hexagon:2610.00207v2

Cross-Energy of Dirichlet Partial Sums. Part 35: How long can an off-line zero survive the de Bruijn–Newman heat flow?

An explicit lower bound for Λ forced by a zero off the critical line

Pedro Caceres
Universidad Europea de Valencia
pedrojesus.caceres@universidadeuropea.es

November 2026 (renumbered 2 Nov 2026, formerly Part 15; v2, 2 Nov 2026: exact cluster absorption (Hermite model) and the barrier for a linear bound; v3, 3 Nov 2026: the area of the quadruples under the heat flow, Section 11)

Part 35 of the book-series *Cross-Energy of Dirichlet Partial Sums* (forty-seven parts, Parts 0–46; block VI: The de Bruijn–Newman constant). Part 0 is the guide. In book version 1 (2 November 2026) this was Part 32; the present numbering is that of book version 2 (10 November 2026).

Abstract

Let Λ be the de Bruijn–Newman constant; the Riemann hypothesis (RH) is equivalent to $\Lambda \le 0$, and Rodgers and Tao proved $\Lambda \ge 0$. We ask how small Λ can be if ζ has a zero $\frac{1}{2} + \delta + i\gamma$ with $\delta > 0$. Using de Bruijn's strip, the zero dynamics of the Polymath project, and an explicit local zero count for the deformations H_τ that we prove here, we show, with no further hypothesis, that such a zero forces

$$\Lambda \;\ge\; \frac{\delta^2}{70\,\log\gamma\,\log\log\gamma},$$

the numerical steps of the proof are certified in ball arithmetic. This is an implication from a hypothetical zero: under RH it says nothing, and the only unconditional lower bound for Λ remains $\Lambda \ge 0$. The proof bounds Λ, not the survival time of the given zero. For a single pair, the interaction term of the exact equation for its imaginary part is, at $\tau = 0$, the energy slope $\partial_\sigma \log|\xi|$ of the other zeros at the zero itself; under local isolation and regular spacing the bound becomes linear in δ. In an exact polynomial model, a pair over a cluster of K real zeros (K even) is absorbed at $2\delta^2/(K+1)$, and clusters of the size the unconditional zero count allows keep the rate at $\delta^2/\log\gamma$ in the simulated dynamics; this suggests that a linear bound needs a local count not known even under RH. On the 21 off-line zeros of the Davenport–Heilbronn function with $\delta > 0.005$ and height below 800, freezing the energy slope predicts the survival time to within 7%, and we show by a computer-assisted argument that its constant satisfies $0.228 < \Lambda_f \le 3.125$. Finally, we prove a heat-flow Landau formula for fixed n and $0 < t \le \frac{1}{2}$: under the flow the memory of a prime decays like $p^{-(t/4)\log(\gamma/2\pi p)}$ and rotates, while composite numbers acquire a memory governed by a deformed von Mangoldt function Λ_t, which is positive for prime powers, for all squarefree n, and for small t when n has two prime factors (numerically in all cases tested); the main term dominates when $\frac{t}{4}\log n < 1$.

1 Introduction and results

For real τ let

$$H_\tau(z) = \int_0^\infty e^{\tau u^2} \Phi(u) \cos(zu)\, du, \qquad \Phi(u) = \sum_{n \geq 1} \big(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\big) e^{-\pi n^2 e^{4u}},$$

so that $H_0(z) = \frac{1}{8}\xi(\frac{1}{2} + \frac{iz}{2})$ and $\partial_\tau H_\tau = -\partial_{zz} H_\tau$. All zeros of H_τ are real if and only if $\tau \geq \Lambda$ [dB, Ne]. RH is equivalent to $\Lambda \leq 0$; Rodgers and Tao proved $\Lambda \geq 0$ [RT], Polymath proved $\Lambda \leq 0.22$ [Pm], the verification of RH up to $3 \cdot 10^{12}$ gives $\Lambda \leq 0.2$ [PT], and smaller upper bounds were claimed in 2026 [GoJ, Gor], [CE33] (none refereed at the time of writing). Ki, Kim and Lee [KKL] proved $\Lambda < \frac{1}{2}$ and that for $t > 0$ all but finitely many zeros of H_t are real and simple. A zero $\frac{1}{2} + \delta + i\gamma$ of ζ corresponds to the zeros $2\gamma \pm 2i\delta$ of H_0.

We ask how *small* Λ can be if ζ has an off-line zero. De Bruijn's argument gives the opposite inequality: the zero of maximal imaginary part y satisfies $\partial_\tau y \leq -1/y$, so an isolated maximal pair reaches the axis by time $y^2/2 = 2\delta^2$ [dB, Tao]. Planat [Pl] keeps the interaction term to sharpen *upper* bounds for Λ, and Tao remarks that de Bruijn's bound can only be sharp for an isolated zero [Tao]. We have not found a lower bound for Λ in terms of (δ, γ) in the literature.

Theorem 1.1. *If $\zeta(\frac{1}{2} + \delta + i\gamma) = 0$ with $\delta > 0$ and $\gamma > 0$, then $\gamma > 3 \cdot 10^{12}$ and*

$$\Lambda \;\geq\; \frac{\delta^2}{70\, \log\gamma\, \log\log\gamma}.$$

More precisely, $\Lambda \geq T(\gamma)\, \delta^2$, where $T(\gamma)$ is given by the explicit formula (3)*; numerical values are in Table 1.*

Corollary 1.2 (distance to the line in terms of Λ)**.** *Every zero $\beta + i\gamma$ of ζ with $\gamma \geq 3 \cdot 10^{12}$ satisfies*

$$|\beta - \tfrac{1}{2}| \leq \big(70\, \Lambda\, \log\gamma\, \log\log\gamma\big)^{1/2}.$$

Corollary 1.2 is Theorem 1.1 solved for δ, using the symmetry $\beta \leftrightarrow 1 - \beta$. It recovers RH from $\Lambda = 0$, as it must. With every known upper bound for Λ it is vacuous: for $\Lambda \leq 0.2$ the right side is 36.8 at $\gamma = 3 \cdot 10^{12}$ and larger above, against the trivial bound $\frac{1}{2}$; it becomes non-trivial only for $\Lambda < \big(280 \log\gamma \log\log\gamma\big)^{-1}$, that is $\Lambda < 3.7 \cdot 10^{-5}$ at $\gamma = 3 \cdot 10^{12}$. In general a quantitative upper bound $\Lambda \leq \varepsilon$ only gives a zero-free region of width $\asymp \sqrt{\varepsilon \log\gamma \log\log\gamma}$, which shrinks to nothing as $\varepsilon \to 0$ at a fixed height but widens with the height. If the given zero has maximal δ among all zeros of ζ, Theorem 1.1 and de Bruijn's bound bracket Λ: $\delta^2/(70 \log\gamma \log\log\gamma) \leq \Lambda \leq 2\delta^2$.

Two times should be kept apart: the time τ_{abs} at which the given zero (or rather the zero of H_τ continuing it) reaches the real axis, and Λ, the time after which *no* nonreal zero remains. The proof bounds Λ only: it shows that a weighted sum over all the zeros of H_τ in the upper half-plane near $x_0 = 2\gamma$ stays positive, so *some* nonreal zero survives, not necessarily the one we started from. Simulations and the Davenport–Heilbronn test (Sections 8 and 10) measure τ_{abs} of individual zeros, which can be smaller than Λ.

All the numerical inequalities used in the proof of Theorem 1.1 (the final step of the local zero count, Lemma 3.2, and the maximisation over p and γ that gives the constant 70) are certified in ball arithmetic (Appendix A), and the integration of the differential inequality across collisions of zeros rests on a monotonicity lemma for continuous functions with a countable exceptional set (Appendix B).

The ingredients are:

- an explicit local zero count for H_τ, uniform for small τ (Lemma 3.2); the Polymath project proves such a count with unspecified constants [Pm, Thm. 1.5];
- a weighted functional $F = \sum \phi(\operatorname{Re} z)(\operatorname{Im} z)^p$ over the zeros in the upper half-plane, whose derivative is bounded below by a sum of local terms (Proposition 5.2);
- a choice $p \asymp \log\log\gamma$ that balances a Hölder inequality (Section 6).

Section 4 gives the exact equation for a single pair and its link with the energy slope of [CE17]. Section 7 proves a sharper bound, linear in δ, under local isolation and regular spacing. Section 8 compares both bounds with simulations. Section 9 computes exactly the absorption of a pair over a cluster of K zeros, $\tau^* = 2\delta^2/(K+1)$ for K even (Proposition 9.1), measures worst-case clusters with the exact zero dynamics, and suggests why a bound linear in δ cannot come from the zero dynamics and a local count alone (Remark 9.2). Section 11 reads the dynamics through the area $4\delta\gamma$ of the quadruple of an off-line zero, whose square the self term of the flow consumes at the rate $2\operatorname{Re} z^2$ (approximately constant), and rewrites the definition of Λ as the time at which a squared-area gap closes. Section 10 tests the dynamics on the genuine off-line zeros of the Davenport–Heilbronn function. Section 12 studies the arithmetic memory of the zeros under the flow: the memory is erased at a power rate (Proposition 12.1), and numerically it follows a deformed Landau law (Conjecture 12.2). Section 13 proves that law as a Landau formula for H_t (Theorem 13.1), proves that the deformed von Mangoldt function is positive at prime powers, at squarefree integers, and for small t at integers with two prime factors (Proposition 13.8), and tests the formula on exactly computed zeros of H_t.

Place in the book. This is Part 35 of the series, in block VI (the de Bruijn–Newman constant). It builds on Parts 1 and 2 (the cross-energy and the transition at the critical line), Part 17 (the energy slope, which reappears here in the exact equation) and is compared with Part 33 (the upper bound for Λ). It is used by Parts 4, 8 and 17, and by Part 20, whose detection thresholds compare with the survival time proved here. Part 19 [CE19], in its version with Turán's cluster detection, meets the same local-count obstruction as Section 9.

2 Known inputs

We use the following facts, valid for $0 \le \tau \le \frac{1}{2}$. Zeros are counted with multiplicity and written $z_k = x_k + iy_k$.

(P1) *Strip.* Every zero of H_τ satisfies $|\operatorname{Im} z| \le 1$ [dB, Thm. 13].

(P2) *Symmetry.* H_τ is even and real on $\mathbb{R}$, so its zeros are symmetric under $z \mapsto -z$ and $z \mapsto \bar{z}$; H_τ is entire of order 1 and $H_\tau(0) = \int_0^\infty e^{\tau u^2}\Phi \ge H_0(0) = \xi(\frac{1}{2})/8 > 0.062$, since $\Phi > 0$ [Pm, Sect. 3].

(P3) *Dynamics.* If $z_k(\tau_0)$ is a simple zero of H_{τ_0}, real or not, it lies on a smooth branch $z_k(\tau)$ with

$$\partial_\tau z_k = 2{\sum_{m\neq k}}' \frac{1}{z_k - z_m},$$

the sum taken over the symmetric pairs $\pm z_m$. A zero of multiplicity $r \ge 2$ splits into r distinct zeros $z_0 + \sqrt{2(\tau-\tau_0)}\lambda_i + O(|\tau-\tau_0|)$, with λ_i the roots of the probabilists' Hermite polynomial He_r [Pm, Prop. 3.1].

(P4) *Gaussian representation.* For $\tau > 0$,

$$H_\tau(z) = \int_{\mathbb{R}} G_\tau(w)\, H_0(z + iw)\, dw, \qquad G_\tau(w) = (4\pi\tau)^{-1/2} e^{-w^2/4\tau}.$$

Indeed, $H_0(z) = \frac{1}{2}\int_{\mathbb{R}} \Phi(|u|) e^{izu} du$ and $\int G_\tau(w) e^{-wu} dw = e^{\tau u^2}$; Fubini applies because Φ decays double exponentially.

(P5) *Verification of RH.* All zeros of ζ with $0 < \gamma \le 3 \cdot 10^{12}$ lie on the critical line [PT].

In the variable $s = \frac{1}{2} + \frac{iz}{2}$, the point $z = x + iy$ corresponds to $s = \frac{1}{2} - \frac{y}{2} + \frac{ix}{2}$. By the functional equation $|\xi(\sigma + it)| = |\xi(1 - \sigma + it)|$, so $|H_0(x + iy)| = \frac{1}{8}|\xi(\frac{1}{2} + \frac{|y|}{2} + \frac{ix}{2})|$.

3 Explicit zero counting for H_τ

Lemma 3.1 (bounds for ξ)**.** *Let $t \ge 50$, $\sigma \ge \frac{1}{2}$ and $s = \sigma + it$. Put*

$$A_t(\sigma) = \frac{\sigma}{2}\log\frac{|s|}{2\pi} + \log|s-1| + \frac{1}{2}\log|s| + \frac{1}{2}\log\pi - \frac{\pi t}{4}.$$

Then

$$\log|\xi(s)| \le U_t(\sigma) := A_t(\sigma) + \frac{1}{3t} + \log\min\big(4\sqrt{t},\ \zeta(\sigma)\big),$$

where $\zeta(\sigma)$ is used only for $\sigma > 1$. If $\sigma > 1$, then

$$\log|\xi(s)| \ge A_t(\sigma) - \frac{\sigma^3}{6t^2} - \frac{1}{3t} + \log\frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$

Proof. Write $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Stirling's formula with error term [Ol, Ch. 8, Sect. 4] gives, for $\operatorname{Re} z > 0$, $\log\Gamma(z) = (z - \frac{1}{2})\log z - z + \frac{1}{2}\log 2\pi + \theta/(6|z|)$ with $|\theta| \le 1$. With $z = s/2$ and $\arg(s/2) = \frac{\pi}{2} - \arctan\frac{\sigma}{t}$,

$$\log|\Gamma(\tfrac{s}{2})| = \Big(\frac{\sigma}{2} - \frac{1}{2}\Big)\log\frac{|s|}{2} - \frac{\pi t}{4} + \Big(\frac{t}{2}\arctan\frac{\sigma}{t} - \frac{\sigma}{2}\Big) + \frac{1}{2}\log 2\pi + \frac{\theta}{3|s|}.$$

The bracket lies in $[-\sigma^3/(6t^2), 0]$. Collecting terms gives $\log|\xi(s)| = A_t(\sigma) + (\text{bracket}) + \theta/(3|s|) + \log|\zeta(s)|$.

For ζ, the Euler–Maclaurin formula with $N = \lceil t \rceil$ gives $|\zeta(s)| \le \sum_{n \le N} n^{-\sigma} + \frac{N^{1-\sigma}}{|s-1|} + \frac{1}{2}N^{-\sigma} + \frac{|s|}{2\sigma N^\sigma}$ [Ti, Sect. 4.11]. For $\frac{1}{2} \le \sigma \le 2$ and $t \ge 50$ the right side is at most $3\sqrt{t} + 2 \le 4\sqrt{t}$; for $\sigma > 1$ also $|\zeta(s)| \le \zeta(\sigma)$, and $\zeta(\sigma) \le \zeta(2) < 4\sqrt{t}$ for $\sigma \ge 2$. The lower bound uses $|\zeta(s)| \ge \zeta(2\sigma)/\zeta(\sigma)$ for $\sigma > 1$. □

Lemma 3.2 (explicit local count)**.** *Let $X \ge 200$, $\ell_X = \log\frac{X+1}{4\pi}$ and $0 \le \tau \le 1/(100\,\ell_X)$. The number of zeros of H_τ, with multiplicity, with $X \le \operatorname{Re} z \le X + 1$ is at most*

$$B(X) := 2.07\,\ell_X + 2.5.$$

The same holds for $-X - 1 \le \operatorname{Re} z \le -X$.

Proof. By (P1) the zeros in question lie in the box $[X, X+1]\times[-1,1]$. Put $Y_0 = 2$, $c = X + \frac{1}{2} - iY_0$, $r = (\frac{1}{4} + (Y_0+1)^2)^{1/2} = 3.0414$ and $R = e\,r = 8.2673$. The box lies in $|z-c| \le r$, so Jensen's formula gives

$$\#\{\text{zeros in the box}\} \le \frac{1}{\log(R/r)} \log \frac{\max_{|z-c|=R} |H_\tau(z)|}{|H_\tau(c)|} = \log \frac{\max_{|z-c|=R} |H_\tau(z)|}{|H_\tau(c)|}. \tag{1}$$

Write $t_c = \frac{1}{2}(X + \frac{1}{2}) \ge 100$ and $s_c = \sigma_c + it_c$ with $\sigma_c = \frac{1}{2} + \frac{Y_0}{2} = \frac{3}{2}$; then $H_0(c+iw) = \frac{1}{8}\xi(s_c - \frac{w}{2})$. For $\tau = 0$ the argument below applies with $G_0 = \delta_0$, so assume $\tau > 0$.

Upper bound on the circle. Let $z = x+iy$ with $|z-c| = R$, and put $t = x/2 \in [t_c - 4.14, t_c + 4.14]$ and $\sigma_y = \frac{1}{2} + \frac{|y|}{2}$. Let $\tilde{U}_t(\sigma) = \max_{\frac{1}{2} \le \sigma' \le \sigma} U_t(\sigma')$, which is nondecreasing. By (P4), the remark after (P5) and Lemma 3.1,

$$|H_\tau(z)| \le \frac{1}{8} \int G_\tau(w)\, e^{\tilde{U}_t(\frac{1}{2} + \frac{|y+w|}{2})}\, dw.$$

For $\sigma \le \sigma_y + \frac{1}{2} \le 6.2$ one has $\partial_\sigma A_t \le \frac{1}{2}\log\frac{t}{2\pi} + (0.75\sigma^2 + 1.5\sigma - 1)/t^2 \le \frac{1}{2}\log\frac{t}{2\pi} + 37.4/t^2$, and the ζ term is nonincreasing in σ. Hence for $|w| \le 1$

$$\tilde{U}_t\Big(\tfrac{1}{2} + \tfrac{|y+w|}{2}\Big) \le \tilde{U}_t(\sigma_y) + \kappa_1 |w|, \qquad \kappa_1 = \frac{1}{4}\log\frac{t_c + 4.14}{2\pi} + \frac{18.7}{(t_c - 4.14)^2}.$$

Since $\int G_\tau e^{\kappa_1|w|} dw \le 2e^{\kappa_1^2 \tau}$, and the region $|w| > 1$ contributes, relative to $e^{\tilde{U}_t(\sigma_y)}$, an amount η_{up} that is bounded in closed form in Appendix A ($\eta_{\mathrm{up}} \le 2 \cdot 10^{-31}$; its Gaussian weight is at most $e^{-25\ell_X}$, while $\tilde{U}_t$ grows only like $\frac{|w|}{4}\log(t + |w|)$), we get

$$\log|H_\tau(z)| \le \tilde{U}_t(\sigma_y) + \log 2 + \kappa_1^2\tau - \log 8 + \tfrac{1}{2}\eta_{\mathrm{up}}.$$

Lower bound at the centre. Since ξ has no zeros in $\operatorname{Re} s > 1$, $L = \log\xi$ is analytic there. Put $\lambda = \frac{1}{2}L'(s_c) = a + ib$ and write $\xi(s_c - \frac{w}{2}) = \xi(s_c)e^{-\lambda w}\rho(w)$. Completing the square, $G_\tau(w)e^{-\lambda w} = e^{\lambda^2\tau} G_\tau(w + 2\lambda\tau)$, so

$$H_\tau(c) = \frac{\xi(s_c)}{8}\, e^{\lambda^2\tau}\Big(1 + \int G_\tau(w + 2\lambda\tau)\,(\rho(w) - 1)\, dw\Big), \qquad |G_\tau(w + 2\lambda\tau)| = e^{b^2\tau} G_\tau(w + 2a\tau).$$

Here $L' = M'/M + \zeta'/\zeta$ with $M(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})$. Since $|\zeta'/\zeta(s_c)| \le |\zeta'/\zeta(\frac{3}{2})| = 1.5053$, we have $a_- \le a \le a_+$ and $|b| \le b_+$, with $a_\pm = \frac{1}{2}(\operatorname{Re}\frac{M'}{M}(s_c) \pm 1.5053)$ and $b_+ = \frac{1}{2}(|\operatorname{Im}\frac{M'}{M}(s_c)| + 1.5053)$. Numerically $a_- \ge \frac{1}{4}\ell_X - 0.76$ and $b_+ \le 1.15$.

For $|w| \le 0.6$ the segment $[s_c - \frac{w}{2}, s_c]$ stays in $1.2 \le \operatorname{Re} s \le 1.8$. There $|L''| \le |(M'/M)'| + (\log\zeta)''(1.2) \le K_2 := 24.832 + \frac{\pi}{4t_c} + \frac{3}{t_c^2}$, because $|(\zeta'/\zeta)'(s)| \le \sum \Lambda(n)\log n\, n^{-\operatorname{Re} s} = (\log\zeta)''(\operatorname{Re} s)$ and $|(M'/M)'(s)| \le |s|^{-2} + |s-1|^{-2} + \frac{1}{4}|\psi'(\frac{s}{2})|$ with $|\psi'(z)| \le \sum_{n\ge 0}|z+n|^{-2} \le (\operatorname{Im} z)^{-2} + \pi/(2\operatorname{Im} z)$. Taylor's formula then gives $|\rho(w) - 1| \le \exp(K_2 w^2/8) - 1$. For $|w| > 0.6$ we use $|\rho(w)| \le \exp\big(\tilde{U}_{t_c}(\frac{1}{2} + |\sigma_c - \frac{w}{2} - \frac{1}{2}|) - \Psi + a_\pm w\big)$, where Ψ is the lower bound of Lemma 3.1 at s_c. This gives

$$|H_\tau(c)| \ge \frac{e^{\Psi}}{8}\, e^{((a_-)_+^2 - b_+^2)\tau}(1 - \varepsilon),$$

with $\varepsilon := \sup_{a_- \le a' \le a_+} e^{b_+^2\tau} \int G_\tau(w + 2a'\tau)\, h_{a'}(w)\, dw$, where $h_{a'}$ is the majorant of $|\rho - 1|$ just described (the tail majorant uses a' in place of a). Because $\tau \le 1/(100\ell_X)$, the Gaussian has variance at most $0.02/\ell_X$ and centre within 0.011 of 0. The inner part is bounded in closed form

by the Gaussian moment $\int \mathcal{N}(m,v)(e^{\beta w^2}-1)\,dw = (1-2\beta v)^{-1/2}e^{\beta m^2/(1-2\beta v)}-1$ with $\beta = K_2/8$, $v = 2\tau$ and $|m| \le 2\max(|a_-|,a_+)\tau$, and the tail $|w| > 0.6$ by a Gaussian tail estimate; Appendix A gives $\varepsilon \le 0.0237$ for every $X \ge 200$.

Conclusion. Inserting both bounds in (1), the count is at most

$$\max_{|z-c|=R} \tilde{U}_t(\sigma_y) - \Psi + \log 2 + (\kappa_1^2 - (a_-)_+^2 + b_+^2)\tau - \log(1-\varepsilon) + \tfrac{1}{2}\eta_{\mathrm{up}}.$$

Up to terms that are $O(1/t_c)$ or $O(\tau)$, the leading part is $\frac{\sigma_{\max}-\sigma_c}{2}\log\frac{t_c}{2\pi} = \frac{R}{4}\ell_c = 2.06683\,\ell_c$, $\ell_c = \log\frac{t_c}{2\pi} \le \ell_X$, where $\sigma_{\max} = \frac{1}{2} + \frac{Y_0+R}{2}$ is attained at the bottom of the circle. In Appendix A we write the whole expression as $\frac{R}{4}\ell_c + g$ and bound g by majorants that are nondecreasing in $u = 1/t_c$ and nonincreasing in ℓ_c; so a single evaluation at $X = 200$, in ball arithmetic, bounds g for every $X \ge 200$ and every admissible τ. It gives $g \le 2.4067$, against $2.5 + (2.07 - \frac{R}{4})\ell_c \ge 2.5088$. Hence the count is at most $2.0668\,\ell_X + 2.4067 \le B(X)$ (computer-assisted, `certify_lemmaC.py`). The case $\operatorname{Re} z < 0$ follows from (P2). □

Lemma 3.3 (global count)**.** *For $0 \le \tau \le \frac{1}{2}$ and $\rho \ge 200$, the number $N_\tau(\rho)$ of zeros of H_τ with $|\operatorname{Re} z| \le \rho$ satisfies $N_\tau(\rho) \le \rho\log\rho$. Moreover $N_\tau(200) \le 750$.*

Proof. The zeros counted lie in $|z| \le \rho+1$. By Jensen's formula with centre 0 and radii $\rho+1$ and $e(\rho+1)$, and by (P2), $N_\tau(\rho) \le \log\big(I(e(\rho+1))/H_0(0)\big)$, where $I(r) = \int_0^\infty e^{u^2/2}\Phi(u)\cosh(ru)\,du \ge \max_{|z|=r}|H_\tau(z)|$. From $\Phi(u) \le 2.01\pi^2 e^{9u-\pi e^{4u}}$ and $\cosh(ru) \le e^{ru}$, $I(r) \le 2.01\pi^2\int_0^\infty e^{f(u)}du$ with the concave $f(u) = \frac{u^2}{2} + (r+9)u - \pi e^{4u}$. Its maximiser satisfies $4\pi e^{4u^*} = r + 9 + u^*$, so $u^* \le \frac{1}{4}\log(\frac{r}{4\pi}+2)$ for $r \ge 200$, $f(u^*) \le (r+9+\frac{u^*}{2})u^* - \frac{r+9}{4}$, and $f'(u) \le -(e^4-1)(r+9)$ for $u \ge u^*+1$. Hence $\int_0^\infty e^f \le (u^*+1+\frac{1}{(e^4-1)(r+9)})e^{f(u^*)}$, and since $\log(2.01\pi^2(u^*+2)) < \frac{r+9}{4}$, $\log I(r) \le \frac{r+10}{4}\log(\frac{r}{4\pi}+2)+3$. With $H_0(0) > 0.062$, $r = e(\rho+1)$, $e(\rho+1)+10 \le e(\rho+5)$ and $\frac{e(\rho+1)}{4\pi}+2 \le \rho+2$. This gives $N_\tau(\rho) \le \frac{e}{4}(\rho+5)\log(\rho+2)+3+\log\frac{1}{0.062} \le \frac{e}{4}(\rho+5)\log(\rho+2)+6 \le \rho\log\rho$, and at $\rho = 200$ the middle term is 745.6, so $N_\tau(200) \le 750$. (Numerical quadrature gives the sharper value 389.8, `lowcount.py`; it is not used, since only the order of magnitude of this count enters, through terms below 10^{-20}.) □

4 The exact equation and the energy slope

Lemma 4.1. *Let $z = x+iy$, $y > 0$, be a simple zero of H_τ. Then*

$$\partial_\tau y = -\frac{1}{y} - 2y\sum_{x_k \text{ real}} \frac{1}{(x-x_k)^2+y^2} - \sum_{w \text{ nonreal, } w\neq z,\bar z} \frac{2(y-\operatorname{Im} w)}{|z-w|^2}.$$

Proof. Take imaginary parts in (P3): $\operatorname{Im}\frac{2}{z-\bar z} = -\frac{1}{y}$, $\operatorname{Im}\frac{2}{z-x_k} = -\frac{2y}{(x-x_k)^2+y^2}$, and $\operatorname{Im}\frac{2}{z-w} = -\frac{2(y-\operatorname{Im} w)}{|z-w|^2}$. □

Remark 4.2 (the energy slope). At $\tau = 0$ write $z = 2\gamma+2i\delta$ and $x_k = 2\gamma_k$. Then $2y\sum_k \frac{1}{(x-x_k)^2+y^2} = \sum_k \frac{\delta}{\delta^2+(\gamma-\gamma_k)^2}$. Whether or not the other zeros of ζ are on the line, Lemma 4.1 at $\tau = 0$ reads

$$\partial_\tau \delta = -\frac{1}{4\delta} - \frac{1}{2}S_{\mathrm{rest}}\big(\tfrac{1}{2}+\delta+i\gamma\big),$$

where $S(\sigma,t) = \partial_\sigma \log|\xi(\sigma+it)| = \sum_\rho \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}$ is the horizontal logarithmic derivative of the completed zeta function (the sum converging in symmetric order, or absolutely after pairing ρ

with $1-\bar\rho$). In [CE17, Sect. 3] it is called the *energy slope*, because it is the slope in σ of $\log|\xi|$, whose mean controls the cross-energy C_2 there; but here we use only its definition as a logarithmic derivative, and no identity between energies is claimed. Here $S_{\rm rest}$ omits only the pair $\frac12\pm\delta+i\gamma$; the mirror pair $\frac12\pm\delta-i\gamma$ is included and matches the mirror terms exactly. Thus the horizontal logarithmic derivative of ξ, with the singular contribution of the pair removed, determines the collective contribution of the remaining zeros to the initial velocity of the zero under the heat flow: the flow pulls an off-line zero towards the line at a speed equal to half of it, plus the self-interaction $\frac{1}{4\delta}$. The mean of S over t is π times the density of zeros, that is $\frac12\log\frac{\gamma}{2\pi}$.

5 The weighted functional

Notation. Throughout this section $\gamma\ge 3\cdot 10^{12}$, $x_0=2\gamma$, and

$$W=\log(3+x_0),\qquad \hat B=B(4x_0),\qquad \tau_C=\frac{1}{200\log x_0}.$$

For $X\le x_0^2$ one has $\ell_X<2\log x_0$, so by Lemma 3.2 *every closed unit interval* $[X,X+1]$ *with* $200\le|X|\le x_0^2$ *contains at most* $B(|X|)$ *zeros of* H_τ *for* $0\le\tau\le\tau_C$, and at most $\hat B$ if $|X|\le 4x_0$; for negative intervals $[X,X+1]$ the relevant height is $|X+1|$, and intervals within $|x|\le 200$ are covered by Lemma 3.3. Since nonreal zeros come in conjugate pairs, *such an interval contains at most* $\hat B/2$ *zeros in the open upper half-plane.* Let

$$\psi(x)=\operatorname{sech}\Big(\frac{x-x_0}{W}\Big).$$

Then $0<\psi\le 1$, $|\psi'|\le\psi/W$, $|\psi''|\le\psi/W^2$, $\psi(x')\le e^{|x'-x|/W}\psi(x)$, $\int\psi=\pi W$, and, since ψ is unimodal, $\sum_{j\in\mathbb Z}\sup_{[j,j+1]}\psi\le\pi W+2$. For $R\ge x_0$ let χ_R be 1 on $|x-x_0|\le R$, 0 on $|x-x_0|\ge 2R$, and the cubic smoothstep in between, so $|\chi_R'|\le\frac{3}{2R}$ and χ_R' is Lipschitz with constant $6/R^2$. Put $\phi=\psi\chi_R$ and

$$F_R(\tau)=\sum_{k:\ y_k>0}\phi(x_k)\,y_k^p,\qquad p\ge 2,$$

a finite sum over the zeros of H_τ in the upper half-plane. We call a zero *central* if $|x_k-x_0|\le x_0/2$.

Lemma 5.1 (explicit sums)**.** *Let* $0\le\tau\le\tau_C$, *let* z_k *be a central zero,* $y\in(0,1]$, *and write* $\Delta x=x_k-x_m$ *and* $\eta=10^{-9}\hat B$. *Let* $s_2=2+2\sum_{j\ge1}\frac{1}{1+j^2}=4.1533$, $s_c=2+8\sum_{j\ge1}\frac{1}{4+j^2}=7.2832$, *and let* H_n *be the harmonic numbers. Summing over* all *zeros* z_m *with multiplicity,*

$$\textit{(i)}\quad \sum_m\frac{1}{\Delta x^2+y^2}\le\hat B\Big(\frac{2}{y^2}+2\zeta(2)\Big)+\eta.$$

Summing only over the zeros z_m *in the open upper half-plane,*

$$\textit{(ii)}\quad \sum_m\frac{1}{1+\Delta x^2}\le\tfrac12 s_2\hat B+\eta,\qquad \textit{(iii)}\quad \sum_m\frac{4}{4+\Delta x^2}\le\tfrac12 s_c\hat B+\eta,$$

$$\textit{(iv)}\quad \sum_{|\Delta x|\le W}\min\Big(\frac12,\frac{1}{|\Delta x|}\Big)\le H_{\lceil W\rceil-1}\hat B,\qquad \textit{(v)}\quad \sum_{|\Delta x|>W}\Delta x^{-2}\le\big(\lfloor W\rfloor^{-2}+\lfloor W\rfloor^{-1}\big)\hat B+\eta.$$

Finally, over all zeros,

$$\textit{(vi)}\quad \Big|{\sum_{m:\,|\Delta x|>1}}'\operatorname{Re}\frac{2}{z_k-z_m}\Big|\le E_3\hat B,\qquad E_3=4H_{\lceil1.5x_0\rceil+3}+4+\frac83\Big(1+\frac{2.1}{\hat B}\Big)(1+10^{-3}).$$

Proof. Group the zeros by the closed unit intervals $\Delta x \in [j, j+1]$ and $\Delta x \in [-j-1, -j]$, $j \geq 0$.

Zeros with $|\Delta x| \leq x_k$. These have $0 \leq x_m \leq 2x_k \leq 3x_0$. Each interval with $x_m \geq 200$ holds at most $\hat{B}$ of them, and at most $\hat{B}/2$ upper ones. The at most 750 zeros with $|x_m| \leq 200$ (Lemma 3.3) lie at distance $\geq x_0/2 - 200$ and contribute less than 10^{-20}. On the two intervals with $j = 0$ the weights in (i), (ii), (iii) are at most y^{-2}, 1, 1; on the others at most j^{-2}, $(1+j^2)^{-1}$, $4(4+j^2)^{-1}$. For (iv) and (v), $\min(\frac{1}{2}, 1/|\Delta x|) \leq \frac{1}{2}$ for $|\Delta x| \leq 2$ and $\leq 1/j$ on the j-th interval, and $\Delta x^{-2} \leq j^{-2}$ there.

Zeros with $|\Delta x| > x_k$. The weights are at most Δx^{-2}. Counting with Lemma 3.2 up to height x_0^2 (at most $B(X) \leq 2.07 \cdot 2 \log x_0 + 2.5$ per unit interval) gives at most $2\sum_{j > x_k} (4.2 \log x_0 + 2.5) j^{-2} \leq (9 \log x_0 + 5)/x_k$. Beyond x_0^2, partial summation with Lemma 3.3 gives at most $2\int_{x_0^2/2}^{\infty} \rho \log \rho \cdot 2\rho^{-3} d\rho \leq 8 \log x_0 / x_0^2$. Both are below η for $x_0 \geq 6 \cdot 10^{12}$. This proves (i)–(v).

For (vi), split the zeros into three groups.

- $0 \leq x_m \leq 2x_k + 2$, $|\Delta x| > 1$. Each term is at most $2/|\Delta x| \leq 2/j$ on the j-th interval, $1 \leq j \leq \lceil x_k \rceil + 2$, giving at most $4H_{\lceil 1.5x_0 \rceil + 3}\hat{B}$ (plus 10^{-9} for $x_m \leq 200$).
- $-2x_k - 2 \leq x_m < 0$. There are at most $(2x_k + 2)\hat{B} + 750$ of them, each at distance $\geq x_k$, giving at most $4\hat{B}(1 + 1/x_k) + 10^{-9}$; the excess $4\hat{B}/x_k$ is absorbed by the factor $1 + 10^{-3}$ below.
- The symmetric set $\{|x_m| > 2x_k + 2\}$. Here $|z_m| \geq 2|z_k|$, so pairing z_m with $-z_m$ gives $\left|\frac{2}{z_k - w} + \frac{2}{z_k + w}\right| = \frac{4|z_k|}{|z_k^2 - w^2|} \leq \frac{16|z_k|}{3|w|^2}$. Summing over pairs with $x_m > 2x_k + 2$, using Lemma 3.2 up to x_0^2 and Lemma 3.3 beyond, gives at most $\frac{16}{3}(x_k + 1)\frac{B(2x_k) + 2.1}{2x_k}(1 + 10^{-9}) \leq \frac{8}{3}(\hat{B} + 2.1)(1 + 10^{-3})$.

□

Proposition 5.2 (explicit differential inequality)**.** *Let $0 \leq \tau \leq \tau_C$ be such that all zeros of H_τ with $|\operatorname{Re} z - x_0| \leq 2R + 1$ are simple. Then*

$$\frac{dF_R}{d\tau} \;\geq\; -\sum_{k:\; y_k > 0} \kappa_k \, \phi(x_k) \, y_k^{p-2} \;-\; E_R(\tau),$$

where, for every central zero, $\kappa_k = \kappa$ with

$$\kappa = p + \hat{B}\Big[s_2 p(p-1) + \big(4 + 4\zeta(2) + s_c + d_W\big)p + e^{1/W}\Big(\frac{2}{W^2} + \frac{p}{W}\Big) + \frac{2 + E_3}{W} + 10^{-6}(p^2 + p + 1)\Big], \quad (2)$$

$d_W = \frac{2e}{W} H_{\lceil W \rceil - 1} + 2(\lfloor W \rfloor^{-2} + \lfloor W \rfloor^{-1})$. For every other zero κ_k is finite and bounded by a polynomial in $|x_k|$. The error satisfies $\sup_{0 \leq \tau \leq \tau_C} |E_R(\tau)| \to 0$ as $R \to \infty$ (Lemma 5.3). Informally (this simplification is not used by the certification), for $\gamma \geq 3 \cdot 10^{12}$ one has $\kappa \leq p + \hat{B}(4.154p^2 + 14.55p + 4.44)$.

Proof. By (P3),

$$\frac{dF_R}{d\tau} = \sum_{y_k > 0} \big(p\phi_k y_k^{p-1} \dot{y}_k + \phi_k' y_k^p \dot{x}_k\big), \qquad \dot{z}_k = \sum_m{}' \frac{2}{z_k - z_m}.$$

Every other zero z_m is either $\bar{z}_k$, real, an upper zero, or the conjugate of another upper zero. All zeros have $|y| \leq 1$ by (P1). We call a quantity *negligible* if it is supported on zeros with $|x - x_0| \geq R - 1$; such zeros carry $\psi \leq 2e^{-(R-1)/W}$. These quantities form E_R; Lemma 5.3

below bounds E_R uniformly in τ, also near collisions. For a central zero we bound each kind of contribution as follows.

(a) The conjugate $\bar z_k$ contributes $-p\phi_k y_k^{p-2}$.

(b) Real zeros contribute $-2p\phi_k y_k^p \sum \frac{1}{\Delta x^2+y_k^2} \ge -(4+4\zeta(2))p\,\hat B\,\phi_k y_k^{p-2} - 2p\eta\phi_k y_k^p$, by Lemma 5.1(i) and $y_k \le 1$.

(c) Conjugates $\bar z_n$ *of other upper zeros.* With $u = y_k + y_n \in [y_k, 2]$, each contributes $-2p\phi_k y_k^{p-1}\frac{u}{\Delta x^2+u^2} \ge -2p\phi_k y_k^{p-2}\frac{4}{4+\Delta x^2}$. By Lemma 5.1(iii), applied to the real parts x_n of the upper zeros z_n, the total is at least $-s_c p\,\hat B\,\phi_k y_k^{p-2} - 2p\eta\phi_k y_k^{p-2}$.

(d) Pairs of upper zeros $\{k,m\}$. Label the pair so that $y_k \ge y_m$, and put $D = |z_k - z_m|$, $\Delta y = y_k - y_m$. The two contributions add up to $T_{km} = -2p\,\Delta y\,(\phi_k y_k^{p-1} - \phi_m y_m^{p-1})/D^2$, which is nonnegative unless $\phi_k y_k^{p-1} > \phi_m y_m^{p-1}$. In that case

$$T_{km} \ge -2p(p-1)\phi_k y_k^{p-2}\frac{\Delta y^2}{D^2} - 2p\chi_k(\psi_k-\psi_m)_+\frac{\Delta y}{D^2}y_m^{p-1} - 2p\psi_m(\chi_k-\chi_m)_+\frac{\Delta y}{D^2}y_m^{p-1}.$$

(d1) Since $\Delta y \le 1$, $\Delta y^2/D^2 \le 1/(1+\Delta x^2)$. Charged to k, the first term gives at least $-s_2 p(p-1)\hat B\phi_k y_k^{p-2} - 2p^2\eta\phi_k y_k^{p-2}$ by Lemma 5.1(ii).

(d2) For $|\Delta x| \le W$: $(\psi_k-\psi_m)_+ \le |\Delta x|\sup|\psi'| \le \frac{e}{W}|\Delta x|\psi_k$ and $|\Delta x|\Delta y/D^2 \le \min(\frac12, 1/|\Delta x|)$. For $|\Delta x| > W$: $(\psi_k-\psi_m)_+\Delta y/D^2 \le \psi_k\Delta x^{-2}$. With $y_m^{p-1} \le y_k^{p-2}$ and Lemma 5.1(iv),(v), the second term gives at least $-d_W p\hat B\phi_k y_k^{p-2} - 2p\eta\phi_k y_k^{p-2}$.

(d3) The third term vanishes unless $\chi_m < 1$, that is $|x_m - x_0| > R$; it is negligible, because $(\chi_k-\chi_m)_+\Delta y/D^2 \le \min(\frac{3}{4R}, \Delta x^{-2})$.

(e) The horizontal term $\phi_k' y_k^p \dot x_k$. Write $\phi' = \chi\psi' + \chi'\psi$. The part with χ' is negligible: pairing the terms of two upper zeros as in (e1) bounds it by a sum over zeros in the transition zone. For $g = \chi\psi'$ we have $|g| \le \phi/W$. On a segment of length ≤ 1 that does not meet $|x - x_0| \ge R-1$ we have $\chi \equiv 1$, so there $|g'| = |\psi''| \le e^{1/W}\psi_h/W^2$ and $|g| \le e^{1/W}\psi_h/W$ for either endpoint h. Segments that do meet that region give negligible terms.

(e1) For upper zeros k, m with $|\Delta x| \le 1$, let h be the higher one. The pair contributes $\frac{2\Delta x}{D^2}(g_k y_k^p - g_m y_m^p)$, and $|g_k y_k^p - g_m y_m^p| \le \sup|g'|\,|\Delta x|\,y_h^p + \sup|g|\,p\,\Delta y\,y_h^{p-1}$. Using $2\Delta x^2 \le 2D^2$ and $2|\Delta x|\Delta y \le D^2$, the pair contributes at least $-e^{1/W}(\frac{2}{W^2}+\frac{p}{W})\phi_h y_h^{p-2}$. A zero has at most $\hat B$ upper partners with $|\Delta x| \le 1$ (two unit intervals).

(e2) For a non-upper zero with $|\Delta x| \le 1$, $|\operatorname{Re}\frac{2}{z_k - z_m}| = \frac{2|\Delta x|}{\Delta x^2+(y_k-y_m)^2} \le \frac{1}{y_k}$, since $y_m \le 0$. There are at most $2\hat B$ of them, contributing at least $-\frac{2}{W}\hat B\phi_k y_k^{p-1}$.

(e3) For $|\Delta x| > 1$, Lemma 5.1(vi) gives at least $-\frac{E_3}{W}\hat B\phi_k y_k^p$.

The η-terms total at most $10^{-8}\hat B p^2\phi_k y_k^{p-2}$ and are absorbed into the last term of (2). Adding (a)–(e) and using $y_k \le 1$ gives (2) for central zeros.

For a non-central zero the same argument applies with $\hat B$ replaced by the counts of Lemmas 3.2 and 3.3 near its own height. Every unit interval meeting $|x| \le \rho$ holds at most $N_\tau(\rho+1) \le (\rho+1)\log(\rho+1)$ zeros, so $\kappa_k \le Cp^2(|x_k|+3)^2$ with an absolute C; any value $C \le 10^{10^9}$ suffices below, because these terms are damped by $e^{-x_0/2W} \le e^{-10^{11}}$.

For the numerical form (an informal simplification, not used in the certified proof of Appendix A): $\hat B \ge 61$ and $W \ge 29.42$, and on $\gamma \ge 3\cdot 10^{12}$ the coefficients of p and of 1 in (2), divided

by $\hat{B}$, are at most their values 14.548 and 4.434 at $\gamma = 3 \cdot 10^{12}$. They decrease with W apart from jumps below 0.004 at integer values of W; on a dense scan of $29.42 \le W \le 200$ the maximum is attained at $W = 29.42$, and the decrease continues beyond. □

Lemma 5.3 (the cut-off error)**.** *In Proposition 5.2,* $|E_R(\tau)| \le C\,p^2\,(x_0+2R+2)^3\,e^{-(R-1)/W}$ *for every* $\tau \in [0,\tau_C]$ *at which the zeros with* $|\operatorname{Re} z - x_0| \le 2R+1$ *are simple, with an absolute constant* C*. In particular* $\sup_\tau |E_R(\tau)| \to 0$ *as* $R \to \infty$*; letting* $R \to \infty$ *is not essential, a fixed large* R *suffices.*

Proof. The negligible pieces are: the terms (a)–(e) of zeros with $|x_k - x_0| \ge R-1$; the third term (d3) of (d); the parts of (e) with χ'; and the segments in (e1) that meet $|x - x_0| \ge R-1$. Each is bounded *after* the pairing used in the proof, never through an individual term of the equation of motion: (a) by $\phi_k y_k^{p-2}$; (b) by $y_k^p/(\Delta x^2 + y_k^2) \le y_k^{p-2}$ per real partner; (c) by $y_k^{p-1}u/(\Delta x^2 + u^2) \le y_k^{p-2}$; (d1) with $\Delta y^2/D^2 \le 1$; (d2) with $|\Delta x|\Delta y/D^2 \le \frac12$; (d3) with $(\chi_k - \chi_m)_+ \Delta y/D^2 \le \min(\frac{3}{4R}, \Delta x^{-2})$ and $\psi_m \le 2e^{-(R-1)/W}$; the χ' parts of (e) by pairing upper zeros with $|\Delta x| \le 1$ (only the Lipschitz constant $6/R^2$ of χ' is used), by $|\operatorname{Re}\frac{2}{z_k - z_m}| \le 1/y_k$ for non-upper partners, and, for $|\Delta x| > 1$, by the $\pm$ pairing of Lemma 5.1(vi). So every negligible piece is at most $Cp^2\psi\, y^{p-2}$ times the number of partners within distance 1, plus convergent far-field sums, with $\psi \le 2e^{-(R-1)/W}$ (the factor p^2 comes from (d1) and (e1)). The zeros involved have $|x - x_0| \le 2R+1$, so $|x| \le x_0 + 2R + 1$. For $|x| \le x_0^2$ the unit intervals are counted by Lemma 3.2 ($\tau_C \le 1/(100\ell_X)$ there); beyond x_0^2, where Lemma 3.2 is not available for $\tau \le \tau_C$, Lemma 3.3 gives at most $\rho \log \rho$ zeros with $|\operatorname{Re} z| \le \rho$. Both counts are polynomial in $x_0 + 2R$, which gives the bound. □

6 Proof of Theorem 1.1

By (P5) we may assume $\gamma > 3 \cdot 10^{12}$. Since $|\operatorname{Im} z| \le 1$ at $\tau = 0$, we have $\delta \le \frac12$. Take $x_0 = 2\gamma$ and $y_0 = 2\delta$; then $F_R(0) \ge \phi(x_0) y_0^p = y_0^p$. Let κ be as in (2) and put

$$\mu = \tfrac12 \hat{B}\,(\pi W + 2).$$

Step 1: Hölder. With exponents $\frac{p}{2}$ and $\frac{p}{p-2}$,

$$\sum_k \kappa_k \phi_k y_k^{p-2} \le \Big(\sum_k \phi_k \kappa_k^{p/2}\Big)^{2/p} F_R^{1-2/p},$$

the sums running over the upper zeros. The central upper zeros contribute at most $\kappa^{p/2}\sum_{\text{central}} \psi_k \le \kappa^{p/2}\mu$, since each unit interval holds at most $\hat{B}/2$ of them and $\sum_j \sup_{[j,j+1]} \psi \le \pi W + 2$. Every other zero has $\psi_k \le 2e^{-|x_k - x_0|/W} \le 2e^{-x_0/(2W)}$, while κ_k and the number of zeros grow only polynomially (Proposition 5.2, Lemma 3.3). So the rest contribute less than $10^{-10^{10}}\kappa^{p/2}\mu$. Hence $\big(\sum_k \phi_k \kappa_k^{p/2}\big)^{2/p} \le (1 + 10^{-100})\,\kappa\,\mu^{2/p}$.

Step 2: differential inequality. Let $G_R = F_R^{2/p}$. Fix $\vartheta \in (0,1)$ and let τ_1 be the first time in $[0,\tau_C]$ with $G_R \le \vartheta y_0^2$ (or $\tau_1 = \tau_C$). On $[0,\tau_1]$, Proposition 5.2 and Step 1 give

$$\frac{dG_R}{d\tau} = \frac{2}{p} F_R^{2/p-1}\frac{dF_R}{d\tau} \;\ge\; -\frac{2}{p}(1 + 10^{-100})\,\kappa\,\mu^{2/p} - \frac{2}{p}(\vartheta y_0^2)^{1-p/2} E_R$$

at every $\tau \in [0,\tau_1]$ outside the set $\mathcal{E}_R$ of times at which H_τ has a multiple zero with $|\operatorname{Re} z - x_0| \le 2R+1$. By Lemma B.2, $\mathcal{E}_R$ is finite. At every $\tau \notin \mathcal{E}_R$ the zeros near the support of ϕ move on smooth branches, so G_R is differentiable there and the inequality holds for its derivative.

Step 3: integration across collisions. By Lemma B.3, F_R, and hence G_R (since $F_R \geq (\vartheta y_0^2)^{p/2} > 0$ on $[0, \tau_1]$), is continuous on $[0, \tau_1]$. Lemma B.1, a mean value inequality for continuous functions whose upper right Dini derivative is bounded below outside a *countable* set, applied with the finite set $\mathcal{E}_R$, gives on $[0, \tau_1]$

$$G_R(\tau) \geq y_0^2 - \Big(\frac{2}{p}(1 + 10^{-100})\kappa\mu^{2/p} + o_R(1)\Big)\tau.$$

Countability is essential: for the Cantor function c, the continuous function $-c$ has derivative 0 outside the Cantor set, which is uncountable, and yet decreases. Here the exceptional set is finite. Let $R \to \infty$ and then $\vartheta \to 0$. For every $\tau < \min\big(\tau_C,\, y_0^2 p/(2(1+10^{-100})\kappa\mu^{2/p})\big)$ we get $F_R(\tau) > 0$, so H_τ has a zero in the upper half-plane and $\tau < \Lambda$. Hence, for every integer $p \geq 2$,

$$\Lambda \;\geq\; T(\gamma)\,\delta^2, \qquad T(\gamma) = \max_{p \geq 2} \frac{2p}{(1 + 10^{-100})\,\kappa(p)\,\mu^{2/p}}, \tag{3}$$

provided $T(\gamma)\delta^2 \leq \tau_C$; this side condition is proved in Step 4.

Step 4: the constant. Put $L = \log\gamma$ and $K_0(\gamma) = 1/(T(\gamma)L\log L)$. Appendix A certifies in ball arithmetic (`certify_constant.py`):

- on $\log(3 \cdot 10^{12}) \leq L \leq 10^6$, split into 10463 cells of relative width 10^{-3}, with the integer p that is optimal at the left end of each cell and every ingredient of κ and μ taken at the unfavourable end of the cell, $K_0 \leq 68.653$, the maximum being attained in the first cell;
- for $L \geq 10^6$, with $p = \lceil 2\log\mu \rceil$ (so $\mu^{2/p} \leq e$), an explicit majorant that is decreasing in L gives $K_0 \leq 52.79$; as $L \to \infty$ the bound tends to $e \cdot 2.07 \cdot 2s_2 = 46.74$.

The side condition $T(\gamma)\delta^2 \leq \tau_C$ holds for every $\gamma \geq 3 \cdot 10^{12}$: since $\kappa \geq \hat{B} s_2 p^2$ and $\min_p p\,\mu^{2/p} = 2e\log\mu$, we have $T(\gamma) \leq 1/(e\,s_2\hat{B}\log\mu)$, while $\delta^2 \leq \frac{1}{4}$, $\hat{B} \geq 2.07\log(2\gamma)$ and $e\,s_2 \cdot 2.07\log\mu \geq 186 > 50$, which gives $T(\gamma)\delta^2 \leq \frac{1}{4}T(\gamma) \leq 1/(200\log x_0) = \tau_C$. Hence $K_0(\gamma) < 68.66 < 70$ for all $\gamma \geq 3 \cdot 10^{12}$, which is Theorem 1.1. □

γ	W	$\hat{B}$	p	$T(\gamma)$	$K_0(\gamma)$
$3 \cdot 10^{12}$	29.42	61.04	19	$1.512 \cdot 10^{-4}$	68.57
10^{15}	35.23	73.06	20	$1.219 \cdot 10^{-4}$	67.08
10^{20}	46.74	96.89	21	$8.707 \cdot 10^{-5}$	65.12
10^{30}	69.77	144.56	22	$5.430 \cdot 10^{-5}$	62.95
10^{50}	115.82	239.88	24	$3.005 \cdot 10^{-5}$	60.90
10^{100}	230.95	478.20	27	$1.356 \cdot 10^{-5}$	58.87
10^{200}	461.21	954.84	30	$6.169 \cdot 10^{-6}$	57.40

Table 1: The explicit constants: $W = \log(3 + 2\gamma)$, $\hat{B} = B(8\gamma)$, the optimal p, $T(\gamma)$ from (3) (so $\Lambda \geq T(\gamma)\delta^2$), and $K_0 = 1/(T\log\gamma\log\log\gamma)$.

Remark 6.1 (where the constant comes from). The dominant term of κ is $s_2 p^2 \hat{B} \approx 4.15 p^2 \hat{B}$ from (d1), the vertical interaction of pairs of upper zeros at distances $\lesssim 1$. With $p \approx 2\log\mu \approx 4\log\log\gamma$ and $\hat{B} \approx 2.07\log\gamma$ this gives $T \approx 2/(e\,s_2\,p\,\hat{B})$. The factor $\log\gamma$ comes from the Jensen count, which is about 26 times the true density $\frac{1}{4\pi}\log\frac{\gamma}{2\pi}$ of zeros of H_0 per unit length. The factor $\log\log\gamma$ comes from allowing arbitrary clusters of other off-line zeros near the given one. Both margins are thin by design, and both are certified: 68.65 against 70 in K_0, and 2.41 against 2.51 in Lemma 3.2.

One may ask whether the quadratic functional $\sum w(x_k)y_k^2$ gives a simpler argument, since its interactions might take a symmetric form. They do: for $p = 2$ and a constant weight, the dynamics (P3) give exactly

$$\frac{d}{d\tau}\sum_{y_k>0} y_k^2 = -2N_+ - \sum_{y_k>0}\sum_{x_m \text{ real}} \frac{4y_k^2}{(x_k-x_m)^2+y_k^2} - \sum_{\{k,m\}} \Big(\frac{4(y_k-y_m)^2}{|z_k-z_m|^2} + \frac{4(y_k+y_m)^2}{|z_k-\bar z_m|^2}\Big),$$

with N_+ the number of upper zeros and the last sum over pairs of upper zeros. Every term has a sign, but the wrong one for a lower bound, and the right side is controlled by the number of upper zeros rather than by the functional itself. This is why we take $p > 2$ and use Hölder's inequality, which converts the counts into a power of F_R.

7 A sharper bound under local regularity

For a single pair, Lemma 4.1 can be integrated with a finer count. Let $z(\tau) = x(\tau) + iy(\tau)$, $y(0) = y_0 \le 1$, and let τ_{abs} be the first time with $y = 0$. On $[0, \tau_{\mathrm{abs}})$ assume:

(I) $z, \bar z$ are the only nonreal zeros with $|\operatorname{Re} w - x(\tau)| \le U$, and the other nonreal zeros w satisfy $\sum 4/|\operatorname{Re} w - x|^2 \le \varepsilon_1$;

(C) the number of real zeros in $[x(\tau)-u, x(\tau)+u]$ is at most $2\varrho u + A$ for $0 < u \le U$, and the real zeros with $|x_k - x| > U$ satisfy $\sum (x-x_k)^{-2} \le \varepsilon_2$.

At $\tau = 0$, (C) holds with $A = O(\log\gamma)$ by Lemma 3.2; numerically $A = 1.7$–2.2 at heights 2000 and 6000 (Section 8). Theorem 7.1 is a statement about a whole trajectory: (I) and (C) are hypotheses on the configuration of the zeros at every $\tau < \tau_{\mathrm{abs}}$, and they cannot be verified from the configuration at $\tau = 0$ alone, since the other zeros move and may approach. Evaluating (I) and (C) at $\tau = 0$ only, as in the simulations below, gives a heuristic prediction, not a bound. Section 9 shows that (C) with $A = O(1)$ at the scale of δ is exactly what the linear regime needs, and that an excess $A \asymp \log\gamma$, which the unconditional count allows, brings the rate back to $\delta^2/\log\gamma$.

Theorem 7.1 (conditional on (I), (C))**.** *Put*

$$a = 1 + 2A, \qquad c = 2\pi\varrho + 2\Big(\frac{2\varrho U + A}{U^2} + \varepsilon_2\Big) + \varepsilon_1, \qquad T(y_0) = \frac{1}{c}\Big(y_0 - \frac{a}{c}\log\Big(1 + \frac{cy_0}{a}\Big)\Big).$$

Then $\tau_{\mathrm{abs}} \ge T(y_0)$ and $\Lambda \ge T(y_0)$. In terms of the zero of ζ, with $\varrho = \frac{1}{4\pi}L$ and $L = \log\frac{\gamma}{2\pi}$,

$$\Lambda \;\ge\; \frac{4\delta}{L} - \frac{4a}{L^2}\log\Big(1 + \frac{L\delta}{a}\Big) \quad (1 + o(1)).$$

Proof. Integration by parts against the counting function gives $\sum_k \frac{1}{(x-x_k)^2+y^2} \le \frac{\pi\varrho}{y} + \frac{A}{y^2} + \frac{2\varrho U + A}{U^2} + \varepsilon_2$. With Lemma 4.1 this gives $\partial_\tau y \ge -a/y - c$ on $[0, \tau_{\mathrm{abs}})$. Put $\varphi(y) = \int_0^y \frac{v\,dv}{a+cv}$. Then $\frac{d}{d\tau}\varphi(y) \ge -1$, so $\varphi(y(\tau)) \ge T(y_0) - \tau$, and letting $\tau \to \tau_{\mathrm{abs}}$ gives $\tau_{\mathrm{abs}} \ge T(y_0)$. For $\tau < \tau_{\mathrm{abs}}$ the function H_τ has a nonreal zero, so $\tau < \Lambda$. □

The bound is linear, $\approx 4\delta/L$, when $L\delta \gg a$, and quadratic, $\approx 2\delta^2/a$, when $L\delta \ll a$. In the linear regime the speed $L/4$ at which δ decreases equals the speed at which the heat flow shifts the effective critical line of the deformed partial sums. Under the flow the n-th Dirichlet term is multiplied by $e^{(\tau/4)\log^2 n}$, the centre term of the cross-energy C_2 of [CE1] becomes $b_n^2 n^{1-2\sigma}/(1 - s + \frac{\tau}{2}\log n)$, and with the gamma factor the shift at height γ is $\frac{\tau}{4}(\log\frac{\gamma}{2\pi} - \log n)$ (Figure 2d; the mechanism is that of [Pm]).

8 Numerical test

Zeros and method. We computed the 10142 zeros of ζ up to height 10^4, with Riemann–Siegel and the C_1 correction; the Lehmer pair is reproduced to $2 \cdot 10^{-6}$. We inserted an off-line pair $x_0 \pm iy_0$ halfway between two consecutive zeros, at heights $\gamma \approx 2000$ and 6000, at five positions each, with $y_0 = 0.4, 0.8, 1.6$. The last value lies outside the strip and serves as a stress test. We integrated (P3) with the 300 nearest zeros on each side exactly and the rest as a frozen field.

Results.

- In all 30 runs, the bound of Theorem 7.1 (with A measured at $\tau = 0$), the simulated absorption time and $y_0^2/2$ come in that order (Figure 1a). The bound is 0.19–0.54 of the simulated time.
- The energy slope S_{rest} at $\tau = 0$ predicts the simulated times with a correlation of 0.998 between logarithms (Figure 1b). The times exceed the prediction by 7–29%, because the pair repels its neighbours and lowers S_{rest} as τ grows.
- The explicit bound of Theorem 1.1 is smaller than the simulated times by a factor of order 10^3 at these heights. The factor $\log \gamma$ is the price of not knowing the local spacing; the factor $\log \log \gamma$ is the price of clusters. Section 9 refines this: the Jensen count costs only a constant, and the obstruction to a linear bound is the local excess of zeros below the mean spacing.

Clusters. Theorem 1.1 allows other off-line zeros near the given one, and the simulations show why this matters. With $y_0 = 0.8$ at $\gamma \approx 2000$, an isolated inserted pair becomes real at $\tau = 0.135$. When two or five further pairs ($y_0 = 0.2$–0.7, within 2.4 units of x) are inserted around it, all zeros become real by $\tau = 0.090$ and 0.086. The other upper zeros attract the top one vertically, and once absorbed they add to the density of real zeros. These times still exceed the conditional bound for the isolated pair (≈ 0.05), but they show that clusters can speed up absorption. Section 9 makes this exact (Proposition 9.1) and shows that it is the local excess of zeros under the pair, real or not, that sets the rate.

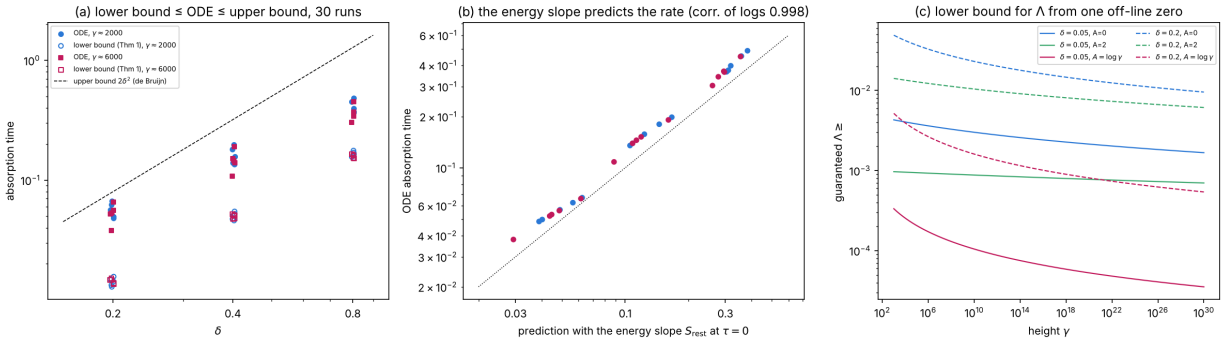


Figure 1: (a) Lower bound of Theorem 7.1 (open), simulated absorption time (filled) and $2\delta^2$, 30 runs. (b) Simulated times against the prediction from the energy slope S_{rest} at $\tau = 0$. (c) The lower bound of Theorem 7.1 for Λ as a function of the height of a hypothetical off-line zero, for $\delta = 0.05, 0.2$ and local excess $A = 0$, 2, $\log \gamma$.

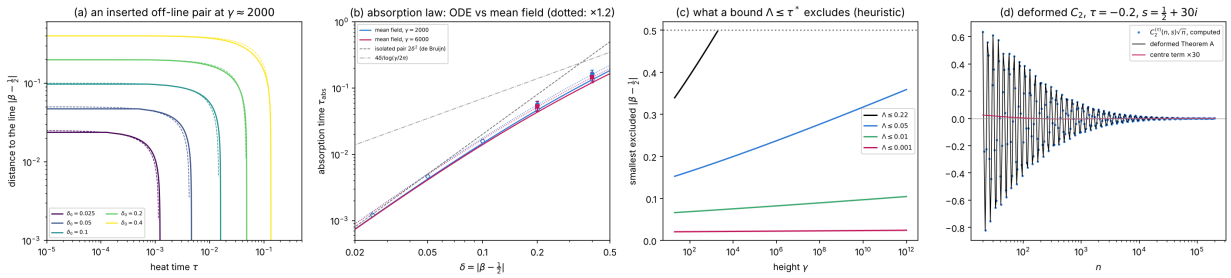


Figure 2: Context. (a) Trajectories of inserted off-line pairs at $\gamma \approx 2000$; dashed: $\partial_\tau y = -1/y - L/2$. (b) Absorption times against the formula of Theorem 7.1 with $A = 0$; dotted: $\times 1.2$, the average excess of the simulations. (c) What an upper bound $\Lambda \le \tau^*$ would exclude by Theorem 7.1 with that factor. (d) The cross-energy C_2 of the deformed partial sums $\sum_{k \le n} e^{(\tau/4)\log^2 k} k^{-s}$ follows the deformed expansion, $\tau = -0.2$.

9 Clusters exactly: a Hermite model and the limits of a linear bound

Theorem 1.1 loses two factors against the linear law of Theorem 7.1: $\log\gamma$, attributed in Section 8 to the Jensen count, and $\log\log\gamma$, attributed to clusters. This section decides which of them is a genuine obstruction to a bound $\Lambda \ge c\,\delta/\log\gamma$. The answer is neither as stated: the Jensen count costs only a constant factor, clusters of off-line zeros behave like clusters of real zeros, and what fixes the rate is the *local* zero count below the mean spacing, which unconditionally allows an excess of order $\log\gamma$. The section contains one exact proposition, a set of measurements with the exact zero dynamics, and an informal remark about proof strategies; it is not a theorem about ζ. Figure 3 collects the data.

The constant of the Jensen count. The certified local count of Lemma 3.2 enters $\hat{B} \approx 2.07\log\gamma$, about 26 times the density of zeros of H_0 per unit length (Section 6). Trudgian's bound $|S(T)| \le 0.112\log T + 0.278\log\log T + 2.510$ [Tr] gives, for the number of zeros of ζ in a short interval $[t-h, t+h]$,

$$\#\{\gamma' \in [t-h, t+h]\} \;\le\; \frac{h}{\pi}\log\frac{t}{2\pi} + A(t), \qquad A(t) = 2\sup_{|t'-t|\le h}|S(t')| + o(1) \approx 0.22\log t,$$

so a sharper count only improves a constant: the excess A is of order $\log\gamma$ in either case. The question is therefore what an excess of order $\log\gamma$ *at scales below the mean spacing* does to the absorption time.

9.1 An exact model: a pair over a cluster

Proposition 9.1 (a pair over a cluster)**.** *Let $K \ge 0$, $y_0 > 0$ and $P(z) = z^K(z^2 + y_0^2)$. The polynomial $e^{-\tau\partial_z^2}P$ has only real zeros if and only if*

$$\tau \;\ge\; \tau_K^* = \frac{y_0^2}{2\,s_K^*}, \qquad s_K^* = \begin{cases} K+1, & K \text{ even}, \\ K+2, & K \text{ odd}, \end{cases} \qquad \text{that is} \qquad \tau_K^* = \frac{y_0^2}{2\,(2\lceil K/2\rceil + 1)}.$$

In particular, for a zero $\frac{1}{2}+\delta+i\gamma$ of ζ ($y_0 = 2\delta$) sitting over K real zeros, K even, the pair reaches the axis at $\tau^* = 2\delta^2/(K+1)$.

Proof. The heat operator acts on monomials through the Hermite polynomials He_n: $e^{-\tau\partial^2} z^n = (2\tau)^{n/2} He_n(z/\sqrt{2\tau})$. With $u = z/\sqrt{2\tau}$ and $s = y_0^2/(2\tau)$ the flowed polynomial is a constant multiple of

$$Q_s = He_{K+2} + s\, He_K = He_K\left(u^2 - (K+1) + s\right) - K\, u\, He_{K-1},$$

by the recurrence $He_{n+1} = uHe_n - nHe_{n-1}$ applied twice. Let ζ_j be the zeros of He_K; they are real, simple and symmetric about 0. From $He_K' = KHe_{K-1}$ and $u\, He_K'/He_K = K + \sum_j \zeta_j/(u-\zeta_j)$, dividing Q_s by He_K and grouping $\pm\zeta_j$ gives, for $v = u^2$ (when K is odd, after removing the factor u common to all terms),

$$v + s - 2K - 1 = \psi(v), \qquad \psi(v) = \sum_{\zeta_j > 0} \frac{2\zeta_j^2}{v - \zeta_j^2}.$$

The function ψ is strictly decreasing between consecutive poles ζ_j^2, so $D(v) = v + s - 2K - 1 - \psi(v)$ is strictly increasing there and has exactly one zero in each gap between poles, one beyond the last pole and one in $(-\infty, \zeta_1^2)$, where ζ_1 is the smallest positive zero. This accounts for all the zeros of the equation in v, which are therefore real. The zeros $u = \pm\sqrt{v}$ are all real if and only if the smallest v is nonnegative, that is, if and only if $D(0) \le 0$. Since $\psi(0) = -2\#\{\zeta_j > 0\}$, which equals $-K$ for even K and $-(K-1)$ for odd K, the condition $D(0) \le 0$ reads $s \le K+1$, respectively $s \le K+2$. Since $s = y_0^2/(2\tau)$ this is the stated inequality for τ. □

The Hermite form of the heat flow on monomials is classical. We did not find Proposition 9.1 in [dB], [Pm] or [RT]; it is elementary, and we record it as not found rather than as new. Two checks (`local_model.py`, `cluster_hermite.py`): the threshold agrees with the smallest τ at which the tridiagonal (Jacobi) matrix of the flowed polynomial has real spectrum for every $K \le 24$, and spreading the cluster over a width $w \le 0.3\, y_0$ instead of placing it at a single point increases τ^* by 0.2–1.6% (9–16% at $w = y_0$).

Three readings of the proposition.

- *The naive estimate is wrong by a factor of two.* If the K zeros stayed at the origin they would pull the pair down with force $2K/y$, and $y\,\dot y = -(2K+1)$ would give $\tau = y_0^2/(2(2K+1))$. The true time is about twice as long, because the cluster opens while the pair descends.

- *Off-line clusters behave like real ones.* Replacing the $K = 2m$ real zeros by m low nonreal pairs (heights $(0.05\text{–}0.3)\, y_0$) lowers τ^* by 4–7%; placing them high ($(0.7\text{–}0.95)\, y_0$) makes absorption 2.0–2.8 times slower. So the cluster of off-line zeros allowed in Theorem 1.1 is a special case of a local excess of zeros, real or not, under the pair: the conjugates act like real zeros.

- *The rate.* With an excess $K = A$ of the size allowed unconditionally, $A \asymp \log\gamma$, the proposition gives $\tau^* \asymp \delta^2/\log\gamma$, not $\delta/\log\gamma$. This is the exact form of the observation of Section 8 that clusters speed up absorption.

9.2 The cluster inside the critical-line background

Proposition 9.1 ignores the other zeros. To include them we integrated the exact zero dynamics $\dot z_k = \sum_{j\ne k} 2/(z_k - z_j)$ of a polynomial whose zeros are an off-line pair $\pm i y_0$, $y_0 = 2\delta$, a cluster of K real zeros spread over a width $0.1\, y_0$ under it, and a lattice of real zeros of density $\varrho = \frac{1}{4\pi}L$, $L = \log\frac{\gamma}{2\pi}$, with 220–300 lattice points on each side (`ode_lattice.py`, `ode_scaling.py`). Truncating the lattice removes far zeros that would pull the pair down, so the measured times are

upper bounds for the corresponding infinite configurations. We used L up to 400, far beyond the heights of the simulations of Section 8, to separate the regimes $\delta L \ll 1$ and $\delta L \gg 1$. The cluster size is $K = \kappa L$ with $\kappa = 0.22$, the size that the bound $|S(T)| \le 0.112 \log T + \dots$ does not exclude at large height.

L	K	δ	δL	$\tau_{\mathrm{abs}} L/\delta$	local model $2\delta L/(K+1)$
50	0	0.05	2.5	2.70	–
400	0	0.05	20	3.71	– (linear law: 4)
50	12	0.02	1	0.148	0.154
400	88	0.02	8	0.169	0.180
50	12	0.05	2.5	0.368	0.385
400	88	0.05	20	0.421	0.449

An isolated pair approaches the linear law $\tau_{\mathrm{abs}} \approx 4\delta/L$ of Theorem 7.1 as δL grows. With the cluster, $\tau_{\mathrm{abs}} L/\delta$ stays near $2\delta/\kappa$ for every L, in agreement with Proposition 9.1 to within 4–7%, *including the regime* $\delta L \gg 1$ where the isolated pair is linear. Taking $\delta = \varepsilon$ and $L \ge \varepsilon^{-2}$ makes $\delta L \to \infty$ while $\tau_{\mathrm{abs}} L/\delta \to 0$. Status: numerical, double precision (`scipy` ODE solver).

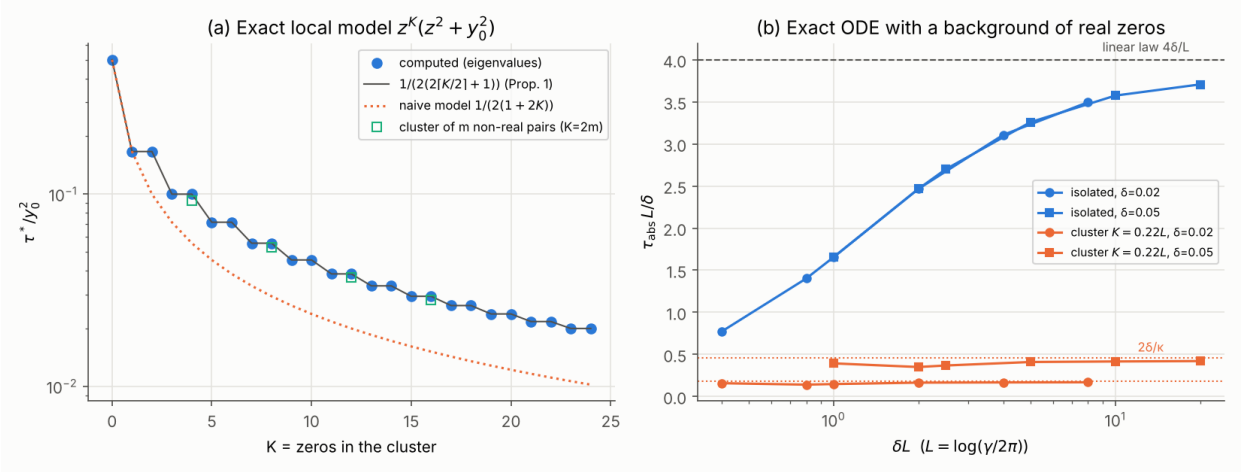


Figure 3: (a) Exact local model: absorption time τ^*/y_0^2 of the pair $\pm iy_0$ over K real zeros, computed from the eigenvalues (dots) against Proposition 9.1 (line) and the naive estimate (dotted); squares: m low nonreal pairs in place of $K = 2m$ real zeros. (b) Exact zero dynamics with a background of real zeros of density $L/4\pi$: an isolated pair approaches the linear law $4\delta/L$; a cluster of $K = 0.22L$ zeros keeps $\tau_{\mathrm{abs}} L/\delta$ at about $2\delta/\kappa$ (dotted) at every δL.

9.3 What a linear bound would need

Remark 9.2 (a barrier for proof strategies; informal). The inputs of Theorem 1.1 are the zero dynamics and a local zero count. Proposition 9.1 and the table show configurations that satisfy the dynamics exactly and a local count with excess $A = \kappa \log \gamma$, and in which the pair is absorbed in time $\asymp \delta^2/\log\gamma$. So no argument that uses only these two inputs can give $\Lambda \ge c\,\delta/\log\gamma$ with an absolute constant c; the best rate compatible with them is $\delta^2/\log\gamma$, which Theorem 1.1 attains up to the factor $\log\log\gamma$. This is a statement about what a proof can use, supported by an exact model and by measurements; it is not a theorem about the zeros of ζ, and it says nothing about whether such clusters occur.

A linear bound would need a local count with bounded excess at the scale δ below the mean spacing, along the whole trajectory of the flow, which is hypothesis (C) of Theorem 7.1 with $A = O(1)$. This is not known even under RH, where the best bound is $S(T) = O(\log T/\log\log T)$ [Ti, Thm. 14.13]. The zeros of ζ themselves are far from the adversarial configuration: among the 4520 zeros below height 5000, no window of length 0.1 contains more than 2 zeros (largest excess 1.91, at $t = 1977.2$), and no window of length 1 more than 3 (excess 2.04 at $t = 2668.3$), while Trudgian's bound allows an excess up to $2(0.112\log T + 0.278\log\log T + 2.510) \approx 8.1$ at $T = 5000$ (`zeta_local_excess.json`). That is data, not a theorem.

Nothing is lost for zero-free regions. Solved for δ as in Corollary 1.2, a linear bound would give $|\beta - \frac{1}{2}| \le C\,\Lambda\log\gamma$, which is trivial (larger than $\frac{1}{2}$) for large γ with any known upper bound, $\Lambda \le 0.2$ [PT], or with the claimed $\Lambda \le 0.174$ of [CE33]. So the barrier costs nothing that a linear bound would have given towards RH. The one improvement left open by this analysis is to remove the factor $\log\log\gamma$ from Theorem 1.1, reaching the rate $\delta^2/\log\gamma$; it would sharpen this Part, not bring RH closer.

The same local-count obstruction appears in Part 19 [CE19], where Turán's power-sum method detects a cluster of off-line zeros in a Weil window at a price exponential in the number of zeros of the cluster, and where a cluster of the size allowed unconditionally pushes the detection length to a power of γ.

10 Genuine off-line zeros: the Davenport–Heilbronn function

The simulations of Section 8 use pairs inserted by hand among the zeros of ζ. To test the dynamics on zeros that really are off the line, we use the Davenport–Heilbronn function

$$f(s) = \tfrac{1}{2}(1 - i\kappa_0)L(s,\chi) + \tfrac{1}{2}(1 + i\kappa_0)L(s,\bar\chi), \qquad \kappa_0 = \frac{\sqrt{10 - 2\sqrt{5}} - 2}{\sqrt{5} - 1},$$

with χ the character mod 5 with $\chi(2) = i$ [DH]. Its Dirichlet coefficients are real. The completed function $\xi_f(s) = (\pi/5)^{-(s+1)/2}\Gamma(\frac{s+1}{2})f(s)$ satisfies $\xi_f(s) = \xi_f(1-s)$, yet f has infinitely many zeros off the critical line, including infinitely many in $\operatorname{Re} s > 1$ [DH], [Ti, Sect. 10.25]; numerical zeros are listed in [Sp, BS]. We put $H_0^f(z) = \xi_f(\frac{1}{2} + \frac{iz}{2})$, which is even, real on $\mathbb{R}$ and of order 1, and define H_τ^f by the Gaussian representation (P4).

The energy-slope time. For a zero $\rho = \beta + i\gamma$ with $\delta = \beta - \frac{1}{2} > 0$, the interaction term of Lemma 4.1 at $\tau = 0$ can be computed from ξ_f alone:

$$S_{\mathrm{rest}}(\rho) = \operatorname{Re}\frac{\xi_f''(\rho)}{2\xi_f'(\rho)} - \frac{1}{2\beta - 1},$$

which is $\partial_\sigma \log|\xi_f(s)/((s-\rho)(s-1+\bar\rho))|$ at $s = \rho$. Freezing S_{rest} in $\partial_\tau\delta = -\frac{1}{4\delta} - \frac{1}{2}S_{\mathrm{rest}}$ gives the *energy-slope time*

$$T_S(\rho) = \int_0^\delta \frac{u\,du}{\frac{1}{4} + \frac{1}{2}S_{\mathrm{rest}}u} = \frac{2\delta}{S_{\mathrm{rest}}} - \frac{1}{S_{\mathrm{rest}}^2}\log\big(1 + 2S_{\mathrm{rest}}\delta\big),$$

which tends to de Bruijn's $2\delta^2$ as $S_{\mathrm{rest}} \to 0$.

Proposition 10.1 (comparison). *Let $z(\tau) = x(\tau) + iy(\tau)$ be a nonreal zero of H_τ (for ξ or ξ_f) that stays simple and obeys Lemma 4.1 on $[0, \tau_{\mathrm{abs}})$. Write that lemma as $\partial_\tau y = -1/y - J(\tau)$ and put $S_{\mathrm{rest}}(\tau) := J(\tau)$, so that $S_{\mathrm{rest}}(0)$ is the quantity above and $\partial_\tau \delta = -\frac{1}{4\delta} - \frac{1}{2} S_{\mathrm{rest}}(\tau)$ with $\delta = y/2$. Assume $\frac{1}{4} + \frac{1}{2} S_{\mathrm{rest}}(0)\, u > 0$ for $0 < u \le \delta(0)$ (automatic if $S_{\mathrm{rest}}(0) \ge 0$). If the interaction term does not increase along the flow, $S_{\mathrm{rest}}(\tau) \le S_{\mathrm{rest}}(0)$, then $\tau_{\mathrm{abs}} \ge T_S(\rho)$.*

Proof. $\partial_\tau \delta \ge -\frac{1}{4\delta} - \frac{1}{2} S_{\mathrm{rest}}(0)$, and the solution of the frozen equation reaches 0 at time T_S. Compare the two as in the proof of Theorem 7.1. □

The hypothesis is natural: a zero leaving the line pushes its neighbours away, which lowers S_{rest}. We do not prove it in general. Note that de Bruijn's bound $2\delta^2$ and T_S differ only through S_{rest}.

Method. We located all zeros of f with $0.505 < \mathrm{Re}\, s < 1.75$ and $1 < \mathrm{Im}\, s < 800$ by the argument principle on unit boxes, and refined them by Newton's method; in every box the winding number equals the number of zeros found. There are no zeros with $\mathrm{Re}\, s \ge 1.75$, since $\sum_{n \ge 2} |a_n| n^{-\sigma} < 1$ there; zeros with $0 < \delta \le 0.005$ are not covered by the scan. The search finds 21 zeros, the first four of which agree with [Sp, BS]. For each zero we tracked the corresponding zero of H_τ^f by Newton's method on the Gaussian integral, with no truncation of the zero set and no frozen field, until it reached the real axis. We obtained τ_{abs} by extrapolating y^2 linearly over the last steps. The measured initial velocity agrees with Lemma 4.1; at $\gamma = 85.70$ it is $\partial_\tau y = -1.90$ against $-1/(2\delta) - S_{\mathrm{rest}} = -1.88$. We also counted the zeros of $\xi_f(\frac{1}{2} + it)$ in $|t - \gamma| < 3$ and compared them with the expected number $\frac{6}{2\pi} \log \frac{5\gamma}{2\pi}$. As a control we did the same in 207 random windows away from off-line zeros: the mean difference is 0.01 and the standard deviation 0.55.

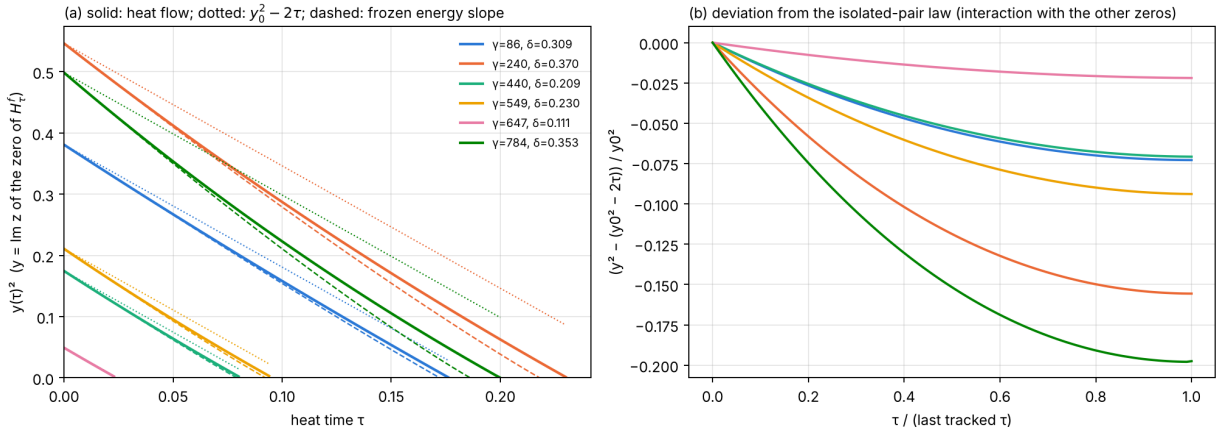


Figure 4: $y(\tau)^2$ for six of the off-line zeros of the Davenport–Heilbronn function under the heat flow, with $y = \mathrm{Im}\, z$ of the corresponding zero of H_τ^f ($y_0 = 2\delta$). (a) The flow (solid), the isolated-pair law $y^2 = y_0^2 - 2\tau$ (dotted) and the frozen energy-slope law $\partial_\tau y = -1/y - S_{\mathrm{rest}}(0)$ (dashed). (b) The relative deviation from the isolated-pair law, which measures the interaction with the other zeros.

Results. Table 2 lists, for each zero, δ, the initial interaction term S_{rest}, the frozen-slope prediction T_S, the computed time τ_{abs} and their ratio; the relative error of the prediction is $1 - T_S/\tau_{\mathrm{abs}} \in [0, 6.8\%]$. For a pair alone, $y^2 = y_0^2 - 2\tau$ exactly; Figure 4 shows $y(\tau)^2$ for six zeros,

γ	δ	$S_{\mathrm{rest}}/\bar{S}$	T_S	τ_{abs}	T_S/τ_{abs}	$2\delta^2$	ΔN
85.70	0.3085	0.123	0.1722	0.1765	0.976	0.1904	−2.0
114.16	0.1508	0.065	0.0442	0.0445	0.993	0.0455	−2.3
166.48	0.0744	0.033	0.0110	0.0110	0.998	0.0111	−1.7
176.70	0.2243	0.115	0.0928	0.0946	0.980	0.1006	−2.7
240.40	0.3695	0.201	0.2180	0.2304	0.946	0.2731	−3.0
320.88	0.3195	0.167	0.1712	0.1786	0.958	0.2042	−2.3
331.05	0.2682	0.164	0.1239	0.1287	0.963	0.1439	−1.3
366.64	0.1285	0.078	0.0318	0.0321	0.991	0.0330	−2.4
411.80	0.3159	0.196	0.1618	0.1705	0.949	0.1996	−2.5
440.48	0.2089	0.128	0.0791	0.0811	0.976	0.0873	−2.6
520.94	0.0159	0.011	0.0005	0.0005	1.000	0.0005	−1.8
531.28	0.3470	0.196	0.1901	0.2011	0.945	0.2408	−1.8
548.91	0.2295	0.154	0.0924	0.0954	0.968	0.1054	−2.8
566.51	0.2866	0.194	0.1345	0.1413	0.952	0.1642	−1.8
595.02	0.0829	0.047	0.0135	0.0136	0.996	0.0137	−1.9
611.78	0.1283	0.146	0.0306	0.0312	0.981	0.0329	−1.9
646.99	0.1108	0.065	0.0238	0.0240	0.993	0.0245	−2.0
657.11	0.2606	0.186	0.1134	0.1186	0.956	0.1358	−2.0
692.89	0.2887	0.188	0.1364	0.1432	0.952	0.1667	−2.0
737.77	0.2774	0.198	0.1253	0.1320	0.949	0.1539	−2.1
783.65	0.3530	0.232	0.1862	0.1998	0.932	0.2492	−2.1

Table 2: All zeros $\beta + i\gamma$ of the Davenport–Heilbronn function with $0.505 < \beta < 1.75$ and $1 < \gamma < 800$ (one of each symmetric quadruple). Here $\delta = \beta - \frac{1}{2}$; $\bar{S} = \frac{1}{2}\log\frac{5\gamma}{2\pi}$ is the mean value of the energy slope; T_S is the energy-slope time; τ_{abs} is the time at which the zero of H_τ^f reaches the real axis; and ΔN is the number of zeros of $\xi_f(\frac{1}{2} + it)$ in $|t - \gamma| < 3$ minus the expected number.

and its deviation from that straight line, 2–20% of y_0^2 at the end, is the interaction with the other zeros. The frozen-slope law follows the computed curves closely, and always lies below them. The prediction from the initial slope and the exact integration of the dynamics are different things: the first uses only $S_{\mathrm{rest}}(0)$, the second follows all the zeros. For all 21 zeros:

- *The energy slope predicts the survival time:* $0.932 \le T_S/\tau_{\mathrm{abs}} \le 1.000$ (mean 0.969), and always $T_S \le \tau_{\mathrm{abs}}$, as Proposition 10.1 predicts when the interaction weakens along the flow.
- *The isolated-pair time is nearly attained:* $\tau_{\mathrm{abs}} \le 2\delta^2$ for each zero, with $0.80 \le \tau_{\mathrm{abs}}/2\delta^2 \le 0.9995$. For a zero that is not the one of maximal imaginary part this is not de Bruijn's theorem; it reflects $S_{\mathrm{rest}} > 0$ along the flow.
- *Every off-line pair sits in a hole.* S_{rest} is only 1–23% of its mean. Within 3 units of γ the line holds on average 2.15 fewer zeros than expected (sample standard deviation 0.42), against 0.01 ± 0.55 in the control; the difference is 17.8 standard errors with the control's spread. Since the expected count includes the off-line pair itself, this says that the *total* count is normal: the pair occupies the two places that two zeros on the line would occupy, as if two zeros had collided and left the line. Consequently the mean-field law of Theorem 7.1, which assumes regular spacing, underestimates the survival time by factors 0.52–0.94. These off-line zeros are isolated, and their dynamics is that of an isolated pair.
- *The bound of Theorem 1.1 is far from these times.* The bound $\delta^2/(70\log\gamma\log\log\gamma)$, which is calibrated for ζ and shown here only for comparison, lies 860–1650 times below τ_{abs}. A

cluster of off-line zeros, which the proof must allow, is the only mechanism we know that could shorten the survival time towards it.

- The tracked zero at $\gamma = 240.40$ reaches the axis at $\tau = 0.2304$, so numerically $\Lambda_f \geq 0.230$; Theorem 10.2 below turns this into a computer-assisted bound.

The de Bruijn–Newman constant of f. Let Λ_f be the infimum of the τ for which all zeros of H_τ^f are real (the generalized de Bruijn–Newman constant of f; for Dirichlet L-functions see [St], and for function fields [AC]).

Theorem 10.2 (computer-assisted)**.** *The de Bruijn–Newman constant of the Davenport–Heilbronn function satisfies*

$$0.228 < \Lambda_f \leq 3.125.$$

Proof. Kernel. Each $L(s,\chi)$ with χ odd has the theta representation $(\pi/5)^{-(s+1)/2}\Gamma(\frac{s+1}{2})L(s,\chi) = \int_0^\infty \sum_n n\chi(n)e^{-\pi n^2 x/5}x^{(s+1)/2}\frac{dx}{x}$. Substituting $x = e^{4u}$ and $s = \frac{1}{2} + \frac{iz}{2}$ gives $H_0^f(z) = \int_{\mathbb{R}} \Phi_f(u)e^{izu}du$, with $\Phi_f(u) = 4\sum_n a_n n\, e^{3u-\pi n^2 e^{4u}/5}$ (the a_n being the real Dirichlet coefficients of f). This kernel decays like $\exp(-\frac{\pi}{5}e^{4u})$ as $u \to +\infty$. The theta function $\theta_f(x) = \sum_n a_n n\, e^{-\pi n^2 x/5}$ of the self-dual combination f satisfies $\theta_f(1/x) = x^{3/2}\theta_f(x)$ (with the root number $+1$; this is equivalent to $\xi_f(s) = \xi_f(1-s)$), which says exactly that Φ_f is even. Hence the same decay holds as $u \to -\infty$. Therefore $H_\tau^f(z) = \int e^{\tau u^2}\Phi_f(u)e^{izu}du$ is even, real on $\mathbb{R}$ and of order 1 for every τ, and agrees with the Gaussian representation (P4).

Upper bound. $|f(s) - 1| \leq \sum_{n\geq 2}|a_n|n^{-\sigma} < 1$ for $\sigma \geq 1.75$, so by the functional equation all zeros of ξ_f satisfy $-0.75 < \operatorname{Re} s < 1.75$. Equivalently, all zeros of H_0^f satisfy $|\operatorname{Im} z| < 2.5$. De Bruijn's theorem [dB, Thm. 13] applies to $\int \Phi(u)e^{izu}du$ with Φ real, even, integrable and $\Phi(u) = O(\exp(-|u|^{2+\varepsilon}))$, which Φ_f satisfies. It gives that H_τ^f has only real zeros for $\tau \geq 2.5^2/2 = 3.125$.

Lower bound. No exclusion of zeros is needed: we exhibit one nonreal zero of $H_{\tau_0}^f$ with a rigorous Rouché argument, and every error (discretisation of the Gaussian integral, truncation, rounding) is enclosed. Suppose $H_{\tau_1}^f$ had only real zeros for some $\tau_1 \leq \tau_0 = 0.228$. Being even, real and of order 1, it would lie in the Laguerre–Pólya class. That class is preserved by $e^{-\tau\partial_z^2}$, $\tau \geq 0$ [Po], [dB, Thm. 13 with $\Delta = 0$], so $H_{\tau_0}^f$ would have only real zeros. We show that $H_{\tau_0}^f$ has a zero in the rectangle $Q = [480.6116, 480.7716] \times [0.0289, 0.1156]$, which lies in the open upper half-plane. Since the set of τ with only real zeros is closed (Hurwitz), it follows that $\Lambda_f > \tau_0$.

Let τ_0 be the double-precision number nearest to 0.228, namely $0.22800000000000000822\ldots > 0.228$. Let $A(z) = h\sum_{|jh|\leq 10} G_{\tau_0}(jh)H_0^f(z + ijh)$ be the trapezoidal approximation to (P4), with exact step $h = 1/16$. All computations below use interval arithmetic with outward rounding (script `ia.py`), and ξ_f is evaluated only in $\operatorname{Re} s \geq \frac{1}{2}$, using the functional equation.

- *Enclosures of A.* The Hurwitz zeta functions are enclosed by Euler–Maclaurin summation with $M = 11$ Bernoulli terms. The remainder bound $|B_{2M}|/(2M)! \cdot |(s)_{2M}|(N + a)^{1-\sigma-2M}/(\sigma + 2M - 1)$ follows from $|\tilde{B}_{2M}| \leq |B_{2M}|$ (compare [Jo]). $\log\Gamma$ is enclosed by Stirling's series with $K = 5$ terms and the remainder bound of [DLMF, Sect. 5.11(ii)].

- $|H_{\tau_0}^f - A|$. The discrete tail $|jh| > 10$ is at most $5 \cdot 10^{-43}$. The trapezoidal error is at most $2M_0/(e^{2\pi/h} - 1) \leq 10^{-40}$ [TW], where $M_0 = \sup_{|b|\leq 1}\int |G_{\tau_0}(w + ib)H_0^f(z - b + iw)|\,dw$ is bounded using $|G_\tau(w + ib)| \leq e^{b^2/4\tau}G_\tau(w)$. Both bounds use a first-order Euler–Maclaurin majorant of $|H_0^f|$ on boxes, and both hold uniformly on ∂Q.

- *Lipschitz control.* By the maximum principle and the majorant, $|A| \le 386.1$ on $Q_{0.15}$ (Q enlarged by 0.15), so $|A'| \le 3861$ on $\partial Q_{0.05}$ by Cauchy's estimate. Enclosures of A at 894 points of $\partial Q_{0.05}$ (spacing 10^{-3}) then give $|A| \le 2.951$ on $\partial Q_{0.05}$, hence $|A'| \le 59.01$ on ∂Q.
- *Winding.* On 330 points of ∂Q with spacing $1.5 \cdot 10^{-3}$, every enclosure of A has modulus $\ge 0.0994 > 59.01 \cdot 1.5 \cdot 10^{-3} = 0.0885$. Therefore A maps each boundary segment into a disc that does not contain 0, and the winding number of A along ∂Q, read off from the enclosures, is exactly 1.
- *Rouché.* On ∂Q, $|A| \ge 0.0994 - 0.0885/2 > 0.055$, while $|H^f_{\tau_0} - A| \le 1.2 \cdot 10^{-14}$. Hence $H^f_{\tau_0}$ has exactly one zero in Q.

The interval arithmetic assumes that the platform's elementary functions (exp, log, cos, sin, arctan) are accurate to 10^{-12}, relative for exp and log and absolute for the others. This is far above their documented accuracy of a few ulp; a spot check against 50-digit references gives errors below $1.2 \cdot 10^{-16}$ (script `libm_check.py`). The tails $10 < |w| \le 60$ and the bound for M_0 are computed in floating point with a relative safety factor, which is negligible against margins of order 10^{40}. Beyond $|w| = 60$ the contributions are below e^{-3400}, since there $G_{\tau_0}(w)\,\Gamma(\frac{\sigma+1}{2}) \le e^{-w^2}$. The whole verification runs in about 100 seconds (script `certify_ia.py`).

Independent verification in Arb. We repeated the whole verification in the Arb ball-arithmetic library (python-flint), which encloses the Hurwitz zeta and $\log\Gamma$ functions rigorously and takes $\tau_0 = 0.228$ exactly (script `certify_flint.py`). That run makes no assumption on elementary functions, and its margins are larger:

- $|A| \le 6.50$ on $\partial Q_{0.15}$, from ball enclosures over boundary segments;
- $|A| \le 1.053$ on $\partial Q_{0.05}$, hence $|A'| \le 21.05$ on ∂Q;
- $21.05 \cdot 1.5 \cdot 10^{-3} = 0.0315 < \min_{\partial Q} |A| = 0.0994$;
- $|H^f_{\tau_0} - A| \le 2 \cdot 10^{-40}$;
- winding number 1.

It runs in about six minutes. □

What this suggests for ζ. If ζ had an off-line zero and behaved like f, the zero would sit in a hole of two missing on-line zeros. For the off-line zero of maximal δ one would then expect $\Lambda \gtrsim 0.9 \cdot 2\delta^2$, far above the worst-case bound of Theorem 1.1. Closing that gap rigorously would need a local zero count with bounded excess at the scale of δ along the flow (Section 9, Remark 9.2), which is not known even under RH.

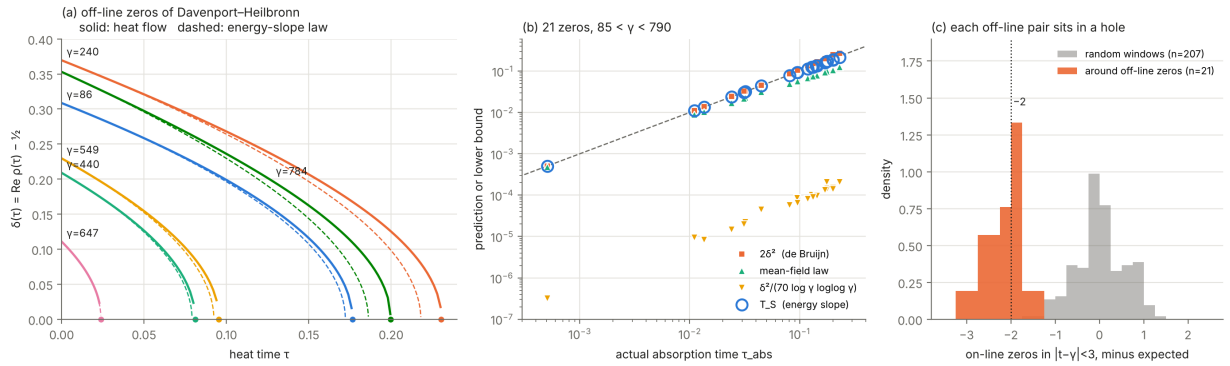


Figure 5: The Davenport–Heilbronn test. (a) Six of the 21 off-line zeros under the heat flow (solid) and the energy-slope law with S_{rest} frozen at $\tau = 0$ (dashed); dots mark τ_{abs}. (b) Predictions and bounds against the measured absorption time; the dashed line is equality. (c) Number of zeros on the line within 3 units of each off-line zero, minus the expected number, against random windows.

11 The area of the quadruples under the heat flow

The quadruple and its area. A zero $\rho = \frac{1}{2} + \delta + i\gamma$ ($\delta, \gamma > 0$) of ζ comes with $\bar{\rho}$, $1-\rho$ and $1-\bar{\rho}$: the four points $\frac{1}{2} \pm \delta \pm i\gamma$ are the vertices of a rectangle centred at $\frac{1}{2}$, of width 2δ and height 2γ, and area

$$A_\rho = 4\delta\gamma = 2\,|\operatorname{Im} \lambda_\rho|, \qquad \lambda_\rho = \rho(1-\rho) = \tfrac{1}{4} + \gamma^2 - \delta^2 - 2i\delta\gamma.$$

The Davenport–Heilbronn function has the same double symmetry: its Dirichlet coefficients are real, so its zeros are symmetric under $s \mapsto \bar{s}$, and $\xi_f(s) = \xi_f(1-s)$, so they are symmetric under $s \mapsto 1-s$. Its off-line zeros therefore also come in rectangles. In the variable of H_τ the quadruple is $\pm z, \pm\bar{z}$ with $z = 2\gamma + 2i\delta$, and since H_τ stays even and real on $\mathbb{R}$ (P2), the zeros of H_τ keep coming in such quadruples for every τ. We define the area of a nonreal zero $z = x + iy$ of H_τ with $x, y > 0$ by

$$A(z) = xy = \tfrac{1}{2} \operatorname{Im}(z^2),$$

which at $\tau = 0$ is A_ρ. In the variable $w = z^2$, which is the variable of Ξ as an entire function of order $\frac{1}{2}$ (and equals $4t_\rho^2$, with $t_\rho := (\rho - \frac{1}{2})/i$, in the notation of Part 11 [CE11]), the quadruple is the conjugate pair $w, \bar{w}$, and its area is $A = \frac{1}{2} \operatorname{Im} w$, a quarter of the distance between w and $\bar{w}$. Under RH every area is zero, and by (P5) the areas of ζ vanish below height $3 \cdot 10^{12}$. In Theorem 1.1, since $\delta = A_\rho/(4\gamma)$, an off-line quadruple of area A at height γ forces

$$\Lambda \;\geq\; \frac{A^2}{1120\,\gamma^2 \log\gamma\, \log\log\gamma}.$$

The area staircase. For the Davenport–Heilbronn function the cumulative area $\sum_{0<\gamma<T} A_\rho$ is a staircase with a step at each off-line zero (Figure 6(a)). Since δ stays in $(0, 0.37)$ while γ grows, the steps grow with the height: the 21 quadruples below 800 (Table 2) have areas from 33.1 (at 520.94, $\delta = 0.0159$) to 1106.5 (at 783.65), mean 414.9, and total 8712. For ζ the staircase is identically zero as far as it has been computed.

The area equation. Write $w_k = z_k^2$, one value for each pair $\pm z_k$.

Proposition 11.1 (the area under the heat flow)**.** *Let $z = x + iy$, $x, y > 0$, be a simple zero of H_τ with all zeros simple, and $w = z^2 = u + iv$. Then*

$$\partial_\tau w = 2 + 8\,w \sum_k{}' \frac{1}{w - w_k},$$

the sum over the values $w_k \neq w$, absolutely convergent. Consequently

$$\frac{d}{d\tau}A^2 = -2\,\mathrm{Re}(z^2) \;-\; 16A^2 \sum_{x_k>0 \text{ real}} \frac{x_k^2}{|z^2 - x_k^2|^2} \;+\; 8A \sum_{\substack{\text{other quadruples} \\ w_k,\ \bar w_k}} \mathrm{Im}\Big(\frac{w_k}{w - w_k} + \frac{\bar w_k}{w - \bar w_k}\Big).$$

The first term comes from the conjugate partner $\bar z$, the second from the real zeros, and the third from the other off-line quadruples.

Proof. By (P3), $\partial_\tau z = 2\sum'(z - z_m)^{-1}$. The zero $-z$ contributes $\frac{1}{z}$, and each other pair $\pm z_m$ contributes $\frac{2}{z - z_m} + \frac{2}{z + z_m} = \frac{4z}{w - w_m}$. Hence $\partial_\tau w = 2z\partial_\tau z = 2 + 8w\sum'\frac{1}{w - w_m}$. The partner $\bar w$ gives $8w/(2iv)$, whose imaginary part is $-4u/v$. For any other w_k, $\frac{w}{w - w_k} = 1 + \frac{w_k}{w - w_k}$, and for real $w_k = x_k^2$, $\mathrm{Im}\,\frac{w_k}{w - w_k} = -\frac{x_k^2 v}{|w - w_k|^2}$. Since $A^2 = v^2/4$, $\frac{d}{d\tau}A^2 = \frac{v}{2}\,\mathrm{Im}\,\partial_\tau w$, which gives the three terms with $v = 2A$. Absolute convergence follows from $|w - w_k| \asymp x_k^2$ for large x_k and $\sum x_k^{-2} < \infty$. □

Proposition 11.1 is Lemma 4.1 written in the variable $w = z^2$, and contains the same information; what changes is the reading.

- *The self term is a clock.* For a quadruple alone, $\frac{d}{d\tau}A^2 = -2\,\mathrm{Re}(z^2) = -2(x^2 - y^2)$, and $\mathrm{Re}\,z^2$ moves at a speed of order 1 while $\mathrm{Re}\,z^2 \approx 4\gamma^2$, so on the time scale δ^2 it is constant. The area squared is consumed at the constant rate $2\,\mathrm{Re}\,z_0^2 \approx 8\gamma^2$ and vanishes at $A_0^2/(2\,\mathrm{Re}\,z_0^2) = 2\delta^2\gamma^2/(\gamma^2 - \delta^2)$, which is de Bruijn's $2\delta^2$ up to a factor $1 + O(\delta^2/\gamma^2)$.
- *Every zero on the line destroys area.* The second term is negative, with weight $x_k^2/|z^2 - x_k^2|^2$; no real zero, however placed, can slow the collapse.
- *Only off-line quadruples can protect it.* The third term is the only one that can be positive. This is the cluster mechanism of Section 9 in the language of the area.

A global gap. Since H_τ is even and of order 1, $w \mapsto H_\tau(\sqrt{w})$ is entire of order $\frac{1}{2}$ and $H_\tau = c_0(\tau)\prod_k(1 - w/w_k)$ with $H_\tau(z) = \sum_m c_m(\tau)z^{2m}$, $c_m(\tau) = \frac{(-1)^m}{(2m)!}\int_0^\infty u^{2m}e^{\tau u^2}\Phi(u)\,du$. Put

$$\mathcal{A}(\tau) = \frac{1}{2}\sum_k \Big(\frac{1}{|w_k|^2} - \mathrm{Re}\,\frac{1}{w_k^2}\Big) = \sum_k \Big(\mathrm{Im}\,\frac{1}{w_k}\Big)^2 = \sum_k \frac{4A_k^2}{|w_k|^4} \;\geq 0,$$

the sum over all w_k (both members of each conjugate pair). Then:

- the holomorphic half is explicit: $\sum_k w_k^{-2} = (c_1/c_0)^2 - 2c_2/c_0$, a function of τ given by three moments of $e^{\tau u^2}\Phi$;
- H_τ has no zeros on $i\mathbb{R}$, since $H_\tau(iy) = \int_0^\infty e^{\tau u^2}\Phi(u)\cosh(yu)\,du > 0$; so $\mathcal{A}(\tau) = 0$ if and only if all zeros of H_τ are real, and

$$\Lambda = \inf\{\tau :\ \mathcal{A}(\tau) = 0\}, \qquad \mathrm{RH} \iff \mathcal{A}(0) = 0.$$

The proof is the Hadamard factorization in w (genus 0), whose logarithm gives $\sum w_k^{-1} = -c_1/c_0$ and $\sum w_k^{-2} = (\sum w_k^{-1})^2 - 2c_2/c_0$, and the identity $|q|^2 - \operatorname{Re} q^2 = 2(\operatorname{Im} q)^2$. At $\tau = 0$ this is the gap of Part 11 [CE11, Prop. 30] (with $w = 4t_\rho^2$, so the two gaps differ by the factor 16). So Λ is the time at which the squared-area gap closes; since $\mathcal{A}(\tau) = 0$ exactly when all zeros are real, this restates the definition of Λ and gains nothing by itself. Of its two halves, one is an explicit function of τ and the other, $\sum |w_k|^{-2}$, is the only one that requires the zeros; this is where the difficulty sits.

Numerical test. We integrated the zero dynamics in the variable z (fourth-order Runge–Kutta, step $\min(2 \cdot 10^{-3}, 0.02y^2)$) for the Davenport–Heilbronn function with its 204 zeros on the line and its 5 off-line quadruples below height 300, and recorded the three terms of Proposition 11.1. The truncation of the zero set does not matter at this precision: the absorption times are 0.1766, 0.0445, 0.0110, 0.0947, 0.2311 against 0.1765, 0.0445, 0.0110, 0.0946, 0.2304 from the exact flow of Section 10.

- The area squared decreases almost linearly (Figure 6(b)); it collapses at 0.85–0.99 of the isolated time $A_0^2/(2\operatorname{Re} z_0^2)$. Over the 21 quadruples below 800 the ratio $\tau_{\text{abs}}/(A_0^2/2\operatorname{Re} z_0^2)$ lies between 0.80 and 0.99.
- At $\tau = 0$ the other zeros add about 16.0%, 4.4%, 1.2%, 12.8% and 39.0% to the self term (the ratio is $2\delta S_{\text{rest}}$, from Table 2), and the other quadruples 0.02% to 0.19% (Figure 6(c)).
- Along all five flows the third term stays negative (its largest value is -0.001): in the Davenport–Heilbronn function no quadruple protects another, and the flow consumes every area at least at the isolated rate. This is consistent with the hole of two missing zeros around each off-line pair found in Section 10: the quadruples are isolated.

Status: Proposition 11.1 and the gap identity are proved (elementary, given (P2)–(P3)); the measurements are numerical (`dh_area_flow.py`, `area_fig32.py`).

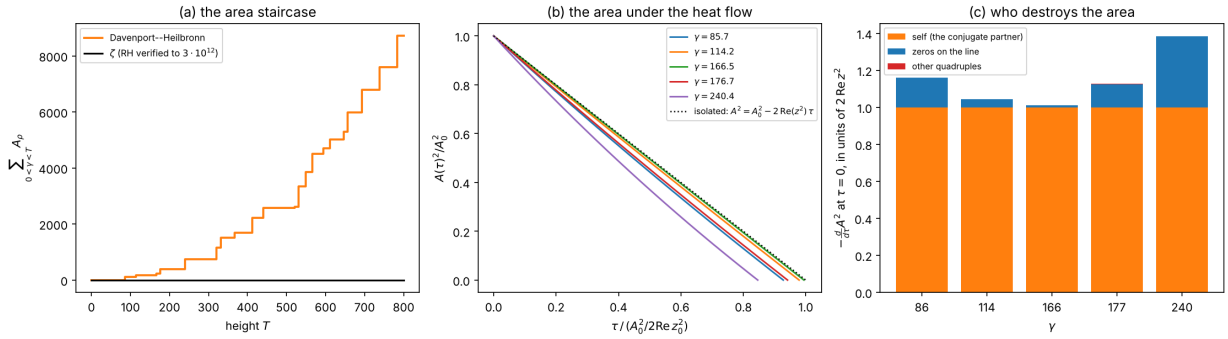


Figure 6: The area of the off-line quadruples. (a) The area staircase $\sum_{0<\gamma<T} 4\delta\gamma$ of the Davenport–Heilbronn function (21 quadruples below 800), and that of ζ, identically zero as far as computed. (b) $A(\tau)^2/A_0^2$ under the heat flow for the five quadruples below 300, against τ in units of the isolated time $A_0^2/(2\operatorname{Re} z_0^2)$; dotted, the isolated law. (c) The three terms of Proposition 11.1 at $\tau = 0$, in units of the self term $2\operatorname{Re} z^2$: self (orange), real zeros (blue), other quadruples (red, too small to see).

12 The arithmetic memory of the zeros under the heat flow

Landau's formula. For a fixed integer $n \geq 2$, Landau's formula [La], in Gonek's uniform form [Go], gives

$$\sum_{T<\gamma\leq T+H} n^{i\gamma} = -\frac{H}{2\pi}\frac{\Lambda(n)}{\sqrt{n}} + O_n(\log T), \qquad 1 \leq H \leq T,$$

the sum running over the zeros $\frac{1}{2}+i\gamma$ of ζ (we write it for zeros on the line; it holds for $\sum n^{\rho-1/2}$ in general). Averaged over the $\sim \frac{H}{2\pi}\log\frac{T}{2\pi}$ zeros in the window, $\langle n^{i\gamma}\rangle \approx -\Lambda(n)/(\sqrt{n}\log\frac{T}{2\pi})$. Geometrically, $n^{i\gamma}$ is the complex conjugate of the position of the zero on the circle traced by the local Euler factor $p^{-s}/(1-p^{-s})$ on the critical line when $n=p$. Landau's formula says that this angle is biased, and only at prime powers: the zeros remember the primes and nothing else. We ask what happens to this memory under the heat flow. Write $x_k(t)$ for the zeros of H_t and $\gamma_k(t) = x_k(t)/2$, so that $\gamma_k(0) = \gamma_k$.

Proposition 12.1 (the memory is erased)**.** *Let $0 < t \leq \frac{1}{2}$ and $n \geq 2$. There are absolute constants $C, c > 0$ such that, for $T \geq \max(e^{C/t}, 2\pi n^2)$ and $1 \leq H \leq T$,*

$$\sum_{T<\gamma_k(t)\leq T+H} n^{i\gamma_k(t)} \;\ll\; \frac{\log T}{\log n} + H\log n\,\log T\,T^{-ct}.$$

In particular, the average of $n^{i\gamma_k(t)}$ over $T < \gamma_k(t) \leq 2T$ is $O(\log n \cdot T^{-ct})$, whereas at $t=0$ it is $\sim -\Lambda(n)/(\sqrt{n}\log T)$, which is nonzero only for prime powers n.

Proof. Let $g(x,t) = \frac{x}{4\pi}\log\frac{x}{4\pi} - \frac{x}{4\pi} + \frac{11}{8} + \frac{t}{16}\log\frac{x}{4\pi}$, and for large m let ξ_m be the solution of $g(\xi_m, t) = m$. By [Pm, Thm. 1.5(i)], for $x \geq e^{C_0/t}$ every zero of H_t with real part x is real and lies within $O(x^{-ct})$ of some ξ_m, where C_0 is the absolute constant of [Pm]. By [Pm, Thm. 1.5(ii)], for each index $m \geq e^{C_0/t}$ there is exactly one zero in the disk $|z-\xi_m| \leq c/\log\xi_m$. Since $g(x,t) \geq e^{C_0/t}$ for $x \geq 4\pi e^{C_0/t}$, and $4\pi \leq e^{2/t}$, the second statement applies for $x \geq e^{C/t}$ with $C = C_0 + 2$. Hence for $x \geq e^{C/t}$ every zero of H_t is real and simple, and the zeros are in one-to-one correspondence with the ξ_m, with $x_k = \xi_m + O(x^{-ct})$. Replacing each x_k in $(2T, 2T+2H]$ by its ξ_m changes $\sum e^{ix_k\log n/2}$ by $\ll \log n \sum_k x_k^{-ct} \ll H\log T\log n\,T^{-ct}$, since the window contains $\ll H\log T$ zeros [Pm, Thm. 1.5(iv)]. It changes the range of m by $O(1)$ terms.

For the smooth sum, write $e^{i\xi_m\log n/2} = e^{2\pi i F(m)}$ with $F(m) = \xi_m\log n/(4\pi)$. Then

$$F'(m) = \frac{\log n}{4\pi\,\partial_x g(\xi_m,t)} = \frac{\log n}{\log(\xi_m/4\pi) + \pi t/(4\xi_m)},$$

which is decreasing in m and lies in $[\log n/(2\log T), \frac{1}{2}]$ since $T \geq 2\pi n^2$. By the Kusmin–Landau inequality [GK, Thm. 2.1], $\left|\sum_m e^{2\pi i F(m)}\right| \leq \cot(\pi\lambda/2) \ll 1/\lambda$ with $\lambda = \log n/(2\log T)$. □

Proposition 12.1 is qualitative: c is not explicit, and it gives no main term. The numerics below suggest that the memory decays at an exact rate depending on n, with a rotation, and that it *appears* at composite n.

The conjectured law. Put $\alpha_*(\gamma) = \frac{1}{2}\log\frac{\gamma}{2\pi} + \frac{i\pi}{4}$, which is the conjugate of the leading term of $\alpha(s) = (\log M_0)'(s)$ of [Pm] at $s = 1 - i\gamma$. Define the deformed coefficients

$$c_m(t,\gamma) = b_m^t\, m^{-(t/2)\alpha_*(\gamma)} = \exp\Big(\frac{t}{4}\log^2 m - \frac{t}{2}\,\alpha_*(\gamma)\log m\Big),$$

and the deformed von Mangoldt function $\lambda_t(n;\gamma)$ as the formal coefficients of $-D'/D = \sum_n \lambda_t(n;\gamma)n^{-s}$ for $D(s)=\sum_m c_m m^{-s}$. The series diverges for every s, since $b_m^t = m^{(t/4)\log m}$; it is meaningful only truncated at $m \lesssim (\gamma/2\pi)^{1/2}$, as in [Pm], and for fixed $n \ll \gamma$ the coefficient $\lambda_t(n;\gamma)$ depends only on c_m with $m \mid n$:

$$\lambda_t(n;\gamma) = c_n \log n - \sum_{d\mid n,\ d<n} \lambda_t(d;\gamma)\, c_{n/d}.$$

Conjecture 12.2 (heat-flow Landau law)**.** *For fixed $0 < t \le \frac12$ and $n \ge 2$, and $T \to \infty$ with $1 \le H \le T$,*

$$\sum_{T<\gamma_k(t)\le T+H} n^{i\gamma_k(t)} = -\frac{1}{2\pi}\int_T^{T+H} \frac{\lambda_t(n;\gamma)}{\sqrt{n}}\, d\gamma + O_{n,t}(\log^2 T).$$

Conjecture 12.2 is proved in Section 13 (Theorem 13.1), with the error term $O_{n,t}(\log T + \frac{H}{T}\log^2 T)$. We keep the conjecture and its heuristic derivation here, as they were found first.

Consequences.

- At $t=0$, $c_m = 1$ and $\lambda_0 = \Lambda$, which is Landau's formula.
- For a prime, $\lambda_t(p;\gamma) = \log p \cdot p^{-(t/4)(\log(\gamma/2\pi p)+i\pi/2)}$. The memory of p decays like $p^{-(t/4)\log(\gamma/2\pi p)}$, a power $T^{-(t/4)\log p}$ of the height consistent with Proposition 12.1, and it rotates by the angle $-\frac{\pi}{8}t\log p$.
- The weights $b_m^t = e^{(t/4)\log^2 m}$ are not multiplicative: $c_{ab} = c_a c_b\, e^{(t/2)\log a \log b}$. So composite numbers acquire a memory that ζ does not have (provably for squarefree n and, for small t, for n with two prime factors; numerically in all cases tested; Proposition 13.8). For example $\lambda_t(6;\gamma) = c_2c_3\log 6\,(e^{(t/2)\log 2\log 3}-1)$, which is $\approx \frac{t}{2}\log 2\log 3\log 6$ for small t. Under the flow, the zeros see the arithmetic of the deformed Dirichlet series $\sum b_m^t m^{-s}$ rather than that of the Euler product.

Heuristic derivation. By [Pm, Thm. 1.3], $H_t(x+iy)/B_t(x+iy) = \sum_{m\le N} b_m^t m^{-s_*} +$ (mirror term) + (error) for $0 \le y \le 1$, with $s_* = \frac{1+y-ix}{2} + \frac{t}{2}\alpha(\frac{1+y-ix}{2})$. Integrate $(H_t'/H_t)(z)\, n^{iz/2}$ around the rectangle $[2T, 2T+2H]\times[-1,1]$, which contains every zero with $T<\gamma_k(t)\le T+H$, since $|\operatorname{Im} z_k| < 1$ for $t>0$.

- *Top edge.* Because H_t is real on $\mathbb{R}$, $(H_t'/H_t)(\bar z) = \overline{(H_t'/H_t)(z)}$. On the top edge the integrand therefore oscillates like $e^{ix(\log n+\log m)/2}$ and contributes $O(\log T)$.
- *Bottom edge.* Expanding $\log\sum_m b_m^t m^{-s_*} = \sum \kappa_m m^{-s_*}$, the term $m=n$ is stationary. After conjugation it contributes $-\frac{1}{2\pi}\int \lambda_t(n;\gamma)n^{-1/2}d\gamma$, with $\lambda_t(n) = \log n\cdot\kappa_n\cdot n^{-(t/2)\bar\alpha}$, which is the recursion above. The other terms of the bottom edge oscillate.
- *Vertical edges.* The usual Borel–Carathéodory argument gives $O(\log^2 T)$.

From the heuristic to a proof. The main term is of size $H\,T^{-(t/4)\log n}$, so the error has to be controlled to a relative precision $o(T^{-(t/4)\log n})$. This needs no new saddle-point analysis. On the line $y=1$ the relative error of [Pm, Thm. 1.3] is already $O_t(x^{-1}\log^2 x)$ (by [Pm, (23), (24)]; [Pm, p. 9] states only the cruder bound $e_A, e_B, e_{C,0} = O(x^{-ct})$, uniform in t, while for fixed t (23) gives $e_A + e_B \ll_t \ell^2/x$ and (24) gives $e_{C,0} \ll x^{-1/2}e^{-(t/16)\ell^2}$, with $\ell = \log\frac{x}{4\pi}$, which decays faster than

any power of x; the slow rate x^{-ct} in [Pm, Thm. 1.5] comes from $f_t - 1$, the deviation of the main sum [Pm, (14)] from 1, not from the approximation error), and the mirror term is $O(x^{-1/2})$ and oscillates. Integrating by parts on the horizontal sides, the error enters only through $\log(H_t/B_t)$, so no bound for its derivative is needed. The details are in Section 13.

Numerical evidence. We moved the 10142 zeros of ζ below height 10^4 forward under the dynamics (P3) up to $t = 0.8$. This is an approximation: the 40 nearest neighbours of each zero are treated exactly and the far field is frozen. We then compared the averages of $n^{i\gamma_k(t)}$ over $1000 < \gamma_k < 9000$ with the averaged prediction $\langle -\lambda_t(n;\gamma)/(\sqrt{n}\log\frac{\gamma}{2\pi})\rangle$. The real and imaginary parts agree within 10^{-3} for all $n \in \{2,3,4,5,6,7,8,10,11,12,13,15,30\}$ and $t \in \{0.05, 0.1, 0.2, 0.4, 0.8\}$ (Figure 7 and Table 3). In particular, the memory of the composites $6, 10, 12, 15$ is exactly zero at $t = 0$ and grows as predicted.

n	t	measured $\langle n^{i\gamma_k(t)}\rangle$	Conjecture 12.2
2	0.8	$-0.0325 + 0.0069i$	$-0.0326 + 0.0072i$
5	0.4	$-0.0481 + 0.0124i$	$-0.0484 + 0.0125i$
7	0.8	$-0.0150 + 0.0107i$	$-0.0158 + 0.0111i$
13	0.4	$-0.0363 + 0.0154i$	$-0.0365 + 0.0155i$
6	0.4	$-0.0066 + 0.0019i$	$-0.0065 + 0.0019i$
10	0.4	$-0.0078 + 0.0030i$	$-0.0079 + 0.0030i$
15	0.4	$-0.0105 + 0.0048i$	$-0.0104 + 0.0047i$
12	0.8	$-0.0032 + 0.0031i$	$-0.0032 + 0.0032i$

Table 3: The heat-flow Landau law, measured on the zeros of H_t coming from the zeros of ζ with $1000 < \gamma < 9000$. The values of t exceed the range $t \le \frac{1}{2}$ of Conjecture 12.2 and Theorem 13.1 only in the numerics.

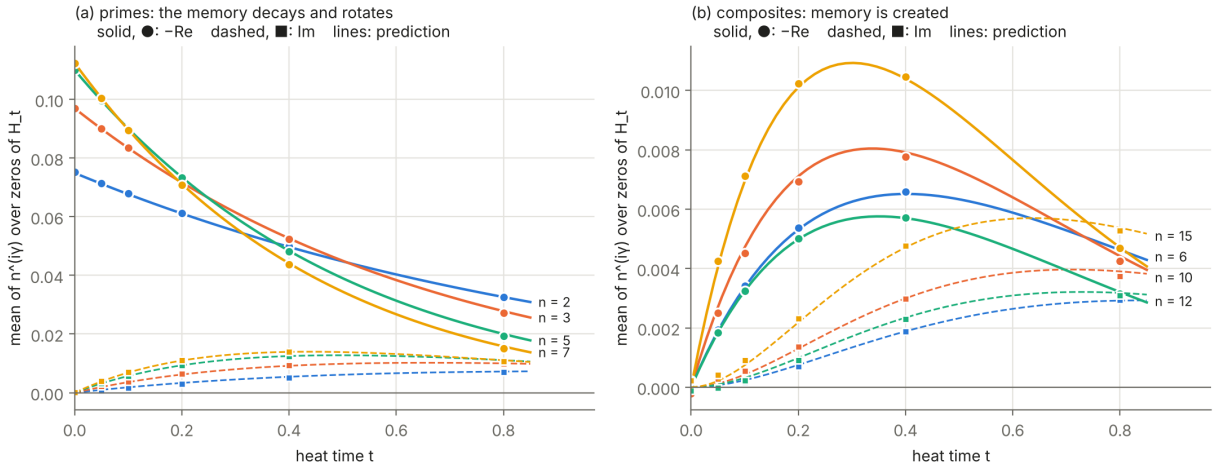


Figure 7: Arithmetic memory of the zeros under the heat flow. (a) Primes: $-\mathrm{Re}$ (solid, dots) and Im (dashed, squares) of $\langle p^{i\gamma_k(t)}\rangle$, measured, and Conjecture 12.2 (lines). (b) The same for the composites $6, 10, 12, 15$, whose memory is created by the flow.

Remark 12.3 (backwards in time). Heuristically, for $t < 0$ the factor $p^{-(t/4)\log(\gamma/2\pi p)}$ amplifies the memory of the primes $p \approx (\gamma/2\pi)^{1/2}$. Their predicted bias exceeds 1 once $\log\frac{\gamma}{2\pi} \gtrsim 4/|t|$, which is

impossible if all the zeros are real. This is consistent with the theorem of Rodgers and Tao that H_t has nonreal zeros for every $t < 0$ [RT]. It is not a proof: Theorem 13.1 is proved only for $0 < t \le \frac{1}{2}$, and this regime is far beyond its range.

13 A Landau formula for H_t

In this section we prove Conjecture 12.2. Throughout, $0 < t \le \frac{1}{2}$ is fixed, $z = x + iy$, $\ell = \log \frac{x}{4\pi}$, $b_m = e^{(t/4)\log^2 m}$ and $g(z) = n^{iz/2} = e^{iz \log n/2}$. We use the notation of [Pm, (6)–(11), (14)–(19)]:

$$M_0(s) = \frac{s(s-1)}{16}\,\pi^{-s/2}\sqrt{2\pi}\,e^{(\frac{s}{2}-\frac{1}{2})\log\frac{s}{2}-\frac{s}{2}}, \qquad \alpha(s) = \frac{1}{2s} + \frac{1}{s-1} + \frac{1}{2}\log\frac{s}{2\pi},$$

$M_t(s) = e^{\frac{t}{4}\alpha(s)^2} M_0(s)$, and $B_t(z) = M_t(\frac{1-iz}{2})$. Note that $\alpha = M_0'/M_0$ and $\mu := M_t'/M_t = \alpha + \frac{t}{2}\alpha\alpha'$. On the line $y = 1$ we write

$$s = 1 - \tfrac{ix}{2}, \quad s' = 1 - s = \tfrac{ix}{2}, \quad s_* = s + \tfrac{t}{2}\alpha(s), \quad s_*' = s' + \tfrac{t}{2}\alpha(s'), \quad \Gamma = \frac{M_t(s')}{M_t(s)}.$$

(In [Pm] the ratio Γ is denoted γ [Pm, (16)], and the second sum below is written $\sum n^y b_n n^{-\overline{s_*}-\kappa_{\mathrm{Pm}}}$, with κ_{Pm} the quantity of [Pm, (18)], so that $n^y n^{-\overline{s_*}-\kappa_{\mathrm{Pm}}} = n^{-s_*'}$. We write Γ and κ_{Pm} to avoid a clash with the height variable γ and with the coefficients κ_n below.)

The deformed von Mangoldt function. Define $\Lambda_t(n)$, $n \ge 1$, by $\Lambda_t(1) = 0$ and

$$b_n \log n = \sum_{d \mid n} \Lambda_t(d)\, b_{n/d}. \tag{4}$$

Equivalently, $\log D_t(s) = \sum_{n\ge 2} \kappa_n n^{-s}$ with $\kappa_n = \Lambda_t(n)/\log n$, as formal Dirichlet series, where $D_t(s) = \sum_m b_m m^{-s}$. The values $\Lambda_t(n)$ are real, $\Lambda_0 = \Lambda$, $\Lambda_t(p) = b_p \log p$, and κ_n depends only on the b_d with $d \mid n$. Since $m \mapsto m^{-(t/2)\alpha_*}$ is completely multiplicative, multiplying (4) by $n^{-(t/2)\alpha_*(\gamma)}$ gives the recursion of Section 12, so that

$$\lambda_t(n;\gamma) = \Lambda_t(n)\, n^{-(t/2)\alpha_*(\gamma)} = \Lambda_t(n)\, n^{-i\pi t/8}\Big(\frac{\gamma}{2\pi}\Big)^{-\frac{t}{4}\log n}. \tag{5}$$

Theorem 13.1 (heat-flow Landau formula)**.** *Let $0 < t \le \frac{1}{2}$ and $n \ge 2$. There is $T_0 = T_0(n,t)$ such that for $T \ge T_0$ and $1 \le H \le T$*

$$\sum_{T < \mathrm{Re}\, z_k/2 \le T+H} n^{iz_k/2} = -\frac{\Lambda_t(n)\, n^{-i\pi t/8}}{2\pi\sqrt{n}} \int_T^{T+H} \Big(\frac{\gamma}{2\pi}\Big)^{-\frac{t}{4}\log n} d\gamma + O_{n,t}\Big(\log T + \frac{H}{T}\log^2 T\Big),$$

where z_k runs over the zeros of H_t, with multiplicity. For $2T \ge e^{C/t}$, with C the absolute constant of [Pm, Thm. 1.5(i)], these zeros are real, so the left side is $\sum_{T<\gamma_k(t)\le T+H} n^{i\gamma_k(t)}$. In particular, Conjecture 12.2 holds.

By (5), the main term is $-\frac{1}{2\pi}\int_T^{T+H} \lambda_t(n;\gamma) n^{-1/2} d\gamma$. It has size $\ll_{n,t} H\, T^{-(t/4)\log n}$, and $\asymp_{n,t} H\, T^{-(t/4)\log n}$ when $\Lambda_t(n) \ne 0$. Theorem 13.1 is an identity with an error term for every n and t; whether the main term dominates the error depends on the exponent $\theta = \frac{t}{4}\log n$.

Corollary 13.2 (when the memory is detected)**.** *Let $0 < t \le \frac{1}{2}$, $n \ge 2$ with $\Lambda_t(n) \ne 0$, and $\theta = \frac{t}{4}\log n$.*

1. *If $\theta < 1$ and $H = H(T) \le T$ satisfies $H/(T^\theta \log T) \to \infty$, the main term of Theorem 13.1 dominates: the sum equals the main term times $1 + o(1)$.*

2. *If $\theta \ge 1$, then for every $1 \le H \le T$ the main term is $O_{n,t}(T^{1-\theta}) = O_{n,t}(1)$, below the error term $O_{n,t}(\log T)$, and the theorem does not detect the memory of n.*

Proof. The main term is $\asymp HT^{-\theta}$. In (1), $HT^{-\theta}/\log T \to \infty$, and $HT^{-\theta}/(\frac{H}{T}\log^2 T) = T^{1-\theta}/\log^2 T \to \infty$. In (2), $HT^{-\theta} \le T^{1-\theta} \le 1$. □

For $t \le \frac{1}{2}$ the detection range $\theta < 1$ is $n < e^{4/t}$, which contains $n < e^8 \approx 2981$ for every admissible t. In particular, in the range of Corollary 13.2(1), the theorem is an asymptotic formula. The case $n = 1$ of the proof (Remark 13.6) gives $(\frac{1}{2\pi}\int_T^{T+H} \log\frac{\gamma}{2\pi} d\gamma)(1+o(1))$ zeros in the window, so in that range (and $\Lambda_t(n) \ne 0$) the mean of $n^{i\gamma_k(t)}$ is $-\lambda_t(n;T)/(\sqrt{n}\log\frac{T}{2\pi})\,(1+o(1))$ when also $H = o(T)$. The first two consequences listed after Conjecture 12.2 are therefore proved: the decay of the memory of p by the factor $p^{-(t/4)\log(\gamma/2\pi p)}$ and the rotation by $-\frac{\pi}{8}t\log p$. The third, a nonzero memory of a composite n, is proved wherever $\Lambda_t(n) > 0$ is proved: for prime powers and for all squarefree n (Proposition 13.8). For the other composites the positivity of $\Lambda_t(n)$, hence the memory, is numerical ($n \le 2 \cdot 10^4$). The constant T_0 is effective but astronomically large: the proof needs $\log\frac{T}{2\pi} \ge 4\log(nK) + 8$ with $K \ge e^{16/t}$, that is $T \ge 2\pi e^8 (nK)^4 \ge 2\pi n^4 e^{8+64/t}$, besides $2T \ge x_1(t,K)$ of Lemma 13.3. This T_0 is effective for the sum over $\operatorname{Re} z_k$; the rewriting as $\sum n^{i\gamma_k(t)}$ also needs $2T \ge e^{C/t}$ and so inherits the unspecified constant C of [Pm]. The numerics below test the formula far below that range. The rate in Proposition 12.1 becomes the exact exponent $\frac{t}{4}\log n$ for fixed n; Proposition 12.1 remains of interest because it is uniform in n.

Lemma 13.3 (the line $y = 1$)**.** *Let $K \ge e^{16/t}$ be an integer and $A_K = \sum_{m\le K} b_m m^{-s_*}$. Write $\frac{1}{\sum_{m\le K} b_m m^{-w}} = \sum_{M\ge 1} \beta_M M^{-w}$ for $\operatorname{Re} w$ large, with real β_M. There is $x_1(t,K)$ such that for $x \ge x_1$ the following hold, with $\sigma = \operatorname{Re} s_*$. First, $\sigma \ge 1 + \frac{t}{4}\ell - \frac{t}{x^2}$, $\operatorname{Re} s_*' \ge 0$, $|\Gamma| \le 2(\frac{x}{4\pi})^{-1/2}$, $\sum_{2\le M\le K} |\kappa_M| M^{-\sigma} + \sum_{M\ge 2} |\beta_M| M^{-\sigma}(1+\log M) \le \frac{1}{4}$, and $|A_K - 1| \le \frac{1}{4}$. Second, $F(x) = H_t(x+i)/B_t(x+i)$ satisfies $|F(x) - 1| \le \frac{1}{2}$, and its principal logarithm $L(x) = \log F(x)$ satisfies*

$$L(x) = \sum_{2\le M\le K} \kappa_M M^{-s_*} + \Gamma \sum_{m\le K} \sum_{M\ge 1} b_m \beta_M\, m^{-s_*'} M^{-s_*} + O_{t,K}\Big(\frac{\ell^2}{x}\Big), \tag{6}$$

where the first sum is finite, the double series converges absolutely, and κ_M are the coefficients of $\log D_t$. (The full series $\sum_{M\ge 2} |\kappa_M| M^{-\sigma}$ diverges, since $\kappa_p = b_p$ grows faster than any power of p; only $M \le K$ is used below.)

Proof. Theorem 1.3 of [Pm] holds in the region $0 < t \le \frac{1}{2}$, $0 \le y \le 1$, $x \ge 200$ [Pm, (5)]; we use it at $y = 1$ and take $x_1 \ge 200$. At $y = 1$ it gives $F = A + \Gamma B + E$ with $A = \sum_{m\le N} b_m m^{-s_*}$, $B = \sum_{m\le N} b_m m^{-s_*'}$, $N = \lfloor\sqrt{x/4\pi + t/16}\rfloor$ [Pm, (19)], and $|E| \le e_A + e_B + e_{C,0}$. The bounds for σ and $|\Gamma|$ are [Pm, (20), (21)], and $\operatorname{Re} s_*' = \frac{t}{2}\operatorname{Re}\alpha(\frac{ix}{2}) = \frac{t}{4}\ell - \frac{t}{2}(1+\frac{x^2}{4})^{-1} \ge 0$. The explicit bounds [Pm, (23), (24)] give $e_A + e_B \ll (1+t^2\ell^2)x^{-1}\sum_{m\le N} b_m m^{-\sigma}$ and $e_{C,0} \ll x^{-1/2}e^{-(t/16)\ell^2}$, so $|E| \ll_t \ell^2/x$.

For $m \le N$ we have $\log m \le \frac{\ell}{2} + x^{-1/3}$, hence $b_m m^{-\sigma} \le m^{-1-\eta}$ and $b_m m^{-\operatorname{Re} s_*'} \le m^{-\eta}$, with $\eta = \frac{t}{8}\ell - x^{-1/3}$. Since $\eta\log K \ge 2\ell - 1$, the terms with $K < m \le N$ contribute $\le \sum_{m>K} m^{-1-\eta} \ll x^{-2}$ to A and $\ll x^{-1/2}\sum_{m>K} m^{-\eta} \ll_K x^{-2}$ to ΓB. Let $u = \sum_{2\le m\le K} b_m m^{-\sigma}(1+$

$\log m) \le \sum_{m\ge 2} m^{-1-\eta}(1+\log m)$, which tends to 0 as $x\to\infty$. Then $|A_K-1|\le u$, $|\Gamma B_K|\le 2K^{1+(t/4)\log K}(\frac{x}{4\pi})^{-1/2}$ with $B_K=\sum_{m\le K} b_m m^{-s'_*}$, and

$$F=A_K(1+w),\qquad w=\frac{\Gamma B_K+E_1}{A_K},\qquad |E_1|\ll_t \frac{\ell^2}{x},\qquad |w|\ll_{t,K} x^{-1/2}.$$

So $|F-1|\le\frac12$ for large x, and $L=\log A_K+\log(1+w)=\log A_K+\Gamma B_K/A_K+O(|E_1|+|w|^2)$. Next, $\log A_K=\sum_{j\ge1}\frac{(-1)^{j-1}}{j}(A_K-1)^j=\sum_{M\ge2}\kappa_M^{(K)}M^{-s_*}$, where $\kappa_M^{(K)}=\kappa_M$ for $M\le K$, because the coefficient of M^{-s} in the logarithm of a Dirichlet series with constant term 1 depends only on the coefficients at the divisors of M. Every product $b_{m_1}\cdots b_{m_j}(m_1\cdots m_j)^{-\sigma}$ with $2\le m_i\le K$ is at most $(m_1\cdots m_j)^{-1-\eta}$, so

$$\sum_{M>K}|\kappa_M^{(K)}|M^{-\sigma}\le K^{-\eta/2}\sum_{j\ge1}\frac1j\big(\zeta(1+\tfrac\eta2)-1\big)^j\le 3(2K)^{-\eta/2}\ll_K x^{-1},$$

using $\zeta(w)-1\le 3\cdot 2^{-w}$ for $w\ge2$ and $\frac{t}{16}\log(2K)\ge1$. The same argument with u bounds the sum of $|\kappa_M|M^{-\sigma}$ over $2\le M\le K$ (where $\kappa_M=\kappa_M^{(K)}$) by $\sum_j j^{-1}u^j$, and the β_M sum by $\sum_{j\ge1}u^j$, since $1/A_K=\sum_{j\ge0}(1-A_K)^j$ and $1+\log(m_1\cdots m_j)\le\prod(1+\log m_i)$. Finally $\Gamma B_K/A_K$ is the double series in (6). □

Lemma 13.4 (local structure)**.** *There are $x_0(t)$ and $C(t)$ with the following properties for $a\ge x_0$ and $z_0=a+i$. (i) H_t has at most $C\log a$ zeros in $|z-z_0|\le4$. (ii) For $|z-z_0|\le2$, $(H_t'/H_t)(z)=\sum_{|z_k-z_0|\le4}(z-z_k)^{-1}+O(C\log a)$. (iii) If no zero of H_t has real part a, then $\big|\int_{-1}^1(H_t'/H_t)(a+iy)\,g(a+iy)\,dy\big|\le C_n\log a$, with C_n depending on n and t.*

Proof. Let $|z-z_0|\le8$, so $|y|\le9$ and $|x-a|\le8$, and let $A_\tau(\sigma)$ be the function of Lemma 3.1, whose subscript is the height. Elementary estimates give $A_\tau(\sigma)\le A_\tau(1)+\frac{\sigma-1}{2}\log\frac{\tau}{2\pi}+\frac{(1+\sigma)^2}{\tau}$ for $\sigma\ge\frac12$. By (P4), the remark after (P5) and Lemma 3.1 with $\tau=\frac{x}{2}$ and $\sigma=\frac12+\frac{|y+w|}{2}$,

$$|H_t(z)|\le\frac18\int G_t(w)\exp\Big(A_{x/2}(1)+\frac{|y+w|-1}{4}\ell+\frac{(3+|y+w|)^2}{x}+\log(4\sqrt{x})+1\Big)dw.$$

Since $\int G_t(w)e^{c|w|+2w^2/x}dw\le 4e^{c^2t(1+16t/x)}$ for $x\ge32$, this gives $\log|H_t(z)|\le-\log8+A_{x/2}(1)+\frac{|y|-1}{4}\ell+\frac{t}{16}\ell^2+\frac12\log x+6$ for large x. On the other hand, Lemma 13.3 gives $|H_t(z_0)|\ge\frac12|M_t(1-\frac{ia}{2})|$. Here $\log|M_0(\sigma+i\tau)|=A_\tau(\sigma)-\log8+\theta$ with $-\sigma^3/(6\tau^2)\le\theta\le0$, as in the proof of Lemma 3.1, since M_0 is $\frac18\cdot\frac12 s(s-1)\pi^{-s/2}$ times the main term of Stirling's formula for the Gamma function at $\frac{s}{2}$; and $\frac{t}{4}\operatorname{Re}\alpha(1-\frac{ia}{2})^2=\frac{t}{16}\ell_a^2-\frac{t\pi^2}{64}+O(\ell_a/a)$ with $\ell_a=\log\frac{a}{4\pi}$. As $|A_{x/2}(1)-A_{a/2}(1)|\le\pi+1$ for $|x-a|\le8$, we get $\log|H_t(z)/H_t(z_0)|\le2\ell+\frac12\log x+12\le3\log a$ on $|z-z_0|\le8$. Jensen's formula then gives (i) with $C=3/\log2$, and Lemma α of [Ti, Sect. 3.9] gives (ii).

For (iii), the segment $\{a+iy:|y|\le1\}$ lies in $|z-z_0|\le2$. By (ii) it suffices to bound $\int_{-1}^1 g(a+iy)(a+iy-z_k)^{-1}dy$ for each of the $O(\log a)$ zeros $z_k=u+iv$ in $|z-z_0|\le4$, where $u\ne a$ and $|v|\le v_0:=\sqrt{1-2t}$ by [dB, Thm. 13]. Since $|g(a+iy)-g(a+iv)|\le\frac12 n^{1/2}\log n\,|y-v|\le\frac12 n^{1/2}\log n\,|a+iy-z_k|$, the integral is $g(a+iv)\int_{-1}^1\frac{dy}{a-u+i(y-v)}+O(n^{1/2}\log n)$. The last integral is $-i\log\frac{a-u+i(1-v)}{a-u-i(1+v)}$, with the principal branch, because the path stays in a half-plane $\pm\operatorname{Re}>0$. Its modulus is at most $\pi+\log\frac{1+v_0}{1-v_0}$. □

Lemma 13.5 (oscillatory integrals)**.** *If $\Psi \in C^2[a,b]$ and $|\Psi'| \ge \lambda > 0$, then $\big|\int_a^b e^{\Psi}dx\big| \le \sup_{[a,b]}|e^{\Psi}|\,\big(\frac{2}{\lambda} + \int_a^b \frac{|\Psi''|}{|\Psi'|^2}dx\big)$. If $\psi \in C^1[a,b]$ and $\lambda \in \mathbb{R}\setminus\{0\}$, then $\big|\int_a^b \psi e^{i\lambda x}dx\big| \le (2\sup|\psi| + \int_a^b |\psi'|)/|\lambda|$.*

Proof. Integrate by parts: $\int e^{\Psi} = [e^{\Psi}/\Psi']_a^b + \int e^{\Psi}\Psi''/\Psi'^2$, and similarly for $\psi e^{i\lambda x}$. □

Proof of Theorem 13.1. Fix n and t and let $K = \max(n, \lceil e^{16/t}\rceil)$. Let T be large, $1 \le H \le T$, $a = 2T$ and $b = 2T + 2H$. Moving a and b to the right by an arbitrarily small amount, we may assume that no zero has real part a or b; the bounds below do not depend on this. By [dB, Thm. 13] (restated as [Pm, Thm. 3.2]) every zero satisfies $|\operatorname{Im} z_k| \le \sqrt{1-2t} < 1$, so by the argument principle

$$\sum_{a<\operatorname{Re} z_k<b} g(z_k) = \frac{1}{2\pi i}\oint_{\partial R} \frac{H_t'}{H_t}(z)\,g(z)\,dz, \qquad R = [a,b]\times[-1,1].$$

By Lemma 13.4(iii) the vertical sides contribute $O_{n,t}(\log T)$.

The horizontal sides. Put $\varphi(x) = (B_t'/B_t)(x+i) = -\frac{i}{2}\mu(s)$, so that $(H_t'/H_t)(x+i) = \varphi(x) + L'(x)$ with L as in Lemma 13.3. Since H_t is real on $\mathbb{R}$, $(H_t'/H_t)(x-i) = \overline{\varphi(x)} + \overline{L'(x)}$. Hence the top and bottom sides contribute

$$I_+ = -\frac{1}{2\pi i}\int_a^b (\varphi + L')\,g(x+i)\,dx, \qquad I_- = \frac{1}{2\pi i}\int_a^b (\overline{\varphi} + \overline{L'})\,g(x-i)\,dx.$$

Here $g(x\pm i) = n^{\mp 1/2}e^{i\lambda x}$ with $\lambda = \frac{1}{2}\log n$, $|\varphi| \ll \ell$ and $|\varphi'| \ll \ell/x$, so Lemma 13.5 bounds the φ terms by $O_n(\log T)$. In the L' terms we integrate by parts:

$$\int_a^b L'g(x+i)\,dx = \big[Lg(x+i)\big]_a^b - i\lambda\int_a^b L\,g(x+i)\,dx,$$

$$\int_a^b \overline{L'}g(x-i)\,dx = \big[\overline{L}g(x-i)\big]_a^b - i\lambda\int_a^b \overline{L}\,g(x-i)\,dx.$$

The boundary terms are $O(n^{1/2})$ since $|L| \le 1$. We insert (6). The error term of (6) contributes $\ll_{n,t} \int_a^b \ell^2 x^{-1}dx \ll \frac{H}{T}\log^2 T$. Each remaining term is e^{Ψ} for an explicit Ψ. We use $\frac{d}{dx}s_* = -\frac{i}{2}(1 + \frac{t}{2}\alpha'(s))$, $\frac{d}{dx}s_*' = \frac{i}{2}(1+\frac{t}{2}\alpha'(s'))$, $\frac{d}{dx}\log\Gamma = \frac{i}{2}(\mu(s)+\mu(s'))$, together with $\alpha'(s), \alpha'(s') = O(1/x)$, $\alpha'' = O(x^{-2})$ and $\mu(s)+\mu(s') = \ell + O(\ell/x)$, which hold because $ss' = \frac{x^2}{4} + \frac{ix}{2}$.

(a) $\kappa_M M^{-s_*}g(x+i)$, $2 \le M \le K$: $\Psi' = \frac{i}{2}\log(nM) + O(\frac{\log M}{x})$ and $\Psi'' = O(\frac{\log M}{x^2})$. Lemma 13.5 gives $O_{n,t}(1)$ in total.

(b) $\kappa_M\overline{M^{-s_*}}g(x-i)$, $2 \le M \le K$, $M \ne n$: $\Psi' = \frac{i}{2}\log\frac{n}{M} + O(\frac{\log M}{x})$, and $|\log\frac{n}{M}| \ge \frac{1}{2K}$. Again $O_{n,t}(1)$ in total.

(c) $\Gamma\, b_m\beta_M m^{-s_*'}M^{-s_*}g(x+i)$ and the conjugate terms times $g(x-i)$, $m \le K$, $M \ge 1$: $\Psi' = \pm\frac{i}{2}\big(\ell + \log\frac{Mn^{\pm 1}}{m}\big) + O(\frac{\ell+\log M}{x})$, so $|\Psi'| \ge \frac{\ell}{4}$ once $\ell \ge 4\log(nK) + 8$, and $\Psi'' = O_K(\frac{1}{x} + \frac{\log M}{x^2})$, since $\mu'(s), \mu'(s') = O(1/x)$ and $\alpha'' = O(x^{-2})$. By Lemma 13.5 each term is $\ll |\Gamma|b_m|\beta_M|M^{-\sigma}n^{1/2}(1+\log M)(\ell^{-1} + \frac{H}{T})$. Summing with Lemma 13.3 gives $O_{n,t}(T^{-1/2})$.

(d) The diagonal term $M = n$ in $\overline{L}\, g(x-i)$. Since $\overline{s_*} = 1+\frac{ix}{2}+\frac{t}{2}\alpha(1+\frac{ix}{2})$, we have $\kappa_n \overline{n^{-s_*}} g(x-i) = \kappa_n n^{-1/2} n^{-(t/2)\alpha(1+ix/2)}$, which does not oscillate. Its contribution to I_- is

$$\frac{-i\lambda}{2\pi i}\int_a^b \kappa_n n^{-\frac{1}{2}-\frac{t}{2}\alpha(1+\frac{ix}{2})}dx = -\frac{\Lambda_t(n)}{2\pi\sqrt{n}}\int_T^{T+H} n^{-\frac{t}{2}\alpha(1+i\gamma)}d\gamma.$$

Finally $\alpha(1+i\gamma) = \alpha_*(\gamma) + O(1/\gamma)$, so replacing $\alpha(1+i\gamma)$ by $\alpha_*(\gamma)$ costs $O_{n,t}(H/T)$, and (5) gives the main term.

Adding up, $\sum_{a<\operatorname{Re} z_k<b} g(z_k)$ equals the main term plus $O_{n,t}(\log T + \frac{H}{T}\log^2 T)$. □

Remark 13.6 (the count). With $g \equiv 1$ the same argument applies: the L terms give $L(b) - L(a) = O(1)$ and the φ terms give $-\frac{1}{\pi}\int_a^b \operatorname{Im}\varphi\, dx = \frac{1}{2\pi}\int_a^b \operatorname{Re}\mu(s)\, dx$, and $\operatorname{Re}\mu(s) = \frac{\ell}{2} + O(\frac{\ell}{x})$. Hence the number of zeros with $T < \operatorname{Re} z_k/2 \le T+H$ is $\frac{1}{2\pi}\int_T^{T+H}\log\frac{\gamma}{2\pi}d\gamma + O_t(\log T + \frac{H}{T}\log T)$. This is consistent with the zero count of [Pm]; for $X \ge e^{C/t}$, with the unspecified absolute constant C, [Pm, Thm. 1.5(iii)] gives the sharper $N_t(X) = g(X,t) + O(1)$ for the number $N_t(X)$ of zeros with real part in $(0, X]$, where $g(x,t)$ is the function of the proof of Proposition 12.1 (not the weight g above).

Remark 13.7 (why $t > 0$ helps). For $t = 0$ the line $y = 1$ is the line $\operatorname{Re} s = 1$, where $\log\zeta$ does not converge absolutely, and Landau's proof moves the contour to $\operatorname{Re} s > 1$. For $t > 0$ the effective variable s_* has $\operatorname{Re} s_* \approx 1 + \frac{t}{4}\log\frac{x}{4\pi}$, which tends to infinity with the height. That is why the logarithm of the deformed Dirichlet sum can be expanded on the line $y = 1$ itself, and why its coefficients carry the factor $n^{-(t/4)\log(x/4\pi)}$: the decay of the memory is the drift of the effective abscissa. The price is that the constants degenerate as $t \to 0$ ($K \ge e^{16/t}$).

13.1 The sign of the deformed von Mangoldt function

After the rotation, the memory has the sign of $-\Lambda_t(n)$. The classical Λ vanishes off the prime powers, while $\Lambda_t(n)$, $t > 0$, is in general nonzero at composite n; so the flow changes the arithmetic structure of the coefficients.

Proposition 13.8 (positivity)**.** *Let $t > 0$.*

1. *$\Lambda_t(p^k) > 0$ for every prime power.*
2. *For squarefree $n = p_1 \cdots p_k$, $\Lambda_t(n)$ is given by the connected-graph expansion* (7) *below, and $\Lambda_t(n) > 0$.*
3. *For every $n \ge 2$, $\Lambda_t(n) = \Lambda(n) + \frac{t}{4}\log n\,\big(\Lambda(n)\log n + (\Lambda * \Lambda)(n)\big) + O_n(t^2)$. Hence, if $n = p^a q^b$ has exactly two distinct prime factors, $\Lambda_t(n) = \frac{t}{2}\log n \log p \log q + O_n(t^2) > 0$ for $0 < t < t_0(n)$; if n has at least three distinct prime factors and is not squarefree, the first-order term vanishes.*

Proof of (1). Indeed, $k \mapsto b_{p^k} = e^{(t/4)k^2\log^2 p}$ is log-convex and equals 1 at $k = 0$, so by Kaluza's theorem [Ka] $1/\sum_k b_{p^k}X^k = 1 - C(X)$ with C having nonnegative coefficients and linear coefficient b_p. Hence $\sum_k \kappa_{p^k}X^k = -\log(1-C(X)) = \sum_{j\ge1} C(X)^j/j$, whose coefficient of X^k is at least b_p^k/k. □

Proof of (2). The sign follows from a cluster expansion. Let $n = p_1\cdots p_k$ with distinct primes and, for $S \subseteq [k] = \{1,\dots,k\}$, $d_S = \prod_{i\in S} p_i$. Since $\log^2 d_S = \sum_{i\in S}\log^2 p_i + 2\sum_{i<j\in S}\log p_i \log p_j$,

$$b_{d_S} = \prod_{i\in S} b_{p_i} \prod_{i<j\in S}(1 + w_{ij}), \qquad w_{ij} = e^{(t/2)\log p_i \log p_j} - 1 > 0.$$

We claim that

$$\Lambda_t(n) = \log n \cdot \prod_{i=1}^{k} b_{p_i} \cdot \sum_{G} \prod_{ij \in G} w_{ij}, \tag{7}$$

the sum running over the connected graphs G on the vertex set $[k]$ (for $k = 1$, the graph with one vertex and empty product 1). In particular $\Lambda_t(n) > 0$ for every squarefree $n \geq 2$ and $t > 0$, and $k = 2$ gives $\Lambda_t(pq) = b_p b_q \log(pq)\,(e^{(t/2)\log p \log q} - 1)$.

Proof of (7). By (4), $\sum_n \Lambda_t(n) n^{-s} = -D_t'/D_t = -(\log D_t)'$, so $\Lambda_t(n) = \kappa_n \log n$ with κ_n the coefficient of n^{-s} in $\log D_t = \sum_{j \geq 1} \frac{(-1)^{j-1}}{j}(D_t - 1)^j$. The Dirichlet series supported on the integers that do not divide n form an ideal, and for squarefree n the quotient algebra is isomorphic to $R = \mathbb{R}[X_1, \ldots, X_k]/(X_1^2, \ldots, X_k^2)$ via $d_S^{-s} \mapsto X^S = \prod_{i \in S} X_i$: indeed $d_S d_{S'}$ divides n exactly when $S \cap S' = \emptyset$, which is when $X^S X^{S'} \neq 0$. Hence κ_n is the coefficient of $X_1 \cdots X_k$ in $\log Z$, where

$$Z = \sum_{S \subseteq [k]} b_{d_S} X^S = \sum_{S} \prod_{i \in S} b_{p_i} X_i \sum_{G \text{ on } S} \prod_{ij \in G} w_{ij},$$

the inner sum running over all graphs on the vertex set S. Grouping each graph by its connected components, which partition S, gives $Z = \sum_\pi \prod_{B \in \pi} c_B X^B$, summed over the sets π of pairwise disjoint nonempty blocks, with $c_B = \prod_{i \in B} b_{p_i} \sum_{G \text{ connected on } B} \prod_{ij \in G} w_{ij}$. In R a product $X^{B_1} \cdots X^{B_r}$ vanishes unless the blocks are disjoint, and each set of r disjoint blocks occurs $r!$ times in $(\sum_B c_B X^B)^r$; so $Z = \exp\big(\sum_{B \neq \emptyset} c_B X^B\big)$ in R (the exponential formula, or Möbius inversion on the Boolean lattice). Thus $\log Z = \sum_B c_B X^B$ and $\kappa_n = c_{[k]}$, which is (7). A check in 50-digit arithmetic against the recursion (4) for $n = 6, 30, 210, 2310$ (1, 4, 38 and 728 connected graphs) at $t = 0.1$ and $t = 0.5$ gives agreement to relative error below 10^{-48} (`lam_sign.py`).

Proof of (3). Since $b_m = 1 + \frac{t}{4}\log^2 m + O_m(t^2)$, $\log D_t = \log \zeta + \frac{t}{4}\frac{\sum_m \log^2 m\, m^{-s}}{\zeta(s)} + O(t^2)$ coefficientwise, so $\kappa_n = \frac{\Lambda(n)}{\log n} + \frac{t}{4}(\mu * \log^2)(n) + O_n(t^2)$, and $(\mu * \log^2)(n) = \Lambda(n)\log n + (\Lambda * \Lambda)(n)$. For $n = p^a q^b$ the only factorizations $n = de$ with $\Lambda(d)\Lambda(e) \neq 0$ are $d = p^a$, $e = q^b$ and the reverse, so $(\Lambda * \Lambda)(n) = 2\log p \log q$; it vanishes when n has three or more distinct prime factors. $\square$

Numerics and an open question. The cases not covered by (1)–(3) are the n that are neither squarefree nor prime powers, at all t for three or more prime factors and at large t for two. In 60-digit arithmetic, $\Lambda_t(n) > 0$ for all such $n \leq 2 \cdot 10^4$ at 12 values $0.001 \leq t \leq 2$, and for all 7-smooth $n \leq 10^{12}$ (which have many repeated factors) at $t = 0.01, 0.1, 0.5, 1, 2$, in 150-digit arithmetic; the first-order law (3) is confirmed at $t = 10^{-4}$ (`lam_sign_ext.py`; the earlier double-precision check `lam_sign.py` covers $n \leq 2 \cdot 10^4$ at five values of t). We have found no sign change. On this evidence the zeros of H_t lean toward the angle $\pi - \frac{\pi}{8} t \log n$ on the circle of every $n \geq 2$, whereas the zeros of ζ lean only on the circles of the prime powers. Whether $\Lambda_t(n) > 0$ for all $n \geq 2$ and $t > 0$ is open; a proof would need an expansion that, unlike (7), handles repeated prime factors, where the quotient algebra is $\prod_i \mathbb{R}[X_i]/(X_i^{k_i+1})$ and the weights $e^{(t/2)a_i a_j \log p_i \log p_j}$ depend on the exponents.

Numerical test with exact zeros. The numerics of Section 12 use an approximate dynamics. To test Theorem 13.1 directly we computed the zeros of H_t on $200 \leq x \leq 18200$ for $t = 0.05, 0.1, 0.2, 0.4$ from the Gaussian representation (P4), with 48-point Gauss–Hermite quadrature in w and ζ by Euler–Maclaurin (`htzeros.py`; relative accuracy 10^{-7} against `mpmath`; at $t = 0$ the routine reproduces the zeros of ζ to 10^{-7}). For each t the routine finds 9065 zeros on this interval, the same number as for ζ, consistent with all of them being real and simple; the smallest gaps are 0.62, 0.85, 1.09 and 1.34. The approximate dynamics of Section 12 puts the zeros within $4 \cdot 10^{-4}$, $9 \cdot 10^{-4}$, $2.2 \cdot 10^{-3}$ and $5.9 \cdot 10^{-3}$ of these (maximum over the zeros; the medians are ten

times smaller), which supports Table 3; rerunning `flow_fwd.py` and `heat_landau.py` reproduces that table. Table 4 compares the sums $S = \sum_{1000<\gamma_k(t)\le 9000} n^{i\gamma_k(t)}$ with the main term M of Theorem 13.1. Over this window the main terms reach 837 in modulus, while $|S - M| \le 3$ for $n \le 15$ and $|S - M| \le 6.7$ for $n = 30$, at all four values of t. Landau's prediction for ζ, $-\frac{H}{2\pi}\Lambda(n)/\sqrt{n}$, misses by up to 582, and the main term without the rotation factor $n^{-i\pi t/8}$ misses by up to 118. For the windows $(1000, 1000 + H]$ with $250 \le H \le 8000$ the largest error over these n lies between 3.2 and 10.1 and does not grow with H, as the error term $O(\log T)$ predicts (`landau_check.py`). The composites $6, 10, 12, 15$ follow the theorem as closely as the primes do. For $n = 30$ the main term is the smallest, $|M| = 1.7$ and 5.7 at $t = 0.05$ and 0.1, and there the error $|S - M| = 6.63$ and 5.96 exceeds $|S| = 5.09$ and 2.04, the distance to Landau's prediction 0. So at these heights the window does not resolve the memory of 30 for small t; only at $t = 0.2$ and 0.4 is S clearly closer to M than to 0 ($|S - M| = 4.89$, 3.40 against $|S| = 11.30$, 27.98). *Status:* numerical, in double precision; the heights are far below $T_0(n, t)$.

	$t = 0.1$		$t = 0.4$	
n	measured S	Theorem 13.1	measured S	Theorem 13.1
2	$-565.7 + 13.3i$	$-563.7 + 15.3i$	$-416.1 + 43.3i$	$-413.9 + 45.2i$
3	$-695.1 + 31.6i$	$-694.9 + 30.0i$	$-438.6 + 78.0i$	$-438.4 + 76.4i$
5	$-750.0 + 47.1i$	$-750.2 + 47.5i$	$-402.8 + 104.0i$	$-402.9 + 104.1i$
7	$-745.7 + 58.1i$	$-746.9 + 57.2i$	$-366.1 + 116.5i$	$-367.4 + 115.9i$
4	$-377.3 + 21.1i$	$-377.3 + 20.6i$	$-230.9 + 51.4i$	$-230.9 + 51.1i$
8	$-265.2 + 22.6i$	$-264.3 + 21.6i$	$-150.5 + 51.9i$	$-149.9 + 50.8i$
9	$-386.6 + 34.4i$	$-387.5 + 33.5i$	$-205.8 + 74.7i$	$-206.8 + 74.3i$
6	$-28.4 + 1.4i$	$-28.1 + 2.0i$	$-54.8 + 15.6i$	$-54.3 + 15.7i$
10	$-37.6 + 4.6i$	$-39.3 + 3.6i$	$-64.6 + 24.8i$	$-65.9 + 24.9i$
12	$-27.0 + 1.8i$	$-27.0 + 2.6i$	$-47.6 + 19.0i$	$-47.5 + 19.5i$
15	$-59.2 + 7.6i$	$-57.8 + 6.2i$	$-87.3 + 39.8i$	$-86.6 + 39.2i$
30	$-0.4 - 2.0i$	$-5.7 + 0.8i$	$-24.4 + 13.7i$	$-27.0 + 15.9i$

Table 4: Theorem 13.1 on the exactly computed zeros of H_t: $S = \sum_{1000<\gamma_k(t)\le 9000} n^{i\gamma_k(t)}$ against the main term $-\frac{\Lambda_t(n) n^{-i\pi t/8}}{2\pi\sqrt{n}} \int_{1000}^{9000} (\frac{\gamma}{2\pi})^{-\frac{t}{4}\log n} d\gamma$. Rows: primes, prime powers, composites; for the composites Landau's formula at $t = 0$ predicts no bias. The same agreement holds at $t = 0.05$ and $t = 0.2$ (`landau_check.out`).

14 Status

- *Proved without unproved hypotheses (implications from a hypothetical zero), explicit, with certified numerical steps:* Lemmas 3.1–3.3, Proposition 5.2, Theorem 1.1 and Corollary 1.2. They use only (P1)–(P5) from [dB, Pm, PT], Stirling's formula and the Euler–Maclaurin formula. The final inequality of Lemma 3.2 and the maximisation over p and γ in (3) are certified in Arb ball arithmetic (Appendix A); the bound $N_\tau(200) \le 750$ is analytic. The integration of the differential inequality across collisions of zeros uses the monotonicity lemma of Appendix B.

- *Unconditional, qualitative:* Proposition 12.1, the heat flow erases the Landau memory at a power rate (from [Pm, Thm. 1.5]).

- *Unconditional, asymptotic:* Theorem 13.1, the heat-flow Landau formula, which proves Conjecture 12.2 for fixed $n \geq 2$ and $0 < t \leq \frac{1}{2}$, with error $O_{n,t}(\log T + \frac{H}{T}\log^2 T)$. It uses [Pm, Thm. 1.3, (20)–(24)], de Bruijn's strip, Lemma 3.1, Jensen's formula and Lemma α of [Ti, Sect. 3.9]. The constant $T_0(n,t)$ is effective for the sum over $\operatorname{Re} z_k$ but astronomically large ($T_0 \geq 2\pi n^4 e^{8+64/t}$); the form with $\gamma_k(t)$ uses the reality of the zeros for $2T \geq e^{C/t}$, with the unspecified constant C of [Pm, Thm. 1.5]. The statement is not uniform in n. The main term dominates only when $\frac{t}{4}\log n < 1$ and the window is long enough (Corollary 13.2). The positivity of the deformed von Mangoldt function, and hence the nonzero memory of n, is proved for prime powers, for all squarefree n, and for small t when n has exactly two distinct prime factors; for the other composites it is numerical (Proposition 13.8 and the checks after it); whether it holds for all n and t is open. The test on the exactly computed zeros of H_t (Table 4) is numerical.

- *Conditional on (I), (C):* Theorem 7.1, the linear regime.

- *Proved, elementary:* Proposition 9.1, the exact absorption time $y_0^2/(2(2\lceil K/2\rceil + 1))$ of a pair over a cluster of K zeros (not found in the literature; the Hermite identity behind it is classical).

- *Numerical:* the worst-case cluster table of Section 9 (exact zero dynamics with a truncated background; the times are upper bounds), and the local excess of the zeros of ζ below height 5000.

- *Informal:* Remark 9.2, no bound $\Lambda \geq c\,\delta/\log\gamma$ can follow from the zero dynamics and a local count with excess $\asymp \log\gamma$; it is a statement about proof strategies, not about ζ. The improvement left open is the removal of the factor $\log\log\gamma$ in Theorem 1.1.

- *Computer-assisted:* Theorem 10.2, $0.228 < \Lambda_f \leq 3.125$ for the Davenport–Heilbronn function, verified independently with outward-rounded interval arithmetic and with Arb ball arithmetic.

- *Numerical, on genuine off-line zeros:* the energy-slope law $T_S \leq \tau_{\mathrm{abs}}$ (with $\tau_{\mathrm{abs}} \leq 2\delta^2$ observed for every zero) and the hole of two missing zeros around each off-line zero of the Davenport–Heilbronn function (Section 10); Proposition 10.1 is conditional on the interaction term not increasing.

- *Heuristic:* the deformed cross-energy picture and the identification of the drift $L/4$ with the shift of the effective critical line.

- *Not used:* a zero-free half-plane $\operatorname{Re} s > \frac{7}{8}$, claimed in an unverified manuscript [OAI], would give (P1) with $\frac{3}{4}$ in place of 1 and slightly better constants; nothing in this paper depends on it.

- *Relation to the literature:* the upper-bound version of the dynamics is used in [Pl, Tao]. The equation of motion (P3) of the zeros goes back to Csordas, Smith and Varga [CSV], whose Lehmer-pair method is the classical route to lower bounds for Λ from configurations of zeros on the line. For the Davenport–Heilbronn function, Dobner's theorem [Do] on Newman's conjecture for the extended Selberg class is the natural reference for $\Lambda_f \geq 0$. As far as we know, lower bounds for Λ in terms of a hypothetical off-line zero, and the link with the energy slope, are new.

- *What this does not do:* prove RH. The bound shows that Λ dominates a distance $\asymp \delta^2/(\log\gamma\log\log\gamma)$ from each zero to the critical line, a distance that tends to 0 with the height. The rigidity proved by Rodgers and Tao, $\Lambda \geq 0$, points the other way.

A Certified numerical inequalities

A.1. The final step of Lemma 3.2 (`certify_lemmaC.py`, Arb through python-flint, 128 bits, about two seconds). Put $u = 1/t_c$, $\ell_c = \log\frac{t_c}{2\pi}$ and $\sigma_{\max} = \frac{1}{2} + \frac{Y_0+R}{2}$, so that $\frac{\sigma_{\max}-\sigma_c}{2} = \frac{R}{4}$. For a point $z = c + Re^{i\theta}$ write $dx = R\cos\theta$, $t = t_c + \frac{dx}{2}$, $\lambda = \log\frac{t}{t_c} \leq \frac{u\,dx}{2}$. For $\frac{1}{2} \leq \sigma' \leq \sigma_y$, Lemma 3.1 gives

$$U_t(\sigma') - \Psi = \frac{R}{4}\ell_c - c(\sigma')\ell_c + m_0(\sigma') + q(\sigma')\lambda - \frac{\pi}{8}dx + Q(\sigma',u) + \frac{1}{3t} + \frac{u}{3} + \frac{\sigma_c^3u^2}{6} - \log\frac{\zeta(3)}{\zeta(3/2)} - N,$$

where $N \geq 0$ collects the terms of Ψ in $|s_c|/t_c$ and $|s_c-1|/t_c$, and where either $m_0 = \log\zeta(\sigma')$, $c = \frac{\sigma_{\max}-\sigma'}{2}$, $q = \frac{\sigma'}{2}+\frac{3}{2}$ (used for $\sigma' \geq 1.1$), or $m_0 = \log 4 + \frac{1}{2}\log 2\pi$, $c = \frac{\sigma_{\max}-\sigma'}{2} - \frac{1}{2}$, $q = \frac{\sigma'}{2}+2$ (from $\log 4\sqrt{t}$, used for $\sigma' \leq 1.1$); in both cases $c \geq 0$, and $Q \leq \big((\sigma'^3+\sigma'^2)/4 + (\sigma'-1)^2/2\big)u^2/(1-\frac{Ru}{2})^2$. Hence, for $\ell_c \geq \ell_a$ and $u \leq u_a$, $-c\,\ell_c \leq -c\,\ell_a$, and $q\lambda - \frac{\pi}{8}dx \leq -(\frac{\pi}{8} - \frac{q_{\max}u_a}{2})dx$ for $dx \geq 0$ and $\leq \frac{\pi}{8}|dx|$ for $dx < 0$. The σ'-range $[\frac{1}{2}, \sigma_{\max}]$ is split into cells of length 10^{-3} (with $\log\zeta$ at the left end, since it decreases), and the circle into 20000 arcs, each with the upper end of σ_y and the lower end of dx; the maximum of $\tilde{U}_t(\sigma_y)$ over each arc is bounded by the running maximum over the σ'-cells below σ_y. The remaining terms are bounded as follows, all by functions nondecreasing in u and nonincreasing in ℓ_c (using $\tau \leq 1/(100\ell_X) \leq 1/(100\ell_c)$):

- $\kappa_1 \leq \frac{\ell_c}{4} + \frac{Ru}{8} + 18.7\frac{u^2}{(1-Ru/2)^2}$, from $\partial_\sigma A_t \leq \frac{1}{2}\log\frac{t}{2\pi} + (0.75\sigma^2+1.5\sigma)/t^2$ for $\sigma \leq \sigma_{\max}+\frac{1}{2}$;
- $a_- \geq \frac{\ell_c}{4} - \frac{1}{2}|\frac{\zeta'}{\zeta}(\frac{3}{2})| - 0.488u^2$, $a_+ \leq \frac{\ell_c}{4} + \frac{1}{2}|\frac{\zeta'}{\zeta}(\frac{3}{2})| + 1.5u^2$ and $b_+ \leq \frac{1}{2}(\frac{\pi}{4} + 2.5u + 0.226u^2 + |\frac{\zeta'}{\zeta}(\frac{3}{2})|)$, from Binet's formula $|\psi(z) - \log z + \frac{1}{2z}| \leq 0.113/(\operatorname{Im} z)^2$ for $\operatorname{Re} z = \frac{3}{4}$, $\operatorname{Im} z \geq 50$;
- $K_2 \leq (\log\zeta)''(1.2) + 3u^2 + \frac{\pi}{4}u$, from $|\psi'(z)| \leq (\operatorname{Im} z)^{-2} + \pi/(2\operatorname{Im} z)$;
- $\varepsilon \leq e^{b_+^2\tau}\big((1-4\beta\tau)^{-1/2}e^{\beta m^2/(1-4\beta\tau)} - 1 + I_{\text{tail}}\big)$, with $\beta = K_2/8$, $m = 2\max(|a_-|, a_+)\tau$, and I_{tail} the Gaussian tail over $|w| > 0.6$ of $|\rho(w)|+1$, where $|\rho(w)| \leq \exp\big(\tilde{U}_{t_c}(\frac{3}{2}+\frac{|w|}{2}) - \Psi + a|w|\big)$ and $\partial_\sigma A_t \leq \frac{1}{2}\log\frac{|s|}{2\pi} + \frac{1}{2} + \frac{3}{4}u$; the tail integrals are bounded by the tangent line of the concave exponent at the end point;
- η_{up}, the region $|w| > 1$ on the circle, in the same way.

At $t_a = 100.25$ ($X = 200$): the σ'-dependent part is at most 0.8978, $-\log\frac{\zeta(3)}{\zeta(3/2)} = 0.7762$, $\log 2$, the u-terms 0.0069, the τ-terms 0.0087, $\varepsilon \leq 0.0237$ and $\eta_{\text{up}} \leq 2\cdot 10^{-31}$, so $g \leq 2.4067 < 2.5 + (2.07 - \frac{R}{4})\ell_a = 2.5088$. Since every bound is monotone in the right direction, this one evaluation covers all $X \geq 200$; at $X = 2\cdot 10^3$, $2\cdot 10^6$ and $2\cdot 10^{12}$ the same script gives $g \leq 2.014$, 1.725 and 1.614.

A.2. The constant 70 (`certify_constant.py`, about one second). With $L = \log\gamma$, every ingredient of κ and μ in (2) and Step 4 is monotone in L: $\hat{B}$, μ, E_3 increase; W increases, so $1/W$, $e^{1/W}/W$ and $1/W^2$ decrease; d_W is bounded with $H_n \leq \log n + \gamma_E + \frac{1}{2n}$, $\lceil W\rceil - 1 \leq W$ and $\lfloor W\rfloor \geq W-1$. On each cell $[L_a, L_b]$ the script evaluates κ and μ with each ingredient at the unfavourable

end, for the integer p that is optimal at L_a (chosen in floating point), and divides by $2pL_a \log L_a$. On $\log(3 \cdot 10^{12}) \le L \le 10^6$ (10463 cells) the maximum is 68.653, in the first cell. For $L \ge 10^6$, with $p = \lceil 2 \log \mu \rceil \le 4 \log L + 2 \log C_3 + 1$, $\mu \le C_3 L^2$, $\hat{B} \le (2.07 + 1.566/L)L$, d_W and $(2 + E_3)/W$ bounded by decreasing majorants, $K_0 \le \frac{e}{2L \log L} + \frac{e\,\hat{B}/L}{2 \log L}\big((s_2 + 10^{-6})(4 \log L + 2 \log C_3 + 1) + c_1 + \frac{c_0}{2}\big)$, which decreases in L and is 52.79 at $L = 10^6$.

A.3. The global count. $N_\tau(200) \le 750$ follows from the analytic bound of Lemma 3.3 and needs no computation.

B Integration across collisions

Lemma B.1 (monotonicity with a countable exceptional set)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous, $E \subset [a, b]$ countable and $c \in \mathbb{R}$. If $D^+ f(\tau) = \limsup_{h \downarrow 0} \frac{f(\tau+h) - f(\tau)}{h} \ge -c$ for every $\tau \in [a, b) \setminus E$, then $f(b) - f(a) \ge -c(b - a)$.*

Proof. Let $\eta > 0$, $g(\tau) = f(\tau) + (c + \eta)\tau$, and suppose $g(b) < g(a)$. Since $g(E)$ is countable, there is $y \in (g(b), g(a)) \setminus g(E)$. Let $\tau^* = \sup\{\tau \in [a, b] : g(\tau) \ge y\}$. By continuity $g(\tau^*) = y$, so $\tau^* < b$ and $\tau^* \notin E$, and $g(\tau) < y$ for $\tau^* < \tau \le b$. Hence $D^+ g(\tau^*) \le 0$, while $D^+ g(\tau^*) = D^+ f(\tau^*) + c + \eta \ge \eta$. This contradiction gives $g(b) \ge g(a)$, and $\eta \to 0$ gives the claim. □

Countability cannot be dropped: minus the Cantor function is continuous, has derivative 0 outside the Cantor set, and decreases.

Lemma B.2 (collision times)**.** *For fixed R, the set $\mathcal{E}_R$ of $\tau \in [0, \tau_C]$ at which H_τ has a multiple zero with $|\operatorname{Re} z - x_0| \le 2R + 1$ is finite.*

Proof. Fix τ_0. By (P1) the zeros in question lie in a compact box; choose a rectangle Q containing $\{|\operatorname{Re} z - x_0| \le 2R + 1,\ |\operatorname{Im} z| \le 1\}$, with horizontal sides on $|\operatorname{Im} z| = 2$ (free of zeros for every $\tau \in [0, \frac{1}{2}]$ by (P1)) and vertical sides that avoid the zeros of H_{τ_0}. Since $H_\tau(z)$ is entire in $(\tau, z) \in \mathbb{C}^2$ (by the double exponential decay of Φ), for τ near τ_0 no zero lies on ∂Q, and the power sums $\frac{1}{2\pi i} \oint_{\partial Q} z^k H_\tau'(z)/H_\tau(z)\, dz$ of the zeros in Q are analytic in τ; so is the discriminant $\Delta(\tau) = \prod_{i<j}(z_i - z_j)^2$ of these zeros, a polynomial in the power sums. $\Delta \not\equiv 0$. Otherwise, by Puiseux expansion of the zeros in Q, there would be an interval of τ on which some zero $z(\tau)$ has constant multiplicity $r \ge 2$ and depends analytically on τ; differentiating $H_\tau^{(r-2)}(z(\tau)) = 0$ and using $\partial_\tau H_\tau = -H_\tau''$ and $H_\tau^{(r-1)}(z(\tau)) = 0$ gives $H_\tau^{(r)}(z(\tau)) = 0$, which contradicts multiplicity r. (This is also the content of the splitting statement in (P3).) So the zeros of Δ are isolated near τ_0. Compactness of $[0, \tau_C]$ gives finiteness. □

Lemma B.3 (continuity of the functional)**.** *For fixed R and $p \ge 2$, F_R is continuous on $[0, \tau_C]$. Near a collision at τ_0 the colliding zeros satisfy $z_j(\tau) = z_0 + O(|\tau - \tau_0|^{1/2})$, so F_R is Hölder continuous of exponent $\frac{1}{2}$ there; it has no jumps.*

Proof. $F_R(\tau) = \sum g(z_k)$ with $g(z) = \phi(\operatorname{Re} z)(\operatorname{Im} z)_+^p$, a continuous function that vanishes outside $\{|\operatorname{Re} z - x_0| < 2R\}$ and on the real axis. With Q as above, Hurwitz's theorem shows that the zeros of H_τ in Q, as a multiset, depend continuously on τ; hence so does the finite sum $\sum_{z_k \in Q} g(z_k)$, and the zeros outside Q contribute 0. The Hölder statement follows from the splitting in (P3) and the Lipschitz continuity of g on $|\operatorname{Im} z| \le 1$, $p \ge 2$. □

C Computational reproducibility

The ancillary files (`anc/README.md`) list every script with its section, input, output and run time. Certified in ball arithmetic (Arb via python-flint): Lemma 3.2 (`certify_lemmaC.py`), the constant 70 (`certify_constant.py`) and Theorem 10.2 (`certify_flint.py`, and independently `certify_ia.py` with outward rounding). Computed in double or multiple precision, not certified: the simulations of Section 8, the Davenport–Heilbronn trajectories and Table 2, the numerics of Sections 12 and 13, and the sign checks of Λ_t (60- and 150-digit `mpmath`). The scripts of Section 9 (`cluster_hermite.py`, `local_model.py`, `ode_lattice.py`, `ode_scaling.py`, `fig_en.py`, with their JSON outputs) are in the second ancillary archive `Caceres_CE35_ancillary_front32a.zip`; they run in double precision. The floating-point scripts `lemmaC.py`, `theorem_constants.py`, `asymptotic.py` and `lowcount.py` of the earlier version are kept for comparison; the certified computations replace them.

Use of AI tools. The computations and the drafting of this paper were carried out with the assistance of an AI system (Claude, Anthropic), under the author's direction. The author checked the results and takes responsibility for them.

References

[CE11] P. Caceres, Cross-Energy of Dirichlet Partial Sums. Part 11: A two-sided criterion for the Riemann hypothesis from weighted prime sums, verified up to $2 \cdot 10^{29}$, preprint, 2026; an earlier version is hexagon:2610.00661.

[AC] J. C. Andrade, A. Chang, S. J. Miller, Newman's conjecture in function fields, *J. Number Theory* 157 (2015), 154–169;

[BS] E. P. Balanzario, J. Sánchez-Ortiz, Zeros of the Davenport–Heilbronn counterexample, *Math. Comp.* 76 (2007), 2045–2049.

[DH] H. Davenport, H. Heilbronn, On the zeros of certain Dirichlet series I, II, *J. London Math. Soc.* 11 (1936), 181–185 and 307–312.

[dB] N. G. de Bruijn, The roots of trigonometric integrals, *Duke Math. J.* 17 (1950), 197–226.

[CSV] G. Csordas, W. Smith, R. S. Varga, Lehmer pairs of zeros, the de Bruijn–Newman constant Λ, and the Riemann hypothesis, *Constr. Approx.* 10 (1994), 107–129.

[Do] A. Dobner, A proof of Newman's conjecture for the extended Selberg class, *Acta Arith.* 201 (2021), 29–62.

[KKL] H. Ki, Y.-O. Kim, J. Lee, On the de Bruijn–Newman constant, *Adv. Math.* 222 (2009), 281–306.

[Jo] F. Johansson, Rigorous high-precision computation of the Hurwitz zeta function and its derivatives, *Numer. Algorithms* 69 (2015), 253–270.

[DLMF] NIST Digital Library of Mathematical Functions, `https://dlmf.nist.gov`, Sect. 5.11.

[GoJ] J. Gomila, de Bruijn–Newman constant: $\Lambda \leq 0.1787854$, online announcement and audit repository (August 2026), `https://judegomila.com/posts/riemann-lambda-0.1787854`; not peer reviewed.

[Gor] S. Gordon, A computer-assisted upper bound for the de Bruijn–Newman constant, preprint and code (2026), `https://github.com/stefangordon/dbn-upper-bound`; not peer reviewed.

[Go] S. M. Gonek, An explicit formula of Landau and its applications to the theory of the zeta-function, *Contemp. Math.* 143 (1993), 395–413.

[GK] S. W. Graham, G. Kolesnik, *Van der Corput's Method of Exponential Sums*, LMS Lecture Note Series 126, Cambridge, 1991.

[Ka] T. Kaluza, Über die Koeffizienten reziproker Potenzreihen, *Math. Z.* 28 (1928), 161–170.

[La] E. Landau, Über die Nullstellen der Zetafunktion, *Math. Ann.* 71 (1912), 548–564.

[Ne] C. M. Newman, Fourier transforms with only real zeros, *Proc. Amer. Math. Soc.* 61 (1976), 245–251.

[OAI] OpenAI, manuscript claiming a zero-free half-plane $\operatorname{Re} s > \frac{7}{8}$ for ζ and Dirichlet L-functions, dated 30 September 2026, released 6 October 2026, with a partial Lean formalization, `https://github.com/openai/math`; not peer reviewed.

[Ol] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, 1974.

[Pl] M. Planat, Collective contraction in the de Bruijn–Newman heat flow: off-zero logarithmic derivatives and certified barrier refinements, preprint, arXiv:2609.37164 (2026).

[PT] D. J. Platt, T. S. Trudgian, The Riemann hypothesis is true up to $3 \cdot 10^{12}$, *Bull. London Math. Soc.* 53 (2021), 792–797.

[Po] G. Pólya, Über trigonometrische Integrale mit nur reellen Nullstellen, *J. Reine Angew. Math.* 158 (1927), 6–18.

[Pm] D. H. J. Polymath, Effective approximation of heat flow evolution of the Riemann ξ function, and a new upper bound for the de Bruijn–Newman constant, *Res. Math. Sci.* 6 (2019), 31; equation numbers as in arXiv:1904.12438v2.

[RT] B. Rodgers, T. Tao, The de Bruijn–Newman constant is non-negative, *Forum Math. Pi* 8 (2020), e6.

[St] J. Stopple, Notes on low discriminants and the generalized Newman conjecture, *Funct. Approx. Comment. Math.* 51 (2014), 23–41.

[Sp] R. Spira, Some zeros of the Titchmarsh counterexample, *Math. Comp.* 63 (1994), 747–748.

[Tao] T. Tao, Polymath15, first thread: computing H_t, asymptotics, and dynamics of zeroes, blog post, 27 January 2018.

[Tr] T. S. Trudgian, An improved upper bound for the argument of the Riemann zeta-function on the critical line II, *J. Number Theory* 134 (2014), 280–292.

[TW] L. N. Trefethen, J. A. C. Weideman, The exponentially convergent trapezoidal rule, *SIAM Rev.* 56 (2014), 385–458.

[Ti] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford, 1986.

[CE1] P. Caceres, Cross-Energy of Dirichlet Partial Sums. Part 1: The C-transform and the cross-energy, preprint, 2026.

[CE17] P. Caceres, Cross-Energy of Dirichlet Partial Sums. Part 17: Jensen's formula, the energy slope and Weil's functional, preprint, 2026.

[CE19] P. Caceres, Cross-Energy of Dirichlet Partial Sums. Part 19: Weil positivity at the windows of the zeros, preprint, 2026.

[CE33] P. Caceres, Cross-Energy of Dirichlet Partial Sums. Part 33: Optimised mollifiers and the bound $\Lambda \leq 0.174$ for the de Bruijn–Newman constant, preprint, 2026.