

STRONGLY REGULAR GRAPHS ON 160, 232, 261 AND 280 VERTICES FROM FUSIONS OF ORBITALS OF $\text{P}\Gamma\text{L}(2, 8)$ AND $2^4:(S_3 \times S_3)$

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ABSTRACT. We construct strongly regular graphs with parameters $(160, 54, 18, 18)$, $(232, 63, 14, 18)$, $(261, 64, 14, 16)$ and $(280, 63, 14, 14)$, and hence also their complements. All four parameter sets were marked “?” (existence unknown) in Brouwer’s online table on 11 and 12 October 2026; they are the parameters of the hypothetical partial geometries $\text{pg}(9, 5, 3)$, $\text{pg}(7, 8, 2)$, $\text{pg}(8, 7, 2)$ and $\text{pg}(9, 6, 2)$. We give eight pairwise non-isomorphic graphs: three on 160 vertices, one on 232, two on 261 and two on 280. None of them is geometric. As far as we can determine, they are the first graphs known with these parameters.

Five of the graphs have a short description over the projective line $\text{PG}(1, 8)$ and its quadratic extension $\text{PG}(1, 64)$. The vertices are pairs of points of $\text{PG}(1, 8)$ or conjugate pairs of points of $\text{PG}(1, 64) \setminus \text{PG}(1, 8)$, each carrying a label in $\mathbb{F}_8^*$, together with a small orbit of points, pairs, or conjugate pairs. Adjacency is decided by cross ratios: for two labelled vertices on disjoint pairs, the quotient (or the product) of the two labels must equal the cross ratio of the two pairs or its inverse; for pairs sharing a point it must equal 1, and a labelled pair and a conjugate pair are adjacent when their cross ratio is a primitive cube root of unity. The “quotient” graphs on 232, 261 and 280 vertices have the automorphism group $(\text{SL}(2, 8) \times C_7) \rtimes C_6 \cong \text{GL}(2, 8) \rtimes C_6$ of order 21168; the two “product” graphs have $C_2 \times \text{P}\Gamma\text{L}(2, 8)$, of order 3024. One graph on 160 vertices lives on the 16 cells of a 4×4 grid and three copies of the 48 flags of its even transversals; its automorphism group is $C_3 \times 2^4:(S_3 \times S_3)$, of order 1728. The other two graphs on 160 vertices have automorphism groups of orders 1728 and 576.

The graphs were found by a computer search over fusions of orbitals of groups acting with two or three orbits, using the GAP library of tables of marks; the search reported five graphs, and enumerating all fusions of the same group actions gave three more. We do not classify the graphs with these parameters. The same constructions over $\text{PG}(1, q)$ give, for $q = 4$, the classical skew-lines graph of $\text{PG}(3, 2)$ and a second known $\text{srg}(35, 16, 6, 8)$, and no strongly regular graph for $q = 16$ or $q = 32$ under the groups searched (complete solver verdicts): within this model the graphs for $q = 8$ appear to be sporadic. Strong regularity of every graph is checked by an exact, standalone, standard-library Python program, the constructions are re-implemented from the text of this paper, and all computer results were reproduced by an independent audit with separately written code and checked again by a referee with its own code.

1. INTRODUCTION

A *strongly regular graph* $\text{srg}(v, k, \lambda, \mu)$ is a k -regular graph on v vertices, neither complete nor edgeless, in which two adjacent vertices have exactly λ common neighbours and two distinct non-adjacent vertices exactly μ . Equivalently, its adjacency matrix A satisfies $A^2 = kI + \lambda A + \mu(J - I - A)$. Brouwer maintains a table of feasible parameter sets with $v \leq 1300$ and their status [2]; the book of Brouwer and Van Maldeghem [3] contains the same table as of its publication.

A partial geometry $\text{pg}(s, t, \alpha)$ has lines of size $s + 1$, $t + 1$ lines on every point, and for every non-incident point–line pair exactly α points of the line are collinear with the point. Its collinearity graph is an $\text{srg}((s + 1)(st + \alpha)/\alpha, s(t + 1), s - 1 + t(\alpha - 1), \alpha(t + 1))$, and a strongly regular graph with these parameters is called *pseudo-geometric*. It is *geometric* if it is the collinearity graph of such a geometry [3, Ch. 1]. The four parameter sets of this paper are pseudo-geometric (Table 1). For $(160, 54, 18, 18)$ the table records that $\text{pg}(9, 5, 3)$ does not exist, so a graph with

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these parameters cannot be geometric. For the other three sets the table asks whether the partial geometry exists. The table recorded no graph for any of the four sets.

TABLE 1. The parameter sets. “Comment” is the comment in [2] as read on 12 October 2026, where each row was marked “?”. The eigenvalues follow from the parameters. “Del.” is the Delsarte clique bound $1 - k/s$ and “Hoff.” the Hoffman coclique bound $v(-s)/(k - s)$.

parameters	spectrum	complement	Del.	Hoff.	comment in [2]
(160, 54, 18, 18)	$54^1 6^{75} (-6)^{84}$	(160, 105, 68, 70)	10	16	pg(9,5,3) does not exist (no 2-graph* for line graph)
(232, 63, 14, 18)	$63^1 5^{144} (-9)^{87}$	(232, 168, 122, 120)	8	29	pg(7,8,2)?
(261, 64, 14, 16)	$64^1 6^{144} (-8)^{116}$	(261, 196, 147, 147)	9	29	pg(8,7,2)?
(280, 63, 14, 14)	$63^1 7^{135} (-7)^{144}$	(280, 216, 166, 168)	10	28	pg(9,6,2)?

Theorem 1.1.

- (1) *There exist strongly regular graphs with parameters (160, 54, 18, 18), (232, 63, 14, 18), (261, 64, 14, 16) and (280, 63, 14, 14). On these parameter sets there are at least 3, 1, 2 and 2 pairwise non-isomorphic graphs, listed in Table 2.*
- (2) *The graphs Γ_{232} , Γ_{261}^Q , Γ_{261}^P , Γ_{280}^Q , Γ_{280}^P of Construction 2.3 are defined over $\text{PG}(1, 8) \subset \text{PG}(1, 64)$ by cross ratios. The group $\Gamma_8 = (\text{SL}(2, 8) \times C_7) \rtimes (C_3 \times C_2)$ of order 21168 acts on the three “quotient” graphs Γ^Q and is their full automorphism group; $C_2 \times \text{P}\Gamma\text{L}(2, 8)$ acts on the two “product” graphs Γ^P and is their full automorphism group.*
- (3) *The graph $\Gamma_{160}^{(1)}$ of Construction 2.8 lives on the cells of a 4×4 grid and three copies of the flags of its even transversals; $\text{Aut}\Gamma_{160}^{(1)} \cong C_3 \times 2^4 : (S_3 \times S_3)$, of order 1728. The graphs $\Gamma_{160}^{(2)}$ and $\Gamma_{160}^{(3)}$ are fusions of orbitals of $2^4 : (S_3 \times S_3)$ acting on $96 + 48 + 16$ points, with automorphism groups of orders 1728 and 576.*
- (4) *None of the eight graphs is geometric. $\Gamma_{160}^{(i)}$, Γ_{280}^Q and Γ_{280}^P have no Delsarte clique at all; Γ_{261}^P and Γ_{261}^Q have only 46 resp. 100 Delsarte cliques, fewer than the 232 lines of a $\text{pg}(8, 7, 2)$; and no subfamily of the 513 Delsarte cliques of Γ_{232} partitions its edge set.*

What is proved by hand, and what by computer. The constructions, the valencies, the invariance of each graph under the named group (hence the lower bounds on $|\text{Aut}|$), the orbit structure of that group, the local graph of a point of Γ_{261}^Q and several cliques and cocliques are proved by hand (Section 2). We have not written a complete hand proof of the values of λ and μ ; strong regularity of every graph is established by exact computation, by several programs written independently of each other (Section 7), among them one written from the text of Section 2 alone. The orders of the automorphism groups, the non-isomorphism, the clique and coclique counts behind “not geometric” and the family results of Section 6 are computer results. Table 7 lists every claim.

Scope. Every completeness statement in this paper concerns all unions of orbitals of one stated group action (“all fusions of one action”). We make no classification claim for any of the four parameter sets: there may be further graphs with these parameters, with other groups or with no symmetry at all.

Relation to earlier work. In earlier papers we found strongly regular graphs as unions of orbitals of transitive permutation groups [7] and of groups acting with two orbits [8]. The present graphs come from the next step of the same programme: the search of [8] extended to three orbits and to direct products $G \times C_m$ (Section 5). None of the eight graphs is vertex-transitive: each

automorphism group has at least two orbits on vertices (Table 2). So no search over transitive actions could have found them.

Conventions. An $\text{srg}(v, k, \lambda, \mu)$ has the eigenvalues $k > r > s$ with $r, s = \frac{1}{2}(\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})$. A clique C has $|C| \leq 1 - k/s$ (Delsarte bound); a clique attaining it is a *Delsarte clique*, and every vertex outside it has exactly $|C|(k - |C| + 1)/(v - |C|)$ neighbours in it. A coclique S has $|S| \leq v(-s)/(k - s)$ (Hoffman bound); a coclique attaining it is a *Hoffman coclique*, and every vertex outside it has exactly $k|S|/(v - |S|)$ neighbours in it [3, Ch. 1]. If Γ is the collinearity graph of a partial geometry, its lines are cliques whose size equals the Delsarte bound of Γ , i.e. Delsarte cliques, and they partition the edge set; so a pseudo-geometric graph whose Delsarte cliques do not contain a partition of the edge set is not geometric. Groups are named as in the ATLAS and as in GAP’s **StructureDescription**; “.” is a split extension. TomLib class numbers refer to the GAP package TomLib [12] version 1.2.11 with GAP 4.12.1 [5]. Notation: $\mathcal{P}$, $\mathcal{L}$, $\mathcal{E}$ are the sets of points, pairs and conjugate pairs of Section 2.1; Γ_q (q a power of 2) denotes a group (Proposition 2.4 and Section 6), while Γ_{232} , Γ_{261}^* , Γ_{280}^* and $\Gamma_{160}^{(i)}$, indexed by the number of vertices, denote graphs; W denotes the group $2^4:(S_3 \times S_3)$ of Section 2.3.

TABLE 2. The eight graphs ($\mathcal{P}$, $\mathcal{L}$, $\mathcal{E}$: notation of Section 2.1). “No.” is the label used in the data files (Data availability). “Orbits” are the orbits of Aut on vertices. All entries are computer results except the constructions; Section 2 proves $|\text{Aut}| \geq$ the order listed for Γ_{232} , Γ^P , Γ^Q and $\Gamma_{160}^{(1)}$ by hand.

No.	graph	construction	$ \text{Aut} $	Aut	orbits
G160	$\Gamma_{160}^{(1)}$	grid and flags (Constr. 2.8)	1728	$C_3 \times 2^4:(S_3 \times S_3)$	144 + 16
G160b	$\Gamma_{160}^{(2)}$	fusion, $2^4:(S_3 \times S_3)$ on $96 + 48 + 16$	1728	$C_3:(2^4:(S_3 \times S_3))$	144 + 16
G160c	$\Gamma_{160}^{(3)}$	fusion, $2^4:(S_3 \times S_3)$ on $96 + 48 + 16$	576	$2^4:(S_3 \times S_3)$	96 + 48 + 16
G232	Γ_{232}	quotient rule on $\mathcal{E} \times \mathbb{F}_8^* \cup \mathcal{L}$	21168	$\text{GL}(2, 8) \rtimes C_6$	196 + 36
G261a	Γ_{261}^P	product rule on $\mathcal{L} \times \mathbb{F}_8^* \cup \mathcal{P}$	3024	$C_2 \times \text{PTL}(2, 8)$	216 + 36 + 9
G261b	Γ_{261}^Q	quotient rule on $\mathcal{L} \times \mathbb{F}_8^* \cup \mathcal{P}$	21168	$\text{GL}(2, 8) \rtimes C_6$	252 + 9
G280a	Γ_{280}^P	product rule on $\mathcal{L} \times \mathbb{F}_8^* \cup \mathcal{E}$	3024	$C_2 \times \text{PTL}(2, 8)$	216 + 36 + 28
G280b	Γ_{280}^Q	quotient rule on $\mathcal{L} \times \mathbb{F}_8^* \cup \mathcal{E}$	21168	$\text{GL}(2, 8) \rtimes C_6$	252 + 28

2. THE CONSTRUCTIONS

2.1. Cross ratios over $\mathbb{F}_8 \subset \mathbb{F}_{64}$. Let $\mathbb{F}_{64} \supset \mathbb{F}_8$, and write $\sigma: x \mapsto x^8$ for the involutory automorphism of $\mathbb{F}_{64}$ over $\mathbb{F}_8$. Points of $\text{PG}(1, 64)$ are written $\langle u \rangle$ with $u \in \mathbb{F}_{64}^2 \setminus \{0\}$; for $u = (u_1, u_2)$ and $w = (w_1, w_2)$ put $[u, w] = u_1 w_2 + u_2 w_1$ (the determinant; signs do not matter in characteristic 2, so $[u, w] = [w, u]$). We use three sets of objects:

- $\mathcal{P} = \text{PG}(1, 8)$, the 9 points of the subline;
- $\mathcal{L}$, the 36 unordered pairs of distinct points of $\mathcal{P}$;
- $\mathcal{E}$, the 28 pairs $\{e, e^\sigma\}$ with $e \in \text{PG}(1, 64) \setminus \text{PG}(1, 8)$ (these are the 56 points off the subline, paired by σ).

Both $\mathcal{L}$ and $\mathcal{E}$ consist of 2-subsets of $\text{PG}(1, 64)$, and any two distinct members of $\mathcal{L} \cup \mathcal{E}$ are disjoint unless both lie in $\mathcal{L}$ and share a point. For disjoint 2-subsets $X = \{a, b\}$ and $Y = \{c, d\}$ put

$$\chi(X, Y) = \frac{[a, c][b, d]}{[a, d][b, c]},$$

computed with any representatives; the value does not depend on them, since numerator and denominator are homogeneous of the same degree in each of a, b, c, d .

Lemma 2.1. *Let X, Y be disjoint 2-subsets of $\text{PG}(1, 64)$.*

- (1) *Swapping a and b , or c and d , replaces χ by χ^{-1} , and $\chi(Y, X) = \chi(X, Y)$. So the unordered pair $\{\chi(X, Y), \chi(X, Y)^{-1}\}$ depends only on $\{X, Y\}$.*
- (2) *$\chi(X, Y) \notin \{0, 1\}$.*
- (3) *If $X, Y \in \mathcal{L}$ or $X, Y \in \mathcal{E}$, then $\chi(X, Y) \in \mathbb{F}_8^* \setminus \{1\}$. If $X \in \mathcal{L}$ and $Y \in \mathcal{E}$, then $\chi(X, Y)^9 = 1$.*
- (4) *$\chi(gX, gY) = \chi(X, Y)$ for every $g \in \text{GL}(2, 64)$, and $\chi(X^\phi, Y^\phi) = \chi(X, Y)^2$ for the Frobenius map $\phi: (u_1, u_2) \mapsto (u_1^2, u_2^2)$.*

Proof. (1) is immediate from $[u, w] = [w, u]$. (2) The denominator and numerator are non-zero because the four points are distinct. The Plücker relation $[a, b][c, d] + [a, c][b, d] + [a, d][b, c] = 0$ (valid in characteristic 2 without signs) shows $[a, c][b, d] = [a, d][b, c]$ iff $[a, b][c, d] = 0$, which is false. (3) Apply σ coordinatewise. Points of $\mathcal{P}$ are fixed and σ swaps the two points of a member of $\mathcal{E}$. For $X, Y \in \mathcal{L}$ nothing moves, for $X, Y \in \mathcal{E}$ both pairs are swapped, so in both cases $\chi^\sigma = \chi$ by (1), i.e. $\chi \in \mathbb{F}_8$. For $X \in \mathcal{L}$, $Y \in \mathcal{E}$ only c, d are swapped, so $\chi^8 = \chi^{-1}$. (4) $[gu, gw] = \det(g)[u, w]$ and $\det g$ cancels; ϕ is a field automorphism applied to all coordinates. $\square$

Fix a generator ω of $\mathbb{F}_8^*$ and the two primitive cube roots of unity $\zeta, \zeta^2 \in \mathbb{F}_4 \subset \mathbb{F}_{64}$. Since $3 \mid 9$, they are among the values allowed for $\chi(X, Y)$ with $X \in \mathcal{L}$, $Y \in \mathcal{E}$ by Lemma 2.1(3).

Lemma 2.2.

- (1) *For $\pi \in \mathcal{L}$ and $x \in \mathbb{F}_8^* \setminus \{1\}$, exactly 7 pairs $\pi' \in \mathcal{L}$ disjoint from π have $\chi(\pi, \pi') \in \{x, x^{-1}\}$.*
- (2) *For $\pi \in \mathcal{L}$, exactly 7 elements $Y \in \mathcal{E}$ have $\chi(\pi, Y) \in \{\zeta, \zeta^2\}$; for $Y \in \mathcal{E}$, exactly 9 pairs $\pi \in \mathcal{L}$ have this property.*

Proof. $\text{SL}(2, 8)$ is 2-transitive on $\mathcal{P}$ and preserves χ , so we may take $\pi = \{0, \infty\}$ with $0 = \langle(0, 1)\rangle$, $\infty = \langle(1, 0)\rangle$. For a point $\langle(a, 1)\rangle$ we have $[0, a] = a$ and $[\infty, a] = 1$. (1) For $\pi' = \{a, b\} \subset \mathbb{F}_8^*$, $\chi(\pi, \pi') = a/b$, and the unordered pairs $\{a, b\}$ with $\{a/b, b/a\} = \{x, x^{-1}\}$ are the 7 pairs $\{a, a/x\}$ (note $x \neq x^{-1}$). (2) For $Y = \{\langle(\varepsilon, 1)\rangle, \langle(\varepsilon^8, 1)\rangle\}$, $\chi(\pi, Y) = \varepsilon/\varepsilon^8 = \varepsilon^{-7}$. This is a primitive cube root of unity iff the order of ε divides 21 but not 7. There are $21 - 7 = 14$ such ε , none in $\mathbb{F}_8$, closed under σ ; they form 7 elements of $\mathcal{E}$. The number of such π is the same for every $Y \in \mathcal{E}$, since $\text{SL}(2, 8)$ is transitive on $\mathcal{E}$ (Proposition 2.4(3), whose proof does not use this lemma) and preserves χ ; double counting, $36 \cdot 7 = 28 \cdot 9$, gives 9. $\square$

2.2. The graphs on 232, 261 and 280 vertices. Vertices of the form (X, c) carry a *label* $c \in \mathbb{F}_8^*$. For two labelled vertices $(X, c), (X', c')$ the *quotient rule* uses $d = c'/c$ and the *product rule* uses $d = c'c$; in both cases d is compared with $\chi(X, X')^{\pm 1}$, so by Lemma 2.1(1) the conditions below are symmetric in the two vertices (for the quotient rule, swapping the vertices replaces d by d^{-1}).

Construction 2.3. Let $\star \in \{Q, P\}$ select the quotient rule (Q) or the product rule (P).

- (1) Γ_{261}^* has vertex set $(\mathcal{L} \times \mathbb{F}_8^*) \cup \mathcal{P}$, $252 + 9$ vertices.
 - $(\pi, c) \sim (\pi', c')$ for $c \neq c'$: the 7 labelled copies of a pair form a clique.
 - For $|\pi \cap \pi'| = 1$: $(\pi, c) \sim (\pi', c')$ iff $d = 1$.
 - For $\pi \cap \pi' = \emptyset$: $(\pi, c) \sim (\pi', c')$ iff $d \in \{\chi(\pi, \pi'), \chi(\pi, \pi')^{-1}\}$.
 - $(\pi, c) \sim p$ iff $p \in \pi$; the 9 points form a clique.
- (2) Γ_{280}^* has vertex set $(\mathcal{L} \times \mathbb{F}_8^*) \cup \mathcal{E}$, $252 + 28$ vertices.
 - Two copies of the same pair are not adjacent.
 - For $|\pi \cap \pi'| = 1$ and for $\pi \cap \pi' = \emptyset$: as in (1).
 - $(\pi, c) \sim Y$ iff $\chi(\pi, Y) \in \{\zeta, \zeta^2\}$; $\mathcal{E}$ is a coclique.
- (3) $\Gamma_{232} = \Gamma_{232}^Q$ has vertex set $(\mathcal{E} \times \mathbb{F}_8^*) \cup \mathcal{L}$, $196 + 36$ vertices.
 - $(Y, c) \sim (Y', c')$ iff $Y \neq Y'$ and $c'/c \in \{\chi(Y, Y'), \chi(Y, Y')^{-1}\}$.
 - $(Y, c) \sim \pi$ iff $\chi(\pi, Y) \in \{\zeta, \zeta^2\}$.
 - $\pi \sim \pi'$ iff $|\pi \cap \pi'| = 1$; on $\mathcal{L}$ this is the triangular graph $T(9)$.

The product rule in (3) does not give a strongly regular graph (it has two values of λ and of μ ; Section 5). The choice of ω , ζ and of the model of $\mathbb{F}_{64}$ does not matter: any two choices differ by a field isomorphism, which maps one graph onto the other. Replacing $\chi^{\pm 1}$ by $\chi^{\pm 2}$ in the rules gives isomorphic graphs: the map $(X, c) \mapsto (X, c^2)$ squares d and leaves χ unchanged, so it maps the graph with exponent ± 1 onto the graph with exponent ± 2 , and likewise for every exponent $\pm m$ with $7 \nmid m$. The exponent 2 was also checked by computer. Section 5.3 shows that the exponents ± 1 , ± 2 , ± 3 account exactly for the multiplicities of the fusions found.

Proposition 2.4.

- (1) Γ_{261}^* is 64-regular, and Γ_{280}^* and Γ_{232} are 63-regular.
- (2) Let $\text{SL}(2, 8)$ act on points (and so on $\mathcal{L}$, $\mathcal{E}$), t on labels by $c \mapsto \omega c$, ϕ by the Frobenius map on coordinates and $c \mapsto c^2$, and ι by $c \mapsto c^{-1}$. These generate a group $\Gamma_8 = (\text{SL}(2, 8) \times \langle t \rangle) \rtimes (\langle \phi \rangle \times \langle \iota \rangle)$ of order $504 \cdot 7 \cdot 3 \cdot 2 = 21168$, which acts faithfully on each vertex set and preserves the graphs Γ_{261}^Q , Γ_{280}^Q and Γ_{232} . Its subgroup $\langle \text{SL}(2, 8), \phi, \iota \rangle \cong C_2 \times \text{P}\Gamma\text{L}(2, 8)$, of order 3024, preserves Γ_{261}^P and Γ_{280}^P .
- (3) The orbits of Γ_8 are $\mathcal{L} \times \mathbb{F}_8^*$ and $\mathcal{P}$, resp. $\mathcal{E}$, resp. $\mathcal{E} \times \mathbb{F}_8^*$ and $\mathcal{L}$. The orbits of $C_2 \times \text{P}\Gamma\text{L}(2, 8)$ on $\mathcal{L} \times \mathbb{F}_8^*$ are $\mathcal{L} \times \{1\}$ (36) and $\mathcal{L} \times (\mathbb{F}_8^* \setminus \{1\})$ (216).

Proof. (1) In Γ_{261}^* a vertex (π, c) has 6 neighbours among the copies of π , one neighbour among the copies of each of the 14 pairs meeting π in a point, two among the copies of each of the 21 pairs disjoint from π (Lemma 2.1(2): $\chi \neq \chi^{-1}$), and the 2 points of π : $6 + 14 + 42 + 2 = 64$. A point has 8 point neighbours and the $8 \cdot 7 = 56$ copies of the pairs through it. In Γ_{280}^* a vertex (π, c) has $14 + 42 + 7 = 63$ neighbours (Lemma 2.2(2)), and $Y \in \mathcal{E}$ has the 7 copies of each of 9 pairs. In Γ_{232} , (Y, c) has two neighbours among the copies of each of the other 27 elements of $\mathcal{E}$ and 9 neighbours in $\mathcal{L}$; a pair has 14 neighbours in $T(9)$ and $7 \cdot 7 = 49$ in $\mathcal{E} \times \mathbb{F}_8^*$.

(2) The adjacency conditions involve only intersection sizes, the cross ratio, membership in $\{\zeta, \zeta^2\}$ and d . $\text{SL}(2, 8)$ fixes labels and preserves χ (Lemma 2.1(4)); it commutes with σ , so it permutes $\mathcal{E}$. t fixes $d = c'/c$. ϕ squares χ and d , and preserves $\{\zeta, \zeta^2\}$; it commutes with σ , so it permutes $\mathcal{E}$. ι inverts d under both rules. So Γ_8 preserves the quotient graphs, and $\langle \text{SL}(2, 8), \phi, \iota \rangle$ preserves the product graphs (t multiplies $d = cc'$ by ω^2 and is not used).

ϕ has order 3 on the vertices. ϕ^3 is $x \mapsto x^8$ on coordinates, i.e. σ ; it fixes every point of $\mathcal{P}$ (hence every member of $\mathcal{L}$), maps each member $\{e, e^\sigma\}$ of $\mathcal{E}$ to itself, and fixes every label ($c^8 = c$ for $c \in \mathbb{F}_8^*$). So ϕ^3 acts trivially on each vertex set, while ϕ itself moves labels ($\omega \mapsto \omega^2$).

The order of Γ_8 . Every generator maps a labelled vertex (X, c) to $(\bar{\gamma}X, \hat{\gamma}c)$ and an unlabelled vertex X to $\bar{\gamma}X$, where $\bar{\gamma}$ is the induced permutation of $\mathcal{P}$ (and so of $\mathcal{L}$ and $\mathcal{E}$) and $\hat{\gamma}$ that of the labels. On $\mathcal{P}$ the generators induce $\text{PSL}(2, 8)$ and the Frobenius map, which generate $\text{P}\Gamma\text{L}(2, 8)$ (t and ι act trivially). On the exponents e of labels $c = \omega^e$, the maps t , ϕ , ι act as $e \mapsto e + 1$, $e \mapsto 2e$, $e \mapsto -e$, and $\text{SL}(2, 8)$ acts trivially; these generate $\text{AGL}(1, 7)$, of order 42. So $\gamma \mapsto (\bar{\gamma}, \hat{\gamma})$ is a homomorphism of Γ_8 into $\text{P}\Gamma\text{L}(2, 8) \times \text{AGL}(1, 7)$, and it is injective, since an element with $\bar{\gamma} = 1$ and $\hat{\gamma} = 1$ fixes every vertex. Let $\epsilon_1: \text{P}\Gamma\text{L}(2, 8) \rightarrow \text{P}\Gamma\text{L}(2, 8)/\text{PSL}(2, 8) \cong C_3$ and $\epsilon_2: \text{AGL}(1, 7) \rightarrow (\mathbb{Z}/7)^*/\{\pm 1\} \cong C_3$ (the class of the linear part), identified so that the Frobenius map and multiplication by 2 go to the same generator. Each generator lies in the fibre product $F = \{(x, y) : \epsilon_1(x) = \epsilon_2(y)\}$, of order $1512 \cdot 42/3 = 21168$. Conversely the image contains $\text{PSL}(2, 8) \times 1$ (from $\text{SL}(2, 8) \cong \text{PSL}(2, 8)$), $1 \times \langle e \mapsto e + 1, e \mapsto -e \rangle \cong 1 \times D_{14}$, and the element (Frobenius, $e \mapsto 2e$), which maps onto a generator of $F/(\text{PSL}(2, 8) \times D_{14}) \cong C_3$; so the image is F and $|\Gamma_8| = 504 \cdot 7 \cdot 3 \cdot 2 = 21168$. The action on each vertex set is faithful: an element acting trivially fixes every label, so $\hat{\gamma} = 1$, and fixes every member of $\mathcal{L}$ (resp. $\mathcal{E}$), so $\bar{\gamma}$ lies in the kernel of the action of $\text{P}\Gamma\text{L}(2, 8)$ on $\mathcal{L}$ (resp. $\mathcal{E}$). That kernel is trivial: on $\mathcal{L}$ because a point is the intersection of two pairs; on $\mathcal{E}$ because the kernel is a normal subgroup meeting the simple group $\text{PSL}(2, 8)$ trivially ($\text{PSL}(2, 8)$ moves elements of $\mathcal{E}$), hence it centralises $\text{PSL}(2, 8)$, whose

centraliser in $\text{P}\Gamma\text{L}(2, 8)$ is trivial. The same argument with $\hat{\gamma}$ restricted to $\langle e \mapsto 2e, e \mapsto -e \rangle$ gives $|\langle \text{SL}(2, 8), \phi, \iota \rangle| = 504 \cdot 3 \cdot 2 = 3024$. In this group ι is central, $\langle \text{SL}(2, 8), \phi \rangle \cong \text{P}\Gamma\text{L}(2, 8)$, and $\iota \notin \langle \text{SL}(2, 8), \phi \rangle$ (the elements of $\langle \text{SL}(2, 8), \phi \rangle$ that fix every point have $\epsilon_1 = 1$, hence lie in $\text{SL}(2, 8)$, which fixes every label), so the group is $C_2 \times \text{P}\Gamma\text{L}(2, 8)$.

(3) $\text{SL}(2, 8)$ is transitive on $\mathcal{L}$ (it is 3-transitive on $\mathcal{P}$) and on $\text{PG}(1, 64) \setminus \text{PG}(1, 8)$ (a point stabiliser is a non-split torus C_9 , of index 56), hence on $\mathcal{E}$; t is transitive on labels. For the product graphs the label 1 is fixed by ϕ and ι , and $\langle \phi, \iota \rangle$ is transitive on $\mathbb{F}_8^* \setminus \{1\}$. $\square$

For Γ_{261}^* we can check the rows of the points by hand.

Proposition 2.5. *In Γ_{261}^* , $\star \in \{Q, P\}$, two adjacent vertices of which at least one is a point have exactly 14 common neighbours, and a point and a non-adjacent vertex have exactly 16. The local graph of a point in Γ_{261}^Q is the 8×8 rook's graph $L_2(8)$, an $\text{srg}(64, 14, 6, 2)$.*

Proof. Two points p, p' : the other 7 points and the 7 copies of $\{p, p'\}$. A point p and (π, c) with $p \in \pi$: the other point of π , the other 6 copies of π , and for each of the 7 pairs $\pi' \ni p$, $\pi' \neq \pi$, the unique copy (π', c') with $d = 1$. A point p and (π, c) with $p \notin \pi$: the 2 points of π ; the 2 pairs $\{p, a\}$, $a \in \pi$, with one copy each; and the 6 pairs $\{p, b\}$, $b \notin \pi$, disjoint from π , with two copies each: $2 + 2 + 12 = 16$. Under the quotient rule, $(\{p, a\}, c) \sim (\{p, b\}, c')$ for $a \neq b$ iff $c = c'$. So the 64 neighbours of p form an 8×8 array with rows indexed by $a \in \mathcal{P} \setminus \{p\}$ (row a : the point a and the 7 copies of $\{p, a\}$) and columns indexed by $\mathbb{F}_8^* \cup \{\text{point}\}$, and two neighbours are adjacent iff they share a row or a column. $\square$

Computation 2.6. The five graphs of Construction 2.3 with $\star$ as stated are strongly regular, with parameters $(261, 64, 14, 16)$, $(280, 63, 14, 14)$ and $(232, 63, 14, 18)$. By Proposition 2.4 it suffices to compute the rows of A^2 for one vertex in each orbit of the group of Proposition 2.4(3) (two or three vertices, plus one per label class for the product graphs); we have also computed the full matrices A^2 with exact integer arithmetic, with several independent programs (Section 7). The product rule on $(\mathcal{E} \times \mathbb{F}_8^*) \cup \mathcal{L}$ is 63-regular with $\lambda \in \{14, 18\}$, hence not strongly regular.

Remark 2.7. Several cliques and cocliques can be read off. In both Γ_{261}^* the 9 points, and the 7 copies of a pair together with its 2 points, are cliques of size $9 = 1 - k/s$, i.e. Delsarte cliques; in Γ_{261}^Q so is a point p together with the 8 copies $(\{p, a\}, c)$ with one fixed label c (a column of the grid of Proposition 2.5). In Γ_{280}^* the set $\mathcal{E}$ is a coclique of size $28 = v(-s)/(k - s)$, a Hoffman coclique. In Γ_{232} the 8 pairs through a point form a Delsarte 8-clique of $T(9)$.

2.3. The first graph on 160 vertices: a grid and flags. Let $R = \{1, 2, 3, 4\}$ and consider the 16 cells $(i, j) \in R \times R$ of a 4×4 grid. Two distinct cells are *collinear* if they share a row or a column. For an even permutation $s \in A_4$ the *transversal* $T_s = \{(j, s(j)) : j \in R\}$ is a set of four pairwise non-collinear cells; there are 12 even transversals. A *flag* is a pair (T, f) with T an even transversal and $f \in T$; there are 48 flags.

Construction 2.8. $\Gamma_{160}^{(1)}$ has as vertices the 16 cells and the 144 triples (T, f, z) with (T, f) a flag and $z \in \mathbb{Z}/3$. Cells are pairwise non-adjacent. A cell x is adjacent to (T, f, z) iff $x \notin T$ and x is not collinear with f . For two triples (T, f, z) , (T', f', z') put $\delta = z' - z$. They are adjacent in exactly the following cases:

- (1) $|T \cap T'| = 1$, $f \neq f'$, f and f' collinear, $f \notin T'$, $f' \notin T$, and $\delta = 0$;
- (2) $|T \cap T'| = 1$, $f \neq f'$ not collinear, $f \notin T'$, $f' \notin T$, and $\delta \neq 0$;
- (3) $|T \cap T'| = 1$, $f \neq f'$ not collinear, and exactly one of $f \in T'$, $f' \in T$ holds. Order the two triples so that $f \in T'$ and $f' \notin T$; then they are adjacent iff $\delta = z' - z = 2$. Equivalently, for the ordering with $f \notin T'$ and $f' \in T$, they are adjacent iff $\delta = 1$;
- (4) $T \cap T' = \emptyset$, f and f' collinear, and $\delta \neq 0$.

All other pairs of triples are non-adjacent; in particular two triples with $T = T'$, or with $|T \cap T'| = 1$ and $f = f'$ (the common cell), are non-adjacent.

Two even transversals meet in 4, 1 or 0 cells, since a non-trivial element of A_4 fixes one point or none. If $|T \cap T'| = 1$ and $f \neq f'$, then $f \in T'$ and $f' \in T$ cannot both hold (the two transversals would share f and f'), so (2) and (3) cover all non-collinear $f \neq f'$. The rule is symmetric: exchanging the two triples negates δ and exchanges the two memberships, so the two formulations of (3) describe the same unordered pairs. Exchanging the values 2 and 1 in (3) gives an isomorphic graph: it is the image under $z \mapsto -z$, which negates δ and preserves (1), (2) and (4). A cell lies on 3 even transversals. So 9 even transversals avoid a given cell x ; each of them meets the row and the column of x in two different cells, so it has exactly 2 cells not collinear with x , and x has $9 \cdot 2 \cdot 3 = 54$ neighbours. The cells form a coclique of size 16, which is the Hoffman bound.

Let $W \leq S_4 \wr S_2$ be the group of maps $(i, j) \mapsto (i^a, j^b)$ with $a, b \in S_4$ of the same sign, together with the transposition $\tau: (i, j) \mapsto (j, i)$; $|W| = 2 \cdot 24 \cdot 24/2 = 576$. On flags, (a, b) maps $(T_s, (i, s(i)))$ to $(T_{a^{-1}sb}, (i^a, i^{sb}))$ (the sign of $a^{-1}sb$ is that of s), and τ maps it to $(T_{s^{-1}}, (s(i), i))$. Let C_3 act by $z \mapsto z + 1$ on triples and trivially on cells.

Proposition 2.9. *$W \times C_3$, of order 1728, acts on $\Gamma_{160}^{(1)}$ as a group of automorphisms with the two orbits “cells” (16) and “triples” (144). W is isomorphic to the group $2^4:(S_3 \times S_3)$ of TomLib, and acts primitively with rank 3 on the cells, with cell stabiliser $S_3 \times S_3$.*

Proof. The rule uses only incidence, collinearity, intersection sizes and δ , which W preserves (W acts trivially on z), and δ is invariant under $z \mapsto z + 1$. W is transitive on the cells, on the 12 even transversals (by $s \mapsto a^{-1}sb$) and on the four cells of a transversal, hence on flags, and C_3 is transitive on z . The cell stabiliser has the three orbits “equal”, “collinear” (6) and “non-collinear” (9), so the rank on cells is 3. The isomorphism with the TomLib group, the primitivity and the cell stabiliser were checked in GAP (`paper_grid_group.g`). $\square$

Computation 2.10. $\Gamma_{160}^{(1)}$ is an $\text{srg}(160, 54, 18, 18)$ and $\text{Aut}\Gamma_{160}^{(1)} = W \times C_3$. The rule of Construction 2.8 was extracted from the orbital fusion found by the search (Section 5) by computing, for every ordered pair of vertices, the invariants used in the rule: adjacency is a function of these invariants, with no conflict over all $160 \cdot 159$ ordered pairs (`rules160.g`). The graph built from the rule alone has the same canonical form as the search hit. The independent audit (Section 7) built the graph from the rule with both orientations of (3) and obtained the canonical form of the search hit in both cases.

2.4. The other two graphs on 160 vertices. The group $W = 2^4:(S_3 \times S_3)$ (TomLib table $2^4:(S3 \times S3)$) has subgroups C_6 (class 25), D_{12} and $S_3 \times S_3$ (class 105), of indices 96, 48 and 16. There are two classes of subgroups D_{12} , numbers 60 and 66, both of index 48. In the model of Section 2.3 the action on the cosets of $S_3 \times S_3$ is the action on the cells. The search action that gave $\Gamma_{160}^{(1)}$ uses three copies of W/D_{12} with D_{12} of class 60; the same action with class 66 gives $\Gamma_{160}^{(1)}$ as well (Table 6). The subgroup C_6 of class 25 lies in exactly one conjugate of a D_{12} of class 60, with index 2, and in no conjugate of class 66; so the action of W on $W/C_6 \sqcup W/D_{12} \sqcup W/(S_3 \times S_3)$, with D_{12} of class 60, has a 96-point orbit that is a double cover of the 48-point orbit (`c25.10g`). On these $96 + 48 + 16 = 160$ points W has 43 symmetrised non-diagonal orbitals, and exactly four unions of them are $\text{srg}(160, 54, 18, 18)$. They fall into two isomorphism classes, $\Gamma_{160}^{(2)}$ and $\Gamma_{160}^{(3)}$, with two fusions each. The automorphism group of $\Gamma_{160}^{(3)}$ is W itself, with the three orbits 96, 48, 16; that of $\Gamma_{160}^{(2)}$ has order 1728, contains W with index 3, and fuses the orbits of lengths 96 and 48.

We have not extracted a prose rule for these two graphs. They are defined as follows, so that they can be rebuilt from TomLib without our files. Number the 160 points as in the GAP code below (GAP 4.12.1, TomLib 1.2.11): points 1–96, 97–144 and 145–160 are the right cosets of

the representatives (**RepresentativeTom**) of the classes 25, 60 and 105, each block in the order returned by **RightCosets**. A graph is given by a list of representative pairs and is the union of the orbits of W on unordered pairs of points that contain one of them:

$$\begin{aligned}\Gamma_{160}^{(2)} &: \{1, 4\}, \{1, 6\}, \{1, 9\}, \{1, 12\}, \{1, 97\}, \{1, 103\}, \{1, 110\}, \{1, 147\}, \{97, 109\}, \{97, 150\}; \\ \Gamma_{160}^{(3)} &: \{1, 6\}, \{1, 9\}, \{1, 12\}, \{1, 14\}, \{1, 99\}, \{1, 103\}, \{1, 104\}, \{1, 145\}, \{97, 109\}, \{97, 150\}.\end{aligned}$$

Each list contains one pair from each of the 10 symmetrised orbitals of the union.

```
LoadPackage("tomlib");
tom := TableOfMarks("2^4:(S3xS3)");; W := UnderlyingGroup(tom);;
cos := Concatenation(List([25, 60, 105], i -> RightCosets(W, RepresentativeTom(tom, i))));;
P := Group(List(GeneratorsOfGroup(W), g -> PermList(List(cos, c -> Position(cos, c * g)))));;
graph := reps -> Set(Concatenation(List(reps, r -> Orbit(P, Set(r), OnSets))));;
G2 := graph([[1,4],[1,6],[1,9],[1,12],[1,97],[1,103],[1,110],[1,147],[97,109],[97,150]]);;
G3 := graph([[1,6],[1,9],[1,12],[1,14],[1,99],[1,103],[1,104],[1,145],[97,109],[97,150]]);;
```

The script **paper_160bc.g** consists of these lines, the two other fusions and a graph6 writer; its outputs have the canonical forms of **G160b.g6** and **G160c.g6** and pass the exact check (**canon_160bc.log**). The graphs were also rebuilt from the orbital lists of the fusion enumeration in Section 7, and the independent audit rebuilt the action from the TomLib table with its own orbital extractor, enumerated all fusions with its own solver model, and again found exactly four fusions in two classes, with the canonical forms of **G160b.g6** and **G160c.g6**.

3. THE GROUPS

3.1. **PTL(2, 8) and Γ_8** . TomLib contains the table of marks of $G = L_2(8):3 \times 2 = C_2 \times \text{PTL}(2, 8)$, of order 3024; in this section G always denotes this group. Its central involution generates the factor C_2 ; in the model of Section 2.2 it is ι . The subgroups that occur are listed in Table 3 (**groups.log**). The point stabiliser $C_2 \times \text{AFL}(1, 8)$ gives $\mathcal{P}$, the normalisers of the split and non-split tori give $\mathcal{L}$ and $\mathcal{E}$ (each element of $\mathcal{E}$ is the pair of fixed points in $\text{PG}(1, 64)$ of a Sylow 3-subgroup C_9 of $\text{PSL}(2, 8)$, a non-split torus), and the dihedral subgroups D_{14} and D_{18} of $\text{PSL}(2, 8)$, which do not contain the centre, give 6 copies of $\mathcal{L}$ resp. $\mathcal{E}$, permuted regularly by $G/\text{PSL}(2, 8) \cong C_6$.

TABLE 3. Subgroups of $G = C_2 \times \text{PTL}(2, 8)$ (TomLib table **L2(8):3x2**) used by the actions, and the coset spaces. “ Z ” is the centre of G .

class	H	remark	index	G/H
57	$C_2 \times \text{AFL}(1, 8)$	point stabiliser	9	$\mathcal{P}$ (Z acts trivially)
53	$C_2 \times (C_7:C_6)$	normaliser of a split torus	36	$\mathcal{L}$
54	$C_2 \times (C_9:C_6)$	normaliser of a non-split torus	28	$\mathcal{E}$
27	$D_{14} \leq \text{PSL}(2, 8)$	$Z \not\leq H$; in class 53	216	$C_6 \times \mathcal{L}$
31	$D_{18} \leq \text{PSL}(2, 8)$	$Z \not\leq H$; in class 54	168	$C_6 \times \mathcal{E}$

The three actions of G found by the search are on $G/H_{31} \sqcup G/H_{53} \sqcup G/H_{54}$ ($168 + 36 + 28 = 232$), $G/H_{27} \sqcup G/H_{53} \sqcup G/H_{57}$ ($216 + 36 + 9 = 261$) and $G/H_{27} \sqcup G/H_{53} \sqcup G/H_{54}$ ($216 + 36 + 28 = 280$). Identify the 1 + 6 copies of $\mathcal{L}$ (or $\mathcal{E}$) with $\mathcal{L} \times \mathbb{F}_8^*$ (or $\mathcal{E} \times \mathbb{F}_8^*$): the copy with label 1 is the orbit G/H_{53} (or G/H_{54}), and the six copies with labels $c \neq 1$ are permuted by $G/\text{PSL}(2, 8) \cong C_6$ as ϕ and ι permute $\mathbb{F}_8^* \setminus \{1\}$. This gives the subgroup $\langle \text{SL}(2, 8), \phi, \iota \rangle$ of Γ_8 of Proposition 2.4; the label shift t is the extra symmetry that fuses the orbit of label 1 with the six others. A fusion search on Γ_8 and on this subgroup in the explicit model (**gamma_family.g**) re-finds all five classes with the same canonical forms.

3.2. Automorphism groups. The orders of the automorphism groups were computed with `dreadnaut` [10]; the generators were checked to be automorphisms in GAP, which also computed the structure (`aut_struct.log`, `aut_struct_160bc.log`).

The quotient graphs. $|\text{Aut}\Gamma_{232}| = |\text{Aut}\Gamma_{261}^Q| = |\text{Aut}\Gamma_{280}^Q| = 21168 = |\Gamma_8|$. Since $\Gamma_8 \leq \text{Aut}$ (Proposition 2.4(2), which also proves $|\Gamma_8| = 21168$), $\text{Aut} = \Gamma_8$ for all three. As a further check, the audit built Γ_8 from independently written generators and verified in GAP that it is isomorphic to $\text{Aut}\Gamma_{261}^Q$. GAP gives the structure $C_7:(C_2 \times (\text{PSL}(2, 8):C_3))$, derived subgroup $\text{GL}(2, 8) \cong \text{SL}(2, 8) \times C_7$ and trivial centre; the three groups are isomorphic, as they must be. Abstractly $\text{Aut} \cong \text{GL}(2, 8) \rtimes C_6$, where C_6 is generated by the field automorphism and by the inversion of the scalars; in our model the scalars C_7 act on labels, not on points.

The product graphs. $|\text{Aut}\Gamma_{261}^P| = |\text{Aut}\Gamma_{280}^P| = 3024$, so $\text{Aut} = C_2 \times \text{P}\Gamma\text{L}(2, 8)$, the TomLib group itself (again because this group, of order 3024 by Proposition 2.4(2), is contained in Aut).

The graphs on 160 vertices. $\text{Aut}\Gamma_{160}^{(1)} = W \times C_3 \cong C_3 \times 2^4:(S_3 \times S_3)$ with centre C_3 , acting with rank 3 on the cells (unfaithfully: the C_3 fixes every cell). $\text{Aut}\Gamma_{160}^{(2)}$ has order 1728 and structure $C_3:(2^4:(S_3 \times S_3))$ with trivial centre; it is not isomorphic to $\text{Aut}\Gamma_{160}^{(1)}$ (GAP's `IsomorphismGroups`). $\text{Aut}\Gamma_{160}^{(3)} \cong 2^4:(S_3 \times S_3)$, of order 576.

Table 4 lists the vertex stabilisers.

TABLE 4. Vertex stabilisers in Aut and the rank of Aut on each of its vertex orbits.

graph	orbit: stabiliser (rank of Aut on the orbit)
$\Gamma_{160}^{(1)}$	144: D_{12} (27); 16: $C_3 \times S_3 \times S_3$ (3)
$\Gamma_{160}^{(2)}$	144: D_{12} (19); 16: $((C_3 \times C_3):C_2) \times S_3$ (3)
$\Gamma_{160}^{(3)}$	96: C_6 (20); 48: D_{12} (9); 16: $S_3 \times S_3$ (3)
Γ_{232}	196: $C_2 \times (C_9:C_6)$ (6); 36: $(C_7 \times C_7):(C_6 \times C_2)$ (3)
Γ_{261}^P	216: D_{14} (30); 36: $C_2 \times (C_7:C_6)$ (3); 9: $C_2 \times \text{A}\Gamma\text{L}(1, 8)$ (2)
Γ_{261}^Q	252: $C_2 \times (C_7:C_6)$ (8); 9: $((C_3^2):C_7) \times D_{14}:C_3$ (2)
Γ_{280}^P	216: D_{14} (30); 36: $C_2 \times (C_7:C_6)$ (3); 28: $C_2 \times (C_9:C_6)$ (2)
Γ_{280}^Q	252: $C_2 \times (C_7:C_6)$ (8); 28: $C_7:(C_2 \times (C_9:C_6))$ (2)

4. STRUCTURE

Table 5 collects invariants that separate the graphs and decide geometricity. All entries are computer results (`structure.py`, `pranks.g`, `geom.py`, `geom_check2.py`, `set_orbits.g`). The independent audit recomputed every entry with its own programs (`b4_invariants.py`, `cliques.c`) and found the same values; it also supplied the complete count 0 for the Hoffman cocliques of Γ_{261}^P and the complete counts 37 and 289 for the two 280 graphs. The referee's enumerator reproduced all coclique counts once more.

4.1. Delsarte cliques and geometricity. The Delsarte cliques were enumerated completely: every clique of size $c = 1 - k/s$ is an edge $\{u, w\}$ together with a $(c - 2)$ -clique in the common neighbourhood of u and w , and `geom.py` lists all of them and checks that no clique of size $c + 1$ exists. Whether a subfamily partitions the edges was decided twice, by CP-SAT (`geom.py`) and by an exact-cover search with Knuth's Algorithm X (`geom_check2.py`). The results:

- $\Gamma_{160}^{(i)}$, Γ_{280}^P , Γ_{280}^Q (Delsarte bound 10) have no clique of size 10 at all. So no edge lies in a Delsarte clique, and none of these graphs is geometric. For the 160 graphs this also follows from the non-existence of $\text{pg}(9, 5, 3)$.

TABLE 5. Invariants. $\text{rk}_2(M)$ is the rank of M over $\mathbb{F}_2$. For $(160, 54, 18, 18)$ and $(261, 64, 14, 16)$ we have $A^2 \equiv 0$, for $(232, 63, 14, 18)$ and $(280, 63, 14, 14)$ we have $A^2 \equiv I \pmod{2}$, so in each row one of A , $A + I$ is invertible over $\mathbb{F}_2$ (it squares to I) and the other carries the information. rk_3 resp. rk_7 is the rank of $A - rI$ over $\mathbb{F}_3$ resp. $\mathbb{F}_7$ (equal to that of $A - sI$ when $p \mid r - s$). “Del.” is the number of Delsarte cliques, with its decomposition into Aut-orbits; “Hoff.” the number of Hoffman cocliques, with the number of Aut-orbits or their lengths (all counts complete). “loc.” lists the number of distinct eigenvalues of the local graph of a vertex in each Aut-orbit, in the order of Table 2.

graph	$\text{rk}_2(A)$	$\text{rk}_2(A+I)$	rk_3/rk_7	Del.	Hoff.	loc.
$\Gamma_{160}^{(1)}$	54	160	$\text{rk}_3 = 60$	0	158 (7 orbits)	34, 10
$\Gamma_{160}^{(2)}$	34	160	$\text{rk}_3 = 60$	0	242 (10 orbits)	36, 6
$\Gamma_{160}^{(3)}$	44	160	$\text{rk}_3 = 60$	0	186 (12 orbits)	36, 36, 18
Γ_{232}	232	64	$\text{rk}_7 = 59$	$513 = 504 + 9$	0	6, 5
Γ_{261}^P	116	261	$\text{rk}_7 = 87$	$46 = 36 + 9 + 1$	0	40, 15, 5
Γ_{261}^Q	44	261	$\text{rk}_7 = 87$	$100 = 63 + 36 + 1$	$252 = 126 + 126$	14, 3
Γ_{280}^P	280	136	$\text{rk}_7 = 88$	0	$37 = 36 + 1$	36, 12, 12
Γ_{280}^Q	280	64	$\text{rk}_7 = 88$	0	$289 = 126 + 126 + 36 + 1$	14, 8

- Γ_{261}^P has 46 cliques of size 9: the 9 points; the 36 sets “7 copies of a pair and its 2 points”; and 9 sets “a point p and the 8 pairs $(\{p, a\}, 1)$ ”. Γ_{261}^Q has 100 cliques of size 9: the 9 points; the 36 sets “7 copies of a pair and its 2 points”; and the 63 columns “a point p and the 8 pairs $(\{p, a\}, c)$ ” of Proposition 2.5. A $\text{pg}(8, 7, 2)$ on 261 points has $261 \cdot 8/9 = 232$ lines, all of them Delsarte cliques of its collinearity graph. Since $46 < 232$ and $100 < 232$, neither graph is geometric. (Equivalently, 6804 resp. 5292 of the 8352 edges lie in no Delsarte clique.)
- Γ_{232} has 513 cliques of size 8, in two Aut-orbits: the 9 “stars” of $T(9)$ (the 8 pairs through a point) and 504 cliques consisting of 7 vertices of $\mathcal{E} \times \mathbb{F}_8^*$ and one pair, with setwise stabiliser $C_7:C_6$. Together they cover every edge: 252 edges once and 7056 twice. This alone does not exclude a partition: a $\text{pg}(7, 8, 2)$ would need $232 \cdot 9/8 = 261$ of the cliques partitioning the 7308 edges, and such a subfamily must contain the unique clique through each of the 252 edges covered once, but the counts do not decide whether the remaining edges can be covered exactly once. This was decided by the exact-cover searches: CP-SAT (`geom.py`) and Algorithm X (`geom_check2.py`), both run to completion, and independently the audit’s CP-SAT model, all report INFEASIBLE, i.e. no subfamily of the 513 cliques partitions the edge set. So Γ_{232} is not geometric.

The same programs, run on the collinearity graphs of $\text{GQ}(2, 2)$, $\text{GQ}(2, 4)$ and on the lattice graph $L_2(5)$, find the lines and the partition (`geom_controls.log`). For $(160, 54, 18, 18)$ the conclusion was forced, since $\text{pg}(9, 5, 3)$ does not exist [2]. The graphs therefore do not decide the existence of $\text{pg}(7, 8, 2)$, $\text{pg}(8, 7, 2)$ and $\text{pg}(9, 6, 2)$.

4.2. Hoffman cocliques. In $\Gamma_{160}^{(i)}$ the cells, and in Γ_{280}^* the set $\mathcal{E}$, are Hoffman cocliques by construction. The Hoffman cocliques were enumerated with CP-SAT [11] in all-solutions mode, with a limit of 1800s per graph. The model asks for a coclique of size $h = v(-s)/(k - s)$ in which every outside vertex has exactly $kh/(v - h)$ neighbours, a condition every Hoffman coclique satisfies (`structure.py`):

- for $\Gamma_{160}^{(1)}$, $\Gamma_{160}^{(2)}$, $\Gamma_{160}^{(3)}$ and Γ_{261}^Q the enumeration finished (status **OPTIMAL**) with 158, 242, 186 and 252 cocliques;
- for Γ_{232} the model is **INFEASIBLE**: Γ_{232} has no coclique of size 29;
- for Γ_{261}^P CP-SAT returned **UNKNOWN** after 1800s, without finding a coclique of size 29. The independent audit decided this case with its own branch-and-bound enumerator (`cliques.c`, which prunes with the Hoffman neighbour count): the search finished completely, after 1 552 209 nodes, with no coclique of size 29 (`out/audit_twoorbit8_fable/cocl_more.log`). The same program reproduces the complete counts 158, 242, 186, 252 and 0 above. So Γ_{261}^P has no Hoffman coclique;
- for Γ_{280}^P and Γ_{280}^Q CP-SAT did not finish within 1800s (status **FEASIBLE**, 25 resp. 47 cocliques listed). The audit's enumerator finished both cases completely (8 431 987 resp. 12 635 254 nodes, `cocl_more.log`): Γ_{280}^P has exactly 37 and Γ_{280}^Q exactly 289 Hoffman cocliques. Under Aut they form the orbits $36+1$ resp. $126+126+36+1$ (`cocl280_orbits.g`, on the complete lists); the orbit of length 1 is $\{\mathcal{E}\}$ in both graphs.

The referee's own enumerator, a third program, reproduced all these counts completely: 158, 242, 186, 0, 0, 252, 37 and 289 for $\Gamma_{160}^{(1)}$, $\Gamma_{160}^{(2)}$, $\Gamma_{160}^{(3)}$, Γ_{232} , Γ_{261}^P , Γ_{261}^Q , Γ_{280}^P , Γ_{280}^Q .

In Γ_{261}^Q each of the 252 cocliques contains exactly one point and 28 labelled pairs, one on each pair avoiding that point. By hand: the points form a clique, so a Hoffman coclique S contains at most one point. If it contains none, each of the 9 points has exactly $kh/(v-h) = 64 \cdot 29/232 = 8$ neighbours in S , but each of the 29 labelled vertices of S is adjacent to only 2 points, and $2 \cdot 29 = 58 < 72$. If S contains the point p , its 28 labelled vertices are non-adjacent to p , so they lie on the 28 pairs avoiding p , and at most one on each pair, since the copies of a pair form a clique.

4.3. Local graphs. The local graph of a vertex x (the subgraph induced on its neighbours) is λ -regular. In Γ_{261}^Q the local graph of a point is the rook's graph $L_2(8)$ (Proposition 2.5); it is the only local graph with three eigenvalues; all others have between 5 and 40 distinct eigenvalues and are not strongly regular. The numbers of distinct eigenvalues are in Table 5. The smallest ones are 5 (the points of Γ_{261}^P : 14, 6, 0, -2 , -8 ; the pairs of Γ_{232}), 6 (the cells of $\Gamma_{160}^{(2)}$: 18, 6, 3, 0, -3 , -6 ; the vertices of $\mathcal{E} \times \mathbb{F}_8^*$ in Γ_{232}) and 8 (the vertices of $\mathcal{E}$ in Γ_{280}^Q). In Γ_{232} the local graph of a pair has the spectrum $14^1 5^{14} (2.873)^6 (-2)^{36} (-4.873)^6$, and that of a vertex of $\mathcal{E} \times \mathbb{F}_8^*$ the spectrum $14^1 5^{16} 0^9 (-1)^{12} (-3)^{18} (-4)^7$.

4.4. Non-isomorphism. Within each row the graphs are pairwise non-isomorphic: their nauty canonical forms differ (Section 7), and so do the invariants of Table 5. For (160, 54, 18, 18) the 2-ranks 54, 34, 44 already distinguish the three graphs; for 261 and 280 the orders of the automorphism groups do, and so do the numbers 0 and 252 (for 261) and 37 and 289 (for 280) of Hoffman cocliques; for 280 also the 2-ranks $\text{rk}_2(A+I) = 136$ and 64. The graphs were found by different search runs and are not related by switching as far as we know; we have not tested Godsil–McKay or other switchings between them.

Remark 4.1. For $\lambda = \mu$, i.e. for $\text{srg}(160, 54, 18, 18)$ and $\text{srg}(280, 63, 14, 14)$, the adjacency matrix is the incidence matrix of a symmetric 2-(160, 54, 18) resp. 2-(280, 63, 14) design with a polarity without absolute points. We have not studied these designs.

5. HOW THE GRAPHS WERE FOUND

5.1. Tables of marks. The *table of marks* of a finite group G , introduced by Burnside [4], is the square matrix, indexed by the conjugacy classes of subgroups, whose entry (H, K) is the number of fixed points of K on the coset space G/H . It determines, without constructing any permutation action, the lengths of the suborbits of G on $G/H \times G/K$: the stabiliser H of a point

of G/H acts on G/K , and its orbit lengths can be read off from products of rows of the table (Ivanov, Klin, Tsaranov and Shpektorov [6]). GAP's TomLib [12] contains the tables of marks of 414 groups, among them $C_2 \times \text{PTL}(2, 8)$ ($\text{L2}(8):3\text{x}2$) and $2^4:(S_3 \times S_3)$ ($2^4:(\text{S3xS3})$), with the function `DecomposedFixedPointVector` that performs this computation.

5.2. The search. In [8] we searched for strongly regular graphs that are unions of symmetrised orbitals of a group G acting on $G/H_1 \sqcup G/H_2$. The search behind the present paper ran on three servers (3×48 cores) from 11 October 2026, in a public Docker image built for this purpose, and covered four kinds of actions, for all 414 TomLib groups and all 1393 parameter sets marked “?” in [2] on 10 October 2026 (for the two-orbit kinds, the 1323 with integral eigenvalues):

- *P2*: two orbits $G/H_i \sqcup G/H_j$, at higher ranks than in [8];
- *T3*: three orbits $G/H_i \sqcup G/H_j \sqcup G/H_k$, $i \leq j \leq k$, subgroups at any depth;
- *C2*, *C3*: $G \times C_m$, $m \in \{2, 3\}$, on two orbits, each of which is G/H with C_m acting trivially, m copies of G/H permuted cyclically by C_m , or G/H_0 with C_m acting through $N_G(H_0)/H_0$.

Each (action, parameter set) pair was decided exactly in three steps. First, the partition into G -orbits of a G -invariant graph is equitable, and its quotient matrix must have integer entries, row sums k , entries that are sums of suborbit lengths, and eigenvalues other than k in $\{r, s\}$; for three orbits the trace and the sum of the 2×2 principal minors are checked. Second, an exact subset-sum test asks whether the suborbit lengths can produce the required degrees. Third, for the survivors the permutation action and all structure constants p_{ab}^c of the coherent configuration are built, and the quadratic system

$$\sum_{a,b} x_a x_b p_{ab}^c = \lambda x_c + \mu(1 - x_c) \quad \text{for every non-diagonal class } c,$$

with a 0/1 variable x_a per symmetrised orbital and the degree equations, is solved by the CP-SAT solver of OR-Tools [11] with a limit of 30s; every solution is expanded to the full matrix and checked by exact integer arithmetic. Each start of the image first runs must-find controls: the graph $\text{srg}(576, 253, 112, 110)$ of [8] from two different actions, $\text{srg}(1024, 495, 238, 240)$, the Higman–Sims graph from a two-orbit action of S_7 and from the three-orbit action of M_{22} on $1 + 22 + 77$ points, the complement of the Schläfli graph, and a planted non-strongly-regular graph that must be rejected. The sweep starts only if all controls pass.

The search is not complete: actions are limited by a rank bound, groups without a TomLib table are not covered, and neither are four or more orbits apart from the $G \times C_m$ cases. Nothing in this paper depends on its completeness.

5.3. The hits, and all fusions of the same actions. The servers reported five graphs on the four open rows: $\text{srg}(232, 63, 14, 18)$, $\text{srg}(261, 64, 14, 16)$ and $\text{srg}(280, 63, 14, 14)$ from the T3 actions of $C_2 \times \text{PTL}(2, 8)$ of Table 3; $\text{srg}(160, 54, 18, 18)$ from the C3 action of $2^4:(S_3 \times S_3) \times C_3$ on three copies of W/D_{12} and $W/(S_3 \times S_3)$ ($3 \cdot 48 + 16$); and a second $\text{srg}(160, 54, 18, 18)$ from the T3 action of $2^4:(S_3 \times S_3)$ on $96 + 48 + 16$ points (Section 2.4).

The decider stops at the first solution of each (action, parameter set) pair. We therefore rebuilt every hit action from scratch in GAP (orbitals as orbits on ordered pairs, without the search code) and enumerated *all* unions of symmetrised orbitals with the given parameters, with CP-SAT in all-solutions mode; each solution was checked exactly and classified by canonical form (Table 6). This gave three graphs that the servers had not reported: $\Gamma_{160}^{(3)}$, Γ_{261}^Q and Γ_{280}^Q . The product rule and the quotient rule of Construction 2.3 were then found by computing, for each fusion, the adjacency as a function of the intersection size, the cross ratio and the labels (`rules_q8.g`).

The multiplicity 3 of the $q = 8$ classes. For $m \in \{1, 2, 3\}$ let $\Gamma_v^*(m)$ be the graph of Construction 2.3 with $\chi^{\pm 1}$ replaced by $\chi^{\pm m}$ (so $\Gamma_v^*(1) = \Gamma_v^*$), on the same vertex set; every exponent $\pm m$

TABLE 6. All fusions of the hit actions. “Sym.” is the number of symmetrised non-diagonal orbitals; all enumerations finished (status OPTIMAL, i.e. complete).

group	action	v	sym.	fusions	classes
$C_2 \times \text{PFL}(2, 8)$	T3, classes 31, 53, 54	$168 + 36 + 28$	28	3	Γ_{232}
$C_2 \times \text{PFL}(2, 8)$	T3, classes 27, 53, 57	$216 + 36 + 9$	31	6	$\Gamma_{261}^P(3), \Gamma_{261}^Q(3)$
$C_2 \times \text{PFL}(2, 8)$	T3, classes 27, 53, 54	$216 + 36 + 28$	33	6	$\Gamma_{280}^P(3), \Gamma_{280}^Q(3)$
$2^4:(S_3 \times S_3) \times C_3$	C3, classes 60 (copies), 105	$3 \cdot 48 + 16$	22	2	$\Gamma_{160}^{(1)}$
$2^4:(S_3 \times S_3) \times C_3$	C3, classes 66 (copies), 105	$3 \cdot 48 + 16$	22	2	$\Gamma_{160}^{(1)}$
$2^4:(S_3 \times S_3)$	T3, classes 25, 60, 105	$96 + 48 + 16$	43	4	$\Gamma_{160}^{(2)}(2), \Gamma_{160}^{(3)}(2)$
Γ_8	$\mathcal{L} \times \mathbb{F}_8^* \cup \mathcal{P}$	$252 + 9$	10	3	Γ_{261}^Q
Γ_8	$\mathcal{L} \times \mathbb{F}_8^* \cup \mathcal{E}$	$252 + 28$	10	3	Γ_{280}^Q
Γ_8	$\mathcal{E} \times \mathbb{F}_8^* \cup \mathcal{L}$	$196 + 36$	9	3	Γ_{232}

with $7 \nmid m$ is one of these three up to sign. (i) $\Gamma_v^*(m)$ is isomorphic to Γ_v^* via $(X, c) \mapsto (X, c^m)$ (Section 2.2). (ii) The proof of Proposition 2.4(2) holds verbatim for every exponent: $\text{SL}(2, 8)$ and t preserve χ^m and d, ϕ squares both, and ι inverts d ; so $\Gamma_v^Q(m)$ is invariant under Γ_8 , and $\Gamma_v^Q(m)$ and $\Gamma_v^P(m)$ are invariant under its subgroup $C_2 \times \text{PFL}(2, 8)$, i.e. each is a union of orbitals of the corresponding action. (iii) The three graphs $\Gamma_v^*(1), \Gamma_v^*(2), \Gamma_v^*(3)$ are pairwise distinct as labelled graphs on the common vertex set, for each of the five pairs $(v, \star)$ (computer check by the referee with its own builder, `r2_build_q8.log`, and canonical forms in `r2c_canon.log`). The Γ_8 -actions of Table 6 have exactly 3 fusions, and the $C_2 \times \text{PFL}(2, 8)$ -actions on $216 + 36 + 9$ and $216 + 36 + 28$ points exactly 3 + 3 (Section 3 identifies these TomLib actions with the explicit model). By (i)–(iii) the graphs $\Gamma_v^*(m)$, $m = 1, 2, 3$, are three distinct fusions of each class, so they are all of them: the fusions of each $q = 8$ class are exactly the graphs with exponents $\pm 1, \pm 2, \pm 3$. For the 232-type action of $C_2 \times \text{PFL}(2, 8)$ the product rule gives no strongly regular graph, which is why that line of the table has 3 fusions only. The audit’s own enumeration of the same actions gave the same numbers of fusions and classes.

Scope of these enumerations, and why three graphs were missed. Each line of Table 6 is a statement about one action only: these are all unions of orbitals of that group on that vertex set with the given parameters. The servers missed $\Gamma_{160}^{(3)}, \Gamma_{261}^Q$ and Γ_{280}^Q only because the decider stops at the first solution of each (action, parameter set) pair; the hit action of each of them also has a solution in another class, which was the one reported.

6. THE FAMILY QUESTION

The vertex numbers of Construction 2.3 are polynomials in $q = 8$: $\frac{1}{2}(q+1)(q^2 - q + 2) = 261$, $\frac{1}{2}q(q-1)(q+2) = 280$ and $\frac{1}{2}q(q^2 - q + 2) = 232$, and the parameters are those of $\text{pg}(q, q-1, 2)$, $\text{pg}(q+1, q-2, 2)$ and $\text{pg}(q-1, q, 2)$. Over $\text{PG}(1, q)$, $q = 2^e$, the construction makes sense with labels in $\mathbb{F}_q^*$ and the group $\Gamma_q = (\text{SL}(2, q) \times C_{q-1}) \rtimes (\langle \phi \rangle \times \langle \iota \rangle)$. We tested whether it continues.

Computation 6.1. In the explicit model over $\text{PG}(1, q)$, with the group Γ_q , with its subgroup without the label shift t (isomorphic to $C_2 \times \text{PFL}(2, q)$), and for $q = 16$ also with Γ_{16} without ϕ , all unions of symmetrised orbitals on the three vertex sets $(\mathcal{L} \times \mathbb{F}_q^*) \cup \mathcal{P}$, $(\mathcal{L} \times \mathbb{F}_q^*) \cup \mathcal{E}$ and $(\mathcal{E} \times \mathbb{F}_q^*) \cup \mathcal{L}$ were searched with CP-SAT:

- (1) $q = 4$ ($v = 35, 36, 28$): with no parameters prescribed (only $k \leq (v-1)/2$ and $1 \leq \mu < k$), over 75 combinations of orbit types, the only strongly regular graphs on these vertex sets are on $(\mathcal{L} \times \mathbb{F}_4^*) \cup \mathcal{P}$: two classes of $\text{srg}(35, 16, 6, 8)$. The quotient rule gives the complement of the Grassmann graph $J_2(4, 2)$, i.e. the skew-lines graph of $\text{PG}(3, 2)$, with

$|\text{Aut}| = 8!$ (same canonical form); the product rule gives a graph with $|\text{Aut}| = 1440$. The parameter set is classical: all $\text{srg}(35, 16, 6, 8)$ are known, they are the complements of the 3854 descendants of the 227 regular two-graphs on 36 vertices [9]. There is no strongly regular graph of the 280- or 232-type.

- (2) $q = 16$ ($v = 2057, 2160, 1936$) with the target parameters $(2057, 256, 30, 32)$, $(2160, 255, 30, 30)$, $(1936, 255, 30, 34)$: **INFEASIBLE** (complete) under Γ_{16} , under $C_2 \times \text{PTL}(2, 16)$ (five orbits, 175, 185, 164 symmetrised orbitals) and under Γ_{16} without ϕ . The TomLib-type actions of $C_2 \times \text{PTL}(2, 16)$, with $1 + 2e = 9$ copies of $\mathcal{L}$ or $\mathcal{E}$ (one for the label 1 and $2e = 8$ for the other labels) in place of the $q - 1 = 15$ copies of the explicit model (so $v = 9 \cdot 136 + 17 = 1241$, $9 \cdot 136 + 120 = 1344$, $9 \cdot 120 + 136 = 1216$), are **INFEASIBLE** (complete) with no parameters prescribed.
- (3) $q = 32$ ($v = 16401, 16864, 15904$) with the target parameters: **INFEASIBLE** (complete) under Γ_{32} only. The subgroup of Γ_{32} without t was not searched at $q = 32$, so for that group the question is open.
- (4) The monomial rules $d \in \{\chi^m, \chi^{-m}\}$, for every exponent m and both rules, are not strongly regular for $q = 16$ (48 runs, exact rows of A^2).

The same code re-finds all five classes of Construction 2.3 for $q = 8$ and the skew-lines graph for $q = 4$. Here **INFEASIBLE** (complete) is the solver status of an exhaustive search: no union of orbitals of the stated group has the stated parameters. Each verdict quoted is this status, not **UNKNOWN** or a timeout.

The independent audit re-ran (1), the Γ_{16} and $C_2 \times \text{PTL}(2, 16)$ cases of (2) and the Γ_{32} case of (3) with its own generators and its own fusion enumerator, controlled by re-finding the five $q = 8$ classes and the skew-lines graph at $q = 4$; all its verdicts were **INFEASIBLE** (complete), and at $q = 4$ it found the same two classes. It did not re-run Γ_{16} without ϕ , the TomLib-type actions at $q = 16$ or the monomial rules of (4); the latter are subsumed by its Γ_{16} verdict.

Two observations explain part of this. First, the rules for the 280- and 232-types need a primitive cube root of unity among the values $\chi(\pi, Y)$, $Y \in \mathcal{E}$, whose order divides $q + 1$ (Lemma 2.1(3) for general q); so $3 \mid q + 1$, i.e. e odd, and for $q = 4, 16$ the cross-type edges of the rules are empty. Second, the TomLib action of $C_2 \times \text{PTL}(2, q)$ on $1 + 2e$ copies matches $\mathcal{L} \times \mathbb{F}_q^*$ only if $2e + 1 = q - 1$, i.e. $q = 8$. For $q = 8$ the group $(C_2 \times \text{PTL}(2, 8))/\text{PSL}(2, 8) \cong C_6$ coincides with $\text{Aut}(\mathbb{F}_8^*)$, and the label shift t of Γ_8 has no counterpart in TomLib. We conclude that within these models the graphs for $q = 8$ are sporadic; the 261-type rule has a classical member at $q = 4$ and the new one at $q = 8$ only. Computation 6.1 says nothing about other constructions with these parameters for $q \geq 16$.

For $(160, 54, 18, 18)$ we know of no natural family: candidate groups such as $2^6:S_3^3$ or groups of shape $3^4: \dots$ have no table of marks in TomLib, and we did not find a canonical generalisation of the grid-and-flags model. We did not pursue this further.

7. VERIFICATION

7.1. The standalone verifier. Appendix A lists `verify_srg.py`, the verifier of [8], a program of about 60 lines that uses only the Python standard library. It decodes `graph6` itself, rejects malformed input, checks that the graph is loopless and k -regular, and computes the number of common neighbours of every pair of vertices with integer bitsets; the graph is accepted iff these numbers take one value λ on edges and one value μ on non-edges. The complete run log on the eight files:

```
# 2026-10-12 Python 3.12.3 verify_srg.py sha256 26954cc37ba0bc89
G160.g6:1 n=160 OK srg(160,54,18,18); complement srg(160,105,68,70)
G160b.g6:1 n=160 OK srg(160,54,18,18); complement srg(160,105,68,70)
G160c.g6:1 n=160 OK srg(160,54,18,18); complement srg(160,105,68,70)
G232.g6:1 n=232 OK srg(232,63,14,18); complement srg(232,168,122,120)
G261a.g6:1 n=261 OK srg(261,64,14,16); complement srg(261,196,147,147)
G261b.g6:1 n=261 OK srg(261,64,14,16); complement srg(261,196,147,147)
```

```
G280a.g6:1 n=280 OK srg(280,63,14,14); complement srg(280,216,166,168)
G280b.g6:1 n=280 OK srg(280,63,14,14); complement srg(280,216,166,168)
```

7.2. Independent programs. Strong regularity of each graph was checked by the following programs. They were written at different times, decode graph6 independently and use different arithmetic; the rebuild (3) reuses the orbital-labelling routine of the search code, but not its decider, and the programs `paper_rules.py` and `verify_srg.py` share no code with any of the others.

- (1) the search decider, which expands each CP-SAT solution to the full matrix and checks A^2 in 64-bit integers, and the recorder inside the image, which re-checks the graph6 output with its own decoder;
- (2) at home, `verify_g6.py` (own graph6 decoder, set intersections) and `dn_check.py` (own decoder, row-by-row integer A^2 , connectivity, `dreadnaut` for $|Aut|$ and canonical form);
- (3) the rebuild from TomLib (`orbitals.g`, `fuse.py`, `fuse_job.py`): orbitals computed in GAP, all fusions enumerated, each solution checked by exact A^2 ;
- (4) `rule_family.py`, which builds the graphs of Construction 2.3 from the rules (NumPy), and `paper_rules.py` (Appendix B), written for this paper from the wording of Section 2 alone, in pure Python with its own field arithmetic, which builds Γ_{261}^P , Γ_{261}^Q , Γ_{280}^P , Γ_{280}^Q , Γ_{232} and $\Gamma_{160}^{(1)}$ and checks them;
- (5) `structure.py`, which computes the exact integer spectrum, and `verify_srg.py` above.

$\Gamma_{160}^{(2)}$ and $\Gamma_{160}^{(3)}$ have no prose construction and are covered by (1)–(3) and (5), and by the TomLib snippet of Section 2.4 (`paper_160bc.g`).

Independent audit. Finally, a blind audit by a session of a different model family (see the Acknowledgements) rebuilt everything from the prose of the analysis note, without importing or calling any of the programs above: its own arithmetic in $\mathbb{F}_{2^m}$ and cross ratios, its own graph6 codec, its own GAP orbital extractor and CP-SAT fusion enumerator, and its own C clique and coclique enumerator (`scripts/audit_twoorbit8_fable/`). All eight graphs rebuilt this way (the five $q = 8$ graphs from the cross-ratio rules, $\Gamma_{160}^{(1)}$ from the grid-and-flags rule, $\Gamma_{160}^{(2)}$ and $\Gamma_{160}^{(3)}$ from TomLib) have the canonical forms of the published files (nauty `labelg`, hashed with SHA-256), and pass the exact check of A^2 . The audit also confirmed the orders and structures of the automorphism groups, the vertex orbits, the p -ranks, the clique, coclique and local-graph data of Section 4, and the family verdicts of Computation 6.1 as stated there. Its report and outputs are `docs/audit_twoorbit8_fable_2026_10_12.md` and `out/audit_twoorbit8_fable/`.

Referee. The second version was then refereed by a further session of the same model family as the audit, with its own code and without reading the code of the analysis or of the audit (`scripts/referee_twoorbit8_fable/`, `out/referee_twoorbit8_fable/`). It re-implemented the constructions from the wording of Section 2, with a different model of $\mathbb{F}_{64}$ ($\mathbb{F}_2[x]/(x^6 + x^5 + 1)$), a different vertex order and a different encoding of the grid, and obtained the canonical forms of the six published files with a prose construction (both orientations of rule (3) of Construction 2.8); it reproduced the SHA-256 sums, parameters, spectra, $|Aut|$ and vertex orbits of all eight files, the Delsarte clique counts and the exact-cover result (own Algorithm X, with controls), the Hoffman coclique counts (own enumerator, complete), the p -ranks, the local spectra, the orbital counts of Table 6, the TomLib data, the group orders 21168 and 3024 on its own generators and the $q = 4$ family result. It supplied the computation behind the multiplicity statement of Section 5.3.

7.3. Canonical forms and checksums. `dn_check.py` runs `dreadnaut` [10] on each file and hashes the canonically relabelled adjacency lists (SHA-256 of the canonical graph, not nauty's short invariant hash). A second canonical labelling with nauty's `labelg` (`canon.sh`) gives the

same partition into isomorphism classes. The graphs built by `paper_rules.py` from Section 2 have the same canonical hashes as the files:

file	graph	Aut	canonical hash (16 hex)	also built by
G160.g6	$\Gamma_{160}^{(1)}$	1728	44ad249c73f95631	Constr. 2.8
G160b.g6	$\Gamma_{160}^{(2)}$	1728	91d66c190396528a	fusion only
G160c.g6	$\Gamma_{160}^{(3)}$	576	b1b59aabf7ed5c02	fusion only
G232.g6	Γ_{232}	21168	fffe11ff01375faa	Constr. 2.3(3)
G261a.g6	Γ_{261}^P	3024	da5644a8f394d82f	Constr. 2.3(1), P
G261b.g6	Γ_{261}^Q	21168	a9bb68285b05e576	Constr. 2.3(1), Q
G280a.g6	Γ_{280}^P	3024	40842d043ca5021f	Constr. 2.3(2), P
G280b.g6	Γ_{280}^Q	21168	b6a29c47126db021	Constr. 2.3(2), Q

8. NOVELTY

We claim that the four parameter sets had no known construction, as far as we can determine, and list what we checked.

Brouwer’s table. On 11 and 12 October 2026 we read the live pages of [2]. The rows were ? 160 54 18 18 (comment “pg(9,5,3) does not exist (no 2-graph* for line graph)”), ? 232 63 14 18 (“pg(7,8,2)?”), ? 261 64 14 16 (“pg(8,7,2)?”) and ? 280 63 14 14 (“pg(9,6,2)?”); the complements are on the same rows and carry no further comment. The references page of the table (`srgtabrefs.html`) mentions none of the four parameter sets. The audit re-read these pages independently on 12 October 2026 with the same result. None of the four parameter sets belongs to a regular two-graph or to a descendant of one ($v - 1 - 2k \notin \{-1 - 2r, -1 - 2s\}$ and $k \neq 2\mu$), so no construction can be hidden behind a two-graph entry of the table.

Literature. The groups $\text{PSL}(2, 8)$, $\text{PTL}(2, 8)$ and their actions have been used for many constructions of strongly regular and related graphs by the Rijeka group; the closest item we found is a directed strongly regular graph with parameters $(63, 11, 8, 1, 2)$ and automorphism group $L_2(8):3$ [1]. Their methods construct vertex-transitive graphs or graphs from designs, and none of our graphs is vertex-transitive. Web searches (12 October 2026) for each parameter set with and without the prefix “srg”, for the complements, for “pg(7, 8, 2)”, “pg(8, 7, 2)”, “pg(9, 6, 2)”, for constructions from $\text{PTL}(2, 8)$ and $\text{PSL}(2, 8)$, and for recent preprints announcing new strongly regular graphs found no construction of a graph with any of these parameter sets; the audit repeated these searches independently and also searched arXiv for 2025 and 2026. The closest recent items concern other parameters: directed strongly regular graphs from linear groups (Crnković, Švob and coauthors), pseudo-geometric graphs with a regular point (van Dam and Guo, parameter sets on 85 and 156 vertices), and several 2026 arXiv preprints on other parameter sets. MathSciNet and zbMATH were not searched, the full text of [3] was not searched, and a construction inside a paper on another subject could have been missed.

Claim. The eight graphs are new as far as we can determine: the four parameter sets had no known construction in Brouwer’s live table (including `srgtabrefs.html`) on 12 October 2026, and none was found by the web and arXiv (2025–2026) searches described above; MathSciNet, zbMATH and the full text of [3] were not searched.

Why $q = 8$. The rules of the 280- and 232-types need a primitive cube root of unity among the cross ratios of a pair of $\text{PG}(1, q)$ and an element of $\mathcal{E}$, whose order divides $q + 1$; so $3 \mid q + 1$, i.e. $q = 2^e$ with e odd (Section 6). Hence $q = 4$ and $q = 16$ are excluded by the rule itself, and $q = 32$ is the only larger value that was tested.

TABLE 7. How each claim is established.

Claim	Established by
Lemmas 2.1, 2.2; Proposition 2.4 (valencies, Γ_8 and $C_2 \times \text{P}\Gamma\text{L}(2, 8)$ act, $ \Gamma_8 = 21168$, $ C_2 \times \text{P}\Gamma\text{L}(2, 8) = 3024$, ϕ of order 3, orbits); Proposition 2.5 (point rows of Γ_{261}^* , local graph $L_2(8)$)	hand
$W \times C_3$ preserves $\Gamma_{160}^{(1)}$; valency of a cell; cells and $\mathcal{E}$ are Hoffman cocliques; the listed Delsarte cliques are cliques	hand
$W \cong \text{TomLib } 2^4:(S_3 \times S_3)$, primitive on cells	computer (GAP)
the eight graphs are strongly regular (Computations 2.6, 2.10)	computer: exact check by the programs of Section 7 (for $\Gamma_{160}^{(2)}, \Gamma_{160}^{(3)}$ without the rule builders)
$ \text{Aut} $, structure of Aut , orbits, ranks	computer (nauty; GAP); $\text{Aut} = \Gamma_8$ for the quotient graphs from $\Gamma_8 \leq \text{Aut}$ and $ \text{Aut} = \Gamma_8 $; audit: $\Gamma_8 \cong \text{Aut}\Gamma_{261}^Q$ in GAP
pairwise non-isomorphism within each row	computer (canonical forms, two tools; invariants)
the rule of Construction 2.8 reproduces the search hit	computer (exhaustive comparison; canonical form)
numbers of Delsarte cliques; not geometric	computer (complete enumeration); 160, 280: no 10-clique; 261: 46, $100 < 232$ lines; 232: exact cover INFEASIBLE (complete, three programs)
Hoffman cocliques: counts for $\Gamma_{160}^{(i)}, \Gamma_{261}^Q$; none in Γ_{232}	computer (CP-SAT, complete; reproduced by the audit's and the referee's enumerators)
Hoffman cocliques: none in Γ_{261}^P ; exactly 37 in Γ_{280}^P and 289 in Γ_{280}^Q	computer (audit enumerator, complete; CP-SAT incomplete; reproduced by the referee's enumerator); orbit decomposition: GAP
structure of the 252 Hoffman cocliques of Γ_{261}^Q (one point, one copy of each pair avoiding it)	hand
fusions of each $q = 8$ class are the exponents $\pm 1, \pm 2, \pm 3$ (Section 5.3)	hand, from the fusion counts and the referee's check that the three exponent graphs are distinct
p -ranks, local graphs	computer (exact)
all fusions of the hit actions (Table 6)	computer (CP-SAT, complete enumeration)
family results (Computation 6.1)	computer (CP-SAT, INFEASIBLE (complete) for the stated groups only; $q = 32$ without t not searched)
$\Gamma_{160}^{(2)}, \Gamma_{160}^{(3)}$ are the graphs of the orbital lists of Section 2.4	computer (GAP snippet; canonical forms; exact check)
all computer results above	reproduced by the independent audit, except where stated in Section 7 and Computation 6.1; most of them reproduced again by the referee (Section 7)
status in [2] on 11 and 12 October 2026	reading the live pages (twice, independently)

9. STATUS OF THE CLAIMS

DATA AVAILABILITY

The eight graph6 files, the list of their SHA-256 checksums (SHA256SUMS_8graphs.txt), the standalone verifiers, the builder of Appendix B, the TomLib snippet `paper_160bc.g` of Section 2.4, the rule scripts (`rule_family.py`, `rules_q8.g`, `rules160.g`, `gamma_family.g`), the coclique orbit script `cocl280_orbits.g`, and the scripts and logs of the independent audit and of the referee are deposited at Zenodo in the archive `new_srgs_twoorbit_data.zip`, doi:10.5281/zenodo.XXXXXXX [placeholder: the DOI is assigned on deposit and will be inserted in the next version]. Each graph file contains one graph6 line, ending in LF. The files `G160.g6`, `G232.g6`, `G261a.g6`, `G280a.g6` are the search hits in the vertex order of the search, `G160b.g6` is the canonical form of the second hit on 160 vertices, and `G160c.g6`, `G261b.g6`, `G280b.g6` are fusions in the vertex order of the rebuild; `paper_rules.py` writes the graphs of Section 2 in the vertex order of its listing. SHA-256 checksums:

```
G160.g6  ac81a833841a785343eb48eed7d42a3ec6dde096a6ddcf9fc386339713ab3f35
G160b.g6  aea444beb1e20fb50dfff81c82151630ec5b148d4a62838eebca54119f2cd0cd
G160c.g6  b8b704026038b6ea811b4fe942ba47e9cb45f6d7c1e3962a4d2d90a5eee6f82e
G232.g6   1329c8c0786a3866f8a46611f33c1b3808c0616ac78db2376594e7b59eabe57a
G261a.g6  a617f1302d361592aae059c062f089a619999b432228ae80603191bbd694f986
G261b.g6  d7b9b95afd8db2194d7b2eadf7602c0ec497fd5cefef25c6cc194af28ee31c3e
G280a.g6  fe5ea67d817b5855f044e48eb3e28ca68e3ea0d7a114a6ff9a95ab793ad6b565
G280b.g6  8a9cbb0b9cb9d9faaa72e8d17f00e016bbaa983d78c869647db21673dcf6a0353
```

The analysis programs (GAP and Python) and their logs are in `scripts/new_srgs_twoorbit_2026_10_11/` and `out/new_srgs_twoorbit_2026_10_11/` of the author's repository and are available from the author; the search image is described in `docs/remote_search_image_2026_10_11.md`. The audit's programs and outputs are in `scripts/audit_twoorbit8_fable/` and `out/audit_twoorbit8_fable/`, its report in `docs/audit_twoorbit8_fable_2026_10_12.md`; the referee's programs and logs are in `scripts/referee_twoorbit_fable/` and `out/referee_twoorbit_fable/`, its report in `docs/referee_new_srgs_twoorbit_v2_fable_2026_10_12.md`.

ACKNOWLEDGEMENTS

Use of AI systems. This work was done with substantial assistance from AI systems. They are not authors; the author is responsible for all content and is the mathematical expert who must accept every claim. Claude (Anthropic) built the search image with its controls (one session); a coordinating session re-verified the server hits with independent checkers; an analysis session rebuilt the actions from scratch, enumerated all fusions, found the explicit rules, the group Γ_8 and the grid-and-flags model, computed the invariants and ran the family searches; a further session wrote this paper and, as a check of the text, re-implemented the constructions from the wording of Section 2 (Appendix B). These sessions used the same model family, so their agreement is evidence of correct computation but not an independent mathematical review. The first version was then audited blind by a session of a different model family (Claude Fable), which re-implemented every construction and computation from the prose of the analysis note with its own code (Section 7), confirmed all eight graphs and all structural and family claims, decided the Hoffman cocliques of Γ_{261}^P and of the two 280 graphs, and listed the corrections made in the second version: the labelling of the 2-ranks, the symmetric statement of rule (3) of Construction 2.8, the local graph data, the per-row non-geometry arguments, the status words and scope of the family and fusion results, a proof of $|\Gamma_8|$, and the novelty wording. The second version was then reviewed by a referee session of the same model family as the audit (Claude Fable), with its own code written from the text of the paper (Section 7); it re-derived the hand proofs, reproduced the computations, confirmed the exact coclique counts of the 280 graphs with a third program, supplied the check

behind the multiplicity statement of Section 5.3, and asked for the corrections made in this version: a gap in the wording of the proof of Lemma 2.2(2), the notation W , the printed orbital lists of $\Gamma_{160}^{(2)}$ and $\Gamma_{160}^{(3)}$, the two classes of D_{12} , the citation for $\text{srg}(35, 16, 6, 8)$, the exact coclique counts, the bibliography and the wording of the abstract. Every statement is established as recorded in Table 7.

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APPENDIX A. THE STANDALONE VERIFIER

```

1  #!/usr/bin/env python3
2  """Standalone exact verifier (Python 3 standard library only).
3  For every graph6 line in the given files: decode it, check that the graph is simple and k-regular, and check
4  A^2 = k I + lambda A + mu (J - I - A) entry by entry in exact integers (common-neighbour counts of all pairs,
5  computed with Python integers as bitsets). Prints the parameters and OK, or the first failure.
6  Usage: verify_srg.py FILE.g6 [...] """
7  import sys
8
9  def decode(line):
10     b = [ord(c) - 63 for c in line.strip()]
11     if any(x < 0 or x > 63 for x in b): raise ValueError('not graph6')
12     if b[0] < 63: n, pos = b[0], 1
13     elif b[1] < 63: n, pos = (b[1] << 12) | (b[2] << 6) | b[3], 4
14     else: raise ValueError('n too large')
15     need = n * (n - 1) // 2
16     if len(b) - pos != (need + 5) // 6: raise ValueError('wrong length')
17     rows = [0] * n
18     q = 0
19     for j in range(1, n):
20         for i in range(j):
21             if (b[pos + q // 6] >> (5 - q % 6)) & 1:
22                 rows[i] |= 1 << j; rows[j] |= 1 << i

```

```

23         q += 1
24     return n, rows
25
26 def check(n, rows):
27     if any(rows[x] >> x & 1 for x in range(n)): return 'FAIL loop'
28     deg = {bin(r).count('1') for r in rows}
29     if len(deg) != 1: return 'FAIL not regular: degrees %s' % sorted(deg)
30     k = deg.pop()
31     if not 0 < k < n - 1: return 'FAIL complete or empty'
32     lam = mu = None
33     for x in range(n):
34         rx = rows[x]
35         for y in range(x + 1, n):
36             c = bin(rx & rows[y]).count('1')
37             if rx >> y & 1:
38                 if lam is None: lam = c
39                 elif c != lam: return 'FAIL adjacent pair (%d,%d): %d common neighbours, expected %d' % (x,
y, c, lam)
40             else:
41                 if mu is None: mu = c
42                 elif c != mu: return 'FAIL non-adjacent pair (%d,%d): %d common neighbours, expected %d' % (
x, y, c, mu)
43     if mu is None: return 'FAIL no non-adjacent pair'
44     # A^2 = kI + lam A + mu(J-I-A): diagonal entries are the degrees (= k), off-diagonal entries checked
above.
45     return 'OK srg(%d,%d,%d,%d); complement srg(%d,%d,%d,%d)' % (n, k, lam, mu, n, n - k - 1, n - 2 - 2 * k +
mu, n - 2 * k + lam)
46
47 if __name__ == '__main__':
48     bad = 0
49     for fn in sys.argv[1:]:
50         with open(fn) as fh:
51             for ln, line in enumerate(fh, 1):
52                 if not line.strip(): continue
53                 n, rows = decode(line)
54                 res = check(n, rows)
55                 bad += not res.startswith('OK')
56                 print('%s:%d n=%d %s' % (fn, ln, n, res), flush=True)
57     sys.exit(1 if bad else 0)

```

APPENDIX B. THE BUILDER WRITTEN FROM SECTION 2

The program below builds the six graphs of Constructions 2.3 and 2.8 (and the non-strongly-regular product variant of the 232-type) from the rules as stated, checks λ and μ for all pairs, and writes graph6 files. $\mathbb{F}_{64}$ is $\mathbb{F}_2[x]/(x^6 + x + 1)$ with $g = x$, $\mathbb{F}_8^* = \langle g^9 \rangle$, $\omega = g^9$ and $\{\zeta, \zeta^2\} = \{g^{21}, g^{42}\}$; cells and transversals are 0-based.

```

1 """Builds the graphs of Constructions 2.3 and 2.8 of docs/paper/new_srgs_twoorbit.tex from the rules as
worded there.
2 GF(64) = GF(2)[x]/(x^6+x+1), g = x; F8 = {0} u <g^9>; omega = g^9; zeta = g^21. Variant 'B' = quotient rule
(Q),
3 variant 'A' = product rule (P). Checks lambda and mu on all pairs and writes paper_*.g6 to the current
directory."""
4 import itertools, sys
5 P = 0b1000011
6 exp = [0]*126; log = [0]*64; x = 1
7 for i in range(63):
8     exp[i] = x; log[x] = i; x <= 1

```

```

9     if x & 64: x ^= P
10    for i in range(63, 126): exp[i] = exp[i-63]
11    def mul(a, b): return 0 if a == 0 or b == 0 else exp[log[a]+log[b]]
12    def inv(a): return exp[(63-log[a]) % 63]
13    def pw(a, k): return 0 if a == 0 else exp[(log[a]*k) % 63]
14    F8 = [0] + [exp[9*i] for i in range(7)]
15    inF8 = lambda a: a == 0 or log[a] % 9 == 0
16    om = exp[9]
17    lab = [pw(om, i) for i in range(7)]      # c = omega^i
18    INF = 'inf'
19    pts = [INF] + F8
20    def vec(p): return (1, 0) if p == INF else (p, 1)
21    def br(a, b):
22        a, b = vec(a), vec(b); return mul(a[0], b[1]) ^ mul(a[1], b[0])
23    def chi(A, B):
24        a, b = A; c, d = B
25        return mul(mul(br(a, c), br(b, d)), inv(mul(br(a, d), br(b, c))))
26    P2 = [frozenset(p) for p in itertools.combinations(pts, 2)]
27    E = sorted({frozenset((e, pw(e, 8))) for e in range(1, 64) if not inF8(e)}, key=sorted)
28    assert len(E) == 28
29    cube = {exp[21], exp[42]}
30    def tup(s): return tuple(sorted(s, key=str))
31    def g6(n, adj, fn):
32        bits = [1 if j in adj[i] else 0 for j in range(1, n) for i in range(j)]
33        bits += [0]*((6-len(bits)) % 6) % 6
34        hdr = [n+63] if n < 63 else [126, 63+(n >> 12 & 63), 63+(n >> 6 & 63), 63+(n & 63)]
35        open(fn, 'w').write(bytes(hdr+[63+int(''.join(map(str, bits[i:i+6])), 2) for i in range(0, len(bits), 6)
36        ]).decode()+'\n')
37    def check(name, n, adj):
38        ks = {len(a) for a in adj}; lam = set(); mu = set()
39        for u in range(n):
40            assert u not in adj[u]
41            for w in range(u+1, n):
42                assert (w in adj[u]) == (u in adj[w])
43                (lam if w in adj[u] else mu).add(len(adj[u] & adj[w]))
44            print(name, 'v', n, 'k', ks, 'lambda', lam, 'mu', mu, flush=True)
45    def build(kind, var):
46        V = []
47        if kind in ('261', '280'):
48            V = [(pc, p, c) for p in P2 for c in lab]
49            V += [(pt, p, None) for p in pts] if kind == '261' else [(e, e, None) for e in E]
50        else:
51            V = [(ec, e, c) for e in E for c in lab] + [(pair, p, None) for p in P2]
52        n = len(V); adj = [set() for _ in range(n)]
53        def rel(X, Y):
54            tx, ox, cx = X; ty, oy, cy = Y
55            if tx in ('pc', 'ec') and ty == tx:
56                d = mul(cy, inv(cx)) if var == 'B' else mul(cy, cx)
57                if ox == oy: return kind == '261' and cx != cy
58                s = len(ox & oy)
59                if s == 1: return d == 1
60                h = chi(tup(ox), tup(oy)); assert inF8(h) and h not in (0, 1)
61                return d in (h, inv(h))
62            if {tx, ty} == {'pc', 'pt'}:
63                p, pt = (ox, oy) if tx == 'pc' else (oy, ox); return next(iter(pt)) in p
64            if tx == ty == 'pt': return True
65            if tx == ty == 'e': return False
66            if {tx, ty} in ({'pc', 'e'}, {'ec', 'pair'}):

```

```

66         p, e = (ox, oy) if tx in ('pc', 'pair') else (oy, ox)
67         h = chi(tup(p), tup(e)); assert pw(h, 9) == 1
68         return h in cube
69         if tx == ty == 'pair': return len(ox & oy) == 1
70         raise Exception((tx, ty))
71     V = [(t, (frozenset([o]) if t == 'pt' else o), c) for t, o, c in V]
72     for i in range(n):
73         for j in range(i+1, n):
74             if rel(V[i], V[j]): adj[i].add(j); adj[j].add(i)
75     return n, adj
76 for kind, var in [('261', 'A'), ('261', 'B'), ('280', 'A'), ('280', 'B'), ('232', 'B'), ('232', 'A')]:
77     n, adj = build(kind, var); check(f'{kind}{var}', n, adj); g6(n, adj, f'paper_{kind}{var}.g6')
78 # 160 #1
79 A4 = [s for s in itertools.permutations(range(4)) if sum(1 for i in range(4) for j in range(i) if s[j] > s[i
    ]) % 2 == 0]
80 cells = [(r, c) for r in range(4) for c in range(4)]
81 V = [('cell', x) for x in cells] + [('flag', s, i, z) for s in A4 for i in range(4) for z in range(3)]
82 T = lambda s: {(j, s[j]) for j in range(4)}
83 col = lambda a, b: a != b and (a[0] == b[0] or a[1] == b[1])
84 def r160(X, Y):
85     if X[0] == 'cell' and Y[0] == 'cell': return False
86     if X[0] == 'flag' and Y[0] == 'cell': X, Y = Y, X
87     if X[0] == 'cell':
88         x = X[1]; s, i = Y[1], Y[2]; f = (i, s[i])
89         return x not in T(s) and not col(x, f) and x != f
90     s, i, z = X[1:]; s2, i2, z2 = Y[1:]
91     t, t2 = T(s), T(s2); f, f2 = (i, s[i]), (i2, s2[i2]); m = len(t & t2); dl = (z2 - z) % 3
92     if m == 1:
93         a, b = f in t2, f2 in t
94         if col(f, f2) and not a and not b: return dl == 0
95         if f != f2 and not col(f, f2):
96             if not a and not b: return dl != 0
97             if a and not b: return dl == 2
98             if b and not a: return dl == 1
99         return False
100     if m == 0: return col(f, f2) and dl != 0
101     return False
102 n = len(V); adj = [set() for _ in range(n)]
103 for i in range(n):
104     for j in range(i+1, n):
105         if r160(V[i], V[j]): adj[i].add(j); adj[j].add(i)
106 check('160#1', n, adj); g6(n, adj, 'paper_160.g6')

```

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