

AXIOMATIC DEGREE THEORY ON TOPOLOGICAL MANIFOLDS

STOYAN APOSTOLOV

ABSTRACT. In this note we study extension of standard Euclidean degree in nonlinear analysis to the setting of general oriented topological manifolds. The axiomatization of Euclidean degree is adapted to this setting. Existence and uniqueness are proven. The existence does not rely on smooth approximations as in the Euclidean case – this is irrelevant for manifolds which do not admit a smooth structure and could be ambiguous for manifolds admitting many smooth structures. Rather, we rely on a cohomological approach using only topological information. When compared to the Euclidean case, we think this approach makes the proofs simpler and more transparent. The standard notion of a degree of maps between closed oriented manifolds studied in algebraic topology is recovered as a special case. A generalization of the notion of winding number is developed and related to the notion of degree. The familiar composition formula of the Euclidean degree is also obtained in this context.

1. INTRODUCTION

The notion of degree in nonlinear analysis and topology (both algebraic and differential) could be traced back to the notion of winding number in complex analysis, discovered by Cauchy. It was around the beginning of the 20th century when Poincare, Kronecker and Brouwer realized the potential of this notion in analysis and topology. For a very nice exposition of the history of degree theory, one may consult [19].

Roughly speaking, degree theory gives a means of assigning an integer to a continuous map between subsets of Euclidean spaces (or more generally – manifolds), satisfying some natural properties, as additivity over domains and homotopy invariance.

It was clearly from the very beginning, that the degree theory has important applications both to what would later become the fields of algebraic and differential topology, and also to the field of differential equations. Regarding the latter, the degree was extended to certain maps between infinite dimensional spaces. These developments continue to be active area of research currently as well.

During the 20-th century, the systematic study of the notion of degree was taken up by analysts, forming a large part of the areas currently known

FACULTY OF MATHEMATICS AND INFORMATICS, SOFIA UNIVERSITY, BULGARIA
Email address: `sapostolov@fmi.uni-sofia.bg`.

as nonlinear analysis and fixed-point theory. A coherent theory of degree on Euclidean spaces was studied and developed, building on an axiomatic approach. See [4, 19] for beautiful expositions of this approach.

However, these developments were somewhat independent of the notion of degree typically studied in topology. General developments covering degree theory in algebraic topology could be found in [2, 6]. The viewpoint of differential topology could be found in [14, 17].

A book which aims to present both the approach taken in analysis and in differential topology is [19]. Still, the book does not provide a general framework unifying both approaches.

The aim of our study is to provide such a framework. We adapt the axioms for degree used in nonlinear analysis to the setting of oriented topological manifolds. We prove uniqueness and existence. Thus we generalize the axiomatic approach to degree theory from Euclidean spaces to topological manifolds. We show how this notion of degree also generalizes the notion of degree for maps between manifolds introduced in algebraic topology.

The existence construction we provide differs from the one used in Euclidean spaces, which relies heavily on smooth approximations. Thus we rely purely on topological information. Recall that there exist many topological manifolds which do not admit smooth structure, see [12]. There are also many manifolds (as simple as S^7) which have more than one smooth structure, see [16]. Thus relying on smooth approximations to define degree on general topological manifolds could be either inaccessible or inadequate.

We formulate a generalization of winding numbers and provide relations with the previously defined notions of degree. Finally, we provide a relevant composition formula.

Axiomatic approaches to the related notions of fixed-point index and coincidence number could be found in [3, 8]. These works are surely strongly related to the present note. There is also close relation to the constructions in [6]. However, we think our work is still of some value to the theory.

It is our opinion that, even in the Euclidean case, the construction and proofs regarding the present approach to degree theory, although relying on the machinery of algebraic topology, are more transparent and natural than the analytical ones.

AI disclosure: AI systems were used in the exploration and development of the note. They were also used for proofreading.

2. NOTATION AND PRELIMINARIES

All functions considered will be assumed to be continuous.

By a topological manifold M of dimension n (for brevity – n -manifold), we mean a Hausdorff and second countable topological space such that for each point $p \in M$ there is an open neighbourhood of p which is homeomorphic to some open subset of $\mathbb{R}^n$. We would assume $n \geq 1$. In particular, when we say "manifold", we mean manifold without boundary. A neighbourhood U

of p together with a homeomorphism $\varphi : U \rightarrow \hat{U}$, where $\hat{U}$ is an open subset of $\mathbb{R}^n$, is called a chart around p . If $\hat{U} = \mathbb{R}^n$ we call (U, φ) a standard chart. Every point is contained in a standard chart.

All the homology and cohomology groups considered will be with coefficients in $\mathbb{Z}$. By H_k and H^k we denote respectively singular homology and cohomology groups. By $\check{H}^n$ we denote Čech cohomology groups. For the latter groups we work with the following definition given in [6], Definition 6.1, Chapter VIII: given a topological manifold M , for locally compact subsets $A \subseteq B \subseteq M$ we define

$$\check{H}^p(B, A) := \varinjlim_{(A,B) \subseteq (U,V), U \subseteq V, U \text{ and } V \text{ are open}} H^p(V, U)$$

(see also [2], Chapter VI, section 8). The definition is very useful, as it leads to the fact that Čech cohomology groups satisfy a sort of "continuity property" with respect to direct limits. The reader may consult, for example, [6], 6.18 in Chapter VIII.

We should keep in mind that such continuity property does not hold for singular cohomology. Both cohomology theories coincide for somewhat regular sets, for example, locally contractible compact subsets of topological manifolds. In general, however, they differ. In particular, relevant to the upcoming discussion, it may happen that $\check{H}^n(\bar{\Omega}, \partial\Omega) \not\cong H^n(\bar{\Omega}, \partial\Omega)$, for relatively compact open subset Ω of a topological n -manifold. The classical example is Ω being the bounded component of the complement of the Warsaw circle in $\mathbb{R}^2$ in which case (dimension 2) $\check{H}^2(\bar{\Omega}, \partial\Omega) \cong \mathbb{Z}$, $H^2(\bar{\Omega}, \partial\Omega) = 0$. See, for example, [15].

Recall that for a manifold M the group $H^n(M, M - x)$ is isomorphic to $\mathbb{Z}$ for any $x \in M$. An orientation of M (if it exists) is a consistent continuous choice of generators of these groups varying $x \in M$.

For a compact connected oriented n -manifold M we have $H^n(M) \cong \mathbb{Z}$, with isomorphism uniquely determined by the orientation of M .

For compact connected oriented manifolds M and N and map $f : M \rightarrow N$, we have the induced map $f^* : H^n(N) \rightarrow H^n(M)$, and the degree usually studied in topology is defined as $f^*(1)$, where we have used the natural identification of the two groups to $\mathbb{Z}$ through the orientations.

Now let us formulate the axiomatic definition of degree, adapted from the one used in nonlinear analysis.

Given connected oriented manifolds M and N of dimension n , consider the set of admissible triples

$$\mathcal{A}_{M,N} := \left\{ (f, \Omega, y) \left| \begin{array}{l} \Omega \text{ is a relatively compact open subset of } M, \\ f : \bar{\Omega} \rightarrow N \text{ is continuous and } y \in N - f(\partial\Omega) \end{array} \right. \right\}.$$

When $M = N$ we denote $\mathcal{A}_M := \mathcal{A}_{M,N}$.

Definition 2.1. A map $d : \mathcal{A}_{M,N} \rightarrow \mathbb{Z}$ is called *degree* if it satisfies the following three properties, also called axioms:

(D 1) (normalization) For any orientation preserving homeomorphism $f : U \rightarrow V$ between open subsets of M and N respectively, for any relatively compact open set Ω with $\bar{\Omega} \subseteq U$ and any $y \in f(\Omega)$ it holds that $d(f, \Omega, y) = 1$.

(D 2) (additivity) Given admissible triple (f, Ω, y) and disjoint open sets $\Omega_1, \Omega_2 \subseteq \Omega$ with $y \in N - f(\bar{\Omega} - (\Omega_1 \cup \Omega_2))$, it holds

$$d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$$

(D 3) (homotopy invariance) If $h : [0, 1] \times \bar{\Omega} \rightarrow N$ and $y : [0, 1] \rightarrow N$ are continuous and $y(t) \in N - h(t, \partial\Omega)$ for all $t \in [0, 1]$, then $d(h(t, \cdot), \Omega, y(t))$ is constant, independent of t .

Next we list some equivalent axioms and direct consequences, whose proofs we omit (they are the same as in the Euclidean case).

Proposition 2.2. *Let $d : \mathcal{A}_{M,N} \rightarrow \mathbb{Z}$ be a degree. Then*

(D 4) (surjectivity) $f^{-1}(y) = \emptyset$ implies $d(f, \Omega, y) = 0$.

(D 5) (excision) $d(f, \Omega, y) = d(f, \Omega_1, y)$ whenever $\Omega_1 \subseteq \Omega$ and $y \in N - f(\bar{\Omega} - \Omega_1)$.

Proposition 2.3. *A map $d : \mathcal{A}_{M,N} \rightarrow \mathbb{Z}$ satisfies (D 2) if and only if it satisfies (D 4) and*

(D 2)' Given admissible triple (f, Ω, y) and disjoint open sets $\Omega_1, \Omega_2 \subseteq \Omega$ with $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$ and $y \in N - f(\bar{\Omega} - (\Omega_1 \cup \Omega_2)) = N - f(\partial\Omega_1 \cup \partial\Omega_2)$, it holds

$$d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y).$$

The following result is the product of the works of many authors, most notably Nagumo [18], Heinz [11] regarding the existence and Fuhrer [7] and Amann-Weiss [1] regarding the uniqueness. See also the books Deimling [4] and Outerelo-Ruiz [19] for a thorough exposition and historical references.

Theorem 2.4. *There exists a unique degree $d : \mathcal{A}_{\mathbb{R}^n} \rightarrow \mathbb{Z}$.*

One aim of this note is to extend this result for general connected oriented manifolds M and N and degree $d : \mathcal{A}_{M,N} \rightarrow \mathbb{Z}$.

3. UNIQUENESS

Using Dugundji's characterization of metric absolute neighbourhood retracts (ANR), [5, Theorem 7.1], particularly 7.12, combined with the fact that manifolds are ANRs, [9, Theorem 3.3], as well as metrizable (via Urysohn metrization theorem) we have the following

Theorem 3.1. *Let M and N be manifolds. Let $A \subset M$ be closed, and let $f : A \rightarrow N$ be continuous. Then there exists an open subset $W \subseteq M$ with $A \subset W$, and a continuous map $\tilde{f} : W \rightarrow N$ such that $\tilde{f}|_A = f$.*

The following result is a corollary from topological transversality machinery, as developed by Kirby-Siebenmann [13, Essay III, Theorem 1.1] and Quinn [20, Theorem 2.4.1].

Theorem 3.2. *Let X and Y be n -manifolds, $f : X \rightarrow Y$, $V \subseteq X$ is open, $D \subseteq X$ is closed and $D \subset V$. Let ρ be a metric on Y , compatible with the topology. Let $y_0 \in Y$. Then for any continuous $\epsilon : X \rightarrow (0, \infty)$ there exists a homotopy $h : [0, 1] \times X \rightarrow Y$ such that*

- (1) $h(0, \cdot) \equiv f$ and $h(t, z) = f(z)$ for all $(t, z) \in [0, 1] \times (X - V)$,
- (2) for $g := h(1, \cdot)$, the set $g^{-1}(y_0) \cap D$ is discrete.
- (3) $\rho(h(t, z), f(z)) < \epsilon(z)$ for all $(t, z) \in [0, 1] \times X$.

Corollary 3.3. *Let X and Y be n -manifolds, $f : X \rightarrow Y$ and $V \subseteq X$ be an open neighbourhood of $f^{-1}(y_0) := \{x \in X \mid f(x) = y_0\}$. Then there exists $g : X \rightarrow Y$ which is homotopic to f relative to $X - V$ and such that the set $g^{-1}(y_0)$ is discrete.*

Proof. Let ρ be any compatible metric on Y . Let W be open neighbourhood of $f^{-1}(y_0)$, such that $\overline{W} \subset V$. Define $\epsilon(z) := \rho(y_0, f(z))$ for $z \in X - W$. Thus $\epsilon : X - W \rightarrow (0, \infty)$. Now extend via Tietze extension theorem continuously to $\epsilon : X \rightarrow (0, \infty)$. Applying Theorem 3.2 with $D = \overline{W}$, we obtain homotopy h . Put $g := h(1, \cdot)$. Then 1. implies that $g \simeq f \text{ rel } X - V$. Moreover 2. implies $g^{-1}(y_0) \cap \overline{W}$ is discrete. Finally, 3. implies that for $z \in X - W$, $\rho(g(z), f(z)) < \rho(y_0, f(z))$, so that $g^{-1}(y_0) \subseteq W$, which shows that $g^{-1}(y_0)$ is discrete. $\square$

We say that two maps $f, g : X \rightarrow Y$ are homotopic relative to a subspace $A \subseteq X$, denoted by $f \simeq g \text{ rel } A$, if there exists a homotopy between them, which coincides with f on A , i.e., there exists a continuous $h : X \times [0, 1] \rightarrow Y$, such that $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ for all x and $h(a, t) = f(a)$ for all $a \in A$ and all $t \in [0, 1]$.

Proposition 3.4. *Let M and N be n -manifolds and $U \subseteq M$ be a relatively compact open subset. Let $f : \overline{U} \rightarrow N$ be continuous and let $y_0 \in N - f(\partial U)$. Then there exists $g : \overline{U} \rightarrow N$ such that $g \simeq f \text{ rel } \partial U$ and $g^{-1}(y_0)$ is finite set.*

Proof. Let $K = f^{-1}(y_0) \subseteq U$. As K is closed, it is compact. Let V be open set such that $K \subseteq V \subseteq \overline{V} \subset U$. Then $C := \overline{U} - V$ is closed and $f|_C : C \rightarrow N - \{y_0\}$. Using Theorem 3.1 we obtain open set W with $C \subset W$ and function $r : W \rightarrow N - \{y_0\}$, such that $r|_C = f|_C$. Now define $\tilde{f} : W_1 := W \cup \overline{U} \rightarrow N$ as follows

$$\tilde{f}(z) = \begin{cases} f(z), & z \in \overline{U} \\ r(z), & z \in W - V \end{cases}$$

The definition is correct, as both f and r agree on $\overline{U} \cap (W - V) = C$. Hence $\tilde{f}$ is continuous as well. Moreover $\tilde{f}^{-1}(y_0) = K$. Applying Corollary 3.3 for $X = W_1$, $Y = N$ and V we obtain $g : W_1 \rightarrow N$, $g \simeq \tilde{f} \text{ rel } W_1 - V$ and

$g^{-1}(y_0)$ is discrete. In particular $g|_{\bar{U}} \simeq f \text{ rel } \partial U$ and $g|_{\bar{U}}^{-1}(y_0)$ is discrete. As this is a closed set it must be compact as well, hence it is finite. $\square$

Lemma 3.5. *Let X be an oriented n -manifold that is contained in one standard oriented chart, i.e., there exists a homeomorphism $\varphi : X \rightarrow \mathbb{R}^n$. Let Y be connected oriented n -manifold. Then there exists at most one degree $d : \mathcal{A}_{X,Y} \rightarrow \mathbb{Z}$.*

Proof. Let d_1, d_2 be two degrees $\mathcal{A}_{X,Y} \rightarrow \mathbb{Z}$. Fix any standard oriented chart (V, ψ) in Y . Given $(f, \Omega, y) \in \mathcal{A}_{\mathbb{R}^n}$ we have admissible triple

$$(\psi^{-1} \circ f \circ \varphi, \varphi^{-1}(\Omega), \psi^{-1}(y)) \in \mathcal{A}_{X,Y}$$

and we define

$$\hat{d}_i(f, \Omega, y) := d_i(\psi^{-1} \circ f \circ \varphi, \varphi^{-1}(\Omega), \psi^{-1}(y)), \quad i = 1, 2.$$

Then we directly check that $\hat{d}_1$ and $\hat{d}_2$ are both degrees on $\mathcal{A}_{\mathbb{R}^n}$, and thus by the uniqueness part of Theorem 2.4 we obtain that

$$d_1(\psi^{-1} \circ f \circ \varphi, \varphi^{-1}(\Omega), \psi^{-1}(y)) = d_2(\psi^{-1} \circ f \circ \varphi, \varphi^{-1}(\Omega), \psi^{-1}(y)) \quad (1)$$

for any standard chart (ψ, V) on Y .

Now take any $(f_0, \Omega_0, y_0) \in \mathcal{A}_{X,Y}$. If $f_0^{-1}(y_0) = \emptyset$, then (D4) implies $d_1(f_0, \Omega_0, y_0) = 0 = d_2(f_0, \Omega_0, y_0)$. Otherwise, fix a standard chart (ψ, V) around y_0 in Y and relatively compact open V' with $y_0 \in V' \subseteq \bar{V}' \subseteq V$ and $\Omega_1 := f_0^{-1}(V') \subseteq \Omega_0$. We have $\bar{\Omega}_1 \subseteq f_0^{-1}(\bar{V}')$. Then set $\Omega = \varphi(\Omega_1)$, $y = \psi(y_0)$ and $f : \bar{\Omega} \rightarrow \mathbb{R}^n$ defined by $f = \psi \circ f_0|_{\bar{\Omega}_1} \circ \varphi^{-1}$ – the definition is correct, as $f_0(\bar{\Omega}_1) \subseteq \bar{V}' \subset V$. We have

$$f(\partial\Omega) = f \circ \varphi(\partial\Omega_1) = \psi \circ f_0(\partial\Omega_1) \subseteq \psi(\partial V'),$$

hence $y = \psi(y_0) \notin f(\partial\Omega)$ as ψ is homeomorphism. Thus we have admissible triple $(f, \Omega, y) \in \mathcal{A}_{\mathbb{R}^n}$. Taking into account $\psi^{-1} \circ f \circ \varphi = f_0|_{\bar{\Omega}_1}$, equality (1) gives

$$d_1(f_0, \Omega_1, y_0) = d_2(f_0, \Omega_1, y_0).$$

Applying (D5) we obtain

$$d_1(f_0, \Omega_0, y_0) = d_1(f_0, \Omega_1, y_0) \quad \text{and} \quad d_2(f_0, \Omega_0, y_0) = d_2(f_0, \Omega_1, y_0)$$

which finishes the proof. $\square$

Theorem 3.6. *Let M and N be connected oriented topological manifolds of dimension n . Then there exists at most one degree $d : \mathcal{A}_{M,N} \rightarrow \mathbb{Z}$.*

Proof. Let d_1 and d_2 be two degrees $\mathcal{A}_{M,N} \rightarrow \mathbb{Z}$. Let $(f, \Omega, y) \in \mathcal{A}_{M,N}$. We need to prove $d_1(f, \Omega, y) = d_2(f, \Omega, y)$. According to Proposition 3.4 there exists $g : \bar{\Omega} \rightarrow N$ such that $g \simeq f \text{ rel } \partial\Omega$ and $g^{-1}(y)$ is a finite set. Then (D3) implies that $d_i(f, \Omega, y) = d_i(g, \Omega, y)$, $i = 1, 2$. If $g^{-1}(y) = \emptyset$, then (D4) implies that $d_i(g, \Omega, y) = 0$ and the conclusion follows. Otherwise, let $g^{-1}(y) = \{x_1, \dots, x_m\} \subset \Omega$. For each x_j fix a standard chart (V_j, φ_j)

containing x_j so that $V_j \subseteq \Omega$ and $V_j \cap V_{j'} = \emptyset$ for $j \neq j'$. Moreover fix open neighbourhood V'_j such that $\overline{V'_j} \subset V_j$. Now (D 2) (extended inductively for more than two sets) implies

$$d_i(g, \Omega, y) = \sum_{j=1}^m d_i(g, V'_j, y) \quad \text{for } i = 1, 2.$$

Then applying Lemma 3.5 for $X = V_j$, $Y = N$ and the restriction of d_i to $\mathcal{A}_{V_j, N}$ (which trivially satisfies the degree axioms), we see that $d_1(g, V'_j, y) = d_2(g, V'_j, y)$ for all $j = 1, \dots, m$. Thus $d_1(g, \Omega, y) = d_2(g, \Omega, y)$ and consequently $d_1(f, \Omega, y) = d_2(f, \Omega, y)$. $\square$

4. EXISTENCE

We will need the following form of Poincaré-Alexander-Lefschetz duality for Čech cohomology (see, e.g. [2], Lemma 8.1 and Theorem 8.3 of Chapter VI, and [10], Theorem 3.44 (for singular cohomology version))

Theorem 4.1. *Let M be oriented n -manifold and $L \subseteq K \subseteq M$ be compact subsets. Then the following diagram with exact rows is commutative*

$$\begin{array}{ccccccc} \check{H}^{n-p}(K, L) & \longrightarrow & \check{H}^{n-p}(K) & \longrightarrow & \check{H}^{n-p}(L) & \xrightarrow{\delta} & \check{H}^{n-p+1}(K, L) \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ H_p(M-L, M-K) & \rightarrow & H_p(M, M-K) & \rightarrow & H_p(M, M-L) & \xrightarrow{\partial} & H_{p-1}(M-L, M-K) \end{array}$$

Here the horizontal rows are the long exact sequence of a pair and of a triple (for the lower row). The vertical arrows, induced by cap product with the locally finite fundamental top homology class, are isomorphisms. Here the connecting homomorphism δ is modified to $(-1)^{n-p+1}$ times the usual connecting homomorphism δ of the long exact sequence of a pair.

In particular, $\check{H}^{n-p}(K) \cong H_p(M, M-K)$.

The isomorphisms will be denoted by $\mathcal{D}$.

Lemma 4.2. *Let U be a relatively compact open subset of an oriented n -manifold M and $C \subseteq \overline{U}$ be closed with $\partial U \subseteq C$. Then we have natural isomorphisms*

$$\check{H}^n(\overline{U}, C) \cong H_0(U - C).$$

In particular, when $C = \partial U$

$$\check{H}^n(\overline{U}, \partial U) \cong H_0(U).$$

Proof. Apply Theorem 4.1 with $K = \overline{U}$, $L = C$ and $p = 0$. We obtain

$$\check{H}^n(\overline{U}, C) \cong H_0(M - C, M - \overline{U}).$$

Since $M - C$ is the disjoint union of $U - C$ and $M - \overline{U}$, on the level of chains we have

$$C_*(M - C) \cong C_*(U - C) \oplus C_*(M - \overline{U}).$$

Hence

$$C_*(M - C, M - \overline{U}) = \frac{C_*(M - C)}{C_*(M - \overline{U})} \cong C_*(U - C).$$

Then, on the level of homology we have

$$H_0(M - C, M - \overline{U}) \cong H_0(U - C),$$

which finishes the proof. $\square$

Lemma 4.3. *Let U be relatively compact open subset of an oriented n -manifold M and $C \subseteq \overline{U}$ be closed with $\partial U \subseteq C$. Let $\{U_j\}_{j \geq 1}$ be the connected components of $U - C$. Then the inclusions $(\overline{U}_j, \partial U_j) \hookrightarrow (\overline{U}, C)$ induce isomorphism*

$$\check{H}^n(\overline{U}, C) \cong \bigoplus_{j \geq 1} \check{H}^n(\overline{U}_j, \partial U_j).$$

Proof. Consider the following diagram induced by inclusions

$$\begin{array}{ccc} \check{H}^n(\overline{U}, C) & \dashrightarrow & \bigoplus_{j \geq 1} \check{H}^n(\overline{U}_j, \partial U_j) \\ \cong \downarrow & & \downarrow \cong \\ H_0(U - C) & \xrightarrow{\cong} & \bigoplus_{j \geq 1} H_0(U_j) \end{array}$$

The vertical arrows are isomorphisms induced by duality from Lemma 4.2. The horizontal arrows are induced by inclusions. Thus by naturality of duality with respect to inclusions, the diagram is commutative. The lower arrow is clearly isomorphism. Commutativity of the diagram then implies that the dashed arrow is isomorphism as well. $\square$

Lemma 4.4. *Let U relatively compact connected open subset of an oriented n -manifold M and $x \in U$. Then the inclusion $(\overline{U}, \partial U) \hookrightarrow (\overline{U}, \overline{U} - x)$ induces isomorphism*

$$\check{H}^n(\overline{U}, \partial U) \cong \check{H}^n(\overline{U}, \overline{U} - x).$$

Proof. Take an open coordinate ball $B \subseteq U$ centered at x . Consider the following diagram induced by inclusions

$$\begin{array}{ccc} \check{H}^n(\overline{U}, \overline{U} - B) & \dashrightarrow & \check{H}^n(\overline{U}, \partial U) \\ \cong \downarrow & & \downarrow \cong \\ H_0(B) & \xrightarrow{\cong} & H_0(U) \end{array}$$

As in the proof of Lemma 4.3, we see that the diagram is commutative and the vertical arrows are isomorphisms. The lower horizontal arrow is isomorphism as well, as U is connected. Thus the dashed arrow is isomorphism.

Finally, observe that $\overline{U} - x$ deformation retracts onto $\overline{U} - B$, so that the inclusion $(\overline{U}, \overline{U} - B) \hookrightarrow (\overline{U}, \overline{U} - x)$ induces isomorphism

$$\check{H}^n(\overline{U}, \overline{U} - B) \cong \check{H}^n(\overline{U}, \overline{U} - x).$$

□

Remark 4.5. The orientation on M specifies a generator of $\check{H}^n(\overline{U}, \overline{U} - x)$ and consequently an isomorphism $\check{H}^n(\overline{U}, \overline{U} - x) \cong \mathbb{Z}$. Then for U as in Lemma 4.4, we would consider a preferred generator of $\check{H}^n(\overline{U}, \partial U)$ the one obtained from the orientation generator of $\check{H}^n(\overline{U}, \overline{U} - x)$ via the isomorphism in Lemma 4.4.

Now we proceed with the construction of the degree. Let M and N be oriented topological n -manifolds and $(f, \Omega, y) \in \mathcal{A}_{M,N}$. Let $\Omega_\alpha, \alpha \in A$ be the connected components of Ω . Then, using Lemma 4.2 and Lemma 4.3 with $C = \partial\Omega$ we get

$$\check{H}^n(\overline{\Omega}, \partial\Omega) \cong \bigoplus_{\alpha \in A} H_0(\Omega_\alpha) \cong \bigoplus_{\alpha \in A} \check{H}^n(\overline{\Omega}_\alpha, \partial\Omega_\alpha) \cong \bigoplus_{\alpha \in A} \mathbb{Z}.$$

For each $\alpha \in A$, let μ_α be the preferred generator of $\check{H}^n(\overline{\Omega}_\alpha, \partial\Omega_\alpha)$ as described in Remark 4.5.

Let σ_y be the generator of $\check{H}^n(N, N - y) \cong \mathbb{Z}$ corresponding to the orientation on N . The map of pairs $f : (\overline{\Omega}, \partial\Omega) \rightarrow (N, N - y)$ induces homomorphism

$$f^* : \check{H}^n(N, N - y) \rightarrow \bigoplus_{\alpha \in A} \check{H}^n(\overline{\Omega}_\alpha, \partial\Omega_\alpha).$$

Thus for any $\alpha \in A$ there exists unique $p_\alpha \in \mathbb{Z}$, such that $(f|_{\overline{\Omega}_\alpha})^*(\sigma_y) = p_\alpha \mu_\alpha$. We define

$$d(f, \Omega, y) := \sum_{\alpha \in A} p_\alpha. \quad (2)$$

Equivalently, identifying $\check{H}^n(N, N - y)$ with $\mathbb{Z}$ and $\bigoplus_{\alpha \in A} \check{H}^n(\overline{\Omega}_\alpha, \partial\Omega_\alpha)$ with

$\mathbb{Z}^A := \bigoplus_{\alpha \in A} \mathbb{Z}$ via the chosen generators, we define

$$d(f, \Omega, y) := s^A \circ f^*(1),$$

where $s_A : \mathbb{Z}^A \rightarrow \mathbb{Z}$ is "summation of components" map.

Let us see that p_α is nonzero for finitely many α so that the definition is meaningful. The set $f^{-1}(y)$ is compact as it is closed subset of the compact $\overline{\Omega}$. But $f^{-1}(y) \cap \partial\Omega = \emptyset$, hence $f^{-1}(y)$ is covered by the components $\Omega_\alpha, \alpha \in A$ of Ω . As $f^{-1}(y)$ is compact, finitely many components Ω_α cover $f^{-1}(y)$. Let $\Omega_{\alpha'}$ be a component such that $\Omega_{\alpha'} \cap f^{-1}(y) = \emptyset$. Then $f|_{\overline{\Omega}_{\alpha'}}$ factors through

$$(\overline{\Omega}_{\alpha'}, \partial\Omega_{\alpha'}) \xrightarrow{f} (N - y, N - y) \hookrightarrow (N, N - y).$$

Hence $(f|_{\overline{\Omega}_{\alpha'}})^* = 0$ as $\check{H}^n(N - y, N - y) = 0$, and thus $p_{\alpha'} = 0$.

This discussion also implies

Corollary 4.6. (D 4) holds for d .

Proposition 4.7. Let M and N be oriented topological n -manifolds, $\Omega \subseteq M$ be relatively compact open subset and $f : \overline{\Omega} \rightarrow N$. Then the map

$$N - f(\partial\Omega) \rightarrow \mathbb{Z} : y \mapsto d(f, \Omega, y),$$

where d is defined by (2), is locally constant and hence constant on each component of $N - f(\partial\Omega)$.

Proof. As $\overline{\Omega}$ is compact, so is $f(\partial\Omega)$. Therefore, given a point $y_0 \in N - f(\partial\Omega)$, there exists open neighbourhood V of y_0 , such that $V \subseteq N - f(\partial\Omega)$. We may assume V is a coordinate ball. For any $y \in V$, the map of pairs $f_y := f : (\overline{\Omega}, \partial\Omega) \rightarrow (N, N - y)$ factors through

$$(\overline{\Omega}, \partial\Omega) \xrightarrow{f} (N, N - V) \hookrightarrow (N, N - y).$$

As $N - y$ deformation retracts onto $N - V$, the inclusion at the end induces isomorphism on relative cohomology groups independent of y . After fixing generators as above and for $\check{H}^n(N, N - V)$, $f_y^* : \check{H}^n(N, N - y) \rightarrow \check{H}^n(\overline{\Omega}, \partial\Omega)$ coincides as a map $\mathbb{Z} \rightarrow \mathbb{Z}^A$ with the map $f^* : \check{H}^n(N, N - V) \rightarrow \check{H}^n(\overline{\Omega}, \partial\Omega)$, hence is independent of $y \in V$. Going back to the definition of d , the map $y \mapsto d(f, \Omega, y)$ is constant on V . $\square$

In the next three propositions we check that d satisfies the axioms for degree. We assume M and N are connected oriented topological n -manifolds.

Proposition 4.8. The map $d : \mathcal{A}_{M,N} \rightarrow \mathbb{Z}$ defined by (2) satisfies (D 1).

Proof. As f is homeomorphism, there is only one $\alpha_0 \in A$ such that $\Omega_{\alpha_0} \cap f^{-1}(y) \neq \emptyset$. As we saw in the argument that d is well defined, $d(f, \Omega, y)$ is defined only by the map

$$f^* : \check{H}^n(N, N - y) \rightarrow \check{H}^n(\overline{\Omega}_{\alpha_0}, \partial\Omega_{\alpha_0}).$$

Consider the diagram with vertical arrow induced by inclusion and f_d, f_h induced by f :

$$\begin{array}{ccc} \check{H}^n(N, N - y) & \xrightarrow{f_h^*} & \check{H}^n(\overline{\Omega}_{\alpha_0}, \overline{\Omega}_{\alpha_0} - f^{-1}(y)) \\ & \searrow f_d^* & \downarrow i \\ & & \check{H}^n(\overline{\Omega}_{\alpha_0}, \partial\Omega_{\alpha_0}) \end{array}$$

It is clearly commutative. After identifying the three groups with $\mathbb{Z}$ via the chosen generators, the horizontal $f_h^*(1) = 1$, as f is orientation preserving, and $i(1) = 1$ as well, due to the choice of generator in $\check{H}^n(\overline{\Omega}_{\alpha_0}, \partial\Omega_{\alpha_0})$, via Remark 4.5. Commutativity implies that $f_d^*(1) = 1$, hence $d(f, \Omega, y) = 1$. $\square$

Proposition 4.9. *The map $d : \mathcal{A}_{M,N} \rightarrow \mathbb{Z}$ defined by (2) satisfies (D 2).*

Proof. In view of Proposition 2.3 and Corollary 4.6, it is enough to prove d satisfies (D 2)'.

Let Ω_1, Ω_2 be disjoint open subsets of Ω and $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and $y \in N - f(\partial\Omega_1 \cup \partial\Omega_2)$. Clearly $\partial\Omega \subseteq \partial\Omega_1 \cup \partial\Omega_2$. The map f induces several maps of pairs: $f : (\bar{\Omega}, \partial\Omega) \rightarrow (N, N - y)$, $\hat{f} : (\bar{\Omega}, \partial\Omega_1 \cup \partial\Omega_2) \rightarrow (N, N - y)$ and $f_k : (\bar{\Omega}_k, \partial\Omega_k) \rightarrow (N, N - y)$, $k = 1, 2$. Thus f factors through

$$(\bar{\Omega}, \partial\Omega) \xhookrightarrow{i} (\bar{\Omega}, \partial\Omega_1 \cup \partial\Omega_2) \xrightarrow{\hat{f}} (N, N - y), \quad (3)$$

and f_k factors through

$$(\bar{\Omega}_k, \partial\Omega_k) \xhookrightarrow{i_k} (\bar{\Omega}, \partial\Omega_1 \cup \partial\Omega_2) \xrightarrow{\hat{f}} (N, N - y). \quad (4)$$

Applying the relative version of Mayer-Vietoris long exact sequence of Čech cohomology for the pairs $(\bar{\Omega}_1, \partial\Omega_1), (\bar{\Omega}_2, \partial\Omega_2)$ we have

$$\begin{aligned} \dots &\longrightarrow \check{H}^k(\bar{\Omega}, \partial\Omega_1 \cup \partial\Omega_2) \xrightarrow{\phi} \check{H}^k(\bar{\Omega}_1, \partial\Omega_1) \oplus \check{H}^k(\bar{\Omega}_2, \partial\Omega_2) \\ &\longrightarrow \check{H}^k(\bar{\Omega}_1 \cap \bar{\Omega}_2, \partial\Omega_1 \cap \partial\Omega_2) \end{aligned}$$

where $\phi([\theta]) := (i_1^*[\theta], i_2^*[\theta])$. As $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \partial\Omega_1 \cap \partial\Omega_2$, the corresponding relative groups vanish, so the map ϕ is an isomorphism.

Let A_k , $k = 1, 2$, index the components of Ω_k .

Consider the following diagram:

$$\begin{array}{ccccc} \check{H}^n(N, N - y) & \xrightarrow{\hat{f}^*} & \check{H}^n(\bar{\Omega}, \partial\Omega_1 \cup \partial\Omega_2) & \xrightarrow{i^*} & \check{H}^n(\bar{\Omega}, \partial\Omega) \\ & \searrow (f_1^*, f_2^*) & \cong \downarrow \phi & & \downarrow s^A \\ & & \check{H}^n(\bar{\Omega}_1, \partial\Omega_1) \oplus \check{H}^n(\bar{\Omega}_2, \partial\Omega_2) & \xrightarrow{s^{A_1} + s^{A_2}} & \mathbb{Z} \\ & & \cong \mathbb{Z}^{A_1} \oplus \mathbb{Z}^{A_2} & & \end{array}$$

The left triangle is commutative as $\phi = (i_1^*, i_2^*)$ and the factorization in (4). Now observe that if $\Omega_{\alpha_k} \subseteq \Omega_\alpha$ and e_{α_k} is a basis element of $\mathbb{Z}^{A_k}$, then $i^* \circ \phi^{-1}(e_{\alpha_k}) = e_\alpha$. Indeed, by naturality of duality with respect to inclusions, the following diagram is commutative:

$$\begin{array}{ccc} \check{H}^n(\bar{\Omega}, \partial\Omega_1 \cup \partial\Omega_2) & \xrightarrow{i^*} & \check{H}^n(\bar{\Omega}, \partial\Omega) \\ \mathcal{D} \downarrow \cong & & \cong \downarrow \mathcal{D} \\ H_0(\Omega_1 \cup \Omega_2) & \longrightarrow & H_0(\Omega) \end{array}$$

As the lower arrow sends the generator of $H_0(\Omega_{\alpha_k})$ to the generator of $H_0(\Omega_\alpha)$, $i^* \circ \phi^{-1}$ sends the preferred generator of $\check{H}^n(\bar{\Omega}_{\alpha_k}, \partial\Omega_{\alpha_k})$ to the preferred generator of $\check{H}^n(\bar{\Omega}_\alpha, \partial\Omega_\alpha)$. Since $(s^{A_1} + s^{A_2})(e_{\alpha_k}) = 1 = s^A(e_\alpha)$, we obtain that the right rectangle is commutative on basis elements, hence it is commutative.

Starting with 1 at $\check{H}^n(N, N - y)$ and moving along the upper row and then down to $\mathbb{Z}$ we end up with $d(f, \Omega, y)$ because of the factorization in (3). Alternatively, moving down right to $\mathbb{Z}$ we end up with $d(f, \Omega_1, y) + d(f, \Omega_2, y)$. Commutativity of the diagram finishes the proof. $\square$

Proposition 4.10. *The map $d : \mathcal{A}_{M,N} \rightarrow \mathbb{Z}$ defined by (2) satisfies (D3).*

Proof. Let $h : [0, 1] \times \bar{\Omega} \rightarrow N$ and $y : [0, 1] \rightarrow N$ be such that the triple $(h(t, \cdot), \Omega, y(t))$ is admissible for all $t \in [0, 1]$. Fix $t_0 \in [0, 1]$. Equip N with metric compatible with the topology. Let $\varepsilon = \text{dist}(h(t_0, \partial\Omega), y(t_0)) > 0$. Then, using $\partial\Omega$ is compact, there exists $\delta > 0$ such that for all $t \in J := (t_0 - \delta, t_0 + \delta) \cap [0, 1]$ the point $y(t)$ belongs to the same connected component of $N - h(t_0, \partial\Omega)$ as $y(t_0)$, $\text{dist}(y(t_0), y(t)) < \varepsilon/2$ and $\text{dist}(h(t, \partial\Omega), y(t_0)) > \varepsilon/2$. Thus Proposition 4.7 implies

$$d(h(t_0, \cdot), \Omega, y(t_0)) = d(h(t_0, \cdot), \Omega, y(t)) \quad (5)$$

for all $t \in J$. Moreover, for $s, t \in J$ holds $y(t) \notin h(s, \partial\Omega)$, as

$$\text{dist}(h(s, \partial\Omega), y(t_0)) > \frac{\varepsilon}{2} > \text{dist}(y(t_0), y(t)).$$

Thus for fixed $t \in J$ we observe that $h(t_0, \cdot) : (\bar{\Omega}, \partial\Omega) \rightarrow (N, N - y(t))$ is homotopic, as a map of pairs, to $h(t, \cdot) : (\bar{\Omega}, \partial\Omega) \rightarrow (N, N - y(t))$ via the homotopy $h(s, \cdot)$. Thus the induced maps $h(t_0, \cdot)^* : \check{H}^n(N, N - y(t)) \rightarrow \check{H}^n(\bar{\Omega}, \partial\Omega)$ and $h(t, \cdot)^* : \check{H}^n(N, N - y(t)) \rightarrow \check{H}^n(\bar{\Omega}, \partial\Omega)$ coincide. Hence

$$d(h(t_0, \cdot), \Omega, y(t)) = d(h(t, \cdot), \Omega, y(t)),$$

which, combined with (5) shows that the map $t \mapsto d(h(t, \cdot), \Omega, y(t))$ is locally constant. Hence it is constant on $[0, 1]$. $\square$

Combining the previous three propositions with Theorem 3.6 we obtain

Theorem 4.11. *Given topological connected oriented n -manifold M and N , there exists unique degree $d : \mathcal{A}_{M,N} \rightarrow \mathbb{Z}$.*

From now on, we denote the unique degree between manifolds M and N by $d_{M,N}$ or just d if M and N are clear from the context.

We conclude this section with the following

Proposition 4.12. *Let M and N be connected oriented n -manifolds with N being noncompact. Let $\Omega \subseteq M$ be relatively compact open set, $y \in N$ and $f : \partial\Omega \rightarrow N - y$. If $f_1, f_2 : \bar{\Omega} \rightarrow N$ are such that $f_1|_{\partial\Omega} \equiv f_2|_{\partial\Omega} \equiv f$, then*

$$d(f_1, \Omega, y) = d(f_2, \Omega, y).$$

Proof. The map $f_k : (\bar{\Omega}, \partial\Omega) \rightarrow (N, N - y)$ ($k = 1, 2$) induces the following diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \check{H}^{n-1}(N - y) & \xrightarrow{\delta_N} & \check{H}^n(N, N - y) & \longrightarrow & \check{H}^n(N) \longrightarrow \dots \\ & & f^* \downarrow & & f_k^* \downarrow & & \downarrow \\ \dots & \longrightarrow & \check{H}^{n-1}(\partial\Omega) & \xrightarrow{\delta_\Omega} & \check{H}^n(\bar{\Omega}, \partial\Omega) & \longrightarrow & \check{H}^n(\bar{\Omega}) \longrightarrow \dots \end{array}$$

The naturality of long exact sequence of cohomology means that the diagram is commutative. As N is noncompact, $\check{H}^n(N) = 0$. Thus δ_N is surjective. Hence for any $\alpha \in \check{H}^n(N, N - y)$ there exists $\alpha' \in \check{H}^{n-1}(N - y)$ such that $\delta_N(\alpha') = \alpha$. Commutativity of the diagram implies

$$f_k^*(\alpha) = \delta_\Omega \circ f^*(\alpha').$$

As this holds for both $k = 1$ and $k = 2$, we obtain $f_1^* = f_2^*$ hence the equality between degrees follows. $\square$

The following example shows, that in general the above proposition does not hold for compact manifold N .

Example 4.13. Let $M = \mathbb{R}^n$, $N = S^n$, $\Omega = B^n$ – the open unit ball in $\mathbb{R}^n$. Let $f_1 : \bar{\Omega} \rightarrow S^n$ be the constant "north pole" p map and $f_2 : \bar{\Omega} \rightarrow S^n$ be the quotient map, sending $\partial\Omega$ to the north pole p of S^n . Now for $y \in N$, $y \neq p$ we have $d(f_1, \Omega, y) = 0$ by (D4). However, the map f_2 is homeomorphism between Ω and $S^n - p$, so that $d(f_2, \Omega, y) = 1$.

5. BROUWER DEGREE AND WINDING NUMBER OF A MAP

Proposition 5.1. *Let M and N be connected oriented n -manifolds and M is compact. Then for any $f : M \rightarrow N$ and any $y \in N$ the triple (f, M, y) is admissible and the number $d(f, M, y)$ is independent of y . We denote this number by $\deg_{M,N}(f)$. We have the following:*

- If N is noncompact, then $\deg_{M,N}(f) = 0$ for any map $f : M \rightarrow N$.
- If N is compact, after identifying $H^n(M)$ and $H^n(N)$ with $\mathbb{Z}$ via the respective orientations, f induces homomorphism $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}$ and then

$$\deg_{M,N}(f) = \varphi(1).$$

Proof. The set M is open (as a subset of M) and relatively compact (as it is compact). We have $\partial M = \emptyset$ hence $y \in N - f(\partial M)$ and so the triple (f, M, y) is admissible. Moreover $N - f(\partial M)$ has one connected component, so Proposition 4.7 implies that $d(f, M, y)$ is independent of y .

If N is noncompact then, as $f(M)$ is compact, we have $N - f(M) \neq \emptyset$. Thus for $y \in N - f(M)$ by (D4) we obtain $d(f, M, y) = 0$ and the conclusion follows because, as we saw, this integer is independent of y .

Now assume N is compact. Going back to the definition of degree in (2), we need to examine the map $f^* : H^n(N, N - y) \rightarrow H^n(M, \emptyset)$. Consider the diagram of long exact sequences of cohomology groups constructed by the map induced by $f : (M, \emptyset) \rightarrow (N, N - y)$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^n(N, N - y) & \xrightarrow{q_1} & H^n(N) & \longrightarrow & H^n(N - y) \longrightarrow \cdots \\ & & \downarrow f^* & & \downarrow \varphi & & \downarrow \\ \cdots & \longrightarrow & H^n(M, \emptyset) & \xrightarrow{q_2} & H^n(M) & \longrightarrow & H^n(\emptyset) \longrightarrow \cdots \end{array}$$

As $N - y$ is noncompact, $H^n(N - y) = 0$, so q_1 is surjective. As both groups are isomorphic to $\mathbb{Z}$, the map q_1 is isomorphism. Similarly, q_2 is isomorphism. Moreover, after identifying all the groups with $\mathbb{Z}$ via the respective orientations, $q_1(1) = 1$ and $q_2(1) = 1$. By naturality of long exact sequences of cohomology, the above diagram is commutative, hence

$$\varphi(1) = \varphi(q_1(1)) = q_2(f^*(1)) = f^*(1).$$

As $d(f, M, y)$ is defined via f^* , the equality follows. $\square$

Observe that in view of the compact case, the noncompact is consistent with the fact that top cohomology vanishes for noncompact manifolds.

The number $\varphi(1)$ is usually called Brouwer degree. In the previous proposition we showed how we could recover it from the degree constructed earlier.

Now we continue with the construction of a generalized winding number for a function $f : \partial\Omega \rightarrow S^{n-1}$, where Ω is relatively compact open subset of a connected oriented n -manifold M with $n \geq 2$. In what follows S^{n-1} is assumed embedded in $\mathbb{R}^n$ in the standard way. As such, it is equipped with the boundary orientation of the unit ball, which inherits its orientation from the natural orientation of $\mathbb{R}^n$.

Theorem 4.1 gives the map $\mathcal{D} : \check{H}^{n-1}(\partial\Omega) \rightarrow H_1(M, M - \partial\Omega)$. The long exact sequence of the pair $(M, M - \partial\Omega)$ gives

$$\widetilde{H}_1(M, M - \partial\Omega) \xrightarrow{\tilde{\partial}} \widetilde{H}_0(M - \partial\Omega).$$

If $\{C_i\}_{i \in I}$ are the connected components of $M - \partial\Omega$, take the class $[C_i]$ having as representative the 0-cycle of an arbitrary point in C_i . The set of classes $\{[C_i]\}_{i \in I}$ generate $H_0(M - \partial\Omega)$ and specify the isomorphism

$$\widetilde{H}_0(M - \partial\Omega) \cong \mathbb{Z}_0^I := \left\{ (d_i)_{i \in I} \in \mathbb{Z}^I \left| \begin{array}{l} d_i \neq 0 \text{ for finitely many } i \\ \text{and } \sum_{i \in I} d_i = 0 \end{array} \right. \right\}.$$

For a connected oriented n -manifold M , relatively compact open subset Ω and $f : \partial\Omega \rightarrow S^{n-1}$, we have homomorphism $f^* : \check{H}^{n-1}(S^{n-1}) \rightarrow \check{H}^{n-1}(\partial\Omega)$. Denote by $s_\Omega^I : \mathbb{Z}_0^I \rightarrow \mathbb{Z}$ the map

$$s_\Omega^I((d_i)_{i \in I}) = \sum_{j : C_j \subseteq \Omega} d_j.$$

Then we define a generalized winding number as

$$\text{wind}(f) := (-1)^n s_\Omega^I \circ \tilde{\partial} \circ \mathcal{D} \circ f^*(1).$$

Given a map $f : C \rightarrow S^{n-1}$ where C is a closed subset of a manifold M , the Dugundji extension theorem implies that f could be extended (non-uniquely) to a function $\hat{f} : M \rightarrow \mathbb{R}^n$. If $C = \partial\Omega$ for some relatively compact open subset of M , then Proposition 4.12 implies that $d_{M, \mathbb{R}^n}(\hat{f}, \Omega, 0)$ is independent of the extension $\hat{f} : \overline{\Omega} \rightarrow \mathbb{R}^n$.

The following result could be viewed as a generalization of Proposition 4.5 in [19], p. 158.

Proposition 5.2. *Let M be connected oriented n -manifold ($n \geq 2$). Let Ω be relatively compact open subset of M and $f : \partial\Omega \rightarrow S^{n-1}$.*

a) *If $\hat{f} : \bar{\Omega} \rightarrow \mathbb{R}^n$ is any extension of f , then*

$$\text{wind}(f) = d_{M, \mathbb{R}^n}(\hat{f}, \Omega, 0).$$

b) *Assume $\bar{\Omega}$ is a connected manifold with boundary, $\Omega = \text{int } \bar{\Omega}$ and $\partial\Omega$ is also connected. If $\partial\Omega$ is equipped with the induced boundary orientation then*

$$\text{wind}(f) = \deg_{\partial\Omega, S^{n-1}}(f).$$

Proof. Consider the following diagram

$$\begin{array}{ccccccc} \check{H}^n(\mathbb{R}^n, \mathbb{R}^n - 0) & & & & & & \\ \downarrow \cong & & & & & & \\ \check{H}^n(\mathbb{R}^n, S^{n-1}) & \xrightarrow{\hat{f}^*} & \check{H}^n(\bar{\Omega}, \partial\Omega) & \xrightarrow[\cong]{\mathcal{D}} & H_0(M - \partial\Omega, M - \bar{\Omega}) & \xrightarrow{s^A} & \mathbb{Z} \\ \delta \uparrow \cong & & \uparrow \delta_\Omega & & \partial_1 \uparrow & & \uparrow s_\Omega^I \\ \check{H}^{n-1}(S^{n-1}) & \xrightarrow{f^*} & \check{H}^{n-1}(\partial\Omega) & \xrightarrow[\cong]{\mathcal{D}} & H_1(M, M - \partial\Omega) & \xrightarrow[\tilde{\partial}]{} & \widetilde{H}_0(M - \partial\Omega) \\ & & & & & & \mathbb{Z}_0^I \end{array}$$

The leftmost upper isomorphism follows by the diagram of long exact sequences in cohomology induced by the inclusion $(\mathbb{R}^n, S^{n-1}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n - 0)$, as $\mathbb{R}^n$ is contractible and $\mathbb{R}^n - 0$ deformation retracts onto S^{n-1} .

The left square is part of the diagram of long exact sequences in cohomology induced by the map $\hat{f} : (\bar{\Omega}, \partial\Omega) \rightarrow (\mathbb{R}^n, S^{n-1})$. In particular δ is isomorphism (as $\mathbb{R}^n$ is contractible) and naturality implies that this square is commutative. After identifying the three leftmost groups with $\mathbb{Z}$ via the standard orientation of $\mathbb{R}^n$, the isomorphisms between them send 1 to 1.

The middle square is obtained from Theorem 4.1 with $p = 1$, $L = \partial\Omega$, $K = \bar{\Omega}$. In particular, it is commutative up to sign $(-1)^n$.

Let us see that the right square is commutative. The connecting homomorphism ∂_1 is described as follows: starting with a relative cycle γ , whose boundary $\partial\gamma$ belongs to $M - \partial\Omega$, collect the parts of $\partial\gamma$ which belong to $(M - \partial\Omega) - (M - \bar{\Omega}) = \Omega$. Applying s^A just sums the coefficients which remained. On the other hand $\tilde{\partial}$ takes all the boundary of the cycle γ . And s_Ω^I only sums the coefficients of elements of Ω .

Regarding a), starting with 1 from the group $\check{H}^n(\mathbb{R}^n, \mathbb{R}^n - 0)$ and moving along the upper row we obtain $d_{M, \mathbb{R}^n}(\hat{f}, \Omega, 0)$ at the end. Starting with 1 from $\check{H}^{n-1}(S^{n-1})$ and moving along the lower row, and then up via s_Ω^I , we obtain $(-1)^n \text{wind}(f)$ at the end. As pointed earlier, the 1's on the leftmost three groups are sent to one another by the two isomorphisms. The sign-commutativity of the diagram concludes the equality.

$$(-1)^n \text{wind}(f). \quad \square$$

Ω be relatively compact open subset of M . Then

$$\check{H}^{n-1}(\partial\Omega) \cong \widetilde{H}_0(M - \partial\Omega).$$

Proof. Applying Theorem 4.1 for $p = 1$ and $K = \partial\Omega$, we obtain

$$\check{H}^{n-1}(\partial\Omega) \cong H_1(M, M - \partial\Omega).$$

Write the long exact sequence of the pair $(M, M - \partial\Omega)$ in reduced homology:

$$\cdots \longrightarrow \widetilde{H}_1(M) \longrightarrow \widetilde{H}_1(M, M - \partial\Omega) \xrightarrow{\partial} \widetilde{H}_0(M - \partial\Omega) \longrightarrow \widetilde{H}_0(M).$$

By assumption $\widetilde{H}_1(M) = H_1(M) = 0$. As M is connected, $\widetilde{H}_0(M) = 0$. Thus the map $\tilde{\partial}$ is isomorphism, which finishes the proof. The isomorphism is $\tilde{\partial} \circ \mathcal{D}$. $\square$

Homotopic maps clearly have the same winding numbers, as they induce the same maps on cohomology. The converse is also true in some situations.

Proposition 5.4. *Let M be connected oriented n -manifold ($n \geq 2$) with $H_1(M) = 0$. Let Ω be relatively compact and connected open subset of M such that $M - \overline{\Omega}$ is connected and let $f_1, f_2 : \partial\Omega \rightarrow S^{n-1}$. If $\text{wind}(f_1) = \text{wind}(f_2)$, then f_1 is homotopic to f_2 .*

Proof. This follows directly from Hopf's classification theorem (see, e.g. [10, 22]), which states that if X is compact Hausdorff of covering dimension at most m , then there is a bijection between the homotopy classes of maps $X \rightarrow S^m$ and $\check{H}^m(X)$, given by $[f] \mapsto f^*\omega$, where ω is the orientation class generator of $H^m(S^m)$. Consider the following row ($k = 1, 2$):

$$\check{H}^{n-1}(S^{n-1}) \xrightarrow{f_k^*} \check{H}^{n-1}(\partial\Omega) \xrightarrow[\tilde{\partial \circ \mathcal{D}}]{\cong} \widetilde{H}_0(M - \partial\Omega) \xrightarrow[\mathbf{s}_O^I]{\cong} \mathbb{Z}$$

The second arrow is isomorphism by Lemma 5.3 and the third arrow is isomorphism as $M - \partial\Omega$ has two connected components. After identification with $\mathbb{Z}$, starting with 1 from $H^{n-1}(S^{n-1})$ we end up with $(-1)^n \text{wind}(f_k)$ by the definition of winding number. As $\text{wind}(f_1) = \text{wind}(f_2)$ and the second and third arrows are isomorphisms, we obtain $f_1^*(1) = f_2^*(1)$. Now, Hopf's theorem applied with $m = n - 1$ and $X = \partial\Omega$ (which has covering dimension at most $n - 1$, see [21]) imply that f_1 and f_2 are homotopic. $\square$

6. COMPOSITION FORMULA

Theorem 6.1. *Let M , N and P be connected oriented topological n -manifolds, Ω is relatively compact open subset of M and Ψ is relatively compact open subset of N . Assume $f : \bar{\Omega} \rightarrow N$ is such that $f(\bar{\Omega}) \subseteq \Psi$. Let $g : \bar{\Psi} \rightarrow P$. Let $\{\Psi_j\}_{j \geq 1}$ be the connected components of the open set $\Psi - f(\partial\Omega)$. Choose arbitrary point $z_j \in \Psi_j$ for all $j \geq 1$. Let $y \in P - g(f(\partial\Omega) \cup \partial\Psi)$. Then*

$$d_{M,P}(g \circ f, \Omega, y) = \sum_j d_{M,N}(f, \Omega, z_j) d_{N,P}(g, \Psi_j, y).$$

Proof. It is easy to observe that the relevant triples are admissible, and that the sum is finite. The composition $g \circ f$ factors as

$$(\bar{\Omega}, \partial\Omega) \xrightarrow{f} (\bar{\Psi}, f(\partial\Omega) \cup \partial\Psi) \xrightarrow{g} (P, P - y).$$

On the level of cohomology we have

$$\check{H}^n(\bar{\Omega}, \partial\Omega) \xleftarrow{f^*} \check{H}^n(\bar{\Psi}, f(\partial\Omega) \cup \partial\Psi) \xleftarrow{g^*} \check{H}^n(P, P - y)$$

Applying Lemma 4.3 for $C = f(\partial\Omega) \cup \partial\Psi$ we get

$$\check{H}^n(\bar{\Psi}, f(\partial\Omega) \cup \partial\Psi) \cong \bigoplus_{j \geq 1} \check{H}^n(\bar{\Psi}_j, \partial\Psi_j). \quad (6)$$

Observe that the inclusion $(\bar{\Psi}_j, \partial\Psi_j) \hookrightarrow (N, N - z_j)$ is the identity map on Ψ_j hence (D 1) and (D 4) imply that induced homomorphism

$$\check{H}^n(N, N - z_j) \rightarrow \check{H}^n(\bar{\Psi}_j, \partial\Psi_j) \begin{cases} \text{isomorphism,} & j = j' \\ \text{trivial,} & j \neq j' \end{cases}. \quad (7)$$

Consider the diagram (with fixed j):

$$\begin{array}{ccccccc} \check{H}^n(P, P - y) & \xrightarrow{g_1^*} & \check{H}^n(\bar{\Psi}, f(\partial\Omega) \cup \partial\Psi) & \xrightarrow{f_1^*} & \check{H}^n(\bar{\Omega}, \partial\Omega) & \xrightarrow{s^A} & \mathbb{Z} \\ & \searrow g_2^* & \downarrow r_1 & \swarrow r & \uparrow f_2^* & & \\ & & \check{H}^n(\bar{\Psi}_j, \partial\Psi_j) & \xleftarrow[r_2]{\cong} & \check{H}^n(N, N - z_j) & & \end{array}$$

The maps r, r_1, r_2 are induced by inclusion, hence the corresponding triangle is commutative. Moreover, by (7), r_2 is isomorphism. The map $g : (\bar{\Psi}_j, \partial\Psi_j) \rightarrow (P, P - y)$ factors as

$$(\bar{\Psi}_j, \partial\Psi_j) \hookrightarrow (\bar{\Psi}, f(\partial\Omega) \cup \partial\Psi) \xrightarrow{g} (P, P - y),$$

hence the upper left triangle is commutative. Similarly, the map $f : (\bar{\Omega}, \partial\Omega) \rightarrow (N, N - z_j)$ factors as

$$(\bar{\Omega}, \partial\Omega) \xrightarrow{f} (\bar{\Psi}, f(\partial\Omega) \cup \partial\Psi) \hookrightarrow (N, N - z_j),$$

hence the upper right triangle is commutative.

Choosing generators induced by orientations, we may identify certain groups with $\mathbb{Z}$, as indicated on the diagram. Moreover, by (6) we may identify $\check{H}^n(\bar{\Psi}, f(\partial\Omega) \cup \partial\Psi)$ with $\bigoplus_{j \geq 1} \mathbb{Z}$. As the isomorphism is induced by inclusion,

we obtain that r_1 is projection onto the j -th summand.

Thus $d_{M,P}(g \circ f, \Omega, y) = s^A \circ f_1^* \circ g_1^*(1)$. Moreover, commutativity of the upper left triangle shows $r_1 \circ g_1^*(1) = g_2^*(1) = d_{N,P}(g, \Psi_j, y)$, hence $g_1^*(1) = (d_{N,P}(g, \Psi_j, y))_{j \geq 1}$. Next, commutativity of the lower triangle and (7) (also with Ψ_j replaced by $\Psi_{j'}$) implies that $r(1) = (0, \dots, 0, \underbrace{1}_j, 0, \dots)$. And

commutativity of the upper right triangle implies

$$s^A \circ f_1^*(0, \dots, 0, \underbrace{1}_j, 0, \dots) = s^A \circ f_2^*(1) = d_{M,N}(f, \Omega, z_j).$$

Combining these observations we obtain

$$\begin{aligned} d_{M,P}(g \circ f, \Omega, y) &= s^A \circ f_1^* \circ g_1^*(1) = s^A \circ f_1^*(d_{N,P}(g, \Psi_j, y))_{j \geq 1} \\ &= \sum_{j \geq 1} d_{N,P}(g, \Psi_j, y) s^A \circ f_1^*(0, \dots, 0, \underbrace{1}_j, 0, \dots) \\ &= \sum_{j \geq 1} d_{N,P}(g, \Psi_j, y) d_{M,N}(f, \Omega, z_j), \end{aligned}$$

□

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