

The Truncated Octahedral Conjecture

Thomas Hales and Lark Song

Abstract

Among three-dimensional parallelohedra of fixed volume, the Archimedean truncated octahedron uniquely minimizes surface area, affirming a conjecture of K. Bezdek from 2006. One inequality in the proof is verified by an exact computer-assisted certificate. The simplex, frame, and graphical-zonotope inequalities used in the proof hold in arbitrary dimension.

1 Introduction

A *parallelohedron* is a convex polyhedron that admits a face-to-face tiling of Euclidean three-space by translations. In 1885, Fedorov proved that there are five combinatorially distinct types: parallelepipeds, hexagonal prisms, rhombic dodecahedra, elongated dodecahedra, and truncated octahedra [Fed85]. Their structure was developed further by Minkowski and Venkov [Min97, Ven54]. Venkov proved, and McMullen later proved independently, that the face-to-face hypothesis can be dropped [Ven54, McM80]. A useful introduction is [Val25].

For a convex body $\mathcal{Z} \subset \mathbb{R}^n$, $n \geq 2$, write its isoperimetric quotient as

$$\iota_n(\mathcal{Z}) := \frac{\text{area}(\partial\mathcal{Z})}{\text{vol}_n(\mathcal{Z})^{(n-1)/n}},$$

where area denotes $(n-1)$ -dimensional boundary measure.

The Archimedean truncated octahedron is, up to scale, the Voronoi cell of the body-centered cubic lattice. Fix a representative $\mathcal{Z}_{p,3}$ of this similarity class. The subscript p refers to the symmetric permutohedron, whose n -dimensional version $\mathcal{Z}_{p,n}$ is defined in Section 4. K. Bezdek conjectured in 2006 that it minimizes the isoperimetric quotient among parallelohedra [Bez06, Conjecture 7.5]; see also [Lán22, Conjecture 1]. Lángi proved the analogous result for mean width [Lán22]. Joós and Lángi obtained the sharp surface area bound for rhombic dodecahedra [JL23].

Cesaroni and Novaga recently proved that the body-centered cubic lattice is a strict local minimizer among three-dimensional lattice Voronoi cells [CN26a]. Voronoi conjectured that every parallelohedron is affinely equivalent to a lattice Voronoi cell [Vor08], which is known to hold through dimension five [Gar25]. Since the isoperimetric quotient is not affine invariant, local minimality does not reduce to lattice Voronoi cells. Song extended the local minimality from the lattice-Voronoi subclass to the full parameter space of parallelohedra [Son26]. Shortly after our preprint was posted to the arXiv, Cesaroni and Novaga announced an independent proof of the main result, Theorem 1.1 [CN26b].

The Kelvin problem asks for a least-area partition of space into cells of equal volume. Kelvin conjectured that a perturbation of the Archimedean truncated-octahedral tiling is

optimal; Weaire and Phelan later found a counterexample [Tho87, WP94]. Various revised conjectures add assumptions on the structure of the foam that make the Kelvin cell a plausible candidate for the optimal partition [SM96]. Bezdek’s conjecture can be interpreted as a modification of the original Kelvin problem that restricts the equal-volume partitions of space to tilings by parallelotopes.

Although Bezdek’s conjecture provides the clearest statement, the conjecture has historical precedents. Because of the way it provides the scaffolding for the Kelvin cell, the Archimedean truncated octahedron has made recurrent appearances in isoperimetric investigations. Lewis supplied polyhedral calculations and comparisons [Lew25]; Matzke can arguably be interpreted as asserting the optimality of the Archimedean truncated octahedron among space-filling convex polyhedra [Mat27, p. 341]; in 1943 Sharp stated, under the additional assumption of equal-edge lengths, that a uniform equal-volume polyhedral division of minimum surface area has cells that are Archimedean truncated octahedra [Sha43]. D’Arcy Thompson likewise described the Archimedean truncated octahedron as the homogeneous space-filler enclosing a given volume with least surface area [Tho42, p. 734]. In Kelvin’s terminology, following Bravais, a homogeneous division has congruent translational cells and hence, in the convex polyhedral case, consists of parallelotopes [Tho94].

This article gives a proof of Bezdek’s conjecture in partnership with AI:

Theorem 1.1 (truncated octahedral conjecture). *Every parallelotope $\mathcal{Z} \subset \mathbb{R}^3$ satisfies*

$$\iota_3(\mathcal{Z}) \geq \iota_3(\mathcal{Z}_{p,3}) = \frac{3(1 + 2\sqrt{3})}{4^{2/3}}. \quad (1)$$

Equality holds if and only if $\mathcal{Z}$ is an Archimedean truncated octahedron.

Outline of the proof

Fedorov’s classification reduces the main theorem to five combinatorial types. The parallelepiped and hexagonal-prism cases follow from the classical sharp inequality for polygonal prisms. The remaining arguments have a common higher-dimensional formulation in terms of cofactors and spanning trees.

Section 2 proves a weighted simplex inequality and its reciprocal balanced-frame form. Applied to the adjugate of a zone matrix, the latter gives the sharp bound for the $(n + 1)$ -generator rhombic class and an elongated family with one additional subset-sum generator. Their three-dimensional specializations exceed the Archimedean truncated-octahedral value.

For the weighted K_{n+1} -graphical family, the zone matrix B_β has determinant equal to the volume. Its adjugate records the facet coefficients, indexed by cuts of the graph. The canonical affine map $d^{1/(2n)} B_\beta^{-1/2}$, where $d = \det B_\beta$, determines a surface-area tensor H_β . A trace estimate removes the arbitrary special-linear deformation, and one calibrated Jensen inequality gives

$$\text{area}(\partial \mathcal{Z}(\beta, M)) \geq 2n(\det H_\beta)^{1/n} \geq 2nd^{(n-1)/n} \left(\frac{\varphi^3}{\psi} \right)^{1/(2n)}.$$

Here φ and ψ are explicit polynomial scalars. Equality in Jensen forces all normalized quadratic values on the cut normals to agree, making B_β scalar and all zone weights equal. The affine equality condition then forces M to be orthogonal.

Section 6 specializes these results to the five Fedorov types. For the truncated-octahedral type the one remaining input is $f_3 = \varphi^3 - \kappa_3^2 \psi \geq 0$. This homogeneous degree-18 inequality in six variables is proved in the appendix by exact Bernstein coefficients in $\mathbb{Q}(\sqrt{3})$ on a finite cover. Nonnegativity alone suffices, since geometric uniqueness has already been established by the analytic argument.

Notation

Bold lowercase Roman letters denote vectors in $\mathbb{R}^n$; the main letters are **u, v, w, n** for balanced frames, simplex vertices, reciprocal edges, and normals. We use $\mathbf{c}_I$ for a cofactor vector indexed by a set of generating vectors, reserving $\mathbf{w}_{ij}$ for a reciprocal edge indexed by its two endpoints. Uppercase Roman letters denote matrices, except that I, J, S are also used for finite index sets. We write I_m for the $m \times m$ identity matrix, $\mathbf{e}_i$ for the i th standard basis vector, and $\mathbf{1}$ for the column vector all of whose entries are one; the ambient dimension is specified by the context. Thus $\mathbf{1}^T$ is the corresponding row vector. The Kronecker delta δ_{ij} is one when $i = j$ and zero otherwise; it is distinct from the positive determinant scalar $\delta_{\mathbf{u}}$ used for a balanced frame. We write $\text{adj } M$ for the adjugate. The symbol $\mathcal{Z}$ denotes a zonotope or polyhedron. Uppercase sans serif letters denote graphs and combinatorial collections. Discrete indices include i, j, k, ℓ, b, e , and the dimension is n . Lowercase Roman and Greek letters denote real scalars and functions. Superscripts T and c denote transpose and set complement. Hats indicate the specified rescaling, not necessarily unit length; a circle indicates canonical affine position. We use $|\cdot|$ for absolute value, Euclidean norm, and cardinality. A *zone* is a direction of generating segments. In dimension three, subscripts h and s refer to hexagonal and square facets.

2 Cofactors, simplices, and balanced frames

Throughout Sections 2–5, the dimension is $n \geq 2$. Let K_{n+1} be the complete graph on vertex set $[n+1] = \{1, \dots, n+1\}$, with edge set $E = \binom{[n+1]}{2}$, and spanning trees T . For edge weights $\rho_e \geq 0$, write

$$\tau(\rho) := \sum_{J \in T} \prod_{e \in J} \rho_e, \quad c_n := \frac{n}{\sqrt{2}} (n+1)^{1/(2n)}. \quad (2)$$

2.1 Determinant and zonotope formulas

For vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^n$ and $I = \{i_1 < \dots < i_{n-1}\} \subset [m]$, define the cofactor vector $\mathbf{c}_I$ by

$$\mathbf{c}_I \cdot \mathbf{x} = \det(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_{n-1}}, \mathbf{x}) \quad (\mathbf{x} \in \mathbb{R}^n). \quad (3)$$

Its coordinates are signed minors. In dimension three it is $\mathbf{v}_{i_1} \times \mathbf{v}_{i_2}$. Reordering I changes only its sign. If $a_i \geq 0$ and Y has columns $\sqrt{a_i} \mathbf{v}_i$, Cauchy–Binet gives

$$\det(YY^T) = \sum_{|I|=n} \left(\prod_{i \in I} a_i \right) \det(\mathbf{v}_i : i \in I)^2. \quad (4)$$

The corresponding identity for the adjugate is equally direct.

Lemma 2.1. *With the notation above,*

$$\operatorname{adj}(YY^T) = \sum_{|J|=n-1} \left(\prod_{j \in J} a_j \right) \mathbf{c}_J \mathbf{c}_J^T. \quad (5)$$

Proof. Absorb the constants $\sqrt{a_i}$ into $\mathbf{v}_i$ and $\mathbf{c}_I$, and assume without loss of generality that $a_i = 1$ for all i . The right side of (5) then has the form CC^T , where C has columns $\mathbf{c}_I$. Abbreviate $d_I = \det(\mathbf{v}_i : i \in I)$, and let Y_I be the submatrix of Y with columns I . We have for appropriate signs $\sigma_{j,J} = \pm 1$, using the cofactor formula for the adjugate of Y_I :

$$\begin{aligned} YY^T CC^T &= \sum_{j,J} \mathbf{v}_j (\mathbf{v}_j \cdot \mathbf{c}_J) \mathbf{c}_J^T = \sum_{|I|=n} d_I \sum_{J \cup \{j\} = I} \sigma_{j,J} \mathbf{v}_j \mathbf{c}_J^T \\ &= \sum_{|I|=n} d_I Y_I \operatorname{adj} Y_I = \sum_{|I|=n} d_I^2 I_n = \det(YY^T) I_n. \end{aligned}$$

The last equality is Cauchy–Binet. If Y has rank n , this identity proves (5). For the remaining cases, first pad Y with zero columns if $m < n$. Both sides are polynomial in the entries of Y , and full-rank matrices are dense, so the same identity holds without the rank assumption. $\square$

For the full-dimensional zonotope $\mathcal{Z} = \sum_{i=1}^m [-a_i \mathbf{v}_i/2, a_i \mathbf{v}_i/2]$, the standard volume and surface-area formulas are

$$\operatorname{vol}_n(\mathcal{Z}) = \sum_{|I|=n} \left(\prod_{i \in I} a_i \right) |\det(\mathbf{v}_i : i \in I)|, \quad (6)$$

$$\frac{1}{2} \operatorname{area}(\partial \mathcal{Z}) = \sum_{|I|=n-1} \left(\prod_{k \in I} a_k \right) |\mathbf{c}_I|. \quad (7)$$

See [She74] and [JL23, Corollary 1]. Thus the same cofactors occur in the adjugate and in surface area. Under an invertible linear map M , they transform by $\mathbf{c}_I \mapsto \det(M) M^{-T} \mathbf{c}_I$.

2.2 A weighted simplex inequality

Theorem 2.2 (weighted simplex inequality). *Let $\mathbf{v}_1, \dots, \mathbf{v}_{n+1} \in \mathbb{R}^n$ be affinely independent, and put*

$$\ell_{ij} := |\mathbf{v}_i - \mathbf{v}_j|, \quad \delta := |\det(\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_{n+1} - \mathbf{v}_1)|.$$

For nonnegative edge weights ρ_e with $\tau(\rho) > 0$,

$$\sum_{e \in E} \rho_e \ell_e \geq c_n (\delta \tau(\rho))^{1/n}. \quad (8)$$

Equality holds if and only if the simplex is regular and all the edge weights are equal.

Proof. Put $s := \sum_e \rho_e \ell_e$, and set

$$\hat{\rho}_e := \frac{\rho_e \ell_e}{s}, \quad \hat{\mathbf{v}}_{ij} := \frac{\mathbf{v}_i - \mathbf{v}_j}{\ell_{ij}}, \quad G := \sum_e \hat{\rho}_e \hat{\mathbf{v}}_e \hat{\mathbf{v}}_e^T.$$

The hypothesis $\tau(\rho) > 0$ means that the positive-weight edges contain a spanning tree. In particular $s > 0$, $\sum_e \hat{\rho}_e = 1$, and the vectors with positive coefficients in G span $\mathbb{R}^n$. Hence G is positive definite, and $\text{tr } G = \sum_e \hat{\rho}_e |\hat{\mathbf{v}}_e|^2 = 1$.

A set of n simplex edges is independent precisely when it is a spanning tree. A cycle gives a linear dependence; an acyclic graph on $n + 1$ vertices with n edges is a tree. Conversely, successively deleting leaves, or performing the corresponding elementary column operations, shows that the determinant of a tree's edge vectors has the same absolute value δ as that of the star at vertex 1. Cauchy–Binet therefore expands the determinant over exactly these trees. Applying trace AM–GM next gives

$$\sum_{J \in \mathcal{T}} \left(\prod_{e \in J} \hat{\rho}_e \right) \left(\frac{\delta}{\prod_{e \in J} \ell_e} \right)^2 = \det G \leq \left(\frac{\text{tr } G}{n} \right)^n = n^{-n}. \quad (9)$$

Equality in this determinant bound holds precisely when $G = I_n/n$.

Let L be the weighted Laplacian on $[n + 1]$ with weights $\hat{\rho}_e$:

$$L_{ij} := \begin{cases} -\hat{\rho}_{ij}, & i \neq j, \\ \sum_{k \neq i} \hat{\rho}_{ik}, & i = j. \end{cases}$$

Each row sums to zero, so $L\mathbf{1} = 0$. Moreover, for $\mathbf{z} = (z_1, \dots, z_{n+1})^T$,

$$\mathbf{z}^T L \mathbf{z} = \sum_{i < j} \hat{\rho}_{ij} (z_i - z_j)^2.$$

Since the positive-weight edges connect all vertices, this expression vanishes exactly when $\mathbf{z}$ is constant. Thus $\ker L = \mathbb{R}\mathbf{1}$: one eigenvalue is zero and the remaining n eigenvalues $\lambda_1, \dots, \lambda_n$ are positive. Their sum is $\text{tr } L = 2 \sum_e \hat{\rho}_e = 2$. The eigenvalue form of the weighted matrix-tree theorem gives $\tau(\hat{\rho}) = (n + 1)^{-1} \lambda_1 \cdots \lambda_n$ [KS19, DKM09]. Scalar AM–GM yields

$$\tau(\hat{\rho}) = \frac{1}{n + 1} \prod_{i=1}^n \lambda_i \leq \frac{1}{n + 1} \left(\frac{\lambda_1 + \cdots + \lambda_n}{n} \right)^n = \frac{1}{n + 1} \left(\frac{2}{n} \right)^n. \quad (10)$$

Cauchy–Schwarz, followed by the two determinant bounds, now gives

$$\frac{\delta \tau(\rho)}{s^n} = \sum_{J \in \mathcal{T}} \left(\prod_{e \in J} \hat{\rho}_e \right) \frac{\delta}{\prod_{e \in J} \ell_e} \leq \sqrt{\tau(\hat{\rho}) \det G} \leq \frac{2^{n/2}}{\sqrt{n + 1} n^n}.$$

Rearranging proves (8).

Suppose equality holds. Equality in (10) forces equal eigenvalues $2/n$. Since L is symmetric, it acts as zero on $\mathbb{R}\mathbf{1}$ and as multiplication by $2/n$ on its orthogonal complement $\mathbf{1}^\perp$. The orthogonal projection onto that complement is $I_{n+1} - (n + 1)^{-1} \mathbf{1}\mathbf{1}^T$. Therefore

$$L = \frac{2}{n} \left(I_{n+1} - \frac{1}{n + 1} \mathbf{1}\mathbf{1}^T \right), \quad \hat{\rho}_e = \frac{2}{n(n + 1)} \quad (e \in \mathbf{E}),$$

where the second identity follows by comparing off-diagonal entries. In particular, every tree has positive weight. Equality in Cauchy–Schwarz then says that $\delta / \prod_{e \in J} \ell_e$, and hence $\prod_{e \in J} \ell_e$, is independent of $J \in \mathcal{T}$. For distinct i, j, k , compare the tree with edges ij, jk and

every remaining vertex attached to k with the tree obtained by replacing ij by ik . Cancelling the common positive factors gives $\ell_{ij} = \ell_{ik}$. Varying the three vertices shows that all edge lengths are equal. Since $\hat{\rho}_e = \rho_e \ell_e / s$, all the original weights ρ_e are equal as well.

Conversely, suppose the simplex is regular, with common edge length ℓ , and the weights are equal. To verify the geometric equality explicitly, translate the centroid to zero and let $Q = (\mathbf{q}_1, \dots, \mathbf{q}_{n+1})$ be the centered vertex matrix. The equal edge lengths and $\sum_i \mathbf{q}_i = 0$ give

$$Q^T Q = \frac{\ell^2}{2} \left(I_{n+1} - \frac{1}{n+1} \mathbf{1}\mathbf{1}^T \right), \quad QQ^T = \frac{\ell^2}{2} I_n.$$

The second identity follows because Q has rank n and QQ^T has the same nonzero eigenvalues as $Q^T Q$. Using the balance identity once more,

$$\sum_{i < j} (\mathbf{q}_i - \mathbf{q}_j)(\mathbf{q}_i - \mathbf{q}_j)^T = (n+1)QQ^T = \frac{(n+1)\ell^2}{2} I_n.$$

Thus $\hat{\rho}_e = 2/(n(n+1))$ indeed gives $G = I_n/n$. The Laplacian has the required equal positive eigenvalues, and all tree edge-length products are equal. Every inequality above is therefore an equality. $\square$

There is an exact isoperimetric interpretation. The sum $\sum_e \rho_e \ell_e$ is the first intrinsic volume of $\mathcal{Z}_\rho = \sum_{i < j} [0, \rho_{ij}(\mathbf{v}_i - \mathbf{v}_j)]$, and $\text{vol}_n(\mathcal{Z}_\rho) = \delta\tau(\rho)$. The first intrinsic volume is proportional to mean width [JL23, Corollary 1 and Remark 1]. Thus the inequality is a sharp mean-width-volume inequality for weighted simplex-edge zonotopes. In dimension three it also follows from Lángi's theorem [Lán22].

2.3 Balanced frames and reciprocal edges

A *balanced* $(n+1)$ -frame is a tuple $(\mathbf{u}_i)_{i=1}^{n+1}$ in $\mathbb{R}^n$ whose sum is zero and for which any n vectors are independent. Its maximal determinants have a common absolute value $\delta_{\mathbf{u}} := |\det(\mathbf{u}_1, \dots, \mathbf{u}_n)| > 0$. The frame is *regular* if, for some $a > 0$,

$$\mathbf{u}_i \cdot \mathbf{u}_j = a((n+1)\delta_{ij} - 1). \quad (11)$$

For $i \neq j$, define the reciprocal edge vector $\mathbf{w}_{ij}$ by

$$\mathbf{u}_k \cdot \mathbf{w}_{ij} = \delta_{\mathbf{u}}(\delta_{ki} - \delta_{kj}) \quad (k \in [n+1]). \quad (12)$$

The balance relation makes these equations consistent: the sum of the right sides is zero, just as the sum of the left sides is zero. Any n of the equations determine $\mathbf{w}_{ij}$ uniquely, and the remaining equation follows from their sum. In particular, $\mathbf{w}_{ji} = -\mathbf{w}_{ij}$. In terms of the earlier cofactor convention, $\mathbf{w}_{ij} = \pm \mathbf{c}_{[n+1] \setminus \{i,j\}}$. The sign is fixed by (12), but it will be immaterial in formulas below.

Corollary 2.3 (balanced-frame inequality). *For a balanced frame and nonnegative weights ρ_{ij} , put*

$$A := \sum_{i < j} \rho_{ij} \mathbf{w}_{ij} \mathbf{w}_{ij}^T.$$

If A is positive definite, then

$$\sum_{i < j} \rho_{ij} |\mathbf{w}_{ij}| \geq c_n \left(\frac{\det A}{\delta_{\mathbf{u}}^{n-1}} \right)^{1/n}. \quad (13)$$

Equality holds if and only if the frame is regular and all ρ_{ij} are equal.

Proof. Write $U = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ and choose simplex vertices

$$\mathbf{v}_{n+1} = 0, \quad \mathbf{v}_i = \delta_{\mathbf{u}} U^{-T} \mathbf{e}_i \quad (1 \leq i \leq n).$$

For $k \leq n$, pairing with $\mathbf{u}_k$ gives $\mathbf{u}_k \cdot \mathbf{v}_i = \delta_{\mathbf{u}} \delta_{ki}$. The equation for $\mathbf{u}_{n+1} = -\sum_{k=1}^n \mathbf{u}_k$ then follows by balance. Thus the differences satisfy (12), and uniqueness gives $\mathbf{w}_{ij} = \mathbf{v}_i - \mathbf{v}_j$. Their simplex determinant is

$$\delta_{\mathbf{v}} = |\det(\delta_{\mathbf{u}} U^{-T})| = \frac{\delta_{\mathbf{u}}^n}{|\det U|} = \delta_{\mathbf{u}}^{n-1}.$$

As in the preceding proof, every tree of reciprocal edges has this absolute determinant. Cauchy–Binet therefore gives $\det A = \delta_{\mathbf{v}}^2 \tau(\rho)$. Since A is positive definite, $\tau(\rho) > 0$, and Theorem 2.2 applies. Substituting $\delta_{\mathbf{v}} \tau(\rho) = \det A / \delta_{\mathbf{v}}$ gives (13).

For equality, the reciprocal simplex is regular precisely when its edge-column matrix $V = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ satisfies $V^T V = b(I_n + \mathbf{1}\mathbf{1}^T)$ for some $b > 0$. Since

$$V^T V = \delta_{\mathbf{u}}^2 (U^T U)^{-1}, \quad (I_n + \mathbf{1}\mathbf{1}^T)^{-1} = I_n - \frac{1}{n+1} \mathbf{1}\mathbf{1}^T,$$

the regular-simplex condition is equivalent to

$$U^T U = \frac{\delta_{\mathbf{u}}^2}{b} \left(I_n - \frac{1}{n+1} \mathbf{1}\mathbf{1}^T \right).$$

Its entries give (11) for $i, j \leq n$, with $a = \delta_{\mathbf{u}}^2 / (b(n+1))$. The pairings involving $\mathbf{u}_{n+1}$ follow from $\mathbf{u}_{n+1} = -\sum_{i=1}^n \mathbf{u}_i$. The converse follows by reversing the calculation. Hence the frame is regular exactly when the reciprocal simplex is regular, and the equality statement follows from Theorem 2.2. $\square$

3 Rhombic and elongated zonotopes

The balanced-frame inequality applies directly to the adjugate of a zone matrix. It gives the sharp surface-area bound for the $(n+1)$ -generator rhombic class and for an elongated extension with one additional subset-sum direction.

3.1 The rhombic class

Let $\mathcal{Z}_{r,n}$ denote a zonotope generated by the vectors of a regular balanced $(n+1)$ -frame, with equal weights. Its similarity class does not depend on the frame or the common weight. This is the projection of an $(n+1)$ -cube along a main diagonal, up to similarity; for $n=3$ it is Kepler's regular rhombic dodecahedron.

Theorem 3.1 (rhombic zonotopes). *Let $(\mathbf{u}_i)_{i=1}^{n+1}$ be a balanced frame and $\beta_i > 0$. Then*

$$\mathcal{Z} := \sum_{i=1}^{n+1} [-\beta_i \mathbf{u}_i/2, \beta_i \mathbf{u}_i/2]$$

satisfies

$$\iota_n(\mathcal{Z}) \geq 2c_n = n\sqrt{2}(n+1)^{1/(2n)}. \quad (14)$$

Equality holds if and only if the frame is regular and all β_i are equal; equivalently, $\mathcal{Z}$ is similar to $\mathcal{Z}_{r,n}$.

This is the surface-area case of a stronger intrinsic-volume theorem of Joós and Lángi [JL23, Theorem 3]. We give a tree-polynomial proof, since the same argument will also treat the elongated family.

Proof. Set $B_\beta := \sum_i \beta_i \mathbf{u}_i \mathbf{u}_i^T$ and $d := \det B_\beta$. All maximal minors of the frame have absolute value $\delta_{\mathbf{u}}$. By (4) and (6),

$$d = \delta_{\mathbf{u}} \text{vol}_n(\mathcal{Z}). \quad (15)$$

With $\mathbf{w}_{ij}$ as in (12), put $\gamma_{ij} := \prod_{k \notin \{i,j\}} \beta_k$. The adjugate and surface-area formulas give

$$\text{adj } B_\beta = \sum_{i < j} \gamma_{ij} \mathbf{w}_{ij} \mathbf{w}_{ij}^T, \quad \frac{1}{2} \text{area}(\partial \mathcal{Z}) = \sum_{i < j} \gamma_{ij} |\mathbf{w}_{ij}|.$$

Applying Corollary 2.3 to $\text{adj } B_\beta$ yields

$$\frac{1}{2} \text{area}(\partial \mathcal{Z}) \geq c_n \left(\frac{\det(\text{adj } B_\beta)}{\delta_{\mathbf{u}}^{n-1}} \right)^{1/n} = c_n \text{vol}_n(\mathcal{Z})^{(n-1)/n}. \quad (16)$$

Equality forces a regular frame and equal γ_{ij} . Comparing two products whose omitted pairs share an index shows that all β_i are equal. The converse follows from the equality case of Corollary 2.3. $\square$

Every configuration of $n+1$ generators with every n independent can be written in this form: orient the generators so that their unique dependence has positive coefficients, and absorb these coefficients into a balanced frame. Reversing a centered generating segment does not change the zonotope.

3.2 One additional subset-sum direction

Theorem 3.2 (elongated zonotopes). *Let $n \geq 3$, let $(\mathbf{u}_i)_{i=1}^{n+1}$ be a balanced frame, and let $S \subset [n+1]$ satisfy $|S|, |S^c| \geq 2$. Put*

$$\mathbf{u}_S := \sum_{i \in S} \mathbf{u}_i = - \sum_{j \in S^c} \mathbf{u}_j.$$

For $\beta_i > 0$ and $t \geq 0$, the zonotope

$$\mathcal{Z}_t := \sum_{i=1}^{n+1} [-\beta_i \mathbf{u}_i/2, \beta_i \mathbf{u}_i/2] + [-t\mathbf{u}_S/2, t\mathbf{u}_S/2] \quad (17)$$

satisfies $\iota_n(\mathcal{Z}_t) \geq 2c_n$. Equality holds if and only if $t = 0$, the frame is regular, and all β_i are equal. For $t > 0$ the bound is strict and is sharp on the closure of this family.

Proof. Set $B_\beta := \sum_i \beta_i \mathbf{u}_i \mathbf{u}_i^T + t \mathbf{u}_S \mathbf{u}_S^T$. Every nonzero maximal minor of the unweighted configuration $(\mathbf{u}_1, \dots, \mathbf{u}_{n+1}, \mathbf{u}_S)$ has absolute value $\delta_{\mathbf{u}}$. To see this for a minor involving $\mathbf{u}_S$, let i, j be the two omitted core indices and put $W_{ij} := \text{span}\{\mathbf{u}_k : k \notin \{i, j\}\}$. The balance relation says that $\mathbf{u}_j$ is congruent to $-\mathbf{u}_i$ modulo the other $n - 1$ core vectors:

$$\mathbf{u}_j \equiv -\mathbf{u}_i \pmod{W_{ij}}, \quad \mathbf{u}_S \equiv \begin{cases} 0, & i, j \text{ on the same shore,} \\ \mathbf{u}_i, & i \in S, j \in S^c, \\ -\mathbf{u}_i, & i \in S^c, j \in S \end{cases} \pmod{W_{ij}}.$$

Adding a vector of W_{ij} to the remaining column does not change the determinant. The minor is therefore zero in the first case and has absolute value $\delta_{\mathbf{u}}$ in the other two cases. Cauchy–Binet (4) sums the squares of these minors, whereas the volume formula (6) sums their absolute values, with the same weight products. Consequently

$$\det B_\beta = \delta_{\mathbf{u}} \text{vol}_n(\mathcal{Z}_t). \quad (18)$$

Nor does the extra zone introduce a new cofactor direction. A cofactor involving $\mathbf{u}_S$ contains the core vectors with indices in some set K of size $n - 2$. Put $W_K := \text{span}\{\mathbf{u}_k : k \in K\}$ and let $\{i, j, \ell\} = [n + 1] \setminus K$. Then

$$\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_\ell \equiv 0 \pmod{W_K}, \quad \mathbf{u}_S \equiv \sum_{k \in S \setminus K} \mathbf{u}_k \pmod{W_K}.$$

If the three omitted indices are all on the same shore, $\mathbf{u}_S$ is congruent to zero modulo W_K . Otherwise it is congruent to the uniquely omitted vector on one shore, or to the negative of that vector, modulo W_K . Multilinearity of the cofactor now shows that every nonzero new cofactor is $\pm \mathbf{w}_{ij}$ for a core pair ij . Grouping equal cofactor lines gives

$$\text{adj } B_\beta = \sum_{i < j} \gamma_{ij} \mathbf{w}_{ij} \mathbf{w}_{ij}^T, \quad \frac{1}{2} \text{area}(\partial \mathcal{Z}_t) = \sum_{i < j} \gamma_{ij} |\mathbf{w}_{ij}|, \quad (19)$$

where

$$\gamma_{ij} = \left(\prod_{k \notin \{i, j\}} \beta_k \right) \begin{cases} 1 + t \sum_{k \in S^c} \beta_k^{-1}, & i, j \in S, \\ 1 + t \sum_{k \in S} \beta_k^{-1}, & i, j \in S^c, \\ 1, & i, j \text{ on different shores.} \end{cases} \quad (20)$$

For example, if $i, j \in S$, the vector $\mathbf{u}_S$ is a sum of the S^c -generators up to sign, and in a nonzero cofactor it can replace exactly one of them. This gives the first line; the other cases follow in the same way. Applying Corollary 2.3 to (19) and using (18) gives the chain (16) with $\mathcal{Z}_t$ in place of $\mathcal{Z}$.

For equality, the frame is regular and every γ_{ij} is the same. Comparing cross-shore pairs first gives $\beta_i = a$ on S and $\beta_j = b$ on S^c . Comparing within-shore pairs with cross-shore pairs then gives

$$a = b + |S^c|t, \quad b = a + |S|t.$$

Thus $(n+1)t = 0$ and $a = b$. These conditions are sufficient by Theorem 3.1. Taking equal core weights in a regular frame and letting $t \downarrow 0$ proves sharpness. $\square$

For $n = 3$ both shores have two elements, giving the usual elongated dodecahedral direction. The theorem concerns the subset-sum family (17), not arbitrary $(n+2)$ -generator zonotopes.

4 Graphical zonotopes and canonical position

4.1 The regular frame and the zone matrix

Fix a regular balanced frame $\mathbf{n}_1, \dots, \mathbf{n}_{n+1} \in \mathbb{R}^n$ normalized by $|\det(\mathbf{n}_1, \dots, \mathbf{n}_n)| = 1$. Its Gram matrix is

$$\mathbf{n}_i \cdot \mathbf{n}_j = (n+1)^{-(n-1)/n}((n+1)\delta_{ij} - 1). \quad (21)$$

Its reciprocal edge vectors, as defined in (12), are characterized by

$$\mathbf{n}_k \cdot \mathbf{w}_{ij} = \delta_{ki} - \delta_{kj}, \quad \mathbf{w}_{ij} = (n+1)^{-1/n}(\mathbf{n}_i - \mathbf{n}_j). \quad (22)$$

For $\beta = (\beta_e)_{e \in E} > 0$ and $M \in \text{SL}(n, \mathbb{R})$, consider

$$\mathcal{Z}(\beta, M) := \sum_{ij \in E} [-\beta_{ij} M \mathbf{w}_{ij} / 2, \beta_{ij} M \mathbf{w}_{ij} / 2]. \quad (23)$$

This is a weighted K_{n+1} -graphical zonotope with a volume-preserving affine deformation. Write $\mathcal{Z}_{p,n} := \mathcal{Z}(\mathbf{1}, I_n)$ for the symmetric permutohedron; for $n = 3$ this is the Archimedean truncated octahedron. Define

$$B_\beta := \sum_{e \in E} \beta_e \mathbf{w}_e \mathbf{w}_e^T, \quad d = d(\beta) := \det B_\beta > 0. \quad (24)$$

The facet pairs are indexed by the nontrivial unordered bipartitions $b = \langle S \mid S^c \rangle$ of $[n+1]$; denote this set by $\mathbf{F}$. For each b , choose either shore and put

$$\mathbf{n}_b := \sum_{i \in S} \mathbf{n}_i, \quad \gamma_b := \tau_S(\beta) \tau_{S^c}(\beta). \quad (25)$$

Here τ_S is the spanning-tree polynomial of the complete graph induced on S , and $\tau_{\{i\}} = 1$. Complementing S changes only the sign of $\mathbf{n}_b$, which will not affect any formula. The corresponding *cut* is the set of edges with one endpoint in each shore; we call $\mathbf{n}_b$ a *cut normal*. A *singleton cut* has one shore $\{i\}$; we abbreviate it by $\langle i \rangle$ and may choose its normal to be $\mathbf{n}_i$. There are $2^n - 1$ facet-normal lines. Moreover,

$$|\mathbf{n}_b|^2 = (n+1)^{-(n-1)/n} |S| |S^c|. \quad (26)$$

Proposition 4.1 (tree and cut formulas). *For the zonotope (23),*

$$\text{vol}_n(\mathcal{Z}(\beta, M)) = d = \tau(\beta), \quad (27)$$

$$\text{adj } B_\beta = \sum_{b \in F} \gamma_b \mathbf{n}_b \mathbf{n}_b^T, \quad (28)$$

$$\frac{1}{2} \text{area}(\partial \mathcal{Z}(\beta, M)) = \sum_{b \in F} \gamma_b |M^{-T} \mathbf{n}_b|. \quad (29)$$

The area of one facet in pair b is $\gamma_b |M^{-T} \mathbf{n}_b|$. Finally, the quadratic value of B_β on a cut normal is the cut weight:

$$r_b := \mathbf{n}_b^T B_\beta \mathbf{n}_b = \sum_{i \in S, j \in S^c} \beta_{ij}. \quad (30)$$

Proof. Put $U = (\mathbf{n}_1, \dots, \mathbf{n}_n)$. By (22), $\mathbf{w}_{i,n+1} = U^{-T} \mathbf{e}_i$ and $\mathbf{w}_{ij} = U^{-T}(\mathbf{e}_i - \mathbf{e}_j)$ for $i, j \leq n$. Thus a set of n generators is independent exactly when it is a spanning tree. Its determinant has absolute value one, since the reduced incidence matrix of a tree has determinant ± 1 and $|\det U| = 1$. Equations (4) and (6) prove (27).

An independent set of $n - 1$ generators is a forest with two components, say S and S^c . Its cofactor is parallel to $\mathbf{n}_b$: both are orthogonal to every internal edge of these components. Adding an edge across the cut makes a spanning tree, and its pairing with $\mathbf{n}_b$ is ± 1 . The cofactor is therefore exactly $\pm \mathbf{n}_b$. The weighted sum of the forests with these components is $\tau_S(\beta) \tau_{S^c}(\beta)$. Grouping the terms in (5) and (7) by the cut gives (28) and (29), including the individual facet formula. Finally $\mathbf{n}_b \cdot \mathbf{w}_{ij}$ is zero for an internal edge and is ± 1 for an edge crossing the cut, proving (30). $\square$

The volume, adjugate, and total surface-area formulas remain valid for nonnegative weights whenever the zonotope is full dimensional; a cut whose coefficient is zero need no longer index an actual facet. The entries of B_β and the polynomials d, γ_b, r_b are homogeneous in β of degrees $1, n, n - 1, 1$, respectively.

4.2 Canonical affine position

The zone matrix selects the determinant-one map and canonical representative

$$\mathring{M}_\beta := d^{1/(2n)} B_\beta^{-1/2}, \quad \mathring{\mathcal{Z}}_\beta := \mathcal{Z}(\beta, \mathring{M}_\beta). \quad (31)$$

For its facet pair b , let $\hat{\mathbf{n}}_b$ be the unit normal and $\hat{\gamma}_b$ the area of one facet. By Proposition 4.1,

$$\hat{\mathbf{n}}_b = \frac{B_\beta^{1/2} \mathbf{n}_b}{\sqrt{r_b}}, \quad \hat{\gamma}_b = d^{-1/(2n)} \gamma_b \sqrt{r_b}. \quad (32)$$

The associated surface-area tensor is

$$H_\beta := \sum_{b \in F} \hat{\gamma}_b \hat{\mathbf{n}}_b \hat{\mathbf{n}}_b^T. \quad (33)$$

Proposition 4.2 (affine reduction). *For every $\beta > 0$ and $M \in \text{SL}(n, \mathbb{R})$,*

$$\text{area}(\partial \mathcal{Z}(\beta, M)) \geq 2n (\det H_\beta)^{1/n}. \quad (34)$$

Proof. Put $A := MM_\beta^{-1}$ and $R := (A^{-1}A^{-T})^{1/2}$. Then R is symmetric positive definite and $\det R = 1$. For every vector $\mathbf{v}$,

$$|A^{-T}\mathbf{v}|^2 = \mathbf{v}^T A^{-1} A^{-T} \mathbf{v} = |R\mathbf{v}|^2.$$

The determinant-one map A carries the canonical representative $\mathring{\mathcal{Z}}_\beta$ to $\mathcal{Z}(\beta, M)$. Its action on facet areas, using (32), therefore gives

$$\frac{1}{2} \text{area}(\partial\mathcal{Z}(\beta, M)) = \sum_{b \in \mathbf{F}} \hat{\gamma}_b |A^{-T} \hat{\mathbf{n}}_b| = \sum_{b \in \mathbf{F}} \hat{\gamma}_b |R \hat{\mathbf{n}}_b|.$$

For a unit vector $\mathbf{v}$, Cauchy–Schwarz gives $|R\mathbf{v}| = |\mathbf{v}||R\mathbf{v}| \geq \mathbf{v}^T R \mathbf{v}$, with equality exactly when $\mathbf{v}$ is an eigenvector of R . The tensor H_β is positive definite because its positively weighted normals span $\mathbb{R}^n$. We may therefore apply trace AM–GM to the positive definite matrix $R^{1/2} H_\beta R^{1/2}$:

$$\begin{aligned} \frac{1}{2} \text{area}(\partial\mathcal{Z}(\beta, M)) &\geq \sum_{b \in \mathbf{F}} \hat{\gamma}_b \hat{\mathbf{n}}_b^T R \hat{\mathbf{n}}_b = \text{tr}(R H_\beta) = \text{tr}(R^{1/2} H_\beta R^{1/2}) \\ &\geq n \det(R^{1/2} H_\beta R^{1/2})^{1/n} = n(\det H_\beta)^{1/n}. \end{aligned} \quad (35)$$

This proves (34). Equality in trace AM–GM holds if and only if $R^{1/2} H_\beta R^{1/2}$ is scalar. Equality in the first estimate additionally requires every canonical facet normal to be an eigenvector of R . $\square$

5 Weighted averaging and equality

The normal lengths supply the calibration for a single application of Jensen’s inequality. Define two positive polynomial scalars by

$$\varphi := \det \left(\sum_{b \in \mathbf{F}} \frac{\gamma_b}{|\mathbf{n}_b|} \mathbf{n}_b \mathbf{n}_b^T \right), \quad \psi := d^{2n-3} \det \left(\sum_{b \in \mathbf{F}} \frac{\gamma_b r_b}{|\mathbf{n}_b|^3} \mathbf{n}_b \mathbf{n}_b^T \right). \quad (36)$$

The polynomials are homogeneous in β of degrees $n(n-1)$ and $3n(n-1)$, respectively.

Theorem 5.1 (graphical zonotope inequality). *For every positive zone vector β and every $M \in \text{SL}(n, \mathbb{R})$,*

$$\iota_n(\mathcal{Z}(\beta, M)) \geq 2n \left(\frac{\varphi(\beta)^3}{\psi(\beta)} \right)^{1/(2n)}. \quad (37)$$

Equality holds if and only if all zone weights are equal and $M \in \text{SO}(n)$.

Proof. Let $\mathbf{B}$ be the n -element subsets of $\mathbf{F}$ whose normals are independent. For $I \in \mathbf{B}$, set

$$p_I := \det(\mathbf{n}_b : b \in I)^2 \prod_{b \in I} \frac{\gamma_b}{|\mathbf{n}_b|}, \quad \hat{r}_b := \frac{r_b}{|\mathbf{n}_b|^2}, \quad \hat{r}_I := \prod_{b \in I} \hat{r}_b. \quad (38)$$

By Cauchy–Binet,

$$\varphi = \sum_{I \in \mathbf{B}} p_I, \quad \psi = d^{2n-3} \sum_{I \in \mathbf{B}} p_I \hat{r}_I. \quad (39)$$

Consequently $\hat{p}_I := p_I/\varphi$ are positive weights of sum one, and $\sum_I \hat{p}_I \hat{r}_I = \psi/(d^{2n-3}\varphi)$.

Put $T := \sum_b \gamma_b r_b^{-1/2} \mathbf{n}_b \mathbf{n}_b^T$. Equations (32)–(33) give

$$H_\beta = d^{-1/(2n)} B_\beta^{1/2} T B_\beta^{1/2}, \quad \det H_\beta = d^{1/2} \det T.$$

Since $\gamma_b/\sqrt{r_b} = (\gamma_b/|\mathbf{n}_b|)\hat{r}_b^{-1/2}$, Cauchy–Binet expands $\det T$ as $\sum_{I \in \mathbf{B}} p_I \hat{r}_I^{-1/2}$. Jensen’s inequality for $x \mapsto x^{-1/2}$ therefore yields

$$\begin{aligned} \det H_\beta &= d^{1/2} \varphi \sum_{I \in \mathbf{B}} \hat{p}_I \hat{r}_I^{-1/2} \\ &\geq d^{1/2} \varphi \left(\sum_{I \in \mathbf{B}} \hat{p}_I \hat{r}_I \right)^{-1/2} = d^{n-1} \frac{\varphi^{3/2}}{\psi^{1/2}}. \end{aligned} \quad (40)$$

Together with (27) and (34), this proves (37).

Suppose equality holds. In a series of small implications, we work backwards from the equality in Jensen’s inequality and equality in the affine estimate to the equality of the zone parameters β_e and the orthogonality of M .

1. *The constants $\hat{r}_I$ are all equal:* Each p_I is positive, for $I \in \mathbf{B}$, because it is a product of positive factors. Equality in Jensen implies that the basis products $\hat{r}_I$ are all equal.
2. *The constants $\hat{r}_b$ are all equal:* First consider the singleton cuts $\langle i \rangle$, with normals $\mathbf{n}_i$. Every n of these $n+1$ normals form a basis. Call them *singleton bases*. Comparing the products for the bases omitting i and omitting j gives $\hat{r}_{\langle i \rangle} = \hat{r}_{\langle j \rangle}$. Let $a > 0$ be their common value. For any cut $b = \langle S \mid S^c \rangle$, choose $i \in S$ and $j \in S^c$. Using the choice $\mathbf{n}_b = \sum_{k \in S} \mathbf{n}_k$, we have a basis $I = \{b\} \cup \{\langle k \rangle \mid k \notin \{i, j\}\}$:

$$|\det(\mathbf{n}_b, \mathbf{n}_k : k \notin \{i, j\})| = |\det(\mathbf{n}_k : k \neq j)| > 0.$$

Comparing this basis with a singleton basis J : $\hat{r}_I = a^{n-1} \hat{r}_b = a^n = \hat{r}_J$. This gives $\hat{r}_b = a$ for all b .

3. *B_β is a scalar multiple of I_n :* Set $D := B_\beta - aI_n$. The identity $\hat{r}_b = a$ says that $\mathbf{n}_b^T D \mathbf{n}_b = 0$ for every $b \in \mathbf{F}$. Polarization gives

$$2\mathbf{n}_i^T D \mathbf{n}_j = (\mathbf{n}_i + \mathbf{n}_j)^T D (\mathbf{n}_i + \mathbf{n}_j) - \mathbf{n}_i^T D \mathbf{n}_i - \mathbf{n}_j^T D \mathbf{n}_j = 0.$$

Thus all pairings of D on a basis vanish, and $B_\beta = aI_n$.

4. *The zone weights β_{ij} are all equal:* The individual zone weights can be recovered from

$$\mathbf{n}_i^T B_\beta \mathbf{n}_j = -\beta_{ij} \quad (i \neq j). \quad (41)$$

Indeed, in the defining sum for B_β , only edge ij has a nonzero pairing with both $\mathbf{n}_i$ and $\mathbf{n}_j$, and its two pairings have opposite signs by (22). Equation (21) now gives $\beta_{ij} = a(n+1)^{-(n-1)/n}$ for every edge, so all zone weights are equal.

5. *H_β is scalar:* Every permutation σ of the regular frame is realized by an orthogonal map Q_σ : the Gram matrix is unchanged, and the frame spans $\mathbb{R}^n$ with its sole relation $\sum_i \mathbf{n}_i = 0$. This map permutes the cut-normal lines and preserves their facet areas,

since the weights are equal and the tree formulas are invariant under vertex relabelling. Therefore $Q_\sigma H_\beta Q_\sigma^T = H_\beta$ for every σ . It follows that $\mathbf{n}_i^T H_\beta \mathbf{n}_i$ has one common value α , and $\mathbf{n}_i^T H_\beta \mathbf{n}_j$ for $i \neq j$ has one common value η . Pairing with $\sum_j \mathbf{n}_j = 0$ gives $\alpha + n\eta = 0$. These pairings are consequently a common positive multiple of the frame Gram matrix (21). Because the frame spans $\mathbb{R}^n$, this proves $H_\beta = hI_n$ for some $h > 0$.

6. $M \in \text{SO}(n)$: Equality in (35) now requires $R^{1/2} H_\beta R^{1/2} = hR$ to be scalar. Since $\det R = 1$, $R = I_n$. Also $\dot{M}_\beta = I_n$, so $A = M$ and $M^{-1} M^{-T} = R^2 = I_n$. Thus M is orthogonal, and its determinant one gives $M \in \text{SO}(n)$.

Conversely, the regular-frame identities give

$$\sum_{i < j} \mathbf{w}_{ij} \mathbf{w}_{ij}^T = (n+1)^{(n-1)/n} I_n.$$

Equal zone weights therefore make B_β scalar, so $\dot{M}_\beta = I_n$ and all $\hat{r}_b$ are equal. The symmetry argument above makes H_β scalar. If $M \in \text{SO}(n)$, then $R = I_n$, and equality holds in Jensen's inequality and in both affine estimates. $\square$

6 Three-dimensional parallelhedra

We now set $n = 3$. The general results handle the truncated-octahedral and dodecahedral families. Only the scalar inequality in Appendix A and the two prism cases remain.

6.1 The six-zone model

For the normalized regular frame we may take

$$\begin{aligned} \mathbf{n}_1 &= 2^{-2/3}(1, 1, 1), & \mathbf{n}_2 &= 2^{-2/3}(1, -1, -1), \\ \mathbf{n}_3 &= 2^{-2/3}(-1, 1, -1), & \mathbf{n}_4 &= 2^{-2/3}(-1, -1, 1). \end{aligned}$$

Its sum is zero and $\det(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) = 1$. The reciprocal generators in (22) satisfy

$$\mathbf{w}_{ij} = \pm(\mathbf{n}_k \times \mathbf{n}_\ell), \quad \{i, j, k, \ell\} = [4].$$

Thus (23) is the standard six-zone parametrization, with complementary edge labels [Lán22, §2.1]; compare the zone-rescaling lemma [McM75, Lemma 1]. Up to translation, every parallelhedron of truncated-octahedral type has the form $\mathcal{Z}(\beta, M)$; this is essentially [Lán22, §2.1], except that Lángi starts with any four nonzero vectors $\mathbf{v}_i$ spanning $\mathbb{R}^3$ and satisfying $\sum_i \mathbf{v}_i = 0$. Choose an invertible matrix A with $\mathbf{v}_i = A\mathbf{n}_i$. Then

$$\mathbf{v}_i \times \mathbf{v}_j = (\text{adj } A)^T(\mathbf{n}_i \times \mathbf{n}_j).$$

Since $\det((\text{adj } A)^T) = (\det A)^2 > 0$, its positive scalar determinant factor can be absorbed into the zone weights, leaving the matrix in $\text{SL}(3, \mathbb{R})$. The rhombic and elongated cases are obtained by deleting zones. Figure 1 illustrates the six parallel classes.

The seven facet pairs consist of the four singleton cuts $F_h = \{\langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle\}$ and the three two-two cuts $F_s = \{\langle 12 \mid 34 \rangle, \langle 13 \mid 24 \rangle, \langle 14 \mid 23 \rangle\}$. The subscripts h and s refer to the

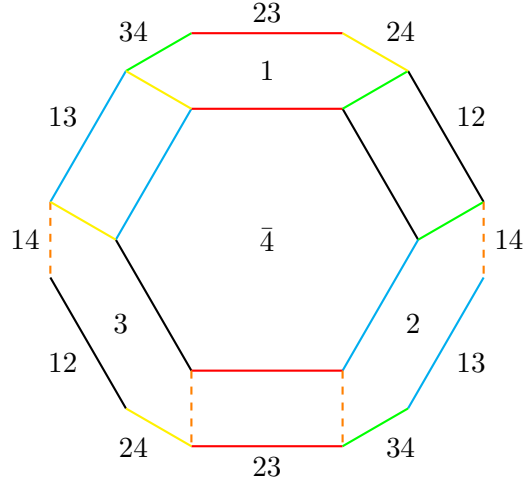


Figure 1: The six zones of a truncated octahedron, labelled by their directions $\mathbb{R}M\mathbf{w}_{ij}$. Each zone contains six edges; hidden edges are omitted. In the symmetric model, the outward normals to facets 1, 2, 3, 4 form a regular tetrahedral frame. Facet 4 is opposite $\bar{4}$.

hexagonal and square facets of the symmetric reference polyhedron; the square-type facets of a general member are parallelograms. For $\{i, j, k, \ell\} = [4]$, the tree formulas become

$$\gamma_{\langle i \rangle} = \beta_{jk}\beta_{j\ell} + \beta_{jk}\beta_{k\ell} + \beta_{j\ell}\beta_{k\ell}, \quad \gamma_{\langle ij|k\ell \rangle} = \beta_{ij}\beta_{k\ell}. \quad (42)$$

For $\beta = \mathbf{1}$ there are 2^4 spanning trees, so $d = 16$, and

$$\gamma_b = \begin{cases} 3, & b \in F_h, \\ 1, & b \in F_s, \end{cases} \quad |\mathbf{n}_b| = \begin{cases} 2^{-2/3}\sqrt{3}, & b \in F_h, \\ 2^{-2/3}2, & b \in F_s. \end{cases} \quad (43)$$

Consequently

$$\text{area}(\partial\mathcal{Z}_{p,3}) = 12(1 + 2\sqrt{3})2^{-2/3}, \quad \iota_3(\mathcal{Z}_{p,3}) = \frac{3(1 + 2\sqrt{3})}{4^{2/3}}.$$

This gives

$$\iota_3(\mathcal{Z}_{p,3}) = 6\kappa^{1/3}, \quad \kappa := \kappa_3 = \frac{(1 + 2\sqrt{3})^3}{128}. \quad (44)$$

6.2 The truncated-octahedral case

The remaining scalar estimate, proved in Appendix A, is

$$\varphi(\beta)^3 \geq \kappa^2 \psi(\beta) \quad (\beta \in \mathbb{R}_{\geq 0}^E, \ n = 3). \quad (45)$$

Theorem 6.1 (truncated-octahedral case). *For every positive zone vector β and every $M \in \text{SL}(3, \mathbb{R})$,*

$$\iota_3(\mathcal{Z}(\beta, M)) \geq \frac{3(1 + 2\sqrt{3})}{4^{2/3}}.$$

Equality holds if and only if all six weights are equal and $M \in \text{SO}(3)$.

Proof. The affine reduction, Jensen's inequality, and (45) give

$$\text{area}(\partial\mathcal{Z}(\beta, M)) \geq 6(\det H_\beta)^{1/3} \geq 6d^{2/3} \left(\frac{\varphi^3}{\psi} \right)^{1/6} \geq 6\kappa^{1/3} d^{2/3}. \quad (46)$$

Since $d = \text{vol}_3(\mathcal{Z}(\beta, M))$, this is the asserted bound. Equality requires equality in Theorem 5.1, which forces equal weights and orthogonal M . Conversely those conditions give $\mathcal{Z}_{p,3}$ up to similarity and equality. $\square$

6.3 The two dodecahedral cases

Corollary 6.2. *Every rhombic-dodecahedral parallelohedron satisfies*

$$\iota_3(\mathcal{Z}) \geq 3 \cdot 2^{5/6},$$

with equality exactly for Kepler's regular rhombic dodecahedron. Every genuine elongated dodecahedron satisfies the strict inequality $\iota_3(\mathcal{Z}) > 3 \cdot 2^{5/6}$; the constant is sharp on its closure.

Proof. In the six-zone model, these classes have, after relabelling,

$$\beta_{34} = 0, \quad \beta_{12} \geq 0, \quad \beta_{13}, \beta_{14}, \beta_{23}, \beta_{24} > 0,$$

with $\beta_{12} = 0$ for the rhombic class and $\beta_{12} > 0$ for the elongated class [Lán22, §2.1]. Orient the four core directions around a cycle:

$$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4) = (M\mathbf{w}_{13}, -M\mathbf{w}_{23}, M\mathbf{w}_{24}, -M\mathbf{w}_{14}).$$

They sum to zero, and every triple has determinant of absolute value one, since it is a spanning tree of K_4 . The remaining direction is $M\mathbf{w}_{12} = \mathbf{u}_1 + \mathbf{u}_2$. Thus Theorems 3.1 and 3.2 apply with $S = \{1, 2\}$, $t = \beta_{12}$, and the corresponding four core weights. Their equality and sharpness statements give the result, since $2c_3 = 3 \cdot 2^{5/6}$. $\square$

6.4 Prisms

The two remaining Fedorov classes are prisms over centrally symmetric polygons. A parallelepiped is a prism over a parallelogram. The hexagonal-prism type has three generating segments in a common plane and one transverse segment. The zonotope is therefore the Minkowski sum of a centrally symmetric planar hexagon and an interval, hence an actual prism. We treat both classes by the same sharp prism inequality.

Steiner solved the prism problem [Ste42]. Pólya treats the right-prism optimization and the comparison with oblique prisms [Pól90, pp. 138–140]; see also Fejes Tóth's discussion of Steiner's optimality condition [FTFTK23, p. 134]. The isoperimetric inequality for polygons goes back to Zenodorus; see also [FT64].

Lemma 6.3 (sharp inequality for polygonal prisms). *Let $\mathcal{Z}_0$ be a convex m -gon in a plane in $\mathbb{R}^3$, and let $\mathcal{Z} = \mathcal{Z}_0 + [-\mathbf{u}/2, \mathbf{u}/2]$ be a nondegenerate, possibly oblique prism. Then*

$$\iota_3(\mathcal{Z}) \geq 6 \left(\frac{m \tan(\pi/m)}{4} \right)^{1/3}. \quad (47)$$

Equality holds if and only if the base is regular, the prism is right, and its height is twice the inradius of the base.

Proof. Let a and p be the area and perimeter of the base $\mathcal{Z}_0$, and let h be the distance between the two base planes. Each lateral parallelogram has area at least h times the length of its base edge. Consequently

$$\text{area}(\partial\mathcal{Z}) \geq 2a + hp, \quad \text{vol}_3(\mathcal{Z}) = ah.$$

Equality for an individual lateral face requires the component of $\mathbf{u}$ parallel to the base plane to be parallel to that base edge. Equality for all lateral faces therefore requires this component to vanish, since a nondegenerate polygon has nonparallel edges. Thus equality in the area estimate holds exactly when the prism is right.

Put $x = hp/a$ and $a_m := m \tan(\pi/m)$, the area of a regular m -gon with inradius one. The polygonal isoperimetric inequality gives $p^2/a \geq 4a_m$, with equality exactly for a regular m -gon. Hence

$$\iota_3(\mathcal{Z}) \geq (2+x)x^{-2/3} \left(\frac{p^2}{a} \right)^{1/3} \geq (2+x)x^{-2/3} (4a_m)^{1/3}.$$

The function of $x > 0$ in this expression has derivative

$$\frac{d}{dx}((2+x)x^{-2/3}) = \frac{x-4}{3x^{5/3}},$$

so its unique minimum is attained at $x = 4$. Substitution gives $\iota_3(\mathcal{Z}) \geq 6(a_m/4)^{1/3}$. Equality requires a right prism, a regular base, and $hp = 4a$. For a regular polygon with inradius r , $a = rp/2$, so the last condition is $h = 2r$. These conditions are also sufficient. $\square$

Proposition 6.4 (two prism classes). *The following sharp inequalities hold.*

(i) *Every parallelepiped $\mathcal{Z}$ satisfies*

$$\iota_3(\mathcal{Z}) \geq 6. \tag{48}$$

Equality holds if and only if $\mathcal{Z}$ is a cube.

(ii) *Every parallelohedron $\mathcal{Z}$ of hexagonal-prism type satisfies*

$$\iota_3(\mathcal{Z}) \geq 3 \cdot 2^{2/3} 3^{1/6}. \tag{49}$$

Equality holds if and only if $\mathcal{Z}$ is a right regular hexagonal prism whose height is twice the inradius of its base.

Proof. Take $m = 4$ and $m = 6$ in Lemma 6.3. For $m = 4$, the regular base is a square and twice its inradius is its side length, so the equality prism is a cube. For $m = 6$, the sharp constant simplifies to $3 \cdot 2^{2/3} 3^{1/6}$, and the equality case is the stated right regular hexagonal prism. $\square$

6.5 Completion of the proof

Proof of Theorem 1.1. The quotient is invariant under translations, rotations, and positive homotheties. Fedorov's classification and the preceding results give

Fedorov type	bound for $\iota_3(\mathcal{Z})$	reference
truncated octahedron	$\geq 3(1 + 2\sqrt{3})/4^{2/3}$	Theorem 6.1
elongated dodecahedron	$> 3 \cdot 2^{5/6}$	Corollary 6.2
rhombic dodecahedron	$\geq 3 \cdot 2^{5/6}$	Corollary 6.2
hexagonal prism	$\geq 3 \cdot 2^{2/3} 3^{1/6}$	Proposition 6.4
parallelepiped	≥ 6	Proposition 6.4

The four non-truncated types have constants strictly greater than $\iota_3(\mathcal{Z}_{p,3})$. In the truncated-octahedral type, Theorem 6.1 gives equality precisely for the Archimedean similarity class. $\square$

A Exact Bernstein certificate in dimension three

Throughout this appendix $n = 3$. We prove (45) by an exact tensor-product Bernstein certificate for

$$f(\beta) := f_3(\beta) = \varphi(\beta)^3 - \kappa^2 \psi(\beta). \quad (50)$$

All certificate arithmetic lies in $\mathbb{Q}(\sqrt{3})$. Only nonnegativity is required: the equality case of the geometric theorem was already settled analytically in Theorem 5.1.

A.1 Reconstructing the polynomial

The seven normal lines in Section 6 realize the rank-three non-Fano matroid [Oxl11]. Their six dependent triples are

$$\{\langle i \rangle, \langle j \rangle, \langle ij \mid k\ell \rangle\}, \quad \{i, j, k, \ell\} = [4], \quad ij \in \mathbf{E}.$$

The other 29 triples form the set $\mathbf{B}$ of normal bases. Of these, 28 have squared determinant 1, while the triple $\mathbf{F}_s$ has squared determinant 4. These values follow by direct computation from the normalized frame.

The graph formulas give $d = \tau(\beta)$, the coefficients (42), and the cut weights (30). For exact reconstruction, put

$$\varepsilon_I := \prod_{b \in I} |\mathbf{n}_b| = 2(\sqrt{3}/2)^{|I \cap \mathbf{F}_h|} \quad (I \in \mathbf{B}).$$

The calibrated data (38) can then be computed directly in $\mathbb{Q}(\sqrt{3})$ as

$$p_I = \frac{\det(\mathbf{n}_b : b \in I)^2}{\varepsilon_I} \prod_{b \in I} \gamma_b, \quad \hat{r}_I = \frac{\prod_{b \in I} r_b}{\varepsilon_I^2}. \quad (51)$$

Thus (39) specializes to

$$\varphi = \sum_{I \in \mathbf{B}} p_I, \quad \psi = d^3 \sum_{I \in \mathbf{B}} p_I \hat{r}_I. \quad (52)$$

The degrees of p_I and $\hat{r}_I$ are 6 and 3; those of φ and ψ are 6 and 18. In particular $f \in \mathbb{Q}(\sqrt{3})[\beta]$ is homogeneous of degree 18. No expanded polynomial is needed as input to the verifier.

Theorem A.1 (direct zone-weight certificate). *For every nonnegative zone vector $\beta \in \mathbb{R}_{\geq 0}^E$,*

$$f(\beta) \geq 0. \quad (53)$$

A.2 Normalization

The construction is invariant under the S_4 -action relabelling the four vertices. For $\beta \neq 0$, homogeneity and edge transitivity allow us to normalize

$$\beta_{34} = 1, \quad 0 \leq \beta_{12}, \beta_{13}, \beta_{14}, \beta_{23}, \beta_{24} \leq 1. \quad (54)$$

Set

$$g(\beta_{12}, \beta_{13}, \beta_{14}, \beta_{23}, \beta_{24}) := f(\beta_{12}, \beta_{13}, \beta_{14}, \beta_{23}, \beta_{24}, 1). \quad (55)$$

The exact expansion of f has 6,083 monomials and multidegree $(6, 6, 6, 6, 6, 6)$. The normalized polynomial g also has 6,083 monomials, minimum total degree 12, maximum total degree 18, and multidegree $(6, 6, 6, 6, 6, 6)$. The verifier reconstructs these data from (51)–(52).

A.3 The finite Bernstein cover

After an affine change from a box to the unit cube, a polynomial of coordinate degrees $m_1, \dots, m_5$ has a tensor-product Bernstein expansion

$$\sum_{0 \leq \alpha_i \leq m_i} b_\alpha \prod_{i=1}^5 \binom{m_i}{\alpha_i} z_i^{\alpha_i} (1 - z_i)^{m_i - \alpha_i}.$$

All the basis functions are nonnegative on the cube. Nonnegative coefficients therefore prove nonnegativity on the box; see [TG19].

Put $a = 1/4$, and split each normalized coordinate into $[0, a]$ and $[a, 1]$. Of the 32 macro-boxes, four are treated separately:

$$\text{all large,} \quad \text{all small,} \quad \{\beta_{13}, \beta_{24}\} \text{ small,} \quad \{\beta_{14}, \beta_{23}\} \text{ small.} \quad (56)$$

The other 28 macro-boxes have nonnegative Bernstein coefficients directly. In each opposite-pair box, split the three large coordinates at $1/2$, giving 8 subboxes for each pair and 16 in all. Their Bernstein coefficients are also nonnegative.

For the all-small box, re-index the five coordinates by $1, \dots, 5$, choose an index j of a largest coordinate, and put

$$\beta_j = t, \quad \beta_i = tu_i \ (i \neq j), \quad 0 \leq t \leq \frac{1}{4}, \quad 0 \leq u_i \leq 1. \quad (57)$$

With $u_j = 1$ suppressed, the minimum degree gives the factorization

$$g(tu) = t^{12} g_j(t, u). \quad (58)$$

Each of the five polynomials g_j has multidegree $(6, 6, 6, 6, 6)$ and nonnegative Bernstein coefficients on its chart box.

For the all-large box, put $z_i = 1 - \beta_i$, choose a largest deviation $z_j = t$, and write $z_i = tu_i$. The chart domain is

$$0 \leq t \leq \frac{3}{4}, \quad 0 \leq u_i \leq 1, \quad u_j = 1. \quad (59)$$

The translated polynomial $g(1 - z)$ has no term of total degree less than two. Hence

$$g(1 - tu) = t^2 h_j(t, u). \quad (60)$$

Each of the five h_j has multidegree $(16, 6, 6, 6, 6)$ and nonnegative Bernstein coefficients on its chart box.

The complete cover is summarized below. For multidegree $(m_1, \dots, m_5)$ there are $\prod_i (m_i + 1)$ coefficients. Every listed coefficient is checked to be nonnegative.

certificate class	rectangles	coefficients per rectangle
ordinary macro-boxes	28	7^5
opposite-pair subboxes	16	7^5
all-small charts	5	7^5
all-large charts	5	$17 \cdot 7^4$
all certificates	54	

A.4 Exact certificate criterion

Proposition A.2 (finite certificate criterion). *Suppose the following checks have been made in exact arithmetic:*

- (a) *the 16 spanning trees and 29 normal bases, with their squared determinants, are reconstructed, and the polynomial f is formed from (51)–(52);*
- (b) *homogeneity, invariance under all 24 vertex permutations, the stated degree data, and the factorizations (58) and (60) are verified;*
- (c) *every Bernstein coefficient on the 54 rectangles in the table is nonnegative.*

Then Theorem A.1 holds.

Proof. The zero zone vector satisfies $f(0) = 0$. Every other nonnegative zone vector can be rescaled and relabelled as in (54). The 28 ordinary boxes and the 16 opposite-pair subboxes cover all but the all-small and all-large boxes. The five maximal-coordinate charts cover the all-small box, and the five maximal-deviation charts cover the all-large box. Their polynomial bounds prove $g \geq 0$ after restoring the factors t^{12} and t^2 , which are nonnegative also at $t = 0$. The cover therefore proves $f \geq 0$ everywhere. No information about the zero faces of the Bernstein expansions is required. $\square$

A.5 The verification package

The package contains `verify_appendix.py` and `exact_bernstein.py`. From its directory, run

```
python verify_appendix.py.
```

A successful run ends with ALL EXACT BETA-BERNSTEIN CHECKS PASSED.

The program reconstructs the graph data, the source polynomials, and every Bernstein coefficient from the formulas above. No expanded coefficient table or saved status file is read

as a proof input. All accepted signs are decided exactly in $\mathbb{Q}(\sqrt{3})$. NumPy is used only to store arrays of Python integers; no floating-point arithmetic, numerical optimization, interval arithmetic, or quantifier elimination is used. File hashes, dependency versions, coefficient totals, and runtime information appear in the package’s `README` file.

Proof of Theorem A.1. The verifier checks conditions (a)–(c) of Proposition A.2. That proposition proves the theorem. $\square$

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AI interactions

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Human interactions

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