
Non-Archimedean Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality

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Abstract: In 1992, Hudzik and Landes [*Math. Ann.*] derived a breakthrough generalization of the triangle inequality for two nonzero elements in normed linear spaces, which was generalized to finitely many nonzero elements independently in 2006 by Dragomir [*Bull. Aust. Math. Soc.*] and in 2007 by Kato, Saito and Tamura [*Math. Inequal. Appl.*]. We derive a non-Archimedean version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura inequality.

Keywords: Normed linear space, Triangle inequality, Non-Archimedean linear space, Ultra-norm.

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1. INTRODUCTION

Let $\mathcal{X}$ be a normed linear space (NLS). From the definition of the norm, we have the triangle inequality

$$(1) \quad \|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathcal{X}.$$

In 1992, Hudzik and Landes derived a breakthrough generalization of Inequality (1) which is valid for any two nonzero elements in a NLS [2].

Theorem 1.1. [2] (*Hudzik-Landes Inequality*) Let $\mathcal{X}$ be a NLS. Then for all $x, y \in \mathcal{X} \setminus \{0\}$,

$$(2) \quad \|x + y\| \leq \|x\| + \|y\| - \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \min\{\|x\|, \|y\|\}.$$

We note that, in 2006, Maligranda independently derived Inequality (2) [4]. It is natural to ask for a generalization of Inequality (2) to more than two non-zero vectors. This is done independently by Dragomir in 2006 [1] and by Kato, Saito and Tamura in 2007 [3].

Theorem 1.2. [1, 3] (*Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality*) Let $\mathcal{X}$ be a NLS and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$, we have

$$\left\| \sum_{j=1}^n x_j \right\| \leq \sum_{j=1}^n \|x_j\| - \left(n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq k \leq n} \|x_k\|.$$

It is natural and important to ask what are non-Archimedean versions of Theorems 1.1 and 1.2? We answer the question by deriving non-Archimedean version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality (Theorem 2.2).

2. NON-ARCHIMEDEAN HUDZIK-LANDES-DRAGOMIR-KATO-SAITO-TAMURA INEQUALITY

Let $\mathbb{K}$ be a field. Recall that a map $|\cdot| : \mathbb{K} \rightarrow [0, \infty)$ is said to be a non-Archimedean valuation if following conditions holds.

- (i) If $\lambda \in \mathbb{K}$ is such that $|\lambda| = 0$, then $\lambda = 0$.
- (ii) $|\lambda\mu| = |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{K}$.
- (iii) (Ultra-triangle inequality) $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.

In this case, $\mathbb{K}$ is called as non-Archimedean valued field [6]. Let $\mathcal{X}$ be a vector space over a non-Archimedean valued field $\mathbb{K}$ with valuation $|\cdot|$. Recall that a map $\|\cdot\| : \mathcal{X} \rightarrow [0, \infty)$ is said to be a non-Archimedean norm if following conditions holds.

- (i) If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
- (ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}$, for all $x \in \mathcal{X}$.
- (iii) (Ultra-norm inequality) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{X}$.

In this case, $\mathcal{X}$ is called as non-Archimedean linear space (NALS) [5]. We first derive non-Archimedean version of Inequality (2).

Theorem 2.1. (Non-Archimedean Hudzik-Landes Inequality) *Let $\mathcal{X}$ be a NALS over $\mathbb{K}$. Then for all $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{K}$ it holds*

$$\|x + y\| \leq \min \left\{ \|x\| \max \left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right\}, \|y\| \max \left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|x\| \right\} \right\}.$$

Proof. Let $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{K}$. Then

$$\begin{aligned} \|x + y\| &= \left\| \|x\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) + \left(1 - \frac{\|x\|}{\|y\|} \right) y \right\| \\ &\leq \max \left\{ \left\| \|x\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\|, \left\| \left(1 - \frac{\|x\|}{\|y\|} \right) y \right\| \right\} \\ &= \max \left\{ \|x\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \|x\| \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right\} \end{aligned}$$

and

$$\begin{aligned} \|x + y\| &= \left\| \left(1 - \frac{\|y\|}{\|x\|} \right) x + \|y\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\| \\ &\leq \max \left\{ \left\| \left(1 - \frac{\|y\|}{\|x\|} \right) x \right\|, \left\| \|y\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\| \right\} \\ &\leq \max \left\{ \|y\| \left| \frac{1}{\|y\|} - \frac{1}{\|x\|} \right| \|x\|, \|y\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right\}. \end{aligned}$$

Therefore

$$(3) \quad \|x + y\| \leq \|x\| \max \left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right\}$$

and

$$(4) \quad \|x + y\| \leq \|y\| \max \left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|x\| \right\}.$$

Inequalities (3) and (4) give

$$\|x + y\| \leq \min \left\{ \|x\| \max \left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right\}, \|y\| \max \left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|x\| \right\} \right\}.$$

□

Note the additional assumption $\|x\|, \|y\| \in \mathbb{K}$ in the previous theorem. The reason is that, since the norm is a real number, we generally do not have a guarantee that it belongs to the given non-Archimedean field. Now we derive non-Archimedean version of Theorem 1.2.

Theorem 2.2. (*Non-Archimedean Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality*)

Let $\mathcal{X}$ be a NALS over $\mathbb{K}$ and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{K}$ it holds

$$\left\| \sum_{j=1}^n x_j \right\| \leq \min_{1 \leq k \leq n} \left\{ \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \max_{1 \leq j \leq n} \left| \frac{1}{\|x_j\|} - \frac{1}{\|x_k\|} \right| \|x_j\| \right\} \right\}.$$

Proof. Let $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{K}$. Let $1 \leq k \leq n$ be fixed. Then

$$\begin{aligned} \left\| \sum_{j=1}^n x_j \right\| &= \left\| \sum_{j=1}^n \frac{\|x_k\| x_j}{\|x_j\|} + \sum_{j=1}^n \left(1 - \frac{\|x_k\|}{\|x_j\|} \right) x_j \right\| \\ &\leq \max \left\{ \left\| \sum_{j=1}^n \frac{\|x_k\| x_j}{\|x_j\|} \right\|, \left\| \sum_{j=1}^n \left(1 - \frac{\|x_k\|}{\|x_j\|} \right) x_j \right\| \right\} \\ &= \max \left\{ \|x_k\| \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \|x_k\| \left\| \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right\} \\ &= \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \left\| \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right\} \\ &\leq \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \max_{1 \leq j \leq n} \left\| \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right\} \\ &= \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \max_{1 \leq j \leq n} \left| \frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right| \|x_j\| \right\}. \end{aligned}$$

By varying k and taking minimum in the right side of previous inequality gives

$$\left\| \sum_{j=1}^n x_j \right\| \leq \min_{1 \leq k \leq n} \left\{ \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \max_{1 \leq j \leq n} \left| \frac{1}{\|x_j\|} - \frac{1}{\|x_k\|} \right| \|x_j\| \right\} \right\}.$$

□

Now we derive continuous version of Theorem 2.2.

Theorem 2.3. Let $\mathcal{X}$ be a NALS over $\mathbb{K}$ and (Ω, μ) be a non-Archimedean measure space. Let $\phi : \Omega \rightarrow \mathcal{X} \setminus \{0\}$ be a measurable function such that $\|\phi(\alpha)\| \in \mathbb{K}$ for every $\alpha \in \Omega$. Then

$$\left\| \int_{\Omega} \phi(\alpha) d\mu(\alpha) \right\| \leq \inf_{\beta \in \Omega} \left\{ \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \sup_{\alpha \in \Omega} \left| \frac{1}{\|\phi(\alpha)\|} - \frac{1}{\|\phi(\beta)\|} \right| \|\phi(\alpha)\| \right\} \right\}.$$

Proof. Let $\phi : \Omega \rightarrow \mathcal{X} \setminus \{0\}$ be a measurable function such that $\|\phi(\alpha)\| \in \mathbb{K}$ for every $\alpha \in \Omega$. Let $\beta \in \Omega$ be fixed. Then

$$\begin{aligned}
 \left\| \int_{\Omega} \phi(\alpha) d\mu(\alpha) \right\| &= \left\| \int_{\Omega} \frac{\|\phi(\beta)\|\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) + \int_{\Omega} \left(1 - \frac{\|\phi(\beta)\|}{\|\phi(\alpha)\|}\right) \phi(\alpha) d\mu(\alpha) \right\| \\
 &\leq \max \left\{ \left\| \int_{\Omega} \frac{\|\phi(\beta)\|\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \left\| \int_{\Omega} \left(1 - \frac{\|\phi(\beta)\|}{\|\phi(\alpha)\|}\right) \phi(\alpha) d\mu(\alpha) \right\| \right\} \\
 &= \max \left\{ \|\phi(\beta)\| \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \left\| \int_{\Omega} \left(\frac{1}{\|\phi(\beta)\|} - \frac{1}{\|\phi(\alpha)\|} \right) \phi(\alpha) d\mu(\alpha) \right\| \right\} \\
 &= \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \left\| \int_{\Omega} \left(\frac{1}{\|\phi(\beta)\|} - \frac{1}{\|\phi(\alpha)\|} \right) \phi(\alpha) d\mu(\alpha) \right\| \right\} \\
 &\leq \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \sup_{\alpha \in \Omega} \left\| \left(\frac{1}{\|\phi(\beta)\|} - \frac{1}{\|\phi(\alpha)\|} \right) \phi(\alpha) \right\| \right\} \\
 &= \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \sup_{\alpha \in \Omega} \left| \frac{1}{\|\phi(\beta)\|} - \frac{1}{\|\phi(\alpha)\|} \right| \|\phi(\alpha)\| \right\}.
 \end{aligned}$$

By varying β and taking infimum in the previous inequality gives

$$\left\| \int_{\Omega} \phi(\alpha) d\mu(\alpha) \right\| \leq \inf_{\beta \in \Omega} \left\{ \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \sup_{\alpha \in \Omega} \left| \frac{1}{\|\phi(\alpha)\|} - \frac{1}{\|\phi(\beta)\|} \right| \|\phi(\alpha)\| \right\} \right\}.$$

□

3. CONCLUSIONS

- (1) In 1992, Hudzik and Landes improved centuries old triangle inequality in normed linear spaces [2].
- (2) In 2006, Dragomir extended Hudzik-Landes inequality for more than two vectors [1].
- (3) In 2007, Kato, Saito and Tamura extended Hudzik-Landes inequality without knowing the work of Dragomir [3].
- (4) In this article, we extended centuries old ultra-norm inequality.

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