y, \qquad t_v = |y_v|_v.$$

We use $\widehat{V}(s) = \int_0^\infty V(x)x^s\,dx/x$. The Bessel integral and absolute Fubini give

$$\widehat{V}_0(s) = C_0 A_0^s \prod_{v|\infty} \prod_{\epsilon=\pm1} \Gamma(a_v + s - i\tau_v + \epsilon/6), \qquad C_0 \neq 0, \quad A_0 = \prod_v 4/c_v^2 > 0. \tag{3.10}$$

Here the finite Mellin factor is the constant $\mathrm{vol}(U)$.

We justify the global identity before dividing these factors. Fix a nonzero row $m \in R$. If $m = \epsilon_m m_0$ with m_0 allowed, put $\mathcal{L}_m(y) = \mathcal{L}(y)H_S(\epsilon_m, y)^{-1}$. This belongs to a fixed finite family with the same infinity type. For $j_\mathfrak{p} = v_\mathfrak{p}(m) \bmod 6$, retain a digit even when $j_\mathfrak{p} = 0$. An additional whole-index coprimality mask is obtained by inserting the zero digit at each of its new primes. The coefficients

$$\alpha_{\mathfrak{p},j}(v) = \begin{cases} \tau_{\mathfrak{p},j}^{-1}\chi_\mathfrak{p}(v)^{-j}, & 1 \le j \le 5,\ v \neq 0, \\ 0, & 1 \le j \le 5,\ v = 0, \\ 1 - q_\mathfrak{p}^{-1}, & j = 0,\ v = 0, \\ -q_\mathfrak{p}^{-1}, & j = 0,\ v \neq 0, \end{cases} \qquad \tau_{\mathfrak{p},j} = \sum_{v\neq0} \chi_\mathfrak{p}(v)^{-j}\psi_\mathfrak{p}(v/p)$$

satisfy $\sum_v \alpha_{\mathfrak{p},j}(v)\psi_\mathfrak{p}(tv/p) = \chi_\mathfrak{p}(t)^j$ with the specified zero extension; here p is the chosen allowed generator of $\mathfrak{p}$. Form the combined digit sums $D_m^+(y)$ and $D_m^-(y)$ by evaluating theta at $g(y, v)$ and $wg(y, v)$, respectively, where g has component $a(y)$ at S and $n(v/p)$ at row primes. Rational automorphy gives $D_m^+ = D_m^-$. They descend under Γ: replacing v by γv gives the rational diagonal $a(\gamma)$ before w, $a(\gamma^{-1})$ after w, and the same exterior compact factor on the far right. These compact factors fix the spherical vectors, and $\alpha(\gamma v) = \alpha(v)$. The constant terms descend by the same root relation and the product formula; this is the exact argument in [17, (6.6)–(6.7)].

Consequently the period

$$\mathcal{P}_m(l) = \int_{X_l/\Gamma} 1_B([y])\mathcal{L}_m(y)^{-1}D_m^\pm(y)\,d_l^\times y$$

is well defined on a compact quotient. Rotation at a complex place with $m_v \neq 0$ kills both combined constant terms: their inducing law is radial, including at the finite right arguments after w, whereas the weight has nontrivial angular character. The signed orbit substitution is

$$\sum_{\gamma\in\Gamma} \int_{\Omega_l} 1_B([y])\mathcal{L}_m(y)^{-1}f(\gamma Ay)\,d_l^\times y = \mathcal{L}_m(A) \int_{X_{q_A l}} 1_B([A]^{-1}[v])\mathcal{L}_m(v)^{-1}f(v)\,d^\times v. \tag{3.11}$$

The sheet measures are invariant under this substitution. On the direct cutoff $[v] = 1$; the torus cocycle is $H_S(\nu, y)^2$, which is 1 there by alternation. The condition $[y] \in B$ then forces $\nu = \gamma A$ with A allowed. Thus the direct expansion is exactly

$$\mathcal{P}_m(l) = \sum_{\mathfrak{c}\ \text{squarefree},\,\mathfrak{n}} \frac{\gamma_2(\mathfrak{c})\vartheta(\mathfrak{c}\mathfrak{n}^3)\chi_{\mathfrak{c}\mathfrak{n}^3}(m)}{q_\mathfrak{c}^{1/2}q_\mathfrak{n}} V_0(q_{\mathfrak{c}\mathfrak{n}^3}l). \tag{3.12}$$

All these interchanges are absolute. At a finite place every fixed smooth Whittaker function vanishes for sufficiently large $|y|_v$. For small $|y|_v$, the ordinary Jacquet module gives a recurrence on triple valuation steps with eigenvalue moduli q_v^{-1}, q_v^{-2}; its polynomial-times-geometric solutions are $O(|y|_v^\delta)$ for every $\delta < 1/3$ [17, (5.13)]. The complex Bessel functions and their Euler derivatives have the corresponding positive small-end power and arbitrary large-end decay. Integrating on norm sheets gives $O(\min(l^\alpha, l^{-A}))$ for every $\alpha < \delta$, also with any fixed unitary character and finite power-class indicator. One can see this directly by taking one complex norm as the dependent coordinate; the remaining logarithmic integrals converge after a smaller positive power is reserved. This is [17, Lemma 5.4]. Together with the convergent exterior coefficient majorant for $\Re s > 1/2$, it proves absolute orbit unfolding.

Expand the reflected digits using $wn(H) = n(H^{-1})a(-H^{-2})\bar{n}(H^{-1})$ and (3.6). The finite coefficients are the exact finite-field identities of [17, Lemmas 6.2–6.3]; their hypotheses are $q_{\mathfrak{p}} \equiv 1 \pmod{12}$ and the retained zero extensions, both valid here. After the finite power-class separation, write the resulting identity as

$$\mathcal{P}_m(l) = \sum_{\kappa} c_{\kappa,m} \sum_{\mathfrak{a}} b_{\kappa,m}(\mathfrak{a}) H_\kappa(\lambda_{\kappa,m} q_{\mathfrak{a}}/l), \qquad \lambda_{\kappa,m} > 0. \tag{3.13}$$

The exterior branch sum is finite for each row. The S-functions H_κ belong to a fixed finite list; all S-valuations are integrated in the sheets. The exterior coefficients and scales in this formula are given by that finite Weyl operation. The raw valuation table and the full expansion with additional divisibility marks are proved separately in Theorem 3.5; the present argument uses only the unmarked expansion of P.

Let $\mathcal{D}_m(s)$ and $\mathcal{D}_{\kappa,m}(s)$ be the Dirichlet series with the respective exterior coefficients in (3.12) and (3.13). The two expansions give rapid decay of $\mathcal{P}_m$ at opposite ends, with all Euler derivatives. Its entire Mellin transform therefore gives

$$\widehat{V}_0(s)\mathcal{D}_m(s) = \sum_{\kappa} c_{\kappa,m} \lambda_{\kappa,m}^s \widehat{H}_\kappa(-s) \mathcal{D}_{\kappa,m}(-s). \tag{3.14}$$

Reciprocal gamma functions continue $\mathcal{D}_m$ to an entire function. On a right line its series is bounded; on a left line (3.14) and Stirling give polynomial growth. Inside the strip its preliminary growth is at most polynomial times $e^{\pi r|t|}$. Dividing by a sufficiently large power of a linear factor with no zero in the strip and applying the maximum principle after multiplication by $e^{\varepsilon s^2}$ proves polynomial strip growth, as in [16, proof of Proposition 5.1, p. 33].

In the reflected sheet the character is $\mathcal{L}_m$, and the complex vector is ϕ_{-m_v}. Consequently

$$R_\kappa(s) = \frac{\widehat{H}_\kappa(s)}{\widehat{V}_0(-s)} = C_\kappa A_0^{2s} B_\kappa(s) \prod_{v|\infty} \prod_{\epsilon=\pm 1} \frac{\Gamma(a_v + s + i\tau_v + \epsilon/6)}{\Gamma(a_v - s - i\tau_v + \epsilon/6)}. \tag{3.15}$$

The finite factor B_κ is holomorphic for $\Re s > -1/3$ and bounded on every closed vertical substrip, by the preceding finite Whittaker bounds. The only finite factor divided out was $\mathrm{vol}(U) \neq 0$. For an annular V, contour shifting in (3.14) now gives the exact transform

$$\Phi_{\kappa,V}(x) = \frac{1}{2\pi i} \int_{(0)} \widehat{V}(-s) R_\kappa(s) x^{-s}\, ds, \tag{3.16}$$

$$\sum_{\mathfrak{a}} a_m(\mathfrak{a}) V(q_{\mathfrak{a}}/X) = \sum_{\kappa} c_{\kappa,m} \sum_{\mathfrak{a}} b_{\kappa,m}(\mathfrak{a}) \Phi_{\kappa,V}(\lambda_{\kappa,m} q_{\mathfrak{a}} X),$$

where a_m denotes the direct coefficient in (3.12). Lemma 3.3 below proves (3.8). Finally define

$$K(T) = \frac{1}{2\pi i} \int_{(\sigma)} \frac{e^{s^2}}{\widehat{V}_0(s)} T^s\, ds, \qquad \widehat{K}(-s) = \frac{e^{s^2}}{\widehat{V}_0(s)}.$$

The quotient is entire and has Gaussian vertical decay times a fixed exponential and polynomial. Moving the line in either direction gives every absolute power moment of K and its Euler derivatives. Absolute Mellin Fubini therefore proves the exact identity

$$\int_0^\infty K(T) V_0(Tx) \frac{dT}{T} = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{\log^2 x}{4}\right).$$

The reflected profile is the equally convergent $\int_0^\infty K(T) H_\kappa(x/T)\, dT/T$. □

Lemma 3.3 (A common gamma-quotient estimate)**.** *Suppose*

$$R(s) = C A_*^s B(s) \prod_{j=1}^{d} \frac{\Gamma(a_j + s + it_j)}{\Gamma(a_j - s - it_j)}, \qquad A_* > 0, \quad a_j \geq 1/3,$$

with fixed parameters, where B is holomorphic for $\Re s > -1/3$ and bounded on closed vertical substrips. For $A \geq 1$, an integer $h \geq 0$ and $J > 2dA + h + 1$, the transform $\mathcal{H}_R V(x) = (2\pi i)^{-1} \int_{(0)} \widehat{V}(-s) R(s) x^{-s}\, ds$ satisfies

$$|(x\partial_x)^h \mathcal{H}_R(V x^{iT})(x)| \ll p_J(V)(1+|T|)^J \min\{x^{1/4}, x^{-A}\}. \tag{3.17}$$

Proof. There is no numerator pole in $\Re s \geq -1/4$. On a fixed vertical line, the opposite imaginary arguments cancel the exponential factors in Stirling's formula, separately in each quotient, giving $|R(\sigma + it)| \ll (1+|t|)^{2d\sigma}$. Shift to $\sigma = -1/4$ or A and integrate by parts J times in the Mellin transform. Euler derivatives insert only $(-s)^h$, and $p_J(V x^{iT}) \ll (1+|T|)^J p_J(V)$. The same bounds justify the horizontal joins. □

For theta take $d = 2r$ and the parameters $a_v \pm 1/6 \geq 1/3$. For ordinary Hecke reflection, if the completed local factors are $\Gamma_{\mathbb{C}}(s + |k_v|/2 + i\tau_v)$, its Mellin quotient is

$$\widehat{W^\sharp}(s) = \widehat{W}(1-s) \prod_{v|\infty} \frac{\Gamma(s + |k_v|/2 - i\tau_v)}{\Gamma(1 - s + |k_v|/2 + i\tau_v)},$$

up to fixed positive exponential scales. Set $U(x) = x^{1/2} W^\sharp(x)$ and $V(x) = x^{1/2} W(x)$. Then $\widehat{U}(s) = \widehat{V}(-s) R(s)$ with $d = r$, $a_v = 1/2 + |k_v|/2$ and $t_v = -\tau_v$. The conductor factor gives dual norm length Q/X, with $Q = |d_K|\, \mathrm{N}\mathfrak{f}$. Apply the common-mask identities before replacing a presentation by its primitive inducing character [16, (18.4)–(18.8)]. If the inducing character is not a pure norm character, the reflection has no residue. A pure norm phase is absorbed into the moving height, and the resulting principal residue is retained in the exceptional calculation. The central small-annulus mass $O(2^{-j/4})$ is summable, which is the only small-end property used in [16, Section 18.1]. Theta retains the precise $x^{1/4}$ envelope used in [16, Lemma 5.3].

3.3 Whole-index marked reflection

The following exterior expansion supplies the coefficient independence needed for the quadratic–cubic norm estimate. The period character in this subsection may be any unitary ϑ unramified outside S whose S-character has a nonzero angular weight. The physical choice is $\bar{\Lambda}\bar{G}\psi$; the canonical choice will be $\bar{\Lambda}\nu$ times a fixed finite-order character.

Write $U_6 = R^\times/\Gamma$, and fix representatives of this finite group. For a row $m = \epsilon_m m_0$, with m_0 allowed, set $\mathcal{L}_v(y) = \mathcal{L}(y) H_S(\epsilon_m, y)^{-1}$, where $v = \epsilon_m \Gamma$. Isotropy of the unit image makes this character trivial on $R^\times$. Its angular and relative radial parameters are those of $\mathcal{L}$.

Lemma 3.4 (Local mark digits)**.** *For an exterior prime p put $Q = q_p$ and $\mu_p(v) = Q^{-1}$ for every $v \in R/p$. Then*

$$\sum_{v \bmod p} \mu_p(v) \psi_p(tv/p) = \mathbf{1}_{p|t}.$$

The zero digit, after reflection, contributes $Q^{-1} W_p(a(\nu))$. For a unit t the nonzero digits contribute

$$\sum_{v \neq 0} \mu_p(v) W_p(a(p^{e-2}t)\, wn(v/p)) = \begin{cases} Q^{-1/2} \gamma_2(p) \chi_p(t)^2, & e = 0, \\ 0, & e \neq 0. \end{cases} \tag{3.18}$$

For comparison, the reflected nonzero-digit part for an ordinary exponent $j \in \{0, \dots, 5\}$ is $s_{p,j,e} \chi_p(t)^{b_{j,e}}$, where

$$\begin{aligned} &b_{j,0} \equiv 8 - j \pmod 6, && s_{p,j,0} = \tau_{p,j}^{-1} \sum_{x \neq 0} \chi_p(x)^{j-4} \psi_p(x/p) && (j \neq 0), \\ &b_{0,0} \equiv 2 \pmod 6, && s_{p,0,0} = -Q^{-1/2} \gamma_2(p), \\ &b_{j,e} \equiv 4 + 2e \pmod 6, && s_{p,j,e} = \mathbf{1}_{j=4} (Q-1) \tau_{p,4}^{-1} W_p(a(p^e)) && (e > 0). \end{aligned} \tag{3.19}$$

All coefficients with $e<0$ vanish. For $e>0$ they also vanish unless $e\equiv 0,1 \pmod 3$. At a row prime with $j=1$ only $e=0$ occurs, and $s_{p,1,0}=\omega_p$ has modulus one. In the $j=0$ ordinary branch the zero digit contributes $(1-Q^{-1})W_p(a(\nu))$.

Proof. The first identity is additive orthogonality. The zero-digit assertions follow from compact invariance of the spherical vector. The local Weyl identity of [17, Lemma 6.2], before choosing digit weights, is

$$\sum_{v\neq 0}\beta(v)W_p(a(p^{e-2}t)wn(v/p))=W_p(a(p^e))\chi_p(t)^{4+2e}\sum_{v\neq 0}\beta(v)\chi_p(v)^4\psi_p(p^{e-1}t/v)$$

for $e\geq 0$, with zero left-hand side for $e<0$. This identity is linear in the weights β. For $\beta=\mu_p$ and $e>0$, χ_p^4 is nonprincipal and its sum is zero. At $e=0$ substitute $x=t/v$. The Gauss sum of $\chi_p^{-4}=\chi_p^2$ contributes $Q^{1/2}\gamma_2(p)$, and the remaining unit power is $\chi_p(t)^8$. This proves (3.18). The same substitution with $\beta=\alpha_{p,j}$ gives (3.19). In particular, for $j=4$ its positive-valuation sum is $(Q-1)/\tau_{p,4}$; for all other j it vanishes. These are raw local factors in the convention of P. Their globally paired cubic exponent is -2, as computed below. □

Fix a factorization $(m)=hd$, where d is squarefree, $(h,d)=1$, and $v_p(m)=1$ for $p\mid d$. The primes of h are frozen, and may also have valuation one. Let r be a squarefree puncture with $(r,d)=1$, put $\mathcal{P}_0=\operatorname{supp}(h)\cup\operatorname{supp}(r)$, and take a finite set of mark primes $\mathcal{Q}$ disjoint from $\operatorname{supp}(d)\cup\mathcal{P}_0$. At $p\in\mathcal{P}_0$ put $j_p=v_p(m)\bmod 6$; a new puncture has $j_p=0$. For a test V define

$$T_{\mathcal{Q}}(X;m,r,V)=\sum_{\substack{c\ \mathrm{sf}\\ n}}\frac{\gamma_2(c)}{q_c^{1/2}q_n}\vartheta(cn^3)\chi_{cn^3}(m)\mathbf{1}_{(cn^3,r)=1}\prod_{q\in\mathcal{Q}}\mathbf{1}_{q\mid cn^3}\,V(q_cq_n^3/X). \tag{3.20}$$

Here c ranges over squarefree exterior ideals and n over all exterior ideals, allowing common factors. If a mark meets m or r, the primal expression is identically zero; such a tuple is removed before making the expansion.

An active branch is a partition $\mathcal{Q}=\mathcal{A}\sqcup\mathcal{I}$; write $P=\prod_{q\in\mathcal{A}}q$. Let f run over squarefree products of primes of $\mathcal{P}_0$ containing every prime with $j_p\neq 0$, and put $I_f=\prod_{p\in\mathcal{P}_0,\,p\nmid f}(1-q_p^{-1})$. Let a' be supported on f, with every exponent 0 or 1 modulo 3. We retain all actual zero coefficients in this convenient larger support. For $p\mid f$ abbreviate $e_p=v_p(a')$, $s_p=s_{p,j_p,e_p}$ and $b_p=b_{j_p,e_p}$. Choose a frequency-unit representative $\epsilon\in U_6$. The variables c',n' satisfy c' squarefree, $(c',f)=1$ and $(n',fP)=1$. Use allowed generators for all displayed ideal products and put

$$\nu_0=\frac{\epsilon a'c'(n')^3}{(dfP)^2},\qquad \kappa=[\epsilon][a'][c'][n']^3[d]^{-2}[f]^{-2}[P]^{-2}\in V. \tag{3.21}$$

The symbol zeros encode the remaining restrictions at d and P.

For $L=\mathcal{L}_v$ and $t=(f,a',n',\epsilon)$ define

$$\Omega_t=I_fL(a')^{-1}L(n')^{-3}L(f)^2\prod_{p\mid f}s_p\chi_p\big(\epsilon(a'/p^{e_p})(n')^3(f/p)^{-2}\big)^{b_p}, \tag{3.22}$$

$$\rho_t(d)=L(d)^2\prod_{p\mid d}\omega_p\chi_p(\epsilon a'(d/p)^{-2}f^{-2})\prod_{p\mid f}\chi_p(d)^{-2b_p}, \tag{3.23}$$

$$\pi_t(P)=L(P)^2\prod_{q\mid P}\gamma_2(q)\chi_q\big(\epsilon a'(P/q)^{-2}f^{-2}\big)^2\prod_{p\mid f}\chi_p(P)^{-2b_p}, \tag{3.24}$$

$$\lambda_t(c')=L(c')^{-1}\prod_{p\mid f}\chi_p(c')^{b_p}\prod_{l\mid c'}\gamma_2(l)\chi_l\big(\epsilon a'(c'/l)f^{-2}\big)^2. \tag{3.25}$$

Extend ρ by zero off squarefree ideals prime to $\mathcal{P}_0$, π by zero off squarefree ideals prime to $\mathcal{P}_0$, and λ by zero off squarefree ideals prime to f. In particular, ρ is independent of P,c',n', π is

independent of d, c', n', and λ is independent of d, P, n'. Their absolute values are at most one. The only dependence of Ω on moving dual variables is allowed inside the dual coefficient.

For absolute majorants set

$$\beta_p(e) = \begin{cases} 1, & e = 0, \\ q_p^{-k}, & e = 1 + 3k, k \geq 0, \\ q_p^{1/2-k}, & e = 3k, k \geq 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \beta_f(a') = \prod_{p|f} \beta_p(e_p). \tag{3.26}$$

Then $|\Omega_t| \leq \beta_f(a')$ by the local table.

Theorem 3.5 (Separated marked expansion)**.** *With the preceding data, fix $\upsilon = \epsilon_m \Gamma$. The exact reflection of (3.20) is*

$$T_{\mathcal{Q}}(X; m, r, V) = \sum_{\mathcal{Q} = \mathcal{A} \sqcup \mathcal{I}} \left(\prod_{q \in \mathcal{I}} q_q^{-1} \right) q_P^{-1/2} \sum_{f, a', n', \epsilon} \sum_{s_d, s_c \in B} \frac{\Omega_t}{q_{n'}} \pi_t(P) \rho_t(d) \mathbf{1}_{[d]=s_d} \mathbf{1}_{(P,n')=1} \mathbf{1}_{(P,d)=1}$$
$$\cdot \sum_{c' \text{ sf}} \frac{\lambda_t(c') \mathbf{1}_{[c']=s_c}}{q_{c'}^{1/2}} \chi_d(c'n')^3 \chi_P(c')^{-2} \Phi_{\upsilon,\kappa,V} \left(\frac{X q_{a'} q_{c'} q_{n'}^3}{(q_d q_f q_P)^2} \right). \tag{3.27}$$

Here the f, a', n', ϵ, c' ranges are those above, and κ is determined by (3.21). The mask $(P, d) = 1$ is redundant under the stated disjointness assumption, but is displayed for use in row norms. The functions $\Phi_{\upsilon,\kappa,V}$ belong to the fixed finite family of (3.16) and satisfy (3.8). The expansion converges absolutely. Besides the common kernel, the only factors coupling different moving kinds of variables are

$$\chi_d(c'n')^3 \chi_P(c')^{-2} \mathbf{1}_{(P,n')=1} \mathbf{1}_{(P,d)=1}. \tag{3.28}$$

The same assertion holds for a test whose Mellin transform is entire and rapidly decreasing on every closed vertical strip, with the required weighted Euler seminorms finite at both ends; this includes the fixed Gaussian used in the physical construction.

Proof. The automorphic identity. In the digit average defining $D_m^{\pm}$, use $\alpha_{p,1}$ at d, α_{p,j_p} at $\mathcal{P}_0$, and μ_q at the marks. The digit weights are invariant under multiplication by Γ. Consequently the same quotient descent as in Proposition 3.2 applies. Both constant terms are radial at every complex place. Integration against L^{-1} kills them by rotation at a place with nonzero angular weight; the relative radial parameters do not affect this cancellation. Unfolding the unreflected side and using Lemma 3.4 gives

$$F(l) = \sum_A a^+(A) V_0(q_A l),$$

where $a^+(A)$ is precisely the arithmetic coefficient of (3.20). The finite expansion acts on the digit functions.

Frequencies and phases. At a row prime the local valuation is -2. At an active ordinary prime it is $e_p - 2$; at an active mark it is -2. Elsewhere the spherical support consists of valuations $3k$ and $3k + 1$. These conditions give exactly the frequencies $\gamma \nu_0$, $\gamma \in \Gamma$, with the indices in (3.21). Their parametrization is unique: reduce valuations modulo 3 outside f, and use allowed-generator uniqueness for the remaining unit. Frequencies with a locally zero s_p still carry coefficient zero.

For $p \mid d$, $p \mid f$, and $q \mid P$, respectively, the unit parts are

$$\epsilon a' c' (n')^3 (d/p)^{-2} f^{-2} P^{-2}, \quad \epsilon (a'/p^{e_p}) c' (n')^3 d^{-2} (f/p)^{-2} P^{-2}, \quad \epsilon a' c' (n')^3 d^{-2} f^{-2} (P/q)^{-2}.$$

At $l \mid c'$ the spherical unit factor is $\chi_l(\epsilon a'(c'/l)(dfP)^{-2})^2$; its cube contribution has exponent divisible by six. The S-character separates as

$$L(\nu_0)^{-1} = L(a')^{-1}L(c')^{-1}L(n')^{-3}L(d)^2L(f)^2L(P)^2,$$

since $L(\epsilon) = 1$. Using reciprocity, the mixed factors are exactly

$$\begin{aligned} d, c' &: \chi_d(c')\chi_{c'}(d)^{-4} = \chi_d(c')^3, \\ d, P &: \chi_d(P)^{-2}\chi_P(d)^{-4} = \mathbf{1}_{(d,P)=1}, \\ P, c' &: \chi_P(c')^2\chi_{c'}(P)^{-4} = \chi_P(c')^{-2}, \\ d, n' &: \chi_d(n')^3, \\ P, n' &: \chi_P(n')^6 = \mathbf{1}_{(P,n')=1}. \end{aligned}$$

These identities retain their zero extensions. The self-interactions and interactions with frozen primes are exactly (3.22)–(3.25). This proves coefficient separation, including its asserted independence.

The S-unfolding gives the sheet function

$$H_{\upsilon,\kappa}(x) = \int_{X_x} \mathbf{1}_B(\kappa[v]^{-1})H_S(\kappa,[v])^2L(v)f_{w,\upsilon}(v)\, d_x^\times v. \tag{3.29}$$

All S-valuations have been integrated here. The exterior coefficient is the product in (3.27) with Φ replaced by $H(|\nu_0|_S/l)$. This gives an exact fixed-scale reflected expansion of $F(l)$. The class multiplier is trivial at complex places, so its Mellin multiplier is precisely (3.15).

Convergence and arbitrary tests. For $\sigma > 1/2$, the absolute reflected Dirichlet majorant, apart from the finitely many row and mark factors, is at most

$$\zeta_K^S(\sigma+1/2)\zeta_K^S(3\sigma+1)\prod_{p\mid f}\left(1+\frac{q_p^{-\sigma}+q_p^{-1/2-3\sigma}}{1-q_p^{-1-3\sigma}}\right). \tag{3.30}$$

This follows by summing (3.26), and by positive extension of the squarefree and cube ranges. It proves absolute convergence of the orbit sums. For each fixed primal row and mark tuple, the direct and reflected expansions make $F(l)$ rapidly decreasing at infinity and at zero, respectively. Thus $\widetilde{F}(s) = \int_0^\infty F(l)l^s\, dl/l$ is entire and bounded on fixed strips. On a right half-plane it is $\widehat{V}_0(s)\mathcal{D}_+(s)$. On a left half-plane it is the sum of $a^-(j)|\nu_0(j)|_S^s\widehat{H}_{\upsilon,\kappa(j)}(-s)$. Since $1/\widehat{V}_0$ is entire, this continues $\mathcal{D}_+$. Its preliminary strip growth is at most $e^{C|\Im s|}$ times a polynomial. On the two boundary lines the preceding majorant and (3.15) give polynomial growth. The strip Phragmén–Lindelöf argument therefore gives polynomial growth throughout the intervening strip.

Mellin inversion of $V(q_A/X)$ may now be shifted from a line $\Re s > 1/2$ to a line $\Re s < -1/2$, without residues. Inserting the reflected series, interchanging absolutely, and putting $w = -s$ gives

$$\frac{1}{2\pi i}\int_{(\sigma)} \widehat{V}(-w)\frac{\widehat{H}_{\upsilon,\kappa}(w)}{\widehat{V}_0(-w)}(X|\nu_0|_S)^{-w}\, dw, \qquad \sigma > -1/3.$$

This is $\Phi_{\upsilon,\kappa,V}(X|\nu_0|_S)$, and proves (3.27). The shifts to $-1/4$ and A and Lemma 3.3 give the claimed seminorm bounds. The argument applies directly to the indicated two-ended test class; for the Gaussian it agrees with the smoothing already constructed. □

3.4 Weighted orbital Poisson summation

For the rest of the section Γ may be replaced by one fixed finite-index subgroup of $(R^\times)^6$ to accommodate fixed sectors. It remains fixed throughout all moving moduli and rows. The following formula keeps finite arithmetic coefficients and fixed unitary backgrounds in their respective variables.

Lemma 3.6 (Weighted masked orbital Poisson)**.** *Let $\chi : K_S^\times \to \mathbb{C}^\times$ be fixed, smooth and unitary, with $\chi(\Gamma) = 1$. For $\phi \in \mathcal{S}(K_S)$ set*

$$W_{\phi,\chi}(x) = \int_{X_x} \chi(y)^{-1}\phi(y)\, d^\times y, \qquad \widetilde{W}_{\phi,\chi}(x) = \int_{X_x} \chi(y)\widehat{\phi}(y)\, d^\times y.$$

Let $f : R/cR \to \mathbb{C}$ be periodic and Γ-invariant. With continuous Fourier sign $e(-xy)$, define $\widehat{f}_d(h) = \sum_{a \bmod c} f(da)e(ha/c)$. For every exterior radical mask $\mathfrak{R}$ and $L > 0$,

$$\begin{aligned}&\sum_{[k]\neq 0} \chi(k)f(k)\mathbf{1}_{(k,\mathfrak{R})=1}W_{\phi,\chi}(q_k/L)\\ &\quad = \frac{L\chi(c)}{q_c}\sum_{d\mid \operatorname{rad}\mathfrak{R}} \frac{\mu(d)\chi(d)}{q_d}\sum_{[h]\neq 0}\bar{\chi}(h)\widehat{f}_d(h)\widetilde{W}_{\phi,\chi}(Lq_h/(q_dq_c)) + Z_{\chi,f}, \qquad (3.31)\end{aligned}$$

$$Z_{\chi,f} = \chi(\ell_L)\mathbf{1}_{\chi|_{T_0}=1}\operatorname{vol}(T_0/\Gamma)\left\{\frac{L\widehat{\phi}(0)}{q_c}\sum_d \frac{\mu(d)}{q_d}\widehat{f}_d(0) - \phi(0)f(0)\sum_d \mu(d)\right\}.$$

Here brackets mean Γ-orbits, $q_k = |k|_S$, and c, d are exterior generators. All zero extensions are retained in f and the mask.

Proof. Integrate the full lattice sum, including its actual zero, over T_0/Γ with weight $\chi(\ell_L)\chi(u)^{-1}$. Expand the mask before the finite transform. Additive Poisson on the self-dual lattice gives exactly

$$\sum_{n\in R} f(dn)\phi(dnu/\ell_L) = \frac{L}{q_dq_c}\sum_{h\in R}\widehat{f}_d(h)\widehat{\phi}(\ell_L h/(dcu)).$$

The finite transform is Γ-invariant by the substitution $a \mapsto \gamma a$. Unfolding the nonzero original orbits gives the left side of (3.31). On the dual side, inversion and Haar translation give the factor $\chi(dc)\bar{\chi}(h)$ and the sheet of norm $Lq_h/(q_dq_c)$. The dual kernel is thus common to all modulus shapes. The integral of $\chi(u)^{-1}$ over the compact quotient is its volume if $\chi|_{T_0} = 1$ and is zero otherwise. Retaining the dual zero and subtracting the included primal zero proves the displayed $Z_{\chi,f}$.

□

Every smooth annular norm test has such a lift: multiply a compact shape function of integral one by the test and by $\chi(y)$. Thus $\chi = 1$ gives the additive interface, while fixed nontrivial χ gives the character-equivariant interface for canonical descendants. The factor $\chi(dc)$ has modulus one; the finite Fourier table remains the table of f. If $f = \xi$ is primitive, its specialization is $\widehat{f}_d(h) = \xi(d)\sqrt{q_c}\gamma(\xi)\bar{\xi}(h)$. This includes the primitive principal case with its actual zero term.

We record two quantitative details. First, a fixed compact unit partition represents orbit sums by lattice sums with compact shape support and finite multiplicity. Second, for a fixed finite local type, the sheet integrals of a Schwartz function and their Euler derivatives have arbitrary power decay at infinity and at most $(1 + |\log x|)^b$ growth at zero, for a fixed $b = b(K, S)$. This follows by bounding by product Schwartz functions: the logarithmic complex norms and the finite valuation shells form a simplex of logarithmic volume near zero. Euler derivatives do not increase its dimension. On polynomial scale ranges these logarithms consume any prescribed positive loss, with only the threshold depending on the fixed data. These conclusions hold for the weighted profiles by unitarity and the fixed infinity parameters.

Lemma 3.7 (Balanced lattice decay)**.** *Let C be a fixed compact subset of K_S^1, and let Φ range over a bounded Schwartz family on K_S^n with fixed finite local type. For $a \in C$, $\rho \geq 1$ and $A > 0$,*

$$\sum_{\mathbf{h}\in R^n\setminus\{0\}} |\widehat{\Phi}(\ell_\rho a\mathbf{h})| \ll_A \rho^{-A}. \qquad (3.32)$$

The bound uses finitely many Schwartz seminorms; their orders may depend on $A[K:\mathbb{Q}]$.

Proof. The finite support and the compact range of a_f put all contributing $\mathbf{h}$ in a fixed fractional $\mathcal{O}_K$-lattice. At infinity ℓ_ρ dilates by $\rho^{1/[K:\mathbb{Q}]}$, and a ranges over a compact family of invertible diagonal maps. The lattice has a fixed positive minimum distance. A Schwartz order greater than $\max\{A[K:\mathbb{Q}], n[K:\mathbb{Q}]\}$ makes its nonzero sum $O(\rho^{-A})$. □

3.5 The full additive Gram estimate

We prove the Gram bound by two successive correlation calculations. For an allowed generator s put

$$g_{\mathfrak{s}}(k) = \sum_{a \bmod s} \chi_{\mathfrak{s}}(a) e(ka/s).$$

Let $c(\mathfrak{s})$ be a fixed finite linear combination of the finite ray coefficients of the additive probe. In particular it may contain a fixed class indicator, a fixed-numerator residue character and the inverse of a fixed character on its unit support. Extend it by zero before evaluating any such inverse. The fixed set S includes all finite coefficient conductors and fixed numerator primes; thus the remaining finite ray restrictions occur only in the fixed local profiles at S. For $W \in C_c^\infty(0,\infty)$, define

$$A_m(Y;\nu) = Y^{-3/2} \sum_{\mathfrak{s}} c(\mathfrak{s}) W(q_{\mathfrak{s}}/Y)(q_{\mathfrak{s}}/Y)^{-1+i\nu} g_{\mathfrak{s}}(-m). \tag{3.33}$$

This is the exact central normalization in [16, Section 6 and Proposition 15.2], now on exterior ideals. The coefficient c is independent of the row and of ν.

Proposition 3.8 (Additive Gram bound)**.** *Let $Q, Y \geq 1$ range in fixed polynomial ranges in Z, and suppose $P_a = Y^2/Q \geq 1$. For every fixed row-ball constant and every $\varepsilon > 0$,*

$$\sum_{\substack{[m]\neq 0 \\ q_m \ll Q}} |A_m(Y;\nu)|^2 \ll_{K,S,c,W,\varepsilon} (1+|\nu|)^J \frac{Q}{Y}\left(1 + P_a^{1/6} + \frac{P_a^2}{Y}\right) Z^\varepsilon. \tag{3.34}$$

The same bound holds for rows in a fixed compact additive shape. The seminorm order J is fixed independently of the moving scales and height. The arithmetic coefficient is fixed independently of the row.

Proof. Choose a nonnegative Schwartz/compact-open majorant whose orbital profile is at least one on the row ball. This is possible by the balanced representatives of Proposition 3.1. The full lattice average includes a nonnegative term at $m = 0$, which may be retained for an upper bound. Expand the square in (3.33), keeping its one joint annular profile, and write $s_i = Cn_i$ with $(n_1, n_2) = 1$.

The complete finite correlation, with every zero extension retained, is

$$F(u,v;j) = \sum_{\substack{x \bmod u,\ y \bmod v \\ vx - uy \equiv j \ (uv)}} \chi_u(x)\overline{\chi_v(y)}. \tag{3.35}$$

Expanding the two Gauss sums and using additive orthogonality shows that their complete row transform is $q_u q_v F(u,v;j)$. The Poisson factor is $Q/(q_u q_v)$, so the exact common prefactor after the central normalization is Q/Y^3. The correlation is zero unless $C \mid j$. For $j = Ck$, reciprocity on the allowed generators gives

$$F(Cn_1, Cn_2; Ck) = \chi_{n_1}(k)\overline{\chi_{n_2}(-k)} L_C(n_1, n_2; k), \tag{3.36}$$

where L_C is the corresponding congruence sum modulo C. At $\mathfrak{p}^a \| C$, when both residual columns are units, its local value is

$$q_{\mathfrak{p}}^{a-1}\chi_{\mathfrak{p}}(n_1/n_2)^a \begin{cases} q_{\mathfrak{p}}-1, & \mathfrak{p} \mid k, \\ -1, & \mathfrak{p} \nmid k,\ 6 \nmid a, \\ q_{\mathfrak{p}}-2, & \mathfrak{p} \nmid k,\ 6 \mid a. \end{cases}$$

If exactly one column is divisible by $\mathfrak{p}$, the value is $q_{\mathfrak{p}}^{a-1}(q_{\mathfrak{p}}-1)\mathbf{1}_{6|a}\mathbf{1}_{\mathfrak{p}\nmid k}$. These formulas follow by reduction to the residue field: in the nonzero-unit case substitute the ratio $(1+t)/t$ and sum the character over the units except 1. Define the local extension to be zero if both residual columns meet $\mathfrak{p}$. The resulting extension satisfies $|L_C| \le q_C$ and is periodic modulo $\operatorname{rad} C$ in each column. On the coprime residual locus it equals the genuine correlation. This proves over every residue field the complete correlation identities used in [16, Lemma 13.3].

Insert the full identity $\mathbf{1}_{(n_1,n_2)=1} = \sum_{d|n_1,\ d|n_2} \mu(d)$ before using this extension, and write $n_i = dm_i$. The residual variables m_1, m_2 may share factors. Nonzero terms have $(d, Ck) = 1$, by the extension and the original character zeros. The fixed sectors are included in fixed local profiles. By (3.4), the remaining exterior coefficient is periodic in the two columns modulo

$$\mathfrak{r} = \operatorname{lcm}(\operatorname{rad} C, \operatorname{rad}(kR)), \qquad q_{\mathfrak{r}} \ll_{\text{fixed}} q_C q_k.$$

Its normalized two-variable Fourier coefficients have magnitude at most $O(q_C)$. Call k nonexceptional when an exterior prime outside C has valuation in kR not divisible by 6. At such a prime one column contains a nonprincipal zero-extended residue character; all other local factors and fixed phases are independent of that residue. Its complete mean is zero. Thus the joint Fourier coefficient at $(0,0)$ vanishes for every nonexceptional k.

The frequency $k = 0$ uses the actual compact-quotient scalar $\operatorname{vol}(T_0/\Gamma)\widehat{\phi}(0)$ from Lemma 3.6. It is exactly diagonal: nonzero residues in (3.35) force $u = v$, and then $F(u,u;0) = \varphi(u)$. There are $O(Y)$ possible C with $q_C \asymp Y$, each contributing at most $O(Y)$. Including Q/Y^3 gives $O(Q/Y)$.

For nonzero k, the first orbital Fourier kernel has argument $Qq_Cq_k/(q_{s_1}q_{s_2}) \asymp q_Cq_k/P_a$. Use dyadic shells $q_Cq_k/P_a \asymp H$, $H = 1, 2, 4, \ldots$, with the first shell containing arguments at most two. Any small-end logarithms of the orbital kernel cost a preliminary Z^{ε} loss. Put $q_C \asymp C_0$, $q_d \asymp D_0$ and $N = Y/(C_0D_0)$. For fixed C, d, k, keep a single joint two-column profile W_H. Its required homogeneous seminorms are $O((1+|\nu|)^J H^{-B})$: derivatives of the first Fourier kernel are absorbed by its rapid large-end decay, and norm-power derivatives cost only fixed powers of $1+|\nu|$. The factor H^{-B} occurs once in the linear two-column Fourier estimate.

Use a compact unit partition in both columns and balance an allowed generator of $\mathfrak{r}$. For a nonexceptional k, two-column Poisson and Lemma 3.7 give

$$\sum_{\mathbf{m}\in R^2} a_{C,d,k}(\mathbf{m}) W_H(\mathbf{m}/\ell_N) \ll (1+|\nu|)^J H^{-B} q_C N^2 \min\{1, (q_{\mathfrak{r}}/N)^A\}. \tag{3.37}$$

Here $A > 2$ is arbitrary. For $N \ge q_{\mathfrak{r}}$ the zero frequency is absent and the lemma gives the displayed decay; for $N < q_{\mathfrak{r}}$ use the compact-shape point count $O(N^2)$. A nonempty subunit annulus has N bounded below by a fixed positive constant. The field degree enters through the fixed derivative order.

Set $K_H = HP_a/C_0$. The ideal count, including the fixed unit sectors, gives $O(K_H)$ nonzero frequency orbits in this shell; the range is empty below a fixed positive lower bound for K_H. There are $O(C_0D_0K_H)$ choices of C, d, k. Multiplying (3.37) by this count and by Q/Y^3 gives

$$\ll (1+|\nu|)^J H^{-B} \frac{Q}{Y} \frac{HP_a}{C_0D_0} \min\{1, (HP_aC_0D_0/Y)^A\}.$$

For $\Lambda_0 = Y/(HP_a)$, the positive dyadic sum is

$$\sum_{i,j\ge 0} 2^{-i-j}\min\{1,(2^{i+j}/\Lambda_0)^A\} \ll_A \frac{1+\log(2+\Lambda_0)}{\Lambda_0}.$$

Split at $2^{i+j} \asymp \Lambda_0$ when $\Lambda_0 \ge 1$; when it is smaller than one the assertion follows from the convergent geometric sum. Summing H with $B > 2$ gives the term $(Q/Y)(P_a^2/Y)Z^\varepsilon$.

An exceptional nonzero frequency has exterior ideal $kR = \mathfrak{a}^6\mathfrak{e}$, where $\mathfrak{e}$ is sixth-power-free and supported on C. There are $6^{\omega(C)} \ll_\varepsilon q_C^\varepsilon$ possibilities for $\mathfrak{e}$ and $O(K_H^{1/6})$ for $\mathfrak{a}$ on a nonempty shell. The fixed unit sectors cost a constant. Use $|L_C| \le q_C$ and the trivial two-column count. Before multiplication by Q/Y^3, the contribution is

$$\ll (1+|\nu|)^J H^{-B} Z^\varepsilon Y^2 (HP_a)^{1/6} C_0^{-1/6} D_0^{-1}.$$

The C_0, D_0 sums converge after taking a smaller preliminary loss, and the H sum converges for the same fixed $B > 2$. This gives $(Q/Y)P_a^{1/6}Z^\varepsilon$. Adding the three frequency types proves (3.34). A fixed compact additive row shape is covered by the same positive majorant. □

In the physical probe the fixed target acts on the completed theta index $\mathfrak{c}\mathfrak{n}^3$, while (3.33) is independent of that target. Its continuous phase is therefore handled by Proposition 3.2; the additive Gram calculation uses $\chi = 1$. The weighted Poisson lemma handles the fixed live backgrounds in Section 4. All complete masks, collision zeros and common joint profiles have been retained. For every prescribed real loss the norm powers in these lemmas are unchanged by K, S or the fixed infinity type. The constants, finite derivative orders and eventual thresholds may depend on those fixed data; increasing the derivative orders supplies the higher-dimensional Poisson decay, and fixed unit logarithms consume only the prescribed loss.

4 Canonical moments

The two estimates needed for row counting are stated first in Theorems 4.1 and 4.2. Their proofs share the coefficient identities, the marked theta terminal, and the profile-propagation lemma. The inverse proof is a two-transform induction on row width (Section 4.5); the ordinary fourth moment adds a same-band comparison and an exceptional-row cancellation (Section 4.8). This order allows the moment statements to be read independently of the recursive bookkeeping.

We use the arithmetic normalization of Proposition 3.1. Thus $K \supset \mu_{12}$ is fixed, $R = \mathcal{O}_{K,S}$ is principal, and an exterior ideal is an integral ideal supported outside S. Its norm is denoted by $q_{\mathfrak{a}}$. Ideal variables in character symbols are represented by allowed generators. Element rows are taken modulo a fixed finite-index subgroup of $(R^\times)^6$. There are only a fixed number of such rows above each exterior ideal. A further finite unit-sector subdivision is allowed in a positive moment estimate, and is recombined afterwards.

For an exterior ideal $\mathfrak{n}$, write $\chi_{\mathfrak{n}}(u)$ for the sextic presentation, and put $\chi_u(\mathfrak{n}) = \chi_{\mathfrak{n}}(u)$. Every character power is extended by zero at a nonunit, including a power divisible by six. The modulus-one character is identically one. All additional coprimality conditions below are imposed before an inverse character is evaluated. An inducing character means the primitive Hecke character, used together with all displayed zero factors.

Fix a finite group Θ of finite-order Hecke characters, with conductors supported in S, containing the finite Gauss and reciprocity characters and the finite characters used in the local sector tables. Membership in Θ refers to primitive inducing characters. A finite-order background is absorbed into Θ. Every global norm twist is put in the smooth tests. The remaining fixed background ω_0 is either 1 or a genuine character modulo global norm twists. Its restriction to a compact local unit group has finite conductor; its possibly infinite-order values on uniformizers and at infinity remain part of the fixed character.

All tests have support in prescribed compact subintervals of $(0,\infty)$. We use Euler seminorms

$$p_J(W)=\max_{0\le j\le J}\sup_{x>0}|(x\partial_x)^jW(x)|.$$

For a tuple of tests we use $1+\sum p_J(W)$. Pure norm powers are normalized as $(q_{\mathfrak{a}}/X)^{it}$ at the stated scale X. Bounds below have the form of a fixed power of a finite test seminorm times a fixed polynomial in the separated heights. The permitted rowwise smooth choices are scale and parameter suprema obtained by logarithmic Fourier inversion and Sobolev's inequality on a fixed finite-dimensional parameter set. Arithmetic column coefficients remain fixed independently of the row.

4.1 Moment statements

Let ψ be a fixed representative oriented as in Proposition 3.2, and put

$$\mathcal{X}_u=\{\theta\chi_u,\theta\bar\chi_u:\theta\in\Theta\}\ \cup\ \{\psi\theta\bar\chi_u,\bar\psi\theta\chi_u:\theta\in\Theta\}. \tag{4.1}$$

For $\lambda_u=\nu\chi_u^{\epsilon_\chi}$ in this family, define

$$M_u(Z^r;W)=Z^{-r/2}\sum_{\mathfrak{n}}\mu_K(\mathfrak{n})\lambda_u(\mathfrak{n})W(q_{\mathfrak{n}}/Z^r).$$

Let I be a fixed finite set of slots. Their underlying prime lists $\mathcal{P}_i$ are disjoint, lie outside S, and satisfy $q_{\mathfrak{p}}\asymp Z^{z_i}$, where $z_i\ge 0$. For coefficients $|a_i(\mathfrak{p})|\le 1$, independent of u, set

$$Q_{u,i}=Z^{-z_i/2}\sum_{\mathfrak{p}\in\mathcal{P}_i}a_i(\mathfrak{p})\lambda_u(\mathfrak{p})W_i(q_{\mathfrak{p}}/Z^{z_i}),\qquad z=\sum_{i\in I}z_i.$$

Empty products equal one. The lists may contain all good primes in the annuli.

Theorem 4.1 (Marked inverse moment)**.** *Fix bounded nonnegative ranges for m,r,z_i, a bound for $|I|$, and $c_1,c_2>0$. If*

$$r+2z\le m-c_1,\qquad 2r+8z\le 3m-c_2,$$

then, for every $\varepsilon>0$,

$$\sum_{0<q_u\ll Z^m}\left|M_u(Z^r;W)\prod_{i\in I}Q_{u,i}\right|^2\ll Z^{m+\varepsilon}. \tag{4.2}$$

The bound is uniform in the prime lists and their bounded coefficients, and holds for every subcollection of the slots and either permitted orientation. No slot mesh is required. The initial masks consist precisely of the fixed exclusions and displayed row-character zeros. The implied constant has the finite-seminorm and polynomial-height dependence specified above.

For the ordinary moment a larger internal class is needed. Write

$$\xi_u(\mathfrak{n})=\vartheta(\mathfrak{n})\prod_j\chi_{\mathfrak{n}}(b_j)^{e_j}\chi_u(\mathfrak{n}),\qquad \lambda_u=\omega_0\xi_u,\qquad M=m+q,$$

where $\vartheta\in\Theta$ and the b_j,e_j are fixed within the current row sum. The union of the supports of their displayed moving factors, before cancellations and reduction modulo six, has norm at most Z^q. Each such prime is a common zero of every column. A redundant factor may be moved to a common radical mask only through an exact factorization preserving its remaining phase. An extra mask $\mathbf{1}_{(\mathfrak{n},\mathfrak{R})=1}$ of polynomial norm is allowed if it is fixed within the row sum and is used in both plain columns and every slot. Cancellations with the varying u remain part of the natural row zeros.

For $X = Z^n$ let

$$S_{\lambda_u,\mathfrak{R}}(n;W) = X^{-1/2}\sum_{\mathfrak{a}} \lambda_u(\mathfrak{a})\mathbf{1}_{(\mathfrak{a},\mathfrak{R})=1}W(q_{\mathfrak{a}}/X).$$

The paired slot coefficient is

$$a_i(\mathfrak{p}) = \omega_0(\mathfrak{p})^{-1}\sum_j c_{ij}\vartheta_{ij}(\mathfrak{p}), \qquad \vartheta_{ij} \in \Theta. \tag{4.3}$$

The finite lists and coefficients are fixed independently of Z, u. Thus the actual character in each prime sum is finite order: $\lambda_u a_i = \xi_u \sum_j c_{ij}\vartheta_{ij}$. In this ordinary moment the slots are the complete smooth sums

$$Q_{u,i,\mathfrak{R}} = Z^{-z_i/2}\sum_{\mathfrak{p}\notin S} a_i(\mathfrak{p})\lambda_u(\mathfrak{p})\mathbf{1}_{(\mathfrak{p},\mathfrak{R})=1}W_i(q_{\mathfrak{p}}/Z^{z_i}),$$

with disjoint underlying supports before the masks. Fixed finite-ray restrictions are encoded in (4.3). Let $\mathcal{R}_0$ consist of the rows for which λ_u^* is nonprincipal; for $z > 0$ let $\mathcal{R}_z$ consist of those for which $\xi_u^* \notin \Theta$. We always also restrict $0 < q_u \ll Z^m$.

Theorem 4.2 (Paired fourth moment)**.** *Let $3/4 \le \kappa \le 1$. If $z > 0$ and $\kappa < 1$, suppose that every finite-order Hecke L-function over K has no zero with real part greater than $(1+\kappa)/2$. No zero-free premise is imposed when $\kappa = 1$. For every $\varepsilon > 0$ and prescribed bounded ranges of all real lengths, there is $\eta > 0$, uniform in $\kappa \in [3/4, 1]$, such that a fixed slot system with $z_i \le \eta$ satisfies*

$$\sum_{u\in\mathcal{R}_z}\left|S_{\lambda_u,\mathfrak{R}}(n_1;W_1)S_{\lambda_u,\mathfrak{R}}(n_2;W_2)\prod_i Q_{u,i,\mathfrak{R}}\right|^2 \ll Z^{M+\varepsilon} \tag{4.4}$$

in the following ranges:

$$\begin{cases} n_1, n_2 \ge 0 \text{ arbitrary in the prescribed bounded ranges}, & z = 0, \\ n_1 + n_2 + 6\kappa z \le M, & z > 0. \end{cases} \tag{4.5}$$

The mesh precedes the fixed number of slots. Constants, finite seminorm orders, and polynomial height degrees may depend on that number and on the fixed arithmetic data, but are uniform in the moving support, common masks, rows, and frozen outer labels.

Theorems 4.1 and 4.2 are the field-and-type versions of [16, Lemmas 17.1 and 18.1], respectively; the internal inverse induction is [16, Lemma 17.2]. Their source statements concern the original finite-character setting. Below we retain those canonical reductions and give the coefficient identities, field-dependent estimates, and termination and loss calculations needed to transfer them.

4.2 Finite coefficients and backgrounds

At an exterior prime of norm P, define the normalized complete sum $\mathcal{G}(\mathfrak{p}^a, h)$ using the original zero-extended character of exponent a and the primitive additive character. Separating the higher residue digits gives, for $a \ge 1$,

$$\begin{aligned} |\mathcal{G}(\mathfrak{p}^a, h)| &= P^{(a-1)/2}\mathbf{1}_{v_{\mathfrak{p}}(h)=a-1}, & 6 \nmid a, \\ \mathcal{G}(\mathfrak{p}^a, h) &= P^{a/2}\mathbf{1}_{\mathfrak{p}^a|h} - P^{a/2-1}\mathbf{1}_{\mathfrak{p}^{a-1}|h}, & 6 \mid a. \end{aligned} \tag{4.6}$$

Indeed the sum over the P^{a-1} lifts vanishes unless $\mathfrak{p}^{a-1} \mid h$. The remaining field sum is a nonprincipal Gauss sum, of absolute value $\sqrt{P}$, or the Ramanujan sum $P\mathbf{1}_{\mathfrak{p}|h/\mathfrak{p}^{a-1}} - 1$. This proves the formulas over every residue field, with all zeros intact.

The Chinese remainder theorem gives the complete two-column correlation of [16, Lemmas 13.3–13.4] with $P = q_{\mathfrak{p}}$. Its diagonal is zero unless the two full moduli agree, and then equals their Euler function. At a common prime of multiplicities $a > b$ the correlation vanishes unless $6 \mid b$ and the divided frequency is a unit; its remaining size is at most P^b. At equal multiplicities a its size is at most P^a, with the stronger P^{a-1} when $6 \nmid a$ and the divided frequency is a unit. These are finite-field identities: solving the two-column congruence leaves either a complete nonprincipal sum or a sum over the units with one value omitted. The finite factors at S are kept in fixed sector tables.

To use the complete-support correlation, first write the full products as Da, Eb, where $(a, b) = 1$ and $(ab, DE) = 1$; D, E contain every common prime with its full multiplicity. After the genuine identity is established, extend its right-hand side to individually allowed residual pairs and insert

$$\mathbf{1}_{(a,b)=1} = \sum_{t \mid (a,b)} \mu_K(t). \tag{4.7}$$

At every stage an extracted prime retains its zero on all affected residual columns.

The calibrated squarefree identities from Proposition 3.1 are

$$\gamma_2(\mathfrak{n})^3 = \mu_K(\mathfrak{n})\Lambda(\mathfrak{n}), \qquad \bar{\Lambda}(\mathfrak{n})\gamma_2(\mathfrak{n})\gamma_1(\mathfrak{n}) = \mu_K(\mathfrak{n})G(\mathfrak{n}), \tag{4.8}$$

where Λ is unitary and G is finite order. The canonical squarefree coefficient is therefore $a_0(\mathfrak{n}) = \bar{\Lambda}(\mathfrak{n})\gamma_2(\mathfrak{n})$. The first primitive transform changes $a_0\nu$ into $\mu_K\nu\vartheta$, and the second changes it back into $a_0\nu\vartheta\vartheta'$, with $\vartheta, \vartheta' \in \Theta$. For example, $\mu_K\bar{\gamma}_1 = a_0\bar{G}$; the remaining factors are the fixed finite-sector coefficients just described.

The identification with the actual theta coefficient, including the cube-completion masks, is made explicit in Lemma 4.3 below.

In the first transform write $n_i = A_i C u_i$. The two column backgrounds satisfy

$$\nu(n_1)\overline{\nu(n_2)} = \nu(A_1)\overline{\nu(A_2)}\,\nu(u_1)\overline{\nu(u_2)},$$

because $|\nu(C)| = 1$. The finite table is expanded while $\nu(u_i)$ remains on its column. The common factor in the second transform cancels in the same way. Cube completion contributes $\nu(b)^3$ on the cube variable; it is a bounded coefficient on that variable, while the live residual character remains ν. At initialization a presentation $\nu\chi_u^{\epsilon_\chi}$ is conjugated in its entirety when $\epsilon_\chi = 1$. Its canonical background is $\nu^{-\epsilon_\chi}$. Thus (4.1) gives exactly the closed class

$$\nu_{\mathrm{can}} \in \Theta \cup \psi\Theta. \tag{4.9}$$

Finite twists preserve the chosen archimedean type and the nonzero angular condition of Proposition 3.2.

For the ordinary reduction each side keeps one background on its whole residual product. Complete extraction contributes only frozen unit scalars. Both finite transforms leave the original individual slot coefficients unchanged, so (4.3) continues to cancel the live background in every actual prime polynomial. A change of row orientation conjugates both plain columns and all slots, coefficients, and tests together. After reflecting one plain column, it alone may be conjugated with its test inside the absolute value of the product. This restores its original character without changing the squared norm.

Lemma 4.3 (The canonical coefficient and cube completion)**.** *On squarefree exterior ideals the coefficient $a_0 = \bar{\Lambda}\gamma_2$ has the exact identities (4.8). Let τ be a completely multiplicative unit-modulus character times any of the original completely multiplicative coprimality indicators. For a whole-index weight d put*

$$\mathscr{T}_{d,\tau}(X) = \sum_{\substack{c \ \mathrm{sf} \\ b}} \frac{\gamma_2(c)\tau(cb^3)}{q_c^{1/2} q_b} d(cb^3) V(q_c q_b^3/X), \qquad d^{[h]}(A) = d(h^3 A).$$

Then, for a compactly supported test,

$$\sum_{c\ \mathrm{sf}} \frac{\gamma_2(c)\tau(c)}{q_c^{1/2}} d(c) V(q_c/X) = \sum_h \frac{\mu_K(h)\tau(h)^3}{q_h} \mathscr{T}_{d^{[h]},\tau}(X/q_h^3). \tag{4.10}$$

Here and below, a puncture is the indicator $\rho(a) = \mathbf{1}_{(a,r_\rho)=1}$, fixed in the current row–label sum. For a fixed squarefree label f and $\tau(a) = \bar{\Lambda}(a)\nu(a)\rho(a)\chi_a(k)\chi_a(f)^4$, the completed sum for one mark tuple is exactly (3.20), with period $\vartheta = \bar{\Lambda}\nu$, row $m = kf^4$, and puncture $\mathrm{rad}(r_\rho)$. Product-form sums of tuples follow by linearity. Fixed finite factors can be absorbed into ν.

Proof. In the cube of the CRT law (3.5) with $j = 2$, both cross exponents are divisible by six. The cross exponents in the product of its $j = 1$ and $j = 2$ laws are also divisible by six. The two squarefree expressions are therefore multiplicative on coprime ideals. Their values at primes, including the minus signs, are (3.3); this proves (4.8). The finite order of G follows from the Weil-index calculation on B.

Expand the right-hand side of (4.10) and put $t = hb$. Every term has common factor $\gamma_2(c)\tau(ct^3)d(ct^3)/(q_c^{1/2}q_t)$ and common test $V(q_c q_t^3/X)$. Its remaining coefficient is $\sum_{h|t} \mu_K(h)$, which is zero unless $t = 1$. All sums are finite for compactly supported V. For the bounded product-form mark weights used here, the Gaussian case also follows by absolute convergence with the weighted divisor majorant. Complete multiplicativity preserves the puncture indicators.

After cube extraction a product-form mark has argument h^3A, with the assignment rule

$$\mathbf{1}_{p|h^3A} = \mathbf{1}_{p|h} + \mathbf{1}_{p\nmid h}\mathbf{1}_{p|A}. \tag{4.11}$$

Expanding this identity for the individual slots leaves either a mark already assigned to h, or one live whole-index mark with its original coefficient and its original exclusion at h. Finally $\tau(cb^3) = \bar{\Lambda}(c)\nu(c)\rho(c)\chi_c(k)\chi_c(f)^4$ times the cube of the same multiplicative factors at b. This is the coefficient in (4.18). To match a centrally normalized squarefree polynomial, take $V_*(y) = y^{1/2}W(y)$: then $q_c^{-1/2}V_*(q_c/X) = X^{-1/2}W(q_c/X)$. This identifies the test and the normalization as well as the arithmetic coefficient in the canonical terminal.

After h is frozen, its extracted phase factors as a row factor $\chi_h(k)^3$, a label factor $\chi_h(f)^{12}$, and a scalar independent of both. Each has absolute value at most one; $\chi_h(f)^{12}$ is still its coprimality indicator. Multiplication by their product is a contraction in the joint row–label Hilbert space. Hence the source's Minkowski step may freeze h before applying the completed estimate. The factor $\nu(h)^3$ in the inversion is on this extracted cube variable; the live theta period remains $\bar{\Lambda}\nu$. □

Lemma 4.4 (Preservation of paired whole-product phases)**.** *In the ordinary canonical reduction, suppose the live plain character is $\omega_0\xi$, with ξ finite, and each live slot has relative coefficient $\omega_0^{-1}\vartheta_i$, $\vartheta_i \in \Theta$. Retain the source's whole-product allocation and common masks. The two finite transforms, extraction, and allowed conjugations preserve this property, up to replacing ξ and ϑ_i by finite twists. Thus only a finite-order character is evaluated in the prime cancellation premise.*

Proof. Write a separated whole index as $A = DA'$. The genuine factor is exactly $\omega_0(D)\omega_0(A')$. Complete extraction freezes the first factor; a common factor in a coefficient and its conjugate cancels by $|\omega_0(C)|^2 = 1$. The complete-support finite Fourier tables of [16, Lemma 18.2] involve only ξ and the residue symbols. They therefore multiply the surviving $\omega_0(A')$ by a finite character.

On a live slot the two genuine factors are $\omega_0(p)$ from the whole product and $\omega_0(p)^{-1}$ from its specified coefficient. They cancel before any prime sum is estimated. If that slot is assigned to an extracted factor, both factors belong to the same frozen allocation; the remaining slots

retain their individual coefficients. This remains true when an index occurs in both rectangles of a centered expression: the allocation is made on the common whole product, before their smooth profiles are separated.

Conjugating a row orientation conjugates both plain factors, every slot, and all of their coefficients and tests. It replaces the pair ω_0, ω_0^{-1} by $\bar{\omega}_0, \bar{\omega}_0^{-1}$. Ordinary reflection changes one plain character to its conjugate; conjugating that one reflected sum and its test inside the product absolute square restores the original orientation. Use this operation for an uncentered product or a separately estimated comparison product. A centered difference retains its common profiles, whole-product character, and residue cancellation until the two rectangles are split. The conductor and common-mask identities are still applied to the original presentation before primitive reduction. These verifications prove the invariant used in the source's canonical reductions. □

4.3 The two sieve inputs

Heath-Brown's quadratic and cubic power-residue large sieves [8, 9] are the classical predecessors of the two bounds used here. The number-field inputs are the quadratic-family theorem of Goldmakher–Louvel [7, Theorem 1.1] and the n-th-order family theorem of Blomer–Goldmakher–Louvel [1, Theorem 1.3], specialized to $n = 3$. The quadratic bound for exterior allowed symbols is the identification in [17, Lemma 6.4]; we check its family hypotheses over K below. In the present notation it says

$$\sum_{\substack{k \text{ sf} \\ q_k \le K_0}} \left| \sum_{\substack{j \text{ sf} \\ q_j \le N_0}} z_j \chi_k(j)^3 \right|^2 \ll_{K,S,\varepsilon} (K_0 N_0)^{\varepsilon} (K_0 + N_0) \sum_j |z_j|^2. \tag{4.12}$$

Here the family hypotheses can be checked directly. Let η_j be the primitive quadratic character of $K(\sqrt{j})/K$, with j represented by an allowed generator. Reciprocity identifies its exterior values with $\chi_\bullet(j)^3$. For squarefree j, every prime dividing j is tamely ramified and every other exterior prime is unramified, so its conductor is $j\mathfrak{s}_j$ with $\mathfrak{s}_j$ supported on S. The finitely many local square classes at S give a fixed common conductor bound there. The class map from exterior ideals to the square-class image of B is a homomorphism, and the reciprocity defect on allowed generators is one. If coprime squarefree j_1, j_2 have the same class, their quotient is a square at S. Thus $\eta_{j_1}\bar{\eta}_{j_2}$ is unramified at S and has exact conductor $j_1 j_2$ outside S. These are the quadratic-family hypotheses of [7, Definition 1 and Theorem 1.1], and prove (4.12). This is also the verification in [17, Lemma 6.4]. All characters here have trivial infinity type. Fixed row restrictions and unimodular row or column factors are harmless. Moving pair-dependent restrictions are treated by their exact Möbius identities below.

Lemma 4.5 (Identification of the cubic sieve family)**.** *For coefficients c_P on squarefree exterior ideals $q_P \le V_0$,*

$$\sum_{\substack{j \text{ sf} \\ q_j \le U_0}} \left| \sum_{P \text{ sf}} c_P \chi_P(j)^{\pm 2} \right|^2 \ll_{K,S,\varepsilon} (U_0 V_0)^{\varepsilon} \{U_0 + V_0 + (U_0 V_0)^{2/3}\} \sum_P |c_P|^2. \tag{4.13}$$

Both variables are prime to S, and the symbols retain their zeros at a common prime.

Proof. We apply [1, Theorem 1.3] with $n = 3$ to the Fisher–Friedberg family described in [1, Section 2, Lemma 2.1 and Appendix II]. Write χ_j^{BGL} for its primitive character. For a fixed modulus $\mathfrak{c}$ supported on S, its reciprocity defect depends only on the two classes in $H_{\mathfrak{c}} \otimes \mathbb{Z}/3$. It has $\chi_{(b)}^{\mathrm{BGL}} = (b/\cdot)_3$ when $b \equiv 1 \pmod{\mathfrak{c}}$, and $\chi_{(1)}^{\mathrm{BGL}} = 1$. For squarefree j its conductor lies between j and $\mathfrak{c}j$.

Choose a modulus $\mathfrak{m}$ supported on S, divisible by $\mathfrak{c}$, so deep that $1+\mathfrak{m}\mathcal{O}_v \subset K_v^{\times 6}$ at every finite $v \in S$. Let $H_{\mathfrak{m}}$ be its ray class group. For a fixed squarefree j compare, on ideals b prime to $\mathfrak{m}j$,

$$\Xi_j(b) = \chi_b(j)^2, \qquad \Xi_j^{\mathrm{BGL}}(b) = \chi_j^{\mathrm{BGL}}(b).$$

Both functions are multiplicative. If (b) is ray-principal modulo $\mathfrak{m}$, then $[b] = 1$ in $K_S^{\times}/K_S^{\times 6}$, so b itself is an allowed generator. Reciprocity gives $\Xi_j((b)) = \chi_j(b)^2 = (b/j)_3$. For the BGL family the reciprocity defect with a trivial ray class is one: use its dependence on classes and the normalization at (1). Hence

$$\Xi_j^{\mathrm{BGL}}((b)) = \chi_{(b)}^{\mathrm{BGL}}(j) = (b/j)_3.$$

It follows that $\xi_j = \Xi_j/\Xi_j^{\mathrm{BGL}}$ descends to a character of $H_{\mathfrak{m}}$. More explicitly, every ray class has a representative prime to j by approximation; the kernel of this restricted ideal-to-class map consists exactly of its ray-principal ideals.

Split the columns P into their finitely many classes $h \in H_{\mathfrak{m}}$. On one such class,

$$\chi_P(j)^2 = \xi_j(h)\chi_j^{\mathrm{BGL}}(P).$$

The factor $\xi_j(h)$ is a unimodular row factor and disappears inside the absolute square. Cauchy over h, followed by the BGL theorem with coefficients $c_P\mathbf{1}_{[P]=h}$, proves the plus sign in (4.13). At a common prime both sides are zero, because the BGL conductor contains squarefree j. Complex conjugation proves the minus sign. All losses from the class splitting are constants for the fixed K, S. □

Lemma 4.6 (Completed quadratic reduction)**.** *Let k, n be squarefree exterior ideals with $q_k \ll K_0$, $q_n \asymp N_0$, and let $q_b \asymp B_0$, with b unrestricted. For $\beta(n,b)$ independent of k, the positive reduction in [16, Lemma 5.5] holds over K with the original zeros. In particular, if $|\beta| \le 1$, then*

$$\sum_k \left| \sum_{n,b} \beta(n,b)\chi_k(nb)^3 \right|^2 \ll_\varepsilon (K_0N_0B_0)^\varepsilon (K_0 + N_0B_0)N_0B_0. \tag{4.14}$$

Proof. Here is the positive form that will also be used with an arbitrary coefficient. Write uniquely $b = gt^2$, with g squarefree, and put $c = (n,g)$, $n = cm$, $g = ch$. Then c, m, h are pairwise coprime and squarefree, while g, t may share factors. Exactly,

$$\chi_k(nb)^3 = \chi_k(mh)^3\mathbf{1}_{(k,ct)=1}.$$

Keep the k–t restriction and expand the full k–c mask by $\sum_{r|(k,c)} \mu_K(r)$. Rowwise divisor Cauchy costs a small power. Put $k = rk'$, $c = rc'$, and separate dyads $q_c \asymp C$, $q_g \asymp G$, $q_t \asymp T$, so $GT^2 \asymp B_0$. Cauchy over t costs $O(T)$, and Cauchy over c' costs $O(C/q_r)$. Group the squarefree product mh, whose norm is $O(N_0G/C^2)$; its factorizations have divisor-bounded multiplicity. Applying (4.12) gives, up to a small power,

$$T \sum_{q_t \asymp T} \sum_{\substack{r \text{ sf} \\ q_r \ll \min(K_0, C)}} \left(\frac{K_0}{q_r} + \frac{N_0G}{C^2} \right) \frac{C}{q_r} \sum_{c',m,h} |\beta(rc'm, rc'ht^2)|^2. \tag{4.15}$$

The last sum keeps its inherited support. The unit factor at r is bounded after taking the square; the row–c' mask has been resolved by the divisor sum. This proves the positive reduction for arbitrary β. Counting its remaining ideal ranges when $|\beta| \le 1$ and summing dyads gives (4.14). All factorizations take place among exterior ideals. □

4.4 The marked quadratic–cubic terminal

Lemma 4.7 (Quadratic–cubic norm)**.** *Let $K_0, N_0, B_0, L_0 \geq 1$. Let k, n, P be squarefree exterior ideals, $q_k \ll K_0$, $q_n \asymp N_0$, $q_P \asymp L_0$, and let b be an exterior ideal with $q_b \asymp B_0$. If $|a(P)|, |\beta(n,b)| \leq 1$, with a independent of k, n, b and β independent of k, P, then*

$$\sum_k \left| \sum_P \frac{a(P)}{\sqrt{q_P}} \mathbf{1}_{(P,k)=1} \sum_{n,b} \beta(n,b) \chi_k(nb)^3 \chi_P(n)^{-2} \mathbf{1}_{(P,b)=1} \right|^2 \ll_\varepsilon (K_0 N_0 B_0 L_0)^\varepsilon (K_0 + N_0 B_0) B_0 \{N_0 + L_0 + (N_0 L_0)^{2/3}\}. \tag{4.16}$$

Fixed row restrictions and fixed coefficient-support restrictions are allowed if these independence conditions are preserved.

Proof. Use (4.12) and Lemma 4.5. The positive completed quadratic reduction for arbitrary coefficients is (4.15).

We spell out the two moving masks in applying these sieves. Expand the first by $\sum_{D|(P,k)} \mu_K(D)$, then use divisor Cauchy and put $P = DP'$, $k = Dk'$. The factors at D join β or the fixed P' coefficient, with the (P,k) mask resolved by the divisor sum. For fixed D apply the completed quadratic reduction, writing $b = gt^2$, $c = (n,g)$, $n = cm$, $g = ch$, and resolving the full row–c mask by its divisor r. Let G, C, T be the dyadic norm scales of g, c, t, respectively. Squarefreeness gives pairwise coprime c, m, h; the factors g, t may overlap. The remaining cubic column is $j = c'm = n/r$, and the exact identity

$$\chi_{P'}(rc'm)^{-2} \mathbf{1}_{(P', rc'ht^2)=1} = \chi_{P'}(r)^{-2} \chi_{P'}(c'm)^{-2} \mathbf{1}_{(P', ht)=1}$$

preserves all missing zeros. Its P' coefficient is independent of j and has squared norm $O_\varepsilon((K_0 N_0 B_0 L_0)^\varepsilon)$ after division by $q_{P'}$. On a block $q_D \asymp D_0$, $q_r \asymp r_0$, the cubic sieve therefore contributes

$$\frac{G}{C} \left\{ \frac{N_0}{r_0} + \frac{L_0}{D_0} + \left(\frac{N_0 L_0}{r_0 D_0} \right)^{2/3} \right\}$$

up to the chosen small power. Here G/C includes the h count. The remaining quadratic and t factors are $T^2 C / r_0$; their product with G/C is B_0/r_0, since $T^2 G = B_0$. There are $O(r_0)$ possible r and $\sum_{D \asymp D_0} q_D^{-1} \ll 1$ up to that power. The quadratic factor is bounded by $K_0 + N_0 B_0$. Summing the divisor and dyadic ranges proves (4.16). This is the factorization proof of [16, Lemma 14.2]; the argument just given verifies its two sieve inputs and its masks over K. □

For a product-form mark write

$$d_I(A) = \sum_{(\mathfrak{p}_i) \in \prod_{i \in I} \mathcal{P}_i} \prod_{i \in I} a_i(\mathfrak{p}_i) \mathbf{1}_{\mathfrak{p}_i | A}. \tag{4.17}$$

The whole-index marked expansion is Theorem 3.5. Its raw local table, mixed phases, and coefficient independence will all be retained in the following norm calculation.

Lemma 4.8 (Marked reflected energy)**.** *Put $\Psi_{k,f}(a) = \nu(a)\rho(a)\chi_a(k)\chi_a(f)^4$. The completed marked sum is*

$$\mathcal{T}_d(X; k, f) = \sum_{\substack{n \\ b}}^{\mathrm{sf}} \frac{a_0(n) \Psi_{k,f}(n)\, \bar{\Lambda}(b)^3 \Psi_{k,f}(b)^3}{q_n^{1/2} q_b} d(nb^3) V(q_n q_b^3 / X), \tag{4.18}$$

where both ideals avoid S but may share primes, V is a fixed annular profile, and d has the product form (4.17). Let the row ball have norm scale Z^M and let the squarefree label f have

scale Z^V. Write $Q = \log_Z q_{\mathrm{rad}(r_\rho)}$ for the fixed puncture length and Z^{N_*} for the completed scale. Freeze the powerful row part of length O, and retain the actual residual squarefree row annulus of length H. Let A_0 be the active nonresidual radical length; S_0 the length of its small Ramanujan and active zero-mask terms; and N_0, B_0 the lengths assigned, respectively, to the dual squarefree variable and to the cube variable alone. Let z_a be the active slot length. On a dual block of squarefree and cube lengths v, ℓ_b, put

$$T_d = 2H + 2A_0 + 2z_a - N_* - N_0 - 3B_0,$$
$$u_0 = \min\{v, z_a, (v + z_a)/3\}, \tag{4.19}$$
$$E_{\mathrm{ref}} = \frac{O}{2} + \max(H, v + \ell_b) - S_0 - B_0 + z_a - u_0 - \ell_b - \frac{1}{2}(T_d - v - 3\ell_b)_+. \tag{4.20}$$

After summing the frozen powerful parts, the squared row norm of this block is $\ll Z^{E_{\mathrm{ref}}+\varepsilon}$, with finite seminorm and fixed polynomial height factors. For every fixed $\tau_{\mathrm{ref}} > 0$, blocks with $v+3\ell_b > T_d+\tau_{\mathrm{ref}}$ have arbitrarily small total size after a fixed further kernel shift. The conductor comparisons on the actual annuli are

$$H \le M - O + O(1/\log Z), \qquad T_d - S_0 - B_0 \le 2M - O + Q + V - N_* + 2z_a + O(1/\log Z). \tag{4.21}$$

The constants in these O-terms depend only on the fixed support intervals and arithmetic data.

Proof. Lemma 4.3 identifies the coefficient, including its unrestricted cube and all of its masks, with Theorem 3.5. We give the resulting norm calculation, keeping track of its coefficient independence.

Rows, collisions, and inactive slots. Write the exterior ideal of a row as $k = k_{\mathrm{pow}} k_{\mathrm{sup}} d$. Here k_{pow} contains the full prime powers of exponent at least two, k_{sup} contains the valuation-one primes supported on $f\,\mathrm{rad}(r_\rho)$, and d contains the remaining valuation-one primes. Fix the first two parts and the unit class of the row, and keep the actual annulus $q_d \asymp Z^H$. To avoid conflict with the canonical label f, denote the active ordinary-prime product in Theorem 3.5 by f_0 in this proof.

A tuple meeting the row, label, or puncture is zero on the primal side. Remove it before reflecting. After fixing all inactive primes, their exclusions from d are a fixed row restriction, and Minkowski in the row Hilbert space removes their sums at cost $\prod_i \sum_{p\in\mathcal{P}_i} q_p^{-1} \ll Z^{\varepsilon_1}$. An inactive prime contributes its scalar weight and remains allowed in the dual index. Active collisions are the explicit mask $\mathbf{1}_{(P,d)=1}$ in (3.27).

Frozen valuations and the full cube. For $p \mid f_0$ write uniquely

$$e_p = e_p^{\min} + 3x_p, \qquad e_p^{\min} = \begin{cases} 0, & e_p = 0, \\ 1, & e_p \equiv 1 \pmod 3, \\ 3, & e_p \ge 3, e_p \equiv 0 \pmod 3. \end{cases}$$

Thus $a' = a_{\min} x^3$ with x supported on f_0. The local table gives

$$|s_{p,4,0}| = q_p^{-1/2}, \quad |s_{p,4,1}| \le 1, \quad |s_{p,4,3}| \le q_p^{-1/2}, \quad |s_{p,0,0}| = q_p^{-1/2}.$$

These are precisely the factors counted by S_0, N_0, B_0. For the remaining valuations the coefficient is bounded by the corresponding minimum-valuation coefficient times q_x^{-1}; the exponents $b_{j,e}$ are unchanged on the allowed positive progressions modulo three. Keep the displayed powers before using any positive majorant.

Put $a_{\min}$ outside the norm by Minkowski, but keep x inside, with the dual cube variable

$$b = xn', \qquad (n', f_0) = 1. \tag{4.22}$$

This decomposition is unique. In the row factor $\chi_d(a') = \chi_d(a_{\min})\chi_d(x)^3$, giving the joint symbol $\chi_d(c'b)^3$. In the mark and squarefree column factors the extra exponent is six, so their x-dependence is one, because P, c' are prime to f_0. The remaining $L(x)^{-3}$ and local coefficients are functions of the dual variables alone. The kernel argument is $Xq_{a_{\min}}q_{c'}q_b^3/(q_dq_{f_0}q_P)^2$. The normalized product of the finitely many local choices has mass at most $C^{\omega(k_{\mathrm{pow}}k_{\mathrm{sup}}fr_\rho)}$ after the assigned powers are extracted; a subsidiary divisor bound absorbs it into Z^{ε_1}. Equivalently one may sum the frozen majorants using (3.30). The per-tuple coefficient exponent on a dual block is consequently

$$-v/2 - \ell_b - (S_0 + B_0 + z_a)/2. \tag{4.23}$$

A common profile on the actual annuli. Fix the finitely many classes of d, c', P, b and the frequency unit. Then the selected κ in (3.21) is fixed. Writing each varying norm as its block scale times a normalized coordinate, the argument of the kernel is

$$Y y_{c'} y_b^3 y_d^{-2} \prod_{i \text{ active}} y_i^{-2}, \qquad Y \asymp Z^{v+3\ell_b - T_d}.$$

All coordinates lie in fixed compact intervals. Apply the common profile separation of [16, Lemma 5.3], using (3.8). Every weighted logarithmic Fourier norm is bounded by finitely many test seminorms times

$$m_{\mathrm{ker}} = Z^{-(T_d - v - 3\ell_b)_+/4} \tag{4.24}$$

(up to fixed support factors). Extract this scalar from the whole linear row vector before the norm inequality or any row extension. The Fourier measure is common to d, P, c', b.

The hybrid norm. At a separated Fourier mode, after the frozen coefficients are extracted, the remaining row vector is a unimodular row factor times

$$\sum_P \frac{a(P)}{q_P^{1/2}} \mathbf{1}_{(P,d)=1} \sum_{c',b} \beta(c',b)\chi_d(c'b)^3 \chi_P(c')^{-2} \mathbf{1}_{(P,b)=1}.$$

The factor $a(P)$ consists of (3.24), the original individual slot coefficients, and slot norm phases. The factor $\beta(c', b)$ consists of (3.25), the remaining cube coefficients, dual norm phases, and the class slices. After (4.22), these have exactly the two independences in Lemma 4.7: a is independent of d, c', b and β is independent of d, P. The mask at P, b equals that at P, n', since P is prime to f_0. Tuple aggregation costs at most a divisor factor.

Apply that lemma with $(K_0, N_0, B_0, L_0) = (Z^H, Z^v, Z^{\ell_b}, Z^{z_a})$. The factor $q_P^{-1/2}$ is already inside the hybrid lemma. The squared coefficient outside its norm is therefore $Z^{-v-2\ell_b-S_0-B_0}$. The hybrid estimate contributes exponent

$$\max(H, v + \ell_b) + \ell_b + \max\{v, z_a, 2(v + z_a)/3\}.$$

Since the last maximum is $v + z_a - u_0$, the total, before the kernel, is $\max(H, v + \ell_b) - S_0 - B_0 + z_a - u_0 - \ell_b$. Squaring (4.24) gives the last term of (4.20).

Tails. For every fixed $\sigma > 1/2$, use (3.30) on the genuine reflected series before Fourier absolutization. Outside the retained range the kernel argument exceeds $Z^{\tau_{\mathrm{ref}}/2}$ once fixed support constants have been absorbed in the threshold. At large-argument order A its total absolute contribution is at most

$$C_A Z^{B_{\mathrm{tail}}} Z^{-(A-\sigma)\tau_{\mathrm{ref}}/2}.$$

The exponent B_{tail} accounts for the bounded row, tuple, and local length ranges and is independent of A; the constant and seminorm order may depend on A. Choosing A large proves the arbitrary power tail assertion. A norm twist changes only a fixed polynomial height factor in these finitely many seminorms.

Frozen row count and conductor comparison. A powerful exterior ideal is uniquely $x_0^2 y_0^3$, with y_0 squarefree. Ideal counting and convergence of $\sum q_{y_0}^{-3/2}$ give $O(Z^{O/2})$ choices in its O-dyad.

The supported part divides $\mathrm{rad}(fr_\rho)$ and has Z^{ε_1} choices. The unit classes are finite. Their positive sum adds $O/2+\varepsilon_1$ once and proves the stated row energy.

For completeness, the contribution to $2A_0-N_0-S_0-4B_0$ is 2 for ordinary digits $j=1,2,3,5$; 1 for active $j=0$, for the $e=0$ part of $j=4$, or for its $e=1$ part; -2 for its $e=3$ part; and 0 for an inactive $j=0$. A residual row prime contributes 2 through $2H$. Prime by prime these contributions are bounded by the valuations of $(k^2/k_{\mathrm{pow}})f\,\mathrm{rad}(r_\rho)$. Taking norms proves (4.21), since $q_k \ll Z^M$ and $q_{k_{\mathrm{pow}}} \asymp Z^O$. The S-variables are contained in the sheet integral (3.29). □

Corollary 4.9 (Unmarked completed moment)**.** *For an empty slot collection and fixed powerful-part dyad O,*

$$\sum_{0<q_k\ll Z^M}{}' |\mathcal{T}_1(Z^{N_*};k,f)|^2 \ll Z^{O/2+\max\{M-O,\,2M-O+Q+V-N_*\}+\varepsilon}. \tag{4.25}$$

The prime restricts the powerful part to that dyad. With slots present one retains z_a, u_0 in (4.20) instead.

Proof. Here $z_a=u_0=0$. On a retained block, $v+3\ell_b \le T_d+\tau_{\mathrm{ref}}$, so directly

$$E_{\mathrm{ref}} \le O/2+\max\{H, T_d-S_0-B_0\}+\tau_{\mathrm{ref}}.$$

Indeed the H branch only subtracts nonnegative terms, and the $v+\ell_b$ branch is at most $T_d-S_0-B_0+\tau_{\mathrm{ref}}$. Insert (4.21) and choose τ_{ref} and the subsidiary dyadic losses within ε. Sum the logarithmically many retained dyads; the tails were proved above. This gives (4.25). □

4.5 Canonical inverse induction

The inverse moment is proved through a slightly larger second-moment family. Besides the row k, it keeps a squarefree label f and a puncture r_ρ. This is the form that is recovered after two Poisson transforms: the first produces inverse coefficients and the second returns the theta-calibrated coefficient a_0. The quantity $F=N+V$ is the combined column-and-label length. The two margins below control, respectively, the puncture budget and the marked reflected terminal.

Proposition 4.10 (Canonical energy)**.** *Fix bounded nonnegative ranges for M,N,V,z_0 and a positive margin c_*. Let f run over squarefree exterior ideals of norm $\asymp Z^V$, with a nonnegative weight $w(f)\le C\tau_K(f)^C$ depending only on f. Let ν belong to (4.9); let $\rho(n)=\mathbf{1}_{(n,r_\rho)=1}$ be fixed within the row and label sum; and let d have the product form (4.17), with total length at most z_0 and coefficients independent of both rows and labels. There is no additional residual coefficient on n. Put $F=N+V$. If*

$$F-M-Q-z_0\ge c_*, \qquad 4F-3M-6z_0\ge c_*, \qquad Q=\log_Z q_{\mathrm{rad}(r_\rho)}, \tag{4.26}$$

then

$$\begin{aligned}\mathcal{E} &= Z^{-V}\sum_f w(f)\sum_{0<q_k\ll Z^M}\left|Z^{-N/2}\sum_{n\ \mathrm{sf}} a_0(n)\nu(n)\rho(n)\chi_n(k)\chi_n(f)^4 d(n)W(q_n/Z^N)\right|^2\\ &\ll Z^{F+\varepsilon}.\end{aligned} \tag{4.27}$$

The statement is simultaneous for all subcollections of the original slots with the same cap z_0, and is uniform in the frozen outer ideals within the stated ranges.

Proof. We induct on the logarithmic row length M. A node consists of the energy (4.27), its lengths (M,N,V), the fixed puncture, and the surviving subset of the original marks. Its

inductive data are the two margins in (4.26); its coefficient class is (4.9). The two-transform reduction of [16, Section 17.5] has the following form over K:

$$\mathcal{E} \xrightarrow{\text{first Poisson}} \text{inverse-column norms} \xrightarrow{\text{second Poisson}} \mathcal{E}_{\text{child}} \quad (M' < M).$$

The identities below specify the coefficients, masks, and norm counts at each arrow. Choose a fixed step $0 < d_0 \le c_*/200$.

Terminal blocks. Cube completion expresses a short block through the marked reflected sum at scale $N_* = N - 3b_h$, where the extracted cube variable has length $b_h < d_0 + O(1/\log Z)$. If $V < d_0$, Lemma 4.8 and (4.26) give the three possible terminal exponents

$$F - c_{\text{node}}, \qquad F - 2c_{\text{node}} + 5d_0, \qquad F - M/4 - 3c_{\text{node}}/4 + 5d_0/2, \tag{4.28}$$

apart from prescribed localization and small-power losses. They correspond to the row branch, the column branch $u_0 = z_a$, and the other column branch. For example the last uses $u_0 \ge v/2$, the retained dual inequality, and the second invariant. All are strictly smaller than F when $c_{\text{node}} \ge c_*/2$. The label sum has weighted mass $\ll Z^{V+\varepsilon}$ and cancels the outside Z^{-V} exactly once.

First transform. For the remaining blocks, reopen the cube completion if necessary. With cube length ℓ and squarefree column length $r = N - 3\ell$, one has $V + \ell \ge d_0$. Define

$$\mathcal{T}_{k,f} = \sum_{b \asymp Z^{\ell}} \beta(b,f)\chi_b(k)^3 \sum_{n \text{ sf}} a_0(n)\nu(n)\rho(n)\chi_n(k)\chi_n(f)^4 d(nb^3) W(q_n/Z^r).$$

The block is

$$Z^{-r-2\ell-V} \sum_f w(f) \sum_k \Phi(q_k/Z^M)|\mathcal{T}_{k,f}|^2,$$

where β is divisor-bounded and independent of k, n. The original cube masks, including $(b, f) = 1$, remain in it. The cube and squarefree column may share primes.

Apply Lemma 3.6 in the row after expanding the square. Write $n_i = A_i C u_i$, with $A_i = (n_i, \operatorname{rad}(b_1 b_2))$ and C the remaining common factor. On the genuine coprime residual locus the primitive row conductor is $u_1 u_2 R_1$, and the remaining zero mask is $C \operatorname{rad}(b_1 b_2)/R_1$. The local digit t_p defining R_1 is

$$t_p = \mathbf{1}_{p|A_1} - \mathbf{1}_{p|A_2} + 3v_p(b_1 b_2) \pmod 6.$$

The primitive Poisson factor is the row scale divided by the mask divisor and the square root of this conductor norm. Formula (4.8) gives the inverse-type residual coefficient $\mu_K \nu \vartheta$. The cube table is particularly useful: write $b_1 b_2 = q_0^2 J_2^2 s$, with s, J_2 coprime and squarefree, and put $J = sJ_2$. The exact leftover column factor is

$$\chi_n(J)^4 \mathbf{1}_{(n, \operatorname{rad} q_0)=1}. \tag{4.29}$$

Thus q_0 is frozen as a moving puncture and J remains an averaged squarefree label.

Mark allocation and the common kernel. For every extracted list $D_1, \dots, D_h$, use the exact priority identity

$$\mathbf{1}_{p|D_1 \cdots D_h n} = \sum_{j=1}^{h} \mathbf{1}_{p|D_j} \prod_{a<j} \mathbf{1}_{p \nmid D_a} + \mathbf{1}_{p|n} \prod_{a \le h} \mathbf{1}_{p \nmid D_a}. \tag{4.30}$$

This holds with shared primes. Assigned marks give divisor-bounded outer coefficients; surviving marks keep their original product coefficients. The two copies of a slot in a square have independent prime choices.

Truncate only a sector-fixed outer frequency ball on the genuine coprime expression. Its complement is contained in a large-argument kernel tail for every supported column pair. Inside

the ball retain the full smooth kernel, then apply (4.7). Logarithmic Fourier inversion uses one joint profile in the bare frequency and all normalized extracted and column norms. Weighted Cauchy on the whole outer sum now gives two positive inverse-column norms. For the first transform's new row $y = hf^2E$, the old-label fibre $f^2E \mid y$ has divisor-bounded mass, so the old f, h disappear at this positive step.

Second transform and the child class. For one such positive norm, write $x_i = g'z_i$ in its expanded square. The second primitive conductor is z_1z_2 and its mask is g'. Apply the same orbital Poisson formula, with divisor $d_2 \mid g'$, and retain the analogous outer ball before the full Möbius extension. Formula (4.8) returns the coefficient $a_0\nu\vartheta\vartheta'$. The new row and squarefree label are, in the notation of [16, (17.42)–(17.55)],

$$k_{\mathrm{new}} = d_k d_2 k'', \qquad f_{\mathrm{new}} = CJd_2v', \qquad \rho_{\mathrm{new}}(n) = \rho(n)\mathbf{1}_{(n,\mathrm{rad}(q_0t'g'/d_2))=1}.$$

The zeros retained in the signed outer scalar force f_{new} to be squarefree. Each child test, background, puncture, and surviving mark is independent of its new row and label. Only after the second weighted Cauchy inequality are these outer zeros bounded by one and the positive row and label ranges enlarged.

At this point the child has all fields of the parent state: a squarefree averaged label, an independent fixed-type background, a common puncture, and marks with their original individual coefficients. The old-label fibres are counted by divisors of the new ideals; fixing a unit sector leaves only a finite multiplicity. This identifies the recursive input before any numerical estimate is applied.

Lengths and normalization. Here are the identities controlling the induction. Use the idealized logarithmic centers of [16, (17.45)–(17.48)], write j for the length of J, and put $D_c = \ell + A_i + t + g - \theta + V$. Then

$$\begin{aligned} F_c &= F - D_c - 2\ell + j, & M_c &= M - 2D_c - 2\ell + j, \\ F_c - M_c &= F - M + D_c, & 4F_c - 3M_c &= 4F - 3M + 2(A_i + t + g - \theta + V) + j. \end{aligned} \tag{4.31}$$

The structural source fibres imply $Q_{\mathrm{new}} \le Q + D_c$, $g \ge \theta$, and $j \le 2\ell$, up to their explicitly bounded annular errors. If each individual support error is at most η_1, a nonempty clipped child satisfies

$$\begin{aligned} F' - M' - Q_{\mathrm{new}} - z_0 &\ge c_{\mathrm{node}} - 5\eta_1, \\ 4F' - 3M' - 6z_0 &\ge c_{\mathrm{node}} - 7\eta_1, \\ M - M' \ge 2(V + \ell) - 8\eta_1, \qquad & F' \le F + 11\eta_1. \end{aligned} \tag{4.32}$$

These estimates use the source witnesses for the frozen triple $(q_0, t', g'/d_2)$.

The reconstruction at fixed triple, new row, and new label has weight at most $D(f_{\mathrm{new}})C(q_0, t', g'/d_2)$, with fixed divisor powers. The first factor belongs only to the child label measure. Normalize the positive measure on distinct frozen triples by its already extracted norm count. Its remaining mass is a prescribed small power, used once after the two Cauchy square roots. The exact prefactor identity is

$$\kappa_i + \lambda_c + C_c + 2F_c = F \qquad [16, (17.64)]. \tag{4.33}$$

Here the four terms are, respectively, the first prefactor, second prefactor, frozen-triple count, and restored child normalization with its asserted exponent. Hence a child loss $\varepsilon_{\mathrm{child}}$ gives parent loss at most $40\eta_1 + \tau_1 + \pi_1 + \varepsilon_{\mathrm{child}}$. The two primitive principal contributions have exponents at most M and $F + 26\eta_1 + \tau_1 + \pi_1$. These calculations are norm identities and the finite digit table; the field change introduces only the fixed sector multiplicity.

Termination and loss allocation. If $D > 2 + M_{max}/d_0$, choose $\eta_1 \le \min(d_0/16, c_*/(14D))$ and $D(40\eta_1+\tau_1+\pi_1) < \varepsilon$. Both margins remain at least $c_*/2$ through depth D, and every nonterminal

child decreases the row length by at least d_0. Every branch therefore reaches a terminal before depth D. The terminal estimates (4.28) and backward induction prove (4.27). The finite-seminorm assertion is proved in Proposition 4.14 below. □

Proof of Theorem 4.1. In a term of $M_u \prod_i Q_{u,i}$ let P be the squarefree product of its slot primes and put $j = (n, P)$, $n = jn_0$, $P = jP_0$. Then $c = n_0P_0$ is squarefree and

$$\mu_K(n_0) = \mu_K(c)\mu_K(P_0), \qquad \lambda_u(n)\lambda_u(P) = \lambda_u(j)^2\lambda_u(c).$$

All zeros are retained. If G is the sum of the assigned slot lengths, the residual cap is $z_0 = z - G$ and its column length is $D' = r+z-2G$. The factor Z^{-G} cancels the $O(Z^G)$ choices of the overlap tuple in the row-space triangle inequality. The scalar $\lambda_u(j)^2$ is a contraction, and the residual column has the fixed puncture $(c, j) = 1$. The signs $\mu_K(P_0)$ are individual prime coefficients. One common profile separates the identity $q_n/Z^r = (q_j/Z^G)(q_c/Z^{D'})/\prod_{i\in I_0}(q_{p_i}/Z^{z_i})$.

Orient the residual polynomial as in (4.9). One masked primitive Poisson transform, followed by the full mutual-gcd expansion, produces the canonical variables

$$k = d'h', \qquad f = d's, \qquad t = C/d', \qquad \rho_{t,j}(n) = \mathbf{1}_{(n,\mathrm{rad}(tj))=1}.$$

The common zero $(s, h') = 1$ is retained through Cauchy, making the new label squarefree. At fixed t, f, k the reconstruction has only divisor multiplicity, and its f-dependent part is the child label weight. Before enclosure losses the two canonical margins are

$$m - r - 2z + 2G, \qquad 3m - 2r - 8z + 10G + 2P_1, \qquad P_1 = \log_Z q_{C/d'} \geq 0. \tag{4.34}$$

Their perturbations are at most $9\eta_1 + \tau_1$ and $22\eta_1 + 3\tau_1$, respectively; see [16, (17.85)]. Choose them within fixed fractions of c_1, c_2 and apply Proposition 4.10. The normalization identity $\kappa_c^{\mathrm{init}} + P_1 + 2F_c^{\mathrm{init}} = m$ then gives $Z^{m+11\eta_1+\pi_1+\varepsilon_{\mathrm{can}}}$. Principal terms have the direct exponent m and the genuine frequency tails have arbitrary saving by a fixed kernel order. Taking the three losses to sum to less than ε proves (4.2), including its arbitrary bounded slot coefficients and the original masks. □

4.6 Ordinary reflection and prime factors

We first remove a fixed extra mask by an exact divisor identity. If $T_u(X)$ denotes the natural plain sum, then

$$S_{u,\mathfrak{R}}(X) = \sum_{d\mid \mathrm{rad}\,\mathfrak{R}} \mu_K(d)\lambda_u(d)q_d^{-1/2}T_u(X/q_d). \tag{4.35}$$

For a prime slot subtract its terms at primes dividing $\mathfrak{R}$. Apply these identities simultaneously to both plains and every slot. Their coefficient masses are bounded by products of $(1-P^{-1/2})^{-1}$ over a polynomial-size radical, hence by Z^{ε} with any prescribed loss. Every resulting length decreases; a nonempty subunit annulus may be replaced by scale one with a bounded dilation. If all slots disappear the unrestricted zero-slot assertion applies.

For a natural presentation let C_u be its primitive conductor scale, including the fixed discriminant factor, and $R_{0,u}$ its redundant natural radical. Tameness and the full declared support give

$$C_u q_{R_{0,u}} \ll Z^{m+q} = Z^M. \tag{4.36}$$

At each exterior prime the primitive conductor or the redundant radical uses that prime once. The components at S and the infinity type have fixed conductor factors. Apply the ordinary Hecke functional equation, and only then restore the deleted redundant factors geometrically. The dual scales have the form

$$Y = \frac{C_u q_{d_0}}{X q_{h_0}}, \qquad d_0 \mid R_{0,u}, \quad \mathrm{rad}(h_0) \mid R_{0,u},$$

with coefficient mass bounded by $\prod_{p|R_{0,u}}(1-P^{-1/2})^{-1}$. Lemma 3.3 gives a summable small-annulus decomposition and arbitrary decay at the large end. Thus, with any fixed length tolerance $\xi > 0$, every retained reflected scale has nonnegative declared length

$$n_{\mathrm{ref}} \le M - n + \xi. \tag{4.37}$$

If the right side is negative there is no retained nonempty annulus. Conjugating that reflected factor and its profile restores the original character inside the absolute product. This is the uncentered operation of Lemma 4.4. The coefficient mass and the permitted rowwise scale supremum cost only a prescribed small power and finite seminorm orders.

For $\kappa < 1$, the zero-free premise in Theorem 4.2 gives, a fixed distance $e > 0$ to the right of $(1+\kappa)/2$, the logarithmic-derivative bound used in [16, (18.11)]. The proof by the completed Hecke L-function and its zeros applies over K: the degree contributes only a fixed multiple of the height logarithm. Mellin inversion of a smooth von Mangoldt sum, shifted to that line, and removal of prime powers give the finite-character prime estimate. On the permitted rows every $\xi_u^*\vartheta_{ij}$ is nonprincipal. Consequently a whole fixed product of slots satisfies

$$\left|\prod_i Q_{u,i}\right|^2 \ll Z^{\kappa z+\varepsilon_1} p_J((W_i))^b (1+|\boldsymbol{t}|)^A. \tag{4.38}$$

The total contribution $2ez$ is included in ε_1. At $\kappa = 1$ the same assertion follows by absolute prime counting. The estimates and their norm-power derivatives are uniform for $3/4 \le \kappa \le 1$. The natural redundant radical deletes only $O(\log Z)$ possible prime labels; its normalized contribution is within the same loss.

4.7 The exceptional terminal

Lemma 4.11 (Equal-product cancellation)**.** *Let $\lambda = \omega_0\vartheta$, $\vartheta \in \Theta$, with global norm twists removed as above. For a common radical $\mathfrak{R}_*$ of norm at most Z^{B_*} put*

$$L_i(X,t) = \sum_{\mathfrak{a}} \lambda(\mathfrak{a})\mathbf{1}_{(\mathfrak{a},\mathfrak{R}_*)=1} q_{\mathfrak{a}}^{it} W_i(q_{\mathfrak{a}}/X).$$

Uniformly for $X > 0$,

$$L_i(X,t) = c_{\lambda,\mathfrak{R}_*} X^{1+it} I_i(t) + O(Z^{\varepsilon_1} p_J(W_i)(1+|t|)^A), \qquad |L_i(X,t)| \ll p_0(W_i)X, \tag{4.39}$$

where $I_i(t) = \int_0^\infty W_i(y) y^{it}\,dy$ and the main coefficient is independent of X, t. It vanishes unless λ is principal. If $U_1U_2 = V_1V_2 = T$ and all four scales are at least Z^a, $a \ge 0$, then

$$T^{-1/2}|L_1(U_1,t)L_2(U_2,t) - L_1(V_1,t)L_2(V_2,t)| \ll Z^{\varepsilon_1} T^{1/2} Z^{-a} p_J((W_1,W_2))^2 (1+|t|)^{2A}. \tag{4.40}$$

Without a nonnegative lower length the same expression is $O(p_0((W_1,W_2))^2 T^{1/2})$.

Proof. First omit the mask. In the Mellin convention used here the exact integral on an absolute line $c > 1$ is

$$L_i^{\mathrm{unmasked}}(X,t) = \frac{X^{it}}{2\pi i}\int_{(c)} L_K^S(s,\lambda)\widehat{W}_i(s+it)X^s\,ds.$$

Hecke continuation [19] therefore gives the residue at $s = 1$ with its stated factor $X^{1+it}I_i(t)$. For $X \ge 1$, shifting to $\operatorname{Re} s = -1$ gives an $O(X^{-1}p_J(W_i)(1+|t|)^A)$ remainder. A norm-reduced genuine background has no pole; the only remaining pole is the principal one. For $X < 1$ bounded ideal counting gives an $O(p_J(W_i))$ remainder. Now insert the common mask by inclusion–exclusion. In a divisor term the factor q_d^{it} cancels the factor q_d^{-it} from the residue at scale X/q_d. Thus

$$c_{\lambda,\mathfrak{R}_*} = \operatorname*{Res}_{s=1} L_K^S(s,\lambda) \sum_{d|\mathrm{rad}\,\mathfrak{R}_*} \frac{\mu_K(d)\lambda(d)}{q_d}.$$

Use a subsidiary divisor loss at most $\varepsilon_1/4$ here, so the products of the resulting errors below still fit within Z^{ε_1}. This proves (4.39); its absolute bound also follows directly from the original ideal sum, including the empty subunit range. The two products have identical residue term $c^2_{\lambda,\mathfrak{R}_*}T^{1+it}I_1(t)I_2(t)$, which cancels before any absolute value. Each cross error is bounded by the largest of the four scales, at most TZ^{-a}; also $1 \le TZ^{-a}$. Division by $T^{1/2}$ proves the assertion. □

4.8 The fourth-moment induction

Proof of Theorem 4.2. Put $A = n_1 + n_2 + z$. The effective row width is $M = m + q$: the logarithmic norm of the declared moving support contributes q in addition to the row length m. We use the finite induction of [16, Sections 18.2–18.8], organized by the following ordered state:

$$(\text{zero or positive slots},\ \text{width band},\ \text{uncentered or centered stage}).$$

All zero-slot bands precede the positive-slot bands. Within a width band the uncentered estimate is proved first. A comparison term returns to that earlier stage in the same band; a two-transform child has strictly smaller width. This gives a finite dependency order.

The coefficient that must be preserved is one whole-product character on each rectangle, with the same norm phase and coprimality mask. Lemma 4.4 preserves that class, and Lemma 4.11 estimates its centered exceptional members. The remaining steps below compute the lengths and the available loss at each reduction.

Base ranges and ordinary reflection. Choose a width floor $\rho > 0$, a step $\sigma > 0$, and a padding $\delta_p > 0$ small compared with both. Divide the bounded width range into bands of length $\sigma/4$. Complete all zero-slot bands first, then the positive-slot bands. In each band prove the uncentered range $A \le 5M/6$ first. For zero slots, (4.37), applied once to each plain longer than $M/2$, reduces all lengths to the padded core $n_i \le M/2 + \xi$, $n_1 + n_2 \le M + \delta_p$. The case $M \le \rho$ follows from absolute counting with loss $\rho + \delta_p$; positive slots add at most $\rho/6$ by (4.38) and the capacity inequality.

Comparison rectangles. For $A > 5M/6$ put $L = M/4$. If one plain is shorter than L, reflect the other. Otherwise subtract the product at comparison scales $Y_1 = Z^L$, $Y_2 = X_1X_2/Y_1$, keeping the same profiles and every slot. Reflecting the longer comparison factor gives total length at most

$$A_{\mathrm{comp}} \le 3M/2 - A + 2z + \xi.$$

For $z = 0$ this has a margin at least $M/6 - \xi$ inside the uncentered range. For $z > 0$, capacity implies $z < M/20$, and both required boundaries have margin at least $M/15 - \xi$; these are the inequalities in [16, (18.15)–(18.16)]. The comparison therefore uses an already proved assertion in the same band. It remains to bound the whole centered difference.

After any frozen allocation b_1, b_2 its coefficient contains

$$D_b(l_1,l_2) = \prod_{i=1}^{2} W_i(q_{b_i}q_{l_i}/X_i) - \prod_{i=1}^{2} W_i(q_{b_i}q_{l_i}/Y_i), \qquad X_1X_2 = Y_1Y_2.$$

The same allocated powers occur in both rectangles. An impossible rectangle is zero but is retained formally. Including the live prime product Π, the arithmetic factor on a separated side is one whole-product character, one norm power, and one common mask:

$$\left(\prod_{i\in I'} a_i(p_i)W_i^{\mathrm{slot}}(q_{p_i}/P_i)\right) D_b(l_1,l_2)\lambda_1(\Pi l_1l_2)q_{\Pi l_1l_2}^{it}\mathbf{1}_{(\Pi l_1l_2,\mathfrak{R})=1}\mathbf{1}_{s\mid \Pi l_1l_2}. \tag{4.41}$$

The last divisibility condition is omitted when $s = 1$. For an uncentered input omit the second rectangle.

First transform and its norm allowance. Apply the first orbital Poisson transform to the nonnegative row norm of the whole expression. Its zero frequency requires the product of the two full indices to be powerful, giving $O(Z^{m+\varepsilon_1})$ after central normalization. For nonzero frequencies extract the complete common support as Ca, Db. Write c, d, p for the lengths of C, D and their common radical, R_1 for the nonzero-exponent radical length, and E_1 for the divisor of the complementary mask. Thus $R_1 + E_1 \le p$. The new displayed moving support has length $\widetilde{q} = q + R_1 + E_1$. The nominal frequency length and the C-side allowance are

$$K_0 = 2A - c - d + R_1 + E_1 - m, \qquad N_C \ll Z^{A-c+\widetilde{q}+B_c-s_0+\varepsilon_G}, \quad B_c = \left(\frac{3c - 5d - R_1}{6}\right)_+, \tag{4.42}$$

where s_0 is the length of the full mutual-gcd divisor. There is a symmetric D-side allowance. The outer prefactor, common-support count, and two Cauchy halves sum to $M + \{B_c + B_d - (c + d - 2p - R_1)\}/2$. This is at most M: at a common prime of multiplicities $i \ge j \ge 1$, with $r_p = \mathbf{1}_{6\nmid i-j}$,

$$\frac{(3i - 5j - r_p)_+}{6} \le i + j - 2 - r_p.$$

Summing with the actual logarithmic prime norms proves the inequality, including every ramified multiplicity.

One prime-pool amplification. We explain how the C-side allowance is obtained. Let K be the retained Gauss-row length, and let $w_o \le w$ be the moving-support and total lengths removed by an amplifier error. Put

$$a_0 = A - c - w, \qquad J = d - c + (K_0 - K) - 2w + w_o.$$

For zero slots enlarge directly by $g = J_+ + \sigma$ and put $\ell_{\mathrm{pool}} = 0$ in the prefactor. For positive slots choose one pool of good primes of length $\ell_* = \sigma/3$, disjoint from every live slot by requiring the mesh $\eta < \sigma/6$. The prime ideal theorem gives $Z^{\ell_*+o(1)}$ such primes. For each row omit its prime divisors and all frozen supports; only $O(\log Z)$ labels are lost. Comparing h with $h\mathfrak{p}^6$ gives a main positive norm with density factor $Z^{-\ell_*+o(1)}$, since each output has boundedly many such representations. Here take $g = J_+ + 2\sigma$ and $\ell_{\mathrm{pool}} = \ell_*$. The only error valuations are $1, 6, 7$, by (4.6). Their squared central coefficients are $P^{-1}, (1 - P^{-1})^2, P^{-1}$, respectively. Thus $(w, w_o) = (i\ell_p, e_i\ell_p)$, with $e_1 = e_7 = 1$, $e_6 = 0$ and $\sigma/6 \le \ell_p \le \sigma/3$. Each error is treated without further amplification, using $g = J_+ + \sigma$ and $\ell_{\mathrm{pool}} = 0$. The entire centered coefficient is used in each of these identities, and the pool average is used once.

Second transform and the width drop. The second transform uses the full finite correlation (4.6). After extracting its genuine complete common supports D_2, E_2, write c_2, d_2 for their lengths, $b_2 = (c_2 + d_2)/2$, p_2 for the common radical length, and g_2 for the length of their gcd. At equal non-six-divisible multiplicities let t_2 be the radical length where the divided frequency is a unit, and V_2 the corresponding nonunit radical length. The local correlation has size at most $Z^{g_2-t_2}$ on its retained partition and is extended by zero off that partition. Writing the frequency as $\gcd(D_2, E_2)V_{\mathrm{id}}h'$ gives

$$\begin{aligned} m' &= 2a_0 - K - g - g_2 - V_2, & q' &= \widetilde{q} + w_o + t_2 + V_2, \\ M' &= m' + q' = M + J - g - g_2 + t_2 \le M - \sigma, \\ \Delta_{\mathrm{child}} &= b_2 - p_2 + w + B_c + \ell_{\mathrm{pool}} \ge w + \ell_{\mathrm{pool}}. \end{aligned} \tag{4.43}$$

The second line uses $g_2 \ge t_2$. The exponent available for the inner products is exactly $M' + \Delta_{\mathrm{child}}$, after the common-support count, Poisson factor, central normalizations, and pool density have each been included once; see [16, (18.32)–(18.33)].

Insert the full residual coprimality sum before separating the children. At a divisor prime allocate its occurrence among the two plains and the live slots by

$$\mathbf{1}_{p\mid\prod_i n_i} = \sum_{\varnothing\ne J} (-1)^{|J|+1} \prod_{i\in J} \mathbf{1}_{p\mid n_i}.$$

Extracting a plain power divides its two rectangle scales by the same norm and preserves the common mask. A selected slot is frozen. Old masks remain common, and a slot selected by two distinct divisor primes gives zero. Both transforms preserve (4.41). The separated child sides have finite inducing-character ratio in Θ, although their smooth norm powers may differ. Hence they have the same finite exceptional status. The live background on each side remains ω_0 up to whole conjugation, and (4.3) remains valid.

On nonexceptional rows first take triangle between the two rectangles, then clip a nonempty subunit plain scale to one. With all actual boundary errors included, the positive-part defect of a child capacity is

$$F_{\mathrm{act}} = \bigl(A_{\mathrm{act}} - M'_{\mathrm{act}} + (6\kappa - 1)z_{\mathrm{act}}\bigr)_+ \le 6(w + \ell_{\mathrm{pool}}) + O(\xi).$$

Remove whole slots by (4.38) until capacity holds or no slot remains. One greedy prefix has at most one overshoot, so the squared cost is

$$\kappa d_z \le F_{\mathrm{act}}/6 + \kappa\eta \le \Delta_{\mathrm{child}} + O(\xi + \eta).$$

The remaining product uses the smaller-width induction; if no slot remains its plain lengths are unrestricted. Divisor allocations, exact extraction coefficients, and paired Cauchy contribute a single aggregate support error.

It remains to justify the exceptional alternative. Every prime in the genuine second common support is disjoint from the old displayed moving support, because its frozen scalar otherwise vanishes before the absolute estimate. If the child finite inducing character belongs to Θ, each exterior row valuation is prescribed modulo six. Thus $(h') = h_0 a^6$ in finitely many fixed unit sectors. At a nonunit common prime of equal multiplicity one the frozen frequency factor has valuation two, forcing $v_p(h') \equiv 4 \pmod 6$. If the radical length of these primes is v_1, put $f_1 = 2v_1$; then $q_{h_0} \ge Z^{2f_1}$ and, on a nonempty range,

$$\#\{h' : q_{h'} \ll Z^{m'_{\mathrm{act}}},\ \xi^*_{h'} \in \Theta\} \ll Z^{(m'_{\mathrm{act}} - f_1)/6 + \varepsilon_1}. \tag{4.44}$$

This count includes all rows admitted by the positive enlargement.

The absolute volumes together with (4.44) have excess $A - 5M/6 - F_1 - F_2$, where the local ledger of [16, (18.42)–(18.44)] gives

$$F_1 \ge \frac{2}{3}(c + w) - 3\sigma - O(\xi), \qquad F_2 \ge \frac{2}{3}b_2.$$

This closes the uncentered range $A \le 5M/6$. For a centered input let $v_* = c + w + \min(c_2, d_2)$. On the side with that smaller extraction, every formal plain scale initially has length at least $L - v_*$. Lemma 4.11 gives saving $(L - v_* - r_1)_+$ after a final divisor extraction of total length r_1 on that side; if this lower length is nonpositive use its absolute-volume alternative. The divisor count and the two raw extraction coefficients retain a saving of $(L - v_*)_+$, since

$$t_- - r_1 - r_2 - (r - r_1)_+ = (t_- - r_2) - \max(r, r_1) \le \omega_2 - r, \qquad r = (L - v_*)_+.$$

Here t_- is the lower divisor-dyad length and ω_2 is the actual frozen-slot ratio error on the other side. Therefore the exceptional excess is at most

$$A - \frac{5}{6}M - \frac{2}{3}v_* - (L - v_*)_+ + 3\sigma + O(\xi) \le (A - M)_+ + 3\sigma + O(\xi),$$

because the maximum in $v_* \ge 0$ occurs at $v_* = L = M/4$. This is at most $3\sigma + O(\xi)$ in the positive-slot region, and $\delta_p + 3\sigma + O(\xi)$ in the padded zero-slot core. Keep the centered scales formal and perform their common residue cancellation before taking absolute values.

The second-transform diagonal has excess at most $5\sigma/3 + O(\xi)$, with an additional δ_p in the padded core; its local diagonal is the full Euler function of the common modulus. Frequency tails are removed as whole common dyads on the genuine sums, before the artificial coprime extension and Fourier absolutization, using the fixed rapid-decay order of the orbital kernels. Equations (4.43) and the specified clipping give a strict width drop at least $\sigma/2$ once ξ is sufficiently small. The number of strict calls is at most $D = 2 + \lceil 2M_{max}/\sigma \rceil$. Within a band, comparison calls the earlier uncentered stage. Each Gauss norm is amplified at most once, and each positive child removes slots once after all capacity defects have been included. Choose the floor and step so that

$$\rho + \delta_p + \rho/6 + 3\sigma + 5\sigma/3 < \varepsilon/4,$$

then choose ξ, the mesh η, and the aggregate local power loss sufficiently small that their sum through D stages is less than $\varepsilon/2$. The backward finite propagation below handles the analytic orders. This proves the natural assertion; the exact mask deletion (4.35) proves the full statement.

The field-dependent inputs in this induction have now all been specified: the complete correlation (4.6), orbital Poisson with one common kernel, the ideal prime pool, and the masked Mellin cancellation. The local multiplicity inequalities use only valuations. Multiplying their contributions by the actual logarithmic prime norms gives the displayed capacities and the strict decrease of M. The finite ideal fibres and sector counts enter the allocated small-power loss; Lemma 4.12 handles the remaining profile logarithms. □

4.9 Uniformity

Lemma 4.12 (Finite propagation with logarithmic profiles)**.** *Consider a finite-depth canonical reduction with the arithmetic powers, normalized outer counting measures, common Fourier measures, and fixed nonnegative Cauchy or Hölder exponents of [16, Lemma 17.3 and Corollary 17.4]. Replace a uniform bound at any fixed Euler or weighted Fourier order by that bound times $(1+\log Z)^A$, where A is finite and may depend on the eventual fixed arithmetic data and on the order. Then the same arithmetic scale exponents propagate through the reduction. At the root there are finite A_*, J_*, H_*, b_* such that its conclusion differs from the original one by at most*

$$C(1+\log Z)^{A_*} p_{J_*}(w)^{b_*} \langle t \rangle^{H_*}. \tag{4.45}$$

Consequently every preassigned additional loss Z^η, $\eta > 0$, absorbs all these logarithms. The threshold and the finite orders, but not η, may depend on those later fixed data.

Proof. Use backward induction on the finite reduction graph. At a terminal the assertion is its finite-order estimate, with a possibly nonzero logarithmic degree. At an edge, the child estimates specify only finitely many seminorm orders and height degrees. Choose the parent Euler order large enough to bound the corresponding weighted L^1 norm of the one common Fourier measure. The modified edge assumption adds a finite logarithmic factor. Products, fixed positive powers, and the finite sums in the edge add or multiply logarithmic degrees by fixed finite numbers. Use the Cauchy exponents attached to each edge. Norm powers in the edge combine exactly as in the original arithmetic estimate. A normalized outer measure has already had its norm count extracted, so its remaining logarithmic mass adds only another such factor.

Iterating gives finite A_*, J_*, H_*, b_* at the root and proves (4.45). Finally $(1+\log Z)^{A_*} \ll_{A_*,\eta} Z^\eta$.

□

Remark 4.13 (Application to norm sheets). The orbital kernels used here are rapidly decreasing at infinity and have at most fixed-degree logarithmic growth at zero, after every fixed Euler derivative. On a retained annular block their nonzero arguments lie in $[cZ^{-C_0}, CZ^{C_0}]$: all

moving norm lengths are bounded, nonzero exterior ideal norms are at least one, and fixed support constants enter only c, C. Therefore each such derivative is bounded by a finite power of $1 + \log Z$ after its assigned central powers and decay factors have been removed. On a product of normalized annuli, derivatives of $K(R_0 \prod_i y_i^{a_i})$ are fixed linear combinations of Euler derivatives of K. Logarithmic Fourier inversion, with a sufficiently large but finite order, gives exactly the weighted edge bounds in Lemma 4.12. Treat the zero orbits separately by Lemma 3.6. The logarithmic propagation lemma then applies to both canonical inductions.

Proposition 4.14 (Uniform real parameters)**.** *For a fixed field and fixed infinity-type orbit, all real losses, strict-width decrements, depth bounds, and the mesh in Theorem 4.2 can be chosen before a later finite-order target twist and its conductor. The fourth-moment mesh is also chosen before the fixed slot count, uniformly for $\kappa \in [3/4, 1]$. Once the target, final S, and slot system are fixed, both moments have finite seminorm orders and fixed polynomial height degrees. An external tail order chosen in a later application does not change those internal degrees.*

Proof. The arithmetic proofs use finite depth and finitely many factorization types. The inverse decrease and its losses are (4.32); the ordinary decrease is (4.43). They are affine identities and positive parts of affine functions of a fixed list of total lengths. In the ordinary capacity the Lipschitz bound is

$$|\Delta M| + |\Delta A| + 5|\Delta z|, \qquad 0 \le 6\kappa - 1 \le 5.$$

It is independent of the number of slots. For a fixed slot system let H_I bound the sum of the absolute logarithms of all its support endpoints and of the finitely many later support enlargements. Every subset then has one total ratio error at most $H_I/\log Z$. Keep the extracted prime norms exact. This makes every stage error at most $C_*(\xi + \eta + \varepsilon_0)$ with a numerical C_* independent of the slot count, after the threshold for $H_I/\log Z$ is imposed. Choose ρ, σ, δ_p, then D, and finally ξ, η, ε_0 so that $C_* D(\xi + \eta + \varepsilon_0) < \varepsilon/2$ and ξ preserves the strict width and comparison margins. These are real choices before the arithmetic data and the slot count.

For those later fixed data, choose the subsidiary divisor, Rankin, and prime-estimate exponents whose aggregate is ε_0. Divisor allocations for I slots obey $C_I^{\omega(\mathfrak{a})} \ll_{I,a} q_{\mathfrak{a}}^a$ for every fixed $a > 0$. Fixed local sector counts are constants. The logarithms introduced by norm-sheet profiles and compact unit representatives have a fixed degree. Every preassigned positive loss absorbs them at a larger threshold, independently of that degree. Fixed conductor and support constants likewise enter lengths only through $O(1/\log Z)$.

Here is the analytic propagation rule, as in [16, Lemma 17.3 and Corollary 17.4]. After its central powers are removed, every coupled smooth factor is one function of normalized norms on a fixed product of annuli. A kernel evaluated at $R_0 \prod y_i^{a_i}$ obeys

$$(y_i \partial_{y_i}) K(R_0 \prod y_j^{a_j}) = a_i (x\partial_x) K(x)\big|_{x = R_0 \prod y_j^{a_j}}.$$

The theta bounds give the assigned decay, and the orbital estimates give Euler bounds with a possible finite logarithmic degree, at every fixed order. Remark 4.13 and Lemma 4.12 propagate these factors with any previously fixed tail decay. Logarithmic Fourier inversion of this whole profile has weighted L^1 norm controlled by a finite larger seminorm. The measure is common to all rows and live labels in the current Hilbert space. Row masks remain outside it.

If a child estimate has height degree H and uses a profile seminorm of order J, the edge's Fourier measure need only be integrated against a fixed weight determined by H, J. Its weighted bound chooses a finite larger parent order. Select these orders backwards through the finite depth; Cauchy edges use the weighted sum of the two child degrees. Normalize the outer counting measures by their extracted norm counts. Lemma 4.12 absorbs the resulting root logarithm in one previously reserved loss. This proves finite seminorm and height dependence at the root. It also justifies the permitted scale suprema by a fixed number of Sobolev derivatives on logarithmic parameter boxes.

After their combined polynomial height degree A_{ht} is fixed, an allotted height loss $\varepsilon_{\mathrm{ht}} > 0$ on a physical range $U \geq Z^{d_{\min}}$ is respected, for example, by

$$0 < \tau \leq \frac{d_{\min}\varepsilon_{\mathrm{ht}}}{20(A_{\mathrm{ht}}+1)}, \qquad T = Z^{\tau}.$$

Indeed, every retained height factor is then at most $Z^{d_{\min}\varepsilon_{\mathrm{ht}}/20}$ up to a fixed constant. One may subsequently increase the external integration-by-parts order to make its discarded tail negligible. That later choice differentiates only the external tests. The internal degree A_{ht}, real losses, and mesh remain fixed. This completes both quantitative assertions. □

For the physical application the declared moving support has $q = 0$ and u is sixth-power-free. If any exterior prime divides u, its valuation is between one and five and its local Kummer character is ramified and nonprincipal there. Hence $\xi_u^* \in \Theta$ is possible only for the finitely many fixed unit-sector rows of exterior norm one. The zero-slot principal exception is contained in these bounded rows. Thus every sufficiently large physical row satisfies precisely the eligibility conditions needed for both moments.

5 Sixth-power amplification

The following amplification keeps the unmarked inverse estimate uniform as the witness length approaches the row length. This is the independent-scale amplification of [16, Lemma 17.6], reproduced with ideal indices and the original zero extensions.

Lemma 5.1. *Let $U, D \geq 1$ be independent scales and let*

$$M_u(D;W) = D^{-1/2}\sum_n \mu(n)\eta(n)\rho_u(n)W(q_n/D)$$

be an inverse presentation admitted by Theorem 4.1. For every fixed $c > 0$ and $\varepsilon > 0$,

$$\sum_{\substack{u \asymp U \\ u \text{ sixth-power-free}}} |M_u(D;W)|^2 \ll \max\{U, U^{1/6}D^{5(1+c)/6}\}(UD)^{\varepsilon}. \tag{5.1}$$

The estimate also holds with the permitted scale and smooth-profile suprema, with their fixed polynomial height factor.

Proof. The retained presentation zeros give

$$\rho_{ua^6}(n) = \rho_u(n)\mathbf{1}_{(n,a)=1}, \tag{5.2}$$

$$M_u(D;W) = \sum_{d \mid \mathrm{rad}(a)} \mu(d)\eta(d)\rho_u(d)q_d^{-1/2}M_{ua^6}(D/q_d;W). \tag{5.3}$$

The second identity follows by separating the part of a squarefree n supported on a.

Set $H \asymp \max\{U, D^{1+c}\}$ and $A \asymp (H/U)^{1/6}$, allowing fixed constants in the row cutoff. Fixed-field ideal counting supplies $\gg A$ permitted ideal multipliers a with $q_a \ll A$, in a fixed allowed sector; for bounded A include the unit multiplier and change the constant. Average (5.3) over these multipliers, apply weighted Cauchy–Schwarz and the ideal divisor bound, and use the permitted supremum in D/q_d. The map $(u,a) \mapsto ua^6$ is injective on sixth-power-free u: reduction of each ideal valuation modulo six recovers u, and division recovers a. Finite class and unit sectors change the bound by a fixed factor.

The no-slot case of Theorem 4.1, with base scale H, has a fixed positive margin because $D \leq H^{1/(1+c)}$. It bounds the resulting row energy by $H(UD)^{\varepsilon}$. Division by the number of multipliers gives

$$H/A \asymp U^{1/6}H^{5/6} \asymp \max\{U, U^{1/6}D^{5(1+c)/6}\}.$$

The scale and profile supremum is the finite-seminorm one from [16, Lemma 4.5]. □

For $D = U^r$ in any fixed bounded range of r, choose c and the preliminary loss in terms of the requested final loss. The effective exponent in (5.1) is

$$e(r) = \max\left\{1, \frac{1+5r}{6}\right\}. \tag{5.4}$$

If an inverse witness has size at least $U^{\delta r/2 - O(\varepsilon)}$, division by its squared size gives

$$\#\mathcal{B} \ll U^{e(r)-\delta r+\varepsilon}(1+T)^A. \tag{5.5}$$

This is the independent-scale argument of [16, Lemma 17.6, equation (19.3), and Remark 19.3]. It applies throughout $1/50 < \delta \le 1$.

6 The physical probe

Fix a target $\psi \in \mathcal{F}_\omega$, with a global norm twist removed and the orientation of Proposition 3.2. Use one final set S and the arithmetic normalization of Proposition 3.1. Put $R = \mathcal{O}_{K,S}$, $T_0 = K_S^1$, and $\Gamma = (R^\times)^6$. All the sums in this section run over exterior integral ideals with their allowed generators, except for the explicitly displayed additive rows in R.

6.1 Definition and Poisson expansion

Let Λ_K and G be the cubic and quadratic Gauss corrections and set

$$\vartheta = \bar{\Lambda}_K \bar{G} \psi.$$

Thus $\gamma_1(\mathfrak{p})\gamma_2(\mathfrak{p})\vartheta(\mathfrak{p}) = -\psi(\mathfrak{p})$ at every good prime, in the normalization of [17, Lemma 3.4]. Define

$$P_G(t) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{(\log t)^2}{4}\right), \qquad \widehat{P}_G(s) = e^{s^2},$$

and the marked completed row

$$\mathcal{B}_{m,\psi;D}(Z) = \sum_{\substack{c \text{ squarefree} \\ n}} \frac{\gamma_2(c)\vartheta(cn^3)}{q_c^{1/2} q_n} \chi_{cn^3}(m) \mathbf{1}_{D \mid cn^3} P_G(q_c q_n^3 / Z). \tag{6.1}$$

The variable n is unrestricted and may share primes with c. The target acts on the whole completed index cn^3. The Gaussian makes the series absolutely convergent; for example, $P_G(t) \ll_\sigma t^{-\sigma}$ with $\sigma > 1/2$ reduces it to $Z^\sigma \zeta_K^S(\sigma + 1/2)\zeta_K^S(3\sigma + 1)$.

Choose $W_{1,\mathrm{add}} \in C_c^\infty((0,\infty))$ nonnegative and nonzero, and a tensor Bruhat–Schwartz function $W_{0,\mathrm{add}}$ compactly supported in $K_S^\times$ such that

$$\widehat{W}_{0,\mathrm{add}}(0) = 0, \qquad \mathcal{M}(1/6) \neq 0, \quad \mathcal{M}(z) = \int_{K_S^\times} \widehat{W}_{0,\mathrm{add}}(y) |y|_S^z \, d^\times y. \tag{6.2}$$

These conditions are compatible. At one complex place subtract two radial bumps so that the additive integral is zero, while their $|y|^{5/6} d^\times y$ pairing is nonzero. At the remaining places take nonnegative annular factors. The local zeta functional equation then gives (6.2), as in [17, equations (7.1)–(7.2)]. Notice that only $W_{1,\mathrm{add}}$ is required to be nonnegative.

For an exterior ideal b put

$$g_b(k) = \sum_{d \bmod b} \chi_b(d) e(kd/b).$$

With the norm section ℓ_x and the quotient measure fixed in Section 3, define

$$I_{\psi;D}(X,Y,Z)=\int_{T_0/\Gamma}\frac{1}{Y}\sum_b\frac{W_{1,\mathrm{add}}(q_b/Y)}{\sqrt{q_bX}}\sum_{m\in R}q_b^{-1/2}g_b(-m)W_{0,\mathrm{add}}\left(\frac{m}{t_0\ell_{q_bX}}\right)\mathcal{B}_{m,\psi;D}(Z)\,dt_0. \tag{6.3}$$

We write $I_\psi=I_{\psi;1}$. Each additive row sum has finite support on a bounded set of quotient representatives, and $m=0$ is absent by the support of $W_{0,\mathrm{add}}$. The expression descends to T_0/Γ. This is the norm-form probe of [17, equation (7.3)], with the stated mark. Finite local sector decompositions, when used to estimate it, are recombined in this definition.

The finite transform underlying its Poisson expansion is worth recording. For $A=cn^3$ and $h\in R$, let

$$\mathscr{C}(b,A,h)=\sum_{\substack{d \bmod b\\ h\equiv Ad\ (b)}}\chi_b(d)g_{\chi_A}\left(A,\frac{h-Ad}{b}\right),\qquad g_{\chi_A}(A,k)=\sum_{v \bmod A}\chi_A(v)e(kv/A).$$

Direct finite Fourier inversion gives

$$\sum_{m \bmod bA}q_b^{-1/2}g_b(-m)\chi_A(m)e(hm/(bA))=q_b^{1/2}\mathscr{C}(b,A,h). \tag{6.4}$$

Indeed, write $m=v+Aj$ and sum first over $j \bmod b$. This calculation also allows common factors of b and A. The lattice index is q_bq_A, whereas the dilation has additive modulus q_bX. After both square-root factors in (6.3), the Poisson multiplier is $X^{1/2}/q_A$. The norm of the Fourier argument is Xq_h/q_A.

Integrate over T_0/Γ, retain the nonzero orbits $h=ur^6$, and apply Lemma 3.6 with trivial continuous row character. The zero frequency vanishes by (6.2). Three Mellin inversions yield

$$\begin{aligned}I_\psi(X,Y,Z)=\frac{1}{(2\pi i)^3}\int_{(4)}\int_{(4)}\int_{(2)}&X^{1/2-z}Z^{s+z-1}Y^{w-1}e^{(s+z-1)^2}\\&\times\mathcal{M}(z)\widehat{W}_{1,\mathrm{add}}(w)\sum_u q_u^{-z}\mathscr{F}_{u,\psi}(s,w,z)\,dz\,dw\,ds,\end{aligned} \tag{6.5}$$

$$\mathscr{F}_{u,\psi}(s,w,z)=\sum_{\substack{c\ \text{squarefree}\\ n,b,r}}\gamma_2(c)\vartheta(cn^3)\mathscr{C}(b,cn^3,ur^6)q_c^{-1/2-s}q_n^{-1-3s}q_b^{-w}q_r^{-6z}. \tag{6.6}$$

The row set u includes the finite unit sectors and has sixth-power-free exterior part. The marked form inserts $\mathbf{1}_{D|cn^3}$ in (6.6). All interchanges on these initial lines are absolute. The sheet integral has the common Mellin transform $\mathcal{M}$. In particular, the exponent of Z in (6.5) is exactly $s+z-1$.

6.2 Compensation

Use the fixed geometry

$$\begin{gathered}\ell=\frac{1}{6},\quad b_0=\frac{1}{8},\quad M=\frac{5}{6},\quad l_x=\frac{17}{48},\quad l_y=\frac{23}{48},\quad h=\frac{13}{16},\\ C(s)=s-\frac{11}{16},\qquad X=Z^{l_x},\quad Y=Z^{l_y}.\end{gathered} \tag{6.7}$$

Fix positive slot lengths ℓ_i with $\sum_i\ell_i=\ell$ and set $P_i=Z^{\ell_i}$. Choose fixed nonnegative nonzero smooth windows W_i with disjoint prime supports before any row masks are inserted. All primes outside S are allowed. At a good prime of norm Q, the compensation is

$$\bar{\psi}(\mathfrak{p})I_{\psi;\mathfrak{p}}(X,Y,ZQ)-Q^{-3/2}I_\psi(X/Q,Y/Q,Z). \tag{6.8}$$

For $J \subseteq \{1,\dots,J_0\}$ write $p_J = \prod_{i\in J} p_i$. The complete finite operator is

$$I_{\psi,\mathrm{mod}}(Z) = \sum_{(p_i)} \prod_{i=1}^{J_0} W_i(q_{p_i}/P_i) \sum_{J\subseteq\{1,\dots,J_0\}} (-1)^{|J|} q_{p_J}^{-3/2} \bar\psi(p_{J^c}) I_{\psi;p_{J^c}}(X/q_{p_J}, Y/q_{p_J}, Zq_{p_{J^c}}). \quad (6.9)$$

Every subset and its original mask is present. The target factors have modulus one.

Let $P_{\mathfrak p}$ denote the local sum in (6.6), and $P_{\mathfrak p}^*$ the part with $\mathfrak p \mid cn^3$. The monomial in (6.5) shows that (6.8) replaces the local factor, after extraction of Q^{z-1}, by

$$\bar\psi(\mathfrak p) Q^s P_{\mathfrak p}^* - Q^{-w} P_{\mathfrak p}.$$

Consequently its leading prime sum is

$$\mathcal{Q}_i(u;z) = -\sum_{\mathfrak p\notin S} W_i(Q/P_i) Q^{z-1} \bar\chi_u(\mathfrak p). \quad (6.10)$$

This is a finite actual prime character, even for an infinite-order target. For example, orient a detector presentation as $\psi\theta\bar\chi_u$. The inverse relative slot coefficient is $(\psi\theta)^{-1}$. For the plain moment conjugate both witnesses and all selected slots; the live character becomes $\bar\psi\bar\theta\chi_u$ and the relative slot coefficient is $\psi\theta$. Thus the actual prime character in Theorem 4.2 is precisely the one required by (6.10).

6.3 Euler factors

At an unramified row prime $\mathfrak p \nmid u$ put

$$V = Q^{-6z}, \quad W = \chi_u(\mathfrak p) Q^{-w}, \quad D = \psi(\mathfrak p)\bar\chi_u(\mathfrak p) Q^{-s}, \quad E = \vartheta(\mathfrak p)^6 Q^{4-6s} V.$$

The finite identity (6.4) and the local Gauss relations give

$$P_{\mathfrak p} = \frac{1}{1-V} + \frac{W}{1-W} + \frac{E}{1-E}\left(\frac{1-Q^{-1}}{1-V} + W\right) - \frac{D}{1-E}\left(1 + \frac{(Q-1)WV}{1-V}\right). \quad (6.11)$$

This is [17, equation (7.19)]. At primes dividing u the six valuation classes use the full finite Gauss sum in (6.4), as in [17, equations (7.21)–(7.22)] and [16, Lemma 7.1]. These ramified factors retain the phase of the local unit part. Multiplication of all local identities gives

$$\mathscr{F}_{u,\psi}(s,w,z) = \zeta_K^S(6z) \frac{L_K^S(w,\chi_u)}{L_K^S(s,\psi\bar\chi_u)} H_{u,\psi}(s,w,z). \quad (6.12)$$

The principal poles are at $w = 1$ and $z = 1/6$.

Using $P_{\mathfrak p} - P_{\mathfrak p}^* = (1-V)^{-1} + W(1-W)^{-1}$, write the full compensated factor as

$$G_{\mathfrak p} = \frac{(\bar\psi(\mathfrak p)Q^s - Q^{-w})(1-V)(1-W)P_{\mathfrak p}^* - Q^{-w}(1-VW)}{1-D}. \quad (6.13)$$

On the principal residue put $v = Q^{-1}$, $D = \psi(\mathfrak p)Q^{-s}$, $E = \vartheta(\mathfrak p)^6 Q^{3-6s}$, and $K_{\mathfrak p} = 1 - vE - (1-v)D$. Formula (6.11) simplifies to

$$P_{\mathfrak p} = \frac{1+v}{1-v}\frac{K_{\mathfrak p}}{1-E}, \qquad P_{\mathfrak p}^* = \frac{(1+v)(E-D)}{1-E}, \quad (6.14)$$

$$H_{\mathfrak p} = \frac{(1-v^2)K_{\mathfrak p}}{(1-E)(1-D)}. \quad (6.15)$$

Lemma 6.1. *The function $H_\psi^S(s) = H_{1,\psi}(s, 1, 1/6)$ is holomorphic and nonzero for $\operatorname{Re} s > 2/3$. On $\sigma = \operatorname{Re} s \geq 3/4$, the principal compensated quotient $B_\mathfrak{p} = G_\mathfrak{p}/H_\mathfrak{p}$ satisfies*

$$B_\mathfrak{p} + 1 = \frac{(1-v)\{E/D - vE - (1-v)D\}}{K_\mathfrak{p}} = O(Q^{-\sigma}), \tag{6.16}$$

uniformly in all local phases and in $\operatorname{Im} s$.

Proof. For $\sigma > 1/2$, both $|D|$ and $|E|$ are less than one, so

$$|vE + (1-v)D| \leq v|E| + (1-v)|D| < 1.$$

Thus each factor in (6.15) is nonzero. On a closed half-plane strictly to the right of $2/3$,

$$H_\mathfrak{p} - 1 \ll Q^{-2} + Q^{-1-\sigma} + Q^{3-6\sigma}.$$

The prime-ideal sum converges normally, proving the assertion for the product. On the principal locus $B_\mathfrak{p} = P_\mathfrak{p}^*/(DP_\mathfrak{p}) - v$. Substitution gives (6.16), and $3 - 5\sigma \leq -\sigma$ when $\sigma \geq 3/4$ proves its bound. □

6.4 The principal normalization

The only physical row whose exterior sextic character is trivial is the principal Kummer class. Indeed, Chebotarev detects every nontrivial Kummer extension. Its sixth-power frequencies are $h = r^6$, with unit ambiguity in Γ; finer finite sectors used for positive estimates are recombined. The double residue of (6.5) has a fixed scalar

$$c_\psi = \frac{1}{6}\big(\operatorname{Res}_{s=1} \zeta_K^S(s)\big)^2 \mathcal{M}(1/6)\widehat{W}_{1,\mathrm{add}}(1) \neq 0. \tag{6.17}$$

The exact theta normalization has already removed its nonzero archimedean scalar. A different fixed measure normalization merely changes c_ψ by a nonzero constant. Its sign is immaterial.

Set

$$S_i(Z) = \sum_{\mathfrak{p} \notin S} W_i(Q/P_i)Q^{-5/6}, \qquad \mathcal{N}(Z) = Z^{-\ell/6}\prod_{i=1}^{J_0} S_i(Z), \tag{6.18}$$

$$J_\psi(Z) = \frac{(-1)^{J_0} I_{\psi,\mathrm{mod}}(Z)}{c_\psi \mathcal{N}(Z)}, \qquad f_\psi(Z) = \frac{1}{2\pi i}\int_{(2)} Z^{C(s)} e^{(s-5/6)^2} \frac{H_\psi^S(s)}{L_K^S(s,\psi)}\, ds. \tag{6.19}$$

The prime ideal theorem gives $S_i(Z) > 0$ eventually and $\mathcal{N}(Z) \asymp (\log Z)^{-J_0}$. Thus its reciprocal costs an arbitrarily small power.

The weights $\mu_{i,Z}(\mathfrak{p}) = W_i(Q/P_i)Q^{-5/6}/S_i(Z)$ are probabilities. Disjointness makes the normalized principal correction exactly

$$H_\psi^S(s)\prod_{i=1}^{J_0}\sum_{\mathfrak{p}} \mu_{i,Z}(\mathfrak{p})(-B_\mathfrak{p}(s)).$$

By Lemma 6.1, this is $H_\psi^S(s)\{1 + O_\psi(Z^{-(7/8)\min_i \ell_i})\}$ on $\operatorname{Re} s > 7/8$. For equal lengths reserve the positive margin

$$\mu_0 = \frac{7}{16}\min_i \ell_i = \frac{7}{96 J_0}. \tag{6.20}$$

All these expressions use the same S and are independent of the analysis height cutoff.

6.5 The low estimate

Proposition 6.2. *For each fixed slot system and every $\varepsilon > 0$, the same physical function in (6.19) satisfies*

$$|J_\psi(Z)| \ll_{\psi,\varepsilon} Z^{3/16+\varepsilon}. \tag{6.21}$$

The exponent is uniform in the slot mesh.

Proof. Fix the rescaled subset J in (6.9), put $d = \sum_{i\in J} \ell_i$, and first hold its prime tuple fixed. Write

$$X' = X/q_{p_J}, \quad Y' = Y/q_{p_J}, \quad M' = \frac{5}{6} - 2d, \quad \ell' = \frac{1}{6} - d, \qquad 0 \le d \le \frac{1}{6}.$$

Sum the surviving marked slots in the completed factor before taking its row norm. Lemma 4.8 gives

$$\sum_{m \text{ in the actual row box}} |\mathcal{B}^J_{m,\psi}(Z)|^2 \ll Z^{M'+\varepsilon}. \tag{6.22}$$

In the reflected energy notation, let O be the powerful row length, H the actual residual row length, and $\Delta_H = M' - O - H \ge -O(1/\log Z)$. The local conductor ledger gives $2A_0 \le O + O(1/\log Z)$, $N_0 \le A_0$, and $z_a \le \ell'$. On a Gaussian annulus the completed length is $N_* = 1 + \ell' + \theta_N$, with $|\theta_N|$ as small as prescribed. The dual length is

$$T_d = 2H + 2A_0 + 2z_a - 1 - \ell' - \theta_N - N_0 - 3B_0, \qquad T_d \le H - 3d + O(|\theta_N| + 1/\log Z).$$

The two branches of the hybrid minimum in the reflected energy are at most

$$M' - O/2 - \Delta_H + O(\varepsilon_0 + |\theta_N|), \qquad M' + \frac{d - 1/6 - 2\Delta_H + \theta_N}{4} + O(\varepsilon_0).$$

Both are bounded by $M'+O(\varepsilon_0+|\theta_N|+1/\log Z)$. This is the algebra in [16, Lemma 15.1], applied to the actual row lengths. The common Gaussian profile has summable homogeneous seminorms over its annuli. Extracting each small profile amplitude before the positive enlargement and taking the preliminary losses smaller proves (6.22).

For the additive factor of (6.3), logarithmic Fourier separation of q_b/Y' leaves one common profile and the polynomials $\mathcal{A}_m(Y')$ of Proposition 3.8. Put $Q_0 \asymp X'Y'$ and $P_a = Y'^2/Q_0 \asymp Z^{1/8}$; fixed arithmetic scale factors are harmless. Since

$$l_y - d - \frac{11}{6}\cdot\frac{1}{8} \ge \frac{1}{12},$$

the Gram bound reduces to

$$\sum_m |\mathcal{A}_m(Y')|^2 \ll (1+|\nu|)^{J_{\mathrm{Gr}}}(Q_0/Y')P_a^{1/6}Z^\varepsilon. \tag{6.23}$$

Cauchy–Schwarz in the same row box and the rapid decay of the separating Fourier density bound this tuple's separation by

$$Q_0^{-1/2}\{(Q_0/Y')P_a^{1/6}\}^{1/2}Z^{M'/2+\varepsilon} \ll (X')^{1/2}P_a^{1/12}Z^\varepsilon.$$

There are at most $Z^{d+\varepsilon}$ rescaled tuples, their coefficient has size $O(Z^{-3d/2})$, and $(X')^{1/2} \asymp X^{1/2}Z^{-d/2}$. Their extra exponent is $d - 3d/2 - d/2 = -d \le 0$. Summing the finitely many subsets gives

$$|I_{\psi,\mathrm{mod}}(Z)| \ll Z^{l_x/2+b_0/12+\varepsilon} = Z^{3/16+\varepsilon}.$$

Divide by (6.18), decreasing the preliminary loss. □

6.6 The high decomposition

We record the precise physical estimates used in the row argument. Their real parameters are chosen before the later finite-order target; their finite height degrees may depend on that target.

Proposition 6.3. *Suppose $B > 7/8$. The fully compensated probe has the row decomposition, zero witnesses, and local-error estimates of [16, Sections 8–10 and 16], with ideal norms and the family (2.2). More explicitly:*

(i) *The detector uses $a \in [51/100, B]$, with $\delta = 2a-1 \in [1/50, 2B-1]$. Its saturated witnesses have $r \le t + O(\varepsilon)$, $r + m \ge t - O(\varepsilon)$, and $0 \le m \le 1/2 + O(\varepsilon)$ for any fixed $t \in [1, 3/2]$. Every live Hecke denominator belongs to $\mathcal{F}_\omega$.*

(ii) *If a moderate row class of norm $U = Z^d$ has cardinality $\ll U^{R+\varepsilon}(1+T)^A$ and aggregate prime-amplitude parameter q, its exponent relative to $C(7/8) = 3/16$ is at most*

$$E(d; R, q) = -\frac{1}{48} + \frac{2}{3}\delta + \frac{q}{6} - h(1-R) + (d-h)\left(R + \frac{\delta}{2} - \frac{17}{50}\right), \tag{6.24}$$

up to the allocated bin, contour, and correction losses. Both selected and unselected prime slots remain in the full correction.

(iii) *The floor bin, small rows, and the absolute large-row contour have the estimates used in Section 7. The principal w and z contours have respective margins $23/960$ and $13/9600$, measured against $C(B)$. The principal slot approximation has margin μ_0.*

All assertions allow arbitrarily high external tail orders after their internal profile and height degrees have been fixed.

Proof. The detector is [16, Section 8], with ordinary Hecke continuation and the fixed-type functional equation. The gamma estimates in Lemma 3.3 give its finite-seminorm height bounds. Its argument allows $a \le 1$; the endpoint $B = 1$ is also included in [16, Lemma 10.5]. The conductor of a sextic row is bounded by its squarefree radical with fixed local factors. Thus the conductor powers in the witness construction are unchanged when element norms are replaced by ideal norms. The live-background verification in Section 4 puts every companion denominator under the same boundary B.

For the local correction use (6.13) and the full ramified Gauss table. The cancellation producing (6.10) is a local identity for every good prime. The remaining phase factors have modulus one. Consequently the two Euler regions and joint error-slot estimates of [16, Section 16] apply with the same powers of Q. On their dynamic contours, for a fixed positive allowance e, the unmarked corrections satisfy

$$|H_{u,\psi,\mathfrak{p}} - 1| \le \begin{cases} CQ^{-10e}, & \mathfrak{p} \mid u, \\ CQ^{-1-10e}, & \mathfrak{p} \nmid u. \end{cases}$$

Choose the usual finite cutoff $P_0(e)$ before splitting these factors. It concerns the nonprincipal dynamic correction and is fully compatible with Lemma 6.1. All masks generated by the local table are retained in the common coefficients of Theorems 4.1 and 4.2. Finite sector multiplicities and the ideal divisor bound change constants and small losses only. This proves the transfer of the full error decomposition, including slots of zero amplitude.

Inserting a row cardinality into the norm-form contour estimate gives (6.24); it is the scale identity in [16, equation (20.4)], with l_x, l_y, ℓ, h from (6.7). The term $q/6$ is measured in base Z. The outer-row comparisons are [16, Lemma 10.6, equations (10.21)–(10.25)]. The principal contour estimates are [16, Lemma 10.5, equations (10.17)–(10.19)], with the numerical substitution in [16, equation (20.3)]. On the principal absolute contour one can use $G_\mathfrak{p} = -H_\mathfrak{p} + O(Q^{-7/8})$ and $H_\mathfrak{p} = 1 + O(Q^{-363/200})$. The selected factors are bounded, the unselected product converges absolutely, and summing each prime window gives the full bound $\ll Z^{\ell \operatorname{Re} z}$.

Finally the initial common contour has $s = B + 20e$ and $w = 1 - a - 6e$, so $\mathrm{Re}(s + w) \geq 1 + (B - a) + 14e$. The final dynamic region still has margin $10e$. Fixed-degree norm-sheet integration uses only the finite-seminorm bounds already proved, and arbitrary integration-by-parts orders give the external tails. □

7 Row counting and the high exponent

Throughout this section we use the common boundary B in (2.3) and assume $B > 7/8$. Thus

$$\Delta = B - \frac{7}{8} \in (0, 1/8], \qquad \kappa = 2B - 1 = \frac{3}{4} + 2\Delta \in (3/4, 1].$$

The fixed geometric constants are

$$h = \frac{13}{16}, \qquad \ell = \frac{1}{6}, \qquad \alpha = \frac{5}{6}, \qquad \delta_0 = \frac{1}{50}, \qquad d_{\min} = \frac{1}{16}. \tag{7.1}$$

Fix $0 < \zeta < 1/48$. A moderate dyadic row range has

$$U = Z^d, \qquad d_{\min} \leq d \leq h + \zeta.$$

Write the physical prime scales as

$$P_i = Z^{\ell_i} = U^{w_i}, \qquad w_i = \frac{\ell_i}{d}, \qquad \ell_i > 0, \qquad \sum_{i=1}^{J} \ell_i = \ell.$$

The slot system is fixed independently of Z; its underlying prime supports are disjoint before any common mask or row zero extension is imposed. Its mesh will be chosen below.

7.1 The analytic inputs and the pointwise subdivisions

We state explicitly the inputs used in the counting argument. A live detector bin has

$$a > \frac{51}{100}, \qquad \delta = 2a - 1 \in (\delta_0, \kappa].$$

The floor bin has $a = 51/100$ and is treated by the total row count. Proposition 6.3, using the buffered detector and saturated witnesses of [16, Lemmas 8.1–8.2 and Proposition 8.3], has the following consequences. For every $1 \leq t \leq 3/2$ and every sufficiently small prescribed loss $\varepsilon_{\mathrm{w}} > 0$, each row in a live bin has centrally normalized inverse and plain witnesses $M_{u,r}$ and $S_{u,m}$, for the same row character, such that

$$r \leq t + O(1/\log U), \qquad r + m \geq t - O(1/\log U), \qquad 0 \leq m \leq \tfrac{1}{2} + O(\varepsilon_{\mathrm{w}}), \tag{7.2}$$

$$|M_{u,r}|^2 \gg U^{\delta r - \varepsilon_{\mathrm{w}}}, \qquad |S_{u,m}|^2 \gg U^{\delta m - \varepsilon_{\mathrm{w}}}. \tag{7.3}$$

Their lengths are U^r and U^m. There are finitely many character presentations and $O((\log U)^2)$ dyadic length pairs. On each such subdivision, their remaining rowwise choices are smooth parameters of common annular profiles. In particular, the inverse base profile has the form

$$W_{\mathrm{base}}(y) = W_1(y) V_{\leq}(U^r y / U^t),$$

whose fixed logarithmic seminorms are bounded uniformly in the displayed scales. Taking the allowed suprema in the smooth parameters costs only a fixed power of $1+T$, where T bounds the retained heights. All height assertions are subject to the single cumulative detector allowance. The inequalities in (7.2) also give $r \geq t - \frac{1}{2} - O(\varepsilon_{\mathrm{w}})$.

The required moment estimates, in row base U, are precisely

$$\sum_u \Big| M_{u,r} \prod_{i\in I} Q_{u,i} \Big|^2 \ll U^{1+\varepsilon_{\mathrm{m}}}(1+T)^A, \qquad r+2z<1, \quad 2r+8z<3, \tag{7.4}$$

$$\sum_u |S_{u,m}|^4 \prod_{i\in I} |Q_{u,i}|^2 \ll U^{1+\varepsilon_{\mathrm{m}}}(1+T)^A, \qquad 2m+6\kappa z \le 1 \quad (z>0), \tag{7.5}$$

$$\sum_u |M_{u,r}|^2 \ll U^{e(r)+\varepsilon_{\mathrm{m}}}(1+T)^A, \qquad e(r)=\max\{1,(1+5r)/6\}. \tag{7.6}$$

Here $z=\sum_{i\in I} w_i$, the two strict inequalities in (7.4) have fixed positive margins, and the selected slot lengths in (7.5) are below its prescribed mesh. Every positive plain slot is the complete smooth sum over the good primes in its annulus, with fixed finite-character coefficients and only the permitted common masks and original row zero extensions.

The marked inverse estimate admits its larger class of fixed bounded coefficients and prime lists; the present application uses complete annuli for both moments. At $z=0$, (7.5) holds without a restriction on the bounded nonnegative length m; its actual inducing character must be nonprincipal after a global norm twist is separated. For $z>0$, the finite inducing character must lie outside the fixed finite family Θ. These are Theorems 4.1 and 4.2, and Lemma 5.1, respectively. The exponent in (7.6) is uniform as r approaches one. The inverse estimates allow their full row families; the plain estimate is summed over the indicated admissible family. Positivity of the moment sums makes each estimate valid on every subset of its row family.

For clarity, the coefficient requirement is preserved when these estimates are applied to physical slots. On a fixed presentation write the common witness character as $\lambda_u=\omega_0\xi_u$, with ω_0 fixed. Each physical prime factor can be written, with a fixed smooth profile V_i, as

$$Q_{u,i}=P_i^{-1/2}\sum_{\mathfrak{p}\notin S}\lambda_u(\mathfrak{p})\,\omega_0(\mathfrak{p})^{-1}b_i(\mathfrak{p})V_i(\mathrm{N}\,\mathfrak{p}/P_i)=P_i^{-1/2}\sum_{\mathfrak{p}\notin S}\xi_u(\mathfrak{p})b_i(\mathfrak{p})V_i(\mathrm{N}\,\mathfrak{p}/P_i). \tag{7.7}$$

The b_i are fixed finite combinations of the allowed finite characters. Thus the actual prime characters remain finite, and the relative coefficients are independent of the moving row. All reciprocity phases, common masks and original zero extensions remain in the displayed factors. If an orientation change is needed, conjugate the whole product, including both copies of the plain witness, all selected prime factors and their profiles. This changes neither its absolute square nor the relevant exceptional-family condition. For a retained sixth-power-free physical row, an exceptional finite inducing character would force every prime outside the fixed set S to have valuation divisible by six. The exceptional physical rows therefore have bounded norm. Apart from the principal row, which is treated separately, they belong to the small-row range below.

The fourth-moment mesh is uniform for $3/4\le\kappa\le 1$, independently of the eventual fixed slot count; see Proposition 4.14 and [16, Lemma 18.1]. If $\kappa<1$, its boundary premise holds because $(1+\kappa)/2=B$ is the common family boundary. At $\kappa=1$ its prime estimate uses absolute counting and requires no zero-free premise.

We next specify the amplitude sets, including their zero bins. Fix the external physical parameters and a subset of error slots only after moving the original fixed row sum to its retained central contours. The shared detector includes the finitely many actual prime-character presentations in (7.7) and their conjugates, as well as the witness presentations; they all belong to the common family defining B. The prime estimate in a detector bin gives, for any prescribed $\varepsilon_{\mathrm{p}}>0$,

$$|Q_{u,i}| \ll U^{\varepsilon_{\mathrm{p}}}P_i^{\delta/2+O(e)} \tag{7.8}$$

on every main slot; this is the annular logarithmic-derivative estimate of [16, Lemma 19.1]. Choose an amplitude width $\vartheta>0$. After the slot system is fixed, e and ε_{p} may be reduced so

that the right side of (7.8) is at most $P_i^{\delta/2+\vartheta}$, for all sufficiently large Z and every slot. For a main slot with $Q_{u,i} \neq 0$, set

$$g_i = \min\left\{\frac{\delta}{2},\, \max\left\{0,\, \vartheta\left\lfloor \frac{\log|Q_{u,i}|}{\vartheta \log P_i}\right\rfloor\right\}\right\}.$$

Set $g_i = 0$ when $Q_{u,i} = 0$, and also for every error slot. Then, on a main slot,

$$|Q_{u,i}| \le P_i^{g_i+\vartheta}, \qquad g_i > 0 \implies |Q_{u,i}| \ge P_i^{g_i}. \tag{7.9}$$

The finitely many vectors (g_i) partition the rows pointwise. On one such set define

$$q = \frac{\sum_i \ell_i g_i}{\ell}, \qquad 0 \le q \le \frac{\delta}{2}. \tag{7.10}$$

Errors are included with their assigned zero amplitudes in this average. Each pointwise set is further subdivided by the witness presentation and dyadic lengths. Within the resulting row sum, the arithmetic coefficients, the amplitude vector and any selected slot indices are common. These pointwise sets bound the already continued integrand.

Lemma 7.1 (Selection of positive amplitudes)**.** *For $0 \le z \le \ell/d$, a subcollection of whole main slots with positive amplitude has total length at most z and satisfies*

$$\prod_{i \text{ selected}} |Q_{u,i}|^2 \ge U^{\,2qz - \delta \max_i w_i}. \tag{7.11}$$

If the requested capacity is first reduced by a fixed $\nu > 0$, the total additional loss in the exponent is at most $2q\nu + \delta \max_i w_i$, whenever $z > \nu$.

Proof. Order the positive g_i decreasingly and fill a length z fractionally in that order. If they fill z, their initial weighted average is at least the average q of all slots. If they do not, their entire gain is $q\ell/d \ge qz$. At most one terminal slot is fractional. Deleting it loses at most $(\delta/2)\max_i w_i$ in the unsquared exponent. Equation (7.9) proves (7.11). Replacing z by $z - \nu$ loses a further $2q\nu$. Only positive amplitudes have been used, so the argument also applies when some or all main factors vanish. □

7.2 A row bound throughout the moderate range

Proposition 7.2. *Given $\zeta \in (0, 1/48)$ and a requested loss $\varepsilon > 0$, a sufficiently fine fixed physical mesh has the following property. For every moderate dyadic range, every retained amplitude set and each witness subdivision of a live detector bin, one has*

$$\#\mathcal{B} \ll U^{R+\varepsilon}(1+T)^A. \tag{7.12}$$

The floor bin admits $R = 1$. For $\alpha \le \delta \le \kappa \le 1$, one may take

$$R = 1 - \delta. \tag{7.13}$$

For $\delta_0 < \delta \le \alpha$, put

$$y = \frac{1}{2} - \frac{q}{\delta}, \qquad j = 185 + (144y - 138)\delta, \qquad t = 1 + \frac{6(7+12y)\delta}{j}, \tag{7.14}$$

and define

$$\overline{R} = 1 - \delta + \frac{(5-6\delta)(7+12y)\delta}{j}. \tag{7.15}$$

Then $70 \le j \le 185$, $1 \le t \le 3/2$, *and*

$$R = \overline{R} + \frac{\Delta}{4} \tag{7.16}$$

is admissible.

The requested loss and the physical mesh can be fixed independently of the later target twist, uniformly over the displayed ranges of δ, q, κ, d. *For each fixed arithmetic datum and slot system, the implied constant, the finite height order* A, *and the lower threshold for* Z *are fixed. The remaining detector and amplitude losses are then chosen sufficiently small, and the retained heights must satisfy the common detector allowance.*

Proof. We first suppress the prescribed witness, moment and rounding losses, and restore them using fixed margins.

For $\delta \ge \alpha$, take $t = 3/2$. The saturated witness lengths give $r \ge 1 - O(\varepsilon_{\mathrm{w}})$. Dividing (7.6) by the inverse spike uses the exact identity

$$e(r) - \delta r = 1 - \delta + \delta(1-r)_+ + (\alpha - \delta)(r-1)_+.$$

Since $\alpha - \delta \le 0$, this is at most $1 - \delta + O(\varepsilon_{\mathrm{w}})$, proving (7.13). This unmarked estimate applies also when $B = 1$. The floor bin uses the total row count $O(U)$.

Now suppose $\delta_0 < \delta < \alpha$, and write $x = q/\delta \in [0, 1/2]$. If $r \ge 1$, the same amplified moment, together with $r \le t + O(\varepsilon_{\mathrm{w}})$, gives

$$R \le L(t) := 1 - \delta + (\alpha - \delta)(t-1). \tag{7.17}$$

If $m \ge 1/2$, the zero-slot plain moment gives $R \le 1 - \delta$. It remains to treat $r < 1$, $m < 1/2$, when both prospective capacities are positive.

The inverse capacity is $z_I = (1-r)/2$. The actual plain capacity is

$$z_P = \frac{1-2m}{6\kappa} = \frac{1-2m}{9/2 + 12\Delta}. \tag{7.18}$$

First compare the inverse count with the plain count at the baseline capacity $2(1-2m)/9$. Division by the individual spikes in (7.3) and the squared prime spike gives the two affine comparison lines

$$A(r) = 1 - \delta[x + (1-x)r], \qquad S_t(r) = 1 - \delta\{4x/9 + (2 - 8x/9)(t-r)\}.$$

The replacement of m by $t - r$ is legitimate up to $O(\varepsilon_{\mathrm{w}})$: both the actual and baseline plain exponents decrease with m, since their derivatives are

$$-2\delta + \frac{4q}{9/2 + 12\Delta} < 0, \qquad -2\delta + \frac{8q}{9} < 0.$$

Choose the inverse estimate when $A(r) \le S_t(r)$, and the plain estimate otherwise. The slopes of A and S_t have opposite signs. Moreover,

$$\begin{aligned} A(3/5) - S_t(3/5) &\ge \delta(1/5 - 14x/45) \ge 2\delta/45 > 0, \\ S_t(t - 1/3) - A(t - 1/3) &= \delta\{(1-x)(t-1) + 5x/27\} \ge 0. \end{aligned}$$

Consequently an inverse selection occurs only for $r \ge 3/5$, while a plain selection occurs only for $r \le t - 1/3$, hence $m \ge 1/3 - O(\varepsilon_{\mathrm{w}})$. In particular,

$$z_I \le \frac{1}{5}, \qquad z_P \le \frac{2}{27} + O(\varepsilon_{\mathrm{w}}). \tag{7.19}$$

Reduce a chosen positive capacity by a fixed small $\nu > 0$. On the inverse side this gives the strict width margins

$$1 - r - 2z \geq 2\nu, \qquad 3 - 2r - 8z = 4(1 - r - 2z) + (2r - 1) \geq 8\nu + \frac{1}{5}.$$

The plain capacity is reduced as well. If $z_I \leq \nu$, use the unmarked inverse count

$$1 - \delta r \leq 1 - \delta + 2\delta\nu.$$

If $z_P \leq \nu$, use the unmarked plain count

$$1 - 2\delta m \leq 1 - \delta + 6\delta\nu,$$

because $1 - 2m \leq 6\kappa\nu \leq 6\nu$.

There is a fixed supply margin for every moderate d:

$$\frac{\ell}{d} \geq \frac{\ell}{h + \zeta} > \frac{1}{5}. \tag{7.20}$$

Lemma 7.1 therefore supplies every requested capacity in (7.19). Its exponent loss is at most

$$2q\nu + \delta \max_i w_i \leq \nu + \frac{\max_i \ell_i}{d_{\min}}.$$

Choose ν within the requested loss, then require $\max_i \ell_i / d_{\min} = 16 \max_i \ell_i$ to be smaller than the rounding allowance and the uniform fourth-moment mesh. The moment contains the selected slots, of total length at most $1/5$. The bounded effective length ranges used to choose its mesh therefore remain fixed independently of the physical slot count.

Replacing the baseline plain capacity by (7.18) increases the count exponent by

$$2q(1 - 2m)\left(\frac{2}{9} - \frac{1}{9/2 + 12\Delta}\right) = \frac{24q(1 - 2m)\Delta}{(9/2)(9/2 + 12\Delta)} \leq \frac{\Delta}{4} + O(\varepsilon_{\mathrm{w}}). \tag{7.21}$$

Here $m \geq 1/3 - O(\varepsilon_{\mathrm{w}})$, $2q \leq 1$, and each denominator is at least four. The estimate is uniform for $0 \leq \Delta \leq 1/8$.

The smaller comparison line is bounded above by the weighted average

$$\frac{(2 - 8x/9)A(r) + (1 - x)S_t(r)}{3 - 17x/9} = 1 - \delta + \delta p(x)(3/2 - t), \qquad p(x) = \frac{(2 - 8x/9)(1 - x)}{3 - 17x/9}.$$

Its r-coefficient is zero. On $0 \leq x \leq 1/2$, use the linear majorant

$$p(x) \leq g(x) := \frac{26 - 24x}{37}, \qquad g(x) - p(x) = \frac{4(1 - 2x)(9 - 14x)}{37(27 - 17x)} \geq 0. \tag{7.22}$$

Put $L_0 = \alpha - \delta + \delta g(x) > 0$ and choose

$$t = 1 + \frac{\delta g(x)}{2L_0}.$$

Then $1 \leq t \leq 3/2$ and

$$\delta g(x)(3/2 - t) = (\alpha - \delta)(t - 1).$$

The short count and the long count (7.17) are therefore both bounded by

$$1 - \delta + \frac{(\alpha - \delta)\delta g(x)}{2(\alpha - \delta + \delta g(x))} + \frac{\Delta}{4},$$

apart from the requested losses. Substitution of $y = 1/2 - x$ gives (7.14)–(7.16), with $j = 222L_0$. The bounds $70 \le j \le 185$ follow directly from $0 \le y \le 1/2$, $0 \le \delta \le 5/6$. At $\delta = \alpha$, the displayed formulas have $t = 3/2$ and $\overline{R} = 1 - \delta$; the already proved unmarked case gives the stronger bound (7.13).

To restore the losses, first fix the moment loss and ν, then the physical mesh and a fixed slot system. The supply gap and the inverse widths above are fixed, and the fourth-moment mesh is uniform for $\kappa \in [3/4, 1]$. Next reduce the witness, detector and amplitude losses. The inequalities $\delta \ge 1/50$ and $j \ge 70$ make all length perturbations and rational expressions uniform; the near-zero capacities have already been covered by unmarked moments. The finitely many amplitude vectors and the $O((\log U)^2)$ witness choices cost at most a further arbitrarily small power. The permitted common-profile calculus supplies the fixed height factor in (7.12). This proves the proposition with the stated order of choices. □

7.3 A fixed numerical margin

In the present geometry,

$$C(s) = s - \frac{11}{16}, \qquad C(7/8) = \frac{3}{16}, \qquad C(B) - C(7/8) = \Delta.$$

For the compensated physical expression, the exponent associated with a pointwise row count $U^{R+\varepsilon}$ is given by (6.24). Relative to $C(7/8)$, it is

$$E(d; R, q) = -\frac{1}{48} + \frac{2\delta}{3} + \frac{q}{6} - h(1 - R) + (d - h)\left(R + \frac{\delta}{2} - \frac{17}{50}\right). \tag{7.23}$$

Thus the central contribution of the original fixed bin is bounded using the supremum of E over its pointwise sets, with the prescribed real losses and a fixed polynomial height factor. In particular, $q/6 = q\ell$ is a base-Z exponent.

Lemma 7.3. *Let $c_* = 49/301920$. Before the adjustable real losses, every moderate bin has an admissible row exponent from Proposition 7.2 such that*

$$E(d; R, q) - \Delta \le -c_* + \zeta. \tag{7.24}$$

This is uniform for $B \in (7/8, 1]$.

Proof. For $0 \le \delta \le 5/6$ and $0 \le y \le 1/2$, let $\overline{E} = E(h; \overline{R}, (1/2 - y)\delta)$, with j and $\overline{R}$ as in (7.14)–(7.15). Expansion gives the exact identity

$$\boxed{1632j(-\overline{E}) = [(204 + 170y)\delta - 79]^2 + 49 + 68y\delta[37 + (48 + 151y)\delta].} \tag{7.25}$$

Every term on the right is nonnegative, and $j \le 185$. Hence

$$-\overline{E} \ge \frac{49}{1632 \cdot 185} = c_* > \frac{1}{10000}.$$

At $d = h$, the capacity cost $\Delta/4$ increases E by $h\Delta/4$. Therefore

$$E(h; \overline{R} + \Delta/4, q) - \Delta \le -c_* - \frac{51}{64}\Delta. \tag{7.26}$$

For $\alpha \le \delta \le 1$, choose $R = 1 - \delta$ and use $q \le \delta/2$. Then

$$E(h; 1 - \delta, q) \le -\frac{1}{48} - \frac{\delta}{16} \le -\frac{7}{96}.$$

In the floor bin, $R = 1$, $\delta = \delta_0$ and $q \le \delta_0/2$ give

$$E(h; 1, q) \le -\frac{1}{48} + \frac{3\delta_0}{4} = -\frac{7}{1200}.$$

It remains to extend these bounds in d. Since $0 < g(x) \le 1$, the selected exponent satisfies

$$1 - \delta \le \overline{R} \le 1 - \delta/2.$$

Using $\Delta/4 \le 1/32$, all the upper exponents just used, including the upper-δ and floor cases, obey

$$\frac{4}{25} \le R + \frac{\delta}{2} - \frac{17}{50} < 1. \tag{7.27}$$

The estimates at h therefore hold for every smaller moderate d, and their extension to $h + \zeta$ costs at most ζ. Both $7/96$ and $7/1200$ exceed c_*, proving (7.24). □

7.4 The two outer ranges

The outer ranges use the full correction before its central main/error decomposition. Let $\mathcal{R}(U; Z)$ denote the contribution of nonprincipal physical rows $U \le \mathrm{N}\, u < 2U$, and let $\mathcal{H}_{u,Z}$ denote their full correction. The quotient-free bounds in the physical expansion are, with $z_0 = 17/50$,

$$|\mathcal{H}_{u,Z}(B + e + it,\, 1/2 + iv,\, z_0 + iw)| \ll Z^{\ell z_0 + \varepsilon} (\mathrm{N}\, u)^{\varepsilon} (1 + |t| + |v| + |w|)^A, \tag{7.28}$$

$$|\mathcal{H}_{u,Z}(2 + it,\, 2 + iv,\, z_\infty + iw)| \ll Z^{\ell z_\infty + \varepsilon} (\mathrm{N}\, u)^{\varepsilon} (1 + |t| + |v| + |w|)^A, \qquad z_\infty > 2. \tag{7.29}$$

These bounds retain every local factor and are valid at its zeros. The contours can thus be shifted for each original dyadic sum.

On the first lines the reciprocal remains on the common global line $\mathrm{Re}\, s = B + e$. The nonprincipal numerator contributes at most $U^{3/5+\varepsilon}$ times a fixed height polynomial. Counting the rows and including their factor $(\mathrm{N}\, u)^{-z_0}$ gives the row power

$$1 + \frac{3}{5} - \frac{17}{50} = \frac{63}{50}.$$

The Z-power outside this row count is $C(B) + h(z_0 - 1/6) - l_y/2 + e$, where $l_y = 23/48$. This proves the outer-row estimate [16, Lemma 10.6] in the present normalization:

$$|\mathcal{R}(U; Z)| \ll Z^{C(B) - 79/800 + e + \varepsilon} U^{63/50 + \varepsilon}. \tag{7.30}$$

The fixed tests integrate the height polynomial. Summing the dyads $1 \le U \le Z^{d_{\min}}$, and decreasing the preliminary powers, gives

$$\sum_{U \le Z^{1/16}} |\mathcal{R}(U; Z)| \ll Z^{C(B) - 1/50 + e + \varepsilon}, \qquad -\frac{79}{800} + \frac{63}{50}\frac{1}{16} = -\frac{1}{50}. \tag{7.31}$$

This estimate includes bounded exceptional rows with nonprincipal numerator. It also applies at $B = 1$, when the reciprocal line is in the region of absolute convergence.

On the absolute lines in (7.29), the scalar L-factors are absolutely bounded. With

$$B_0 = \frac{l_x}{2} + 1 + l_y = \frac{53}{32}, \qquad l_x = \frac{17}{48},$$

the same dyadic calculation gives

$$|\mathcal{R}(U; Z)| \ll Z^{B_0 + h z_\infty + \varepsilon} U^{1 + \varepsilon - z_\infty}. \tag{7.32}$$

Choose the explicit fixed line $z_\infty = 1+3/\zeta$. For $\varepsilon < z_\infty - 1$, geometric summation over $U > Z^{h+\zeta}$ yields

$$\sum_{U>Z^{h+\zeta}} |\mathcal{R}(U;Z)| \ll Z^{B_0+(h+\zeta)(1+\varepsilon)-\zeta z_\infty+\varepsilon} = Z^{-17/32+(1+h+\zeta)\varepsilon}. \tag{7.33}$$

As $C(B) \geq 3/16$, its saving relative to $C(B)$ is at least $23/32 - O(\varepsilon)$.

Equations (7.24), (7.31) and (7.33) cover every nonprincipal physical row. Choose $\zeta < c_*/4$, then choose the requested row-count loss and its uniform moment mesh within fixed fractions of c_*. After fixing the slot system, choose the remaining real detector, amplitude and correction losses so that their total exponent cost is less than $c_*/4$. This leaves a fixed positive nonprincipal saving, independent of B and of the later target. The finite height orders and constants may depend on that target and the fixed slot system. Their absorption, the principal remainder and the common normalization are handled in the final choice of heights.

8 Mellin continuation

8.1 Real parameters and heights

The choices are made in the following order. Steps 1–5 determine a common real saving; Steps 6–9 specialize the estimates to a target.

1. **Field and infinity types.** Fix K and the three types $0, \omega_\infty, \bar{\omega}_\infty$. The final excluded set S is chosen at Step 6, after the target's finite conductor is known.
2. **Geometric margins.** Fix $c = 10^{-4}$, $d_{\min} = 1/16$, and $0 < \zeta < \min\{1/96, c/16\}$. The certificate and the outer-row estimates in Section 7 leave the required positive reserves.
3. **Moment losses and mesh.** Choose the moment exponent losses and capacity decrements within small fixed fractions of c. Proposition 4.14 then gives a prime-slot mesh, uniform on $3/4 \leq \kappa \leq 1$. These choices precede the number of slots because the capacity-error coefficients depend on total lengths.
4. **The finite slot system.** Take $\ell_i = 1/(6J_0)$ and choose J_0 so that $16 \max_i \ell_i$ is smaller than both the mesh and the rounding reserve. Fix nonnegative nonzero tests supported in the disjoint intervals
$$\left(1 + \frac{2i-1}{2J_0+1}, 1 + \frac{2i}{2J_0+1}\right), \qquad 1 \leq i \leq J_0.$$
This fixes the analytic slot system and $\mu_0 = 7/(96J_0) > 0$. The low exponent for this system is the one in Proposition 6.2.
5. **Remaining real losses.** Choose the amplitude bins, detector width, contour shifts, and local correction losses. They may depend on J_0 and are taken within the remaining nonprincipal reserve and the principal margins $23/960$, $13/9600$, and μ_0. The principal losses are chosen separately from the earlier moment losses. Set $z_\infty = 1 + 3/\zeta$. Reserve a common number $m_* > 0$ for the final height estimate.
6. **Target and arithmetic data.** Choose the later finite-order twist and hence a target in $\mathcal{F}_\omega$. Remove its norm twist and use the permitted conjugate, writing ψ for that representative. Fix its finite sectors and final S, including the cutoff $P_0(e)$. This same S is used in the auxiliary function, residue scalar, and Mellin signal.
7. **Internal analytic orders.** Fix the internal seminorm and Sobolev orders and the resulting finite height degrees A_ψ, B_ψ. Propositions 4.14 and 6.3 give, in a positive target-dependent

height range,

$$|J_\psi(Z) - f_\psi(Z)| \ll_{\psi,N} Z^{C(B)-2m_*}(1+T)^{A_\psi} + Z^{B_\psi}T^{-N}. \tag{8.1}$$

The degrees in this estimate are fixed before the external tail order N.

8. **Height and external tail.** Choose $\tau_\psi > 0$ within that height range and with $\tau_\psi \le m_*/[2(A_\psi+1)]$, and set $T = Z^{\tau_\psi}$. Then choose N with $N\tau_\psi > B_\psi - 3/16 + m_*$.
9. **Constants and thresholds.** Choose the implied constants and a lower threshold for Z for these fixed data and orders. They may depend on the target. The common real number m_* was fixed at Step 5.

Since $C(B) \ge 3/16$, both terms in (8.1) are now $\ll_\psi Z^{C(B)-m_*}$. Thus

$$|J_\psi(Z) - f_\psi(Z)| \ll_\psi Z^{C(B)-\eta_0}, \qquad \eta_0 = m_* > 0. \tag{8.2}$$

The saving is common to the later targets and independent of $\Delta = B - 7/8$. Conjugation and vertical translation restore the original target after its chosen representative has been treated.

8.2 Family closure

We use the common-signal continuation principle of [16, Proposition 2.1]. The formulation below retains the maximum of the low exponent and the fixed high-side exponent, and applies it to the fixed-type family of this paper.

Lemma 8.1. *Suppose (8.2) holds for every representative of the fixed family with a common $\eta_0 > 0$, and the same fixed probe satisfies (6.21) for every $\varepsilon > 0$. Then every member of the family is zero-free in*

$$\operatorname{Re} s > \max\{7/8, B - \eta_0\}. \tag{8.3}$$

Consequently $B \le 7/8$.

Proof. Combining the two estimates gives, for every $\varepsilon > 0$,

$$|f_\psi(Z)| \ll_{\psi,\varepsilon} Z^{C(7/8)+\varepsilon} + Z^{C(B)-\eta_0}.$$

As $Z \downarrow 0$, moving the defining line in (6.19) arbitrarily far to the right gives decay faster than any fixed power. The Gaussian supplies absolute vertical convergence on each such line. Therefore

$$F_\psi(s) = \int_0^\infty f_\psi(Z) Z^{-C(s)} \frac{dZ}{Z} \tag{8.4}$$

is holomorphic in (8.3). To verify local uniform convergence, choose ε smaller than the distance of a given compact set from $\operatorname{Re} s = 7/8$.

Mellin inversion on an initial absolute line identifies (8.4) with

$$e^{(s-5/6)^2} \frac{H_\psi^S(s)}{L_K^S(s,\psi)}.$$

The Gaussian and H_ψ^S are nonzero in the asserted half-plane, by Lemma 6.1. A zero of $L_K^S(s,\psi)$ would create a pole, contrary to holomorphy. At a norm-character pole the reciprocal is holomorphic and vanishes. Deleted Euler factors are nonzero in $\operatorname{Re} s > 0$.

If $B > 7/8$, the number $\max\{7/8, B - \eta_0\}$ is strictly smaller than B. By the definition (2.3), some member of $\mathcal{F}_\omega$ has a zero strictly to its right. This contradicts (8.3) and proves $B \le 7/8$. □

Proof of Theorem 1.2. Given F and ψ, take $K = F(\zeta_{12})$ and $\omega = \psi \circ \mathrm{N}_{K/F}$. The arithmetic, reflection, and canonical arguments above apply over K. If its common boundary exceeded $7/8$, the row estimates and the fixed choices just made would give (6.21) and (8.2) for the same probes. Lemma 8.1 rules this out. In particular $L_K(s, \omega)$ is zero-free for $\operatorname{Re} s > 7/8$. Lemma 2.1 gives the assertion over F, including its possible norm-character pole and any additionally deleted Euler factors. □

Remark 8.2. The maximum in (8.3) is necessary when $B - 7/8 < \eta_0$: the low estimate still contributes $Z^{C(7/8)+\varepsilon}$.

9 Weil representations and applications

We deduce Theorem 1.1 and its applications from Theorem 1.2 by unitary Brauer induction and pole counting.

9.1 Unitary induction and Euler factors

We use the classical global Weil group, with its local maps and its finite-index subgroups W_E for finite extensions E/F, as in [14, Chapters 1–2] and [11, Section 3.1]. A finite-dimensional continuous representation factors through a relative Weil group $W_{K/F}$ for some finite Galois extension K/F; this is the no-small-subgroups argument recorded in [11, proof of Proposition 7]. We retain the norm convention from the introduction:

$$\mathcal{N}_F|_{W_E} = \mathcal{N}_E, \qquad \nu_{F,t}|_{W_E} = \nu_{E,t}. \tag{9.1}$$

Indeed a Frobenius at a prime of residue degree f maps to the fth power of the downstairs Frobenius, and its norm is $(\mathrm{N}\,\mathfrak{p})^f$.

Lemma 9.1 (Unitary Brauer induction)**.** *Let r be a finite-dimensional continuous unitary representation of W_F. There are finite extensions E_j/F, continuous unitary Hecke characters ψ_j of E_j, and integers n_j such that*

$$[r] = \sum_{j=1}^{m} n_j \left[\operatorname{Ind}_{W_{E_j}}^{W_F} \psi_j\right]. \tag{9.2}$$

This is an equality in the Grothendieck group of semisimple representations.

Proof. Apply [14, Lemma 2.2] to a relative Weil group through which r factors. Every Hecke quasicharacter has a unique expression $\psi = \psi^0 \mathcal{N}_E^a$, where ψ^0 is unitary and $a \in \mathbb{R}$: its absolute value is trivial on the compact norm-one idele class group and therefore is a real power of the norm. The tensor-induction identity [14, Lemma 2.3] and (9.1) give

$$\operatorname{Ind}_{W_E}^{W_F} \psi = \left(\operatorname{Ind}_{W_E}^{W_F} \psi^0\right) \otimes \mathcal{N}_F^a.$$

Induction from a finite-index subgroup preserves unitarity: use the sum of the invariant inner products on a set of coset representatives. Thus every irreducible constituent of $\operatorname{Ind} \psi^0$ is unitary. If $a \neq 0$, none of its twists by $\mathcal{N}_F^a$ is unitarizable. To see this, choose g with $\mathcal{N}_F(g) > 1$; all eigenvalues of such a twist at g have absolute value $\mathcal{N}_F(g)^a \neq 1$.

The Grothendieck group is freely generated by irreducible classes. Project its underlying abelian group onto the classes of the unitarizable irreducibles. This projection fixes $[r]$ and discards all summands with $a \neq 0$. The remaining inducing characters have $a = 0$, which proves the required refinement of Langlands's lemma. □

The standard induction identity for local L-factors, including ramified primes, gives

$$L_F\Big(s, \operatorname{Ind}_{W_E}^{W_F} \psi\Big) = L_E(s, \psi). \tag{9.3}$$

See [14, Chapter 20, pp. 211–213]. One can read the finite-place calculation directly on inertia invariants: Frobenius permutes the cosets in cycles, and a cycle of residue degree f contributes the determinant $\det(1 - \psi(\mathrm{Frob}_{\mathfrak{P}})q_{\mathfrak{p}}^{-fs} \mid V_{\psi}^{I_{\mathfrak{P}}})$. It is 1 when these invariants vanish. Since $q_{\mathfrak{P}} = q_{\mathfrak{p}}^{f}$, its reciprocal is the upstairs Euler factor. Consequently (9.2) implies

$$L_F(s, r) = \prod_{j=1}^{m} L_{E_j}(s, \psi_j)^{n_j}. \tag{9.4}$$

This identity first holds in the absolute half-plane and then defines the usual meromorphic continuation.

If S is a finite set of finite primes of F and S_E is the set above it, induction also gives $L_F^S(s, \operatorname{Ind}\psi) = L_E^{S_E}(s, \psi)$. Moreover the eigenvalues of Frobenius on the inertia invariants of any unitary representation have absolute value one. Thus

$$L_F^S(s, r) = L_F(s, r) \prod_{\mathfrak{p} \in S} \det\Big(1 - r(\mathrm{Frob}_{\mathfrak{p}})q_{\mathfrak{p}}^{-s} \mid V_r^{I_{\mathfrak{p}}}\Big) \tag{9.5}$$

is multiplication by a holomorphic nowhere-zero function on $\operatorname{Re} s > 0$. Deleting these factors changes neither a zero nor a pole in our half-plane.

9.2 The pole divisor

Proposition 9.2 (Transfer of a common Hecke half-plane)**.** *Let $0 < \beta < 1$. Suppose every continuous unitary Hecke character over every number field has a zero-free L-function on $\operatorname{Re} s > \beta$. Then every finite-dimensional continuous unitary Weil representation r has no zeros there. Away from the points $1 + it$ it is holomorphic, and at those points*

$$\operatorname{ord}_{1+it} L_F(s, r) = -\dim \operatorname{Hom}_{W_F}(\nu_{F,t}, r).$$

The same assertion holds for $L_F^S(s, r)$.

Proof. Use (9.4) with the unitary inducing characters of Lemma 9.1. Each factor is zero-free in the specified half-plane. By the Hecke pole classification [19], its only possible pole is a simple pole at $1 + it$, and this occurs precisely when $\psi_j = \nu_{E_j,t}$. At all other points every factor and its reciprocal are holomorphic and nonzero.

At $1+it$ take meromorphic orders. The multiplicity of a fixed irreducible character is additive on the Grothendieck group of semisimple representations, and Frobenius reciprocity gives

$$\begin{aligned} \operatorname{ord}_{1+it} L_F(s, r) &= -\sum_j n_j \mathbf{1}_{\{\psi_j = \nu_{E_j,t}\}} \\ &= -\sum_j n_j \dim \operatorname{Hom}_{W_{E_j}}(\nu_{F,t}|_{W_{E_j}}, \psi_j) \\ &= -\dim \operatorname{Hom}_{W_F}(\nu_{F,t}, r). \end{aligned} \tag{9.6}$$

The last dimension is nonnegative, so the displayed order is nonpositive. The deleted-factor assertion follows from (9.5). □

Proof of Theorem 1.1. Apply Proposition 9.2 with $\beta = 7/8$ and Theorem 1.2. □

Only finitely many t have $m_t(r) > 0$, and their multiplicities sum to at most $\dim r$. An equivalent form of the result is that

$$L_F^S(s,r) \prod_{t:m_t(r)>0} (s-1-it)^{m_t(r)} \tag{9.7}$$

is holomorphic and nowhere zero on $\operatorname{Re} s > 7/8$.

Remark 9.3 (Completed L-functions). The completed Weil L-function has the same zeros and poles as its finite part on $\operatorname{Re} s > 0$. Indeed the unitary archimedean factors are products of $\Gamma_{\mathbb{R}}(s+\epsilon+it)$, with $\epsilon \in \{0,1\}$, and $\Gamma_{\mathbb{C}}(s+a+it)$, with $a \geq 0$; they and any conductor exponential are holomorphic and nonzero there. This follows from the local formulas of [14, Chapter 20].

9.3 Artin twists and weight-one forms

Corollary 9.4. *Let $\rho : G_F \longrightarrow \mathrm{GL}_d(\mathbb{C})$ be a continuous finite-image representation. Then $L_F(s,\rho)$ is zero-free on $\operatorname{Re} s > 7/8$, and its only possible pole there is at $s = 1$, of order $\dim \rho^{G_F}$. In particular every nontrivial irreducible Artin L-function is holomorphic and nowhere zero in this half-plane. For any continuous unitary Hecke character ψ of F, Theorem 1.1 also applies to $\rho \otimes \psi$, with pole multiplicities $\dim \operatorname{Hom}_{W_F}(\nu_{F,t}, \rho \otimes \psi)$.*

Proof. Average an inner product over the finite image of ρ and pull the representation back to W_F. No $\nu_{F,t}$ with $t \neq 0$ has finite image, so only the trivial character can contribute a pole to the untwisted representation. Tensoring by a unitary character preserves unitarity, and the general assertion follows. □

The finite-image assertion needs only the finite-order subcase of Theorem 1.2 over all number fields: ordinary Brauer induction uses finite-order characters. The infinite-order twists and the general Weil statement use its full Hecke scope. More explicitly, if $[\rho] = \sum_j n_j [\operatorname{Ind}_{G_{E_j}}^{G_F} \chi_j]$ is a Brauer decomposition, then the tensor-induction identity gives

$$L_F(s, \rho \otimes \psi) = \prod_j L_{E_j}\Big(s, \chi_j(\psi \circ \mathrm{N}_{E_j/F})\Big)^{n_j}. \tag{9.8}$$

The source of the induction theorem is [2]; the Weil and L-function compatibilities are those of [14, Lemma 2.3 and Chapter 20]. If ρ is irreducible of dimension at least two, any such twist has no one-dimensional constituent and is holomorphic as well as nonzero on the asserted half-plane.

Corollary 9.5. *For every normalized cuspidal weight-one newform f over $\mathbb{Q}$, its standard L-function is holomorphic and nowhere zero for $\operatorname{Re} s > 7/8$.*

Proof. Deligne–Serre attach a finite-image irreducible representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ and prove $L(s,f) = L_{\mathbb{Q}}(s,\rho_f)$, including the bad finite Euler factors [4, Theorems 4.1 and 4.6]. Apply Corollary 9.4. □

9.4 Effective Chebotarev

Let L/F be a finite Galois extension with group G, and let C be one conjugacy class. Write $c_C = |C|/|G|$, $n_L = [L:\mathbb{Q}]$, $D_L = |\operatorname{disc}(L)|$, and $\mathcal{D}_L = \log(eD_L) + n_L$. Let $\pi_C(x)$ count the unramified prime ideals $\mathfrak{p}$ of F with $q_{\mathfrak{p}} \leq x$ and $\mathrm{Frob}_{\mathfrak{p}} \in C$.

Corollary 9.6 (Prime ideals with a prescribed Frobenius class)**.** *For every $0 < \varepsilon < 1/8$ and $x \geq 2$,*

$$\pi_C(x) = c_C \operatorname{Li}(x) + O_\varepsilon\Big(c_C \mathcal{D}_L x^{7/8+\varepsilon}\Big). \tag{9.9}$$

The implied constant is effectively computable and independent of L, F, C. There is an unramified prime ideal in the class C with

$$q_{\mathfrak{p}} \ll \mathcal{D}_L^8, \tag{9.10}$$

where the implied constant is absolute and effectively computable.

Proof. We use the classical cyclic reduction and the Hecke explicit formula, in the form recorded in [21, Section 3, Lemmas 4–5 and Proposition 6]. Here are the estimates with the zero boundary supplied by Theorem 1.2.

Choose $g \in C$, put $H = \langle g \rangle$ and $E = L^H$, and regard $\chi \in \widehat{H}$ as a finite-order Hecke character of E. Finite-group induction gives $\mathbf{1}_C = c_C \sum_{\chi \in \widehat{H}} \bar{\chi}(g) \operatorname{Ind}_H^G \chi$. Thus the associated logarithmic derivative is

$$D_C(s) = -c_C \sum_{\chi \in \widehat{H}} \bar{\chi}(g) \frac{L_E'}{L_E}(s, \chi). \tag{9.11}$$

Its coefficient at an unramified prime power is $\mathbf{1}_C(\mathrm{Frob}_{\mathfrak{p}}^m) \log q_{\mathfrak{p}}$. At a ramified prime it is the corresponding inertia average, of absolute value at most $\log q_{\mathfrak{p}}$. For $A(\chi) = D_E \mathrm{N}\mathfrak{f}_\chi$, conductor–discriminant and $[L : \mathbb{Q}] = |H|[E : \mathbb{Q}]$ give $\sum_\chi \log A(\chi) = \log D_L$ and $\sum_\chi [E : \mathbb{Q}] = n_L$.

All nontrivial zeros of these Hecke functions have real part at most $7/8$. The local zero count and logarithmic-derivative expansion in [21, Lemma 5, equations (3.18) and (3.21)] therefore give, uniformly for $7/8 + \varepsilon/2 \le \sigma \le 2$,

$$D_C(s) - \frac{c_C}{s-1} \ll_\varepsilon c_C(\log D_L + n_L \log(|t| + 3)), \qquad s = \sigma + it. \tag{9.12}$$

Apply truncated Perron summation at a half-integer x and height $T = x^2$, and move the line to $7/8 + \varepsilon/2$. The Dirichlet coefficient at the integer n is bounded in absolute value by $[F : \mathbb{Q}]\Lambda(n)$, so the truncation error is $O([F : \mathbb{Q}] \log^2(3x))$. The only crossed pole contributes $c_C x$. The new vertical integral is $O_\varepsilon(c_C \mathcal{D}_L x^{7/8+\varepsilon/2} \log^2(3x))$; the horizontal integrals are smaller. Replacing x by the adjacent half-integer preserves the prime-power sum and changes the main term by $O(1)$. Absorbing the logarithms gives the same estimate with exponent $7/8 + \varepsilon$.

The contribution of powers $m \ge 2$ is $O([F : \mathbb{Q}]x^{1/2} \log^2(3x))$. At a ramified prime, the discriminant exponent is at least $|G|/2$, and hence $\sum_{\mathfrak{p}\,\mathrm{ram}} \log q_{\mathfrak{p}} \le 2 \log D_L/|G|$. Their prime-power contribution is $O(|G|^{-1} \log D_L \log(3x))$. Both fit the stated error since $[F : \mathbb{Q}] = n_L/|G| \le c_C n_L$. After removing these terms, partial summation proves (9.9).

For the least-prime bound, fix a nonnegative nonzero $W \in C_c^\infty((1, 2))$ once and for all and write $a_W = \widehat{W}(1) > 0$. Mellin inversion of (9.11), followed by a shift to $\operatorname{Re} s = -1/4$, gives the annular explicit formula

$$\sum_{\mathfrak{p}, m \ge 1} \vartheta_C(\mathfrak{p}^m) \log q_{\mathfrak{p}} W(q_{\mathfrak{p}}^m/x) = c_C a_W x - c_C \sum_{\chi \in \widehat{H}} \bar{\chi}(g) \sum_{\rho_\chi} x^{\rho_\chi} \widehat{W}(\rho_\chi) + O_W(c_C \mathcal{D}_L). \tag{9.13}$$

Here ϑ_C is the inertia-average coefficient just defined, and ρ_χ runs over nontrivial zeros with multiplicity. This is the contour argument of [21, Proposition 6] with the entire Mellin transform of an annular test. The residue at zero has size $O_W([E : \mathbb{Q}])$ for each character, and the functional equation bounds the remaining line by $O_W(x^{-1/4}(\log A(\chi) + [E : \mathbb{Q}]))$.

Uniformly for $0 \le \operatorname{Re} \rho \le 1$, $\widehat{W}(\rho) \ll_W (1 + |\Im \rho|)^{-2}$. The local zero count therefore implies $\sum_{\rho_\chi} |\widehat{W}(\rho_\chi)| \ll_W \log A(\chi) + [E : \mathbb{Q}]$. The zero sum in (9.13) is consequently $O_W(c_C \mathcal{D}_L x^{7/8})$. Removing ramified primes and higher powers as above, using $x^{1/2} \log^2(3x) \ll x^{7/8}$, gives

$$\sum_{\substack{\mathfrak{p}\ \text{unramified} \\ \mathrm{Frob}_{\mathfrak{p}} \in C}} \log q_{\mathfrak{p}}\, W(q_{\mathfrak{p}}/x) = c_C a_W x + O_W(c_C \mathcal{D}_L x^{7/8}).$$

For x a sufficiently large absolute multiple of $\mathcal{D}_L^8$ this sum is positive. Some required prime then has norm between x and $2x$, proving (9.10). All constants in these contour and zero-count estimates are effective; the weight is fixed. □

9.5 Geometric and dihedral consequences

Remark 9.7 (Pure potentially abelian realizations). Let V be a semisimple geometric ℓ-adic representation over F whose restriction to some finite extension is a sum of characters. Here geometric means unramified outside finitely many primes and de Rham above ℓ. Fix compatible coefficient embeddings, and suppose its good Euler roots are pure of weight $w \in \mathbb{Z}$, with $|\alpha_{\mathfrak{p},j}| = q_{\mathfrak{p}}^{w/2}$ for every complex embedding. For S containing the ramified primes and those above ℓ, the algebraic Weil correspondence gives a parameter R with these good factors [11, Theorem 9 and Corollary 10]. Its normalization $r_0 = R \otimes \mathcal{N}_F^{-w/2}$ is unitary: after a finite extension, purity fixes the absolute value of every character to be $\mathcal{N}^{w/2}$, and an invariant inner product descends by averaging over the finite cosets [11, Lemma 14(1)]. Consequently

$$L_F^S(s, V) = L_F^S(s - w/2, r_0), \qquad L_F^S(s, V) \neq 0 \quad (\operatorname{Re} s > w/2 + 7/8). \tag{9.14}$$

The only possible pole there is at $1+w/2$, with order $\dim \operatorname{Hom}_{W_F}(\mathcal{N}_F^{w/2}, R)$. Indeed an algebraic norm constituent $\mathcal{N}_F^{w/2+it}$ has trivial angle at a complex local Weil subgroup, so its two integral exponents agree; this forces $t = 0$ and w even. In particular the function is holomorphic in this half-plane for odd w.

This includes pure CM motives in the category generated by CM abelian varieties and Artin motives using Hodge cycles. Their full local Weil realization gives the same assertion for the complete L-function [6, Corollaries 7.6, 7.9 and Proposition 7.12]. For a motive over F, apply that realization over $\mathbb{Q}$ to its additive restriction of scalars: its realizations are induced, and $L_{\mathbb{Q}}(s, \operatorname{Res}_{F/\mathbb{Q}} M) = L_F(s, M)$ by local induction. Exterior and symmetric powers, tensor products, duals, Tate twists, Artin twists, and direct summands preserve potential abelianity. For CM abelian varieties this yields boundaries $j/2+7/8$ for $H^j(A)$, $m/2+7/8$ for $\operatorname{Sym}^m H^1(A)$, and $15/8$ for $H^1(A) \otimes H^1(B)$.

Remark 9.8 (Dihedral decompositions). Let K/F be quadratic with automorphism c and character $\epsilon = \epsilon_{K/F}$. For a unitary Hecke character θ put $r = \operatorname{Ind}_{W_K}^{W_F} \theta$ and $\delta = \det r$. Index-two induction gives

$$\operatorname{Ad} r \simeq \epsilon \oplus \operatorname{Ind}(\theta/\theta^c), \qquad \operatorname{Ind}\theta_1 \otimes \operatorname{Ind}\theta_2 \simeq \operatorname{Ind}(\theta_1\theta_2) \oplus \operatorname{Ind}(\theta_1\theta_2^c),$$

and

$$\operatorname{Sym}^m r \simeq \bigoplus_{0 \le j < m/2} \operatorname{Ind}_{W_K}^{W_F}(\theta^{m-j}(\theta^c)^j) \oplus \begin{cases} 0, & m \text{ odd}, \\ (\delta\epsilon)^{m/2}, & m \text{ even}. \end{cases} \tag{9.15}$$

The tensor identity is [14, Lemma 2.3]. To check the middle line in the symmetric power, choose $\tau \notin W_K$ and a basis with $r(\tau) = \left(\begin{smallmatrix} 0 & a \\ 1 & 0 \end{smallmatrix}\right)$, $a = \theta(\tau^2)$. The monomials of indices j and $m-j$ form the induced pairs; for $m = 2k$ the middle monomial has τ-value a^k, equal to $(\delta\epsilon)^k(\tau)$. The adjoint decomposition follows from the diagonal trace-zero line and the two off-diagonal lines.

Local induction converts all these decompositions into Hecke factorizations, so Theorem 1.2 applies to every factor. For an induced CM elliptic parameter with $\delta = 1$ and $\theta^c = \theta^{-1}$, for example, $L_F(s, \operatorname{Sym}^4 r) = L_K(s, \theta^4) L_K(s, \theta^2) \zeta_F(s)$. The two Hecke characters retain nonzero angular weights, and the remaining factor gives a simple pole at $s = 1$; in the arithmetic variable it lies at $s = 3$.

AI provenance

The manuscript text and mathematical arguments were developed predominantly with OpenAI language models. The analytic core in Sections 3–8 builds on the AI-generated preprints [16, 17].

A Transfer of the source arguments

The table separates the arithmetic or analytic input that changes with the field from the norm calculation used in the source induction. All ideals in the replacement column are exterior ideals. The set S contains the fixed local data and generators of the class group; $R = \mathcal{O}_{K,S}$ and $\Gamma = (R^\times)^6$.

Table 1: Source inputs, field replacements, and preserved calculations.

Source step	Field-sensitive input	Replacement and reason the exponent is preserved	Location here
Generators and arithmetic indexing; Q; P, §3	Primary generators in the Eisenstein ring and its finite unit group (Q); cyclotomic fields containing μ_{12} (P).	Localize to make R principal. Allowed generators are unique modulo Γ, and residual unit classes form a finite set. Each ideal is counted with its displayed multiplicity; the finite sector count changes only the constant.	Proposition 3.1; start of Section 3.
Gauss calibration and reciprocity; P, Lemma 3.4	Cubic reciprocity, Gauss phases, and the primary-generator convention.	Hilbert reciprocity on an isotropic complement supplies allowed reciprocity. Norm pullback of the cubic Jacobi character and the Weil index supply Λ_K, G. The CRT correlations use the same local powers of $q_{\mathfrak{p}}$.	Proposition 3.1; (4.6).
Orbital Poisson and complete correlations; Q, §§13, 17–18	Additive lattice shape, finite units, and common support of the moduli.	Integrate on the compact quotient T_0/Γ and balance generators. The multiplier is the row scale divided by the lattice index, and the dual length is the conductor norm divided by that scale. Higher dimension changes derivative orders; norm-sheet logarithms enter the finite propagation bound.	Lemmas 3.6–3.7; Lemma 4.12.
Theta and marked reflection; Q, §14; P, §§5–6	The finite-background theta expansion and its archimedean test.	A character-weighted norm-sheet period gives the fixed-type Gamma multiplier. Local mark digits separate the row, mark, and dual coefficients. Their remaining interactions are the quadratic and cubic symbols in (3.28); these preserve the source conductor powers and collision masks.	Proposition 3.2; Theorem 3.5.
Reflected norm and short inverse terminal; Q, §§14, 17	Quadratic and cubic sieve families, powerful rows, and local dual data.	Fixed ray-class splitting identifies the sieve families. Ideal factorization gives the same powerful-row count. Finite-S frequencies are included in the common profile. Combining its squared decay with the explicit local weights gives (4.20) and the three terminal bounds (4.28).	Lemmas 4.5–4.8; Proposition 4.10.

Continued on the next page

Table 1 continued

Source step	Field-sensitive input	Replacement and reason the exponent is preserved	Location here
Two inverse transforms and initialization; Q, §17	Complete Gauss correlations, primary-generator fibres, and finite backgrounds in the two transforms.	The local tables and CRT are unchanged after replacing prime sizes by norms. Allowed-generator fibres have divisor-bounded multiplicity and finitely many unit sectors. The fixed background stays on the columns. The child margins and row decrease are the affine norm identities (4.31)–(4.32); the prefactor identity (4.33) counts each outer measure once.	Proposition 4.10; proof of Theorem 4.1.
Ordinary reflection and the prime premise; Q, §18.2	Rank-one functional equation, fixed gamma factors, and finite characters in the prime sums.	Hecke reflection has dual norm length Q/X for the actual conductor Q. The paired coefficient cancels the infinite-order background at primes. Its small annular terms are summable, including the fixed logarithmic factors. Thus the capacity remains $n_1 + n_2 + 6\kappa z \le m + q$.	Lemma 3.3; (4.3); Section 4.6.
Fourth-moment recursion and exceptional rows; Q, §§18.2–18.8	Finite two-column correlations, the prime pool, exceptional rows, and common-product cancellation.	Ideal valuations give the local inequality for each pair of multiplicities and the same count of sixth-power classes. The prime ideal theorem supplies the pool with its single density gain. A common masked Mellin residue cancels on equal-product rectangles. The resulting child width is (4.43); all remaining losses are positive parts of the displayed log-norm expressions.	Lemma 4.11; Section 4.8; Proposition 4.14.
Physical estimates and Euler compensation; Q, §§5–16; P, §7	Additive Gram estimate, theta terminal, and local Euler algebra.	The Gram proof uses the common two-column orbital kernel. The complete marked reflection supplies its row energy. Local compensation is a rational identity in unit-modulus character values, so it gives the same physical norm monomial and row exponents.	Proposition 3.8; Section 6.
Exponent comparison and closure; Q, §§19–20 and Proposition 2.1	Numerical lengths and the common zero boundary.	The comparison is field-independent algebra. The linear envelope and (7.25) give the fixed positive reserve; the Mellin transform uses the same normalized function on both sides.	Section 7; Lemma 8.1.

The rows on the two inverse transforms and on the fourth-moment recursion invoke the recursive arguments of [16, §§17–18] by locator rather than reproducing them. Checking the transfer amounts to confirming, against those sections, that the inputs in the second column are the only field- or type-dependent ones.

The two recursion rows use three concrete inputs: the complete local correlation, one common smooth kernel, and the explicitly counted ideal fibres. Once those have been supplied, the child lengths, capacities, and normalization factors are identities among ideal norms. The finite-depth estimate in Lemma 4.12 accounts for the additional logarithms of the unit-quotient kernels.

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