

AN ANALOGUE OF THE ERDŐS–KAC THEOREM FOR SIGNED IWASAWA INVARIANTS OF MODULAR FORMS

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ABSTRACT. Erdős and Kac proved that the number of distinct prime factors of an integer has a Gaussian limiting distribution. We prove an analogue for the signed Iwasawa invariants of non-ordinary weight-two modular forms. Under the hypotheses of this paper, we construct infinitely many forms congruent to a given form g that realize every common nonnegative increase of its two signed λ -invariants. We count the levels realizing each increase. In the resulting family, the common increase is asymptotically Gaussian, with mean and variance $\delta_\Pi \log \log X + O_g(1)$ for a positive constant δ_Π .

1. INTRODUCTION

1.1. Background and motivation. A central theme in arithmetic geometry is to understand how arithmetic invariants vary in infinite families. Iwasawa theory encodes the growth of class groups, Selmer groups and related arithmetic objects in p -adic towers in terms of a small collection of algebraic invariants.

The theory originated in Iwasawa’s study of ideal class groups in infinite towers of number fields. Let p be a prime number and K_∞/K be the cyclotomic $\mathbb{Z}_p$ -extension. Let K_n be the unique intermediate field satisfying $[K_n : K] = p^n$. If A_n denotes the p -primary part of the ideal class group of K_n , then the inverse limit of the groups A_n , with respect to the norm maps, is a module over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$. Writing $\#A_n = p^{e_n}$, Iwasawa proved that there are integers $\mu, \lambda \geq 0$ and ν such that

$$e_n = \mu p^n + \lambda n + \nu \quad (1.1)$$

for all sufficiently large n ; see [Iwa59]. This essentially leverages the structure theory of finitely generated and torsion modules over Λ . The integers μ and λ are the Iwasawa invariants of the tower.

Mazur extended this point of view to elliptic curves [Maz72]. Let $E/\mathbb{Q}$ be an elliptic curve and let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$ -extension. The role of the class group is played by the p -primary Selmer group $\text{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)$. For every finite extension $F/\mathbb{Q}$ contained in $\mathbb{Q}_\infty$, the Selmer group fits into the fundamental exact sequence

$$0 \longrightarrow E(F) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \text{Sel}_{p^\infty}(E/F) \longrightarrow \text{III}(E/F)[p^\infty] \longrightarrow 0.$$

When its Pontryagin dual is torsion over the cyclotomic Iwasawa algebra, it has well defined invariants $\mu_p(E)$ and $\lambda_p(E)$ which arise from the structure theory of Λ -modules. These invariants describe the growth of the Selmer groups in the finite layers of $\mathbb{Q}_\infty/\mathbb{Q}$ and are closely related to the growth of Mordell–Weil groups and Tate–Shafarevich groups.

The same construction applies to the Galois representation attached to a modular form. Let h be a normalized newform and let V_h be its p -adic Galois representation. A choice of stable lattice $T_h \subset V_h$ gives a discrete representation $A_h = V_h/T_h$, and one defines a Selmer group over $\mathbb{Q}_\infty$

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by imposing suitable local conditions on A_h . In the ordinary case, the local condition at p is obtained from the ordinary filtration of V_h . Kato's Euler-system shows that the Pontryagin dual of the resulting Selmer group is a finitely generated torsion module over the Iwasawa algebra [Kat04]. Its characteristic power series therefore defines the associated μ - and λ -invariants.

At a non-ordinary prime there is no ordinary filtration giving the required local condition. For elliptic curves with supersingular reduction, Kobayashi introduced the plus and minus Selmer groups [Kob03], while Pollack constructed the corresponding bounded p -adic L -functions when $a_p = 0$ [Pol03]. Signed Coleman maps and signed Selmer groups for modular forms were subsequently developed by Lei, Lei–Loeffler–Zerbes and Sprung [Lei11; LLZ10; LLZ11; Spr12; Spr17]. In the weight-two setting considered here, these constructions give two signed Selmer groups, indexed by $\natural \in \{\sharp, \flat\}$, and hence two pairs of Iwasawa invariants $(\mu^\sharp(h), \lambda^\sharp(h))$ and $(\mu^\flat(h), \lambda^\flat(h))$.

A fundamental question is to understand how Iwasawa invariants vary with respect to congruences of Galois representations. Greenberg and Vatsal proved that, under suitable hypotheses, the vanishing of the Iwasawa μ -invariant is preserved under congruences between ordinary elliptic curves. When the μ -invariants vanish, they expressed the difference of the corresponding λ -invariants as a sum of explicit local correction terms [GV00]. Emerton–Pollack–Weston extended this study to ordinary modular forms in Hida families [EPW06]. They showed that if the algebraic or analytic μ -invariant vanishes for one form, it vanishes throughout the family. They also described how the λ -invariants vary between branches and proved that they are constant along each branch. The algebraic and analytic variation formulae agree; consequently, the main conjecture for one modular form implies it for the other modular forms that are congruent to it. Kim extended the Greenberg–Vatsal comparison to the plus and minus Selmer groups of congruent supersingular elliptic curves [Kim09; Kim13]. Pollack–Weston used congruences to prove Kida-type formulas, including one for supersingular elliptic curves [PW06]. Hatley and Lei then treated congruent non-ordinary modular forms of the same weight. For each sign, they showed that the μ -invariant vanishes for one form if and only if it vanishes for the other; when it vanishes, the difference of the λ -invariants is determined by local terms at primes away from p [HL19].

In [Ray24], we combined the ordinary congruence formula with level raising results of Diamond and Taylor [DT94]. Starting from an ordinary weight-two newform g with $\mu_p(g) = 0$, we showed that, for every $n \geq 0$, there are infinitely many newforms f congruent to g with $\mu_p(f) = 0$ and $\lambda_p(f) = \lambda_p(g) + n$. The present paper extends this construction to non-ordinary modular forms and develops a probabilistic model for the variation of their signed λ -invariants as the tame level varies. In the non-ordinary setting there are two signed λ -invariants. We show that level raising increases them by the same amount, so their difference remains unchanged. We then study the distribution of the common increment as the tame level varies.

1.2. Main results. Let $p \geq 5$ be a prime number. Let

$$g = \sum_{n \geq 1} a_n(g) q^n \in S_2(\Gamma_0(N_g))$$

be a normalized newform of trivial nebentypus, where $p \nmid N_g$, and suppose that g is non-ordinary at p . Write $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and let

$$\bar{\rho}_g : G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\mathbf{k})$$

be the residual Galois representation attached to g at a fixed p -adic place of its coefficient field, where $\mathbf{k}$ is the corresponding residue field. Let $\bar{\chi}_p$ denote the mod- p cyclotomic character, and let

μ_{p^r} denote the group of p^r -th roots of unity. Put

$$H_g = \bar{\rho}_g(G_{\mathbb{Q}}) \subseteq \mathrm{GL}_2(\mathbf{k}).$$

Let L_g be the fixed field of $\ker(\bar{\rho}_g)$. Since g has weight two and trivial nebentypus, $\det \bar{\rho}_g = \bar{\chi}_p$. In particular, $\det(H_g) = \mathbb{F}_p^\times$ and $\mathbb{Q}(\mu_p) \subseteq L_g$. For a prime $q \nmid N_g p$, we write Frob_q for an arithmetic Frobenius element at q .

For $h \in H_g$, put $d(h) = \det(h)$ and define the conjugacy-stable subsets

$$\begin{aligned} \mathcal{C}_g^+ &= \{h \in H_g : d(h) \notin \{\pm 1\}, \mathrm{Tr}(h) = 1 + d(h)\}, \\ \mathcal{C}_g^- &= \{h \in H_g : d(h) \notin \{\pm 1\}, \mathrm{Tr}(h) = -(1 + d(h))\}. \end{aligned}$$

Thus the characteristic polynomial of an element of $\mathcal{C}_g^+$ is $(X - 1)(X - d(h))$, whereas that of an element of $\mathcal{C}_g^-$ is $(X + 1)(X + d(h))$.

For a non-ordinary modular form h and $\natural \in \{\sharp, \flat\}$, let $X^\natural(h)$ denote the Pontryagin dual of its $\natural$ -signed Selmer group over the cyclotomic $\mathbb{Z}_p$ -extension of $\mathbb{Q}$. When $X^\natural(h)$ is torsion, we denote its Iwasawa invariants by $\mu^\natural(h)$ and $\lambda^\natural(h)$. We impose the following hypotheses.

(H1) The representation $\bar{\rho}_g$ is absolutely irreducible, both $\mathcal{C}_g^+$ and $\mathcal{C}_g^-$ are nonempty, and

$$L_g \cap \mathbb{Q}(\mu_{p^2}) = \mathbb{Q}(\mu_p).$$

(H2) The integer N_g is the prime-to- p Artin conductor of $\bar{\rho}_g$.

(H3) For each $\natural \in \{\sharp, \flat\}$, the signed Selmer dual $X^\natural(g)$ is torsion over the cyclotomic Iwasawa algebra and $\mu^\natural(g) = 0$.

The residual-image hypothesis (H1) is weaker than requiring $\mathbf{k} = \mathbb{F}_p$ and $H_g = \mathrm{GL}_2(\mathbb{F}_p)$. In particular, Proposition 3.3 shows that it holds whenever $\mathrm{SL}_2(\mathbf{k}) \subseteq H_g$.

Let Π_g be the set of primes $q \nmid N_g p$ such that $\bar{\rho}_g(\mathrm{Frob}_q) \in \mathcal{C}_g^+$ and $q^{p-1} \not\equiv 1 \pmod{p^2}$. Let Ω_g be the set of primes $\ell \nmid N_g p$ such that $\bar{\rho}_g(\mathrm{Frob}_\ell) \in \mathcal{C}_g^-$. Write δ_Π and δ_Ω for the natural densities of Π_g and Ω_g , respectively. These densities exist by the Chebotarev density theorem, and Proposition 3.4 gives

$$\delta_\Pi = \frac{p-1}{p} \frac{|\mathcal{C}_g^+|}{|H_g|} \quad \text{and} \quad \delta_\Omega = \frac{|\mathcal{C}_g^-|}{|H_g|}. \quad (1.2)$$

Both are positive by (H1). When $\mathbf{k} = \mathbb{F}_p$ and $H_g = \mathrm{GL}_2(\mathbb{F}_p)$, these become

$$\delta_\Pi = \frac{p-3}{p(p-1)} \quad \text{and} \quad \delta_\Omega = \frac{p-3}{(p-1)^2}.$$

The first main result is the following signed analogue of the prescription theorem of [Ray24].

Theorem 1.1. *Assume (H1)–(H3). Let $Q \subset \Pi_g$ and $R \subset \Omega_g$ be finite sets. There exists a normalized weight-two newform f of trivial nebentypus and exact level*

$$N_f = N_g \prod_{q \in Q} q \prod_{\ell \in R} \ell$$

such that $\bar{\rho}_f \simeq \bar{\rho}_g$ and f is non-ordinary at p . For each $\natural \in \{\sharp, \flat\}$, the signed Selmer dual $X^\natural(f)$ is torsion over the cyclotomic Iwasawa algebra and

$$\mu^\natural(f) = 0 \quad \text{and} \quad \lambda^\natural(f) = \lambda^\natural(g) + |Q|.$$

The distinction between the two sets of auxiliary primes is local. A prime in Π_g contributes one to each signed λ -invariant, whereas a prime in Ω_g contributes zero. At the primes dividing N_g , the optimal-level hypothesis implies that the local correction terms for f and g agree. Since these tame corrections do not depend on the signed label, Theorem 1.1 also gives

$$\lambda^\sharp(f) - \lambda^\flat(f) = \lambda^\sharp(g) - \lambda^\flat(g).$$

As both auxiliary sets are infinite, we obtain the following consequence.

Corollary 1.2. *Assume (H1)–(H3). For every $n \geq 0$, there are infinitely many normalized newforms f , of pairwise distinct exact levels, such that $\bar{\rho}_f \simeq \bar{\rho}_g$ and the following assertions hold:*

- (1) $\mu^\sharp(f) = \mu^\flat(f) = 0$,
- (2) $(\lambda^\sharp(f), \lambda^\flat(f)) = (\lambda^\sharp(g) + n, \lambda^\flat(g) + n)$.

Theorem 1.1 determines the signed λ -invariants for each choice of level-raising primes. We now let these choices vary and bound the product of the primes. If the level is raised at $Q \subset \Pi_g$ and $R \subset \Omega_g$, both signed λ -invariants increase by $|Q|$. Their common increment is therefore the number of prime factors of the new level belonging to Π_g .

For comparison, let $\omega(m)$ denote the number of distinct prime divisors of a positive integer m . Hardy and Ramanujan proved that the normal order of $\omega(m)$ is $\log \log m$ [HR17], and Turán gave a proof using the second moment [Tur34]. Erdős and Kac showed that the fluctuations about this normal order have a Gaussian limiting distribution [EK40]. More precisely, for every $x \in \mathbb{R}$,

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \# \left\{ m \leq Y : \frac{\omega(m) - \log \log Y}{\sqrt{\log \log Y}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt.$$

We prove the corresponding limit law for the common signed λ -increment, with $\delta_\Pi \log \log Y$ in place of $\log \log Y$.

Call a positive integer D *admissible* if it is squarefree and every prime divisor of D belongs to $\Pi_g \cup \Omega_g$. Since the two sets are disjoint, we may write $D = D_\Pi D_\Omega$, where D_Π is supported on Π_g and D_Ω is supported on Ω_g . Put

$$\omega_\Pi(D) = \#\{q : q \mid D_\Pi\}.$$

For every admissible D , choose one form f_D supplied by Theorem 1.1. It has exact level $N_g D$, and

$$\lambda^\sharp(f_D) - \lambda^\sharp(g) = \lambda^\flat(f_D) - \lambda^\flat(g) = \omega_\Pi(D). \quad (1.3)$$

Thus the variation of the signed Iwasawa invariants is governed by an additive arithmetic function supported on the Frobenian set Π_g .

Let

$$\mathcal{D}(Y) = \{D \leq Y : D \text{ is admissible}\} \quad \text{and} \quad \mathcal{F}(Y) = \{f_D : D \in \mathcal{D}(Y)\}.$$

Since f_D has exact level $N_g D$, different admissible integers give different members of $\mathcal{F}(Y)$. For $\natural \in \{\sharp, \flat\}$ and $f \in \mathcal{F}(Y)$, set

$$\Delta^\natural(f) = \lambda^\natural(f) - \lambda^\natural(g).$$

If X is a real-valued function on $\mathcal{F}(Y)$, we write $\mathbb{E}_Y[X]$ and $\text{Var}_Y(X)$ for its expectation and variance with respect to the uniform probability measure on $\mathcal{F}(Y)$.

The second main result is the following.

Theorem 1.3 (Analogue of Erdős–Kac theorem). *Let $\natural \in \{\sharp, \flat\}$. Then for every real number x ,*

$$\lim_{Y \rightarrow \infty} \frac{1}{\#\mathcal{F}(Y)} \# \left\{ f \in \mathcal{F}(Y) : \frac{\lambda^\natural(f) - \lambda^\natural(g) - \delta_\Pi \log \log Y}{\sqrt{\delta_\Pi \log \log Y}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt. \quad (1.4)$$

Moreover,

$$\mathbb{E}_Y[\Delta^\natural] = \delta_\Pi \log \log Y + O_g(1), \quad (1.5)$$

$$\text{Var}_Y(\Delta^\natural) = \delta_\Pi \log \log Y + O_g(1). \quad (1.6)$$

The two signed increments agree for every member of the family. Consequently,

$$\lambda^\sharp(f) - \lambda^\flat(f) = \lambda^\sharp(g) - \lambda^\flat(g)$$

for every $f \in \mathcal{F}(Y)$.

The Gaussian law describes the typical size of the common increase in the two signed λ -invariants. We also count the forms for which this increase has a prescribed value. For $n \geq 0$, let $\mathcal{F}_n(X)$ consist of the chosen forms f_D of level at most X for which

$$\lambda^\sharp(f_D) - \lambda^\sharp(g) = \lambda^\flat(f_D) - \lambda^\flat(g) = n.$$

For each fixed n , we prove that

$$\#\mathcal{F}_n(X) \sim \frac{C_{g,n}}{N_g} X(\log X)^{\delta_\Omega - 1} (\log \log X)^n,$$

where $C_{g,n} > 0$.

1.3. Organization of the paper. Including the introduction, the article consists of five sections. Section 2 recalls the signed Selmer groups and the congruence results we use. In Section 3, we construct the auxiliary primes and the level-raised forms. Section 4 computes the local contributions to the signed λ -invariants and proves Theorem 1.1. Finally, Section 5 counts the levels realizing each fixed increase and proves the Erdős–Kac theorem.

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2. SIGNED SELMER GROUPS AND THEIR CONGRUENCES

Fix a prime number $p \geq 5$. Let χ_p be the p -adic cyclotomic character and $\bar{\chi}_p$ be its mod- p reduction. Let $\mathbb{Q}(\mu_{p^\infty})$ be the infinite cyclotomic field generated by the p -power roots of unity μ_{p^∞} . Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$ -extension and set $\Gamma := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. Let $E/\mathbb{Q}_p$ be a finite extension with valuation ring $\mathcal{O}$, uniformizer ϖ and $\mathbf{k} = \mathcal{O}/(\varpi)$. The Iwasawa algebra (with coefficients in $\mathcal{O}$) is defined as the completed group algebra:

$$\Lambda_{\mathcal{O}} := \mathcal{O}[[\Gamma]] = \varprojlim_n \mathcal{O}[\Gamma/\Gamma^{p^n}].$$

We choose a topological generator γ of Γ and identify $\Lambda_{\mathcal{O}}$ with $\mathcal{O}[[T]]$ by setting $T := \gamma - 1$.

2.1. Iwasawa modules. We first describe the properties of modules over the Iwasawa algebra $\Lambda_{\mathcal{O}}$. Here, if M and M' are two modules over $\Lambda_{\mathcal{O}}$, then a pseudoisomorphism is a $\Lambda_{\mathcal{O}}$ -module homomorphism $\varphi : M \rightarrow M'$ whose kernel and cokernel both have finite cardinality. If M is a finitely generated and torsion $\Lambda_{\mathcal{O}}$ -module, there is a pseudo-isomorphism

$$M \longrightarrow \bigoplus_{i=1}^r \Lambda_{\mathcal{O}}/(\varpi^{\mu_i}) \oplus \bigoplus_{j=1}^s \Lambda_{\mathcal{O}}/(F_j(T)^{e_j}), \quad (2.1)$$

where $\mu_i, e_j \geq 1$ and the $F_j(T)$ are irreducible distinguished polynomials. We then set

$$\mu(M) = \sum_{i=1}^r \mu_i \quad \text{and} \quad \lambda(M) = \sum_{j=1}^s e_j \deg F_j.$$

Lemma 2.1. *Let M be a compact $\Lambda_{\mathcal{O}}$ -module, where $\Lambda_{\mathcal{O}} = \mathcal{O}[[T]]$. The following conditions are equivalent.*

- (i) *The quotient $M/\varpi M$ is finite-dimensional over $\mathbf{k}$.*
- (ii) *The module M is finitely generated and torsion over $\Lambda_{\mathcal{O}}$, and $\mu(M) = 0$.*

If these conditions hold and M has no nonzero finite $\Lambda_{\mathcal{O}}$ -submodule, then

$$\lambda(M) = \dim_{\mathbf{k}}(M/\varpi M).$$

Proof. We first assume that $M/\varpi M$ is finite-dimensional over $\mathbf{k}$. Let

$$\mathfrak{n} = (\varpi, T)$$

be the maximal ideal of $\Lambda_{\mathcal{O}}$. Since

$$M/\mathfrak{n}M = (M/\varpi M)/T(M/\varpi M),$$

the quotient $M/\mathfrak{n}M$ is finite-dimensional over $\mathbf{k}$. Choose elements $m_1, \dots, m_r \in M$ whose images generate $M/\mathfrak{n}M$, and let N be the $\Lambda_{\mathcal{O}}$ -submodule of M generated by these elements. Then $M = N + \mathfrak{n}M$. Since M is compact and $\Lambda_{\mathcal{O}}$ is a complete local ring, the topological Nakayama lemma gives $M = N$. Thus M is finitely generated over $\Lambda_{\mathcal{O}}$; see [BH97, Section 3, Corollary following the first theorem].

We may now apply the structure theorem for finitely generated $\Lambda_{\mathcal{O}}$ -modules. There is a homomorphism

$$\phi : M \longrightarrow M'$$

with finite kernel and finite cokernel, where

$$M' = \Lambda_{\mathcal{O}}^{r_0} \oplus \bigoplus_{i=1}^u \Lambda_{\mathcal{O}}/(\varpi^{a_i}) \oplus \bigoplus_{j=1}^v \Lambda_{\mathcal{O}}/(F_j(T)^{e_j}).$$

Here $a_i, e_j \geq 1$, and each $F_j(T)$ is a distinguished irreducible polynomial which is not divisible by ϖ .

It is easy to see that $M'/\varpi M'$ is finite-dimensional over $\mathbf{k}$. This rules out the first two types of elementary factors. Indeed, a free summand $\Lambda_{\mathcal{O}}$ contributes $\mathbf{k}[[T]]$ to $M'/\varpi M'$, which is infinite-dimensional over $\mathbf{k}$. Likewise,

$$(\Lambda_{\mathcal{O}}/(\varpi^{a_i}))/\varpi(\Lambda_{\mathcal{O}}/(\varpi^{a_i})) \simeq \mathbf{k}[[T]],$$

so the presence of a factor $\Lambda_{\mathcal{O}}/(\varpi^{a_i})$ would again make $M'/\varpi M'$ infinite-dimensional. It follows that $r_0 = u = 0$. Therefore M is torsion and $\mu(M) = 0$.

We now prove the converse. Suppose that M is finitely generated and torsion over $\Lambda_{\mathcal{O}}$ and that $\mu(M) = 0$. The structure theorem then gives a pseudo-isomorphism

$$\varphi : M \longrightarrow M' := \bigoplus_{j=1}^v \Lambda_{\mathcal{O}}/(F_j(T)^{e_j}),$$

with finite kernel and cokernel. If $d_j = \deg F_j$, then, since F_j is distinguished,

$$F_j(T) \equiv T^{d_j} \pmod{\varpi}.$$

It follows that $M'/\varpi M'$ is finite-dimensional. Since M is pseudoisomorphic to M' , it is easy to see that $M/\varpi M$ is finite-dimensional.

It remains to prove the formula for the λ -invariant. Assume in addition that M has no nonzero finite $\Lambda_{\mathcal{O}}$ -submodule. In the pseudo-isomorphism φ above, the kernel is finite and is a submodule of M . Hence $K = 0$. We therefore have an exact sequence

$$0 \longrightarrow M \longrightarrow M' \longrightarrow C \longrightarrow 0$$

with C finite.

Each module $\Lambda_{\mathcal{O}}/(F_j(T)^{e_j})$ is finite free over $\mathcal{O}$ of rank $e_j d_j$. Thus M' is a finitely generated and free $\mathcal{O}$ -module of rank

$$\sum_{j=1}^v e_j d_j = \lambda(M).$$

Since M is an $\mathcal{O}$ -submodule of the finite free module M' , it is finitely generated and torsion-free over the discrete valuation ring $\mathcal{O}$, and hence is itself free over $\mathcal{O}$. The quotient $M'/M = C$ is finite, so M and M' have the same $\mathcal{O}$ -rank. Therefore

$$\text{rank}_{\mathcal{O}} M = \lambda(M).$$

Finally, if a finite free $\mathcal{O}$ -module has rank r , then its reduction modulo ϖ has dimension r over $\mathbf{k}$. Hence

$$\dim_{\mathbf{k}}(M/\varpi M) = \text{rank}_{\mathcal{O}} M = \lambda(M),$$

as required. $\square$

2.2. Signed local conditions. Let $h = \sum_{n \geq 1} a_n(h) q^n$ be a normalized newform of weight 2, level prime to p , and trivial nebentypus. Choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and let $E/\mathbb{Q}_p$ be a finite extension containing the image of the coefficient field of h . Associated to h is a 2-dimensional Galois representation over E ; let V_h denote the underlying vector space. Our convention is that the Hodge–Tate weights of V_h are $\{0, -1\}$. Let $T_h \subset V_h$ be Kato’s Galois stable $\mathcal{O}$ -lattice, cf. [Kat04, Section 8.3]. We then set

$$A_h = (V_h/T_h)(1) \quad \text{and} \quad \mathcal{T}_h = T_h(1).$$

The lattice $\mathcal{T}_h$ is crystalline at p , with Hodge–Tate weights $\{0, 1\}$.

For $n \geq 0$, put $K_n = \mathbb{Q}_p(\mu_{p^{n+1}})$ and let $K_{\infty} = \bigcup_{n \geq 0} K_n$. Thus

$$G_{\infty} := \text{Gal}(K_{\infty}/\mathbb{Q}_p) \simeq \Delta \times \Gamma.$$

The completion $\mathbb{Q}_{\infty, p}$ of $\mathbb{Q}_{\infty}$ at the prime above p is K_{∞}^{Δ} . Thus passage to the trivial Δ -component identifies the local conditions below with the corresponding local conditions over $\mathbb{Q}_{\infty, p}$. If T is a $\mathcal{O}$ -lattice equipped with a continuous action of $G_{\mathbb{Q}_p}$, its local Iwasawa cohomology is defined by

$$H_{\text{Iw}}^1(\mathbb{Q}_p, T) := \varprojlim_n H^1(K_n, T), \tag{2.2}$$

where the inverse limit is taken with respect to the corestriction maps. It is naturally a module over

$$\mathcal{O}[[G_{\infty}]] \simeq \Lambda_{\mathcal{O}}(\Delta) := \mathcal{O}[\Delta] \widehat{\otimes}_{\mathcal{O}} \Lambda_{\mathcal{O}}.$$

We apply this construction to the positive crystalline lattice $\mathcal{T}_h$. The construction of Lei–Loeffer–Zerbes [LLZ10] gives, after fixing a suitable basis, two Coleman maps

$$\text{Col}_{h,1}, \text{Col}_{h,2} : H_{\text{Iw}}^1(\mathbb{Q}_p, \mathcal{T}_h) \longrightarrow \Lambda_{\mathcal{O}}(\Delta).$$

We use the notation $\sharp, \flat$ for the signed Coleman maps in the normalization customary in the weight-two non-ordinary setting; see [Spr12; Spr17]. For a character $\eta : \Delta \rightarrow \mathcal{O}^{\times}$, projection to the η -isotypical component gives

$$\text{Col}_{h, \sharp}^{\eta} : H_{\text{Iw}}^1(\mathbb{Q}_p, \mathcal{T}_h)^{\eta} \longrightarrow \Lambda_{\mathcal{O}},$$

where $\mathfrak{b} \in \{\sharp, \flat\}$. In this article we work throughout in the trivial Δ -component, so $\eta = \mathbf{1}$. We shall therefore suppress η from the notation and regard the signed Coleman maps as $\Lambda_{\mathcal{O}}$ -valued maps.

We next recall the local Tate pairing used to define the signed local condition. The representation A_h is the Cartier dual of $\mathcal{T}_h$ in our normalization. Thus there is a perfect $G_{\mathbb{Q}_p}$ -equivariant pairing

$$\mathcal{T}_h \times A_h \longrightarrow E/\mathcal{O}(1).$$

For every finite extension $K/\mathbb{Q}_p$, cup product followed by the local invariant map gives Tate's local pairing

$$H^1(K, \mathcal{T}_h) \times H^1(K, A_h) \longrightarrow H^2(K, E/\mathcal{O}(1)) \xrightarrow{\text{inv}_K} E/\mathcal{O}.$$

If L/K is a finite extension, then

$$\langle \text{cores}_{L/K}(x), y \rangle_K = \langle x, \text{res}_{L/K}(y) \rangle_L.$$

Passing to the inverse limit in the first variable and the direct limit in the second gives a pairing

$$\langle \cdot, \cdot \rangle_p : H_{\text{Iw}}^1(\mathbb{Q}_p, \mathcal{T}_h) \times H^1(K_{\infty}, A_h) \longrightarrow E/\mathcal{O}. \quad (2.3)$$

If $\mathfrak{b} \in \{\sharp, \flat\}$, the signed local condition at p is defined to be the orthogonal complement of the kernel of the corresponding Coleman map:

$$H_{\mathfrak{b}}^1(K_{\infty}, A_h) = (\ker(\text{Col}_h^{\mathfrak{b}}))^{\perp}.$$

This is the local condition at p used in the definition of the signed Selmer group. For a prime $\ell \neq p$, the local condition is the unramified subgroup

$$H_{\text{ur}}^1(\mathbb{Q}_{\infty, v}, A_h) = \ker(H^1(\mathbb{Q}_{\infty, v}, A_h) \longrightarrow H^1(I_v, A_h)),$$

where v is a prime of $\mathbb{Q}_{\infty}$ above ℓ . If Σ contains p, ∞ and all primes at which A_h is ramified, we define

$$\text{Sel}^{\mathfrak{b}}(A_h/\mathbb{Q}_{\infty}) = \ker \left(H^1(G_{\mathbb{Q}, \Sigma}/\mathbb{Q}_{\infty}, A_h) \longrightarrow \frac{H^1(\mathbb{Q}_{\infty, p}, A_h)}{H_{\mathfrak{b}}^1(\mathbb{Q}_{\infty, p}, A_h)} \oplus \bigoplus_{\substack{v|\ell \\ \ell \in \Sigma \setminus \{p\}}} \frac{H^1(\mathbb{Q}_{\infty, v}, A_h)}{H_{\text{ur}}^1(\mathbb{Q}_{\infty, v}, A_h)} \right). \quad (2.4)$$

Its Pontryagin dual is denoted

$$X^{\mathfrak{b}}(h) = \text{Sel}^{\mathfrak{b}}(A_h/\mathbb{Q}_{\infty})^{\vee}.$$

Whenever $X^{\mathfrak{b}}(h)$ is torsion, its Iwasawa invariants are written $\mu^{\mathfrak{b}}(h)$ and $\lambda^{\mathfrak{b}}(h)$.

Let $\Sigma_0 \subseteq \Sigma \setminus \{p, \infty\}$. The non-primitive signed Selmer group $\text{Sel}^{\mathfrak{b}, \Sigma_0}(A_h/\mathbb{Q}_{\infty})$ is obtained from (2.4) by imposing no local condition at the primes above Σ_0 . Set

$$Y_h^{\mathfrak{b}, \Sigma_0} = \text{Sel}^{\mathfrak{b}, \Sigma_0}(A_h/\mathbb{Q}_{\infty})^{\vee}.$$

For $\ell \in \Sigma_0$, put

$$\mathcal{H}_{\ell}(A_h) = \prod_{v|\ell} \frac{H^1(\mathbb{Q}_{\infty, v}, A_h)}{H_{\text{ur}}^1(\mathbb{Q}_{\infty, v}, A_h)}.$$

Under the torsion hypothesis, the Poitou–Tate argument of [HL19, Corollary 2.10] gives an exact sequence

$$0 \longrightarrow \text{Sel}^{\mathfrak{b}}(A_h/\mathbb{Q}_{\infty}) \longrightarrow \text{Sel}^{\mathfrak{b}, \Sigma_0}(A_h/\mathbb{Q}_{\infty}) \longrightarrow \bigoplus_{\ell \in \Sigma_0} \mathcal{H}_{\ell}(A_h) \longrightarrow 0. \quad (2.5)$$

The local modules on the right are cotorsion and have vanishing μ -invariant; see [HL19, Remark 2.11]. If

$$\delta_{\ell, h} := \lambda(\mathcal{H}_{\ell}(A_h)^{\vee}),$$

then (2.5) gives

$$\lambda^{\natural, \Sigma_0}(h) = \lambda^{\natural}(h) + \sum_{\ell \in \Sigma_0} \delta_{\ell, h}. \quad (2.6)$$

2.3. Residual comparison. We next compare the residual signed Selmer groups for two congruent modular forms.

Let f and g be p -non-ordinary modular forms of the same even weight k , and let T_f and T_g be the corresponding $G_{\mathbb{Q}}$ -stable $\mathcal{O}$ -lattices. We assume that the hypotheses (irred), (inv), (BLZ) and (Cong) of Hatley–Lei hold. We recall these conditions explicitly; they are summarized in [HL19, p. 1289].

The hypothesis (irred) requires that $T_f/\varpi T_f$ be irreducible as a $G_{\mathbb{Q}}$ -representation. Since we assume that $T_f/\varpi T_f \simeq T_g/\varpi T_g$, this is equivalent to the corresponding assertion for g . The hypothesis (Cong) is precisely the residual congruence

$$T_f/\varpi T_f \simeq T_g/\varpi T_g \quad (2.7)$$

as $G_{\mathbb{Q}}$ -representations.

The hypothesis (inv) is the vanishing

$$A_h(m)^{G_{\mathbb{Q}_{\infty, p}}} = 0 \quad (2.8)$$

for every $m \geq 0$.

Finally, the hypothesis (BLZ) requires that

$$\text{ord}_p(a_p(f)), \text{ord}_p(a_p(g)) > \left\lfloor \frac{k-2}{p-1} \right\rfloor; \quad (2.9)$$

cf. [HL19, p. 1289] and [BLZ04, pp. 374–376]. When $k = 2$, the right-hand side of (2.9) is zero. Consequently (BLZ) becomes $\text{ord}_p(a_p(f)) > 0$ and $\text{ord}_p(a_p(g)) > 0$, i.e., both forms are non-ordinary at p .

Proposition 2.2. *Let Σ contain p, ∞ , and all primes at which either f or g is ramified, and set $\Sigma_0 = \Sigma \setminus \{p, \infty\}$. For either signed label $\natural$, there is an isomorphism*

$$\text{Sel}^{\natural, \Sigma_0}(A_f/\mathbb{Q}_{\infty})[\varpi] \simeq \text{Sel}^{\natural, \Sigma_0}(A_g/\mathbb{Q}_{\infty})[\varpi].$$

Proof. This is a standard result and the method goes back to Greenberg–Vatsal [GV00, Section 2]. The same argument in the ordinary modular-form setting is recalled and used in [Ray24, Section 4]. In the non-ordinary setting, Hatley–Lei [HL19, Propositions 4.3 and 4.5] identify the ϖ -torsion of the non-primitive signed Selmer group with the residual Selmer group. $\square$

Proposition 2.3. *Let f and g be as in Proposition 2.2. Fix $\natural \in \{\sharp, \flat\}$. If $X^{\natural}(g)$ is torsion and $\mu^{\natural}(g) = 0$, then $X^{\natural}(f)$ is torsion and $\mu^{\natural}(f) = 0$.*

Proof. Take Σ_0 as in Proposition 2.2 and write $Y_h = Y_h^{\natural, \Sigma_0}$. Since $X^{\natural}(g)$ is torsion, (2.5) is an exact sequence for g . The local terms are torsion with vanishing μ -invariant, so Y_g is torsion with $\mu(Y_g) = 0$. Lemma 2.1 gives $\dim_{\mathbf{k}} Y_g/\varpi Y_g < \infty$.

Pontryagin duality identifies

$$(\text{Sel}^{\natural, \Sigma_0}(A_h/\mathbb{Q}_{\infty})[\varpi])^{\vee} \simeq Y_h/\varpi Y_h$$

for $h = g$ or f . Therefore Proposition 2.2 gives

$$Y_f/\varpi Y_f \simeq Y_g/\varpi Y_g.$$

It follows that $Y_f/\varpi Y_f$ is finite dimensional. Applying Lemma 2.1 to Y_f shows that Y_f is finitely generated and torsion over $\Lambda_{\mathcal{O}}$ with $\mu(Y_f) = 0$. The primitive signed Selmer group is a subgroup

of the non-primitive group; after dualizing, $X^\natural(f)$ is a quotient of Y_f . Hence $X^\natural(f)$ is torsion over $\Lambda_{\mathcal{O}}$ with $\mu(X^\natural(f)) = 0$. $\square$

Lemma 2.4. *Let h be a weight-two p -non-ordinary newform of level prime to p and trivial nebentypus. Assume (irred), (inv), (BLZ) and (tor) in the notation of Hatley–Lei. For each $\natural \in \{\sharp, \flat\}$, we have that $\text{Sel}^\natural(A_h/\mathbb{Q}_\infty)$ has no proper $\Lambda_{\mathcal{O}}$ -submodule of finite index.*

Proof. The result follows from [HL19, Theorem 3.1]. $\square$

Proposition 2.5 (Signed congruence formula). *Let f and g satisfy the hypotheses of Proposition 2.2. If $X^\natural(g)$ is torsion and $\mu^\natural(g) = 0$, then $X^\natural(f)$ is torsion, $\mu^\natural(f) = 0$, and*

$$\lambda^\natural(f) - \lambda^\natural(g) = \sum_{\ell \in \Sigma_0} (\delta_{\ell,g} - \delta_{\ell,f}). \quad (2.10)$$

Proof. Proposition 2.3 first supplies the torsion and $\mu = 0$ hypotheses for f . Thus the running hypothesis (tor) of Hatley–Lei is satisfied for both f and g . By Lemma 2.4, both signed Selmer groups have no proper finite-index submodule.

By Proposition 2.2, the residual non-primitive Selmer groups are isomorphic. It follows then from the [HL19, Proof of Theorem 4.6] and [Kim09, Lemma 12] that

$$\lambda^{\natural, \Sigma_0}(f) = \lambda^{\natural, \Sigma_0}(g).$$

Applying (2.6) to f and g and subtracting gives (2.10). $\square$

Theorem 2.6. *Let f and g be as in Proposition 2.5. Assume that $X^\sharp(g)$ and $X^\flat(g)$ are torsion and that $\mu^\sharp(g) = \mu^\flat(g) = 0$. Then*

$$\lambda^\sharp(f) - \lambda^\flat(f) = \lambda^\sharp(g) - \lambda^\flat(g).$$

More precisely, there is an integer $c(f, g)$ determined by the tame local correction terms such that

$$(\lambda^\sharp(f), \lambda^\flat(f)) = (\lambda^\sharp(g) + c(f, g), \lambda^\flat(g) + c(f, g)).$$

Proof. Apply Proposition 2.5 for $\sharp$ and $\flat$. Every term on the right-hand side of (2.10) is attached to a prime $\ell \neq p$ and is therefore the same for the two signed Selmer structures. On subtracting the two identities, we obtain the result. $\square$

3. RESIDUAL IMAGES, NON-OPTIMAL LEVELS AND AUXILIARY PRIMES

In this section, we carry out the global construction of the forms that will be used in the proof of Theorem 1.1. There are three ingredients. We first recall the part of the non-optimal-level theorem of Carayol and Diamond–Taylor that allows one to raise the tame level at several primes simultaneously. We then isolate two conjugacy-stable subsets of the residual image and use the Chebotarev density theorem to produce two corresponding sets of auxiliary primes. Finally, we show that level raising at any finite collection of these primes preserves the residual representation and forces the required local Steinberg signs. The argument is a refinement of the construction in [Ray24, Sections 2, 4–5].

3.1. Artin conductors and non-optimal levels. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For every prime ℓ , choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. This identifies the absolute Galois group $G_\ell = \text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ with a decomposition subgroup of $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We write $I_\ell \subseteq G_\ell$ for the inertia subgroup and choose an arithmetic Frobenius element $\text{Frob}_\ell \in G_\ell/I_\ell$.

Let

$$\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\mathbf{k})$$

be a continuous, odd and absolutely irreducible representation of characteristic p . If $\ell \neq p$, let $a_\ell(\bar{\rho})$ denote the Artin-conductor exponent of the local representation $\bar{\rho}|_{G_\ell}$. The prime-to- p Artin conductor of $\bar{\rho}$ is

$$N(\bar{\rho}) = \prod_{\ell \neq p} \ell^{a_\ell(\bar{\rho})}. \quad (3.1)$$

If h is a modular lift of $\bar{\rho}$ whose tame level is N_h , then $N(\bar{\rho})$ divides N_h ; see [Rib94; Car89]. We say that h has *optimal level* if $N_h = N(\bar{\rho})$. In this terminology, a prime dividing $N_h/N(\bar{\rho})$ records ramification that is present in the characteristic-zero representation but disappears after reduction modulo the maximal ideal.

Suppose that $\ell \nmid pN(\bar{\rho})$. The residual representation is then unramified at ℓ , and Carayol's condition for introducing conductor exponent one at ℓ is

$$\ell \operatorname{Tr}(\bar{\rho}(\operatorname{Frob}_\ell))^2 = (1 + \ell)^2 \det(\bar{\rho}(\operatorname{Frob}_\ell)) \quad \text{in } \mathbf{k}. \quad (3.2)$$

Since $\det(\bar{\rho}(\operatorname{Frob}_\ell)) = \ell$, condition (3.2) is equivalent to

$$\operatorname{Tr}(\bar{\rho}(\operatorname{Frob}_\ell))^2 = (1 + \ell)^2. \quad (3.3)$$

Let f be a weight-two newform with trivial nebentypus, and let $\ell \neq p$ be a prime such that $\ell \parallel N_f$. We say that f is *Steinberg at ℓ* if its local automorphic representation has the form $\pi_{f,\ell} \simeq \operatorname{St} \otimes \xi_\ell$, where ξ_ℓ is an unramified character. Since the nebentypus of f is trivial, one has $\xi_\ell(\operatorname{Frob}_\ell) = \varepsilon_\ell$. The semisimplification of the local representation at ℓ has unramified characters ξ_ℓ and $\xi_\ell \chi_p$. Consequently, after reduction modulo ϖ , the Frobenius eigenvalues are ε_ℓ and $\varepsilon_\ell \ell$.

Theorem 3.1. *Let $p \geq 5$, and let $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \operatorname{GL}_2(\mathbf{k})$ be odd and absolutely irreducible, with determinant $\bar{\chi}_p$. Suppose that $\bar{\rho}$ arises from a normalized newform of weight two, trivial nebentypus and optimal tame level $N(\bar{\rho})$. Let S be a finite set of primes such that every $\ell \in S$ satisfies*

- $\ell \nmid pN(\bar{\rho})$,
- $\ell \not\equiv \pm 1 \pmod{p}$,
- $\operatorname{Tr}(\bar{\rho}(\operatorname{Frob}_\ell))^2 = (1 + \ell)^2$.

Then there is a normalized weight-two newform h of trivial nebentypus, lifting $\bar{\rho}$, whose tame level is

$$N_h = N(\bar{\rho}) \prod_{\ell \in S} \ell. \quad (3.4)$$

The form h is new at every prime in S . More precisely, its local representation at such a prime is a Steinberg representation of sign $\varepsilon_\ell \in \{\pm 1\}$, and the residual Frobenius eigenvalues are ε_ℓ and $\varepsilon_\ell \ell$.

Proof. This is the same non-optimal level argument as [Ray24, Proposition 4.7]. Indeed, for every $\ell \in S$, condition (3.2) is case (1) of Carayol's criterion, so adjoining ℓ increases the conductor exponent by one. Hence

$$M = N(\bar{\rho}) \prod_{\ell \in S} \ell$$

is an admissible non-optimal level for $\bar{\rho}$. Since $p \geq 5$ and the weight is 2, the fixed-nebentypus form of the Diamond–Taylor level-raising theorem applies in the trivial-character Hecke algebra and produces a weight-two newform f of trivial nebentypus lifting $\bar{\rho}$ of level M ; see [Ray24, Theorem 2.3], which recalls [DT94]. Equivalently, one may apply the usual level-raising theorem successively at the primes in S ; at each step the nebentypus remains trivial. This is precisely the argument used in [Ray24, Proposition 4.7].

At every $\ell \in S$, the form f is ramified at ℓ and $\bar{\rho}_f(\text{Frob}_\ell)$ has eigenvalues $\epsilon_\ell \ell$ and ϵ_ℓ , with $\epsilon_\ell \in \{\pm 1\}$. The local calculation of [Ray24, Lemma 4.4] therefore applies. It shows that, after a change of basis,

$$\rho_f(\text{Frob}_\ell) = \begin{pmatrix} \ell x_\ell & 0 \\ 0 & x_\ell \end{pmatrix} \quad \text{and} \quad \rho_f(\tau_\ell) = \begin{pmatrix} 1 & y_\ell \\ 0 & 1 \end{pmatrix},$$

where $x_\ell \equiv \epsilon_\ell \pmod{\varpi}$. Since f has weight two and trivial nebentypus, $\det \rho_f = \chi_p$, and hence $x_\ell^2 = 1$. As p is odd, it follows that $x_\ell = \epsilon_\ell$. Thus the local representation is of Steinberg type with sign ϵ_ℓ . This proves the final assertion. $\square$

3.2. The residual image. We study the residual representation of the fixed form g in more detail. Set

$$H_g = \bar{\rho}_g(G_{\mathbb{Q}}) \subseteq \text{GL}_2(\mathbf{k}),$$

where $L_g = \overline{\mathbb{Q}}^{\ker \bar{\rho}_g}$ is the field cut out by $\bar{\rho}_g$. The representation $\bar{\rho}_g$ identifies H_g with $\text{Gal}(L_g/\mathbb{Q})$. Since g has weight two and trivial nebentypus, one has $\det \bar{\rho}_g = \bar{\chi}_p$. In particular, $\det(H_g) = \mathbb{F}_p^\times$, where $\mathbb{F}_p^\times$ is viewed as a subgroup of $\mathbf{k}^\times$. Moreover, $\ker \bar{\rho}_g$ is contained in $\ker \bar{\chi}_p$, and hence $\mathbb{Q}(\mu_p) \subseteq L_g$.

Proposition 3.2 (Dickson). *Let G be a subgroup of $\text{GL}_2(\mathbb{F}_{p^r})$, and let $\bar{G}$ denote its image in $\text{PGL}_2(\mathbb{F}_{p^r})$. Up to conjugacy, one of the following possibilities occurs:*

- (i) G is contained in a Borel subgroup;
- (ii) G is contained in the normalizer of a Cartan subgroup;
- (iii) $\bar{G}$ is isomorphic to A_4 , S_4 , or A_5 ;
- (iv) there is a subfield $\mathbb{F}_{p^s} \subseteq \mathbb{F}_{p^r}$ such that $\bar{G}$ is conjugate to $\text{PSL}_2(\mathbb{F}_{p^s})$ or $\text{PGL}_2(\mathbb{F}_{p^s})$.

Proof. This is Dickson's classification in the form recalled in [Zyw23, Lemma 5.1]. $\square$

For $u \in H_g$, write $d(u) = \det(u)$ and define

$$\mathcal{C}_g^+ = \{u \in H_g : d(u) \notin \{\pm 1\}, \text{Tr}(u) = 1 + d(u)\}, \quad (3.5)$$

$$\mathcal{C}_g^- = \{u \in H_g : d(u) \notin \{\pm 1\}, \text{Tr}(u) = -(1 + d(u))\}. \quad (3.6)$$

Both conditions depend only on the trace and determinant, so the two sets are stable under conjugation in H_g . If $u \in \mathcal{C}_g^+$, then its characteristic polynomial factors as

$$X^2 - (1 + d(u))X + d(u) = (X - 1)(X - d(u)).$$

Similarly, if $u \in \mathcal{C}_g^-$, its characteristic polynomial is $(X + 1)(X + d(u))$. Since $d(u) \notin \{\pm 1\}$, the two roots in either factorization are distinct. The following proposition shows, in particular, that hypothesis (H1) is automatic under a large-image condition. It also gives explicit densities in that case.

Proposition 3.3. *Suppose that $\text{SL}_2(\mathbf{k}) \subseteq H_g$, and put $q_0 = |\mathbf{k}|$. Then $\mathcal{C}_g^+$ and $\mathcal{C}_g^-$ are nonempty and*

$$L_g \cap \mathbb{Q}(\mu_{p^2}) = \mathbb{Q}(\mu_p). \quad (3.7)$$

Moreover,

$$\frac{|\mathcal{C}_g^+|}{|H_g|} = \frac{|\mathcal{C}_g^-|}{|H_g|} = \frac{p - 3}{(p - 1)(q_0 - 1)}. \quad (3.8)$$

Proof. The determinant map on H_g is surjective onto $\mathbb{F}_p^\times$. It is easy to see that

$$H_g = \{A \in \text{GL}_2(\mathbf{k}) : \det(A) \in \mathbb{F}_p^\times\}. \quad (3.9)$$

For $a \in \mathbb{F}_p^\times \setminus \{\pm 1\}$, the two matrices $\text{diag}(a, 1)$ and $\text{diag}(-a, -1)$ both belong to H_g . They belong to $\mathcal{C}_g^+$ and $\mathcal{C}_g^-$, respectively. Since $p \geq 5$, there is at least one such a , proving that both sets are nonempty.

We next prove (3.7). We already know that $\mathbb{Q}(\mu_p)$ is contained in L_g . Since $[\mathbb{Q}(\mu_{p^2}) : \mathbb{Q}(\mu_p)] = p$, there is no proper intermediate field between these two cyclotomic fields. Thus, if the intersection in (3.7) were larger than $\mathbb{Q}(\mu_p)$, it would contain all of $\mathbb{Q}(\mu_{p^2})$. Restriction would then exhibit the cyclic group $\text{Gal}(\mathbb{Q}(\mu_{p^2})/\mathbb{Q}(\mu_p))$, which has order p , as a quotient of

$$\text{Gal}(L_g/\mathbb{Q}(\mu_p)) = \ker(\det : H_g \longrightarrow \mathbb{F}_p^\times).$$

By (3.9), this kernel is $\text{SL}_2(\mathbf{k})$. For $p \geq 5$, the group $\text{SL}_2(\mathbf{k})$ is perfect. It has no nontrivial abelian quotient and, in particular, no quotient of order p .

It remains to count the two conjugacy-stable subsets. Fix $a \in \mathbb{F}_p^\times \setminus \{\pm 1\}$. Every element of $\mathcal{C}_g^+$ with determinant a has distinct eigenvalues 1 and a , and is therefore conjugate in $\text{GL}_2(\mathbf{k})$ to $\text{diag}(a, 1)$. Its centralizer is the split diagonal torus, of order $(q_0 - 1)^2$. Hence this conjugacy class has cardinality

$$\frac{|\text{GL}_2(\mathbf{k})|}{(q_0 - 1)^2} = q_0(q_0 + 1).$$

The classes obtained from distinct values of a are disjoint because their determinants are distinct. There are $p - 3$ possible values of a , and consequently $|\mathcal{C}_g^+| = (p - 3)q_0(q_0 + 1)$. The same argument, using $\text{diag}(-a, -1)$, gives the identical formula for $|\mathcal{C}_g^-|$. Finally, (3.9) gives $|H_g| = (p - 1)|\text{SL}_2(\mathbf{k})| = (p - 1)q_0(q_0^2 - 1)$. Dividing the preceding cardinalities by this order proves (3.8). $\square$

3.3. The two Frobenian sets. We now define the auxiliary primes. Let Π_g be the set of primes $q \nmid N_gp$ such that $\bar{\rho}_g(\text{Frob}_q) \in \mathcal{C}_g^+$ and $q^{p-1} \not\equiv 1 \pmod{p^2}$. Let Ω_g be the set of primes $\ell \nmid N_gp$ such that $\bar{\rho}_g(\text{Frob}_\ell) \in \mathcal{C}_g^-$. The determinant of $\bar{\rho}_g(\text{Frob}_v)$ is the reduction of v modulo p . Since the determinant of an element of either $\mathcal{C}_g^+$ or $\mathcal{C}_g^-$ is different from ± 1 , every prime in $\Pi_g \cup \Omega_g$ satisfies $v \not\equiv \pm 1 \pmod{p}$. Thus the congruence restriction required by Theorem 3.1 is already built into the definitions.

Proposition 3.4 (Densities of the auxiliary primes). *Assume (H1). The sets Π_g and Ω_g are Frobenian, and their densities are*

$$\delta(\Pi_g) = \frac{p-1}{p} \frac{|\mathcal{C}_g^+|}{|H_g|} \quad \text{and} \quad \delta(\Omega_g) = \frac{|\mathcal{C}_g^-|}{|H_g|}. \quad (3.10)$$

In particular, both sets have positive density and are infinite.

Proof. The set Ω_g is the Chebotarev set in the finite Galois extension $L_g/\mathbb{Q}$ associated with the union of conjugacy classes $\mathcal{C}_g^-$. Removing the finitely many primes that divide N_gp does not change its density. The Chebotarev density theorem therefore gives $\delta(\Omega_g) = |\mathcal{C}_g^-|/|H_g|$.

The additional congruence $q^{p-1} \not\equiv 1 \pmod{p^2}$ in the definition of Π_g requires us to adjoin the p^2 -th roots of unity. Setting $M = \mathbb{Q}(\mu_{p^2})$, we find that both L_g and M contain $\mathbb{Q}(\mu_p)$. Hypothesis (H1) states that their intersection is $\mathbb{Q}(\mu_p)$. Consequently, the Galois group of the compositum can be identified with the fibered product

$$\text{Gal}(L_g M/\mathbb{Q}) \simeq H_g \times_{\mathbb{F}_p^\times} (\mathbb{Z}/p^2\mathbb{Z})^\times. \quad (3.11)$$

Here the maps to $\mathbb{F}_p^\times$ are the determinant on H_g and reduction modulo p on $(\mathbb{Z}/p^2\mathbb{Z})^\times$. Every element of $\mathbb{F}_p^\times$ has exactly p lifts to $(\mathbb{Z}/p^2\mathbb{Z})^\times$, so the group in (3.11) has order $p|H_g|$.

Fix $u \in \mathcal{C}_g^+$ and set $a = d(u)$. Among the p elements of $(\mathbb{Z}/p^2\mathbb{Z})^\times$ which reduce to a , exactly one has $(p - 1)$ -st power equal to 1 modulo p^2 . Indeed, reduction modulo p induces an isomorphism

from the subgroup of $(p-1)$ -st roots of unity in $(\mathbb{Z}/p^2\mathbb{Z})^\times$ onto $\mathbb{F}_p^\times$, and the unique lift in that subgroup is the Teichmüller lift of a . The remaining $p-1$ lifts have $(p-1)$ -st power different from 1 modulo p^2 .

It follows that the conjugacy-stable subset of $\text{Gal}(L_g M/\mathbb{Q})$ defining Π_g has cardinality $(p-1)|\mathcal{C}_g^+|$. The Chebotarev density theorem therefore gives

$$\delta(\Pi_g) = \frac{(p-1)|\mathcal{C}_g^+|}{p|H_g|}.$$

This proves (3.10). Positivity follows from the nonemptiness of $\mathcal{C}_g^+$ and $\mathcal{C}_g^-$ in hypothesis (H1). $\square$

If $v \nmid N_g p$, the characteristic polynomial of $\bar{\rho}_g(\text{Frob}_v)$ is the reduction of $X^2 - a_v(g)X + v$. Hence, for $q \in \Pi_g$, the factorization attached to $\mathcal{C}_g^+$ gives $a_q(g) \equiv 1 + q \pmod{\varpi}$. Likewise, if $\ell \in \Omega_g$, then $a_\ell(g) \equiv -(1 + \ell) \pmod{\varpi}$. In either case, (3.3) holds. Every prime in $\Pi_g \cup \Omega_g$ is therefore a level-raising prime.

3.4. Simultaneous level raising. We may now apply the preceding construction to any finite collection of auxiliary primes.

Proposition 3.5. *Let $Q \subset \Pi_g$ and $R \subset \Omega_g$ be finite sets. Under hypotheses (H1) and (H2), there is a normalized newform f of weight two and trivial nebentypus such that $\bar{\rho}_f \simeq \bar{\rho}_g$ and*

$$N_f = N_g \prod_{q \in Q} q \prod_{\ell \in R} \ell. \quad (3.12)$$

Further, f is new at every prime in $Q \cup R$. Its Steinberg sign is $+1$ at every $q \in Q$ and -1 at every $\ell \in R$.

Proof. The two auxiliary sets are disjoint. Indeed, an element of $\mathcal{C}_g^+$ has trace $1 + d$, while an element of $\mathcal{C}_g^-$ has trace $-(1 + d)$; these values cannot be equal when $d \neq -1$. Thus $Q \cap R = \emptyset$. If both sets are empty, we may simply take $f = g$. Suppose otherwise and put $S = Q \cup R$.

Every prime in S is prime to pN_g , is not congruent to ± 1 modulo p , and satisfies the trace condition (3.3). Hypothesis (H2) identifies N_g with $N(\bar{\rho}_g)$. Theorem 3.1 therefore produces a normalized newform f of trivial nebentypus whose residual representation is isomorphic to $\bar{\rho}_g$ and whose level is the integer in (3.12). The same theorem shows that f is new at every prime in S .

It remains to determine the local signs. Let $v \in S$, and let $\varepsilon_v \in \{\pm 1\}$ be the Steinberg sign of f at v . The residual Frobenius eigenvalues of a Steinberg representation of sign ε_v are $\{\varepsilon_v, \varepsilon_v v\}$, and hence its residual trace is $\varepsilon_v(1 + v)$. If $q \in Q$, the definition of $\mathcal{C}_g^+$ gives residual trace $1 + q$. Since $q \not\equiv -1 \pmod{p}$, the element $1 + q$ is nonzero in $\mathbf{k}$, and therefore $\varepsilon_q = +1$. If $\ell \in R$, the definition of $\mathcal{C}_g^-$ gives residual trace $-(1 + \ell)$, and the same argument gives $\varepsilon_\ell = -1$. $\square$

Lemma 3.6 (Preservation of non-ordinarity). *The newform f furnished by Proposition 3.5 is non-ordinary at p .*

Proof. After enlarging the coefficient field if necessary, choose stable lattices for f and g whose reductions realize the fixed isomorphism $\bar{\rho}_f \simeq \bar{\rho}_g$. Restricting this isomorphism to a decomposition group at p gives

$$\bar{\rho}_f|_{G_{\mathbb{Q}_p}} \simeq \bar{\rho}_g|_{G_{\mathbb{Q}_p}}. \quad (3.13)$$

Since g has weight two, level prime to p , and is non-ordinary, its p -th Fourier coefficient is zero modulo ϖ . The local description of the residual representation in the Fontaine–Laffaille range shows that $\bar{\rho}_g|_{G_{\mathbb{Q}_p}}$ is of niveau two and is irreducible; see [Edi92, Theorem 2.6]. More explicitly, after extending scalars to an algebraic closure of the residue field, the semisimplification of its restriction to inertia has characters ω_2 and ω_2^p , where ω_2 is a fundamental character of niveau

two. These two characters are interchanged by Frobenius, so neither inertia eigenspace is stable under the full group $G_{\mathbb{Q}_p}$. This is precisely the irreducible niveau-two case.

Suppose, to the contrary, that f were ordinary at p . The p -adic representation attached to an ordinary weight-two form has a $G_{\mathbb{Q}_p}$ -stable one-dimensional subspace and an unramified one-dimensional quotient. This gives a rank-one direct summand whose reduction modulo ϖ is a nonzero proper $G_{\mathbb{Q}_p}$ -stable subspace. Thus the residual local representation of f would be reducible. This contradicts (3.13) and the irreducibility of the residual local representation of g . Hence f is non-ordinary at p . $\square$

4. THE LOCAL CORRECTION TERMS AND PROOF OF THE MAIN THEOREM

Let h be a weight-two newform and let $\ell \neq p$. With respect to the notation of [HL19, Section 5], set

$$P_{\ell,h}(X) = \det \left(1 - \text{Frob}_\ell X \mid V_h^{I_\ell} \right) \in \mathcal{O}[X] \quad (4.1)$$

and write $\tilde{P}_{\ell,h}(X)$ for its reduction modulo ϖ . Let $d_{\ell,h}$ be the multiplicity of $X = \ell^{-1}$ as a root of $\tilde{P}_{\ell,h}(X)$. If r_ℓ denotes the number of primes of $\mathbb{Q}_\infty$ above ℓ , then Proposition 5.1 and Corollary 5.2 of [HL19] give

$$\delta_{\ell,h} = r_\ell d_{\ell,h}. \quad (4.2)$$

Indeed, the local λ -invariant at each prime $v \mid \ell$ is $d_{\ell,h}$, and there are r_ℓ such primes.

Let $q \in \Pi_g$. Since $q \nmid N_g p$, the representation V_g is unramified at q . The residual Frobenius eigenvalues are 1 and q , hence

$$\tilde{P}_{q,g}(X) = (1 - X)(1 - qX). \quad (4.3)$$

Note that $X = q^{-1}$ is a root of the second factor. It is not a root of the first factor because $q \not\equiv 1 \pmod{p}$, and thus

$$d_{q,g} = 1. \quad (4.4)$$

Let f be the form given by Proposition 3.5. Since f is new of exponent one at q , its local automorphic representation is Steinberg up to an unramified twist. Consequently $V_f^{I_q}$ is one-dimensional. The positive sign in Proposition 3.5 gives

$$\tilde{P}_{q,f}(X) = 1 - X. \quad (4.5)$$

At $X = q^{-1}$ this is nonzero because $q \not\equiv 1 \pmod{p}$, and hence

$$d_{q,f} = 0. \quad (4.6)$$

The factor r_q is controlled by the additional congruence modulo p^2 . We spell this out. The cyclotomic character gives an identification $\Gamma \simeq 1 + p\mathbb{Z}_p$. The decomposition subgroup of a prime $q \neq p$ is the closed subgroup generated by the pro- p part of q . Equivalently it is generated by $q^{p-1} \in 1 + p\mathbb{Z}_p$. The condition

$$q^{p-1} \not\equiv 1 \pmod{p^2}$$

is equivalent to $v_p(q^{p-1} - 1) = 1$, and an element of $1 + p\mathbb{Z}_p$ with this property is a topological generator. Hence the decomposition subgroup is all of Γ , and therefore

$$r_q = [\Gamma : \Gamma_q] = 1. \quad (4.7)$$

Combining (4.2), (4.4), (4.6) and (4.7) yields

$$\delta_{q,g} - \delta_{q,f} = 1. \quad (4.8)$$

Now let $\ell \in \Omega_g$. The residual Frobenius eigenvalues of g are -1 and $-\ell$, so

$$\tilde{P}_{\ell,g}(X) = (1 + X)(1 + \ell X). \quad (4.9)$$

At $X = \ell^{-1}$ the two factors are $1 + \ell^{-1}$ and 2, both nonzero because $p \geq 5$ and $\ell \not\equiv -1 \pmod{p}$. Therefore $d_{\ell,g} = 0$. The level-raised form has negative Steinberg sign and

$$\tilde{P}_{\ell,f}(X) = 1 + X, \quad (4.10)$$

which is again nonzero at $X = \ell^{-1}$. Hence $d_{\ell,f} = 0$, and (4.2) gives

$$\delta_{\ell,g} - \delta_{\ell,f} = 0 \quad (4.11)$$

without any condition on the number of primes above ℓ in $\mathbb{Q}_\infty$.

Lemma 4.1. *Assume that h is congruent to g and let $\ell \neq p$ be a prime such that the exponent of ℓ in the level of h equals the exponent of ℓ in the Artin conductor of $\bar{\rho}_g$. Setting $\bar{T}_h := T_h / \varpi T_h$, we find that the natural inclusion map*

$$T_h^{I_\ell} / \varpi T_h^{I_\ell} \longrightarrow (\bar{T}_h)^{I_\ell} \quad (4.12)$$

is an isomorphism. Consequently

$$\tilde{P}_{\ell,h}(X) = \det(1 - \text{Frob}_\ell X \mid (\bar{T}_h)^{I_\ell}). \quad (4.13)$$

Proof. This is precisely the argument of [Ray24, Lemma 4.8], applied to the inertia-invariant lattice and its Frobenius characteristic polynomial. $\square$

Lemma 4.2. *Let f be as in Proposition 3.5 and let $\ell \mid N_g$. Then, we have that*

- (1) $\tilde{P}_{\ell,f}(X) = \tilde{P}_{\ell,g}(X)$,
- (2) $d_{\ell,f} = d_{\ell,g}$,
- (3) $\delta_{\ell,f} = \delta_{\ell,g}$.

Proof. The forms f and g have the same ℓ -adic level exponent, which equals the residual Artin-conductor exponent by Hypothesis (H2). Lemma 4.1, together with $\bar{T}_f \simeq \bar{T}_g$, therefore gives $\tilde{P}_{\ell,f}(X) = \tilde{P}_{\ell,g}(X)$. Equality of the multiplicities at $X = \ell^{-1}$ gives $d_{\ell,f} = d_{\ell,g}$, and then (4.2) gives $\delta_{\ell,f} = \delta_{\ell,g}$. $\square$

Proposition 4.3. *Let f be as in Proposition 3.5. Then the following assertions hold:*

- (i) if $q \in Q \subset \Pi_g$, then $\delta_{q,g} - \delta_{q,f} = 1$;
- (ii) if $\ell \in R \subset \Omega_g$, then $\delta_{\ell,g} - \delta_{\ell,f} = 0$;
- (iii) if $\ell \mid N_g$, then $\delta_{\ell,g} = \delta_{\ell,f}$;
- (iv) at every other tame prime the two reduced local Euler polynomials agree, and hence there is no contribution to the transition formula.

Proof. Assertions (i) and (ii) are (4.8) and (4.11). Assertion (iii) is Lemma 4.2. Finally, if $\ell \nmid pN_f$, both characteristic-zero representations are unramified at ℓ and the congruence $\bar{\rho}_f \simeq \bar{\rho}_g$ identifies the reduced Frobenius polynomials. Thus $d_{\ell,f} = d_{\ell,g}$ and the corresponding term is zero. $\square$

We can now prove the main theorem.

Proof of Theorem 1.1. Choose finite disjoint sets $Q \subset \Pi_g$ and $R \subset \Omega_g$. Proposition 3.5 produces a newform f of the asserted level with $\bar{\rho}_f \simeq \bar{\rho}_g$. By Lemma 3.6, the form f is again non-ordinary at p . The hypotheses needed for the signed congruence results hold for f and g : (irred) and (Cong) follow from (H1) and $\bar{\rho}_f \simeq \bar{\rho}_g$, (BLZ) follows from non-ordinarity in weight two, and (inv) follows from [Lei11, Lemma 4.4] since $2 \leq p$. Fix a signed label $\natural \in \{\sharp, \flat\}$. Proposition 2.3 implies that $X^\natural(f)$ is torsion and $\mu^\natural(f) = 0$.

Let Σ contain p , ∞ , and every prime dividing N_f , and put $\Sigma_0 = \Sigma \setminus \{p, \infty\}$. Proposition 2.5 then gives

$$\lambda^\natural(f) - \lambda^\natural(g) = \sum_{\ell \in \Sigma_0} (\delta_{\ell,g} - \delta_{\ell,f}).$$

By Proposition 4.3, every prime of Q contributes one, every prime of R contributes zero, and the contributions at the old primes cancel. The right-hand side is therefore $|Q|$. The calculation is independent of $\mathfrak{h}$, and hence it holds simultaneously for $\sharp$ and $\flat$. $\square$

Proof of Corollary 1.2. Fix $n \geq 0$. Choose a set $Q \subset \Pi_g$ with $|Q| = n$. If $n > 0$, there are already infinitely many possible choices for Q . If $n = 0$, take $Q = \emptyset$ and vary finite nonempty sets $R \subset \Omega_g$. Since both auxiliary sets are infinite, this gives infinitely many distinct tame levels. The result then follows from Theorem 1.1. $\square$

5. QUANTITATIVE AND PROBABILISTIC REALIZATION

Throughout this section we retain the hypotheses and notation of Theorem 1.1. Thus g is a weight-two newform which is non-ordinary at p , both $X^\sharp(g)$ and $X^\flat(g)$ are torsion over the cyclotomic Iwasawa algebra, and $\mu^\sharp(g) = \mu^\flat(g) = 0$. Recall also that a prime in Π_g contributes one to each signed λ -invariant, whereas a prime in Ω_g changes the exact tame level without changing either signed λ -invariant. Write δ_Π and δ_Ω for the positive densities of Π_g and Ω_g , respectively, as in Proposition 3.4. Thus

$$\delta_\Pi = \frac{p-1}{p} \frac{|\mathcal{C}_g^+|}{|H_g|} \quad \text{and} \quad \delta_\Omega = \frac{|\mathcal{C}_g^-|}{|H_g|}.$$

When $\mathbf{k} = \mathbf{F}_p$ and $H_g = \mathrm{GL}_2(\mathbf{F}_p)$, these specialize to

$$\delta_\Pi = \frac{p-3}{p(p-1)}, \quad \delta_\Omega = \frac{p-3}{(p-1)^2}.$$

Since the two sets of primes are disjoint, an admissible integer has a unique factorization $D = D_\Pi D_\Omega$, where D_Π is supported on Π_g and D_Ω is supported on Ω_g . Put

$$\omega_\Pi(D) = \#\{q : q \mid D_\Pi\}.$$

For every admissible D , choose one newform f_D supplied by Theorem 1.1, using the prime divisors of D_Π as the set Q and the prime divisors of D_Ω as the set R . The next proposition records the arithmetic content which will be counted below.

Proposition 5.1. *For every admissible D , the form f_D is a normalized weight-two newform of trivial nebentypus and exact level $N_g D$. It is non-ordinary at p and satisfies $\bar{\rho}_{f_D} \simeq \bar{\rho}_g$. Moreover, for each $\mathfrak{h} \in \{\sharp, \flat\}$, the signed Selmer dual $X^\mathfrak{h}(f_D)$ is torsion over the cyclotomic Iwasawa algebra,*

$$\mu^\mathfrak{h}(f_D) = 0 \quad \text{and} \quad \lambda^\mathfrak{h}(f_D) = \lambda^\mathfrak{h}(g) + \omega_\Pi(D). \quad (5.1)$$

In particular,

$$\lambda^\sharp(f_D) - \lambda^\flat(f_D) = \lambda^\sharp(g) - \lambda^\flat(g) \quad (5.2)$$

for every admissible D .

Proof. Apply Proposition 3.5 with Q and R equal to the sets of prime divisors of D_Π and D_Ω , respectively. This gives all the assertions concerning f_D . Theorem 1.1 then gives, for each $\mathfrak{h} \in \{\sharp, \flat\}$,

$$\mu^\mathfrak{h}(f_D) = 0$$

and

$$\lambda^\mathfrak{h}(f_D) = \lambda^\mathfrak{h}(g) + |Q| = \lambda^\mathfrak{h}(g) + \omega_\Pi(D).$$

Subtracting the formulas for $\sharp$ and $\flat$ proves (5.2). $\square$

We next recall the analytic input in the form needed here. Recall that a set of rational primes $\mathcal{P}$ is called Frobenian if there exists a finite Galois extension $L/\mathbb{Q}$ and a union C of conjugacy classes in $\text{Gal}(L/\mathbb{Q})$ such that, apart from finitely many primes, a prime p belongs to $\mathcal{P}$ precisely when the Frobenius conjugacy class Frob_p lies in C .

Lemma 5.2. *Let $\mathcal{P}$ be a Frobenian set of density δ . There is a constant $c > 0$ and a function $G_{\mathcal{P}}(s)$, holomorphic in the dented zero-free region*

$$\mathcal{R}_c = \left\{ s = \sigma + it : \sigma \geq 1 - \frac{c}{1 + \log(2 + |t|)} \right\} \setminus (-\infty, 1],$$

and continuous at 1, such that

$$\sum_{\ell \in \mathcal{P}} \ell^{-s} = \delta \log \frac{1}{s-1} + G_{\mathcal{P}}(s). \quad (5.3)$$

The constant c may be chosen so that $\exp(G_{\mathcal{P}}(s))$ has at most polynomial growth in $|t|$ on $\mathcal{R}_c$. In particular, $\sum_{\ell \leq Y, \ell \in \mathcal{P}} \ell^{-1} = \delta \log \log Y + O_{\mathcal{P}}(1)$.

Proof. This is the Frobenian Dirichlet-series theorem of Serre [Ser81], together with the standard zero-free region for the finite collection of Artin–Hecke L -functions occurring in its proof. Brauer induction reduces the required continuation and growth bound to Hecke L -functions. Since the extension defining $\mathcal{P}$ is fixed, the region may be chosen uniformly for these finitely many factors. $\square$

We define for $z \in \mathbb{C}$ and $\Re(s) > 1$,

$$F(s, z) = \prod_{\ell \in \Omega_g} (1 + \ell^{-s}) \prod_{q \in \Pi_g} (1 + zq^{-s}). \quad (5.4)$$

Expanding the product gives

$$F(s, z) = \sum_{D \text{ admissible}} \frac{z^{\omega_{\Pi}(D)}}{D^s}. \quad (5.5)$$

We set $\alpha(z) := \delta_{\Omega} + z\delta_{\Pi}$.

Lemma 5.3. *There is an open set of z -values containing the closed interval $[0, 1]$ and a constant $c > 0$ such that*

$$F(s, z) = (s-1)^{-\alpha(z)} H(s, z), \quad (5.6)$$

where $H(s, z)$ is holomorphic in both variables for $s \in \mathcal{R}_c$. For every compact set K of z -values in this open set, $H(s, z)$ has at most polynomial growth in $|\Im(s)|$, uniformly for $z \in K$. Moreover the open set can be chosen so that $H(1, z)$ is nonzero there. Moreover, $H(1, x) > 0$ for $0 \leq x \leq 1$.

Proof. For $\Re(s) > 1$, the expansion $\log(1+w) = w + O(w^2)$ gives

$$\log F(s, z) = \sum_{\ell \in \Omega_g} \ell^{-s} + z \sum_{q \in \Pi_g} q^{-s} + R(s, z),$$

where $R(s, z)$ is holomorphic for $\Re(s) > 1/2$ and locally uniformly holomorphic in z . Lemma 5.2 identifies the singular part with $\alpha(z) \log(1/(s-1))$. After shrinking c if necessary, exponentiating the remaining holomorphic terms gives (5.6) throughout $\mathcal{R}_c$, with the stated polynomial growth uniformly on compact sets of z -values. For real $s > 1$ and real $x \geq 0$, all Euler factors in $F(s, x)$ are positive. Taking $s \rightarrow 1^+$ proves positivity of the resulting nonzero limiting factor. $\square$

For $Y \geq 2$, write

$$S(Y, z) = \sum_{\substack{D \leq Y \\ D \text{ admissible}}} z^{\omega_{\Pi}(D)}.$$

The Selberg–Delange theorem allows us to deduce an asymptotic formula for $S(Y, z)$ from the behavior of $F(s, z)$ near $s = 1$.

Proposition 5.4. *Let K be a compact set of z -values contained in the open set from Lemma 5.3, and suppose that $\inf_{z \in K} \Re \alpha(z) > 0$. Then, uniformly for $z \in K$,*

$$S(Y, z) = Y(\log Y)^{\alpha(z)-1} \left(\frac{H(1, z)}{\Gamma(\alpha(z))} + O_K \left(\frac{1}{\log Y} \right) \right). \quad (5.7)$$

On a slightly smaller compact set, this estimate may be differentiated a fixed number of times with respect to z .

Proof. Put

$$G(s, z) = F(s, z)\zeta(s)^{-\alpha(z)}.$$

Since $(s-1)\zeta(s)$ is holomorphic and nonzero in a sufficiently small region of the form $\mathcal{R}_c$, Lemma 5.3 shows that $G(s, z)$ is holomorphic there and has at most polynomial growth, uniformly for $z \in K$. If $A = \max(1, \sup_{z \in K} |z|)$, then the coefficients of $F(s, z)$ are bounded in absolute value by the nonnegative coefficients of $F(s, A)$. Thus the type hypotheses in the uniform Selberg–Delange theorem are satisfied uniformly for $z \in K$. Theorem II.5.2 of [Ten15, Chapter II.5] gives (5.7). Applying the same uniform estimate on a slightly larger compact set and using Cauchy’s integral formula gives the asserted fixed number of z -derivatives. $\square$

For a fixed integer $n \geq 0$, let

$$\mathcal{D}_n(Y) = \{D \leq Y : D \text{ is admissible and } \omega_{\Pi}(D) = n\},$$

and let $\mathcal{F}_n(X)$ be the family consisting of the chosen forms f_D with $D \in \mathcal{D}_n(X/N_g)$. Thus every member of $\mathcal{F}_n(X)$ has level at most X .

Theorem 5.5. *For every fixed $n \geq 0$,*

$$\#\mathcal{D}_n(Y) \sim C_{g,n} Y(\log Y)^{\delta_{\Omega}-1} (\log \log Y)^n, \quad (5.8)$$

where

$$C_{g,n} = \frac{H(1, 0)}{\Gamma(\delta_{\Omega})} \frac{\delta_{\Pi}^n}{n!} > 0. \quad (5.9)$$

Consequently,

$$\#\mathcal{F}_n(X) \sim \frac{C_{g,n}}{N_g} X(\log X)^{\delta_{\Omega}-1} (\log \log X)^n. \quad (5.10)$$

Every $f \in \mathcal{F}_n(X)$ is non-ordinary at p , satisfies $\bar{\rho}_f \simeq \bar{\rho}_g$, and has torsion signed Selmer duals with

$$\mu^{\sharp}(f) = \mu^{\flat}(f) = 0 \quad \text{and} \quad (\lambda^{\sharp}(f), \lambda^{\flat}(f)) = (\lambda^{\sharp}(g) + n, \lambda^{\flat}(g) + n). \quad (5.11)$$

Proof. For a power series $F(z)$, we write $[z^m]F(z)$ for the coefficient of z^m in its expansion. By definition, the coefficient of z^n in $S(Y, z)$ counts precisely those admissible integers having exactly n prime divisors in Π_g . Thus

$$[z^n]S(Y, z) = \#\mathcal{D}_n(Y).$$

We may therefore recover the required asymptotic by extracting the coefficient of z^n from the uniform Selberg–Delange estimate (5.7).

Choose $\rho > 0$ sufficiently small that $\rho\delta_\Pi < \min\{\delta_\Omega, 1\}$. The first inequality ensures that $\Re(\delta_\Omega + z\delta_\Pi) > 0$ throughout the closed disc $|z| \leq \rho$, so Proposition 5.4 applies uniformly on this disc. Put

$$B(z) = \frac{H(1, z)}{\Gamma(\delta_\Omega + z\delta_\Pi)}.$$

This is holomorphic near the origin, and the Selberg–Delange estimate takes the form

$$S(Y, z) = Y(\log Y)^{\delta_\Omega - 1} e^{z\delta_\Pi \log \log Y} B(z) + E(Y, z),$$

where, uniformly for $|z| = \rho$,

$$E(Y, z) \ll_\rho Y(\log Y)^{\delta_\Omega - 2 + \rho\delta_\Pi}.$$

The additional factor $(\log Y)^{\rho\delta_\Pi}$ in this bound comes from taking the maximum of $|(\log Y)^{z\delta_\Pi}|$ on the circle $|z| = \rho$.

Let $L = \log \log Y$ and write $B(z) = \sum_{j \geq 0} b_j z^j$. Since

$$e^{z\delta_\Pi L} = \sum_{m \geq 0} \frac{(\delta_\Pi L)^m}{m!} z^m,$$

the coefficient of z^n in $B(z)e^{z\delta_\Pi L}$ is

$$\sum_{j=0}^n b_j \frac{(\delta_\Pi L)^{n-j}}{(n-j)!}.$$

Here n is fixed. The term with $j = 0$ is the only term containing L^n ; all the remaining terms contain at most L^{n-1} . Consequently,

$$[z^n](B(z)e^{z\delta_\Pi L}) = B(0) \frac{\delta_\Pi^n}{n!} L^n + O_{g,n}(L^{n-1}).$$

It remains to verify that the error term remains negligible after coefficient extraction. By Cauchy's coefficient formula,

$$|[z^n]E(Y, z)| \leq \rho^{-n} \max_{|z|=\rho} |E(Y, z)| \ll_{\rho,n} Y(\log Y)^{\delta_\Omega - 2 + \rho\delta_\Pi}.$$

Relative to $Y(\log Y)^{\delta_\Omega - 1}(\log \log Y)^n$, this is bounded by a constant multiple of

$$\frac{(\log Y)^{-1 + \rho\delta_\Pi}}{(\log \log Y)^n},$$

which tends to zero because $\rho\delta_\Pi < 1$. We therefore obtain

$$\#\mathcal{D}_n(Y) \sim B(0) \frac{\delta_\Pi^n}{n!} Y(\log Y)^{\delta_\Omega - 1} (\log \log Y)^n.$$

Finally,

$$B(0) = \frac{H(1, 0)}{\Gamma(\delta_\Omega)} > 0,$$

so this is precisely (5.8), with the constant given in (5.9).

To pass from admissible integers to modular forms, take $Y = X/N_g$. The form f_D has exact level $N_g D$, so two different admissible integers give forms of different exact levels. Hence

$$\#\mathcal{F}_n(X) = \#\mathcal{D}_n(X/N_g).$$

Since N_g is fixed, replacing Y by X/N_g does not change the powers of $\log X$ or $\log \log X$ in the asymptotic. This proves (5.10). The assertion (5.11) now follows directly from Proposition 5.1. $\square$

We now allow the number of primes from Π_g to vary. Let $\mathcal{D}(Y)$ denote the set of all admissible integers $D \leq Y$, and consider the corresponding family

$$\mathcal{F}(Y) = \{f_D : D \in \mathcal{D}(Y)\}.$$

Since f_D has exact level $N_g D$, different values of D give different members of $\mathcal{F}(Y)$. In particular, $\#\mathcal{F}(Y) = \#\mathcal{D}(Y)$.

We regard $\mathcal{F}(Y)$ as a finite probability space by giving each of its members equal weight. Thus, if $\mathcal{A} \subseteq \mathcal{F}(Y)$, then

$$\mathbb{P}_Y(\mathcal{A}) = \frac{\#\mathcal{A}}{\#\mathcal{F}(Y)}.$$

If $X : \mathcal{F}(Y) \rightarrow \mathbb{R}$ is a real-valued function, we define its expectation and variance by

$$\mathbb{E}_Y[X] = \frac{1}{\#\mathcal{F}(Y)} \sum_{f \in \mathcal{F}(Y)} X(f)$$

and

$$\text{Var}_Y(X) = \frac{1}{\#\mathcal{F}(Y)} \sum_{f \in \mathcal{F}(Y)} (X(f) - \mathbb{E}_Y[X])^2.$$

Equivalently, $\text{Var}_Y(X) = \mathbb{E}_Y[X^2] - \mathbb{E}_Y[X]^2$.

For $\mathfrak{h} \in \{\sharp, \flat\}$, define the signed increment by

$$\Delta^{\mathfrak{h}}(f_D) = \lambda^{\mathfrak{h}}(f_D) - \lambda^{\mathfrak{h}}(g).$$

Proposition 5.1 gives the pointwise identity

$$\Delta^{\sharp}(f_D) = \Delta^{\flat}(f_D) = \omega_{\Pi}(D). \quad (5.12)$$

Thus the variation of either signed λ -invariant is governed by the elementary arithmetic function $\omega_{\Pi}(D)$. The probabilistic statements below are consequently distribution theorems for the signed Iwasawa invariants in the family $\mathcal{F}(Y)$.

We first record the size of the underlying family. Setting $z = 1$ in Proposition 5.4 removes the weight which records the number of prime divisors in Π_g . We obtain

$$\#\mathcal{F}(Y) = \#\mathcal{D}(Y) \sim \frac{H(1, 1)}{\Gamma(\delta_{\Pi} + \delta_{\Omega})} Y(\log Y)^{\delta_{\Pi} + \delta_{\Omega} - 1}. \quad (5.13)$$

Proposition 5.6. *There is a function*

$$\Phi_g(z) = \frac{H(1, z)}{H(1, 1)} \frac{\Gamma(\delta_{\Pi} + \delta_{\Omega})}{\Gamma(\delta_{\Omega} + z\delta_{\Pi})},$$

which is holomorphic and nonzero in a neighborhood of $z = 1$ and satisfies $\Phi_g(1) = 1$, such that, locally uniformly for z near 1 and for either signed label $\mathfrak{h}$,

$$\frac{1}{\#\mathcal{F}(Y)} \sum_{f \in \mathcal{F}(Y)} z^{\Delta^{\mathfrak{h}}(f)} = \Phi_g(z) \exp(\delta_{\Pi}(z - 1) \log \log Y) \left(1 + O_g\left(\frac{1}{\log Y}\right)\right). \quad (5.14)$$

The error term may be differentiated twice with respect to z on a neighborhood of 1.

Proof. By (5.12), the numerator on the left-hand side of (5.14) is

$$\sum_{D \in \mathcal{D}(Y)} z^{\omega_{\Pi}(D)} = S(Y, z).$$

The denominator is $S(Y, 1)$.

Choose a closed disc centered at 1 on which $\Re(\delta_\Omega + z\delta_\Pi) > 0$. Proposition 5.4 applies uniformly on this disc and gives

$$S(Y, z) = Y(\log Y)^{\delta_\Omega + z\delta_\Pi - 1} \left(\frac{H(1, z)}{\Gamma(\delta_\Omega + z\delta_\Pi)} + O_g \left(\frac{1}{\log Y} \right) \right).$$

At $z = 1$, the same formula gives (5.13). Dividing the two asymptotic formulae, and using the positivity of $H(1, 1)$, we find that

$$\frac{S(Y, z)}{S(Y, 1)} = \frac{H(1, z)}{H(1, 1)} \frac{\Gamma(\delta_\Pi + \delta_\Omega)}{\Gamma(\delta_\Omega + z\delta_\Pi)} (\log Y)^{\delta_\Pi(z-1)} \left(1 + O_g \left(\frac{1}{\log Y} \right) \right).$$

Since

$$(\log Y)^{\delta_\Pi(z-1)} = \exp(\delta_\Pi(z-1) \log \log Y),$$

this is precisely (5.14).

The function Φ_g is holomorphic near 1. It is also nonzero after the neighborhood is made sufficiently small, since $H(1, 1) > 0$ and the Gamma function is finite and nonzero at $\delta_\Pi + \delta_\Omega > 0$. Finally, the Selberg–Delange estimate is uniform and holomorphic in z . Cauchy’s integral formula therefore allows its error term to be differentiated twice on a slightly smaller disc. $\square$

Theorem 5.7. *Let $(Y_m)_{m \geq 1}$ be a sequence tending to infinity, and let*

$$X_m : \mathcal{F}(Y_m) \longrightarrow \mathbb{R}$$

be a sequence of real-valued functions. Suppose that, for every $t \in \mathbb{R}$,

$$\frac{1}{\#\mathcal{F}(Y_m)} \sum_{f \in \mathcal{F}(Y_m)} e^{itX_m(f)} \longrightarrow e^{-t^2/2}.$$

Then, for every $x \in \mathbb{R}$,

$$\frac{1}{\#\mathcal{F}(Y_m)} \#\{f \in \mathcal{F}(Y_m) : X_m(f) \leq x\} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du.$$

Proof. Let

$$\nu_m = \frac{1}{\#\mathcal{F}(Y_m)} \sum_{f \in \mathcal{F}(Y_m)} \delta_{X_m(f)}$$

be the probability distribution of X_m . Its characteristic function is the expression on the left-hand side of the hypothesis. Since this converges pointwise to $e^{-t^2/2}$, Lévy’s continuity theorem [Bil95, Theorem 26.3, p. 349] shows that ν_m converges weakly to the standard normal distribution and the asserted formula follows. $\square$

Proof of Theorem 1.3. We set $L_Y = \delta_\Pi \log \log Y$ and define

$$Z_Y = \frac{\Delta^\natural - L_Y}{\sqrt{L_Y}}.$$

Since $\delta_\Pi > 0$, we have $L_Y \rightarrow \infty$. We shall prove the limiting distribution by showing that the characteristic functions of the variables Z_Y converge to that of a standard Gaussian.

Fix $t \in \mathbb{R}$ and set

$$z_Y = \exp\left(\frac{it}{\sqrt{L_Y}}\right).$$

Then $z_Y \rightarrow 1$, and hence Proposition 5.6 applies for all sufficiently large Y . We find that $z_Y^{\Delta^\natural} = \exp(it\Delta^\natural/\sqrt{L_Y})$. We arrive at

$$\mathbb{E}_Y[e^{itZ_Y}] = \Phi_g(z_Y) \exp\left(L_Y \left(z_Y - 1 - \frac{it}{\sqrt{L_Y}}\right)\right) (1 + o(1)).$$

The first factor tends to one because Φ_g is continuous at 1 and $\Phi_g(1) = 1$. On the other hand, Taylor's formula gives

$$z_Y = 1 + \frac{it}{\sqrt{L_Y}} - \frac{t^2}{2L_Y} + O_t(L_Y^{-3/2}).$$

It follows that

$$L_Y \left(z_Y - 1 - \frac{it}{\sqrt{L_Y}} \right) = -\frac{t^2}{2} + O_t(L_Y^{-1/2}).$$

Consequently, we find that

$$\lim_{Y \rightarrow \infty} \mathbb{E}_Y[e^{itZ_Y}] = e^{-t^2/2}.$$

The function $e^{-t^2/2}$ is the characteristic function of the standard normal distribution. Theorem 5.7 now gives (1.4).

It remains to prove (1.5) and (1.6). We do this by computing the first two moments. Let

$$G_Y(z) = \mathbb{E}_Y[z^{\Delta^\natural}]$$

be the probability generating function of the signed increment. Taking logarithms in (5.14) gives, locally uniformly near $z = 1$,

$$\log G_Y(z) = L_Y(z - 1) + \log \Phi_g(z) + O_g\left(\frac{1}{\log Y}\right).$$

The differentiated assertion in Proposition 5.6 allows us to differentiate this identity twice at $z = 1$.

Since $G_Y(1) = 1$, we have

$$\mathbb{E}_Y[\Delta^\natural] = G'_Y(1) = (\log G_Y)'(1) = L_Y + (\log \Phi_g)'(1) + O_g\left(\frac{1}{\log Y}\right).$$

This proves (1.5). For the variance, the usual identities for a probability generating function give

$$\text{Var}_Y(\Delta^\natural) = G''_Y(1) + G'_Y(1) - G'_Y(1)^2 = (\log G_Y)''(1) + (\log G_Y)'(1).$$

Differentiating the logarithmic asymptotic once more, we obtain

$$\text{Var}_Y(\Delta^\natural) = L_Y + (\log \Phi_g)'(1) + (\log \Phi_g)''(1) + O_g\left(\frac{1}{\log Y}\right),$$

which proves (1.6).

Finally, (5.12) shows that the two signed increments agree for every D , rather than merely having the same limiting distribution. Subtracting the two identities gives

$$\lambda^\sharp(f_D) - \lambda^\flat(f_D) = \lambda^\sharp(g) - \lambda^\flat(g);$$

see (5.2). This completes the proof. $\square$

REFERENCES

- [BH97] P. N. Balister and S. Howson. “Note on Nakayama’s lemma for compact Λ -modules”. In: *Asian Journal of Mathematics* 1.2 (1997), pp. 224–229. DOI: [10.4310/AJM.1997.v1.n2.a2](https://doi.org/10.4310/AJM.1997.v1.n2.a2).
- [Bil95] P. Billingsley. *Probability and Measure*. 3rd ed. Wiley Series in Probability and Mathematical Statistics. New York: John Wiley & Sons, Inc., 1995. ISBN: 0-471-00710-2.
- [BLZ04] L. Berger, H. Li, and H. J. Zhu. “Construction of some families of 2-dimensional crystalline representations”. In: *Mathematische Annalen* 329.2 (2004), pp. 365–377. DOI: [10.1007/s00208-004-0529-y](https://doi.org/10.1007/s00208-004-0529-y).

- [Car89] H. Carayol. “Sur les représentations galoisiennes modulo ℓ attachées aux formes modulaires”. In: *Duke Mathematical Journal* 59.3 (1989), pp. 785–801. DOI: [10.1215/S0012-7094-89-05937-1](https://doi.org/10.1215/S0012-7094-89-05937-1).
- [DT94] F. Diamond and R. Taylor. “Non-optimal levels of mod l modular representations”. In: *Inventiones Mathematicae* 115.3 (1994), pp. 435–462. DOI: [10.1007/BF01231768](https://doi.org/10.1007/BF01231768).
- [Edi92] B. Edixhoven. “The weight in Serre’s conjectures on modular forms”. In: *Inventiones Mathematicae* 109.3 (1992), pp. 563–594. DOI: [10.1007/BF01232041](https://doi.org/10.1007/BF01232041).
- [EK40] P. Erdős and M. Kac. “The Gaussian law of errors in the theory of additive number theoretic functions”. In: *American Journal of Mathematics* 62.1 (1940), pp. 738–742. DOI: [10.2307/2371483](https://doi.org/10.2307/2371483).
- [EPW06] M. Emerton, R. Pollack, and T. Weston. “Variation of Iwasawa invariants in Hida families”. In: *Inventiones Mathematicae* 163.3 (2006), pp. 523–580. DOI: [10.1007/s00222-005-0467-7](https://doi.org/10.1007/s00222-005-0467-7).
- [GV00] R. Greenberg and V. Vatsal. “On the Iwasawa invariants of elliptic curves”. In: *Inventiones Mathematicae* 142.1 (2000), pp. 17–63. DOI: [10.1007/s002220000080](https://doi.org/10.1007/s002220000080).
- [HL19] J. Hatley and A. Lei. “Arithmetic properties of signed Selmer groups at non-ordinary primes”. In: *Annales de l’Institut Fourier* 69.3 (2019), pp. 1259–1294. DOI: [10.5802/aif.3270](https://doi.org/10.5802/aif.3270).
- [HR17] G. H. Hardy and S. Ramanujan. “The normal number of prime factors of a number n ”. In: *Quarterly Journal of Pure and Applied Mathematics* 48 (1917), pp. 76–92.
- [Iwa59] K. Iwasawa. “On Γ -extensions of algebraic number fields”. In: *Bulletin of the American Mathematical Society* 65.4 (1959), pp. 183–226. DOI: [10.1090/S0002-9904-1959-10317-7](https://doi.org/10.1090/S0002-9904-1959-10317-7).
- [Kat04] K. Kato. “ p -adic Hodge theory and values of zeta functions of modular forms”. In: *Astérisque* 295 (2004), pp. 117–290.
- [Kim09] B. D. Kim. “The Iwasawa invariants of the plus/minus Selmer groups”. In: *Asian Journal of Mathematics* 13.2 (2009), pp. 181–190. DOI: [10.4310/AJM.2009.v13.n2.a2](https://doi.org/10.4310/AJM.2009.v13.n2.a2).
- [Kim13] B. D. Kim. “The plus/minus Selmer groups for supersingular primes”. In: *Journal of the Australian Mathematical Society* 95.2 (2013), pp. 189–200. DOI: [10.1017/S1446788713000165](https://doi.org/10.1017/S1446788713000165).
- [Kob03] S. Kobayashi. “Iwasawa theory for elliptic curves at supersingular primes”. In: *Inventiones Mathematicae* 152.1 (2003), pp. 1–36. DOI: [10.1007/s00222-002-0265-4](https://doi.org/10.1007/s00222-002-0265-4).
- [Lei11] A. Lei. “Iwasawa theory for modular forms at supersingular primes”. In: *Compositio Mathematica* 147.3 (2011), pp. 803–838. DOI: [10.1112/S0010437X10005130](https://doi.org/10.1112/S0010437X10005130).
- [LLZ10] A. Lei, D. Loeffler, and S. L. Zerbes. “Wach modules and Iwasawa theory for modular forms”. In: *Asian Journal of Mathematics* 14.4 (2010), pp. 475–528. DOI: [10.4310/AJM.2010.v14.n4.a2](https://doi.org/10.4310/AJM.2010.v14.n4.a2).
- [LLZ11] A. Lei, D. Loeffler, and S. L. Zerbes. “Coleman maps and the p -adic regulator”. In: *Algebra & Number Theory* 5.8 (2011), pp. 1095–1131. DOI: [10.2140/ant.2011.5.1095](https://doi.org/10.2140/ant.2011.5.1095).
- [Maz72] B. Mazur. “Rational points of abelian varieties with values in towers of number fields”. In: *Inventiones Mathematicae* 18.3–4 (1972), pp. 183–266. DOI: [10.1007/BF01389815](https://doi.org/10.1007/BF01389815).
- [Pol03] R. Pollack. “On the p -adic L -function of a modular form at a supersingular prime”. In: *Duke Mathematical Journal* 118.3 (2003), pp. 523–558. DOI: [10.1215/S0012-7094-03-11835-9](https://doi.org/10.1215/S0012-7094-03-11835-9).
- [PW06] R. Pollack and T. Weston. “Kida’s formula and congruences”. In: *Documenta Mathematica* Extra Vol. (2006), pp. 615–630. DOI: [10.4171/DMS/4/18](https://doi.org/10.4171/DMS/4/18).
- [Ray24] A. Ray. “Constructing Galois representations with prescribed Iwasawa λ -invariant”. In: *Bulletin of the London Mathematical Society* 56.5 (2024), pp. 1624–1642. DOI: [10.1112/blms.13015](https://doi.org/10.1112/blms.13015).

- [Rib94] K. A. Ribet. “Report on mod ℓ representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ ”. In: *Motives (Seattle, WA, 1991)*. Vol. 55, Part 2. Proceedings of Symposia in Pure Mathematics. Providence, RI: American Mathematical Society, 1994, pp. 639–676.
- [Ser81] J.-P. Serre. “Quelques applications du théorème de densité de Chebotarev”. In: *Publications Mathématiques de l’IHÉS* 54 (1981), pp. 123–201. DOI: [10.1007/BF02698692](https://doi.org/10.1007/BF02698692).
- [Spr12] F. E. I. Sprung. “Iwasawa theory for elliptic curves at supersingular primes: a pair of main conjectures”. In: *Journal of Number Theory* 132.7 (2012), pp. 1483–1506. DOI: [10.1016/j.jnt.2011.11.003](https://doi.org/10.1016/j.jnt.2011.11.003).
- [Spr17] F. Sprung. “On pairs of p -adic L -functions for weight-two modular forms”. In: *Algebra & Number Theory* 11.4 (2017), pp. 885–928. DOI: [10.2140/ant.2017.11.885](https://doi.org/10.2140/ant.2017.11.885).
- [Ten15] G. Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*. Third. Vol. 163. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2015.
- [Tur34] P. Turán. “On a theorem of Hardy and Ramanujan”. In: *Journal of the London Mathematical Society* s1-9.4 (1934), pp. 274–276. DOI: [10.1112/jlms/s1-9.4.274](https://doi.org/10.1112/jlms/s1-9.4.274).
- [Zyw23] D. Zywina. “Modular forms and some cases of the inverse Galois problem”. In: *Canadian Mathematical Bulletin* 66.2 (2023), pp. 568–586. DOI: [10.4153/S0008439522000534](https://doi.org/10.4153/S0008439522000534).

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