

HILBERT FUNCTIONS OF CHOPPED IDEALS OF GENERAL PLANE POINTS

ABSTRACT. Let Z be a finite set of general points in the projective plane over a field of characteristic zero. We determine the Hilbert function of the ideal generated by the degree- d equations of Z , for every d . This proves the plane case of the Expected Syzygy Conjecture. The proof realizes the quotient of the point ideal by its degree- d equations as the cokernel of a linear block in a Hilbert–Burch matrix, and applies natural cohomology for general plane Steiner bundles.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero, and put

$$S = k[x, y, z], \quad N_t = \begin{cases} \binom{t+2}{2} & t \geq 0, \\ 0 & t < 0. \end{cases}$$

For a finite set $Z \subset \mathbb{P}^2$, write $I(Z)$ for its saturated homogeneous ideal. Following Gesmundo, Kayser, and Telen [3], the ideal

$$J = (I(Z)_d)$$

is called the *chopped ideal in degree d* . Its Hilbert function measures which higher-degree equations of the points are generated by the equations of degree d .

The Expected Syzygy Conjecture of Gesmundo, Kayser, and Telen predicts this Hilbert function for general points. We prove its plane case. The main input is the natural-cohomology theorem for general plane Steiner bundles, in the form recorded by Coskun, Huizenga, and Smith [1, Remark 1.5]. To apply it, we construct a family of Hilbert–Burch matrices with independent quadratic and linear blocks, and show that this family contains reduced point ideals. The quotient $I(Z)/J$ is then presented by the linear block itself.

Here *general* means that the assertion holds on a nonempty Zariski open subset of the space of ordered, pairwise distinct points. All degrees in the following theorem are covered simultaneously on one such open subset. The descent to arbitrary characteristic-zero fields is explained at the end of the proof.

Theorem 1.1. *Let $d \geq 2$, let $Z \subset \mathbb{P}^2$ consist of $p \geq 1$ general points, and set $I = I(Z)$ and $J = (I_d)$. In degrees $t < d$ one has $\dim_k(S/J)_t = N_t$. The remaining Hilbert function is as follows.*

- (1) *If $p \geq N_d$, then $J = 0$.*
- (2) *If $p \leq N_{d-1}$, then $J_t = I_t$ for every $t \geq d$; in particular, $\dim_k(S/J)_t = p$ in those degrees.*
- (3) *Suppose that $N_{d-1} < p < N_d$, and put $r = N_d - p$, so that $1 \leq r \leq d$. For $r = 1$ or $r = 2$, the ideal J is generated by a regular sequence of r forms of degree d , and*

$$(1.1) \quad \text{Hilb}(S/J; u) = \frac{(1 - u^d)^r}{(1 - u)^3}.$$

For $r \geq 3$ and every $e \geq 0$,

$$(1.2) \quad \begin{aligned} \dim_k(S/J)_{d+e} &= \max\{p, N_{d+e} - rN_e\} \\ &= p + \max\left\{\frac{e}{2}(2d + 3 - 3r - (r - 1)e), 0\right\}. \end{aligned}$$

In the last case, the least positive integer e for which $J_{d+e} = I_{d+e}$ is

$$(1.3) \quad e_0 = \max \left\{ 1, \left\lceil \frac{2d+3-3r}{r-1} \right\rceil \right\} < d,$$

and this equality of graded pieces persists for every $e \geq e_0$.

The main case is proved by constructing point ideals through the Hilbert–Burch theorem. In the range where generators of degree $d+1$ are necessary, the quotient $I/(I_d)$ is presented by a general matrix of linear forms. Its Hilbert function is then determined by the natural cohomology of general Steiner bundles on $\mathbb{P}^2$.

2. PRELIMINARIES

Lemma 2.1. *General p points have Hilbert function $\dim_k(S/I(Z))_t = \min\{N_t, p\}$. More generally, if a reduced set Z of p points satisfies $\dim_k(S/I(Z))_q = p$ for some $q \geq 0$, then $I(Z)$ is generated in degrees at most $q+1$.*

Proof. For each fixed t , evaluation at points spans S_t^* : a form annihilated by every evaluation is zero. One can therefore choose successive distinct points whose evaluation functionals are independent until the rank is $\min\{N_t, p\}$. This rank condition defines a nonempty open subset of the irreducible configuration space. If q is the least integer with $N_q \geq p$, intersect the two opens giving injectivity in degree $q-1$ and surjectivity in degree q (omitting the first condition if $q=0$). Injectivity propagates to lower degrees. Choose a linear form λ nonzero at all the points. Multiplication by λ makes evaluation surjective in every degree above q . This proves the Hilbert function assertion.

For the second assertion, put $R = S/I(Z)$. Multiplication by λ is injective on R , and surjectivity of evaluation in degree q again implies $\dim_k R_t = p$ for all $t \geq q$. Consequently,

$$\dim_k(R/\lambda R)_t = \dim_k R_t - \dim_k R_{t-1} = 0 \quad (t \geq q+1).$$

The image $\bar{I}$ of $I(Z)$ in $S/(\lambda) \simeq k[u, v]$ thus contains every form of degree $q+1$ and is generated in degrees at most $q+1$. Lift homogeneous generators of $\bar{I}$ to $I(Z)$. These lifts generate $I(Z)$: after subtracting their multiples from an element of $I(Z)$, the remainder is λh , and $\lambda h \in I(Z)$ implies $h \in I(Z)$. Induction on degree completes the argument. $\square$

We use the following established result in its matrix formulation.

Proposition 2.2 (Plane Steiner bundles). *Let $a > 0$ and $c - a \geq 2$. For a general $a \times c$ matrix L of linear forms in S , the map*

$$(2.1) \quad L : S_t^c \longrightarrow S_{t+1}^a$$

has maximal rank for every $t \geq 0$.

Deduction from natural cohomology. The general matrix defines a surjection of vector bundles

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^2}^c \xrightarrow{L} \mathcal{O}_{\mathbb{P}^2}(1)^a \longrightarrow 0.$$

Indeed, the locus of scalar matrices of rank below a has codimension $c - a + 1 \geq 3$, and general evaluation avoids this locus on $\mathbb{P}^2$. Its dual is the defining sequence of a Steiner bundle. The general Steiner bundle on $\mathbb{P}^2$ has natural cohomology, meaning that each twist has at most one nonzero cohomology group; the result and this dual formulation are recorded in [1, Remark 1.5]. Serre duality gives the same property for K . Twisting the displayed sequence by $t \geq 0$ identifies the kernel and cokernel of (2.1) with $H^0(K(t))$ and $H^1(K(t))$, respectively. They cannot both be nonzero, which is precisely maximal rank. Only finitely many of these open rank conditions will be needed below. $\square$

3. REDUCED POINT SCHEMES FROM MATRICES

We give the geometric justification that permits the linear matrix in Proposition 2.2 to occur inside a point ideal. All matrix entries in the next lemma vary independently.

Lemma 3.1. *Let $\mathcal{P}$ be the affine space of homogeneous matrices defining maps between fixed direct sums of line bundles on $\mathbb{P}^2$, of ranks c and $c + 1$. Suppose that every matrix entry has positive degree, and that one matrix has a finite projective maximal-minor scheme. Then a general matrix has a finite reduced maximal-minor scheme. Its maximal minors generate a saturated ideal of height two, and its graded resolution is the Hilbert–Burch resolution.*

Proof. The universal maximal-minor scheme is closed in $\mathcal{P} \times \mathbb{P}^2$ and is proper over $\mathcal{P}$. The condition that its fibers have dimension at most zero is open. For a matrix in this nonempty open, the homogeneous maximal-minor ideal has height at least two. The determinantal height bound gives height at most two, so the height is exactly two. In particular, the projective scheme cannot be empty: an empty scheme would give an irrelevant-primary ideal of height three. These standard determinantal facts and the Hilbert–Burch theorem may be found in [2, Chapter 20].

At any fixed point of $\mathbb{P}^2$, evaluation of the coefficients surjects onto all scalar $(c + 1) \times c$ matrices, since the line bundles of the entries are globally generated. Locally on $\mathbb{P}^2$, this evaluation is a smooth morphism from $\mathcal{P} \times \mathbb{P}^2$ to the corresponding matrix bundle. The scalar rank- $(c - 1)$ stratum is smooth of codimension two. Its inverse image is therefore smooth, of dimension $\dim \mathcal{P}$. The locus of rank at most $c - 2$ has codimension six (and is empty when $c = 1$). Its universal inverse image has dimension at most $\dim \mathcal{P} - 4$; its proper projection cannot dominate $\mathcal{P}$. A general fiber consequently meets only the smooth rank- $(c - 1)$ stratum. Generic smoothness in characteristic zero, applied to its dominating components, now shows that a general zero-dimensional fiber is smooth, hence reduced. Components that do not dominate are removed by shrinking the parameter open.

Finally, Hilbert–Burch gives a free resolution of the maximal-minor ideal of length one. Thus its quotient has depth one and dimension one. It has no elements annihilated by a power of the irrelevant ideal, which is exactly saturation of the homogeneous ideal. $\square$

4. THE QUOTIENT PRESENTED BY THE LINEAR BLOCK

Assume $N_{d-1} < p < N_d$, $r = N_d - p \geq 3$, and

$$(4.1) \quad a = d + 2 - 2r > 0, \quad c = d + 1 - r.$$

Then $r + a = c + 1$ and $c - a = r - 1 \geq 2$. Consider the full affine space

$$(4.2) \quad \mathcal{P} = \text{Mat}_{r \times c}(S_2) \times \text{Mat}_{a \times c}(S_1), \quad B = \begin{pmatrix} Q \\ L \end{pmatrix}.$$

Deleting a row of Q gives a maximal minor of degree $2(r - 1) + a = d$; deleting a row of L gives one of degree $2r + a - 1 = d + 1$.

To check the hypothesis of Lemma 3.1, number the rows $0, \dots, c$ and give row i degree δ_i , equal to 2 in the first r rows and 1 in the last a rows. Take the bidiagonal matrix with entries

$$B_{i,i} = x^{\delta_i}, \quad B_{i+1,i} = y^{\delta_{i+1}} \quad (0 \leq i < c),$$

and all other entries zero. Two of its maximal minors are x^{d+1} and y^d , up to sign. Its projective maximal-minor scheme is therefore finite. The lemma shows that a general B defines a reduced point ideal I_B with resolution

$$(4.3) \quad 0 \longrightarrow S(-d-2)^c \xrightarrow{B} S(-d)^r \oplus S(-d-1)^a \longrightarrow I_B \longrightarrow 0.$$

In particular,

$$(4.4) \quad \text{Hilb}(S/I_B; u) = \frac{1 - ru^d - au^{d+1} + cu^{d+2}}{(1-u)^3}.$$

Taking coefficients, using (4.1), gives N_t for $t < d$ and the constant value $N_d - r = p$ for $t \geq d$. Thus the reduced scheme consists of exactly p points and has the required Hilbert function.

Let J_B be generated by the r minors of degree d . Since I_B has no elements of lower degree, $J_B = ((I_B)_d)$. Quotienting the middle term of (4.3) by $S(-d)^r$ gives an actual presentation of the quotient module:

$$(4.5) \quad I_B/J_B \simeq \text{coker}(L : S(-d-2)^c \longrightarrow S(-d-1)^a).$$

Consequently,

$$(4.6) \quad \dim_k(I_B/J_B)_{d+e} = \max\{aN_{e-1} - cN_{e-2}, 0\} = \max\left\{\frac{e}{2}(2d+3-3r-(r-1)e), 0\right\}$$

whenever $L : S_{e-2}^c \rightarrow S_{e-1}^a$ has maximal rank. For $e = 0$ both sides are zero; for $e = 1$ both sides equal a ; and for $e \geq 2$ the first equality follows from Proposition 2.2. The second follows by expanding the binomial coefficients.

It remains to choose one reduced point scheme for all the needed ranks at once. Let e_0 be as in (1.3). Maximal rank for $2 \leq e \leq e_0$ defines a nonempty open in the linear factor of (4.2). Its inverse image and the nonempty reduced-point open intersect because $\mathcal{P}$ is irreducible. For a matrix in their intersection, (4.6) holds through $e = e_0$. The module I_B/J_B is generated in degree $d+1$, by (4.5); its vanishing in degree $d+e_0$ therefore implies vanishing in all higher degrees. This proves (4.6) for every $e \geq 0$ for one reduced set of p points.

5. TRANSFER TO GENERAL POINTS AND THE REMAINING CASES

On the open configuration space of p points with the Hilbert function in Lemma 2.1, the kernels I_d of evaluation form a vector bundle of rank r . For each fixed e , multiplication gives a morphism of vector bundles

$$I_d \otimes_k S_e \longrightarrow S_{d+e}.$$

Its rank is lower semicontinuous. On this space one always has

$$(5.1) \quad \dim_k(S/(I_d))_{d+e} \geq \max\{p, N_{d+e} - rN_e\} :$$

the first bound follows from the quotient onto S/I , and the second from the dimension of the multiplication source. The constructed point scheme attains both bounds in their maximum, by (4.6) and (4.4). Hence general points attain this maximum for $0 \leq e \leq e_0$. Lemma 2.1, with $q = d$, says that their ideal is generated in degrees at most $d+1$; it has no elements below degree d . The quotient $I/(I_d)$ is thus generated in degree $d+1$. Vanishing at $d+e_0$ again propagates to all higher degrees. Only finitely many open conditions have been imposed, proving the simultaneous assertion of the theorem when $a > 0$.

Now assume $a = d+2-2r \leq 0$, still with $r \geq 3$. Put

$$\ell = 2r - d - 2, \quad q = d + 1 - r, \quad \ell + q = r - 1, \quad \ell + 2q = d.$$

Take a general $r \times (r-1)$ matrix having ℓ columns of linear forms and q columns of quadrics. If column j has degree $\epsilon_j \in \{1, 2\}$, the bidiagonal matrix with entries x^{ϵ_j} and y^{ϵ_j} in rows j and $j+1$ has maximal minors x^d and y^d . Lemma 3.1 therefore applies. It gives a reduced point ideal with resolution

$$(5.2) \quad 0 \longrightarrow S(-d-1)^\ell \oplus S(-d-2)^q \longrightarrow S(-d)^r \longrightarrow I_B \longrightarrow 0.$$

The Hilbert series has numerator $1 - ru^d + \ell u^{d+1} + qu^{d+2}$; its coefficients give N_t for $t < d$ and $p = N_d - r$ for $t \geq d$. All its generators have degree d , so $(I_B)_d$ generates I_B . Semicontinuity transfers the equality in degree $d + 1$ to general points, and Lemma 2.1 propagates it. Finally,

$$\frac{e}{2}(2d + 3 - 3r - (r - 1)e) \leq 0 \quad (e \geq 1),$$

since its inner linear factor is decreasing and its value at $e = 1$ is $2a \leq 0$. This proves (1.2) in this range.

For either range, the residual expression is positive at a positive integer e precisely when $e < (2d + 3 - 3r)/(r - 1)$. This gives (1.3). Since $r \geq 3$,

$$\frac{2d + 3 - 3r}{r - 1} \leq d - 3, \quad (r - 1)(d - 3) - (2d + 3 - 3r) = (r - 3)d \geq 0.$$

Thus $e_0 \leq \max\{1, d - 3\} < d$, as claimed.

For completeness we prove the two small-generator cases. When $r = 1$, a nonzero generator is a regular sequence of length one. When $r = 2$, a reduced complete intersection supplies the required configuration. Choose pairwise distinct scalars α_i and β_j and set

$$f = \prod_{i=1}^d (x - \alpha_i z), \quad g = \prod_{j=1}^d (y - \beta_j z).$$

Their common zeros are the d^2 distinct affine grid points $[\alpha_i : \beta_j : 1]$, and they have no common zero at infinity. The pair f, g is regular, and its ideal is saturated, by its complete-intersection resolution and the depth argument in Lemma 3.1. Consequently evaluation of $S_d/(f, g)_d$ on the grid is injective, and this space has dimension $p = N_d - 2 \leq d^2$. Choose p grid points whose evaluation coordinates form a basis of its dual. For this subset Z , one has $I(Z)_d = \langle f, g \rangle$. Moreover $I(Z)_{d-1} = 0$: any nonzero form of degree $d - 1$ would give three independent multiples in the two-dimensional space $I(Z)_d$.

The condition that a basis f, g of I_d be a regular pair is open. Indeed, it is equivalent to injectivity of

$$S_{d-1}^{\oplus 2} \longrightarrow S_{2d-1}, \quad (u, v) \longmapsto uf + vg.$$

If f, g are coprime, every syzygy is a multiple of $(g, -f)$ and none occurs in this degree. If they have a common factor of positive degree, the pair obtained by dividing $(g, -f)$ by that factor gives a syzygy of coefficient degree at most $d - 1$, which can be multiplied to coefficient degree $d - 1$. The grid subset therefore proves that the open condition is nonempty. This proves the regular-sequence assertion for general points and (1.1) follows from the Koszul resolution.

Finally, if $p \geq N_d$, general evaluation is injective in degree d , so $J = 0$. If $p \leq N_{d-1}$, evaluation is surjective in degree $d - 1$ and Lemma 2.1 shows that I is generated in degrees at most d . Multiplying these generators shows $I_t = S_{t-d}I_d$ for all $t \geq d$. This includes the boundary $p = N_{d-1}$ and every choice of a higher generating degree, and finishes the proof of Theorem 1.1.

Corollary 5.1. *The Expected Syzygy Conjecture for chopped ideals of general points holds in $\mathbb{P}^2$.*

Proof. Use the formulation in [3, Conjecture 1], which takes $d = \min\{t : N_t \geq p\}$. If $p = N_d$, there are no degree- d equations and $J = 0$, as required. Otherwise $N_{d-1} < p < N_d$. When the predicted quotient dimension never reaches the point count, the cutoff is interpreted as $+\infty$. For $r = 1, 2$ the alternating expression is the regular-sequence series (1.1). For $r \geq 3$, before degree $2d$ its value is $N_{d+e} - rN_e$; terms from the second and higher Koszul modules first occur in degree $2d$. By (1.3), this expression first reaches or falls below the point count p at degree $d + e_0 < 2d$. The conjecture retains the earlier values and the constant value p thereafter, which is exactly (1.2). The cases $d \leq 1$ follow by direct computation. The boundary $p = N_{d-1}$ in Theorem 1.1 also covers chopping one degree later than the convention of the conjecture. $\square$

Remark 5.2. The conclusions hold over any characteristic-zero field in the generic sense. All evaluation and multiplication rank conditions used above are defined over the prime field, and only finitely many are imposed. Their nonemptiness can be tested after algebraic closure; their intersection thus defines a nonempty open over the original field. The degree-one cases are immediate: one point has an ideal generated by two independent linear forms, two general points have a one-dimensional space of linear equations, and at least three general points have no nonzero linear equation. If $p = 0$, with $I(\emptyset) = S$, then $(I(\emptyset)_d) = (x, y, z)^d$.

REFERENCES

- [1] I. Coskun, J. Huizenga, and G. Smith, *Stability and cohomology of kernel bundles on $\mathbb{P}^n$* , Michigan Math. J. **75** (2025), no. 1, 173–198. [doi:10.1307/mmj/20226238](https://doi.org/10.1307/mmj/20226238).
- [2] D. Eisenbud, *Commutative Algebra: with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. [doi:10.1007/978-1-4612-5350-1](https://doi.org/10.1007/978-1-4612-5350-1).
- [3] F. Gesmundo, L. Kayser, and S. Telen, *Hilbert functions of chopped ideals*, J. Algebra **666** (2025), 415–445. [doi:10.1016/j.jalgebra.2024.11.017](https://doi.org/10.1016/j.jalgebra.2024.11.017).