

An order-five Haagerup–Izumi vertex operator algebra at central charge eight

Tzu-Chen Huang

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Abstract

We construct an explicit unitary strongly rational vertex operator algebra U of central charge eight inside $V_{A_4 \oplus A_4}$ and identify $\text{Rep}(U)$ with the Drinfeld center of the positive order-five Haagerup–Izumi category. The construction adjoins one algebraic weight-two field to a diagonal affine $\mathfrak{sl}_2$ at level forty and the ambient conformal vector. Exact finite lattice identities imply strong generation and C_2 -cofiniteness. A finite integral order in the actual Zhu algebra descends semisimplicity from a 1590-dimensional upper presentation over $\mathbb{F}_{10069}$. To determine the modules, we combine the good-prime block bound with lattice character growth and a Wronskian inequality. The resulting full branching has an irreducible congruence complement of dimension at most fifteen; its exact classification forces all fourteen denominator-29 sectors. Finally, the actual holomorphic E_8 extension determines the boundary fusion category and its associator. The proof is computer-assisted, with exact arithmetic certificates separated from exploratory numerical evidence.

1 Statement and construction

The Haagerup–Izumi fusion category $\mathcal{H}_5$ has simple objects $\alpha_g, \alpha_g \rho, g \in \mathbb{Z}/5$, with

$$\alpha_g \alpha_h = \alpha_{g+h}, \quad \rho \alpha_g = \alpha_{-g} \rho, \quad \rho^2 = \mathbf{1} \oplus \bigoplus_g \alpha_g \rho. \quad (1)$$

Its positive dimension is $d = (5 + \sqrt{29})/2$. The chiral realization problem concerns the braided center $Z(\mathcal{H}_5)$, rather than $\mathcal{H}_5$ itself. Evans–Gannon proposed VOA realizations of this family and described their modular targets [1]. Rayhaun formulated symmetry–subalgebra duality in [2, Conjecture D.20]. In this framework, Möller and Rayhaun [3, Proposition^{ph} 7.2 and Example^{ph} 7.6] relate positive strongly rational $c = 8$ VOAs with representation category $Z(\mathcal{F})$ to $\mathcal{F}$ symmetries of $(E_8)_1$. Gannon and Rayhaun [4, Section 4.3, Claim 4.5] describe the corresponding structure for untwisted Haagerup–Izumi categories of every odd order: assuming an $\mathcal{H}_n$ action exists, its invariant algebra is obtained from the $\mathbb{Z}/n$ fixed-point algebra by a two-element hypergroup action. For $n = 5$, the nontrivial element satisfies

$$a \star a = \frac{1}{w} \mathbf{1} + \frac{w-1}{w} a, \quad w = d^2 = \frac{27 + 5\sqrt{29}}{2},$$

and the associated topological-line category has near-group fusion rules $\mathbb{Z}_5 \times \mathbb{Z}_5 + 25$. The present construction gives explicit lattice generators and proves strong rationality and the representation-category identification. The recent order-three lattice constructions [5, 6] motivate the lattice approach.

Theorem 1.1. *The explicitly specified subalgebra*

$$U = \langle E, H_c, F, \omega_B, u \rangle \subset B = V_{A_4 \oplus A_4}$$

is simple, unitary and strongly rational, with $c = 8$, $U_1 \cong \mathfrak{sl}_2$ at level 40, and a ribbon equivalence

$$\text{Rep}(U) \simeq Z(\mathcal{H}_5).$$

It has 28 simple modules and

$$\chi_U = q^{-1/3}(1 + 3q + 38q^2 + 250q^3 + 1575q^4 + 7545q^5 + O(q^6)).$$

The lattice restriction is $B \downarrow_U = U \oplus \pi$, with $h_\pi = 1$, $\dim \pi_1 = 45$, and $\text{qdim } \pi = d^2$.

Here is a complete specification of the generators. Use simple-root coordinates on $A_4 \oplus A_4$, with Cartan matrix G , and cocycle

$$\epsilon(a, b) = (-1)^{\sum_i a_i b_i + \sum_{i>j} a_i G_{ij} b_j}.$$

Set $c = (4, 6, 6, 4, 4, 6, 6, 4)$ and

$$E = \sum_{i=1}^8 e^{\alpha_i}, \quad F = -\sum_{i=1}^8 c_i e^{-\alpha_i}, \quad H_c = \sum_{i=1}^8 c_i \alpha_i (-1) \mathbf{1}.$$

Then $E_{(0)}F = H_c$, $H_{c(0)}E = 2E$, $E_{(1)}F = 40\mathbf{1}$ and $H_{c(1)}H_c = 80\mathbf{1}$. The vector ω_B is the usual rank-eight lattice conformal vector.

In each A_4 write $\gamma_{ij} = e_i - e_j$, $\theta = \gamma_{15}$ and $p = \theta + \gamma_{24}$. The field u is the sum of twelve norm-four exponentials with the following coefficients and momenta, in corresponding order:

$$\begin{array}{l|l} s, t & (p, 0), (0, p) \\ a, b, c_0 & (\gamma_{13}, \theta), (\gamma_{24}, \theta), (\gamma_{35}, \theta) \\ d_0, e_0, f_0, g_0 & (\gamma_{14}, \gamma_{14}), (\gamma_{14}, \gamma_{25}), (\gamma_{25}, \gamma_{14}), (\gamma_{25}, \gamma_{25}) \\ h, i_0, j_0 & (\theta, \gamma_{13}), (\theta, \gamma_{24}), (\theta, \gamma_{35}). \end{array}$$

The dependent coefficients are $b = a + d_0 - e_0$, $c_0 = b + f_0 - g_0$, $i_0 = h + d_0 - f_0$, $j_0 = i_0 + e_0 - g_0$. Put $R_0 = \sqrt{145}$,

$$z = \frac{3\sqrt{29} - 5R_0}{58}, \quad D = \frac{9125 + 125R_0}{20736} + \frac{365\sqrt{5} + 25\sqrt{29}}{6912}, \quad \tau = i\sqrt{D}.$$

All square roots in these formulas are positive before multiplication by i . The independent parameter column is $A + \tau B_0$, where

$$A = \begin{pmatrix} 1 \\ -(73 + R_0)/72 \\ (31\sqrt{5} + 11\sqrt{29})/144 \\ (-3\sqrt{5} + \sqrt{29})/16 \\ -(1 + R_0)/32 - (\sqrt{5} + 5\sqrt{29})/288 \\ -(1 + R_0)/32 + (\sqrt{5} + 5\sqrt{29})/288 \\ -(1 + R_0)/32 + (\sqrt{5} + 5\sqrt{29})/288 \\ -(1 + R_0)/32 - (\sqrt{5} + 5\sqrt{29})/288 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 \\ 0 \\ 1 + z \\ 1 - z \\ -1 \\ z \\ -z \\ 1 \end{pmatrix}. \quad (2)$$

The order is $(s, t, a, h, d_0, e_0, f_0, g_0)$. All computations take place in the degree-eight field $K = \mathbb{Q}(\sqrt{5}, \sqrt{29}, \tau)$.

The vector u is an affine highest vector of horizontal weight 12. In this Chevalley gauge, compact PCT conjugation is

$$\Theta(\alpha_{i_1}(-m_1) \cdots \alpha_{i_r}(-m_r)e^a) = (-1)^r \left(\prod_i c_i^{a_i} \right) \alpha_{i_1}(-m_1) \cdots \alpha_{i_r}(-m_r)e^{-a},$$

with antilinear coefficients. Exact calculation gives $\Theta(u) = F_{(0)}^{12}u/12!$ and $F_{(0)}^{13}u = 0$. Thus the generating set is PCT stable. The positive lattice form restricts to U ; the invariant-form argument with $U_0 = \mathbb{C}\mathbf{1}$ makes U simple and self-contragredient.

2 Finite closure and C_2 -cofiniteness

Let $J = \langle E, H_c, F \rangle$ and let A be the exact weight-two space obtained by closing the current descendants, ω_B and the current multiplet of u under weight-preserving products. With $T = L(-1)$ define

$$\begin{aligned} N_3 &= J_{(-1)}A + TA, & S_3 &= N_3 + A_{(0)}A, \\ N_4 &= J_{(-1)}S_3 + TS_3 + A_{(-1)}A, & W_4 &= N_4 + A_{(0)}S_3, \\ N_5 &= J_{(-1)}W_4 + TW_4 + A_{(-1)}S_3. \end{aligned}$$

These definitions initially give finite subspaces, without asserting that they exhaust the graded pieces of U .

Lemma 2.1 (Recursive closure). *Let $G_1, \dots, G_h$ be finite positive-weight spaces, and let $\mathcal{N}$ be the span of all parenthesized differential normal words in them. Suppose all singular products of $G_1 + G_2$ with the G_i belong to $\mathcal{N}$, and for $i \geq 3$ suppose*

$$G_i \subset (G_2)_{(0)}G_{i-1} + \mathcal{N}_i(G_1 + \cdots + G_{i-1}).$$

Then $\mathcal{N}$ is a vertex subalgebra.

Proof. Zero modes are derivations, so the assumed G_2 zero products first prove $(G_2)_{(0)}\mathcal{N} \subset \mathcal{N}$. Wick induction handles any pair having a derivative or composite input from generator brackets of smaller total weight. For the remaining generator pairs, induct lexicographically on total weight and left weight. A recursive term $s = a_{(0)}b$ satisfies

$$s_{(n)}t = a_{(0)}(b_{(n)}t) - b_{(n)}(a_{(0)}t).$$

The first term has smaller total input weight. In the second, a composite or derivative in the known normal expression of $a_{(0)}t$ is handled by Wick induction; a single generator has the same total weight and smaller left weight. All generator brackets close, and Wick induction finishes. No PBW independence is assumed. $\square$

Proposition 2.2 (Exact lattice certificate). *For the seed (2),*

$$\dim A = 38, \quad \dim S_3 = 250, \quad \dim N_4 = 1286, \quad \dim W_4 = 1575, \quad \dim N_5 = 7545.$$

The hypotheses of Lemma 2.1 hold with $(G_1, G_2, G_3, G_4) = (J, A, S_3, W_4)$.

Exact computational proof. Lattice fields are expanded by their cocycle and the finite truncations of the Heisenberg exponential formula. Current highest vectors reduce the computations to rational Clebsch–Gordan projections over K . Each asserted dependency is checked by reconstruction in the full ambient lattice basis, not only on a selected minor. Independent specialization ranks certify the selected vectors’ independence.

The primitive complements have dimensions $\dim(S_3/N_3) = 112$ and $\dim(W_4/N_4) = 289$, in 14 and 35 current multiplets. The exact positive-product certificate checks all 5550 projected components on these complements. Current products follow from their recursive provenance. The zero-mode derivation and skew-symmetry identities give $A_{(0)}N_4 \subset N_5$. The remaining 1074 projected components of $A_{(0)}(W_4/N_4)$ all belong to the same 7545-dimensional exact normal space. The coverage audit enumerates every required component and checks the producer and witness hashes; see Section 6. Lemma 2.1 now applies in characteristic zero. $\square$

It follows that $U_1 = J$ and $U \neq B$. The graded dimensions through weight five are now dimensions of the actual VOA. In particular, finite vertex generation has not been substituted for strong generation.

Proposition 2.3. *U is C_2 -cofinite.*

Proof. The positive Heisenberg bound is $L_0 \geq H_0^2/160$. If a field has weight g and nonzero charge ℓ , its N th normal power vanishes whenever $N > 160g/\ell^2$. Thus charged classes in $U/C_2(U)$ are nilpotent. In characteristic zero the nilradical is stable under derivations, so all components of nontrivial current multiplets are nilpotent as well.

The strong spaces have only two essential neutral classes: $\omega_C = \omega_B - \omega_{40}$ and a fixed second weight-two singlet x . The affine Sugawara class is a polynomial in currents; the weight-three singlets are translates and the weight-four singlets are normal expressions in lower fields. In the reduced C_2 quotient, two exact singlet projections of $A_{(0)}S_3$ give homogeneous quadratics $q_1(\omega_C, x) = q_2(\omega_C, x) = 0$. Their coefficients come from brackets of nontrivial current multiplets, which vanish in this reduced Poisson quotient. The four linear multiples $\omega_C q_1, xq_1, \omega_C q_2, xq_2$ span all neutral cubics; the exact 4×4 coefficient determinant is nonzero. Reducedness forces both neutral classes to vanish. Strong generation makes the C_2 algebra finitely generated, and each positive-degree generator is nilpotent. Bounded monomials span it. $\square$

3 An integral order and rationality

An affine extension makes the Zhu computation finite and small. Let a be the unique current-highest-weight-eight vector in A and $\lambda = H_c/2 = (2, 3, 3, 2; 2, 3, 3, 2)$. The extremal weight-ten, charge-forty lattice space is $\mathbb{C}e^\lambda$. Exact calculation gives

$$a_{(-1)}^5 \mathbf{1} = \kappa e^\lambda, \quad \kappa \neq 0.$$

One rational coordinate of κ is $-5918853670082299150421875/5210532790841206823544$. It can be computed by five applications of $a_{(3)}$ to $e^{-\lambda}$, using the invariant pairing. PCT and affine fusion therefore embed

$$\mathcal{D} = L_{40}(\mathfrak{sl}_2) \oplus L_{40}(40) \subset U.$$

Its internal grading agrees with the ambient grading on $\mathcal{D}$, so $A(\mathcal{D}) \rightarrow A(U)$ is defined although its Sugawara conformal vector differs from ω_B . Its simple tops have current types $0, 2, \dots, 20$, with two blocks at 20.

Let R be the DVR in K above $p = 10069$ with residues $\sqrt{5} = 2054$, $\sqrt{29} = 531$, $\tau = 515$. All finite closure and recursive-provenance coefficients are integral at this prime. An audit of 10 571 523 rational coordinates finds no denominator divisible by p , and the neutral cubic determinant has residue 7638. Unit-minor checks for the exact highest-vector bases and normal targets prove saturation of the required integral spans; remaining low weight-five memberships have explicit integral witnesses. Let U_R be the resulting generated vertex algebra.

Lemma 3.1. *The R -span C_R of the images of U_R in $A(U_K)$ is a finite free order. Its current highest-line idempotent e is full over R and in the special fiber.*

Proof. Use negative-mode words for integral strong generation. The proof of Lemma 2.1 uses integral binomial identities and divided translations, so requires no arbitrary factorial inversions. Set

$$Q_R = U_R / \sum_{n \geq 2, b, c \in U_R} R b_{(-n)} c.$$

Retaining all higher negative modes is essential in positive characteristic. The integral Borcherds identities make Q_R a commutative graded Poisson algebra, generated by the finite strong fields, with finite graded pieces.

Its special fiber is finite-dimensional. For a highest field of weight $g \leq 4$ and current highest weight $\ell > 0$, put $N = \lfloor 160g/\ell^2 \rfloor + 1$. Its class satisfies $a^N = 0$. Applying the finite lowering exponential and a Vandermonde matrix with unit determinant shows every component is nilpotent of exponent $(\ell + 1)(N - 1) + 1 \leq 481 < p$; for weight-two fields the bound is 241. Here $N\ell \leq 322 < p$, so every factorial used is a unit. If $z^M = 0$ and $M!$ is invertible, then for a derivation D , $D^M(z^M) = 0$ implies $(Dz)^M \in (z)$ and $(Dz)^{M^2} = 0$. This bounded argument replaces the generally false positive-characteristic Poisson-nilradical assertion. It applies to the two neutral bracket identities. Their unit determinant kills the neutral classes in the reduced special fiber. Thus all its algebra generators are nilpotent. Graded Nakayama makes Q_R finite over R .

The Zhu product has integral coefficients. Its higher-mode identities make $Q_R \twoheadrightarrow \text{gr } C_R$ surjective, so C_R is finite. It is torsion-free by construction inside $A(U_K)$, hence free. Finally, current matrix units involve Casimir differences of absolute value at most 440 and factorials of index at most 20. These are units in R , and choosing one highest line in each current type gives the asserted full idempotent. $\square$

The necessary Zhu equations use exact weight-three and weight-four kernel identities and weight-five normal/zero-product consequences. After reduction they give a quiver algebra P surjecting onto $e(C_R/pC_R)e$. They eliminate the current vertices above 12, leaving seven vertices and 98 arrows.

Proposition 3.2. *P is a semisimple algebra of dimension 1590 over $\mathbb{F}_{10069}$. Consequently U is strongly rational.*

Proof. Decreasing rewrite rules of degree at most four have an acyclic normal-word automaton. Their 2580 normal words, with degree counts (7, 98, 768, 1145, 464, 90, 8), span P ; confluence is not assumed. On their free span construct tentative left and right arrow actions. The subspace generated by relation vectors and their left/right commutators, closed under both actions, has dimension 990. On the quotient, exact checks verify all 6698 defining equations, all 9604 left/right commutator matrices, and evaluation of all 2580 words on the unit.

Every reduction is necessary, so this quotient surjects onto P . Conversely, the commuting actions and their equality on the cyclic unit make every defining relation an operator identity: it kills the unit and commutes with the cyclic right action. This gives the reverse surjection and proves the

dimension $2580 - 990 = 1590$. The regular trace Gram matrix has rank 1590, checked in all 28 opposite-endpoint blocks. The Jacobson radical lies in the trace kernel, so P is semisimple.

Its quotient is semisimple, and fullness of e gives semisimplicity of C_R/pC_R . A finite free algebra with separable special fiber has absolutely semisimple generic fiber. Indeed, complete the DVR, split the special fiber by an unramified extension, lift matrix idempotents, and use Nakayama on their corners to obtain matrix algebras over the completed DVR. Faithful field extension gives semisimplicity of $A(U)$. Proposition 2.3, simplicity and self-contragredience now allow McRae's Zhu-semisimplicity criterion [7, Corollary 5.19]. That result also supplies the modular tensor category without assuming positive lowest weights for nonvacuum modules; its modular action agrees with the trace-function action [8, Theorem 3.10]. $\square$

4 The rank bound determines every sector

Label the lattice discriminant group by $(x, y) \in (\mathbb{Z}/5)^2$, with ordinary lattice labels $(i, j) = (x + 2y, 2x + y)$. Its quadratic form is $xy/5$. The twelve nonzero labels modulo simultaneous sign give actual simple modules generated by their complete lattice ground spaces. Their exact zero-mode actions are irreducible and pairwise inequivalent; adjoint closure promotes irreducibility of a top to simplicity of its generated unitary module. The weight-one complement gives π . The known modules are

modules	number	h	top dimension
U	1	0	1
π	1	1	45
$G_{10}, G_{20}, G_{01}, G_{02}$	4	1	50
G_{11}, G_{32}	2	$6/5$	100
G_{21}, G_{12}	2	$2/5$	5
G_{31}, G_{42}	2	$3/5$	10
G_{41}, G_{22}	2	$4/5$	25.

Exact opposite-coset intertwiners give their self-duality.

Lemma 4.1. *There are at most fourteen additional simples. Their rational lowest weights are p -integral, and their residues are $k/29$, at most once each, for*

$$\mathcal{K} = \{8, 11, 12, 14, 15, 17, 18, 19, 21, 26, 27, 31, 32, 39\}. \quad (3)$$

All simple modules are self-dual.

Proof. The semisimple integral order preserves simple-block dimensions, current-idempotent ranks, and the residue of the central conformal element under specialization. The center of P has dimension 30. The known highest-line block dimensions exhaust its six known-weight clusters: 1, 305, 2, 8, 50, 512, at 0, 1, $2/5$, $3/5$, $4/5$, $6/5$. The other sixteen clusters are individual split matrix blocks, as verified by explicit simple left ideals and their arrow actions. Two have only current type zero. The exact scalar Zhu identity

$$h(29h - 11)(6757h^2 - 12429h + 2732) = 0$$

excludes them: their residues 3298, 1979 match neither rational root 0, $11/29$, and the remaining two roots are irrational. The fourteen remaining residues are precisely (3). Different residues imply distinct weights; contragredience preserves the weight, proving self-duality of every additional simple. $\square$

Lemma 4.2. *The vacuum is the unique simple of weight zero, all other lowest weights are positive, and every lowest weight satisfies $h \leq 217/3$.*

Proof. If the smallest lowest weight were negative, it would be unique by Lemma 4.1. Its nonzero S_{0M} entry would give the vacuum character growth $\exp(2\pi(1/3 - h)/t)$ as $t \rightarrow 0^+$. There is no other leading term to cancel it. This contradicts $0 < \chi_U(it) \leq \chi_B(it) \sim \frac{1}{5} \exp(2\pi/(3t))$. The residue exclusion at zero makes the vacuum the unique minimum. The modular transform of each positive ordinary character now gives $S_{M0} > 0$, so the spherical dimensions are positive.

Let $r' \leq 28$ be the dimension of the ordinary-character span. Choose a basis of actual characters containing the desired character; their lowest weights are all nonnegative. The nonzero modular Wronskian has weight $r'(r' - 1)$ and finite-order multiplier. Its order at infinity is at least $\sum_i (h_i - 1/3)$, even when leading exponents coincide. The valence formula gives

$$h \leq \sum_i h_i \leq \frac{r'}{3} + \frac{r'(r' - 1)}{12} \leq \frac{217}{3}.$$

This argument does not require independence of all simple characters. $\square$

Proposition 4.3. *The full lattice restrictions are $B_{0,0} = U \oplus \pi$ and $B_{x,y} = G_{[x,y]}$ for nonzero (x, y) .*

Proof. An additional simple of weight $j/5$ would have $0 \leq j \leq 361$. Its residue requires $j \equiv 5k/29 \pmod{10069}$, whereas the smallest nonnegative representative among these residues is 697. Thus the lattice restrictions contain only known simples. Their ground spaces exhaust the lowest spaces, and their actual weights permit no further summand at a higher integer grade. In the vacuum coset, U_0 and the dimensions $3 + 45 = 48$ at weight one exhaust the possibilities. Rationality ensures these restrictions are finite semisimple sums without arbitrary shifts in conformal weight. $\square$

The positive-dimensional lattice extension is a connected separable commutative algebra, so its branching matrix satisfies $bS_U = S_B b$, $bT_U = T_B b$ [10, 11]. Its row space has dimension thirteen. Let W be its invariant orthogonal complement, of dimension at most fifteen. It has basis $v = (e_0 - e_\pi)/\sqrt{2}$ and the additional labels. Remove the central-charge character from T , leaving the twists.

Lemma 4.4. *W is the nonsquare even Weil representation of level 29. In particular $\dim W = 15$ and $S_{vv} = -1/\sqrt{29}$.*

Proof. All simples are self-dual and have positive dimensions, so S is real orthogonal. The twist-one line in W is uniquely $\mathbb{C}v$. The other twists are distinct: if $h_i - h_j = n \in \mathbb{Z}$, reduction gives $29n - (k_i - k_j) \equiv 0 \pmod{10069}$. Its absolute value is below 10069 by Lemma 4.2, so it is zero, contradicting the distinct residues of (3) modulo 29. Branching gives $S_{\pi j} = -S_{0j}$ for an additional label, so $S_{vj} = \sqrt{2}S_{0j} > 0$. With simple T spectrum, these connections to v force W to be irreducible. The congruence theorem [8] applies to W .

We use the complete Nobs–Wolfart congruence classification as implemented in `SL2Reps` [9]. Among all 6268 irreducibles of dimensions 1 through 15, precisely 113 have $S^2 = 1$, simple T spectrum and exactly one twist one. For each other phase a/N , the exact test

$$h = a/N + n \leq 217/3, \quad n \geq 0, \quad h \equiv k/29 \pmod{10069}$$

leaves only the trivial representation and the nonsquare even Weil representation of level 29. There is at most one integer n for each residue. An independent exact-fraction replay verifies all 113 spectral filters.

The trivial possibility is excluded without guessing any index. For $u_0 = (e_0 + e_\pi)/\sqrt{2}$, branching gives $S_{u_0 u_0} = 1/5$. Positivity of S_{00} and $S_{0\pi}$ implies $|S_{vv}| < 1/5$, whereas the trivial representation has $S_{vv} = 1$. In the nonsquare Weil representation, the Gauss sum gives $S_{vv} = -1/\sqrt{29}$; an independent exact Fourier/character-table calculation checks this identification. $\square$

Thus all fourteen additional simples occur. Their phases together with their preserved residues and the bound determine the actual lowest weights $k/29$, not merely their classes modulo one. Their top dimensions are

$29h$	8	11	12	14	15	17	18
$\dim M_h$	3	1	3	7	5	13	8
$29h$	19	21	26	27	31	32	39
$\dim M_h$	17	21	31	41	62	58	134.

Consequently the actual Zhu algebra has dimension 62388, and its current highest-line corner has dimension 1588. The two excluded scalar blocks account for the difference from $\dim P = 1590$. Finally,

$$S_{00} = \frac{1}{2} \left(\frac{1}{5} - \frac{1}{\sqrt{29}} \right). \quad (4)$$

5 The extension identifies the category

The isotropic discriminant line $L = \{(x, 0)\}$ extends B to $H = V_{E_8}$. Proposition 4.3 identifies its actual algebra object in $\mathcal{C} = \text{Rep}(U)$ as

$$\mathcal{E} = \mathbf{1} \oplus \pi \oplus 2G_{10} \oplus 2G_{20}.$$

It is Lagrangian: its local modules are Vec because H is holomorphic. The center theorem [11] gives $\mathcal{C} \simeq Z(\mathcal{T})$ for the fusion category $\mathcal{T} = \mathcal{C}_{\mathcal{E}}$ of right modules. Use the left action induced by the inverse braiding to fix the center orientation. Positive spherical structures are preserved. This is the extension underlying the symmetry–subalgebra correspondence of [2, Conjecture D.20], [3, Proposition^{ph} 7.2] and [4, Section 4.3, Claim 4.3]; here we determine its fusion category from the actual extension algebra.

Proposition 5.1. *$\mathcal{T}$ has the fusion rules (1), positive dimension $d = (5 + \sqrt{29})/2$, and untwisted pointed subcategory $\text{Vec}_{\mathbb{Z}/5}$.*

Proof. Let $F : Z(\mathcal{T}) \rightarrow \mathcal{T}$ be forgetful and I its adjoint. Since $\mathcal{E} = I(\mathbf{1})$,

$$\text{rank } \mathcal{T} = \dim \text{End}(\mathcal{E}) = 1 + 1 + 4 + 4 = 10.$$

Successive induction from the intermediate algebra $\mathbf{1} \oplus \pi$ embeds five distinct invertibles α_g . The complementary isotropic line $\{(0, y)\}$ gives coherent representatives: the pointed metric category has trivial associator and braiding $c((x, y), (x', y')) = \zeta_5^{xy'}$, so its restriction to that line is trivial. This proves the pointed associator assertion.

For $g \neq 0$, the underlying central object $I(\alpha_g)$ is the sum of the five lattice restrictions in $L + (0, g)$, none occurring in $\mathcal{E}$. Thus $\text{Hom}(\mathcal{E}, I(\alpha_g)) = 0$. Using $FI(\mathbf{1}) = \bigoplus_Y Y \otimes Y^*$, adjunction rules out $\alpha_g Y \simeq Y$ for any simple Y . The ten simples form two free orbits of five. Duality gives free right action and a self-dual representative ρ in the second orbit. Equation (4) gives

$$\text{Dim } \mathcal{T} = S_{00}^{-1} = \frac{145 + 25\sqrt{29}}{2} = 5 + 5d^2,$$

so that orbit is noninvertible.

The commuting left and right actions give $\rho\alpha_g = \alpha_{\sigma(g)}\rho$. The only invertible constituent of ρ^2 is $\mathbf{1}$, once, and comparison with its right translates gives $\sigma^2 = 1$. The multiplicities 1, 1, 2, 2 in $I(\mathbf{1})$ are the irreducible degrees of the complexified Grothendieck algebra [12, Theorem 2.13]. This algebra is noncommutative; hence $\sigma = -1$. Conjugation by α_1 fixes ρ^2 and permutes its five possible noninvertible constituents transitively. Their multiplicities are equal, say m , and $d^2 = 1 + 5md$ forces $m = 1$. $\square$

For completeness, we specify the associator input in the final step. It cannot be replaced by agreement of modular data.

Lemma 5.2 (Associator normalization). *A complex fusion category with (1) and untwisted pointed subcategory is tensor equivalent to an Evans–Gannon category $\mathcal{C}(\mathbb{Z}/5; \pm, \omega, M)$.*

Normalization argument. Choose coherent pointed multiplication and the maps moving an invertible through ρ . The remaining crossing discrepancy is a scalar 2-cocycle, removable because $H^2(\mathbb{Z}/5, \mathbb{C}^\times) = 0$. Tensor words can then be strictified with object monoid $(\mathbb{Z}/5) \rtimes_{-1} \mathbb{N}$; this does not trivialize the simple objects’ associator. The categorical splitting $\rho^2 = \mathbf{1} \oplus \bigoplus_g \alpha_g \rho$ gives inclusions s, t_g and inverse projections s', t'_g obeying the Leavitt relations. They embed faithfully in the padded algebra

$$\mathcal{B} = \bigoplus_{\gamma \in (\mathbb{Z}/5) \rtimes_{-1} \mathbb{Z}} \varinjlim_n \text{Hom}(\rho^n, \gamma \rho^n).$$

Transition maps tensor on the right by ρ and are injective because $\mathbf{1}$ splits from ρ^2 . Multiplication uses common right multiples of tensor words; left tensoring gives endomorphisms. The induced map from the simple Leavitt algebra $L(1, 6)$ is unital and therefore injective [14].

The Evans–Gannon deconstruction [13, Section 4] now takes place in actual categorical Hom spaces. The four relevant spaces for the images of s, s', t_g, t'_g have dimensions 6, 6, 27, 27, as obtained directly from (1). Their splitting bases give all coefficient comparisons used there; no assertion about every intertwiner of an ambient endomorphism algebra is required. The scalar equations normalize to $\delta^2 = 5\delta + 1$, $\omega^3 = 1$ and the matrix equations of [13, Theorem 2]. The resulting endomorphisms preserve $L(1, 6)$. Repeated splitting of the final ρ^2 reduces every tensor word to the ten simple types; the corresponding inclusions and projections express every morphism in that Leavitt algebra. Thus reconstruction is full and faithful as well as essentially surjective. Expanded scalar equations and details of the padding construction are recorded in the companion categorical note; see Section 6 for availability. $\square$

The positive global dimension in Proposition 5.1 selects the positive branch of Lemma 5.2. Evans–Gannon’s complete order-five classification has two positive tensor classes [13, Theorem 3(iii)], which are Morita equivalent [15, Theorem 6.4 and Section 6.6]. One is the standard Izumi category; their positive centers are ribbon equivalent. This proves the categorical assertion of Theorem 1.1. In particular, the argument uses the actual extension multiplication and the recovered associator.

6 Computational scope and availability

The proof uses exact rational arithmetic in a fixed number field, exact prime-field linear algebra, and exact cyclotomic calculations. The exact-arithmetic certificates, source code, full-state witnesses, reproduction instructions and expanded companion notes are available from the author upon request. These materials cover finite closure, C_2 relations, the integral Zhu order, the finite Zhu upper algebra and its regular trace, simple-block profiles, and the modular representation and weight calculations.

The closure audit verifies exhaustive coverage and the recorded full-state witnesses; it is distinct from replaying each producer. The Zhu quotient checks distinguish a spanning set from a basis. The congruence filter is independently replayed with exact fractions, the degree enumeration is checked against the level enumeration and independent Chinese-remainder counts, and the external representation constructors are checked against GAP character tables. The certificates verify finite computational inputs to the written mathematical reductions; they do not constitute a formal verification of the entire proof.

AI Disclosure

AI systems were used in the exploration and development of the note.

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